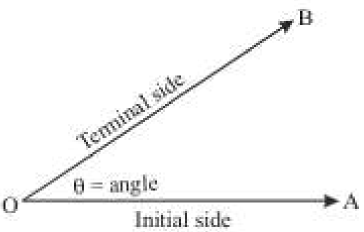
CHAPTER 23

TRIGONOMETRY AND ITS APPLICATION

ANGLE

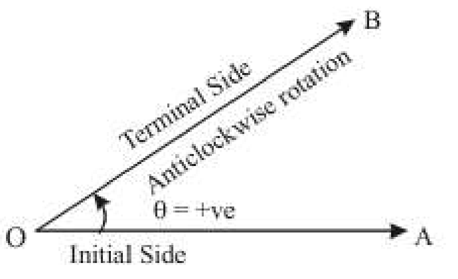
Consider a ray $\overrightarrow{OA}$. If this ray rotates about its end point O and takes the position OB, then the angle $\angle AOB$ has been generated. An angle is considered as the figure obtained by rotating a given ray about its end-point. The initial position OA is called the initial side and the final position OB is called terminal side of the angle. The end point O about which the ray rotates is called the vertex of the angle. The measure of an angle is the amount of rotation performed to get the terminal side from initial side.

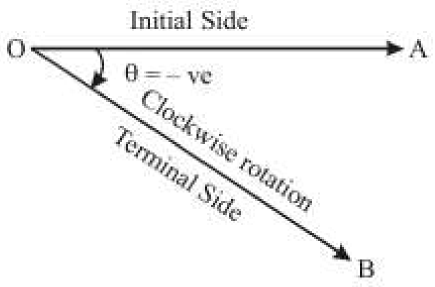


There are several units for measuring angles.
But in this chapter we use degree measure of angle.

Sense of sign of an angle :

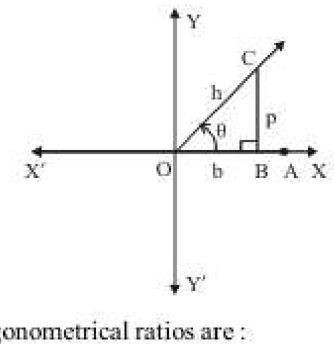
The sense of sign of an angle is said to be positive or negative according as the initial side rotates in anticlockwise or clockwise direction respectively to get the terminal side.





TRIGONOMETRIC RATIOS

In the figure XOX' and YOY' are horizontal and vertical axes respectively.
Horizontal axis $X'OX$ is called X-axis and vertical axis YOY' is called Y-axis.
Let A be a point on OX. Also suppose that the ray OA start rotating in the XY-plane in anti-clockwise direction from the initial position OA about the point O till it reaches its final position OC after some interval of time (See Fig.). Thus, an angle COA is formed with x-axis. Let $\angle COA = \theta$. (θ is a Greek letter, and we read it as "theta"). Draw $CB \perp OX$. Now clearly $\Delta\ CBO$ is a right angled triangled.
In right ΔCBO, OC is the hyponenus. For angle $\theta = \angle COA$, BC and OB are called side opposite to angle θ and adjucent side of angle θ respectively.
Let $CB = p$, $OB = b$ and $OC = h$. We define the different ratios between hypotenus, side opposite to angle θ and adjucent side of angle θ as trigonometric ratios for angle θ.



These trigonometrical ratios are :

$$\text{Sine of } \theta = \frac{\text{Side opposite to angle } \theta}{\text{Hypotenuse}} = \frac{CB}{OC} = \frac{p}{h}$$

$$\text{Cosine of } \theta = \frac{\text{Adjacent side to angle } \theta}{\text{Hypotenuse}} = \frac{OB}{OC} = \frac{b}{h}$$

$$\text{Tangent of } \theta = \frac{\text{Side opposite to angle } \theta}{\text{Adjacent side to angle } \theta} = \frac{CB}{OB} = \frac{p}{b}$$

Cotangent of $\theta = \dfrac{\text{Adjacent side to angle } \theta}{\text{Side opposite to angle } \theta} = \dfrac{OB}{CB} = \dfrac{b}{p}$

