CHAPTER

13

FUNCTIONS

INTRODUCTION

Function in mathematics is an equation or rule that defines a relationship between the two variables; one of them is dependent variable and other is independent variable. This chapter is very important from the point of view of CAT and other equivalent aptitude tests. The number of questions being asked from this topic is almost constant. Basically on an average 3–4 problems are asked from this chapter. A deep understanding of the concepts of this chapter is required to solve the problems.

FUNCTION

A function is a rule which relates two or more than two variables. Out of these variables one is dependent variable and others are independent variables. If y is dependent variable and x is independent variable, then the function is symbolically expressed as

$$y = f(x)$$

$y = f(x)$ is read as y is the function of x. But f denotes the rule by which y varies with x.

In the function $y = f(x)$, there is a unique real value of y for each real value of x. A set D of all real values of x for which the value of y is a unique real value is called domain of the function $y = f(x)$. A set R of all unique real values of y corresponding to each value of x from set D is called Range of the function $y = f(x)$.

The concept of the function can be easily understood by the following examples:

(i) The function between diameter d of a circle and radius r is

$$d = 2r$$

Here d is a dependent variable and r is an independent variable, because d and r both are variable but value of d is dependent upon the value of r.

Here domain is a set of all positive real values, because value of r cannot be non-positive and for each positive real value of r, the value of d is a unique positive real number.

Range is also a set of all positive real values, because the diameter,which is twice the length of the radius will be all the positive real numbers for all positive real value of r.

(ii) The function between the volume V of a cuboid with its side length x is

$$V = x^3$$

Here V is dependent variable and x is independent variable

Domain = Set of all positive real numbers.

Range = Set of all positive real numbers.

(iii) The function between the area A of the circle with its radius r is

$$A = \pi r^2$$

Here A and r are dependent and independent variables respectively.

Since value of r can be any positive real number and for all positive real values of r, values of A will be all positive real numbers, hence

Domain = Set of all positive real numbers.

Range = Set of all positive real numbers.

(iv) For the function $y = x^2$, y is a dependent variable and x is an independent variable,

Domain = Set of all real numbers

But Range = Set of all non-negative real numbers, because value of y cannot be negative for any value of x for the given function.

Illustration 1: If $f(x) = -2x + 7$ and $g(x) = x^2 - 5x + 6$, find $f(3), f(-4), g(2)$, and $g(-1)$.

Solution:

$f(x) = -2x + 7,$	$g(x) = x^2 - 5x + 6$
$f(3) = -2(3) + 7 = 1$	$g(2) = 2^2 - 5(2) + 6 = 0$
$f(-4) = -2(-4) + 7 = 15$	$g(-1) = (-1)^2 - 5(-1) + 6 = 12$

RULES FOR FINDING THE DOMAIN OF A FUNCTION

1. Domain of Algebraic Functions

(i) Denominator should be non-zero

For the function $y = \frac{2x}{x-3}$, the value of x can be any real number but can not be 3, because for $x = 3$, denominator of the function will be zero.

Hence domain of the function is the set of all real numbers except 3 i.e. domain = $R - \{3\}$.

(ii) Expression under the even root (i.e. square root, fourth root, etc.) should be non-negative.
For the function $y = \sqrt{5-x}$,
$$5 - x \geq 0 \Rightarrow x \leq 5$$
Hence domain = Set of all real numbers which are equal or less than 5.

2. Domain of Logarithmic Functions

$\log_b a$ is defined when $a > 0$, $b > 0$ but $b \neq 1$.
For the function $y = \log_2 (x - 4)$
$$x - 4 > 0 \Rightarrow x > 4$$
Hence domain = Set of all real numbers greater than 4.

3. Domain of Exponential Functions

a^x is defined for all real values of x, where $a > 0$.
For the function $y = (3x - 2)^x$,
$$3x - 2 > 0 \Rightarrow x > \frac{2}{3}$$
Hence domain = Set of all real numbers greater than $\frac{2}{3}$.

