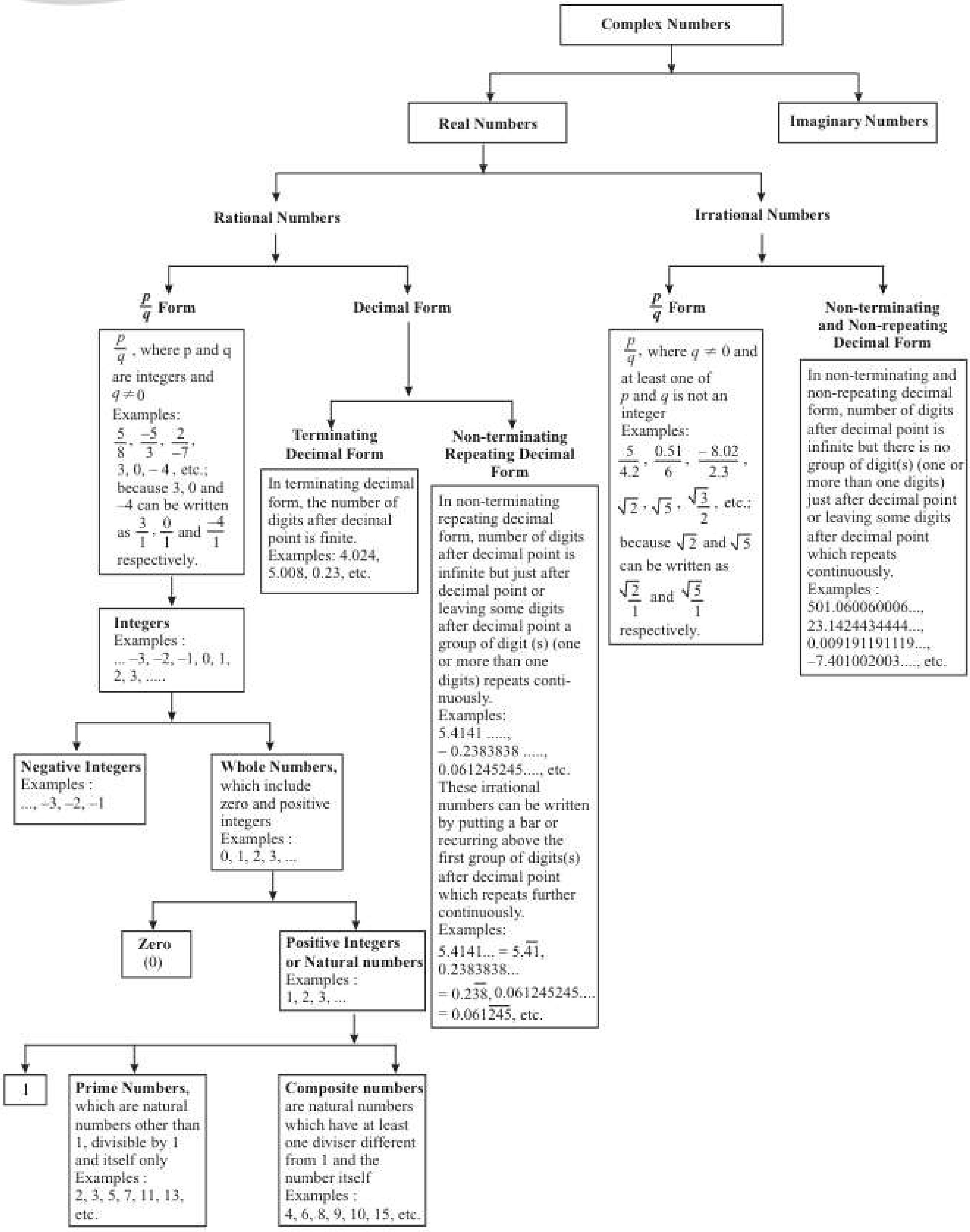
CHAPTER

2

NUMBER SYSTEM

Chart: Classification of Numbers



CONCEPT OF NUMBER LINE (OR NUMBER LINE)

A number line is a straight line from negative infinitive $(-, \infty)$ in left hand side to positive infinitive $(+, \infty)$ in right hand side as given:

$$-\infty \quad -6 \quad -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \infty$$

Each point on the number line represents a unique real number and each real number is denoted by a unique point on the number line.

Symbols of some special sets are:

N : the set of all natural numbers
Z : the set of all integers
Q : the set of all rational numbers
R : the set of all real numbers
Z^+ : the set of positive integers
Q^+ : the set of positive rational numbers, and
R^+ : the set of positive real numbers

The symbols for the special sets given above will be referred to throughout the text.

Even Integers

An integer divisible by 2 is called an even integer. Thus, ..., – 6, – 4, – 2, 0, 2, 4, 6, 8, 10, 12,...., etc. are all even integers. $2n$ always represents an even number, where n is an integer.

For example, by putting $n = 5$ and 8 in $2n$, we get even integer $2n$ as 10 and 16 respectively.

Odd Integers

An integer not divisible by 2 is called an odd integer.

Thus, ..., –5, –3, –1, 1, 3, 5, 7, 9, 11, 13, 15,..., etc. are all odd integers.

$(2n - 1)$ or $(2n + 1)$ always represents an odd number, where n is an integer.

For example by putting $n = 0$, 1 and 5 in $(2n - 1)$, we get odd integer $(2n - 1)$ as – 1, 1 and 9 respectively.

Properties of Positive and Negative Numbers

If n is a natural number then

$$(\text{A positive number})^{\text{natural number}} = \text{A positive number}$$
$$(\text{A negative number})^{\text{even positive number}} = \text{A positive number}$$
$$(\text{A negative number})^{\text{odd positive number}} = \text{A negative number}$$

CONVERSION OF RATIONAL NUMBER OF THE FORM NON-TERMINATING RECURRING DECIMAL INTO THE RATIONAL NUMBER OF THE FORM $\frac{p}{q}$

First write the non-terminating repeating decimal number in recurring form i.e., write

64.20132132132..... as $64.20\overline{132}$

Then using formula given below we find the required $\frac{p}{q}$ form of the given number.