Secant of $\theta = \dfrac{\text{Hypotenuse}}{\text{Adjacent side to angle } \theta} = \dfrac{OC}{OB} = \dfrac{h}{b}$

Cosecant of $\theta = \dfrac{\text{Hypotenuse}}{\text{Side opposite to angle } \theta} = \dfrac{OC}{CB} = \dfrac{h}{p}$

Sine of θ is abbreviated as sin θ, Cosine of θ is abbreviated as cos θ, Tangent of θ is abbreviated as tan θ, Cotangent of θ is abbreviated as cot θ, Secant of θ is abbreviated as sec θ and Cosecant of θ is abbreviated as cosec θ

Now, throughout the study of trigonometry we shall use only abbreviated form of these trigonometric ratios.

Thus,

$$\sin\theta = \frac{p}{h},\ \cos\theta = \frac{b}{h},\ \tan\theta = \frac{p}{b},\ \cot\theta = \frac{b}{p},$$

$$\sec\theta = \frac{h}{b},\ \operatorname{cosec}\theta = \frac{h}{p}$$

Note that sin θ is an abbreviation for "sine of angle θ"and not the product of sin and θ.

VALUE OF TRIGONOMETRIC RATIOS FOR SOME SPECIFIC ANGLES

The values of trigonometric ratios for angles 0°, 30°, 45°, 60° and 90° are quite often used in solving problems in our day-to-day life. Thus the following table is very useful.

IMPORTANT TABLE

(θ) → Trigonometrical ratio ↓	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
$\cot\theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\operatorname{cosec}\theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined

BASIC FORMULAE OR TRIGONOMETRIC IDENTITY

(i) $\sin\theta \cdot \operatorname{cosec}\theta = 1$ or $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$ or $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

(ii) $\cos\theta \cdot \sec\theta = 1$ or $\cos\theta = \dfrac{1}{\sec\theta}$ or $\sec\theta = \dfrac{1}{\cos\theta}$

(iii) $\tan\theta \cdot \cot\theta = 1$ or $\cot\theta = \dfrac{1}{\tan\theta}$ or $\tan\theta = \dfrac{1}{\cot\theta}$

(iv) $\sin^2\theta + \cos^2\theta = 1$

or $\cos^2\theta = 1 - \sin^2\theta$ or $\sin^2\theta = 1 - \cos^2\theta$

(v) $\sec^2\theta - \tan^2\theta = 1$

or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

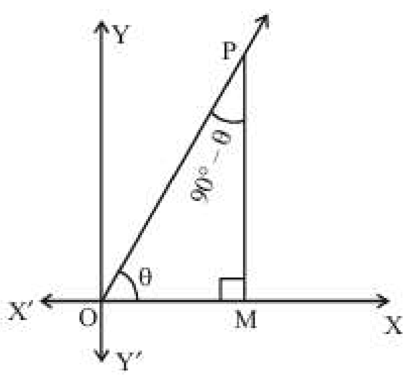
(vi) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

or $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \operatorname{cosec}^2\theta - 1$

(vii) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(viii) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

TRIGONOMETRIC RATIOS FOR COMPLEMENTARY ANGLES



$\sin(90° - \theta) = \frac{OM}{OP} = \cos\theta$, $\cos(90° - \theta) = \frac{PM}{OP} = \sin\theta$,

$\tan(90° - \theta) = \frac{OM}{PM} = \cot\theta$, $\cot(90° - \theta) = \frac{PM}{OM} = \tan\theta$,

$\text{cosec}(90° - \theta) = \frac{OP}{OM} = \sec\theta$ and $\sec(90° - \theta) = \frac{OP}{PM} = \text{cosec}\,\theta$

Illustration 1: If tan A = 1 and sin B = $\frac{1}{\sqrt{2}}$, find the value of cos (A + B) where A and B are both acute angles.

Solution : $\tan A = 1 \Rightarrow A = 45°$ and $\sin B = \frac{1}{\sqrt{2}} \Rightarrow B = 45°$

$\therefore \cos(A + B) = \cos(45° + 45°) = \cos 90° = 0$

Illustration 2: If sin θ – cos θ = 0 and 0 < θ < 90°, find θ.

