

## CONCEPT OF PROBABILITY

If you go to buy 10 kg of sugar at ₹ 40 per kg, you can easily find the exact price of your purchase is ₹ 400. On the other hand, the shopkeeper may have a good estimate of the number of kg of sugar that will be sold during the day, but it is impossible to predict the exact amount, because the number of kg of sugar that the consumers will purchase during a day is random.

There are various phenomenon in nature, leading to an outcome, which cannot be predicted in advance. For example, we cannot exactly predict that (i) a head will occur on tossing a coin, (ii) a student will clear the CAT, (iii) India will win the cricket match against Pakistan, etc. But we can measure the amount of certainty of occurrence of an outcome of a phenomenon. This amount of certainty of occurrence of an outcome of a phenomenon is called probability. For example, on tossing a coin certainty of occurrence of each of a head and a tail are the same. Hence amount of certainty of occurrence of each of a head and a tail is 50% i.e.,

$\frac{50}{100} = \frac{1}{2}$ . Therefore  $\frac{1}{2}$  is the amount of certainty of occurrence of a head (or a tail) on tossing a coin and hence  $\frac{1}{2}$  is the probability of occurrence of a head (or a tail) on tossing a coin. On throwing a dice (a dice is a cuboid having one of the numbers 1, 2, 3, 4, 5 and 6 on each of its six faces) certainty of occurrence of each of the numbers 1, 2, 3, 4, 5 and 6 on its top face are the same.

Therefore certainty of occurrence of each of the numbers 1, 2, 3, 4, 5 and 6 is  $\frac{1}{6}$ .

Therefore  $\frac{1}{6}$  is the amount of certainty of occurrence of each of the numbers 1, 2, 3, 4, 5 or 6 on the top face of the dice on throwing the dice and hence  $\frac{1}{6}$  is the probability of occurrence of each of the numbers 1, 2, 3, 4, 5, or 6 on the top face of the dice on tossing a dice is  $\frac{1}{6}$ .

Therefore  $\frac{1}{6}$  is the amount of certainty of occurrence of each of the numbers 1, 2, 3, 4, 5 or 6 on the top face of the dice on throwing the dice and hence  $\frac{1}{6}$  is the probability of occurrence of each of the numbers 1, 2, 3, 4, 5, or 6 on the top face of the dice on tossing a dice is  $\frac{1}{6}$ .

## BASIC TERMS

- An Experiment:** An action or operation resulting in two or more outcomes is called an experiment. For examples
  - Tossing of a coin is an experiment because there are two possible outcomes head and tail.
  - Drawing a card from a pack of 52 cards is an experiment because there are 52 possible outcomes.
- Sample Space:** The set of all possible outcomes of an experiment is called the sample space, denoted by  $S$ . An element of  $S$  is called a sample point. For examples
  - In the experiment of tossing a coin, the sample space has two points corresponding to head ( $H$ ) and Tail ( $T$ ) i.e.,  $S\{H, T\}$ .
  - When we throw a dice then any one of the numbers 1, 2, 3, 4, 5 and 6 will come up. So the sample space,  $S = \{1, 2, 3, 4, 5, 6\}$
- An Event:** Any subset of a sample space is an event. For example,  
If we throw a dice then  $S = \{1, 2, 3, 4, 5, 6\}$   
Then  $A = \{1, 3, 5\}$ ,  $B = \{2, 4, 6\}$ , the null set  $\phi$  and  $S$  itself are some events of  $S$ , because they all are subsets of set  $S$ .
- Impossible Event:** The null set  $\phi$  is called the impossible event or null event. For example,  
Getting 7 when a dice is thrown is an impossible or a null event.
- Sure Event:** The entire sample space is called sure or certain event. For example,  
Here the event:  
Getting an odd or even number on throwing a dice is a sure event, because the event =  $S$ .
- Complement of an Event:** The complement of an event  $A$  is denoted by  $\bar{A}$ ,  $A'$  or  $A^c$ , is the set of all sample points of the sample space other than the sample points in  $A$ . For example, in the experiment of tossing a fair dice,  
 $S = \{1, 2, 3, 4, 5, 6\}$  If  $A = \{1, 3, 5, 6\}$ , then  $A^c = \{2, 4\}$   
Note that  $A \cup A^c = S$ ,  $A \cap A^c = \phi$ .

**7. Simple (or Elementary) Event:** An event is called a simple event if it is a singleton subset of the sample space  $S$ . The singleton subset means the subset having only one element. For example,

(i) When a coin is tossed, sample space  $S = \{H, T\}$

Let  $A = \{H\}$  = the event of occurrence of head and

$B = \{T\}$  = the event of occurrence of tail.

