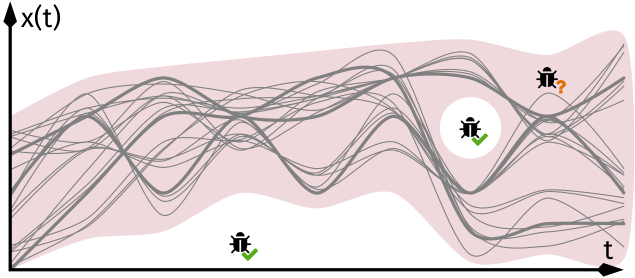


Abstract Interpretation

Software Quality Assurance — Static Code Analysis, II | Florian Sihler | December 10, 2025

Software Engineering
Programming Languages

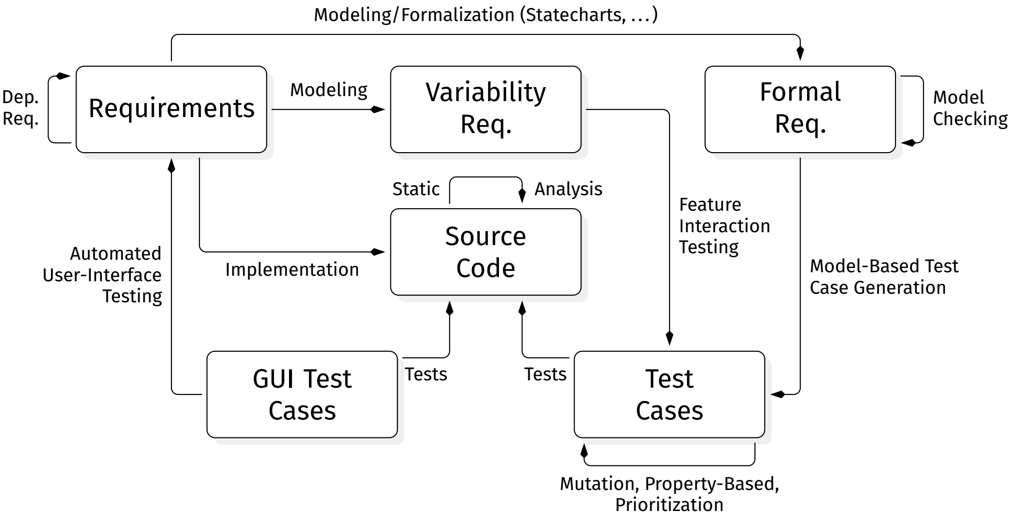
universität ulm

1. The Why

Why am I even here?

Embedding a Landscape

Embedding a Landscape



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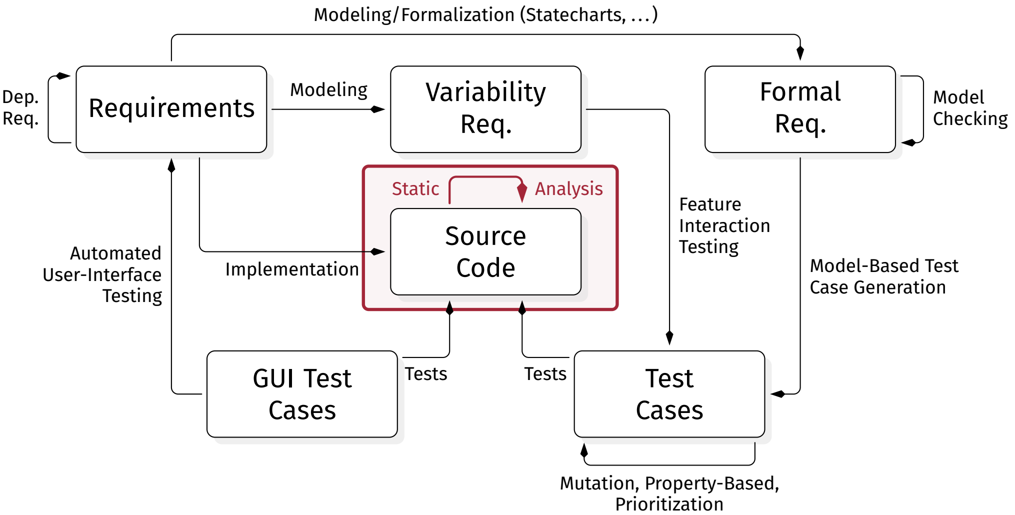
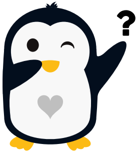
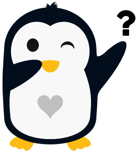




What is static analysis?



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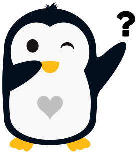
Discover *syntactic/semantic properties* of programs **without** running them.[RY20]



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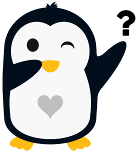

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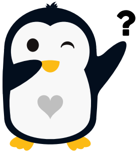

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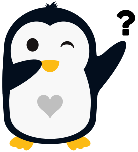

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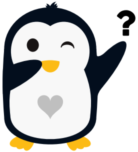

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Today, we learn how to...

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- describe and map language semantics to properties
- deal with the cost of abstraction (and the fun)