Rational number in the form $\frac{p}{q}$

$$= \frac{\begin{bmatrix}\text{Complete number neglecting} \\ \text{the decimal and bar over} \\ \text{repeating digit(s)}\end{bmatrix} - \begin{bmatrix}\text{Non-recurring part of} \\ \text{the number neglecting} \\ \text{the decimal}\end{bmatrix}}{m \text{ times 9 followed by } n \text{ times 0}}$$

where m = number of recurring digits in decimal part
and n = number of non-recurring digits in decimals part

Thus, $\frac{p}{q}$ form of $64.20\overline{132} = \frac{6420132 - 6420}{99900}$

$$= \frac{6413712}{99900} = \frac{534476}{8325}$$

In short; $0.\overline{a} = \frac{a}{9}$, $0.\overline{ab} = \frac{ab}{99}$, $0.\overline{abc} = \frac{abc}{999}$, etc. and

$$0.a\overline{b} = \frac{ab - a}{90},\ 0.a\overline{bc} = \frac{abc - a}{990},\ 0.ab\overline{c} = \frac{abc - ab}{900},$$

$$0.ab\overline{cd} = \frac{abcd - ab}{9900},\ ab \cdot cd\overline{e} = \frac{abcde - abc}{990}, \text{etc.}$$

Illustration 1: Convert $2.\overline{46102}$ in the $\frac{p}{q}$ form of rational number.

Solution: Required $\frac{p}{q}$ form $= \frac{246102 - 2}{99999} = \frac{246100}{99999}$

Illustration 2: Convert $0.167\overline{3206}$ in the $\frac{p}{q}$ form of rational number.

Solution: Required $\frac{p}{q}$ form $= \frac{1673206 - 167}{9999000} = \frac{1673039}{9999000}$

Illustration 3: Convert 31.026415555 ... into $\frac{p}{q}$ form of rational number.

Solution: First write 31.026415555... as $31.02641\overline{5}$

Now required $\frac{p}{q}$ form $= \frac{31026415 - 3102641}{900000} = \frac{27923774}{900000}$

$$= \frac{13961887}{450000}.$$

DIVISION

$$4\overline{)275}(68$$
$$\underline{24}$$
$$35$$
$$\underline{32}$$
$$3$$

Here 4 is the divisor, 275 is the dividend, 68 is the quotient and 3 is the remainder. Remainder is always less than divisor.

Thus, Divisor) Dividend (Quotient
$\underline{abc}$
Remainder

Thus,

Dividend = Divisor × Quotient + Remainder

For example, 275 = 4 × 68 + 3

When quotient is a whole number and remainder is zero, then dividend is divisible by divisor.

TESTS OF DIVISIBILITY

I. Divisibility by 2:

A number is divisible by 2 if its unit digit is any of 0, 2, 4, 6, 8.

Ex. 58694 is divisible by 2, while 86945 is not divisible by 2.

II. Divisible by 3:

A number is divisible by 3 only when the sum of its digits is divisible by 3.

Ex. (i) Sum of digits of the number 695421 = 27, which is divisible by 3.

$\therefore$ 695421 is divisible by 3.

(ii) Sum of digits of the number 948653 = 35, which is not divisible by 3.

$\therefore$ 948653 is not divisible by 3.

III. Divisible by 4:

A number is divisible by 4 if the number formed by its last two digits i.e. ten's and unit's digit of the given number is divisible by 4.

Ex. (i) 6879376 is divisible by 4, since 76 is divisible by 4.

(ii) 496138 is not divisible by 4, since 38 is not divisible by 4.

IV. Divisible by 5:

A number is divisible by 5 only when its unit digit is 0 or 5.

Ex. Each of the numbers 76895 and 68790 is divisible by 5.

V. Divisible by 6:

A number is divisible by 6 if it is simultaneously divisible by both 2 and 3.

Ex. 90 is divisible by 6 because it is divisible by both 2 and 3 simultaneously.

VI. Divisible by 7:

A number is divisible by 7 if and only if the difference of the number of its thousands and the remaining part of the given number is divisible by 7 respectively.

Ex. 473312 is divisible by 7, because the difference between 473 and 312 is 161, which is divisible by 7.

VII. Divisible by 8:

A number is divisible by 8 if the number formed by its last three digits i.e. hundred's, ten's and unit's digit of the given number is divisible by 8.

Ex. (i) In the number 16789352, the number formed by last 3 digits, namely 352 is divisible by 8.

$\therefore$ 16789352 is divisible by 8.

(ii) In the number 576484, the number formed by last 3 digits, namely 484 is not divisible by 8.

$\therefore$ 576484 is not divisible by 8.

VIII. Divisible by 9:

A number is divisible by 9 only when the sum of its digits is divisible by 9.

Ex. (i) Sum of digits of the number 246591 = 27, which is divisible by 9.

$\therefore$ 246591 is divisible by 9.

(ii) Sum of digits of the number 734519 = 29, which is not divisible by 9.

$\therefore$ 734519 is not divisible by 9.

IX. Divisible by 10:

A number is divisible by 10 only when its unit digit is 0.

Ex. (i) 7849320 is divisible by 10, since its unit digit is 0.

(ii) 678405 is not divisible by 10, since its unit digit is not 0.

X. Divisible by 11:

A number is divisible by 11 if the difference between the sum of its digits at odd places from right and the sum of its digits at even places also from right is either 0 or a number divisible by 11.

Ex. (i) Consider the number 29435417.

(Sum of its digits at odd places from right) – (Sum of its digits at even places from right)

$(7 + 4 + 3 + 9) - (1 + 5 + 4 + 2) = (23 - 12) = 11$, which is divisible by 11.

$\therefore$ 29435417 is divisible by 11.

(ii) Consider the number 57463822.

(Sum of its digits at odd places) – (Sum of its digits at even places)

$= (2 + 8 + 6 + 7) - (2 + 3 + 4 + 5) = (23 - 14)$

$= 9$, which is neither 0 nor divisible by 11.

$\therefore$ 57463822 is not divisible by 11.

XI. Divisible by 12:

A number is divisible by 12, if it is simultaneously divisible by both 3 and 4.

Illustration 4: Find the least value of * for which 7* 5462 is divisible by 9.

Solution: Let the required value be x. Then,

$(7 + x + 5 + 4 + 6 + 2) = (24 + x)$ should be divisible by 9.