Here  $A$  and  $B$  are simple events.

(ii) When a dice is thrown then sample space,

$S = \{1, 2, 3, 4, 5, 6\}$

Let  $A = \{5\}$  = the event of occurrence of 5

$B = \{2\}$  = the event of occurrence of 2

Here  $A$  and  $B$  are simple events.

**8. Compound Event:** It is the joint occurrence of two or more simple events. For example,

The event of at least one head appears when two fair coins are tossed is a compound event,

$A = \{HT, TH, HH\}$

**9. Equally Likely Events:** A number of simple events are said to be equally likely if there is no reason for one event to occur in preference to any other event. For example,

In drawing a card from a well shuffled pack of 52 cards, there are 52 outcomes and hence 52 simple events which are equally likely because there is no reason for one event to occur in preference to any other event. For example,

## MATHEMATICAL DEFINITION OF PROBABILITY

If an event  $A$  consists of  $m$  sample points of a sample space  $S$  having  $n$  elements ( $0 \leq m \leq n$ ), then the probability of occurrence

of event  $A$ , denoted by  $P(A)$  is defined to be  $\frac{m}{n}$  i.e.,  $P(A) = \frac{m}{n}$

$$\therefore 0 \leq m \leq n \Rightarrow 0 \leq \frac{m}{n} \leq 1 \Rightarrow 0 \leq P(A) \leq 1$$

If the event  $A$  has  $m$  elements, then  $A'$  has  $(n - m)$  elements.

$$\therefore P(A') = \frac{n - m}{n} = 1 - \frac{m}{n} = 1 - P(A)$$

Let  $S = \{a_1, a_2, \dots, a_n\}$  be the sample space

$$P(S) = \frac{n}{n} = 1, \text{ corresponding to the certain event.}$$

$$P(\phi) = \frac{0}{n} = 0, \text{ corresponding to the null event } \phi \text{ (or impossible event)}$$

If  $A_i = \{a_i\}$ ,  $i = 1, 2, \dots$  or  $n$ ; then  $A_i$  is the event corresponding to a single sample point  $a_i$ , then  $P(A_i) = \frac{1}{n}$ .

**Illustration 1:** Two dice are thrown at a time. Find the probability of the followings:

(i) the numbers shown are equal

(ii) the difference of numbers shown is 1

**Solution:** The sample space in a throw of two dice

$S = \{(1, 1), (1, 2), \dots, (1, 6), (2, 1), (2, 2), \dots, (2, 6), (3, 1), \dots, (3, 6), (4, 1), \dots, (4, 6), (5, 1), \dots, (5, 6), (6, 1), \dots, (6, 6)\}$

$\therefore$  total no. of outcomes,  $n(S) = 36$

(i) Here  $E_1$  = the event of showing equal number on both dice

$= \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

$$\therefore n(E_1) = 6, \Rightarrow P(E_1) = \frac{n(E_1)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(ii) Here  $E_2$  = the event of showing numbers whose difference is 1.

$= \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3), (4, 5), (5, 4), (5, 6), (6, 5)\}$

$$\therefore n(E_2) = 10, \Rightarrow P(E_2) = \frac{n(E_2)}{n(S)} = \frac{10}{36} = \frac{5}{18}$$

**Illustration 2:** If three cards are drawn from a pack of 52 cards, what is the chance that all will be queen?

**Solution:** If the sample space be  $S$ , then  $n(S)$  = the total number of ways of drawing 3 cards out of 52 cards =  ${}^{52}C_3$

Now, if  $A$  = the event of drawing three queens, then

$$n(A) = {}^4C_3$$

$$\therefore P(E) = \frac{n(A)}{n(S)} = \frac{{}^4C_3}{{}^{52}C_3} = \frac{4}{52 \times 51 \times 50} = \frac{1}{5525}$$

**Note that in a pack of playing cards,**

Total number of cards: 52(26 red, 26 black)

Four suits: Heart, Diamond, Spade, Club-13 cards of each suit

Court number of cards: 12(4 kings, 4 queens, 4 jacks)

Face number of cards: 16(4 aces, 4 kings, 4 queens, 4 jacks)

**Illustration 3:** Words are formed with the letters of the word PEACE. Find the probability that 2 E's come together.

**Solution:** Total number of words which can be formed with the letters of the word P E A C E =  $\frac{5!}{2!} = 60$

Number of words in which 2 E's come together =  $4! = 24$

$$\therefore \text{Required prob.} = \frac{24}{60} = \frac{2}{5}$$

**Illustration 4:**  $A$  and  $B$  play a game where each is asked to select a number from 1 to 25. If the two numbers match, both of them win a prize. The probability that they will not win a prize in a single trial is

$$(a) \frac{1}{25}$$

$$(b) \frac{24}{25}$$

$$(c) \frac{2}{25}$$

(d) None of these

**Solution:** (b) Total number of possibilities =  $25 \times 25$   
Favourable cases for their winning = 25

$$\therefore P(\text{they win a prize}) = \frac{25}{25 \times 25} = \frac{1}{25}$$

