CHAPTER

12

LINEAR EQUATIONS

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Many times in mathematics, we have to find the value of an unknown. In this case we represent the unknown by using some letters like p, q, r, x, y etc. These letters are then called as the variable representations of the unknown quantity.

Let's see a problem: A man says, "I am thinking of a number, when I divide it by 3 and then add 5, my answer is twice the number thought of ". Find the number.

Although you do not have the actual number in your mind, you can still move ahead to solve the problem by assuming a variable to represent the number.

The information given in the problem related to the number ultimately will give the value of the unknown i.e., the number in this particular problem. See the process involved in solving the above problem:

Let the number be x.

On dividing the number x by 3, we get $\frac{x}{3}$.

On adding 5 to $\frac{x}{3}$, we get $\frac{x}{3}+5$.

According to the information given in the problem, $\frac{x}{3}+5$ is twice the number i.e., $2x$.

$$\therefore \quad \frac{x}{3}+5 = 2x$$

$$\Rightarrow \quad 5 = 2x-\frac{x}{3}$$

$$\Rightarrow \quad 5 = \frac{6x-x}{3}$$

$$\Rightarrow \quad 5 = \frac{5x}{3}$$

$$\Rightarrow \quad 5\times 3 = 5x$$

$$\Rightarrow \quad \frac{5\times 3}{5} = x \Rightarrow x = 3$$

Hence, required number = 3

Here, '$\frac{x}{3}+5=2x$' is the mathematical statement of equality involving the variable x.

Each mathematical statement of equality involving any number of variables is called an equation. Note that in the above equation there is a single variable x, but according to the given and required information, you may have to suppose more than one variable to move ahead to solve the problem and hence, an equation may have one or more than one variable. If all the variables in the equation are in numerator, no product or quotient (of the expressions including variable(s)) is available in the equation and the power of each variable is unity, then the equation is called linear equation.

Linear equations are commonly used in CAT and Cat like Apptitute tests.

See the following illustration, whose solutions will be found out by converting the statements of the problems into linear equation(s).

Illustration 1: Find the two consecutive even numbers whose sum is 76.

Solution: Let one of the two consecutive even numbers be x.

As we know that the difference between any two consecutive even number is always 2. Therefore the next consecutive even number will be $(x+2)$.

According to the question, sum of the two consecutive even numbers is 76.

$$\therefore \quad x+(x+2)=76$$

$$\Rightarrow \quad 2x+2=76, \Rightarrow 2x=76-2=74$$

$$\Rightarrow \quad x=\frac{74}{2}=37$$

Hence the two consecutive numbers are 37 and 39.

Note that 'the difference between any two consecutive even numbers is always 2' is an information related to the variable x is an extra information because it is not given in the problem, but without this information, we would not form the equation required for solving the problem. Thus you must use the extra information, which helps in formation of equation, if needed.

Illustration 2: Sanjay starts his job with a certain monthly salary and earns a fixed increment every year. If his salary was ₹ 31,000 after four years of service and ₹ 40,000 after 10 years, find his initial salary and annual increment.

Solution: Let the initial salary be ₹ x and fixed increment every year be ₹ y.

$\therefore \quad x + 4y = 31000 \quad ...(1)$

and $\quad x + 10y = 40000 \quad ...(2)$

On subtracting equation (1) from (2), we get

$$6y = 9000 \Rightarrow y = 1500$$

Now putting the value of y in equation (1), we get

$$x + 6000 = 31000 \Rightarrow x = 25000$$

Hence initial salary = ₹ 25000

and fixed annual increment = ₹ 1500.

Illustration 3: If a number is decreased by 4 and divided by 6, the result is 8. What would be the result if 2 was subtracted from the number and then it was divided by 5?