Solution :

$\sin\theta - \cos\theta = 0 \Rightarrow \sin\theta = \cos\theta \Rightarrow \frac{\sin\theta}{\cos\theta} = 1 \Rightarrow \tan\theta = 1$

But $0 < \theta < 90° \Rightarrow \theta = 45°$

Illustration 3: If A , B and C are interrior angles of a triangle ABC, then show that

$$\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$$

Solution : In ΔABC,

$A + B + C = 180° \Rightarrow B + C = 180° - A \Rightarrow \frac{B+C}{2} = 90° - \frac{A}{2}$

$\Rightarrow \sin\left(\frac{B+C}{2}\right) = \sin\left(90 - \frac{A}{2}\right) \Rightarrow \sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$.

Illustration 4: Simplyfy : $\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} + \sin\theta\cos\theta$

Solution :

$$\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} + \sin\theta\cos\theta$$

$$= \frac{(\sin\theta + \cos\theta)(\sin^2\theta + \cos^2\theta - \sin\theta\cos\theta)}{(\sin\theta + \cos\theta)} + \sin\theta\cos\theta$$

$$= \sin^2\theta + \cos^2\theta - \sin\theta\cos\theta + \sin\theta\cos\theta$$

$$= \sin^2\theta + \cos^2\theta = 1$$

Illustration 5: Evaluate $\frac{\cos 43°}{\cos 47°} + \frac{\sec 32°}{\text{cosec}\,58°}$

Solution :

We know that $\cos(90° - \theta) = \sin\theta$

$\sin 47° = \sin(90° - \theta) = \cos 43°$

Also, $\text{cosec}\,58° = \text{cosec}(90° - 32°) = \cos 32°$

$$\therefore \frac{\cos 43°}{\cos 47°} + \frac{\sec 32°}{\text{cosec}\,58°} = \frac{\cos 43°}{\cos 43°} + \frac{\sec 32°}{\sec 32°} = 1 + 1 = 2$$

Illustration 6: Evaluate $\frac{\sec^2 54° - \cot^2 36°}{\text{cosec}^2 57° - \tan^2 33°}$ $+ 2\sin^2 38° \sec^2 52° - \sin^2 45° + \frac{2}{\sqrt{3}}\tan 17° \tan 60° \tan 73°$.

Solution : The given expression is

$$\frac{\sec^2 54° - \cot^2 36°}{\text{cosec}^2 57° - \tan^2 33°} + 2\sin^2 38° \sec^2 52° - \sin^2 45° + \frac{2}{\sqrt{3}}\tan 17° \tan 60° \tan 73°$$

$$= \frac{\sec^2(90° - 36°) - \cot^2 36°}{\text{cosec}^2(90° - 33°) - \tan^2 33°} + 2\sin^2 38° \sec^2 (90° - 38°) - \sin^2 45° + \frac{2}{\sqrt{3}}\tan(90° - 73°)\tan 73° \tan 60°$$

$$= \frac{1}{1} + 2\sin^2 38° \times \frac{1}{\sin^2 38°} - \frac{1}{2} + \frac{2}{\sqrt{3}} \times \frac{1}{\tan 73°} \times \tan 73° \times \sqrt{3}$$

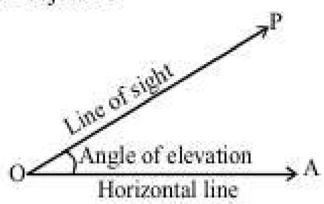
$$[\because \text{cosec}^2\theta - \cot^2\theta = 1,\ \sec^2\theta - \tan^2\theta = 1]$$

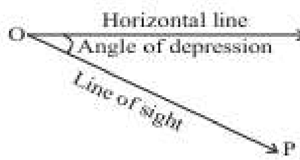
$$= 1 + 2 - \frac{1}{2} + 2 = 5 - \frac{1}{2} = \frac{9}{2}$$

ANGLE OF ELEVATION AND ANGLE OF DEPRESSION

Let an observer at the point O is observing an object at the point P. The line OP is called the LINE OF SIGHT of the point P. Let OA be the horizontal line passing through O. O, A and P be in the same vertical plane.