The Why

```java
public static void main(String[] args) {
    int a = 1;
    double r = Math.random() * 10;
    if (r > 5) {
        a = 2;
    }
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- We want to proof, that a program satisfies certain properties

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```

java

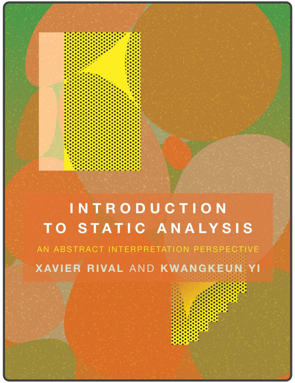
- We want to proof, that a program satisfies certain properties
- Abstract Interpretation is one (/the) technique to do so

Recommended Resources

And for an overview: “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17]

Recommended Resources

Using Analyses [RY20]



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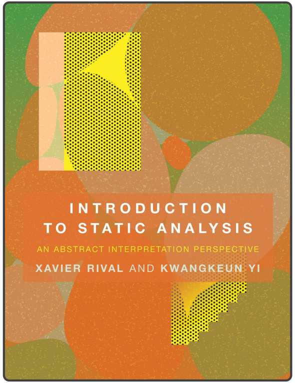
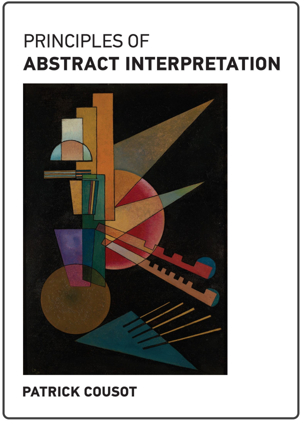


Formal Foundations [Cou21]



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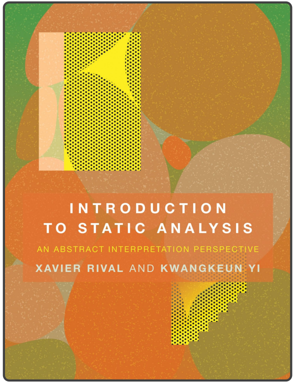

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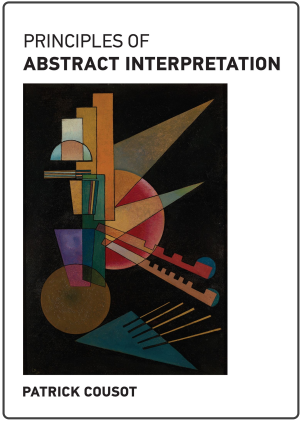
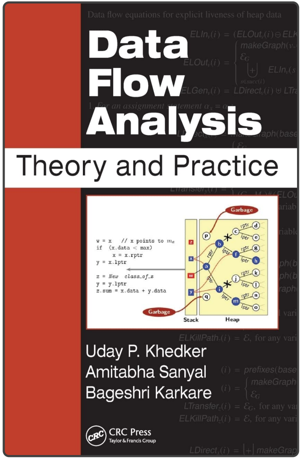


Dataflow Perspective [KSK09]



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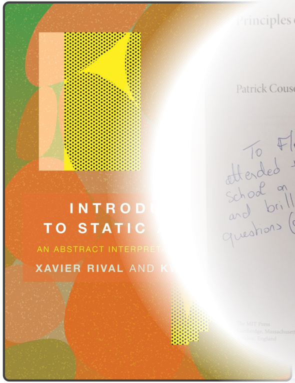

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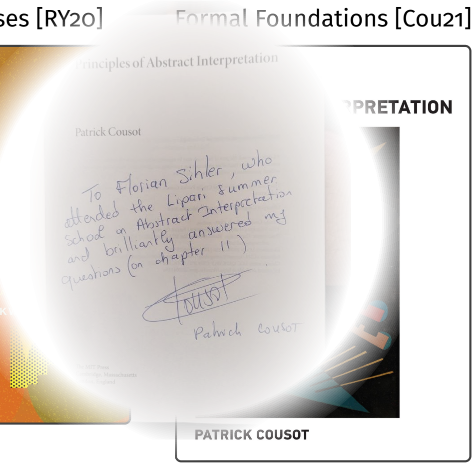

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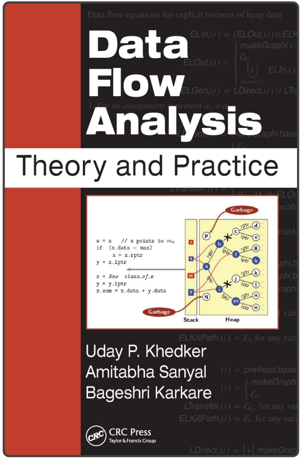

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2. The How

Gimme properties, gimme abstractions!

Abstract Interpretation

See “A casual introduction to Abstract Interpretation” [Cou12]

Concrete Interpretation

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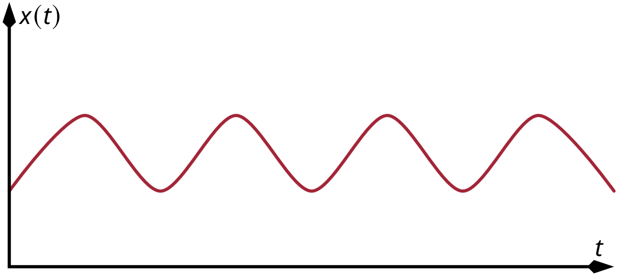

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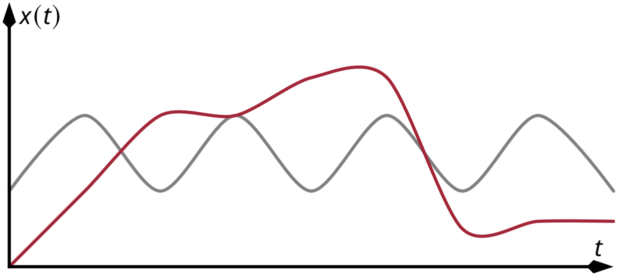

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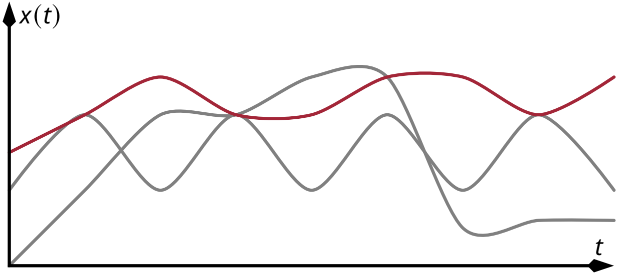

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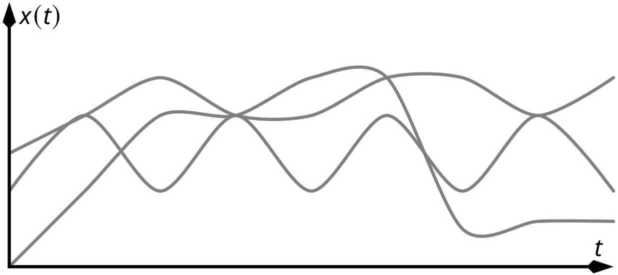

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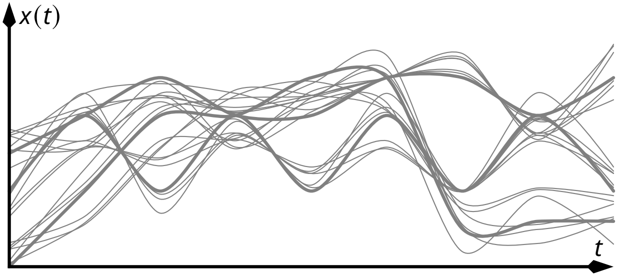

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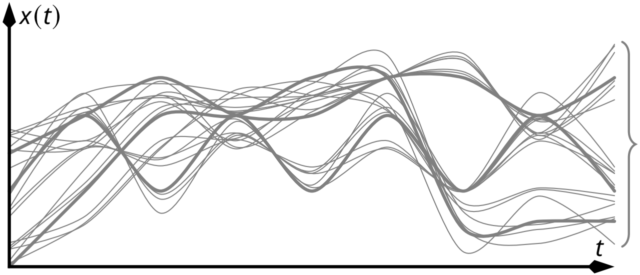

Collecting Semantics[Cou21, p. 91]

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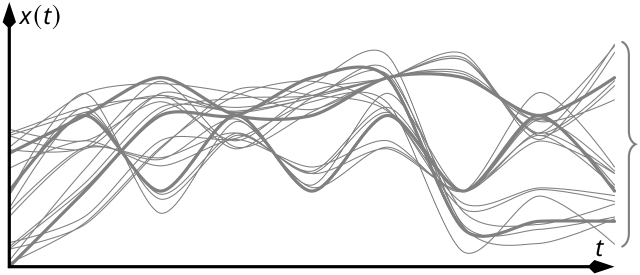

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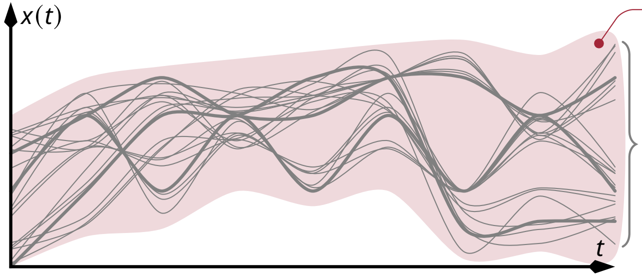

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just one of many
this **must** include all concrete traces!
(over-approximate)

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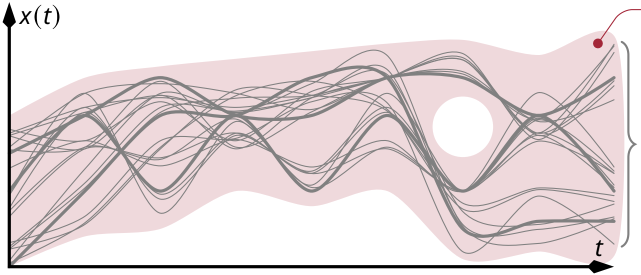

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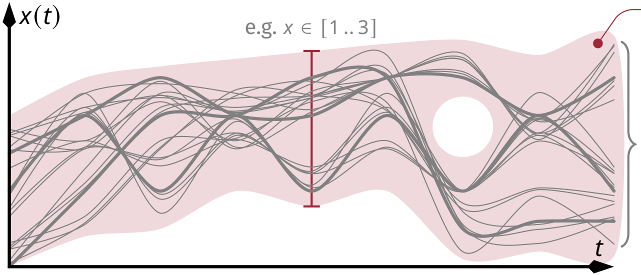

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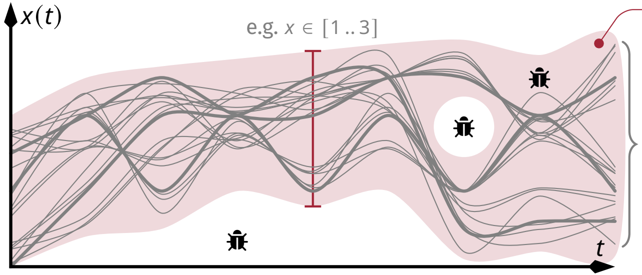

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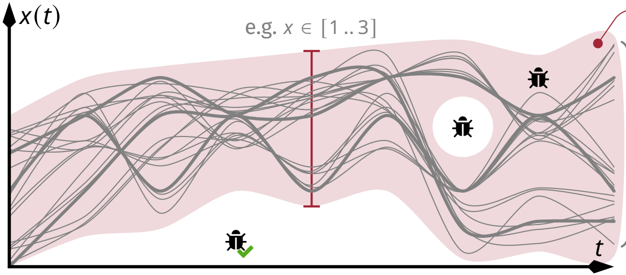

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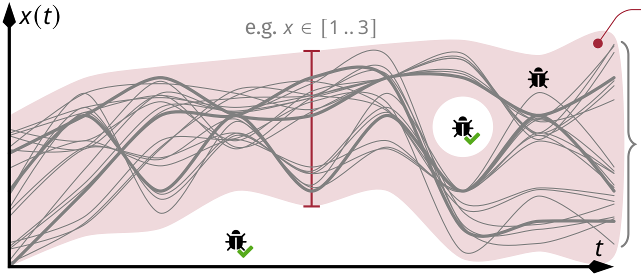

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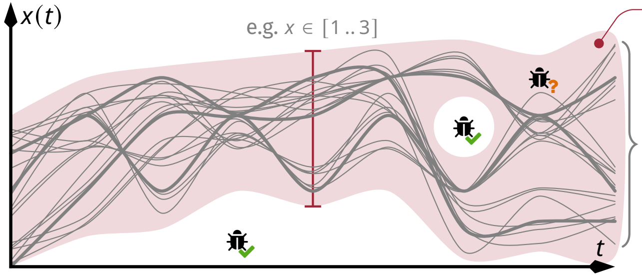

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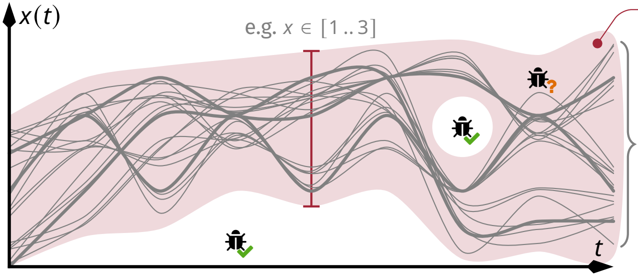
Abstract Interpretation

Radhia Cousot (1947–2014)
Patrick Cousot

Patrick Cousot (1948)
Personal Website



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Terminology

- **Property**

“Principles of Abstract Interpretation” [Cou21, p. 15],“Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 18]

Terminology

- **Property** — Set of states/traces that satisfy that property

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$$\emptyset \subseteq P_1 \subseteq P_2 \subseteq \mathbb{U}$$

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strongest

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- **Partial Order**

“Principles of Abstract Interpretation” [Cou21, p. 15],“Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 18]

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strongest · stronger · weaker · weakest