(a) $9\frac{2}{3}$ **(b)** 10

(c) $10\frac{1}{5}$ **(d)** $11\frac{1}{5}$

Solution: (b) Let the number be x. Then,

$$\frac{x-4}{6} = 8 \Rightarrow x - 4 = 48 \Rightarrow x = 52$$

$$\therefore \quad \frac{x-2}{5} = \frac{52-2}{5} = \frac{50}{5} = 10.$$

Illustration 4: If three numbers are added in pairs, the sums equal 10, 19 and 21. The numbers are

(a) 4, 6, 10 **(b)** 6, 4, 15

(c) 3, 5, 10 **(d)** 2, 5, 15

Solution: (b) Let the numbers be x, y and z. Then,

$x + y = 10$ ..... (1) $y + z = 19$ ..... (2) $x + z = 21$ ..... (3)

Adding (1), (2) and (3), we get : $2(x + y + z) = 50$

or $x + y + z = 25.$

Thus, $x = 25 - 19 = 6; y = 25 - 21 = 4;$

$z = 25 - 10 = 15.$

Hence, the required numbers are 6, 4 and 15.

Illustration 5: If the sum of two numbers is 42 and their product in 437, then find the absolute difference between the numbers.

(a) 4 **(b)** 7

(c) 9 **(d)** Cannot be determined

Solution: (a) Let the numbers be x and y. Then, $x + y = 42$ and $xy = 437$.

$$x - y = \sqrt{(x+y)^2 - 4xy} = \sqrt{(42)^2 - 4 \times 437}$$

$$= \sqrt{1764 - 1748} = \sqrt{16} = 4.$$

$\therefore$ Required difference = 4.

Note that depending upon the number of variables in a problem, a linear equation may have one, two or even more variables. But to get the value of the variables the number of equations should be always equal to the number of variables.

STEPS TO BE FOLLOWED TO SOLVE A WORD PROBLEM USING LINEAR EQUATION(S)

Step (i): Read the problem carefully and note what is/are given and what is/are required.

Step (ii): Denote the unknown quantity by some letters, say p, q, r, x, y etc.

Step (iii): Translate the statements of the problem into mathematical statements i.e., equations using the condition(s) given in the problem and extra information(s) related to the variable(s) derived from the statement(s) in the problem.

Step (IV): Solve the equation(s) for the unknown(s).

Step (V): Check whether the solution satisfies the equation(s).

Most of the time in solving the word problem you get struck. It could be due to one or more of the following four reasons:

Reason (i): You are not able to interpret one or more statements in the problem. In this case you concentrate on developing your ability to decode the mathematical meaning of the statement(s) in the problems.

Reason (ii): You have either not used all the information given in the problem or have used them in the incorrect order.

In such a case, go back to the problem and try to identify each statement and see whether you have utilized it or not. If you have already used all the information, then check whether you have used the information given in the problem in the correct order.

Reason (iii): Even though you might have used all the information given in the problem, you have not utilized some of the information completely.

In such a case, you need to review each part of each information given in the problem and look at whether any additional details can be derived out from the same informations. If derived any additional details, use them in forming or solving the equation(s). Sometimes a statement can be used for more than one perspective. In this case, if you have used that statement for one perspective, then using it in the other perspective will solve the problem.

Reason (iv): You are struck because the problem does not have a solution. In such a case, check the solution once and if it is correct go back to reason (i), (ii) and (iii).

Illustration 6: Find the two odd numbers whose sum is 12.

Solution: Let the two odd numbers are x and y.

Then $x + y = 12$

There is no other information about the two variable x and y.

Hence, there will be no other equation between the variable x and y. So, we can not find the exact solution of the problem. The equation formed above yields a set of possibilities for the value of x and y as (1, 11), (3, 9), (5, 7), (7, 5), (9, 3), (11, 1). One of these possibilities has to be the correct answer.

Illustration 7: A piece of wire is 80 metres long. It is cut into three pieces. The longest piece is 3 times as long as the middle-sized and the shortest piece is 46 metres shorter than the longest piece. Find the length of the shortest piece (in metres).

Solution: Let the length of the longest piece = a metres

Length of middle-sized piece = b metres

Since sum of the length of three pieces of wire = 80 metres

$\therefore$ length of shortest piece = $80 - (a + b)$ metres

Now $\quad a = 3b \qquad ...(1)$

and $\quad 80 - (a + b) = a - 46 \qquad ...(2)$

From (1) and (2),

$$80 - \left(a + \frac{a}{3}\right) = a - 46$$

$$\Rightarrow \quad 80 - \frac{3a + a}{3} = a - 46$$

$$\Rightarrow \quad 80 + 46 = a + \frac{4a}{3}$$

$$\Rightarrow \quad \frac{7a}{3} = 126 \Rightarrow a = 126 \times \frac{3}{7} = 54$$

$$\therefore \quad b = \frac{a}{3} = \frac{54}{3} = 18,$$

and $\quad 80 - (a + b) = 80 - (54 + 18) = 8$

Hence length of shortest piece = 8 metres.

