CHAPTER

# 21

# PERMUTATIONS AND COMBINATIONS

## FUNDAMENTAL PRINCIPLE OF COUNTING

### Multiplication Principle

If an operation can be performed in '$m$' different ways; followed by a second operation performed in '$n$' different ways, then the two operations in succession can be performed in $m \times n$ ways. This can be extended to any finite number of operations.

**Illustration 1: A person wants to go from station $P$ to station $R$ via station $Q$. There are 4 routes from $P$ to $Q$ and 5 routes from $Q$ to $R$. In how many ways can he travel from $P$ to $R$ ?**

**Solution:** He can go from $P$ to $Q$ in 4 ways and $Q$ to $R$ in 5 ways.

So number of ways of travel from $P$ to $R$ is $4 \times 5 = 20$.

**Illustration 2: A college offers 6 courses in the morning and 4 in the evening. Find the possible number of choices with the student if he wants to study one course in the morning and one in the evening.**

**Solution:** The college has 6 courses in the morning out of which the student can select one course in 6 ways.

In the evening the college has 4 courses out of which the student can select one in 4 ways.

Hence the required number of ways $= 6 \times 4 = 24$.

**Illustration 3: In how many ways can 5 prizes be distributed among 4 boys when every boy can take one or more prizes ?**

**Solution:** First prize may be given to any one of the 4 boys, hence first prize can be distributed in 4 ways.

Similarly every one of second, third, fourth and fifth prizes can also be given in 4 ways.

$\therefore$ The number of ways of their distribution

$= 4 \times 4 \times 4 \times 4 \times 4 = 4^5 = 1024$

### Addition Principle

If an operation can be performed in '$m$' different ways and another operation, which is independent of the first operation, can be performed in '$n$' different ways. Then either of the two operations can be performed in $(m + n)$ ways. This can be extended to any finite number of independent operations.

**Illustration 4: A college offers 6 courses in the morning and 4 in the evening. Find the number of ways a student can select exactly one course, either in the morning or in the evening.**

**Solution:** The college has 6 courses in the morning out of which the student can select one course in 6 ways.

In the evening the college has 4 courses out of which the student can select one in 4 ways.

Hence the required number of ways $= 6 + 4 = 10$.

**Illustration 5: A person wants to leave station $Q$. There are 4 routes from station $Q$ to $P$ and 5 routes from $Q$ to $R$. In how many ways can he travel from the station $Q$ ?**

**Solution:** He can go from $Q$ to $P$ in 4 ways and $Q$ to $R$ in 5 ways. To go from $Q$ to $P$ and $Q$ to $R$ are independent to each other. Hence the person can leave station $Q$ in $4 + 5 = 9$ ways.

## FACTORIALS

If $n$ is a natural number then the product of all natural numbers upto $n$ is called factorial $n$ and it is denoted by $n$ ! or $\underline{|n}$

Thus, $n ! = n (n - 1) (n - 2) ..... 3.2.1$

Note that $0! = 1 = 1!$

$$\begin{aligned} n! &= n (n-1)! \\ &= n (n-1)(n-2)! \\ &= n (n-1)(n-2)(n-3)!, \text{ etc.} \end{aligned}$$

For example $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

But $4! = 4 \times 3 \times 2 \times 1$

$\therefore$ $6! = 6 \times 5 \times 4!$ or $6 \times 5 \times 4 \times 3!$

Remember that

$0 ! = 1, 1 ! = 1, 2 ! = 2, 3 ! = 6, 4 ! = 24, 5 ! = 120, 6 ! = 720$, etc.

## MEANING OF PERMUTATION AND COMBINATION

Each of the different arrangements which can be made by taking some or all of a number of things is called a permutation. Note that in an arrangement, the order in which the things arranged is considerable i.e., arrangement $AB$ and $BA$ of two letters $A$ and $B$ are different because in $AB$, $A$ is at the first place and $B$ is at the second place from left whereas in $BA$, $B$ is at the first place and $A$ is at the second place.

The all different arrangements of three letters $A$, $B$ and $C$ are $ABC$, $ACB$, $BCA$, $BAC$, $CAB$ and $CBA$.

Here each of the different arrangements $ABC$, $ACB$, $BCA$, $BAC$, $CAB$ and $CBA$ is a permutation and number of different arrangement i.e. 6 is the number of permutations.