$\Rightarrow \qquad x = 3$

Illustration 5: Find the least value of * for which 4832*18 is divisible by 11.

Solution: Let the digit in place of * be x.

(Sum of digits at odd places from right) – (Sum of digits at even places from right)

$= (8 + x + 3 + 4) - (1 + 2 + 8 = (4 + x)$,

which should be divisible by 11.

$\therefore \qquad x = 7$.

PRIME NUMBERS

A number other than 1 is called a prime number if it is divisible by only 1 and itself.

All prime numbers less than 100 are:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

Note that 2 is the smallest prime number. 2 is the only even prime number.

Smallest odd prime number is 3.

Twin Primes: A pair of prime numbers are said to be twin prime when they differ by 2. For example 3 and 5 are twin primes.

Co-primes or Relative primes: A pair of numbers are said to be co-primes or relative primes to each other if they do not have any common factor other than 1. For example 13 and 21.

Some Properties which Help in Finding Two Co-prime Numbers

(i) Two consecutive natural numbers are always co-prime.

Ex. 8 and 9 are co-prime.

Also 12 and 13 are co-prime.

(ii) Two consecutive odd integers are always co-prime.

Ex. 7, 9; 15, 17; 21, 23; etc.

(iii) Two prime numbers are always co-prime.
Ex. 19 and 23 are co-prime.
Also 29 and 41 are co-prime.

(iv) A prime number and a composite number such that the composite number is not a multiple of the prime number are always co-prime.
Ex. 7 and 15 are co-prime.

(v) Square of two co-prime numbers are always co-prime numbers.

Some Properties which Help in Finding Three Co-prime Numbers

3 numbers are co-prime to each other means all the possible pair of numbers out of these three numbers are co-prime. For example from three numbers 7, 8, 13 three pairs (7, 8), (7, 13) and (8, 13) are formed and each of these pair is a pair of co-prime. Hence, 7, 8, 13 are three co-prime numbers.

Following are some properties helping in finding three co-prime numbers:

(i) Three consecutive odd integers are always co-prime.
Ex. 9, 11, 13 are co-prime.

(ii) Three consecutive natural numbers with first one being odd are always co-primes.
Ex. 7, 8, 9 are co-prime.

(iii) Two consecutive natural numbers along with the next odd numbers are always co-primes.
Ex. 12, 13, 15 are co-prime. Also 17, 18, 19 are co-prime.

(iv) Three prime numbers are always co-prime.
Ex. 3, 11, 13 are co-prime.

To Test Whether a Given Number is Prime Number or Not

In CAT and CAT like competitions you are required to check whether a given number maximum upto 400 is prime number or not.

If you want to test whether any number is a prime number or not, take an integer equal to the square root of the given number but if square root is not an integer then take an integer just larger than the approximate square root of that number. Let it be 'x'. Test the divisibility of the given number by every prime number less than 'x'. If the given number is not divisible by any prime number less than, then the given number is prime number; otherwise it is a composite number.

Square root of 361 is 19. Prime numbers less than 19 are clearly 2, 3, 5, 7, 11, 13 and 17. Since, 361 is not divisible by any of the numbers 2, 3, 5, 7, 11, 13 and 17. Hence, 361 is a prime number.

It is advisable to learn the squared numbers of all integers from 1 to 20, which are very useful to find whether a given number is a prime or not.

From the table it is clear that if any number, say 271 lies between 256 and 289, then its square root lies between 16 and 17, because $16^2 = 256$ and $17^2 = 289$. Thus square root of the given number is not an integer. So, we take 17 as an integer just greater than the square root of the given number. Now all the prime numbers less than 17 are 2, 3, 5, 7, 11 and 13. Since 271 is not divisible by any of the numbers 2, 3, 5, 7, 11 and 13. Hence 361 is a prime number.

$1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
$5^2 = 25$
$6^2 = 36$
$7^2 = 49$
$8^2 = 64$
$9^2 = 81$
$10^2 = 100$
$11^2 = 121$
$12^2 = 144$
$13^2 = 169$
$14^2 = 196$
$15^2 = 225$
$16^2 = 256$
$17^2 = 289$
$18^2 = 324$
$19^2 = 361$
$20^2 = 400$

Illustration 6: Is 171 is a prime number ?

Solution: Square root of 171 lies between 13 and 14, because $13^2 = 169$ and $14^2 = 196$. Therefore, the integer just greater than the square root of 171 is 14.

Now prime numbers less than 14 are 2, 3, 5, 7, 11 and 13.

Since 171 is divisible by 3, therefore 171 is not a prime number.

Illustration 7: Is 167 is a prime number ?

Solution: Square root of 167 lies between 12 and 13, because $12^2 = 144$ and $13^2 = 169$. Therefore the integer just greater than the square root of 167 is 13.

Now prime numbers less than 13 are 2, 3, 5, 7 and 11.

Since 167 is not divisible by any of the prime numbers 2, 3, 5, 7 and 11; therefore 167 is a prime number.

GENERAL OR EXPANDED FORM OF 2 AND 3 DIGITS NUMBERS

(i) In a two digits number AB, A is the digit of tenth place and B is the digit of unit place, therefore AB is written using place value in expanded form as
$AB = 10A + B$
Ex. $35 = 10 \times 3 + 5$

(ii) In a three digits number ABC, A is the digit of hundred place, B is the digit of tenth place and C is the digit of unit place, therefore ABC is written using place value in expanded form as
$ABC = 100A + 10B + C$
Ex. $247 = 100 \times 2 + 10 \times 4 + 7$
These expanded forms are used in forming equations related to 2 and 3 digits numbers.

Illustration 8: In a two digit prime number, if 18 is added, we get another prime number with reversed digits. How many such numbers are possible ?

Solution: Let a two-digit number be pq.

$\therefore\ 10p + q + 18 = 10q + p$

$\Rightarrow\ -9p + 9q = 18 \Rightarrow q - p = 2$

Satisfying this condition and also the condition of being a prime number (pq and qp both), there are 2 numbers 13 and 79.

FACTORISATION

It is a process of representing a given number as a product of two or more prime numbers.

Here each prime number which is present in the product is called a factor of the given number.

For example, 12 is expressed in the factorised form in terms of its prime factors as $12 = 2^2 \times 3$.