Note: If a and b are two real numbers such that $a > b$, then
(i) Interval $[a, b]$ means all real numbers equal or greater than a but equal or less than b.
(ii) Interval $[a, b)$ means all real numbers equal or greater than a but less than b.
(iii) Interval $(a, b]$ means all real numbers greater than a but equal or less than b.
(iv) Interval (a, b) means all real numbers greater than a but less than b.
(v) $(a, b) \cup (c, d)$ means all real numbers greater than a but less than b or greater than c but less than d.

Illustration 2: The domain of the function $f(x) = \dfrac{1}{\sqrt{x^2 - 3x + 2}}$ is

(a) $(-\infty, 1)$ **(b) $(-\infty, 1) \cup (2, \infty)$**
(c) $(-\infty, 1] \cup [2, \infty)$ **(d) $(2, \infty)$**

Solution: (b) For $f(x)$ to be defined, we must have
$x^2 - 3x + 2 = (x - 1)(x - 2) > 0 \Rightarrow x < 1$ or > 2
Domain of $f = (-\infty, 1) \cup (2, \infty)$.

METHODS OF REPRESENTATION OF FUNCTIONS

A function is represented mainly in three ways as given below.

1. Analytical Representation

When a function is represented by a uniform equation for the entire domain or by several equations which are different for different parts of the domain.
For example
(a) $y = 5x^2 + 2x$
This is the uniform function for entire domain
(b) $y = \begin{cases} x^2 + 4, & \text{if } x \leq 2 \\ x - 3, & \text{if } x > 2 \end{cases}$

This is the function which is represented by two equations which are different for different parts of the domain as given above.

2. Tabular Representation

When a function is represented by a sequence of values of the independent variable with the corresponding values of the dependent variable, then this representation is called Tabular representation of the function.
For example,
(a)

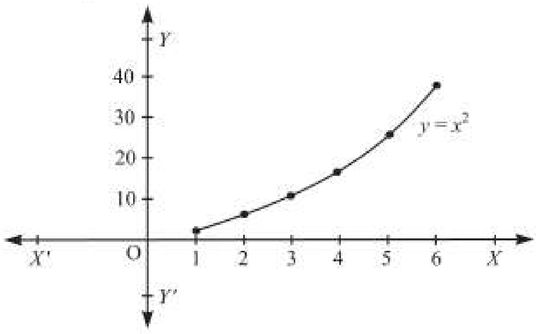
x	1	2	3	4	5	6
y	1	4	9	16	25	36

(b)

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1

3. Graphical Representation

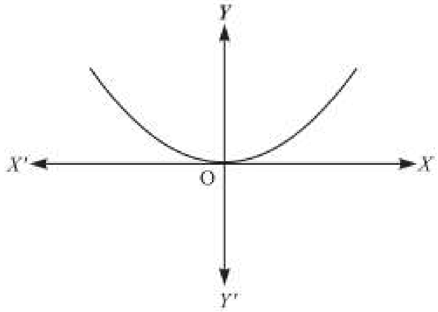
When a function is represented by a graph taking different values of dependent variable along x-axis and corresponding values of independent variable along y-axis in a cartesian plane, then this representation of function is called graphical representation of function.
For example



SOME SPECIAL FUNCTIONS

1. Even and Odd Functions

(i) **Even functions:** If a function $y = f(x)$ be such that $f(-x) = f(x)$, then the function is called an even function. Graph of the even function $y = f(x)$ is symmetrical about the y-axis.
For example the graph of even function $y = x^2$ is symmetrical about y-axis.

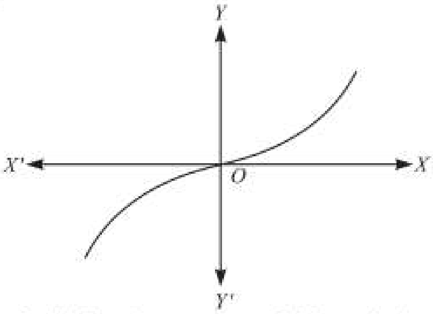


However, if y is independent variable and x is dependent variable, then the even function $x = f(y)$ is symmetrical about the x-axis.