*ABC*, *ACB*, *BCA*, *BAC*, *CAB* and *CBA* are different arrangements of three letters *A*, *B* and *C*, because in each arrangement, order in which the letters arranged, is considered. But if the order in which the things are arranged is not considered; then *ABC*, *ACB*, *BCA*, *BAC*, *CAB* and *CBA* are not different but the same. Similarly *AB* and *BA* are not different but the same.

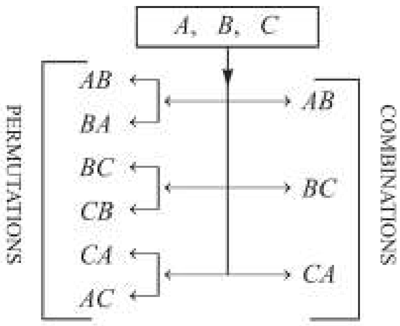
Each of the different selections or groups which can be made by some or all of a number of given things without reference to the order of things in any selection or group is called a combination.

As in selection order in which things are selected is not considered; hence, selections of two letters *AB* and *BA* out of three letters *A*, *B* and *C* are the same. Similarly selections of *BC* and *CB* are the same.

Also selections of *CA* and *AC* are the same.

Hence selection of two letters out of the three letters *A*, *B* and *C* can be made as *AB*, *BC* and *CA* only.

As in arrangements, order in which things are arranged is considered. Hence all arrangements of two letters out of the three letters *A*, *B* and *C* are *AB*, *BA*, *BC*, *CB*, *CA* and *AB*.




Number of permutations (or arrangements) of two letters out of three letters *A*, *B* and $C = 6$.

Number of combinations (or groups) of two letters out of three letters *A*, *B* and $C = 3$.

Permutations of three different letters *A*, *B* and *C* taken two at a time is also understood as selections of any two different letters *AB*, *BC* or *CA* out of *A*, *B* and *C*, then the selected two letters arranged in two ways as

*AB*, *BA* ;  *BC*, *CB*  or  *CA*, *AC*

Hence using multiplication principle, number of permutations of three different letters *A*, *B* and *C* taken two at a time

= (Number of ways to select any two different letters out of the three given letters) × (Number of arrangements of two selected letters)

$= 3 \times 2 = 6$

Thus permutations means selection of some or all of the given things at a time and then arrangements of selected things. In most of the problems, it is mentioned that the problem is of permutation or combination but in some problems it is not mentioned. In the case where it is not mentioned that problem given is of permutation or combination, you can easily identify the given problem is of permutation or combination using the following classifications of problems:

### Problems of Permutations

**(i)** Problems based on arrangements
**(ii)** Problems based on standing in a line
**(iii)** Problems based on seated in a row
**(iv)** Problems based on digits
**(v)** Problems based on arrangement letters of a word
**(vi)** Problems based on rank of a word (in a dictionary)

### Problems of Combinations

**(i)** Problems based on selections or choose
**(ii)** Problems based on groups or committee
**(iii)** Problems based on geometry

If in any problem, it is neither mentioned that the problem is of permutation or combination nor does the problem fall in the categories mentioned above for the problems of permutations or problems of combinations, then do you think whether arrangement (i.e. order) is meaningful or not? If arrangement (i.e., order) is considerable in the given problem, then the problem is of permutation otherwise it is of combination. This will be more clear through the following illustrations:

Suppose you have to select three batsmen out of four batsmen $B_1$, $B_2$, $B_3$ and $B_4$, you can select three batsmen $B_1\ B_2\ B_3$, $B_2\ B_3\ B_4$, $B_3\ B_4\ B_1$ or $B_4\ B_1\ B_2$.

Here order of selections of three batsmen in any group of three batsmen is not considerable because it does not make any difference in the match.

Hence in the selection process; $B_2\ B_3\ B_4$, $B_2\ B_4\ B_3$, $B_3\ B_2\ B_4$, $B_3\ B_4\ B_2$, $B_4\ B_2\ B_3$ and $B_4\ B_3\ B_2$ all are the same.

But for batting, the order of batting is important.

