CHAPTER 17

SET THEORY

SETS

A set is a well- defined collection of different objects.

In everyday life, we often speak about the collection of objects of particular kind such as a cricket team, the rivers of India, the vowels in the English alphabet etc. Each of these collection is well-defined collection of objects in the sense that we can definitely decide whether a given particular object belongs to a given collection or not. For example, we say that 10 does not belongs to the given collection of all odd natural numbers. On the other hand, 15 belongs to this given collection.

Note that

(i) Objects, elements and members of a set are synonymous terms.

(ii) Sets are usually denoted by capital letters A, B, C, D, E, F, etc.

(iii) The elements of a set are represented by small letters a, b, c, d, e, f, etc.

(iv) Each element in a set comes only once i.e. repetition of any element is not allowed.

If a is an element of a set A, we say that "a belongs to A". The Greek symbol $\in$ (epsilon) is used to denote the phrase 'belongs to'.

Thus, we write $a \in A$. If 'b' is not an element of a set A, we write $b \notin A$ and read "b does not belong to A".

If V be the set of vowels of English alphabet, then $a \in V$ but $b \notin V$. In the set P of prime factors of 30, $3 \in P$ but $15 \notin P$.

REPRESENTATIONS OF SETS

There are two methods of representing a set:

(i) Roster or tabular form (ii) Set-builder form.

Roster or Tabular Form

(i) In roster form, all the elements of a set are listed within a bracket { } and separated by commas. For example, the set of all even positive integers less than 7 is described in roster form as $\{2, 4, 6\}$.

(ii) In roster form, the order in which the elements are listed is immaterial.

Set-builder Form

The set $\{a, e, i, o, u\}$ in roster form can be written as set in builder form as $\{x : x \text{ is } a \text{ vowel of English alphabet}\}$. Here the set written in set builder form is read as 'x' is an element of the set such that x is a vowel of English alphabet'. Here the colon (:) read as 'such that'. In set-builder, a common property which posses all the elements of the set is written after colon (:).

Statement	Roster form	Set-builder form
(1) The set of currencies used in USA, England, Japan, Germany and Russia.	{Dollar, Pound, Yen, Euro, Rouble}	{x : x is the currencies used in USA, England, Japan, Germany and Russia }
(2) The set of Capital of Kerala, Karnataka, Tamilnadu, Andhra Pradesh and Gujarat	{Tiruvananthapuram, Banglore, Chennai, Hyderabad and Gandhi Nagar}	{x : x is the capitals of Kerala, Karnataka, Tamilnadu, Andhra Pradesh and Gujarat}
(3) The set of all distinct letters used in the word student.	$\{s, t, u, d, e, n\}$	{x : x is the distinct letters used in the word student.}
(4) The set of all the states of India beginning with the letter A.	{Andhra Pradesh, Arunachal Pradesh, Assam}	{x : x is the state of India beginning with the letter A}
(5) The set of six presidents of India since 1980.	{Neelam Sanjeeva Reddy, Gyani Zail Singh, Radha Swami Venkat Raman, Dr. Shankar Dayal Sharma, K.R. Narayan, A.P.J. Abdul Kalam}	{x : x is the presidents of India since 1980}
(6) The set of all natural numbers between 11 and 15.	$\{12, 13, 14\}$	$\{x : x \in N, 11 < x < 15\}$

Illustration 1: **Write the set $X = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, ...\right\}$ in the set-builder form.**

Solution:

We observe that the elements of set X are the reciprocals of the squares of all natural numbers. So, the set X in set builder form is

$$X = \left\{\frac{1}{n^2} : n \in N\right\}.$$

Illustration 2: **Write the following intervals in set builder form**

(i) (–3, 0) (ii) [6, 12] (iii) (6, 12] (iv) [–23, 5)

Solution:

The following intervals are written in set builder form as :

(i) (–3, 0) is an open interval which does not include both – 3 and 0. So, it can be shown in the set builder form as : $\{x : x \in R, -3 < x < 0\}$.

(ii) [6, 12] is a closed interval which includes both 6 and 12. So it can be shown in the set builder form as $\{x : x \in R, 6 \le x \le 12\}$

(iii) (6, 12] is an interval open at the first end and closed at the second end i.e. it excludes 6 but includes 12. So it is shown in the set builder form as : $\{x : x \in R, 6 < x \le 12\}$

(iv) [–23, 5) is an interval closed at the first end point but open at the second end point. It means that the interval includes –23 but excludes 5. It is written in the set builder form as : $\{x : x \in$ R, $-23 \le x < 5\}$.