Illustration 9: If $N = 2^3 \times 3^7$, then

(a) What is the smallest number that you need to multiply with N in order to make it a perfect square ?

(b) What is the smallest number that you need to divide by N in order to make it a perfect square ?

Solution:

(a) Any perfect square number in its factorised form has prime factors with even powers. So in order to make $2^3 \times 3^7$ a perfect square, the smallest number that we need to multiply it with would be 2×3 i.e. 6. The resulting perfect square will be $2^4 \times 3^8$.

(b) Similarly, in order to arrive at a perfect square by dividing the smallest number, we need to divide the number by 2×3 i.e., 6. The resulting perfect square will be $2^2 \times 3^6$.

NUMBER OF WAYS OF EXPRESSING A COMPOSITE NUMBER AS A PRODUCT OF TWO FACTORS

(i) Number of ways of expressing a composite number N which is not a perfect square as a product of two factors

$$= \frac{1}{2} \times (\text{Number of prime factors of the } N)$$

(ii) Number of ways of expressing a perfect square number M as a product of two factors $= \frac{1}{2}$ [(Number of prime factors of M + 1]

Illustration 10: Find the number of ways of expressing 180 as a product of two factors.

Solution: $180 = 2^2 \times 3^2 \times 5^1$

Number of factors $= (2 + 1)(2 + 1)(1 + 1) = 18$

Since 180 is not a perfect square, hence there are total $\frac{18}{2} = 9$ ways in which 180 can be expressed as a product of two factors.

Illustration 11: Find the number of ways expressing 36 as a product of two factors.

Solution: $36 = 2^2 \times 3^2$

Number of factors $= (2 + 1)(2 + 1) = 9$

Since 36 is a perfect square, hence the number of ways of expressing 36 as a product of two factors

$= \frac{9+1}{2} = 5$, as $36 = 1 \times 36, 2 \times 18, 3 \times 12, 4 \times 9$ and 6×6.

SUM OF FACTORS (OR DIVISORS) OF A COMPOSITE NUMBER

Let N be a composite number in such a way that $N = (x)^a (y)^b (z)^c$ ... where x, y, z... are prime numbers. Then, the sum of factors (or divisors) of $N = \frac{x^{a+1}-1}{x-1} \times \frac{y^{b+1}-1}{y-1} \times \frac{z^{c+1}-1}{z-1} \ldots$

Illustration 12: What is the sum of the divisors of 60 ?

Solution: $60 = 2^2 \times 3 \times 5$

$\Rightarrow$ Sum of the divisors $= \frac{2^3-1}{2-1} \times \frac{3^2-1}{3-1} \times \frac{5^2-1}{5-1} = 168$.

SUM OF UNIT DIGITS

For given n different digits $a_1, a_2, a_3, \ldots, a_n$; the sum of the digits at unit place of all different numbers formed is

$(a_1 + a_2 + a_3 + \ldots + a_n)(n-1)!$ *i.e.*, (Sum of the digits) $(n-1)!$

Illustration 13: Find the sum of unit digits of all different numbers formed from digits 4, 6, 7 and 9.

Solution: Required sum $= (4 + 6 + 7 + 9) - (4 - 1)!$
$= 26 - 3! = 26 - 6 = 20.$

THE LAST DIGIT FROM LEFT (i.e., UNIT DIGIT) OF ANY POWER OF A NUMBER

The last digits (from left) of the powers of any number follow a cyclic pattern i.e., they repeat after certain number of steps. If we find out after how many steps the last digit of the powers of a number repeat, then we can find out the last digit of any power of any number.

Let us look at the powers of 2:

Last digit of 2^1 is 2 . Last digit of 2^6 is 4 .
Last digit of 2^2 is 4 . Last digit of 2^7 is 8 .
Last digit of 2^3 is 8 . Last digit of 2^8 is 6 .
Last digit of 2^4 is 6 . Last digit of 2^9 is 2 .
Last digit of 2^5 is 2 .

Since last digit of 2^5 is the same as the last digit of 2^1, then onwards the last digit will start repeating, i.e., digits of $2^5, 2^6, 2^7, 2^8$ will be the same as those of $2^1, 2^2, 2^3, 2^4$. Then the last digit of 2^9 is again the same as the last digit of 2^1 and so on. Thus, we see that when power of 2 increases, the last digits repeat after every 4 steps.

In above pattern, we can see that whenever the power of 2 is a multiple of 4, the last digit of that number will be the same as the last digit of 2^4.

Suppose we want to find out the last digit of 2^{66}, we should look at a multiple of 4 which is just less than or equal to the power 66 of 2. Since 64 is a multiple of 4, the last digit of 2^{64} will be the same as the last digit of 2^4.

Then the last digits of $2^{65}, 2^{66}$ will be the same as the last digits of $2^1, 2^2$ respectively. Hence the last digit of 2^{66} is the same as the last digit of 2^2 i.e., 4.

Similarly, we can find out the last digit of 3^{75} by writing down the pattern of the powers of 3.

Last digit of 3^1 is 3. Last digit of 3^4 is 1.
Last digit of 3^2 is 9. Last digit of 3^5 is 3.
Last digit of 3^3 is 7. Last digit of $3^6 = 9$
Last digit of $3^7 = 7$
Last digit of $3^8 = 1$
Last digit of $3^9 = 3$

The last digit repeats after 4 steps (like in the case of powers of 2).

Whenever the powers of 3 is a multiple of 4, the last digit of that number will be the same as the last digit of 3^4.

To find the last digit of 3^{75}, we look for a multiple of 4 which is just less than or equal to the power 75 of 3. Since, 72 is multiple of 4, the last digit of 3^{72} will be the same as that of 3^4. Hence the last digit of 3^{75} will be the same as the last digit of 3^3 i.e., 7.

Last Digit (i.e., Unit Digit) of a Product

Last digit of the product $a \times b \times c$ ... is the last digit of the product of last digits of a, b, c, ...

Illustration 14: Find the last digit of $2^{416} \times 4^{430}$.

Solution: Writing down the powers of 2 and 4 to check the pattern of the last digits, we have

We have seen that whenever the power of 2 is a multiple of 4, the last digit of that number will be the same as the last digit of 2^4.