Therefore for batting; $B_2\ B_3\ B_4$, $B_2\ B_4\ B_3$, $B_3\ B_2\ B_4$, $B_3\ B_4\ B_2$, $B_4\ B_2\ B_3$ and $B_4\ B_3\ B_2$, are different because $B_2\ B_3\ B_4$ means batsman $B_2$ batting first then batsman $B_3$ and then batsman $B_4$ whereas $B_2\ B_4\ B_3$ means batsman $B_2$ batting first then batsman $B_4$ and then batsman $B_3$.

## COUNTING FORMULA FOR LINEAR PERMUTATIONS

### Without Repetition

1. Number of permutations of *n* different things, taking *r* at a time is denoted by ${}^nP_r$ or $P(n, r)$, which is given by

$$ {}^nP_r = \frac{n!}{(n-r)!} \ (0 \le r \le n) $$
$$ = n(n-1)(n-2) \ldots\ldots (n-r+1), $$

   where *n* is a natural number and *r* is a whole number.

2. Number of arrangements of *n* different objects taken all at a time is ${}^nP_n = n\,!$

**Note:**

$$ {}^nP_1 = n, \quad {}^nP_r = n.{}^{n-1}P_{r-1}, \quad {}^nP_r = (n-r+1).{}^nP_{r-1}, $$
$$ {}^nP_n = {}^nP_{n-1} $$

**Illustration 6: Find the number of ways in which four persons can sit on six chairs.**

**Solution:** ${}^6P_4 = 6.5.4.3 = 360$

### With Repetition

1. Number of permutations of $n$ things taken all at a time, if out of $n$ things $p$ are alike of one kind, $q$ are alike of second kind, $r$ are alike of a third kind and the rest $n-(p+q+r)$ are all different is
$$\frac{n!}{p!\,q!\,r!}$$
2. Number of permutations of $n$ different things taken $r$ at a time when each thing may be repeated any number of times is $n^r$.

**Illustration 7: Find the number of words that can be formed out of the letters of the word COMMITTEE taken all at a time.**

**Solution:** There are 9 letters in the given word in which two T's, two M's and two E's are identical. Hence the required number of words $= \frac{9!}{2!\,2!\,2!} = \frac{9!}{(2!)^3} = \frac{9!}{8} = 45360$

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## NUMBER OF LINEAR PERMUTATIONS UNDER CERTAIN CONDITIONS

1. Number of permutations of $n$ different things taken all together when $r$ particular things are to be placed at some $r$ given places $= {}^{n-r}P_{n-r} = (n-r)!$
2. Number of permutations of $n$ different things taken $r$ at a time when $m$ particular things are to be placed at $m$ given places $= {}^{n-m}P_{r-m}$.
3. Number of permutations of $n$ different things, taken $r$ at a time, when a particular thing is to be always included in each arrangement, is $r\,.\,{}^{n-1}P_{r-1}$.
4. Number of permutation of $n$ different things, taken $r$ at a time, when $m$ particular thing is never taken in each arrangement is ${}^{n-m}P_r$.
5. Number of permutations of $n$ different things, taken all at a time, when $m$ specified things always come together is
$$m! \times (n-m+1)!$$
6. Number of permutations of $n$ different things, taken all at a time, when $m$ specific things never come together is
$$n! - m! \times (n-m+1)!$$

**Illustration 8: How many different words can be formed with the letters of the word 'JAIPUR' which start with '$A$' and end with '$I$'?**

**Solution:** After putting $A$ and $I$ at their respective places (only in one way) we shall arrange the remaining 4 different letters at 4 places in 4! ways. Hence the required number $= 1 \times 4! = 24$.

**Illustration 9: How many different 3 letter words can be formed with the letters of word 'JAIPUR' when $A$ and $I$ are always to be excluded?**

**Solution:** After leaving $A$ and $I$, we are remained with 4 different letters which are to be used for forming 3 letters words. Hence the required number $= {}^4P_3 = 4 \times 3 \times 2 = 24$.

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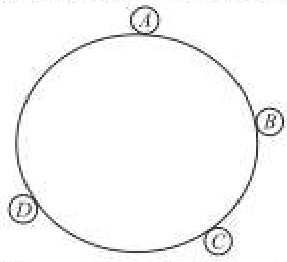
## CIRCULAR PERMUTATIONS

### 1. Arrangement Around a Circular Table

In circular arrangements, there is no concept of starting point (i.e. starting point is not defined). Hence number of circular permutations of $n$ different things taken all at a time is $(n-1)!$ if clockwise and anti-clockwise order are taken as different.