STANDARD SYMBOLS OF SOME SPECIAL SETS

N : Set of all natural numbers
Z : Set of all integers
Q : Set of all rational numbers
R : Set of all real numbers
Z$^+$: Set of all positive integers
Q$^+$: Set of all positive rational numbers, and
R$^+$: Set of all positive real numbers.

The symbols for the special sets given above will be referred throughout the chapter.

TYPES OF SETS

Empty Set

A set which does not contain any element is called an empty set, null set or void set.

The empty set is denoted by the symbol ϕ or {}.

Given below are few examples of empty sets.

(i) If $A = \{x : 1 < x < 2, x$ is a natural number$\}$, then A is the empty set, because there is no natural number between 1 and 2.

(ii) If $B = \{x : x^2 - 2 = 0$ and x is rational number$\}$, then B is the empty set, because the equation $x^2 - 2 = 0$ is not satisfied by any rational value of x.

(iii) If $C = \{x : x$ is an even prime number greater than $2\}$, then C is the empty set, because 2 is the only even prime number.

(iv) If $D = \{x : x^2 = 4, x$ is odd$\}$, then D is the empty set, because the equation $x^2 = 4$ is not satisfied by any odd value of x.

Equal Sets

Two sets A and B are said to be equal if they have exactly the same elements and we write $A = B$. Otherwise, the sets are said to be unequal and we write $A \neq B$.

(i) Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 1, 4, 2\}$, then $A = B$, because elements of both sets are the same. Only order of the elements in the two sets is different but it is not considered in a set.

(ii) Let A be the set of prime numbers less than 6 and P the set of prime factors of 30. Then A and P are equal, since 2, 3 and 5 are the only prime factors of 30 and also these are less the only prime numbers than 6.

Illustration 3: **Find the pairs of equal sets, from the following sets, if any, giving reasons:**

$A = \{0\}$, $B = \{x : x > 15$ and $x < 5\}$, $C = \{x : x - 5 = 0\}$, $D = \{x : x^2 = 25\}$

$E = \{x : x$ is an integral positive root of the equation $x^2 - 2x - 15 = 0\}$.

Solution: We have,

$A = \{0\}$,
$B = \{x : x > 15$ and $x < 5\} = \phi$,
$C = \{x : x - 5 = 0\} = \{5\}$,
$D = \{x : x^2 = 25\} = \{-5, 5\}$,
and $E = \{5\}$.
Clearly, $C = E$.

SUBSETS

Set A is said to be a subset of a set B if every element of set A is also an element of set B. Here set B is called superset of set A. A is a subset of B, is represented ACB. Thus $A \subset B$ if whenever $a \in A$, then $a \in B$. It is often convenient to use the symbol "$\Rightarrow$" which means implies. Using this symbol, we can write the definition of subset as follows: $A \subset B$ if $a \in A \Rightarrow a \in B$.

We read the above statement as "A is a subset of B if a is an element of A implies a is also an element of B". If A is not a subset of B, we write $A \not\subset B$. For example:

(i) The set Q of rational numbers is a subset of the set R of real numbers, so we write $Q \subset R$.

(ii) If A is the set of all divisors of 56 and B the set of all prime divisors of 56, then B is a subset of A so we write $B \subset A$.

(iii) Let $A = \{1, 3, 5\}$ and B $\{x : x$ is an odd natural number less than $6\}$. Then $A \subset B$ and $B \subset A$ and hence $A = B$.

(iv) Let $A = \{a, e, i, o, u\}$ and $B = \{a, b, c, d\}$. Then A is not a subset of B. Also B is not a subset of A.

Important Points about Subsets

(i) Every set is a subset of itself.

B Empty set is a subset of every set.

(iii) Total number of subsets of a finite set containing *n* element is 2^n.

UNIVERSAL SET

If there are some sets under consideration, and out of these sets, there is a set which is the superset of all other given sets i.e., all other sets under consideration are subsets of this set. Such a set is known as the universal set, denoted by U.