Now, Last digit of $4^1 = 4$.
Last digit of $4^2 = 6$.
Last digit of $4^3 = 4$.
Last digit of $4^4 = 6$.

Thus last digit of any power of 4 is 4 for an odd power and 6 for an even power. The last digit of 2^{416} will be the same as 2^4 because 416 is a multiple of 4. So the last digit of 2^{416} is 6.

Last digit of 4^{430} is 6, since the power of 4 is even.

Hence the last digit of $2^{416} \times 4^{430}$ will be equal to the last digit of $6 \times 6 = 6$.

CONCEPT OF REMAINDERS

(I) Suppose the numbers $N_1, N_2, N_3, \ldots$ give quotients $Q_1, Q_2, Q_3, \ldots$ and remainder $R_1, R_2, R_3, \ldots$ when divided by a common divisor D.

Let S be the sum of $N_1, N_2, N_3, \ldots$

Therefore, $S = N_1 + N_2 + N_3 + \ldots$
$= (D \times Q_1 + R_1) + (D \times Q_2 + R_2) + (D \times Q_3 + R_3) + \ldots$
$= D \times K + (R_1 + R_2 + R_3 \ldots)$, ... (1)

where K is some number

Hence the remainder when S is divided by D is the remainder when $(R_1 + R_2 + R_3 \ldots\ldots)$ is divided by D.

(II) Suppose the numbers, $N_1, N_2, N_3, \ldots$ give quotients $Q_1, Q_2, Q_3, \ldots$ and remainders $R_1, R_2, R_3, \ldots$ respectively, when divided by a common divisor D.

Therefore $N_1 = D \times Q_1 + R_1$, $N_2 = D \times Q_2 + R_2$, $N_3 = D \times Q_3 \times R_3 \ldots$ and so on.

Let P be the product of $N_1, N_2, N_3, \ldots$

Therefore,

$P = N_1 N_2 N_3 \ldots$
$= (D \times Q_1 + R_1)(D \times Q_2 + R_2)(D \times Q_3 + R_3) \ldots$
$= D \times K + (R_1 R_2 R_3 \ldots)$, ... (2)

where K is some number

In the above equation, since only the product $(R_1 R_2 R_3 \ldots)$ is free of D, therefore the remainder when P is divided by D is the remainder when the product $(R_1 R_2 R_3 \ldots)$ is divided by D.

Illustration 15: What is the remainder when the product 1991 × 1992 × 2000 is divided by 7 ?

Solution: The remainder when 1991, 1992 and 2000 are divided by 7 are 3, 4 and 5 respectively.

Hence the final remainder is the remainder when the product $3 \times 4 \times 5 = 60$ is divided by 7. Therefore, remainder = 4.

Illustration 16: What is the remainder when 2^{2010} is divided by 7 ?

Solution: 2^{2010} is a product $(2 \times 2 \times 2 \ldots (2010 \text{ times}))$. Since, 2 is a number less than 7, we try to convert the product into product of numbers higher than 7. Notice that $8 = 2 \times 2 \times 2$. Therefore, we convert the product in the following manner

$$2^{2010} = 8^{670} = 8 \times 8 \times 8 \ldots (670 \text{ times.})$$

The remainder when 8 is divided by 7 is 1. Hence the remainder when 8^{670} is divided by 7 is the remainder obtained when the product $1 \times 1 \times 1 \ldots$ (670 times) is divided by 7. Therefore, remainder = 1.

Illustration 17: What is the remainder when 25^{24} is divided by 9 ?

Solution: Again $25^{24} = (18 + 7)^{24} = (18 + 7)(18 + 7) \ldots$ 24 times $= 18K + 7^{24}$.

Hence, remainder when 25^{24} is divided by 9 is the remainder when 7^{24} is divided by 9.

Now, $7^{24} = 7^3 \times 7^3 \times 7^3 \ldots$ (8 times) $= 343 \times 343 \times 343 \ldots$ (8 times)

Now when 343 is divided by 9 the remainder is 1

So, the remainder when dividing $(343)^8$ by 9 means remainder when dividing $(1)^8$ by 9. So the required remainder is 1.

NUMBER OF ZEROES IN AN EXPRESSION LIKE $a \times b \times c \times \ldots$, WHERE $a, b, c, \ldots$ ARE NATURAL NUMBERS

Consider an expression $8 \times 15 \times 20 \times 30 \times 40$.

The expression can be written in the standard form as :

$8 \times 15 \times 20 \times 30 \times 40$
$= (2^3) \times (3 \times 5) \times (2^2 \times 5) \times (2 \times 3 \times 5) \times (2^3 \times 5)$
$= 2^9 \times 3^2 \times 5^4$, in which base of each factor is a prime number.

A zero is formed by the product of 2 and 5 i.e. 2×5. Hence number of zeroes is equal to the number of pair(s) of 2's and 5's formed.

In the above standard form of the product there are 9 twos and 4 fives. Hence number of pairs of 2 and 5 i.e. (2×5) is 4. Hence, there will be 4 zeroes at the end of the final product.

In the same above way, we can find the number of zeroes at the end of any product given in the form of an expression like $a \times b \times c \times \ldots$, where $a, b, c, \ldots$ are natural numbers.

If there is no pair of 2 and 5 i.e. 2×5, then there is no zero at the end of the product. For example, consider the expression $9 \times 21 \times 39 \times 49$.

The given expression in standard form,

$$9 \times 21 \times 39 \times 49 = (3^2) \times (3 \times 7) \times (3 \times 13) \times (7^2) = 3^4 \times 7^3 \times 13$$

There is no pair of 2 and 5 in the standard form of expression given as product, therefore there will be no zero at the end of the final product.

Illustration 18: Find the number of zeroes in the product $1^1 \times 2^2 \times 3^3 \times 4^4 \times 5^5 \times 6^6 \times \ldots\ldots\ldots \times 49^{49}$

Solution: Clearly the fives will be less than the twos. Hence, we need to count only the fives.