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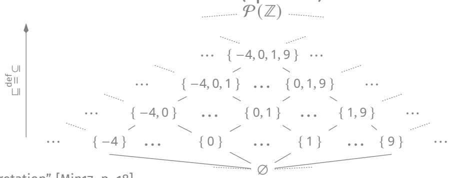
Hasse Diagram of $(\mathcal{P}(\mathbb{Z}), \subseteq)$



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A Task In-Between

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Helmut Hasse (1898–1979)
CC BY SA 2.0 Oberwolfach Photo Collection

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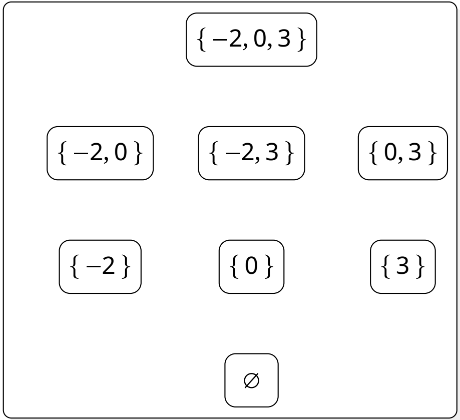

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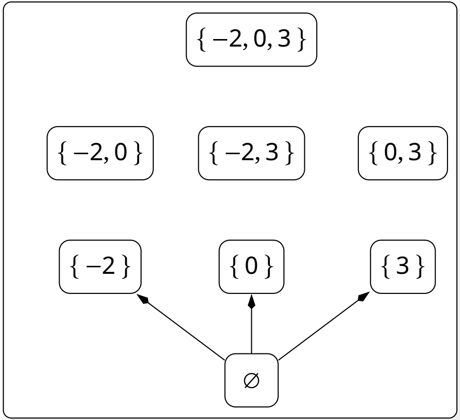

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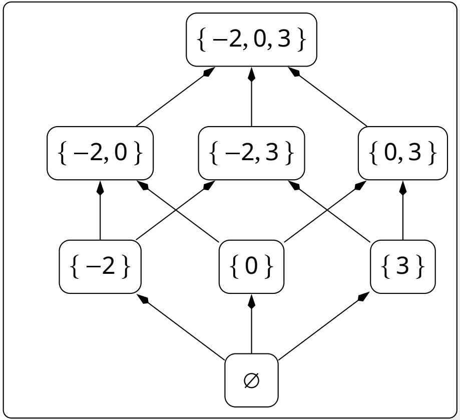

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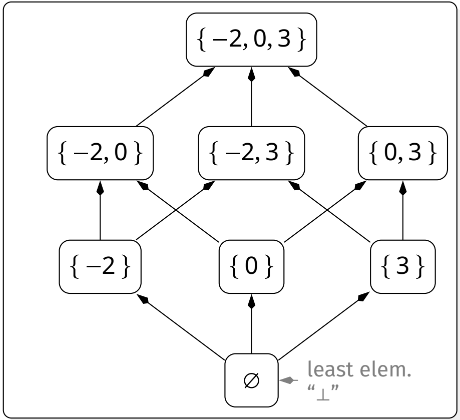

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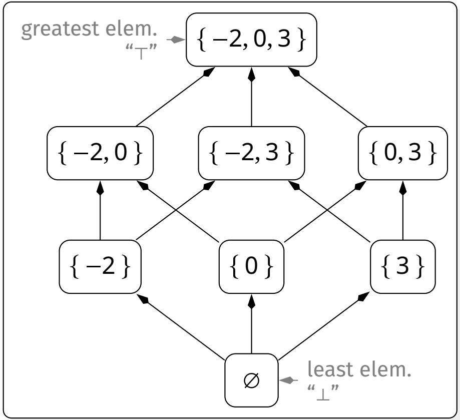

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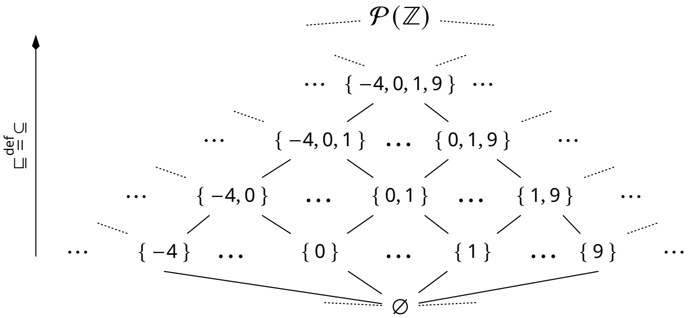
Posets, Lattices, and Chains



"Lattice theory" [Bir67], see also sublattices [Min17, p. 25]

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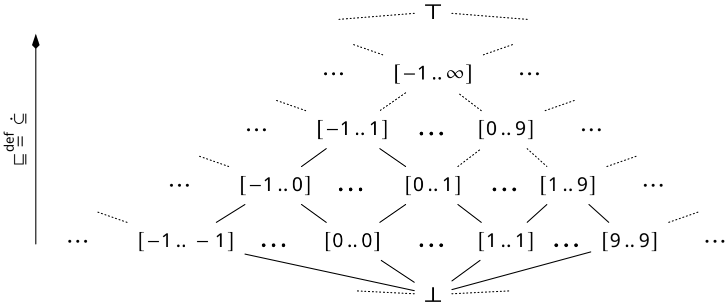

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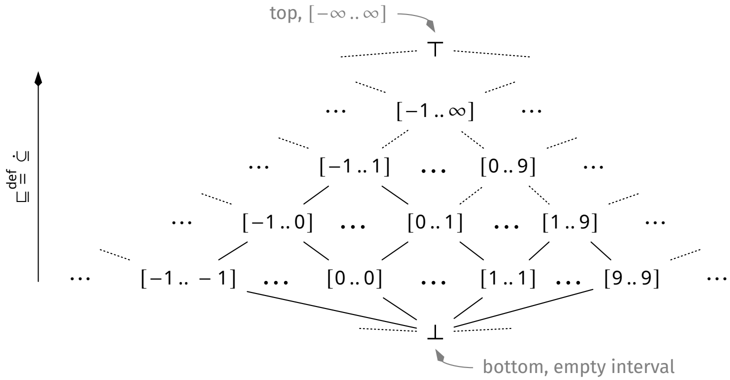

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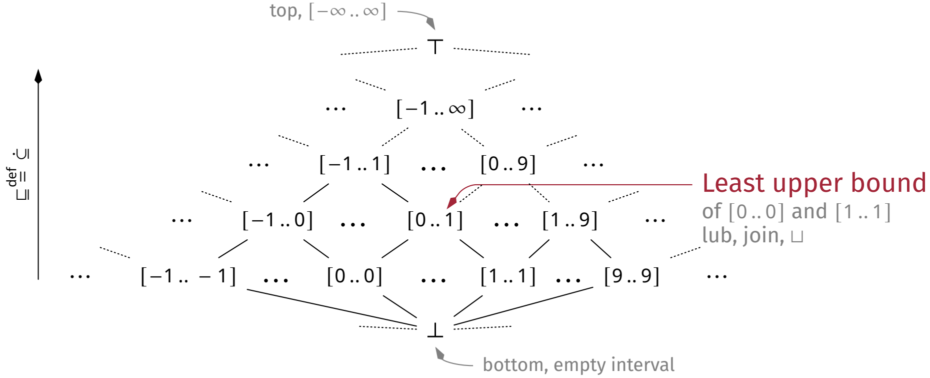

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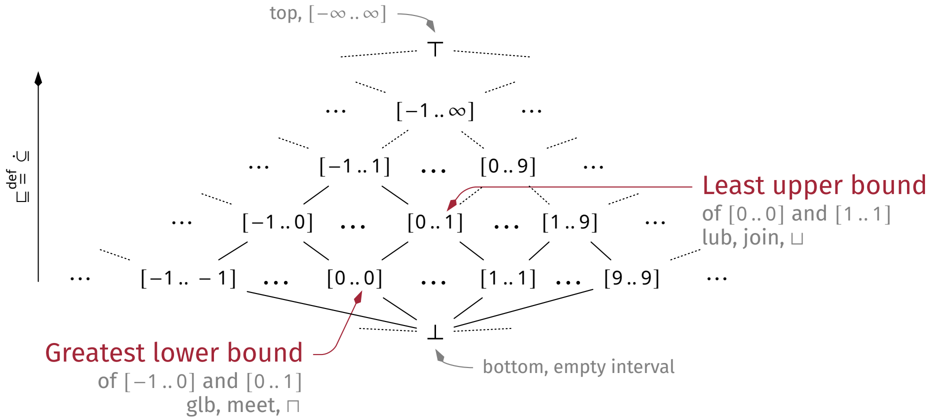

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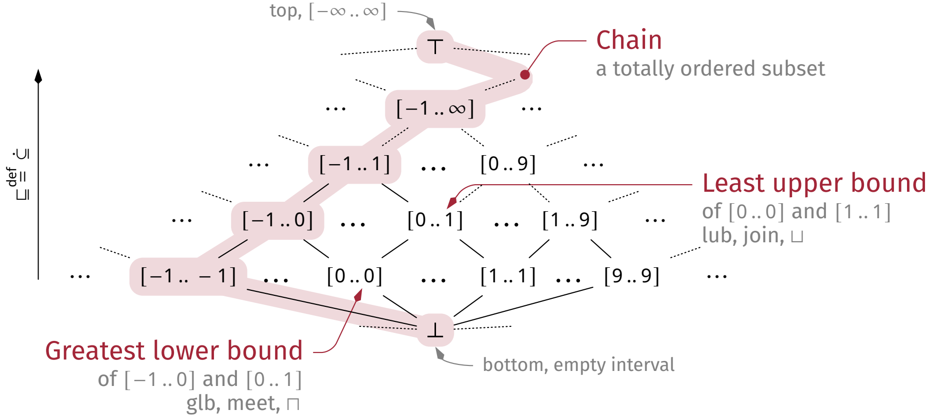

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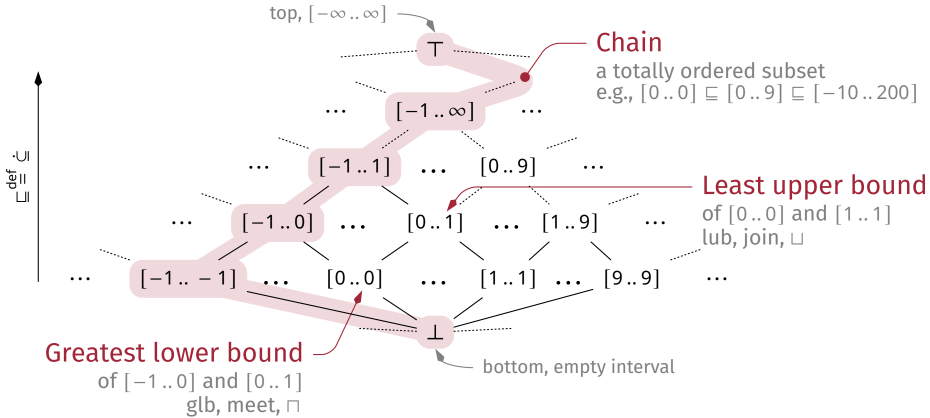

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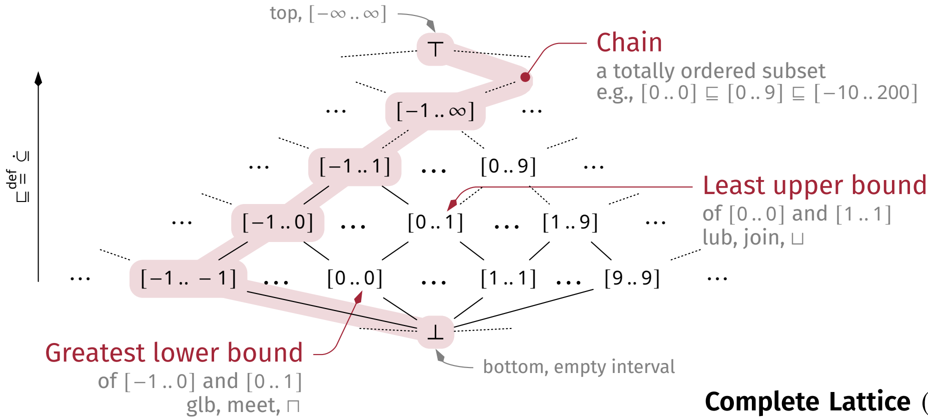

Complete Lattice $(X, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$

"Lattice theory" [Bir67], see also sublattices [Min17, p. 25]

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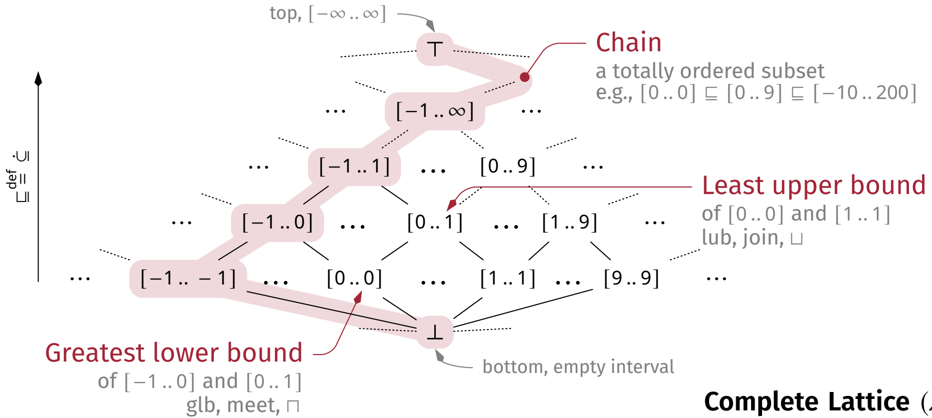

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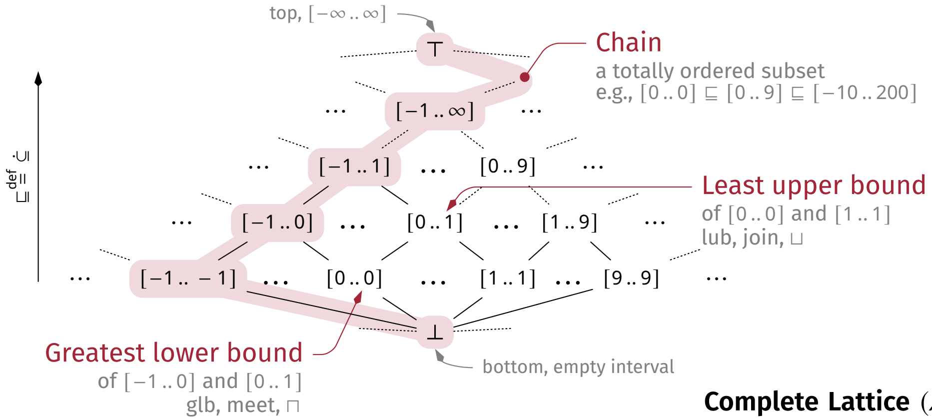

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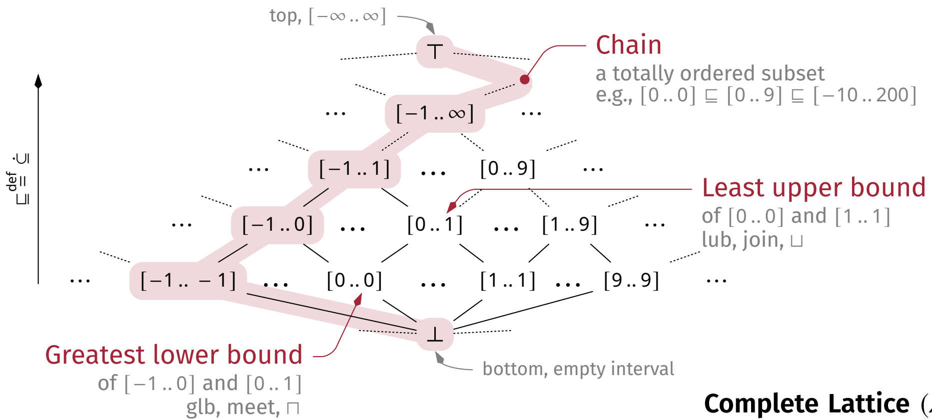

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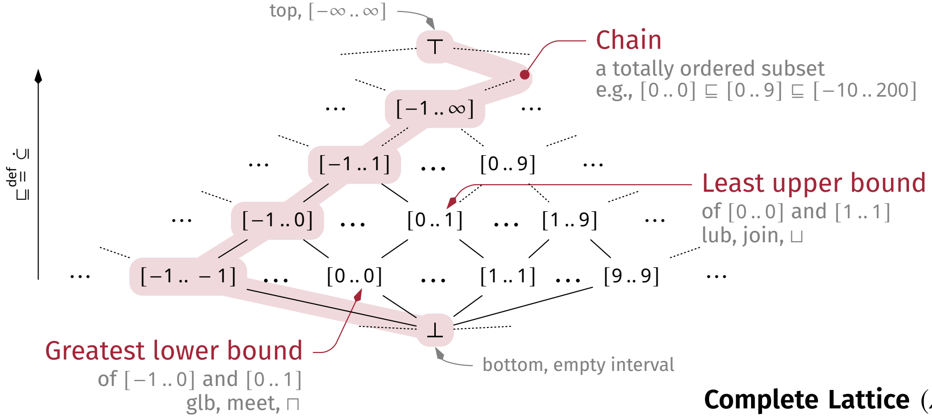

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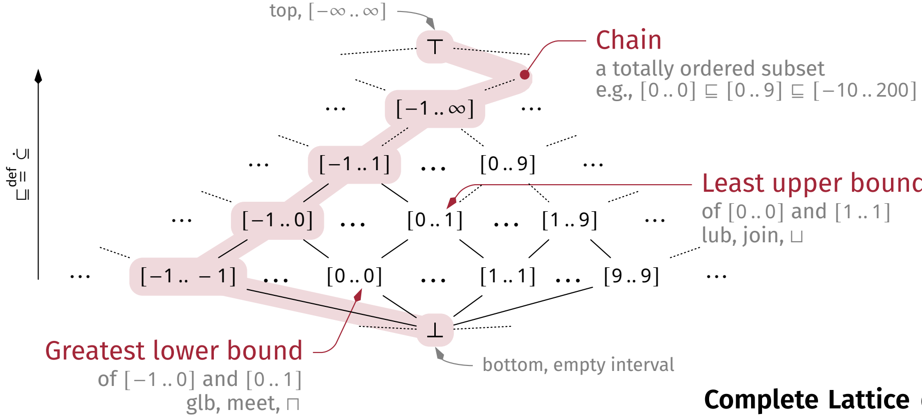

Garrett Birkhoff (1911–96)
Paul Halmos

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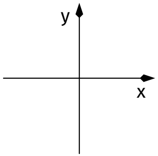
Abstract Domains

Pick Your Favorite Lattice!

Numerical



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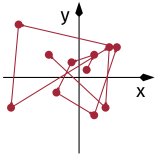

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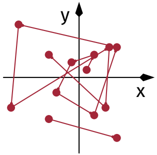

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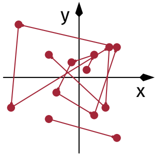

Collecting Semantics

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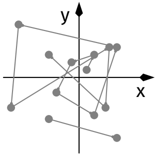

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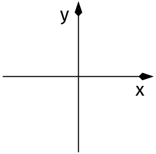

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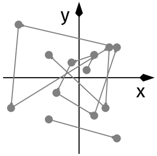

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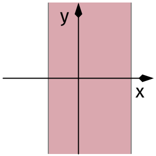

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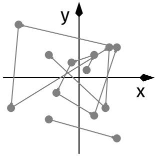

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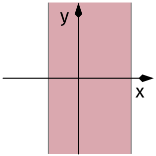

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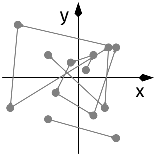

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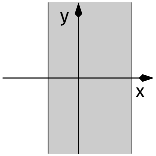

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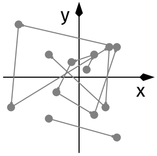

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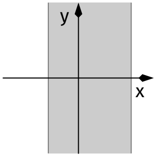

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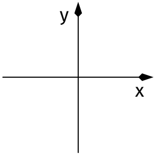

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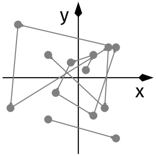