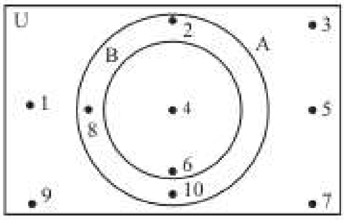
For example,

(i) In the context of human population studies, the universal set consists of all the people in the world.

(ii) If {1, 2, 3, 4}, {2, 5, 6}, {1, 3, 7, 8, 9} and {1, 2, 3, 4, 5, 6, 7, 8, 9} are the sets under consideration, then set {1, 2, 3, 4, 5, 6, 7, 8, 9} can be considered as universal set because all other three sets are the subsets of this set.

VENN DIAGRAMS

In order to illustrate universal sets, subsets and certain operations on sets in a clear and simple way, we use geometric figures. These figures are called Venn-Diagrams. In Venn Diagrams, a universal set is represented by a rectangle and any other set is represented by a circle.



In the Venn-diagrams, the elements of the sets are written in their respective circles.

In the Venn-diagrams, $U = \{1, 2, 3, ..., 10\}$ is the universal set of which $A = \{2, 4, 6, 8, 10\}$ and $B = \{4, 6\}$ are subsets, and also $B \subset A$.

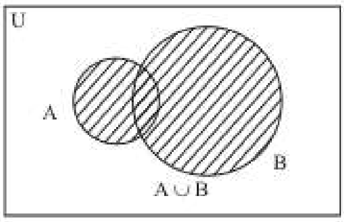
OPERATION ON SETS

Union of Sets

Union of two sets A and B is the set which consists of all those elements which are either in A or in B (including those which are in both sets A and B). In symbols, we write

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

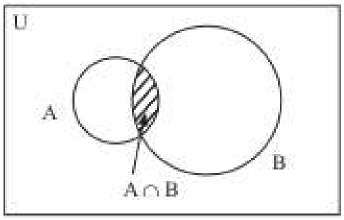
The union of two sets A and B can be represented by a Venn diagram as shown in figure by shaded portion



Intersection of Sets

The intersection of two sets A and B is the set of all those elements which belong to both sets A and B. Symbolically, we write

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$



The shaded portion in figure indicates the intersection of sets A and B.

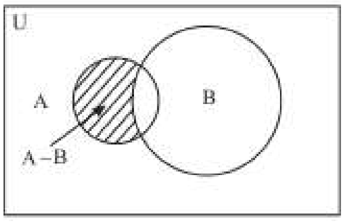
Difference of Sets

The difference of the sets A and B (in the order A minus B) is the set of elements which belong to A but not to B. Symbolically, we write $A - B$ and read as "A minus B".

In the set builder notation, we can write

$$A - B = \{x : x \in A \text{ but } x \notin B\}$$

The difference of two sets A and B is represented in Venn diagram by shaded portion.

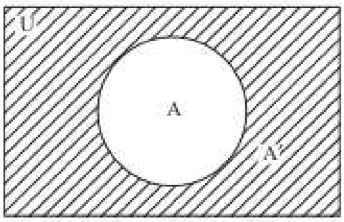


Complements of a Set

Let U be the universal set and A be a subset of U. Then the complement of A is the set of all elements of U which are not the elements of set A. Symbolically, we write A' or A^c to denote the complement of set A.

Thus, $A' = \{x : x \in U \text{ but } x \notin A\}$.

Obviously $A' = U - A$



Complement of set A i.e. A' is represented in Venn diagram by shaded region.

Some Properties of Complement of a Set

1. Complement laws:
   (i) $A \cup A' = U$
   (ii) $A \cap A' = \phi$

2. De Morgan's law:
   (i) $(A \cup B)' = A' \cap B'$
   (ii) $(A \cap B)' = A' \cup B'$

3. Law of double complementation: $(A')' = A$

4. Laws of empty set and universal set: $\phi' = U$ and $U' = \phi$.

These laws can be verified by using Venn diagrams.

Illustration 4: If $A = \{x : x = 3n, n \in Z\}$ and $B = \{x : x = 4n, n \in Z\}$, then find $A \cap B$.

Solution: We have,

$x \in A \cap B \Leftrightarrow x = 3n, n \in Z$ and $x = 4n, n \in Z$

$\Leftrightarrow x$ is a multiple of 3 and x is a multiple of 4

$\Leftrightarrow x$ is a multiple of 3 and 4 both

$\Leftrightarrow x$ is a multiple of 12.

