CHAPTER 14

QUADRATIC AND CUBIC EQUATIONS

QUADRATIC POLYNOMIALS

An expression in the form of $ax^2 + bx + c$, where a,b,c are real numbers but $a \neq 0$, is called a quadratic polynomial. For examples $2x^2 - 5x + 3, -x^2 + 2x, 3x^2 - 7, \sqrt{2}x^2 + 7x + 2$, etc.

QUADRATIC EQUATIONS

A quadratic expression when equated to zero is called a quadratic equation. Hence an equation in the form of $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$, is called a quadratic equation. For examples,

$$2x^2 - 5x + 3 = 0, \; -x^2 + 2x = 0,$$

$$3x^2 - 7 = 0 \text{ and } \sqrt{2}x^2 + 7x + 2 = 0, \text{ etc.}$$

Illustration 1: Which of the following is not a quadratic equation?

(a) $x^2 - 2x + 2(3 - x) = 0$
(b) $x(x + 1) + 1 = (x - 2)(x - 5)$
(c) $(2x - 1)(x - 3) = (x + 5)(x - 1)$
(d) $x^3 - 4x^2 - x + 1 = (x - 2)^3$

Solution: (b) Hint: $x(x + 1) + 1 = (x - 2)(x - 5)$

$\Rightarrow x^2 + x + 1 = x^2 - 7x + 10$

$\Rightarrow 8x - 9 = 0$, which is not a quadratic equation.

Discriminant (D)

For the quadratic equation $ax^2 + bx + c = 0$,

$$D = b^2 - 4ac$$

Here, D is the symbol of discriminant.

Roots or Solution of a Quadratic Equation

(i) If $D > 0$, then the quadratic equation $ax^2 + bx + c = 0$ has two distinct roots given by

$$\alpha = \frac{-b + \sqrt{D}}{2a} \text{ and } \beta = \frac{-b - \sqrt{D}}{2a}$$

Here α and β are symbols of roots of the quadratic equation.

(ii) If $D = 0$, then the quadratic equation $ax^2 + bx + c = 0$ has two equal roots given by

$$\alpha = \beta = -\frac{b}{2a}$$

Illustration 2: If $ax^2 + bx + c = 0$ has equal roots, then $c =$

(a) $-\frac{b}{2a}$ **(b)** $\frac{b}{2a}$

(c) $-\frac{b^2}{4a}$ **(d)** $\frac{b^2}{4a}$

Solution: (d) $ax^2 + bx + c = 0$ has equal roots if disc. $b^2 - 4ac = 0$

$\Rightarrow \quad b^2 = 4ac$

$\Rightarrow \quad c = \frac{b^2}{4a}$

Illustration 3: If $x^2 + 4x + k = 0$ has real roots, then

(a) $k \geq 4$ **(b)** $k \leq 4$
(c) $k \leq 0$ **(d)** $k \geq 0$

Solution: (b) Since $x^2 + 4x + k = 0$ has real roots.

$\therefore$ Disc. $(4)^2 - 4k \geq 0$

$\Rightarrow 16 - 4k \geq 0$

$\Rightarrow 4k \leq 16$

$\Rightarrow k \leq 4$

Properties of Quadratic Equations and Their Roots

(i) If D is a perfect square then roots are rational otherwise irrational.

(ii) If $p + \sqrt{q}$ is one root of a quadratic equation, then their conjugate $p - \sqrt{q}$ must be the other root and vice-versa, where p is rational and $\sqrt{q}$ is a surd.

(iii) If a quadratic equation in x has more than two roots, then it is an identity in x.

SUM AND PRODUCT OF ROOTS

If α and β are the roots of a quadratic equation $ax^2 + bx + c = 0$, Then,

Sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$

Product of roots, $\alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

Illustration 4: Find the sum and product of roots of $-2x^2 + 3x - 5 = 0$.

Solution: Sum of roots $= -\frac{b}{a} = -\frac{3}{-2} = \frac{3}{2}$

Product of roots $= \frac{c}{a} = \frac{-5}{-2} = \frac{5}{2}$

FORMATION OF AN EQUATION WITH GIVEN ROOTS

If α and β are the roots of a quadratic equation, then the quadratic equation will be

$$x^2 - (\alpha + \beta)x + \alpha.\beta = 0$$

i.e., x^2 – (Sum of the roots) x + Product of the roots = 0

Illustration 5: If α and β are the roots of the equation $3x^2 - x + 4 = 0$, then find the quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Solution: $\alpha + \beta = -\frac{-1}{3} = \frac{1}{3}, \alpha.\beta = \frac{4}{3}$

Now, $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{a+b}{ab}$

$$= \frac{\frac{1}{3}}{\frac{4}{3}} = \frac{1}{4}$$

$$\frac{1}{\alpha}\cdot\frac{1}{\beta} = \frac{1}{ab} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

Hence required quadratic equation,

$$x^2 - \frac{1}{4}x + \frac{3}{4} = 0$$

$$\Rightarrow \quad 4x^2 - x + 3 = 0$$