Illustration 8: Mohan took five papers in an examination, where each paper was of 250 marks. His marks in these papers were in the ratio 6 : 8 : 10 : 12 : 15. In all papers together, Mohan obtained 70% of the total marks. Then find the number of papers in which he got more than 80% marks.

Solution: Ratio of marks obtained in five papers are

6 : 8 : 10 : 12 : 15.

Let marks obtained in five papers are $6x$, $8x$, $10x$, $12x$ and $15x$.

$$\therefore \; 6x + 8x + 10x + 12x + 15x = 5 \times 250 \times \frac{70}{100}$$

$$\Rightarrow 51x = 125 \times 7 \Rightarrow x = \frac{125 \times 7}{51} = 17 \text{ (approx.)}$$

$$\text{Now } 80\% \text{ of } 250 = 250 \times \frac{80}{100} = 200$$

Now $10x = 170$, $12x = 12 \times 17 = 204$

Hence $\quad 6x, 8x, 10x < 200$

and $\quad 12x, 15x > 200$

Therefore Mohan got more than 80% in only two subjects.

Illustration 9: The sum of the digits of a two digit number is 16. If the number formed by reversing the digits is less than the original number by 18. Find the original number.

Solution: Let unit digit be x. Then tens digit = $16 - x$

$$\therefore \quad \text{Original number} = 10 \times (16 - x) + x = 160 - 9x.$$

On reversing the digits, we have x at the tens place and $(16 - x)$ at the unit place.

$$\therefore \quad \text{New number} = 10x + (16 - x) = 9x + 16$$

Original number – New number = 18

$$(160 - 9x) - (9x + 16) = 18$$

$$160 - 18x - 16 = 18$$

$$-18x + 144 = 18$$

$$-18x = 18 - 144 \Rightarrow 18x = 126$$

$$\Rightarrow \quad x = 7$$

$\therefore$ In the original number, we have unit digit = 7

Tens digit = $16 - 7 = 9$

Thus, original number = 97

Illustration 10: The denominator of a rational number is greater than its numerator by 4. If 4 is subtracted from the numerator and 2 is added to its denominator, the new number becomes $\frac{1}{6}$. Find the original number.

Solution: Let the numerator be x.

Then, denominator = $x + 4$

$$\therefore \quad \frac{x-4}{x+4+2} = \frac{1}{6}$$

$$\Rightarrow \quad \frac{x-4}{x+6} = \frac{1}{6}$$

$$\Rightarrow \quad 6(x - 4) = x + 6$$

$$\Rightarrow \quad 6x - 24 = x + 6 \Rightarrow 5x = 30$$

$$\therefore \quad x = 6$$

Thus, Numerator = 6, Denominator = 6 + 4 = 10.

Hence the original number = $\frac{6}{10}$.

Illustration 11: A man covers a distance of 33 km in $3\frac{1}{2}$ hours; partly on foot at the rate of 4 km/hr and partly on bicycle at the rate of 10 km/hr. Find the distance covered on foot.

Solution: Let the distance covered on foot be x km.

$\therefore$ Distance covered on bicycle = $(33 - x)$ km

$$\therefore \text{ Time taken on foot} = \frac{\text{Distance}}{\text{Speed}} = \frac{x}{4} \text{ hr.}$$

$$\therefore \text{ Time taken on bicycle} = \frac{33-x}{10} \text{hr.}$$

$$\text{The total time taken} = \frac{7}{2} \text{hr.}$$

$$\frac{x}{4} + \frac{33-x}{10} = \frac{7}{2}$$

$$\frac{5x + 66 - 2x}{20} = \frac{7}{2}$$

$$6x + 132 = 140$$

$$6x = 140 - 132$$
$$6x = 8$$
$$x = \frac{8}{6} = 1.33 \text{ km.}$$

$\therefore$ The distance covered on foot is 1.33 km.

