CHAPTER 15

INEQUALITIES

INEQUALITY

Two real numbers, two algebraic expressions or an algebraic expression and a real number related by the symbol $>$, $<$, $\geq$ or $\leq$ form an inequality.

'$>$' means 'greater than'. Hence $a > b$ read as a is greater than b.

'$<$' means 'less than'. Hence $a < b$ read as a is less than b.

'$\geq$' means 'greater than or equal to'. Hence $a \geq b$ is read as a is greater than or equal to b.

'$\leq$' means 'less than or equal to'. Hence $a \leq b$ is read as a is less than or equal to b.

TYPES OF INEQUALITIES

1. **Numerical Inequalities**
   Inequalities which does not contain any variable are called numerical, inequalities.
2. **Literal Inequalities**
   Inequalities which does not contain any variable are called literal inequalities. For examples, $8 > 6$, $-7 < 0$, etc.
3. An inequality may contain more than one variable. For examples $2xy < 8$, $x + 3y \geq 20$, etc.
   An inequality in one variable may be linear, quadratic or cubic etc. For examples $2x + 5 < 10$, $x^2 + 4x + 3 \geq 0$, $-x^3 + 2x^2 - 4 \leq 8$, etc.
4. **Strict Inequalities**
   Inequalities involving the symbol '$>$' or '$<$' are called strict inequalities.
5. **Slack Inequalities**
   Inequalities involving the symbol '$\geq$' or '$\leq$' are called slack inequalities.

$$a \geq b \text{ means } a > b \text{ or } a = b$$
$$a \leq b \text{ means } a < b \text{ or } a = b$$

Note that simultaneous relation between any three different quantities a, b and c will be either $a < b < c$, $a < b \leq c$, $a \leq b < c$ or $a \leq b \leq c$

SOME PROPERTIES OF INEQUALITY

(i) If $a > b$, then evidently $b < a$ i.e. if the sides of an equality be transposed, the sign of equality must be reversed.

(ii) Sign of inequality does not change when equal numbers added to (or subtracted from) both sides of an inequality.

i.e. $\quad a > b$

$\Rightarrow \quad a + 5 > b + 5$

and also $\quad a - 4 > b - 4$

(iii) Sign of inequality does not change when both sides of an inequality can be multiplied (or divided) by the same positive number. But when both sides are multiplied or divided by a negative number, then the sign of inequality is reversed.

i.e. $\quad a \leq b$

$\Rightarrow \quad 3a \leq 3b$

and also $\quad \frac{a}{5} \leq \frac{b}{5}$.

But $\quad -3a \geq -3b$

and also $\quad \frac{a}{-5} \geq \frac{b}{-5}$

(iv) If $a > b$ and $b > c$, then $a > c$. Since $5 > 4$ and $4 > 2$, therefore $5 > 2$.

(v) If $a > b > 0$ then $\frac{1}{a} < \frac{1}{b}$

Since $6 > 2 > 0$, therefore $\frac{1}{6} < \frac{1}{2}$.

(vi) If $a > b > 0$ and $n > 0$ then $a^n > b^n$ and $(a)^{1/n} > (b)^{1/n}$

Since $3 > 2 > 0$ and $4 > 0$, therefore $(3)^4 > (2)^4$ and also $(3)^{1/4} > (2)^{1/4}$

(vii) If $x > y > 0$ and $a > 1$, then $a^x > a^y$

Since $5 > 3 > 0$ and $6 > 1$, therefore $(6)^5 > (6)^3$

(viii) If $x > y > 0$ and $0 < a < 1$ then $a^x < a^y$

Since $6 > 4 > 0$ and $0 < \frac{2}{3} < 1$, therefore $\left(\frac{2}{3}\right)^6 < \left(\frac{2}{3}\right)^4$.

IMPORTANT RESULTS

(i) Square of any real number is always equal or greater than 0. i.e. if a is a real number, then $a^2 \geq 0$.

(ii) For any real number a,
$$|a| \geq 0$$

(iii) If a is a positive real number and $|x| \leq a$, then
$$-a \leq x \leq a$$

(iv) If a is a positive real number and $|x| \geq a$, then
$$x \leq -a \text{ or } x \geq a$$

(v) $|a+b| \le |a| + |b|$

In general

$|a_1 + a_2 + a_3 + ... + a_n| \le |a_1| + |a_2| + |a_3| + ... + |a_n|$

(vi) $|a-b| \ge |a| - |b|$

(vii) $a^2 + b^2 \ge 2ab$

NOTATION AND RANGES

If a, b, c, d are four numbers such that $a < b < c < d$, then

(i) $x \in (a, b)$ means $a < x < b$
(ii) $x \in [a, b]$ means $a \le x \le b$
(iii) $x \in [a, b)$ means $a \le x < b$
(iv) $x \in (a, b]$ means $a < x \le b$
(v) $x \in (a, b) \cup (c, d)$ means $a < x < b$ or $c < x < d$

SOLUTIONS OF LINEAR INEQUALITIES IN ONE UNKNOWN

Inequalities of the form $ax + b > 0$, $ax + b \ge 0$, $ax + b < 0$ and $ax + b \le 0$ are called linear inequalities.