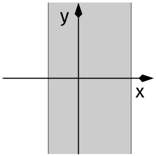
Abstract Domains

Pick Your Favorite Lattice!

Numerical



Collecting Semantics



Intervals $x \in [a\,..\,b]$

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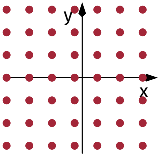

Simple Congruences

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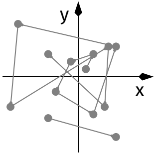

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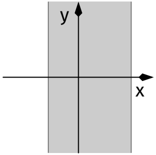

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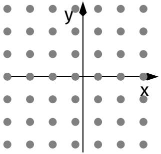

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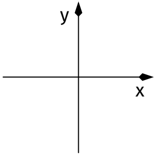

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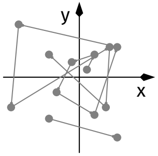

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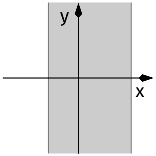

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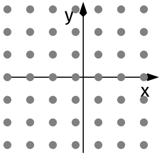

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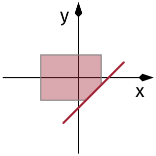

Pentagons

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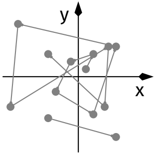

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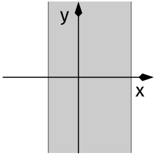

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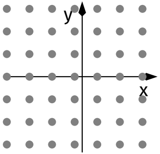

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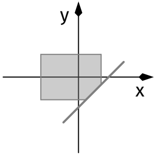

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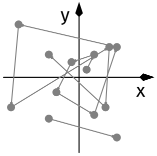

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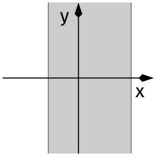

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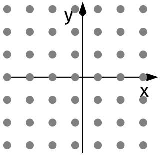

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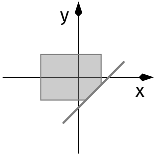

Pentagons

- Octagons

"Principles of Abstract Interpretation" [Cou21], "Pentagons: a weakly relational abstract domain for the efficient validation of array accesses" [LF08, p. 25]

Abstract Domains

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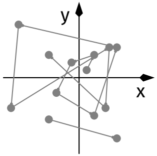
Numerical



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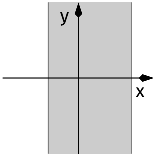

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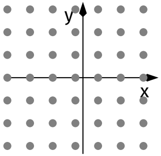

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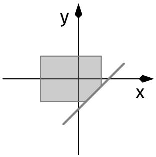

Pentagons

- Octagons
- Ellipses

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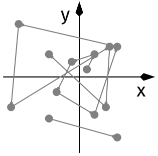

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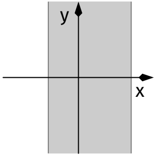

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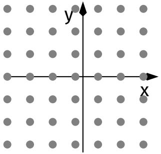

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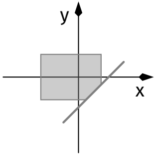

Pentagons

- Octagons
- Ellipses
- Exponentials

"Principles of Abstract Interpretation" [Cou21], "Pentagons: a weakly relational abstract domain for the efficient validation of array accesses" [LF08, p. 25]

Abstract Domains

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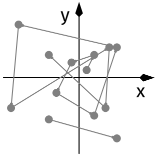

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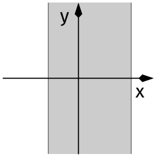

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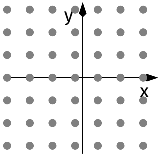

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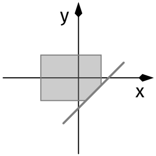


Pentagons

- Octagons
- Ellipses
- Exponentials
- Signs

"Principles of Abstract Interpretation" [Cou21], "Pentagons: a weakly relational abstract domain for the efficient validation of array accesses" [LF08, p. 25]

Sign Analysis

Sign Analysis

Simple Sign Domain

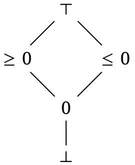
$$
\begin{tikzcd}[column sep=small]
 & \top \arrow[dl, dash] \arrow[dr, dash] & \\
\geq 0 \arrow[dr, dash] & & \leq 0 \arrow[dl, dash] \\
 & 0 \arrow[d, dash] & \\
 & \bot &
\end{tikzcd}
$$

Sign Analysis

Simple Sign Domain

- We still have no program semantics, but we can try…