In the case of four persons $A$, $B$, $C$ and $D$ sitting around a circular table, then the two arrangements $ABCD$ (in clockwise direction) and $ADCB$ (the same order but in anti-clockwise direction) are different.

Hence the number of arrangements (or ways) in which four different persons can sit around a circular table $= (4-1)!$ $= 3! = 6$.

### 2. Arrangement of Beads or Flowers (All Different) Around a Circular Necklace or Garland

The number of circular permutations of $n$ different things taken all at a time is $\frac{(n-1)!}{2}$, if clockwise and anti-clockwise order are taken as the same.

If we consider the circular arrangement, if necklace made of four precious stones $A$, $B$, $C$ and $D$; the two arrangements $ABCD$ (in clockwise direction) and $ADCB$ (the same but in anti-clockwise direction) are the same because when we take one arrangement $ABCD$ (in clockwise direction) and then turn the necklace around (front to back), then we get the arrangement $ADCB$ (the same but in anti-clockwise direction). Hence the two arrangements will be considered as one arrangement because the order of the stones is not changing with the change in the side of observation. So in this case, there is no difference between the clockwise and anti-clockwise arrangements.

Therefore number of arrangements of four different stones in the necklace $= \frac{(n-1)!}{2}$.

### 3. Number of Circular Permutations of $n$ Different Things Taken $r$ at a Time

Case I: If clockwise and anti-clockwise orders are taken as different, then the required number of circular permutations
$$= \frac{{}^nP_r}{r}.$$

Case II: If clockwise and anti-clockwise orders are taken as same, then the required number of circular permutations
$$= \frac{{}^nP_r}{2r}.$$

### 4. Restricted Circular Permutations

When there is a restriction in a circular permutation then first of all we shall perform the restricted part of the operation and then perform the remaining part treating it similar to a linear permutation.

**Illustration 10: In how many ways can 5 boys and 5 girls be seated at a round table so that no two girls may be together ?**

**Solution:** Leaving one seat vacant between two boys, 5 boys may be seated in 4! ways. Then at remaining 5 seats, 5 girls can sit in 5! ways. Hence the required number = 4! × 5!

**Illustration 11: In how many ways can 4 beads out of 6 different beads be strung into a ring ?**

**Solution:** In this case a clockwise and corresponding anticlockwise order will give the same circular permutation. So the required number $= \frac{{}^6P_4}{4.2} = \frac{6.5.4.3}{4.2} = 45$.

**Illustration 12: Find the number of ways in which 10 persons can sit round a circular table so that none of them has the same neighbours in any two arrangements.**

**Solution:** 10 persons can sit round a circular table in 9! ways. But here clockwise and anti-clockwise orders will give the same neighbours. Hence the required number of ways $= \frac{1}{2}9!$.

## COUNTING FORMULA FOR COMBINATION

### 1. Selection of Objects Without Repetition

The number of combinations or selections of $n$ different things taken $r$ at a time is denoted by ${}^nC_r$ or $C\ (n, r)$ or $C\binom{n}{r}$

$$\text{where } {}^nC_r = \frac{n!}{r!(n-r)!};\ (0 \le r \le n)$$

$$= \frac{n(n-1)(n-2)...(n-r+1)}{r(r-1)(r-2)....2.1};$$

where $n$ is a natural number and $r$ is a whole number.

Some Important Results

(i) ${}^nC_n = 1$, ${}^nC_0 = 1$ (ii) ${}^nC_r = \frac{{}^nP_r}{r!}$

(iii) ${}^nC_r = {}^nC_{n-r}$ (iv) ${}^nC_x = {}^nC_y \Rightarrow x + y = n$

(v) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ (vi) ${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}$

(vii) ${}^nC_1 = {}^nC_{n-1} = n$

**Illustration 13: If ${}^{20}C_r = {}^{20}C_{r-10}$, then find the value of ${}^{18}C_r$**

**Solution:** ${}^{20}C_r = {}^{20}C_{r-10} \Rightarrow r + (r - 10) = 20 \Rightarrow r = 15$

$$\therefore\ {}^{18}C_r = {}^{18}C_{15} = {}^{18}C_3 = \frac{18.17.16}{1.2.3} = 816$$

**Illustration 14: How many different 4-letter words can be formed with the letters of the word 'JAIPUR' when *A* and *I* are always to be included ?**

**Solution:** Since $A$ and $I$ are always to be included, so first we select 2 letters from the remaining 4, which can be done in ${}^4C_2 = 6$ ways. Now these 4 letters can be arranged in 4! = 24 ways, so the required number = 6 × 24 = 144.