$\Leftrightarrow x = 12n, n \in Z$

Hence, $A \cap B = \{x : x = 12n, n \in Z\}$.

If A and B are two sets, then $A \cap B = A$, if $A \subset B$ and $A \cap B = B$, if $B \subset A$.

Illustration 5: Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$. Find

(i) A' **(ii)** $(A \cup B)'$

(iii) $(A')'$ **(iv)** $(B - C)'$

Solution:

(i) $\{5, 6, 7, 8, 9\}$ (ii) $\{5, 7, 9\}$

(iii) A (iv) $\{1, 3, 4, 5, 6, 7, 9\}$

Illustration 6: Find the union of each of the following P airs of sets:

(i) $A = \{x : x$ is a natural number and $1 < x \le 6\}$

$B = \{x : x$ is a natural number and $6 < x \le 10\}$

(ii) $A = \{1, 2, 3\}$, $B = \phi$.

Solution:

(i) $A = \{x : x$ is a natural number and $1 < x \le 6\}$

$\Rightarrow \quad A = \{2, 3, 4, 5, 6\}$

$B = \{x : x$ is a natural number and $6 < x \le 10\}$

$\Rightarrow \quad B = \{7, 8, 9, 10\}$

$\therefore \quad A \cup B = \{2, 3, 4, 5, 6\} \cup \{7, 8, 9, 10\}$

$\Rightarrow \quad A \cup B = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(ii) We have, $A = \{1, 2, 3\}$, $B = \phi$

$\Rightarrow \quad A \cup B = \{1, 2, 3\} \cup \phi$

$\Rightarrow \quad A \cup B = \{1, 2, 3\}$

Illustration 7: If $A = \{x : x = 3n, n \in Z\}$ and B $\{x : x = 4n, n \in Z\}$, then find $(A \cap B)$.

Solution:

Let $x \in (A \cap B) \Leftrightarrow x \in A$ and $x \in B$

$\Leftrightarrow x$ is a multiple of 3 and x is a multiple of 4.

$\Leftrightarrow x$ is a multiple of 3 and 4 both

$\Leftrightarrow x$ is a multiple of 12

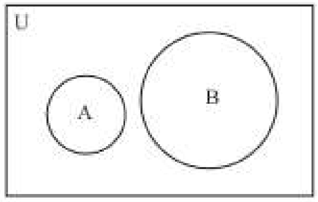
$\Leftrightarrow x = 12n, n \in Z$

Hence $A \cap B = \{x : x = 12n, n \in Z\}$

DISJOINT SETS

If A and B are two sets such that $A \cap B = \phi$, then A and B are called disjoint sets.

For example, let $A = \{2, 4, 6, 8\}$ and $B = \{1, 3, 5, 7\}$. Here A and B are disjoint sets, because there is no element common to both sets A and B.



In the Venn diagram, A and B are disjoint sets.

CARDINAL NUMBER

Number of element in a set A is called cardinal number of set A. It is represented by $n(A)$. If $A = \{a, b, c, d, e, f\}$, then $n(A) = 6$

1. If A and B are finite sets then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

2. If A, B and C are three finite sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

Illustration 8: In a political survey, 78% of the politicians favour at least one proposal, 50% of them are in favour of proposal A, 30% are in favour of proposal B and 20% are in favour of proposal C. 5% are in favour of all three proposals. what is the percentage of people favouring more than one proposal?

(a) 16 **(b) 17**

(c) 18 **(d) 19**

Solution: (b)

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(6A \cap C) + n(A \cap B \cap C)$

or $78 = 50 + 30 + 20 - \Sigma n(A \cap B) + 5$

or $\Sigma n(A \cap B) = 27$

This includes $n(A \cap B \cap C)$ three times.

$\therefore$ Percentage of people favouring more than one proposal $= 27 - 5 \times 2 = 17$

Illustration 9: If X and Y are two sets such that $X \cup Y$ has 50 elements, X has 28 elements and Y has 32 elements, how many elements does $X \cap Y$ have?