```java
int a = 0;
int b = 12;
int c = a + b;
int d = c - b;
```



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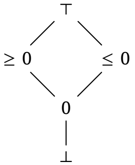

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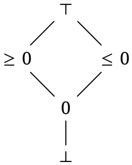

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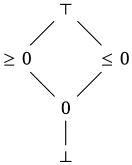

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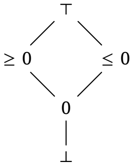

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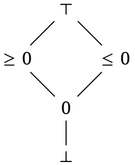

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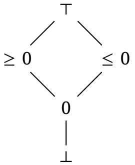

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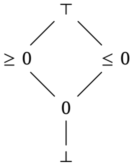

Galois Connection — Linking Posets [Cou21, p. 110] (small detour)

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A tale of two lattices…

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A tale of two lattices…

Concrete Domain
$(C, \sqsubseteq_C)$

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This is the (usually infinite and unknown) ground truth, the reality

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The abstract domain we choose, e.g., intervals

Galois Connection — Linking Posets [Cou21, p. 110] (small detour)

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A tale of two lattices…

α abstraction

Concrete Domain $(C, \sqsubseteq_C)$ $\xrightarrow{\alpha}$ **Abstract Domain** $(A, \sqsubseteq_A)$

This is the (usually infinite and unknown) ground truth, the reality

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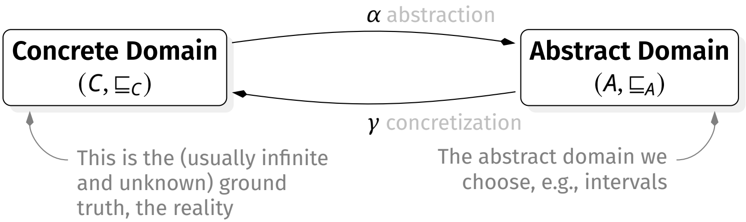

Beware, beware, conversions may be lossy!

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Beware, beware, conversions may be lossy!

- $\forall x \forall y : \alpha(x) \sqsubseteq_A y \iff x \sqsubseteq_C \gamma(y)$
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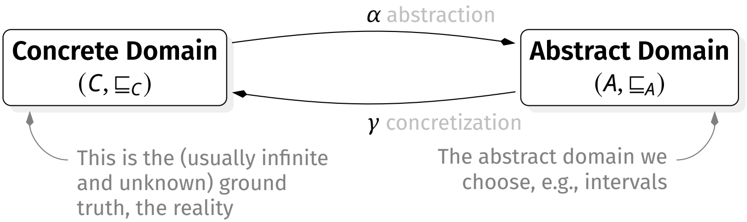

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"We can abstract $x \in \{1, 2\}$ with $\alpha(x) = [1 \mathinner{..} 2]$ or $\alpha(x) = [-5 \mathinner{..} 42]$, but *not* (e.g.) with $\alpha(x) = [1 \mathinner{..} 1]$! "

Galois Connection — Linking Posets [Cou21, p. 110] (small detour)

Évariste Galois (1811–32)
Public Domain
died in a duel for love

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γ concretization

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Happy Abstractions

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- Suppose, you use the interval domain, what are your best abstractions if u is an unknown integer?

Happy Abstractions

- Suppose, you use the interval domain, what are your best abstractions if `u` is an unknown integer?
  - $\alpha($`u`$) =$
  - $\alpha($`u + 1`$) =$
  - $\alpha($`u * 0`$) =$
  - $\alpha($`sin(u)`$) =$

Happy Abstractions

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  - Potential sequences (e.g., `for(i=0; i<n; i++) ...`)
- But, how do we handle loops? Recursion?

Fixpoints

“A lattice-theoretical fixpoint theorem and its applications.” [Tar55], “Introduction to metamathematics” [Kle52], “Principles of Abstract Interpretation” [Cou21, p. 165]

Fixpoints

- For operators $f : X \longrightarrow X$ a **fixpoint** is a $x \in X$ such that $f(x) = x$

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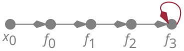
  - 

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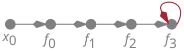
  - 

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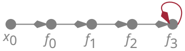
  - 

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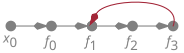
  - 

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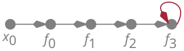
  - 

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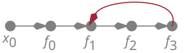
  - 

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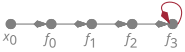
  - 

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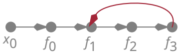
  - 

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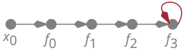
  - 

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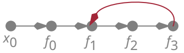
  - 

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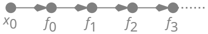
  - 

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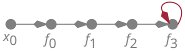
  - 

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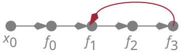
  - 

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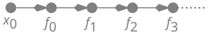
  - 

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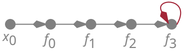
  - 

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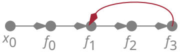
  - 

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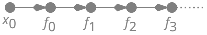
  - 

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for complete, nonempty lattices
Tarski's Theorem

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Alfred Tarski (1901–83)
2.0 Oberwolfach Photo Collection

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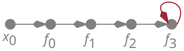
  - 

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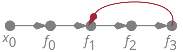
  - 

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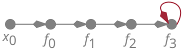
  - 

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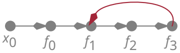
  - 

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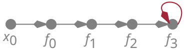
  - 

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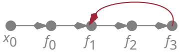
  - 

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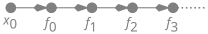
  - 

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- You may know fixpoints from tools like LaTeX or the λ-calculus

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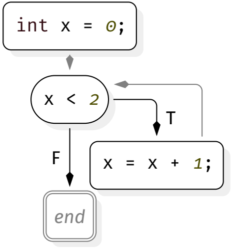
Interval Analysis, I (the intuitive approach)