Sum, difference, product and quotient of even functions are also even.

(ii) **Odd functions:** If a function $y = f(x)$ is such that $f(-x) = -f(x)$, then the function is called an odd function.

For example graph of the odd function $y = x^3$ is shown in the figure.



Graph of odd functions are two-fold graphs i.e., on folding the graph paper twice, once along x-axis and then along y-axis, one part of the graph overlaps the other part of the graph.

Some examples of odd functions are $y = x^3 - 2x$, $y = x^5$, $y = x^3 + \frac{1}{x}$, etc.

- Sum and difference of two odd functions is odd function.
- Product of two odd functions is an even function.
- Sum of even and odd function is neither even nor odd function.
- Product of an even and an odd function is odd function.
- Every function can be expressed as the sum of an even function and an odd function.
- A function may be even, odd or neither even nor odd.

For example $4x^3 + 3x^2 + 5$ is neither an even function nor an odd function.

Illustration 3: The function $f(x) = x\dfrac{a^x - 1}{a^x + 1}$ is odd or even ?

Solution:

Since $f(-x) = -x.\dfrac{a^{-x}-1}{a^{-x}+1} = -x.\dfrac{1-a^x}{1+a^x} = x\dfrac{a^x-1}{a^x+1} = f(x)$

$\therefore f(x)$ is an even function.

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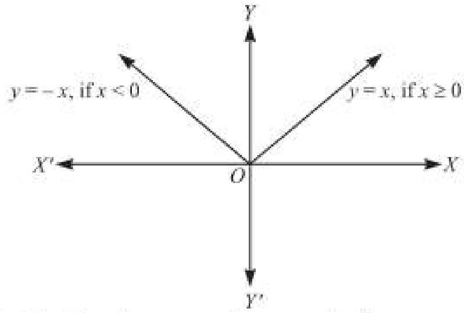
2. Modulus Function

$$f(x) = |x|$$

or $$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Domain = Set of all real numbers

Range = Set of all non-negative real numbers



Note that $|x|$ is always equal or greater than zero

i.e. $|x| \geq 0$

For example,

$|0| = 0$

$|5| = 5$, since $5 > 0$

$|-5| = -(-5) = 5$, since $-5 < 0$

Illustration 4: If $|6x - 4| = 5$, find the value of x.

Solution: **Case–I;** $6x - 4 = 5$, if $6x - 4 \geq 0$

$\Rightarrow \quad x = \frac{3}{2}$, if $x \geq \frac{2}{3}$

Case–II: $-(6x - 4) = 5$, if $6x - 4 < 0$

$\Rightarrow \quad 6x = -1$, if $x < \frac{2}{3}$

$\Rightarrow \quad x = -\frac{1}{6}$, if $x < \frac{2}{3}$

Illustration 5: Find the value of x if $2x^2 + 6|x| + 3 = 0$.

Solution: Since $2x^2$ and $6|x|$ is non-negative and 3 is positive, therefore their sum cannot be equal to zero.

Hence, there is no value of x for which $2x^2 + 6|x| + 3 = 0$

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3. Composite Function

If two or more functions are composed into one function, then the resulting function is called composite function.

For example, if

$y = f(x)$ and $y = g(x)$ are two functions then

$f(g(x))$ and $g(f(x))$ are composite functions

Let $f(x) = 2x - 3$ and $g(x) = -3x^2$

Then $f(g(x)) = 2(-3x^2) - 3 = -6x^2 - 3$

and $g(f(x)) = -3(2x - 3)^2$

$f(g(x))$ and $g(f(x))$ are also written as $f'g(x)$ and $g'f(x)$ respectively

Illustration 6: Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f - g)(x)$. Then evaluate the difference when $x = 2$.

Solution: The difference of the functions f and g is given by

$$(f - g)(x) = f(x) - g(x)$$
$$= (2x + 1) - (x^2 + 2x - 1) = -x^2 + 2.$$

When $x = 2$, the value of this difference is

$$(f - g)(2) = -(2)^2 + 2 = -2.$$