(i) $ax + b > 0$

$\Rightarrow \quad x > -\dfrac{b}{a}$, if $a > 0$

and $x < -\dfrac{b}{a}$, if $a < 0$

(ii) $ax + b \ge 0$

$\Rightarrow \quad x \ge -\dfrac{b}{a}$, if $a > 0$

and $x \le -\dfrac{b}{a}$, if $a < 0$

(iii) $ax + b < 0$

$\Rightarrow \quad x < -\dfrac{b}{a}$, if $a > 0$

and $x > -\dfrac{b}{a}$, if $a < 0$

(iv) $ax + b \le 0$

$\Rightarrow \quad x \le -\dfrac{b}{a}$, if $a > 0$

and $x \ge -\dfrac{b}{a}$, if $a < 0$

Illustration 1: Solve $2(x-3) + 4 \ge 4 - x$

Solution: $2(x-3) + 4 \ge 4 - x$

$\Rightarrow \quad 2x - 6 + 4 \ge 4 - x \Rightarrow 2x + x - 2 \ge 4$

$\Rightarrow \quad 3x \ge 4 + 2 \Rightarrow 3x \ge 6$

$\Rightarrow \quad x \ge \dfrac{6}{3} \Rightarrow x \ge 2$

This solution can also be written as $x \in [2, \infty)$.

Illustration 2: Solve $3(x+4) + 1 < 2(3x+1) + 15$

Solution: $3(x+4) + 1 < 2(3x+1) + 15$

$\Rightarrow \quad 3x + 12 + 1 < 6x + 2 + 15$

$\Rightarrow \quad 3x - 6x < 17 - 13$

$\Rightarrow \quad -3x < 4$

$\Rightarrow \quad x > \dfrac{4}{-3} \Rightarrow x > -\dfrac{4}{3}$

This solution can also be written as $x \in \left(-\dfrac{4}{3}, \infty\right)$.

Illustration 3: Solve the following inequations:

$$\frac{2x+4}{x-1} \ge 5$$

Solution: We have,

$$\frac{2x+4}{x-1} \ge 5$$

$\Rightarrow \quad \dfrac{2x+4}{x-1} - 5 \ge 0$

$\Rightarrow \quad \dfrac{x-3}{x-1} \le 0$ [Dividing both sides by 3]

$\Rightarrow \quad 1 < x \le 3$

$\Rightarrow \quad x \in (1, 3]$

Hence, the solution set of the given inequation is (1,3].

Illustration 4: Solve: $-5 \le \dfrac{2-3x}{4} \le 9$.

Solution: We have,

$$-5 \le \frac{2-3x}{4} \le 9$$

$\Rightarrow \quad \dfrac{22}{3} \ge x \ge \dfrac{-34}{3}$

$\Rightarrow \quad \dfrac{-34}{3} \le x \le \dfrac{22}{3}$

$\Rightarrow \quad x \in [-34/3, 22/3]$

Hence, the interval $[-34/3, 22/3]$ is the solution set of the given system of inequations.

INEQUALITIES CONTAINING A MODULUS

(i)
- If $a > 0$, then $|x| \le a \Rightarrow -a \le x \le a$
- If $a > 0$, then $|x| < a \Rightarrow -a < x < a$

(ii)
- If $a > 0$, then $|x| \ge a \Rightarrow x \le -a$ and $x \ge a$
- If $a > 0$,then $|x| > a \Rightarrow x < -a$ and $x > a$
- If $a < 0$, then $|x| \ge a \Rightarrow x \le a$ and $x \ge -a$
- If $a < 0$, then $|x| > a \Rightarrow x < a$ and $x > -a$

Illustration 5: Solve $|x-3| \ge 4$

Solution: $|x-3| \ge 4$

$\Rightarrow \quad (x-3) \le -4$ and $(x-3) \ge 4$

$\Rightarrow \quad x \le -4 + 3$ and $x \ge 4 + 3$

$\Rightarrow \quad x \le -1$ and $x \ge 7$

i.e. $\quad x \in (-\infty, -1] \cup [7, \infty)$

Illustration 5: Solve $|5 - 4x| < -2$

Solution: $|5 - 4x| < -2$

$\Rightarrow \quad (5-4x) < -2$ and $(5-4x) > 2$

$\Rightarrow \quad -4x < -2 - 5$ and $-4x > 2 - 5$

$\Rightarrow \quad -4x < -7$ and $-4x > -3$

$\Rightarrow \quad 4x > 7$ and $4x < 3$

$\Rightarrow \quad x > \dfrac{7}{4}$ and $x < \dfrac{3}{4}$

i.e. $\quad x \in \left(-\infty, \dfrac{3}{4}\right) \cup \left(\dfrac{7}{4}, \infty\right)$