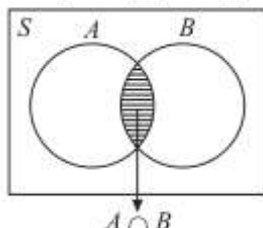
$$\therefore P(\text{they will not win a prize}) = 1 - \frac{1}{25} = \frac{24}{25}$$

## ADDITION THEOREM

If  $A$  and  $B$  are any events in  $S$ , then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

i.e.,  $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$



For three events  $A$ ,  $B$  and  $C$  in  $S$ , we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

### Special Addition Rule

If  $A$ ,  $B$ , and  $C$  are mutually exclusive, then  $P(A \cap B)$ ,  $P(B \cap C)$ ,  $P(C \cap A)$ ,  $P(A \cap B \cap C) = 0$ , hence  $P(A \cup B) = P(A) + P(B)$  and  $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

**Illustration 5:** A bag contains 6 white, 5 black and 4 red balls. Find the probability of getting either a white or a black ball in a single draw.

**Solution:** Let  $A$  = Event that we get a black ball

Two events  $A$  and  $B$  are mutually exclusive.

$$P(A) = \frac{{}^6C_1}{{}^{15}C_1} = \frac{6}{15}, P(B) = \frac{{}^5C_1}{{}^{15}C_1} = \frac{5}{15}$$

$$\text{So, } P(A \cup B) = P(A) + P(B) = \frac{6}{15} + \frac{5}{15} = \frac{11}{15}.$$

**Illustration 6:** One digit is selected from first 20 positive integers. What is the probability that it is divisible by 3 or 4.

**Solution:**

Let  $A$  = Event that the selected number is divisible by 3

$B$  = Event that the selected number is divisible by 4

Here, the events  $A$  and  $B$  are not mutually exclusive because 12 is divisible by both 3 and 4.

$$P(A) = \frac{6}{20}, P(B) = \frac{5}{20}, P(A \cap B) = \frac{1}{20}$$

$$\begin{aligned} \therefore P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{6}{20} + \frac{5}{20} - \frac{1}{20} = \frac{10}{20} = \frac{1}{2}. \end{aligned}$$

**Illustration 7:** The probability that at least one of the events  $A$  and  $B$  occurs is 0.7 and they occur simultaneously with probability 0.2. Then  $P(\bar{A}) + P(\bar{B}) =$

- (a) 1.8                      (b) 0.6  
(c) 1.1                      (d) 0.4

**Solution:** (c) We have  $P(A \cup B) = 0.7$  and  $P(A \cap B) = 0.2$

$$\text{Now, } P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow P(A) + P(B) = 0.9 \Rightarrow 1 - P(\bar{A}) + 1 - P(\bar{B}) = 0.9$$

$$\Rightarrow P(\bar{A}) + P(\bar{B}) = 1.1$$

## INDEPENDENT EVENTS

Two or more events are said to be independent if occurrence or non-occurrence of any of them does not influence the probability of occurrence or non-occurrence of other events.

For example, when two cards are drawn from a pack of 52 playing cards with replacement (i.e., the first card drawn is put back in the pack and then the second card is drawn), then the event of occurrence of a king in the first draw and the event of occurrence of a king in the second draw are independent events because the occurrence or non-occurrence of a king in first draw does not influence the probability of occurrence or non-occurrence of the king in second draw. You can also see that the probability

of drawing a king in the second draw is  $\frac{4}{52}$  whether a king is

drawn in the first draw or not. But if the two cards are drawn without replacement, then the two events are not independent, because in this case probability of drawing a king in the second draw depends on whether a king is drawn in first draw or not. If a king is drawn in first draw, then probability of drawing a king

in second draw will be  $\frac{3}{51}$  but if a king is not drawn in first draw,

then the probability of drawing a king in second draw will be  $\frac{4}{51}$ .

**Illustration 8:** A fair coin is tossed repeatedly. If the tail appears on first four tosses, then the probability of the head appearing on the fifth toss equals

- (a)  $\frac{1}{2}$                       (b)  $\frac{1}{32}$   
(c)  $\frac{31}{32}$                       (d)  $\frac{1}{5}$

**Solution:** (a) The event that the fifth toss results a head is independent of the event that the first four tosses results tails.