**Illustration 15: How many combinations of 4 letters can be made of the letters of the word 'JAIPUR' ?**

**Solution:** Here 4 things are to be selected out of 6 different things.

So the number of combinations $= {}^6C_4 = \frac{6.5.4.3}{4.3.2.1} = 15$

### 2. Selection of Objects With Repetition

The total number of selections of $r$ things from $n$ different things when each thing may be repeated any number of times is ${}^{n+r-1}C_r$

### 3. Restricted Selection

(i) Number of combinations of $n$ different things taken $r$ at a time when $k$ particular things always occur is ${}^{n-k}C_{r-k}$.

(ii) Number of combinations of $n$ different things taken $r$ at a time when $k$ particular things never occur is ${}^{n-k}C_r$.

### 4. Selection From Distinct Objects

Number of ways of selecting at least one thing from $n$ different things is

${}^nC_1 + {}^nC_2 + {}^nC_3 + .....+ {}^nC_n = 2^n - 1$.

This can also be stated as the total number of combination of $n$ different things is $2^n - 1$.

**Illustration 16: Ramesh has 6 friends. In how many ways can he invite one or more of them at a dinner ?**

**Solution:** He can invite one, two, three, four, five or six friends at the dinner. So total number of ways of his invitation

$$= {}^6C_1 + {}^6C_2 + {}^6C_4 + {}^6C_5 + {}^6C_6 = 2^6 - 1 = 63$$

### 5. Selection From Identical Objects

(i) The number of combination of $n$ identical things taking $r$ $(r \le n)$ at a time is 1.

(ii) The number of ways of selecting any number $r$ $(0 \le r \le n)$ of things out of $n$ identical things is $n + 1$.

(iii) The number of ways to select one or more things out of $(p + q + r)$ things; where $p$ are alike of first kind, $q$ are alike of second kind and $r$ are alike of third kind $= (p + 1)(q + 1)(r + 1) - 1$.

**Illustration 17: There are *n* different books and *p* copies of each in a library. Find the number of ways in which one or more than one books can be selected.**

**Solution:** Required number of ways

$$= (p + 1)(p + 1)......n \text{ terms} - 1 = (p + 1)^n - 1$$

**Illustration 18: A bag contains 3 one ₹ coins, 4 five ₹ coins and 5 ten ₹ coins. How many selection of coins can be formed by taking atleast one coin from the bag?**

**Solution:** There are 3 things of first kind, 4 things of second kind and 5 things of third kind, so the total number of selections $= (3 + 1)(4 + 1)(5 + 1) - 1 = 119$

## DIVISION AND DISTRIBUTION OF OBJECTS

1. The number of ways in which $(m + n)$ different things can be divided into two groups which contain $m$ and $n$ things respectively is

$${}^{m+n}C_m\ {}^nC_n = \frac{(m+n)!}{m!n!},\ m \ne n$$

Particular case:

When $m = n$, then total number of ways is

$\frac{(2m)!}{(m!)^2}$, when order of groups is considered and

$\frac{(2m)!}{2!(m!)^2}$, when order of groups is not considered.

**2.** The number of ways in which $(m + n + p)$ different things can be divided into three groups which contain $m$, $n$ and $p$ things respectively is

$$^{m+n+p}C_m \cdot {}^{n+p}C_p \cdot {}^pC_p = \frac{(m+n+p)!}{m!n!p!}, m \neq n \neq p$$

Particular case:

When $m = n = p$, then total number of ways is

$\frac{(3m)!}{(m!)^3}$, when order of groups is considered and

$\frac{(3m)!}{3!(m!)^3}$, when order of groups is not considered.

**3.** (i) Total number of ways to divide $n$ identical things among $r$ person is $^{n+r-1}C_{r-1}$

(ii) Also total number of ways to divide $n$ identical things among $r$ persons so that each gets atleast one is $^{n-1}C_{r-1}$.