If object P be above the horizontal line OA, then the acute angle AOP, between the line of sight and the horizontal line is known as ANGLE OF ELEVATION of object P. If the object P is below the horizontal line OA then the angle AOP, between the line of sight and the horizontal line is known as ANGLE OF DEPRESSION of object P.





TO FIND THE HEIGHT AND THE DISTANCE OF AN INACCESSIBLE TOWER STANDING ON A HORIZONTAL PLANE

Let AB be a tower and B be its foot. On the horizontal line through B, take two points P and Q. Measure the length PQ.

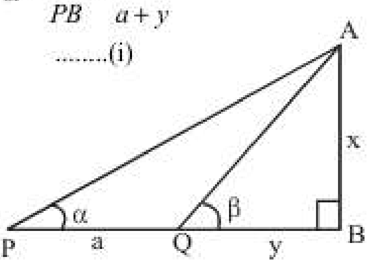
Let PQ = a.

Let the angles of elevation of the top A of the tower as seen from P and Q be respectively α and β ($\beta > \alpha$), then

$\angle APB = \alpha$, $\angle AQB = \beta$. Let $AB = x$, $BQ = y$.

From right angled Δ ABP, $\tan\alpha = \frac{AB}{PB} = \frac{x}{a+y}$

$\therefore a + y = x\cot\alpha.$ ....(i)



From right angled ΔABQ,

$$\tan\beta = \frac{AB}{BQ} = \frac{x}{y}$$

$\therefore \quad y = x\cot\beta$ ....(ii)

From equations (i) and (ii),

$$\therefore a = x\cot\alpha - x\cot\beta.$$

$$\Rightarrow \quad x = \frac{a}{\cot\alpha - \cot\beta}$$

Also $y = x\cot\alpha - a \;\Rightarrow\; y = \frac{a\cot\alpha}{\cot\alpha - \cot\beta} - a$

$$\Rightarrow y = \frac{a\cot\alpha - a(\cot\alpha - \cot\beta)}{\cot\alpha - \cot\beta} \;\Rightarrow\; y = \frac{a\cot\beta}{\cot\alpha - \cot\beta}$$

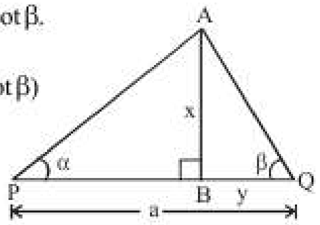
In the above case, P and Q are on the same side of the tower. If the two points are on the opposite sides of the tower then from the adjoining figure, we get

$$\tan\alpha = \frac{x}{PB} \; or \; PB = x\cot\alpha$$

and $\tan\beta = \frac{x}{BQ} \; or \; BQ = x\cot\beta.$

$$\therefore a = PB + BQ = x(\cot\alpha + \cot\beta)$$

$$\therefore x = \frac{a}{\cot\alpha + \cot\beta}$$



and $y = BQ = x\cot\beta$

Note that : Here, all the lines AP, AQ, AB are in the same plane.

Illustration 7: The angle of elevation of a ladder leaning against a wall is 58°, and the foot of the ladder is 9.6 m from the wall. Find the length of the ladder.

Solution : Let AB be the ladder leaning against a wall OB such that $\angle$ OAB = 58° and OA = 9.6 m

In Δ AOB , we have, $\cos 58° = \frac{OA}{AB}$

$$\Rightarrow \quad AB = \frac{OA}{\cos 58°}$$

$$\Rightarrow AB = \frac{9.6}{0.5299} = 18.11 \text{ m}$$

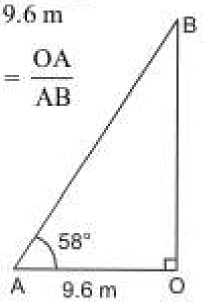
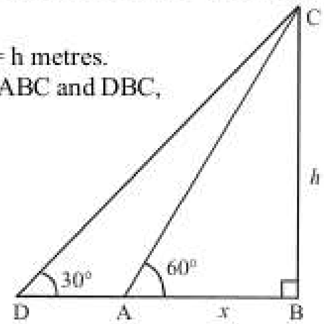


Illustration 8: A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60° ; when he retreates 20m from the bank, he finds the angle to be 30°. Find the height of the tree and the breadth of the river.