Now, $5^5 \times 10^{10} \times 15^{15} \times 20^{20} \times 25^{25} \times 30^{30} \times 35^{35} \times 40^{40} \times 45^{45}$
$= (5)^5 \times (5 \times 2)^{10} \times (5 \times 3)^{15} \times (5 \times 4)^{20} \times (5 \times 5)^{25} \times (5 \times 6)^{30} \times (5 \times 7)^{35} \times (5 \times 8)^{40} \times (5 \times 9)^{45}$

It gives us $5 + 10 + 15 + 20 + 25 \times 25 + 30 + 35 + 40 + 45$ fives *i.e.*, 825 fives

Thus the product has 825 zeroes.

BASE SYSTEM

The number system in which we carry out all calculation is decimal (base 10) system. It is called decimal system because there are 10 digits 0 to 9.

There are other number systems also depending on the number of digits contained in the base system. Some of the most common systems are Binary system, Octal system, and Hexadecimal system. A number system containing two digits 0 and 1 is called binary (base 2) system. Number system containing eight digits 0, 1, 2, 3, ..., 7 is called Octal (base 8) system.

Hexadecimal (base 16) system has 16 digits 0, 1, 2, 3, .., 9, *A*, *B*, *C*, *D*, *E*, *F*; where *A* has a value 10, *B* has a value 11 and so on.

> Let a number *abcde* be written in base *p*, where *a, b, c, d* and *e* are single digits less than *p*. The value of the number abcde in base 10 $= e \times p^0 + d \times p^1 + c \times p^2 + b \times p^3 + a \times p^4$
>
> For example, The number 7368 can be written as
>
> $8 + 6 \times 10 + 3 \times (10)^2 + 7 \times (10)^3 = 7368$ in decimal (base 10) number system.
>
> The number 7368 in base 9 is written in decimal number system as
>
> $8 \times 9^0 + 6 \times 9 + 3 \times 9^2 + 7 \times 9^3 = 5408$

There are mainly two types of operations associated with conversion of bases: First conversion from any base to base ten and second conversion from base 10 to any base.

(i) Conversion From Any Base to Base Ten

The number $(pqrstu)_a$ (i.e., the number *pqrstu* on base *a*) is converted to base 10 by finding the value of the number.

$(pqrstu)_a = u + ta + sa^2 + ra^3 + qa^4 + pa^5$.

Here subscript '*a*' in $(pqrst)_a$ denotes the base of the number *pqrstu*.

Illustration 19: Convert $(21344)_5$ to base 10.

Solution:

$$(21344)_5 = 4 \times 5^0 + 4 \times 5^1 + 3 \times 5^2 + 1 \times 5^3 + 2 \times 5^4$$
$$= 4 + 4 \times 5 + 3 \times 25 + 1 \times 125 + 2 \times 625 = 1474.$$

(ii) Conversion From Base 10 to Any Base

A number written in base 10 can be converted to any base '*a*' by first dividing the number by '*a*' and then successively dividing the quotients by '*a*'. The remainders, written in reverse order, give the equivalent number in base '*a*'.

For example the number 238 in base 3 is found as

$$\begin{array}{r|r|l} 3 & 238 & \\ \hline & 79 & 1 \\ & 26 & 1 \\ & 8 & 2 \\ & 2 & 2 \end{array} \leftarrow \text{Remainders}$$

The remainders in reverse order is 22211.

Hence, 22211 is the required number in base 3.

> **Note:** Value of a single digit number to all bases are the same. For example,
>
> $5_4 = 5_7 = 5_8 = 5_{10}$

Addition, Subtraction and Multiplication in the Same Bases

Illustration 20: Add the numbers $(4235)_7$ and $(2354)_7$.

Solution: The numbers are written as

$$\begin{array}{r} 4\ 2\ 3\ 5 \\ \underline{2\ 3\ 5\ 4} \\ \hline \end{array}$$

The addition of 5 and 4 (first digit from right of both numbers) is 9 which being more than 7 would be written as $9 = 7 \times 1 + 2$. Here 1 is the quotient and 2 is the remainder when 9 is divided by 7. The remainder 2 is placed at the first place from right of the answer and the quotient 1 gets carried over to the second place from the right.

At the second place from the right $3 + 5 + 1$ (carry) $= 9 = 7 \times 1 + 2$

$$\begin{array}{cccc} & & +1 & +1 \\ 4 & 2 & 3 & 5 \\ 2 & 3 & 5 & 4 \\ \hline 6 & 6 & 2 & 2 \\ \hline \end{array}$$

The remainder 2 is placed at the second place from right of the answer and the quotient 1 carry over to the third place from right.

In the same way, we can find the other digits of the answer.

Illustration 21: $(52)_7 + 46_8 = (?)_{10}$

(a) $(75)_{10}$ **(b)** $(50)_{10}$

(c) $(39)_{39}$ **(d)** $(28)_{10}$

Solution: (a) $(52)_7 = (5 \times 7^1 + 2 \times 7^0)_{10} = (37)_{10}$

Also, $(46)_8 = (4 \times 8^1 + 6 \times 8^0)_{10} = (38)_{10}$

Sum $= (37)_{10} + (38)_{10} = (75)_{10}$

Illustration 22: $(11)_2 + (22)_3 + (33)_4 + (44)_5 + (55)_6 + (66)_7 + (77)_8 + (88)_9 = (?)_{10}$

(a) 396 **(b)** 276

(c) 250 **(d)** 342

Solution: (b)

$$\begin{aligned} (11)_2 &= (1 \times 2^1 + 1 \times 2^0)_{10} = (3)_{10} \\ (22)_3 &= (2 \times 3^1 + 2 \times 3^0)_{10} = (8)_{10} \\ (33)_4 &= (3 \times 4^1 + 3 \times 4^0)_{10} = (15)_{10} \\ (44)_5 &= (4 \times 5^1 + 4 \times 5^0)_{10} = (24)_{10} \\ (55)_6 &= (5 \times 6^1 + 5 \times 6^0)_{10} = (35)_{10} \\ (66)_7 &= (6 \times 7^1 + 6 \times 7^0)_{10} = (48)_{10} \\ (77)_8 &= (7 \times 8^1 + 7 \times 8^0)_{10} = (63)_{10} \\ (88)_9 &= (8 \times 9^1 + 8 \times 9^0)_{10} = (80)_{10} \\ \text{Sum} &= (3)_{10} + (8)_{10} + (15)_{10} + (24)_{10} \\ &\quad + (35)_{10} + (48)_{10} + (63)_{10} + (80)_{10} \\ &= (276)_{10} \end{aligned}$$

Illustration 23: Subtract $(247)_8$ from $(345)_8$.