**Illustration 19: In how many ways 20 identical mangoes may be divided among 4 persons if each person is to be given at least one mango?**

**Solution:** If each person is to be given at least one mango, then number of ways will be $^{20-1}C_{4-1} = {}^{19}C_3 = 969$.

**Illustration 20: In how many ways can a pack of 52 cards be divided in 4 sets, three of them having 17 cards each and fourth just one card?**

**Solution:** Since the cards are to be divided into 4 sets, 3 of them having 17 cards each and 4th just one card, so number of ways $= \frac{52!}{1!51!} \cdot \frac{51!}{(17!)^3\,3!} = \frac{52!}{(17!)^3\,3!}$.

## IMPORTANT RESULTS ABOUT POINTS

**1.** If there are $n$ points in a plane of which $m$ ($< n$) are collinear, then

(i) Total number of different straight lines obtained by joining these $n$ points is $^nC_2 - {}^mC_2 + 1$.

(ii) Total number of different triangles formed by joining these $n$ points is $^nC_3 - {}^mC_3$

**2.** Number of diagonals of a polygon of $n$ sides is $^nC_2 - n$ *i.e.*, $\frac{n(n-3)}{2}$.

**3.** If $m$ parallel lines in a plane are intersected by a family of other $n$ parallel lines, then total number of parallelograms so formed is $^mC_2 \times {}^nC_2$ *i.e.*, $\frac{mn(m-1)(n-1)}{4}$.

**4.** Given $n$ points on the circumference of a circle, then

(i) Number of straight lines obtained by joining these $n$ points $= {}^nC_2$

(ii) Number of triangles obtained by joining these $n$ points $= {}^nC_3$

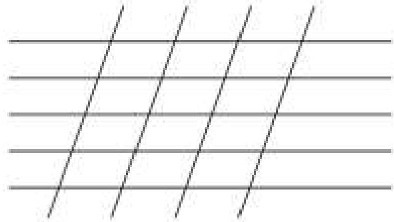
(iii) Number of quadrilaterals obtained by joining these $n$ points $= {}^nC_4$

**Illustration 21: There are 10 points in a plane and 4 of them are collinear. Find the number of straight lines joining any two of them.**

**Solution:** Total number of lines $= {}^{10}C_2 - {}^4C_2 + 1 = 40$.

**Illustration 22: If 5 parallel straight lines are intersected by 4 parallel straight lines, then find the number of parallelograms thus formed.**

**Solution:**



Number of parallelograms $= {}^5C_2 \times {}^4C_2 = 60$.

## FINDING THE RANK OF A WORD

We can find the rank of a word out of all the words with or without meaning formed by arranging all the letters of a given word in all possible ways when these words are listed as in a dictionary. You can easily understand the method to find the above mentioned rank by the following illustrations.

**Illustration 23: If the letters of the word RACHIT are arranged in all possible ways and these words (with or without meaning) are written as in a dictionary, then find the rank of this word RACHIT.**

**Solution:** The order of the alphabet of RACHIT is $A, C, H, I, R, T$.

The number of words beginning with $A$ (i.e. the number of words in which $A$ comes at first place) is $^5P_5 = 5!$.

Similarly, number of words beginning with $C$ is 5!, beginning with $H$ is 5! and beginning with $I$ is also 5!.

So before $R$, four letters $A, C, H, I$ can occur in $4 \times (5!) = 480$ ways.

Now the word RACHIT happens to be the first word beginning with $R$. Therefore the rank of this word RACHIT $= 480 + 1 = 481$.

**Illustration 24: The letters of the word MODESTY are written in all possible orders and these words (with or without meaning) are listed as in a dictionary then find the rank of the word MODESTY.**

**Solution:**

The order of the alphabet of MODESTY is D, E, M, O, S, T, Y.

Number of words beginning with $D$ is $^6P_6 = 6!$

Number of words beginning with $E$ is $^6P_6 = 6!$

Number of words beginning with $MD$ is $^5P_5 = 5!$

Number of words beginning with $ME$ is $^5P_5 = 5!$

Now the first word start with $MO$ is MODESTY.

Hence rank of the word MODESTY

$= 6! + 6! + 5! + 5! + 5! + 1$

$= 720 + 720 + 120 + 120 + 1$

$= 1681$.