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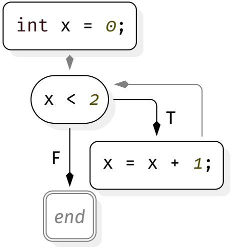

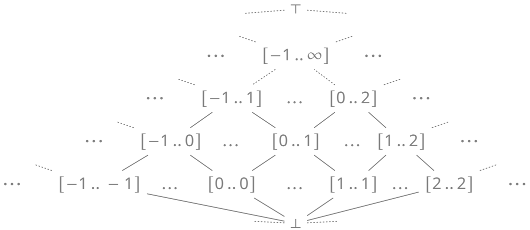

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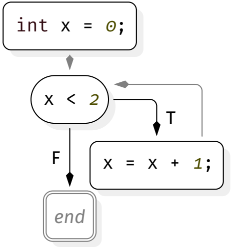

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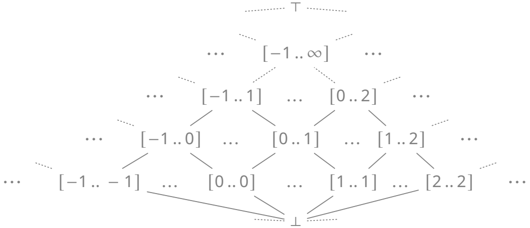

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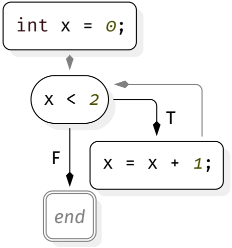

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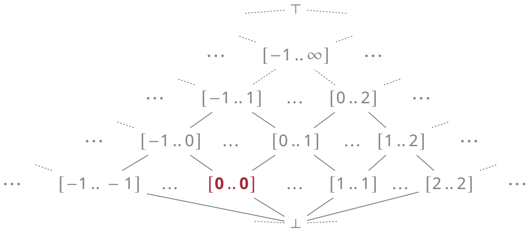

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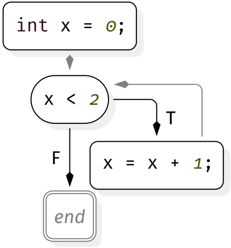

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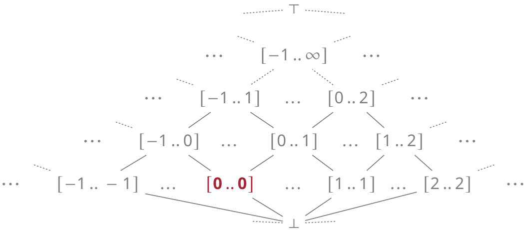

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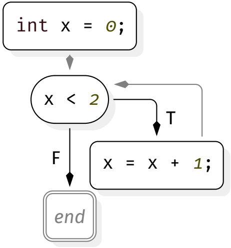

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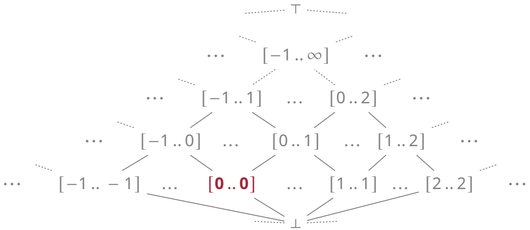
$\wr$ [in] $x_2 \in [0\,..\,0] \quad ([0\,..\,0] \cap (-\infty\,..\,1]) \,\int$

if we are inside the loop we know that $x < 2$ holds!



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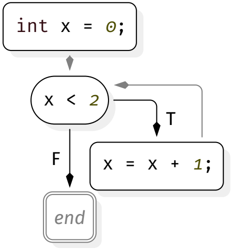

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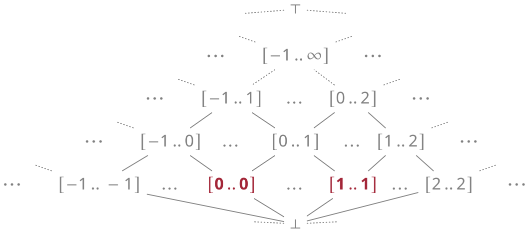
we have to know how to add intervals here



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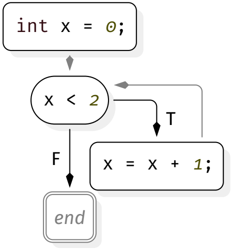

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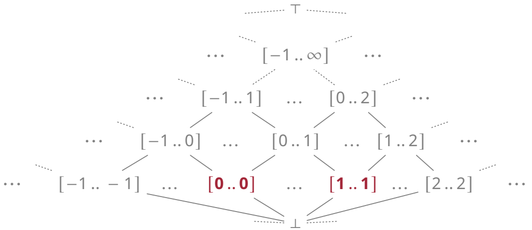
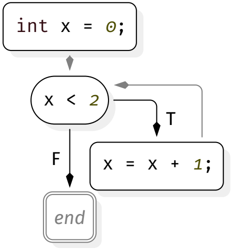
$\{\!\{\ x_3 \in [1\,..\,1] \quad ([0\,..\,0] \oplus [1\,..\,1])\ \}\!\}$

we have to know how to add intervals here



Interval Analysis, I (the intuitive approach)



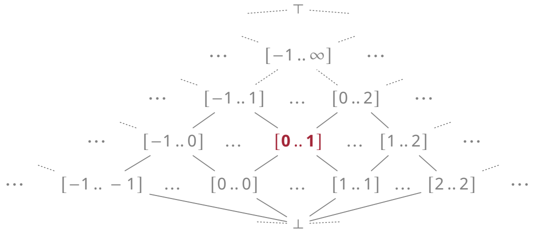
$\wr\; x_0 \in [0\,..\,0] \;\int$

we join on loops because, at this point x can have any of the values

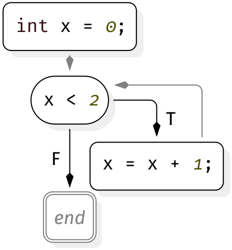
$\wr\;$ [pre] $\boldsymbol{x_1 \in [0\,..\,1]} \quad \boldsymbol{([0\,..\,0] \cup [1\,..\,1])} \;\int$

$\wr\;$ [in] $x_2 \in [0\,..\,0] \quad ([0\,..\,0] \cap (-\infty\,..\,1]) \;\int$

$\wr\; x_3 \in [1\,..\,1] \quad ([0\,..\,0] \oplus [1\,..\,1]) \;\int$



Interval Analysis, I (the intuitive approach)



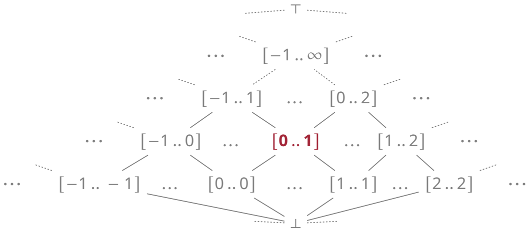
$\{ x_0 \in [0..0] \}$

we join on loops because, at this point x can have any of the values

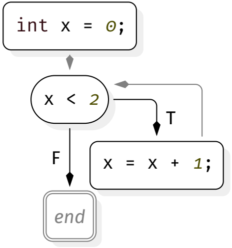
$\{ \text{[pre] } x_1 \in [0..1] \quad ([0..0] \cup [1..1]) \}$

$\{ \text{[in] } \boldsymbol{x_2 \in [0..1] \quad ([0..1] \cap (-\infty..1])} \}$

$\{ x_3 \in [1..1] \quad ([0..0] \oplus [1..1]) \}$



Interval Analysis, I (the intuitive approach)



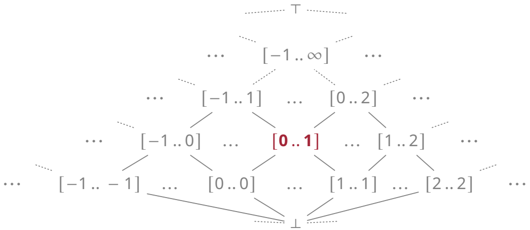
$\{ x_0 \in [0 \mathinner{..} 0] \}$

we join on loops because, at this point x can have any of the values

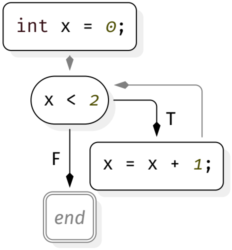
$\{ \text{[pre] } x_1 \in [0 \mathinner{..} 1] \quad ([0 \mathinner{..} 0] \cup [1 \mathinner{..} 1]) \}$

$\{ \text{[in] } x_2 \in [0 \mathinner{..} 1] \quad ([0 \mathinner{..} 1] \cap (-\infty \mathinner{..} 1]) \}$

$\boldsymbol{\{ x_3 \in [1 \mathinner{..} 2] \quad ([0 \mathinner{..} 1] \oplus [1 \mathinner{..} 1]) \}}$



Interval Analysis, I (the intuitive approach)

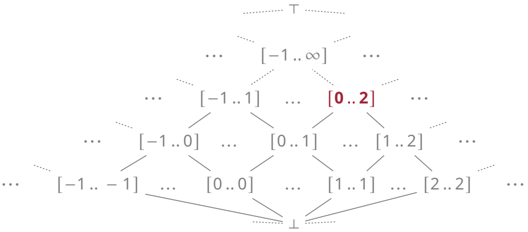


$$\{\!\{\ x_0 \in [0 \mathinner{..} 0]\ \}\!\}$$

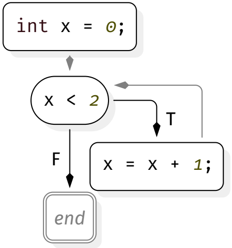
$$\{\!\{\ \text{[pre]}\ \boldsymbol{x_1 \in [0 \mathinner{..} 2]} \quad \mathbf{([0 \mathinner{..} 1] \cup [1 \mathinner{..} 2])}\ \}\!\}$$

$$\{\!\{\ \text{[in]}\ x_2 \in [0 \mathinner{..} 1] \quad ([0 \mathinner{..} 1] \cap (-\infty \mathinner{..} 1])\ \}\!\}$$

$$\{\!\{\ x_3 \in [1 \mathinner{..} 2] \quad ([0 \mathinner{..} 1] \oplus [1 \mathinner{..} 1])\ \}\!\}$$



Interval Analysis, I (the intuitive approach)



$$\{\!\mid x_0 \in [0 \mathinner{..} 0] \mid\!\}$$

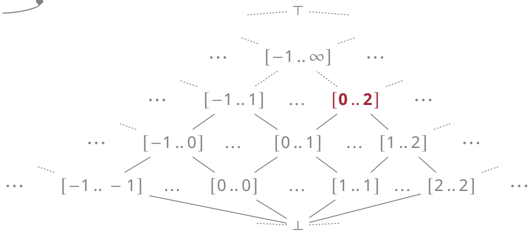
$$\{\!\mid \text{[pre]}\ x_1 \in [0 \mathinner{..} 2] \quad ([0 \mathinner{..} 1] \cup [1 \mathinner{..} 2]) \mid\!\}$$

$$\{\!\mid \text{[in]}\ x_2 \in [0 \mathinner{..} 1] \quad ([0 \mathinner{..} 1] \cap (-\infty \mathinner{..} 1]) \mid\!\}$$

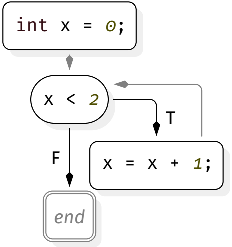
$$\{\!\mid x_3 \in [1 \mathinner{..} 2] \quad ([0 \mathinner{..} 1] \oplus [1 \mathinner{..} 1]) \mid\!\}$$

$$\{\!\mid \text{[post]}\ x_4 \in [2 \mathinner{..} 2] \quad ([0 \mathinner{..} 2] \cap [2 \mathinner{..} \infty)) \mid\!\}$$

after the loop we know that $\neg(x < 2) = x \geq 2$ holds!

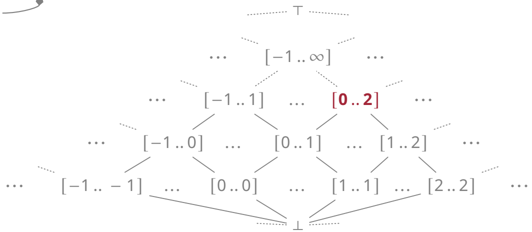


Interval Analysis, I (the intuitive approach)

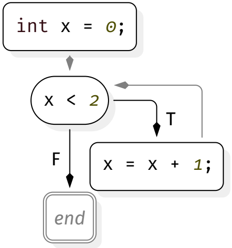


Intervals	Signs
$\{\!\{\ x_0 \in [0 \,..\, 0]\ \}\!\}$	$\{\!\{\ x_0 = 0\ \}\!\}$
$\{\!\{\ \text{[pre]}\ x_1 \in [0 \,..\, 2] \quad ([0 \,..\, 1] \cup [1 \,..\, 2])\ \}\!\}$	$\{\!\{\ \text{[pre]}\ x_1 \geq 0\ \}\!\}$
$\{\!\{\ \text{[in]}\ x_2 \in [0 \,..\, 1] \quad ([0 \,..\, 1] \cap (-\infty \,..\, 1])\ \}\!\}$	$\{\!\{\ \text{[in]}\ x_2 \geq 0\ \}\!\}$
$\{\!\{\ x_3 \in [1 \,..\, 2] \quad ([0 \,..\, 1] \oplus [1 \,..\, 1])\ \}\!\}$	$\{\!\{\ x_3 \geq 0\ \}\!\}$
$\{\!\{\ \text{[post]}\ x_4 \in [2 \,..\, 2] \quad ([0 \,..\, 2] \cap [2 \,..\, \infty))\ \}\!\}$	$\{\!\{\ \text{[post]}\ x_4 \geq 0\ \}\!\}$

after the loop we know that $\neg(x < 2) = x \geq 2$ holds!

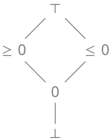


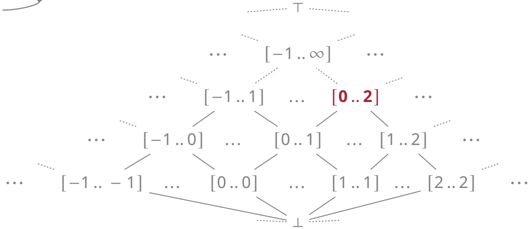
Interval Analysis, I (the intuitive approach)



Intervals	Signs
$\{\!\vert\; x_0 \in [0\,..\,0] \;\vert\!\}$	$\{\!\vert\; x_0 = 0 \;\vert\!\}$
$\{\!\vert\; \text{[pre]}\; x_1 \in [0\,..\,2] \quad ([0\,..\,1] \cup [1\,..\,2]) \;\vert\!\}$	$\{\!\vert\; \text{[pre]}\; x_1 \geq 0 \;\vert\!\}$
$\{\!\vert\; \text{[in]}\; x_2 \in [0\,..\,1] \quad ([0\,..\,1] \cap (-\infty\,..\,1]) \;\vert\!\}$	$\{\!\vert\; \text{[in]}\; x_2 \geq 0 \;\vert\!\}$
$\{\!\vert\; x_3 \in [1\,..\,2] \quad ([0\,..\,1] \oplus [1\,..\,1]) \;\vert\!\}$	$\{\!\vert\; x_3 \geq 0 \;\vert\!\}$
$\{\!\vert\; \text{[post]}\; x_4 \in [2\,..\,2] \quad ([0\,..\,2] \cap [2\,..\,\infty)) \;\vert\!\}$	$\{\!\vert\; \text{[post]}\; x_4 \geq 0 \;\vert\!\}$

after the loop we know that $\neg(x < 2) = x \geq 2$ holds!





3. Semantics

What does my program mean?

Semantics

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Semantics Program Syntax (simplified)