Solution:

(i) 5 is less than 7. So borrow 1 from the previous digit 4. Since, we are working in octal system, so 5 become $5 + 8 = 13$. Subtract 7 from 13, you will get 6.

$$\begin{array}{rrr} 3 & 4 & 5 \\ -2 & 4 & 7 \\ \hline & & 6 \\ \hline \end{array}$$

(ii) Since, we have borrowed 1, the 4 in the first row has now become 3, which is less than the digit (4), just below it in the second row, So borrow 1 from 3 of first row. So, the 4 in first row is now becomes 3 + 8 = 11. Subtracting 4 of second row from 11, we get 7. Hence,

$$\begin{array}{r} 3\ 4\ 5 \\ -2\ 4\ 7 \\ \hline 0\ 7\ 6 \\ \hline \end{array}$$

FACTORS AND MULTIPLES

If one number '*a*' completely divides a second number '*b*' then 1st number '*a*' is said to be a factor of the 2nd number '*b*'. For example 3 completely divides 15, so 3 is a factor of 15; while 4 does not divide 15 completely, so 4 is not a factor of 15.

Factors of 30 are 1, 2, 3, 5, 6, 10, 15 and 30

Factors of 40 are 1, 2, 4, 5, 8, 10, 20 and 40.

If a number '*a*' is exactly divisible by *a* number '*b*' then the 1st number '*a*' is said to be a multiple of 2nd number '*b*'. For example, 35 is exactly divisible by 7, so 35 is a multiple of 7. Multiple of a number '*b*' can be written down as '*nb*' where *n* is a natural number. So multiples of 5 are 5, 10, 15, 20, 25, ...

HIGHEST COMMON FACTOR (HCF) OR GREATEST COMMON DIVISOR (GCD)

The highest (i.e. largest) number that divides two or more given numbers is called the highest common factor (HCF) of those numbers.

Consider two numbers 12 and 15.

Factors of 12 are 1, 2, 3, 4, 6, 12.

Factors of 30 are 1, 2, 3, 5, 6, 10, 15, 30.

We have some common factors out of these factors of 12 and 30, which are 1, 2, 3, 6. Out of these common factors, 6 is the highest common factor. So, 6 is called the Highest Common Factor (HCF) of 12 and 30.

Methods to Find The HCF or GCD

There are two methods to find HCF of the given numbers

(i) Prime Factorization Method

When a number is written as the product of prime numbers, then it is called the prime factorization of that number. For example, $72 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$. Here, $2 \times 2 \times 2 \times 3 \times 3$ or $2^3 \times 3^2$ is called prime factorization of 72.

To find the HCF of given numbers by this methods, we perform the prime factorization of all the numbers and then check for the common prime factors. For every prime factor common to all the numbers, we choose the least index of that prime factor among the given numbers. The HCF is the product of all such prime factors with their respective least indices.

Illustration 24: Find the HCF of $36x^3y^2$ and $24x^4y$.

Solution $36x^3y^2 = 2^2.3^2.x^3.y^2$, $24x^4y = 2^3.3.x^4.y$. The least index of 2, 3, *x* and *y* in the numbers are 2, 1, 3 and 1 respectively. Hence the HCF $= 2^2.3.x^3.y = 12x^3y$.

Illustration 25: The numbers 400, 536 and 646; when divided by a number N, give the remainders of 22, 23 and 25 respectively. Find the greatest such number *N*.

Solution: *N* will be the HCF of (400 – 22), (536 – 23) and (646 – 25). Hence, *N* will be the HCF of 378, 513 and 621. Hence, $N = 27$.

Illustration 26: The HCF of two numbers is 12 and their product is 31104. How many such numbers are possible.

Solution: Let the numbers be $12x$ and $12y$, where *x* and *y* are co-prime to each other.

Therefore, $12x \times 12y = 31104 \rightarrow xy = 216$.

Now we need to find co-prime pairs whose product is 216.

$216 = 2^3 \times 3^3$. Therefore, the co-prime pairs will be (1, 216) and (8, 27). Therefore, (12, 12 × 216) and (8 × 12, 27 × 12) are two possible numbers.

(ii) Division Method

To find the HCF of two numbers by division method, we divide the larger number by the smaller number. Then we divide the smaller number by the first remainder, then first remainder by the second remainder.. and so on, till the remainder becomes 0. The last divisor is the required HCF.

Illustration 27: Find the HCF of 288 and 1080 by the division method.

Solution:

$$\begin{array}{l} 288 \,)\, 1080 \,(\, 3 \\ \quad\ \ \underline{864} \\ \quad\ \ 216 \,)\, 288 \,(\, 1 \\ \quad\quad\quad \underline{216} \\ \quad\quad\quad 72 \,)\, 216 \,(\, 3 \\ \quad\quad\quad\quad \underline{216} \\ \quad\quad\quad\quad\ 0 \end{array}$$

The last divisor 72 is the HCF of 288 and 1080.

Shortcut for Finding HCF or GCD

To find the HCF of any number of given numbers, first find the difference between two nearest given numbers. Then find all factors (or divisors) of this difference. Highest factor which divides all the given numbers is the HCF.

Illustration 28: Find the HCF of 12, 20 and 32.

Solution: Difference of nearest two numbers 12 and 20 = 20 – 12 = 8

All factors (or divisor) of 8 are 1, 2, 4 and 8.

1, 2 and 4 divides each of the three given numbers 12, 20 and 32. Out of 1, 2 and 4; 4 is the highest number. Hence, HCF = 4.

LEAST COMMON MULTIPLE (LCM)

The least common multiple (LCM) of two or more numbers is the lowest number which is divisible by all the given numbers.

Consider two numbers 12 and 15.

Multiples of 12 are 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132,...

While the multiples of 15 are 15, 30, 45, 60, 75, 90, 105, 120, 135, 150,.....

Out of these series of multiples, we have some common multiples like 60, 120, 180, ..., etc. Out of these common multiples, 60 is the lowest, so 60 is called the Lowest Common Multiple (LCM) of 12 and 15.

Methods to Find The LCM

There are two methods to find the LCM.