$\therefore$  Probability of the required event =  $1/2$ .

## CONDITIONAL PROBABILITY

Let  $A$  and  $B$  be two events associated with a random experiment. Then, the probability of occurrence of  $A$  under the condition that  $B$  has already occurred and  $P(B) \neq 0$ , is called the conditional probability of occurrence of  $A$  when  $B$  has already occurred and it is denoted by  $P(A/B)$ .

Thus,  $P(A/B)$  = Probability of occurrence of  $A$ , if  $B$  has already occurred and  $P(B) \neq 0$

$$= \frac{P(A \cap B)}{P(B)} = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}} = \frac{n(A \cap B)}{n(B)}$$

Similarly,  $P(B/A)$  = Probability of occurrence of  $B$ , if  $A$  has already occurred and  $P(A) \neq 0$

$$= \frac{P(A \cap B)}{P(A)} = \frac{n(A \cap B)}{n(A)}$$

### 1. Multiplication Theorem on Probability

If  $A$  and  $B$  are two events associated with a random experiment, then

$$P(A \cap B) = P(A) \cdot P(B/A), \text{ if } P(A) \neq 0$$

$$\text{or } P(A \cap B) = P(B) \cdot P(A/B), \text{ if } P(B) \neq 0.$$

### 2. Multiplication Theorem for Independent Events

If  $A$  and  $B$  are independent events associated with a random experiment, then  $P(A/B) = P(A)$  and  $P(B/A) = P(B)$

$$\therefore P(A \cap B) = P(A) \cdot P(B/A) = P(A) \cdot P(B)$$

i.e., the probability of simultaneous occurrence of two independent events is equal to the product of probability of their individual occurrence.

#### Extension of multiplication theorem for independent events

If  $A_1, A_2, \dots, A_n$  are independent events associated with a random experiment, then

$$P(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n) = P(A_1) P(A_2) \dots P(A_n).$$

### 3. Probability of Occurrence of at Least One of the $n$ Independent Events

If  $p_1, p_2, p_3, \dots, p_n$  be the probabilities of occurrence of  $n$  independent events  $A_1, A_2, A_3, \dots, A_n$  respectively, then

(i) Probability of happening none of them

$$\begin{aligned} &= P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \dots \cap \bar{A}_n) \\ &= P(\bar{A}_1) P(\bar{A}_2) \cdot P(\bar{A}_3) \dots P(\bar{A}_n) \\ &= (1 - p_1)(1 - p_2)(1 - p_3) \dots (1 - p_n) \end{aligned}$$

(ii) Probability of happening at least one of them

$$\begin{aligned} &= P(A_1 \cup A_2 \cup A_3 \dots \cup A_n) \\ &= 1 - P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \dots \cap \bar{A}_n) \\ &= 1 - P(\bar{A}_1) P(\bar{A}_2) P(\bar{A}_3) \dots P(\bar{A}_n) \\ &= 1 - (1 - p_1)(1 - p_2)(1 - p_3) \dots (1 - p_n) \end{aligned}$$

**Illustration 9:** A man and his wife appear for an interview for two posts. The probability of the husband's selection is  $\frac{1}{7}$

and that of the wife's selection is  $\frac{1}{5}$ . The probability that only one of them will be selected is

- |                    |                    |
|--------------------|--------------------|
| (a) $\frac{6}{7}$  | (b) $\frac{4}{35}$ |
| (c) $\frac{6}{35}$ | (d) $\frac{2}{7}$  |

**Solution:** (d) Probability that only husband is selected

$$= P(H) P(\bar{W}) = \frac{1}{7} \left(1 - \frac{1}{5}\right) = \frac{1}{7} \times \frac{4}{5} = \frac{4}{35}$$

Probability that only wife is selected

$$= P(\bar{H}) P(W) = \left(1 - \frac{1}{7}\right) \left(\frac{1}{5}\right) = \frac{6}{7} \times \frac{1}{5} = \frac{6}{35}$$

$\therefore$  Probability that only one of them is selected

$$= \frac{4}{35} + \frac{6}{35} = \frac{10}{35} = \frac{2}{7}$$

**Illustration 10:** A bag contains 4 red and 4 blue balls. Four balls are drawn one by one from the bag, then find the probability that the drawn balls are in alternate colour.

**Solution:**

$E_1$ : Event that first drawn ball is red, second is blue and so on.