Solution : Let AB be the width of the river and BC be the tree which makes an angle of 60° at a point A on the opposite bank. Let D be the position of the person after retreating 20 m from the bank.



Let AB = x metres and BC = h metres.

From right angled triangles ABC and DBC,

we have $\tan 60° = \frac{BC}{AB}$

and $\tan 30° \Rightarrow \sqrt{3} = \frac{h}{x}$

and $\frac{1}{\sqrt{3}} = \frac{h}{x+20} \Rightarrow h = x\sqrt{3}$

and $h = \frac{x+20}{\sqrt{3}} \;\Rightarrow\; x\sqrt{3} = \frac{x+20}{\sqrt{3}}$

$\Rightarrow$ 3x = x + 20 $\Rightarrow x$ = 10 m

Putting x = 10 in h = $\sqrt{3}\,x$, we get

$h = 10\sqrt{3}$ = 17. 32 m

Hence, height of the tree = 17.32 m and the breadth of the river = 10 m.

Illustration 9: The angles of elevation of the top of a tower at the top and the foot of a pole of height 10 m are 30° and 60° respectively. Find the height of the tower.

Solution : Let AB and CD be the pole and tower respectively.

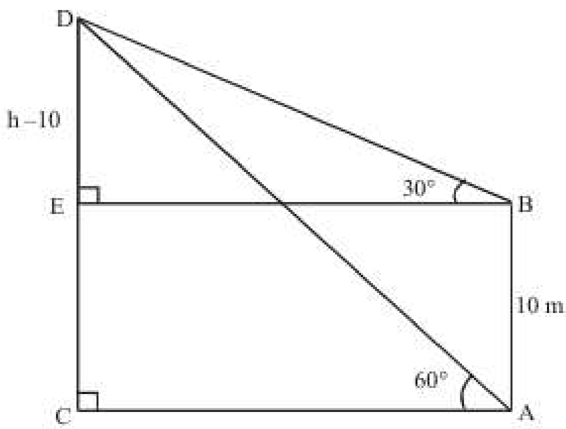
Let CD = h

Then $\angle$ DAC = 60° and $\angle$ DBE = 30°

Now $\frac{CD}{CA}$ = tan 60° = $\sqrt{3}$ $\therefore$ CD = $\sqrt{3}$ CA

$$\Rightarrow \frac{h}{\sqrt{3}} = CA$$

Again $\frac{DE}{BE}$ = tan 30° = $\frac{1}{\sqrt{3}}$



$\therefore$ (h – 10) = $\frac{BE}{\sqrt{3}} = \frac{CA}{\sqrt{3}} = \frac{h/\sqrt{3}}{\sqrt{3}} = \frac{h}{3}$ [$\because$ BE = CA]

$\Rightarrow$ 3h – 30 = h $\Rightarrow$ 2h = 30 $\Rightarrow$ h = 15

Hence, height of the tower = 15 m

Illustration 10: A man is standing on the deck of a ship, which is 8m above water level. He observes the angle of elevation of the top of a hill as 60° and the angle of depression of the base of the hill as 30°. Calculate the distance of the hill from the ship and the height of the hill.

Solution : Let x be the distance of hill from man and $h + 8$ be height of hill which is required.

In rt. ΔACB,

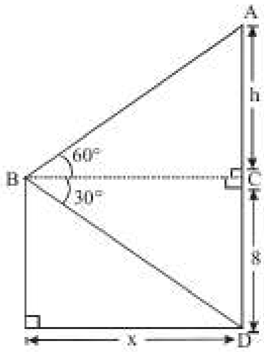
$$\tan 60° = \frac{AC}{BC} = \frac{h}{x}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{x}$$

In rt. ΔBCD,

$$\tan 30° = \frac{CD}{BC} = \frac{8}{x}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{8}{x} \quad \Rightarrow \quad x = 8\sqrt{3}$$