```java
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

java

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

Variable $v \in \mathbb{V}$

```java
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

java

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

Variable $v \in \mathbb{V}$
Assignment

java

```java
int x = 0;

while(x < 2) {

  x = x + 1;

}
```

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Semantics   Program Syntax (simplified)

Variable $v \in \mathbb{V}$

Assignment

```java
int x = 0;

while(x < 2) {

  x = x + 1;

}
```

Numeric Constant $c \in \mathbb{I}$

java

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Semantics Program Syntax (simplified)

Variable $v \in \mathbb{V}$

Assignment

Sequence

Numeric Constant $c \in \mathbb{I}$

java

```java
int x = 0;

while(x < 2) {

   x = x + 1;

}
```

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

Variable $v \in \mathbb{V}$

Assignment

Sequence

Numeric Constant $c \in \mathbb{I}$

Loop

```java
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

java

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

Variable $v \in \mathbb{V}$

Assignment

```java
int x = 0;
```

Sequence

Numeric Constant $c \in \mathbb{I}$

Loop

```java
while(x < 2) {
```

Comparison $\bowtie \in \{\leq, <, \ldots\}$

```java
  x = x + 1;

}
```

java

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

Variable $v \in \mathbb{V}$

Assignment

```java
int x = 0;
```

Sequence

Numeric Constant $c \in \mathbb{I}$

Loop

```java
while(x < 2) {
```

Comparison $\bowtie \in \{\leq, <, \ldots\}$

```java
  x = x + 1;
```

Binary Expression

```java
}
```

java

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

Variable $v \in \mathbb{V}$

Assignment

```
int x = 0;
```

Sequence

Numeric Constant $c \in \mathbb{I}$

Loop

```
while(x < 2) {
```

Comparison $\bowtie \in \{\leq, <, \ldots\}$

```
   x = x + 1;
```

Binary Expression

```
}
```

$$
\begin{array}{llll}
\mathit{stm} & ::= & V \leftarrow \mathit{expr} & \text{(assignment, } V \in \mathbb{V}\text{)} \quad \text{java} \\
 & | & \mathit{stm}_1;\ \mathit{stm}_2 & \text{(sequence)} \\
 & | & \textbf{while}(\mathit{cond})\ \{\ \mathit{stm}\ \} & \text{(loop)} \\
\mathit{expr} & ::= & V & \text{(variable, } V \in \mathbb{V}\text{)} \\
 & | & c & \text{(constant, } c \in \mathbb{I}\text{)} \\
 & | & \mathit{expr}_1 \diamond \mathit{expr}_2 & \text{(bin. expr., } \diamond \in \{+, -, \ldots\}\text{)} \\
\mathit{cond} & ::= & b & \text{(boolean, } b \in \mathbb{B}\text{)} \\
 & | & \mathit{expr}_1 \bowtie \mathit{expr}_2 & \text{(comparison, } \bowtie \in \{\leq, <, \ldots\}\text{)}
\end{array}
$$