Illustration 12: The total age of A and B is 12 years more than the total age of B and C. C is how many years younger than A?

(a) 12 **(b) 24**
(c) C is elder than A **(d) Data inadequate**

Solution: (a) $(A + B) - (B + C) = 12 \Rightarrow A - C = 12.$

C is 12 year younger than A.

Illustration 13: The sum of four numbers is 64. If you add 3 to the first number, 3 is subtracted from the second number, the third is multiplied by 3 and the fourth is divided by 3, then all the results are equal. What is the difference between the largest and the smallest of the original numbers?

(a) 21 **(b) 27**
(c) 32 **(d) Cannot be determined**

Solution: (c) Let the four numbers be A, B, C and D.

Let $A + 3 = B - 3 = 3C = D/3 = x$ (let)

Then, $A = x - 3, B = x + 3, C = x/3$ and $D = 3x$.

$A + B + C + D = 64 \Rightarrow (x - 3) + (x + 3) + x/3 + 3x = 64$

$\Rightarrow 5x + x/3 = 64 \Rightarrow 16x = 192 \Rightarrow x = 12$

Thus, the numbers are 9, 15, 4 and 36.

$\therefore$ Required difference = (36 – 4) = 32.

CONDITION OF CONSISTENCY AND INCONSISTANCY OF A PAIR OF SIMULTANEOUS LINEAR EQUATIONS IN TWO VARIABLES

Let $a_1x + b_1y = c_1$ .... (i)
and $a_2x + b_2y = c_2$ .... (ii)
are a pair of simultaneous linear equations in two various x and y. Here a_1, b_1, c_1 and a_2, b_2, c_2 are coefficient of x, coefficient of y and real constants in equation (i) and (ii) respectively.

(a) Consistent with unique solution :

The two equations (i) and (ii) have an unique solution,

if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

(b) Consistent with infinite many solution :

The two equations (i) and (ii) have infinite many solution,

if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(c) Inconsistent :

The two equations (i) and (ii) have no solution, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

Illustration 14: Find the value of k for which the system of linear equation:
$kx + 4y = k - 4$, $16x + ky = k$, has infinite many solutions.

Solution:

$kx + 4y = k - 4$ .... (1)

$16x + ky = k$ .... (2)

$a_1 = k,\ b_1 = 4,\ c_1 = -(k-4)$

$a_2 = 16,\ b_2 = k,\ c_2 = -k$

Here condition for infinite solutions is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\Rightarrow \frac{k}{16} = \frac{4}{k} = \frac{k-4}{k}$$

$$\Rightarrow \frac{k}{16} = \frac{4}{k} \Rightarrow k^2 = 64 \quad x = \pm 8$$

Also, $\frac{4}{k} = \frac{k-4}{k} \Rightarrow 4k = k^2 - 4k$

$\Rightarrow k^2 - 8k = 0 \Rightarrow k(k-8) = 0$

$\Rightarrow k = 0$ or $k = 8$ but $k = 0$ is not possible otherwise equation will be one variable.

$\therefore k = 8$ is correct value for many solution.

Illustration 15: Check wehether the following given pair of equations has no solution, unique solution or infinite solutions.

$3x + 4y = 8$
$9x + 12y = 24$

Solution:

For these two equations

$a_1 = 3, a_2 = 9, b_1 = 4, b_2 = 12, c_1 = -8, c_2 = -24,$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Since, $\frac{3}{4} = \frac{4}{12} = \frac{-8}{-24}$

The above pair of equations will have infinite solutions.

Illustration 16: Find the value of P for which the given system of equations has only solution (i.e., unique solution)

$Px - y = 2$; $6x - 2y = 3$

Solution:

$Px - y = 2$ .... (1)

$6x - 2y = 3$ .... (2)

$a_1 = P,\ b_1 = -1,\ c_1 = -2$

$a_2 = 6,\ b_2 = -2,\ c_2 = -3$

Condition for unique solution is $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$$\Rightarrow \frac{P}{6} \neq \frac{+1}{+2} \Rightarrow P \neq \frac{6}{2} \Rightarrow P \neq 3$$

$\therefore$ P can have all real values except 3.