(i) Prime Factorization Method

After performing the prime factorization of all the given numbers, we find the highest index of all the prime numbers among the given numbers. The LCM is the product of all these prime numbers with their respective highest indices because LCM must be divisible by all of the given numbers.

Illustration 29: Find the LCM of 72, 288 and 1080.

Solution:

$$72 = 2^3 \times 3^2$$
$$288 = 2^5 \times 3^2$$
$$1080 = 2^3 \times 3^3 \times 5$$

Hence, LCM $= 2^5 \times 3^3 \times 5^1 = 4320$

(ii) Division Method

To find the LCM of 5, 72, 196 and 240, we use the division method in the following way:

Check whether any prime number that divides at least two of all the given numbers. If there is no such prime number, then the product of all these numbers is the required LCM, otherwise find the smallest prime number that divides at least two of the given numbers. Here, we see that smallest prime number that divides at least two given numbers is 2.

Divide those numbers out of the given numbers by 2 which are divisible by 2 and write the quotient below it. The given number(s) that are not divisible by 2 write as it is below it and repeat this step till you do not find at least two numbers that are not divisible by any prime number.

2	5,	72,	196,	240
2	5,	36,	98,	120
2	5,	18,	49,	60
3	5,	9,	49,	30
5	5,	3,	49,	10
	1,	3,	49,	2

After that find the product of all divisors and the quotient left at the end of the division. This product is the required LCM.

Hence, LCM of the given numbers = product of all divisors and the quotient left at the end.

$$= 2 \times 2 \times 2 \times 3 \times 5 \times 3 \times 49 \times 2 = 35280$$

Illustration 30: On a traffic signal, traffic light changes its colour after every 24, 30 and 36 seconds in green, red and orange light. How many times in an hour only green and red light will change simultaneously.

Solution: LCM. of 24 and 30 = 120

So in 1 hr both green and red light will change simultaneously 3600/120 times = 30 times

LCM of 24, 30 and 36 is 360

Hence in 1 hr all three lights will change simultaneously 3600/360 times = 10 times

So in 1 hr only red and green lights will change 30 – 10 = 20 times simultaneously.

Shortcut For Finding LCM

Using idea of co-prime, you can find the LCM by the following shortcut method:

LCM of 9, 10, 15 and 36 can be written directly as $9 \times 10 \times 2$.

The logical thinking that behind it is as follows:

Step 1: If you can see a set of 2 or more co-prime numbers in the set of numbers of which you are finding the LCM, write them down by multiply them.

In the above situation, since we see that 9 and 10 are co-prime to each other, we start off writing the LCM by writing 9×10 as the first step.

Step 2: For each of the other numbers, consider what prime factor(s) of it is/are not present in the LCM (if factorised into primes) taken in step 1. In case you see some prime factors of each of the other given numbers separately are not present in the LCM (if factorised into primes) taken in step 1, such prime factors will be multiplied in the LCM taken in step 1.

Prime factorisation of $9 \times 10 = 3 \times 3 \times 2 \times 5$

Prime factorisation of $15 = 3 \times 5$

Prime factorisation of $36 = 2 \times 2 \times 3 \times 3$

Here we see that both prime factors of 15 are present in the prime factorisation of 9×10 but one prime factor 2 of 36 is not present in the LCM taken in step 1. So to find the LCM of 9, 10, 15 and 36; we multiply the LCM taken in step 1 by 2.

Thus required LCM $= 9 \times 10 \times 2 = 180$

Rule For Finding HCF and LCM of Fractions

(I) HCF of two or more fractions

$$= \frac{\text{HCF of numerator of all fractions}}{\text{LCM of denominator of all fractions}}$$

(II) LCM of two or more fractions

$$= \frac{\text{LCM of numerator of all fractions}}{\text{HCF of denominator of all fractions}}$$

Illustration 31: Find the HCF and LCM of $\frac{4}{5}, \frac{6}{11}, \frac{3}{5}$.

Solution:

$$\text{HCF} = \frac{\text{HCF of 4, 6, 3}}{\text{LCM of 5, 11, 5}} = \frac{1}{55}$$

$$\text{LCM} = \frac{\text{LCM of 4, 6, 3}}{\text{HCF of 5, 11, 5}} = \frac{12}{1} = 12$$

For any two numbers, HCF × LCM = product of the two numbers

This formula is applicable only for two numbers.

For example, HCF of 288 and 1080 is 72 and LCM of these two numbers is 4320.

We can see that $72 \times 4320 = 311040 = 288 \times 1080$.

GREATEST INTEGRAL VALUE

If x be a real number, then $[x]$ indicates greatest integer equal or less than x.

If the given number is an integer, then the greatest integer gives the number itself, otherwise it gives the first integer towards the left of the number x on the number line.

For example $[4] = 4$, $[3.4] = 3$, $[6.8] = 6$, $[-2.3] = -3$, $[-5.6] = -6$ and so on.

Note that -3 is less than -2.3 and -6 is less than -5.6, etc.

Illustration 32: What is the value of

$$\left[\sqrt{1}\right]+\left[\sqrt{2}\right]+\left[\sqrt{3}\right]+\ldots+\left[\sqrt{49}\right]+\left[\sqrt{50}\right]$$

where $[x]$ denotes greatest integer function?

Solution: $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$

Therefore, from $\left[\sqrt{1}\right]$ to $\left[\sqrt{3}\right]$, the value will be 1, from $\left[\sqrt{4}\right]$ to $\left[\sqrt{8}\right]$ the value will be 2, from $\left[\sqrt{9}\right]$ to $\left[\sqrt{15}\right]$ the value will be 3 and so on.

Therefore, the total value

$$= 1 \times 3 + 2 \times 5 + 3 \times 7 + 4 \times 9 + 5 \times 11 + 6 \times 13 + 7 \times 2$$
$$= 3 + 10 + 21 + 36 + 55 + 78 + 14 = 217.$$

Illustration 33: What is the value of x for which $x[x] = 32$?

Solution: If the value of x is 5, $x[x] = 25$, and if the value of x is 6, then $x[x] = 36$

Therefore, the value of x lies between 5 and 6.

If x lies between 5 and 6, then $[x] = 5$.

$$\Rightarrow \quad x = \frac{28}{[x]} = \frac{32}{5} = 6.4\,.$$