$\therefore$ Height of hill $= h + 8 = \sqrt{3}x + 8 = (\sqrt{3})\,(8\sqrt{3}) + 8 = 32$ m

Distance of ship from hill $= x = 8\sqrt{3}$ m

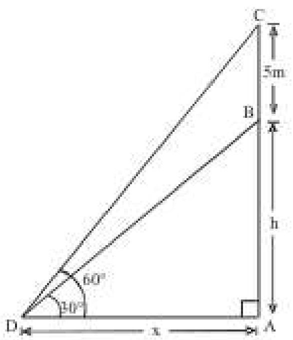
Illustration 11: A vertical to stands on a horizontal plane and is surmounted by a vertical flag staff of height 6 meters. At point on the plane, the angle of elevation of the bottom and the top of the flag staff are respectively 30° and 60°. Find the height of tower.

Solution :

Let AB be the tower of height h meter and BC be the height of flag staff surmounted on the tower.

Let the point of the plane be D at a distance m meter from the foot of the tower.

In ΔABD,



$$\tan 30° = \frac{AB}{BD} \quad \Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{x} \Rightarrow x = \sqrt{3}h \qquad \text{........... (i)}$$

In Δ ADC,

$$\tan 60° = \frac{AC}{AD} \quad \Rightarrow \quad \sqrt{3} = \frac{5+h}{x} \Rightarrow x = \frac{5+h}{\sqrt{3}} \qquad \text{.......... (ii)}$$

From (i) and (ii), $\sqrt{3}h = \dfrac{5+h}{\sqrt{3}} \quad \Rightarrow \quad 3h = 5 + h \Rightarrow \quad 2h = 5$

$$\Rightarrow \; h = \frac{5}{2} = 2.5 \text{ m}$$

So, the height of tower = 2.5 m

Illustration 12: The angles of depressions of the top and bottom of 8m tall building fron the top of a multistoried building are 30° and 45° respectively. Find the height of multistoried building and the distance between the two buildings.

Solution : Let AB be the multistoried building of height h and let the distance between two buildings be x meters.

$\angle XAC = \angle ACB = 45°$ (Alternate angles)

$\angle XAD = \angle ADE = 30°$ (Alternate angles)

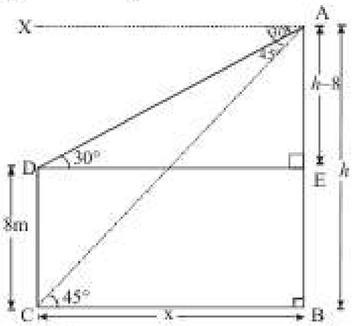
In ΔADE, $\tan 30° = \dfrac{AE}{ED} \quad \Rightarrow \quad \dfrac{1}{\sqrt{3}} = \dfrac{h-8}{x}$

$[\because CB = DE = x]$

$$\Rightarrow \quad x = \sqrt{3}\,(h-8) \qquad \text{........... (i)}$$

In ΔACB,

$$\tan 45° = \frac{h}{x} \quad \Rightarrow \quad 1 = \frac{h}{x} \quad \Rightarrow \; = \text{h} \qquad \text{........... (ii)}$$



From (i) and (ii),

$$\sqrt{3}\,(h-8) = h \quad \Rightarrow \quad \sqrt{3}h - 8\sqrt{3} = h$$

$$\Rightarrow \quad \sqrt{3}h - h = 8\sqrt{3} \quad \Rightarrow \quad h(\sqrt{3}-1) = 8\sqrt{3}$$

$$\Rightarrow \quad h = \frac{8\sqrt{3}}{\sqrt{3}-1} \times \frac{(\sqrt{3}+1)}{\sqrt{3}+1} \quad \Rightarrow \quad h = \frac{8\sqrt{3}(\sqrt{3}+1)}{2}$$

$$\Rightarrow \quad h = 4\sqrt{3}(\sqrt{3}+1) \quad \Rightarrow \quad h = 4(3+\sqrt{3}) \text{ metres}$$

From (ii), $x = h$

So, $x = 4(3+\sqrt{3})$ metres

Hence, height of multistoried building $= 4\,(3+\sqrt{3})$ metres

distance between two building $= 4\,(3+\sqrt{3})$ metres.