Solution:

Given that $n(X \cup Y) = 50$, $n(X) = 28$, $n(Y) = 32$, $n(X \cap Y) = ?$

By using the formula,

$$n(X \cup Y) = n(X) + n(Y) - n(X \cap Y),$$

We find that $n(X \cap Y) = n(X) + n(Y) - n(X \cup Y) = 28 + 32 - 50 = 10$

Illustration 10: In a class of 35 students, 24 like to play cricket and 16 like to play football. Also, each student likes to play at least one of the two games. How many students like to play both cricket and football?

Solution: Let X be the set of students who like to play cricket and Y be the set of students who like to play football. Then $X \cup Y$ is the set of students who like to play at least one of the two games, and $X \cap Y$ is the set of students who like to play both games.

Given $n(X) = 24$, $n(Y) = 16$, $n(X \cup Y) = 35$, $n(X \cap Y) = ?$

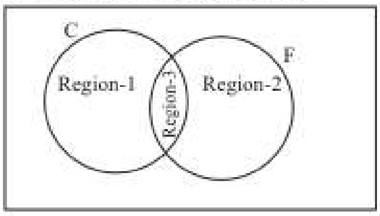
Using the formula $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$,

We get $35 = 24 + 16 - n(X \cap Y)$

Thus, $n(X \cap Y) = 5$ i.e., 5 students like to play both games.

SITUATION BASED VENN DIAGRAMS

1. Suppose set C represents the people who like cricket and F represents the people who like football.



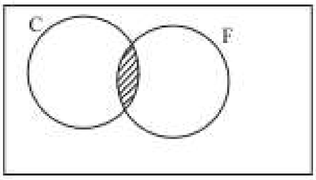
In the above Venn-diagram,

Region- 1: Represents the people who like cricket only (means people who like cricket but not football.)

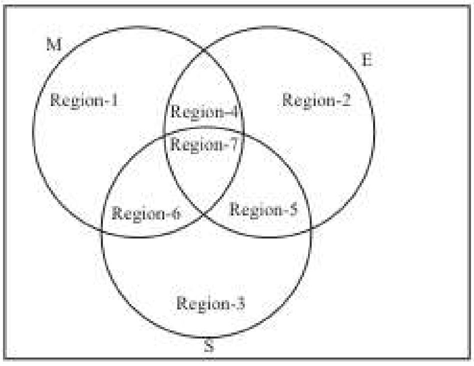
Region- 2: Represents the people who like football only (means people who like football but not cricket.)

Region- 3: Represents the people who like both cricket and football.

The people who like both cricket and football is represented by the common shaded region of set A and set B in the Venn diagram.



2. Let M represent the students who passed in mathematics, E represents the students who passed in English and S represents the students who passed in Science. Then students who passed in both Mathematics and English are represented by common region of the sets M and E.



Students who passed in both English and Science are represented by the common region of set E and S. Students who passed in both Science and Mathematics represented by the common region of set S and M. Students who passed in both Mathematics and English are represented by the common region of sets M and E. Students who passed in all the three subjects, Mathematics, English and Science are represented by common region of all the three sets M, E and S.

Region- 1: Represents the students who passed in Mathematics only (means the students who passed in Mathematics but not passed in English and Science).

Region- 2: Represents the students who passed in English only (means the students who passed in English but not passed in Science and mathematics).

Region- 3: Represents the students who passed in Science only (means the students who passed in science but not passed in Mathematics and English).

Region- 4: Represents the students who passed in both Mathematics and English only (means the students who passed in both Mathematics and English but not in Science).

Region- 5: Represents the students who passed in both English and Science only (means the students who passed in both English and Science but not passed in Mathematics)

Region- 6: Represents the students who passed in both Science and Mathematics only (means the students who passed in both Science and Mathematics, but not passed in English).

Region- 7: Represents the students who passed in all the three subjects Mathematics, English and Science.

Note that

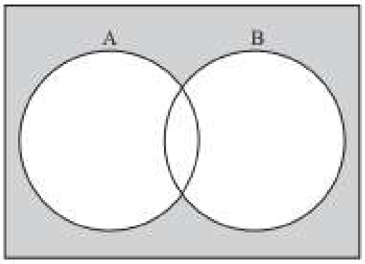
- Students who passed in Mathematics are represented by the sum of the regions 1, 4, 6 and 7.
- Students who passed in English are represented by the sum of the regions 2, 4, 5 and 7.
- Students who passed in Science are represented by the sum of the regions 3, 5, 6 and 7.
- Students who passed in both Mathematics and English are represented by the sum of the regions 4 and 7.
- Students who passed in both English and Science are represented by the sum of the regions 5 and 7.
- Students who passed in both Science and Mathematics are represented by the sum of the regions 6 and 7.