$E_2$ : Event that first drawn ball is blue, second is red and so on.

$$\therefore P(E_1) = \frac{4}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5} \quad \text{and} \quad P(E_2) = \frac{4}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5}$$

$$P(E) = P(E_1) + P(E_2) = 2 \times \frac{4}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5} = \frac{6}{35}$$

**Illustration 11:** A bag contains 5 red and 4 green balls. Four balls are drawn at random then find the probability that two balls are of red colour and two balls are of green.

**Solution:**

$n(S)$  = The total number of ways of drawing 4 balls out of total 9 balls =  ${}^9C_4$ .

If  $A_1$  = The event of drawing 2 red balls out of 5 red balls then  $n(A_1) = {}^5C_2$ .

$A_2$  = The event of drawing 2 green balls out of 4 greens balls then  $n(A_2) = {}^4C_2$ .

Let  $A$  = The event of drawing 2 balls are of red colour and 2 balls are of green colour.

$$\begin{aligned} \therefore n(A) &= n(A_1) \cdot n(A_2) = {}^5C_2 \times {}^4C_2 \\ \therefore P(A) &= \frac{n(A)}{n(S)} = \frac{{}^5C_2 \times {}^4C_2}{{}^9C_4} = \frac{5 \times 4 \times 4 \times 3}{2 \times 2 \times 9 \times 8 \times 7 \times 6} = \frac{10}{21} \end{aligned}$$

**Illustration 12:** Let  $A, B, C$  be 3 independent events such that  $P(A) = \frac{1}{3}, P(B) = \frac{1}{2}, P(C) = \frac{1}{4}$ . Then find the probability of exactly 2 events occurring out of 3 events.

**Solution:**  $P$  (exactly two of  $A, B, C$  occur)

$$\begin{aligned} &= P(A \cap B) + P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C) \\ &= P(A) \cdot P(B) + P(B) \cdot P(C) + P(C) \cdot P(A) - 3P(A) \cdot P(B) \cdot P(C) \\ &= \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{3} - 3 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{4} \end{aligned}$$

**Illustration 13:** A bag contains 3 red, 6 white and 7 blue balls. Two balls are drawn one by one. What is the probability that first ball is white and second ball is blue when first drawn ball is not replaced in the bag?

**Solution:**

Let  $A$  = Event of drawing a white ball in first draw  
and  $B$  = Event of drawing a blue ball in second draw  
Here  $A$  and  $B$  are dependent events.

$$P(A) = \frac{6}{16}, P\left(\frac{B}{A}\right) = \frac{7}{15}$$

$$P(A \cap B) = P(A) \cdot P\left(\frac{B}{A}\right) = \frac{6}{16} \times \frac{7}{15} = \frac{7}{40}$$

**Illustration 14:** Three coins are tossed together. What is the probability that first shows head, second shows tail and third shows head?

**Solution:** Let  $A$  = The event first coin shows head  
 $B$  = The event that second coin shows tail  
 $C$  = The event that third coin shows head

These three events are mutually independent.

$$\text{So, } P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

**Illustration 15:** A problem of mathematics is given to three students  $A$ ,  $B$ , and  $C$ ; whose chances of solving it are  $1/2$ ,  $1/3$ ,  $1/4$  respectively. Then find the probability that the problem will be solved.

**Solution:** Obviously the events of solving the problem by  $A$ ,  $B$  and  $C$  are independent.

The problem will be solved if at least one of the three students will solve the problem.

Therefore required probability

$$= 1 - \left[ \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \right] = 1 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{3}{4}$$

**Illustration 16:** Two dice are thrown simultaneously. Find the probability that the sum of the number appeared on two dice is 8, if it is known that the second dice always exhibits 4.

**Solution:** Let  $A$  be the event of occurrence of 4 always on the second

dice =  $\{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4)\}$ ,  $\therefore n(A) = 6$   
 and  $B$  be the event of occurrences of such numbers on both dice whose sum is 8 =  $\{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$

Thus,  $A \cap B = \{(4, 4)\}$

$$\therefore n(A \cap B) = 1$$

$$\therefore P\left(\frac{B}{A}\right) = \frac{n(A \cap B)}{n(A)} = \frac{1}{6}.$$

**Illustration 17:** A coin is tossed thrice. If  $E$  be the event of showing at least two heads and  $F$  be the event of showing head

in the first throw, then find  $P\left(\frac{E}{F}\right)$ .

**Solution:**

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

$$E = \{HHH, HHT, HTH, THH\}$$

$$F = \{HHH, HHT, HTH, HTT\}$$

$$E \cap F = \{HHH, HHT, HTH\}$$

$$n(E \cap F) = 3, n(F) = 4$$

$$\therefore \text{Reqd prob.} = P\left(\frac{E}{F}\right) = \frac{n(E \cap F)}{n(F)} = \frac{3}{4}.$$