Illustration 13: The angle of elevation of an aeroplane from a point on the ground is 45°. After a flight of 15 sec, the elevation changes to 30°. If the aeroplane is flying at a height of 3000 metres, find the speed of the aeroplane.

Solution :

Let the point on the ground is E which is y metres from point B and let after 15 sec. flight it covers x metres distance

In ΔAEB, $\tan 45° = \dfrac{AB}{EB}$

$$\Rightarrow \quad 1 = \frac{3000}{y} \Rightarrow y = 3000\text{m} \qquad \text{............. (i)}$$

In $\Delta\, CED$, $\tan 30° = \frac{CD}{ED} \Rightarrow \frac{1}{\sqrt{3}} = \frac{3000}{x+y}$

$(\because AB = CD)$

$\Rightarrow\ x + y = 3000\sqrt{3}$ .... (ii)

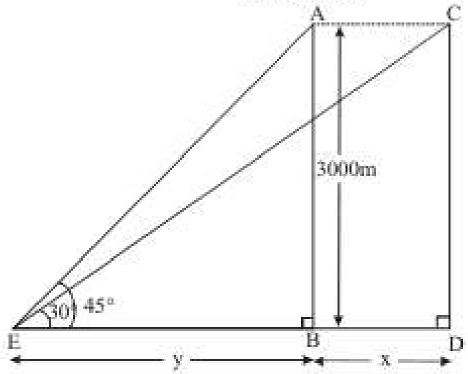
From eqs. (i) and (ii)

$x + 3000 = 3000\sqrt{3} \Rightarrow x = 3000\sqrt{3} - 3000$

$\Rightarrow\ x = 3000\,(\sqrt{3}-1) \Rightarrow x = 3000 \times (1.732 - 1)$

$\Rightarrow\ x = 3000 \times 0.732 \Rightarrow x = 2196\text{m}$

Speed of aeroplane $= \frac{\text{Distance covered}}{\text{Time taken}}$



$= \frac{2196}{15}\,\text{m/sec} = 146.4\ \text{m/sec}$

$= \frac{2196}{15} \times \frac{18}{5}\,\text{km/hr} = 527.04\ \text{km/hr}$

Hence, the speed of aeroplane is 527.04 km/hr.

Illustration 14: A boy is standing on the ground and flying a kite with 100m of string at an elevation of 30°. Another boy is standing on the roof of a 10m high building and is flying his at an elevation of 45°. Both the boys are on opposite sides of both the kites. Find the length of the string that the second boy must have so that the two kites meet.

Solution :

Let the length of second string be x m.

In $\Delta\, ABC$,

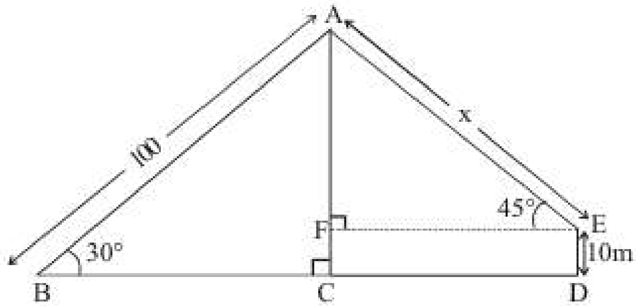
$\sin 30° = \frac{AC}{AB}$ or $\frac{1}{2} = \frac{AC}{100} \Rightarrow AC = 50\text{m}$

In $\Delta\, AEF$,

$\sin 45° = \frac{AF}{AE} \Rightarrow \frac{1}{\sqrt{2}} = \frac{AF - FC}{x}$

$\Rightarrow\ \frac{1}{\sqrt{2}} = \frac{50-10}{x}$ $[\because AC = 50\text{m}, FC = ED = 10\text{m}]$

$\Rightarrow\ \frac{1}{\sqrt{2}} = \frac{40}{x} \Rightarrow x = 40\sqrt{2}\text{m}$



So the length of string that the second boy must have so that the two kites meet $= 40\sqrt{2}\text{m}$