Illustration 11: Draw the appropriate Venn diagram for each of the following:

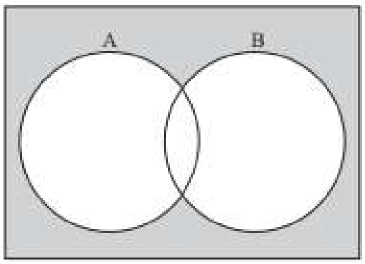
(i) $(A \cup B)'$ (ii) $A' \cap B'$ (iii) $(A \cap B)'$ (iv) $A' \cup B'$

Solution:

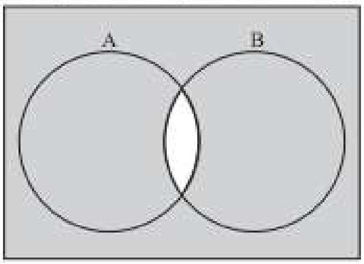
(i) $(A \cup B)'$ is represented by the shaded region.



(ii) $A' \cap B'$ is represented by the shaded region.



(iii) $(A \cap B)'$ is represented by the shaded region.



(iv) $A' \cup B'$ is represented by the shaded region.

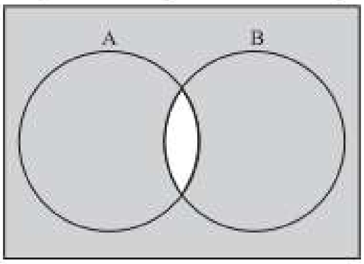


Illustration 12: Out of 10000 people surveyed, 3700 liked city *A*, 4000 liked city *B* and 5000 liked city *C*. 700 people liked *A* and *B* 1200 liked *A* and *C* and 1000, liked *B* and *C*. Each person liked at least one city. Then find

(A) The number of people liking all the three cities.

(B) The number of persons liking at least two cities as a % of number of people liking exactly one city.

(C) The number of persons liking exactly two cities as a percentage of the number of people liking at least one city.

(D) The number of persons liking *A* and *B* but not *C*.

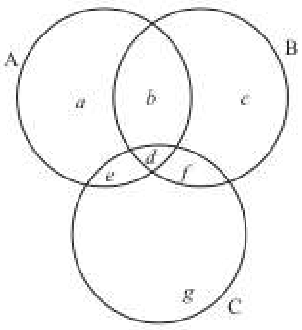
Solution:

Refer the figure given

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B \cap C)$

$\Rightarrow$ $10000 = 3700 + 4000 + 5000 - 700 - 1000 - 1200 + d$

$\Rightarrow$ $d = 200$



Once the value of d is known, all other values will be determined fastly.

e.g. $b + d = 700$ (given) $\Rightarrow$ $b = 500$

Similarly $e = 1000$, $f = 800$, $a = 2000$, $c = 2500$, $g = 3000$

A. $d = 200$.

B. At least two cities $b + d + e + f = 2500$

Exactly one city $a + c + g = 7500$

$\Rightarrow \% = 2500/7500 \times 100\ \% = 33.33\%$

C. Exactly two cities $= b + e + f = 2300$

At least one city = 10000

$\Rightarrow$ Required % = 23 %.

D. $b = 500$.

Illustration 13: In a survey of 100 students, the number of students studying the various languages were found to be: English only 18, English but not Hindi 23, English and Sanskrit 8, English 26, Sanskrit 48, Sanskrit and Hindi 8, no language 24. Find:

(i) How many students were studying Hindi?

(ii) How many students were studying English and Hindi?

Solution:

We have, $a = 18$, $a + b = 23$, $d + e = 8$, $a + b + d + e = 26$, $d + e + f + g = 48$, $e + f = 8$, $a + b + c + d + e + f + g = 100 - 24 = 76$

$\therefore$ $a = 18$, $b = 0$, $c = 10$, $d = 5$, $e = 3$, $f = 5$ and $g = 35$

(i) $n(H) = b + c + e + f = 18$

(ii) $n(H \cap E) = b + e = 3$