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```java
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

java

$$
\begin{array}{llll}
\mathit{expr} & ::= & V & \text{(variable, } V \in \mathbb{V}\text{)} \\
 & | & c & \text{(constant, } c \in \mathbb{I}\text{)} \\
 & | & \mathit{expr}_1 \diamond \mathit{expr}_2 & \text{(bin. expr., } \diamond \in \{+, -, \ldots\}\text{)}
\end{array}
$$

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```java
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

java

$$
\begin{array}{rcll}
\mathit{expr} & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
 & | & c & (\text{constant}, c \in \mathbb{I}) \\
 & | & \mathit{expr}_1 \diamond \mathit{expr}_2 & (\text{bin. expr.}, \diamond \in \{+, -, \ldots\})
\end{array}
$$

- We use an environment $\mathcal{E} \overset{\text{def}}{=} \mathbb{V} \to \mathbb{I}$ to represent the current program state

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```java
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

java

$$
\begin{array}{rcll}
\mathit{expr} & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
 & | & c & (\text{constant}, c \in \mathbb{I}) \\
 & | & \mathit{expr}_1 \diamond \mathit{expr}_2 & (\text{bin. expr.}, \diamond \in \{+, -, \ldots\})
\end{array}
$$

Variable

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \mathbb{V} \to \mathbb{I}$ to represent the current program state

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```java
int x = 0;
while(x < 2) {
  x = x + 1;
}
```
java

$$\begin{array}{llll} expr & ::= & V & (\text{variable}, V \in \mathbb{V}) \\ & \mid & c & (\text{constant}, c \in \mathbb{I}) \\ & \mid & expr_1 \diamond expr_2 & (\text{bin. expr.}, \diamond \in \{+, -, \ldots\}) \end{array}$$

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \underbrace{\mathbb{V}}_{\text{Variable}} \to \underbrace{\mathbb{I}}_{\text{Integer Values}}$ to represent the current program state

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```java
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

java

$$
\begin{array}{rcll}
\mathit{expr} & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
 & | & c & (\text{constant}, c \in \mathbb{Z}) \\
 & | & \mathit{expr}_1 \diamond \mathit{expr}_2 & (\text{bin. expr.}, \diamond \in \{+, -, \ldots\})
\end{array}
$$

- We use an environment $\mathcal{E} \overset{\text{def}}{=} \underbrace{\mathbb{V}}_{\text{Variable}} \to \underbrace{\mathbb{Z}}_{\text{Integer Values}}$ to represent the current program state

$\mathbb{V}$	$\mathbb{Z}$
x	0
c	5

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```java
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

java

expr	::=	V	(variable, $V \in \mathbb{V}$)
	|	c	(constant, $c \in \mathbb{I}$)
	|	$expr_1 \diamond expr_2$	(bin. expr., $\diamond \in \{+, -, \ldots\}$)

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \underbrace{\mathbb{V}}_{\text{Variable}} \rightarrow \underbrace{\mathbb{I}}_{\text{Integer Values}}$ to represent the current program state
- Now we can define evalExpr(*expr*, *env*) for an environment $env \in \mathcal{E}$

$\mathbb{V}$	$\mathbb{I}$
x	0
c	5

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```java
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

java

$$
\begin{array}{lcll}
\mathit{expr} & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
 & | & c & (\text{constant}, c \in \mathbb{I}) \\
 & | & \mathit{expr}_1 \diamond \mathit{expr}_2 & (\text{bin. expr.}, \diamond \in \{+, -, \ldots\})
\end{array}
$$

- We use an environment $\mathcal{E} \overset{\text{def}}{=} \underbrace{\mathbb{V}}_{\text{Variable}} \rightarrow \underbrace{\mathbb{I}}_{\text{Integer Values}}$ to represent the current program state
- Now we can define $\text{evalExpr}(\mathit{expr}, \mathit{env})$ for an environment $\mathit{env} \in \mathcal{E}$

Usually written as $\mathbb{E}[\![\, \mathit{expr} \,]\!]\rho$

$\mathbb{V}$	$\mathbb{I}$
x	0
c	5

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```java
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

java

$$
\begin{array}{rcll}
expr & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
 & | & c & (\text{constant}, c \in \mathbb{I}) \\
 & | & expr_1 \diamond expr_2 & (\text{bin. expr.}, \diamond \in \{+, -, \ldots\})
\end{array}
$$

Variable

Integer Values

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \mathbb{V} \to \mathbb{I}$ to represent the current program state

$\mathbb{V}$	$\mathbb{I}$
x	0
c	5

Usually written as $\mathbb{E}[\![\, expr \,]\!]\rho$

- Now we can define evalExpr($expr$, env) for an environment $env \in \mathcal{E}$

$$\text{evalExpr}(V,\ env) \stackrel{\text{def}}{=} env(V)$$

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```java
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

java

$$
\begin{array}{llll}
expr & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
 & \mid & c & (\text{constant}, c \in \mathbb{I}) \\
 & \mid & expr_1 \diamond expr_2 & (\text{bin. expr.}, \diamond \in \{+, -, \ldots\})
\end{array}
$$

Variable — Integer Values

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \mathbb{V} \to \mathbb{I}$ to represent the current program state

$\mathbb{V}$	$\mathbb{I}$
x	0
c	5

Usually written as $\mathbb{E}[\![\, expr \,]\!]\rho$

- Now we can define evalExpr($expr$, env) for an environment $env \in \mathcal{E}$

$$\text{evalExpr}(V,\ env) \stackrel{\text{def}}{=} env(V)$$

Value of $V \in \mathbb{V}$ in Environment env

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```java
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

java

$$
\begin{array}{rcll}
\mathit{expr} & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
 & \mid & c & (\text{constant}, c \in \mathbb{I}) \\
 & \mid & \mathit{expr}_1 \diamond \mathit{expr}_2 & (\text{bin. expr.}, \diamond \in \{+, -, \ldots\})
\end{array}
$$

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \underbrace{\mathbb{V}}_{\text{Variable}} \rightarrow \underbrace{\mathbb{I}}_{\text{Integer Values}}$ to represent the current program state

$\mathbb{V}$	$\mathbb{I}$
x	0
c	5

- Now we can define evalExpr(expr, env) for an environment $\mathit{env} \in \mathcal{E}$

Usually written as $\mathbb{E}[\![\mathit{expr}]\!]\rho$

$$
\begin{aligned}
\text{evalExpr}(V, \mathit{env}) &\stackrel{\text{def}}{=} \mathit{env}(V) \\
\text{evalExpr}(c, \mathit{env}) &\stackrel{\text{def}}{=} c
\end{aligned}
$$

Value of $V \in \mathbb{V}$ in Environment env

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```java
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

java

$$\begin{array}{rcll} expr & ::= & V & (\text{variable}, V \in \mathbb{V}) \\ & | & c & (\text{constant}, c \in \mathbb{I}) \\ & | & expr_1 \diamond expr_2 & (\text{bin. expr.}, \diamond \in \{+, -, \ldots\}) \end{array}$$

Variable
Integer Values

- We use an environment $\mathcal{E} \overset{\text{def}}{=} \mathbb{V} \to \mathbb{I}$ to represent the current program state

$\mathbb{V}$	$\mathbb{I}$
x	0
c	5

Usually written as $\mathbb{E}[\![\, expr \,]\!]\rho$

- Now we can define $\text{evalExpr}(expr, env)$ for an environment $env \in \mathcal{E}$

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Value of $V \in \mathbb{V}$ in Environment env

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```java
int x = 0;
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```

java

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\begin{array}{rcll}
\mathit{expr} & ::= & V & \text{(variable, } V \in \mathbb{V}\text{)} \\
 & | & c & \text{(constant, } c \in \mathbb{I}\text{)} \\
 & | & \mathit{expr}_1 \diamond \mathit{expr}_2 & \text{(bin. expr., } \diamond \in \{+, -, \ldots\}\text{)}
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Variable

Integer Values

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \mathbb{V} \to \mathbb{I}$ to represent the current program state

$\mathbb{V}$	$\mathbb{I}$
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c	5

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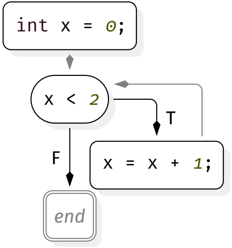
Interval Analysis, II



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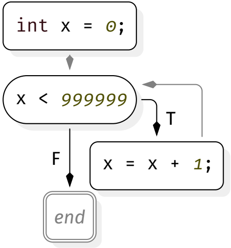

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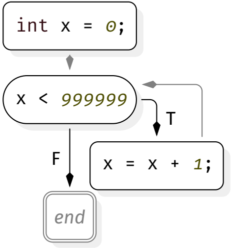

$\wr\, x_0 \in [0\,..\,0]\, \wr$

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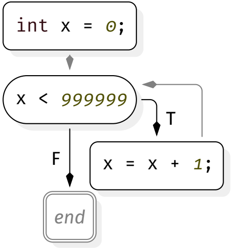

$\wr\, x_0 \in [0\,..\,0] \,\int$

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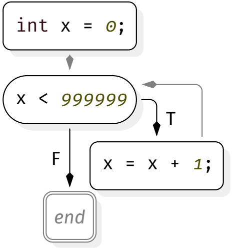

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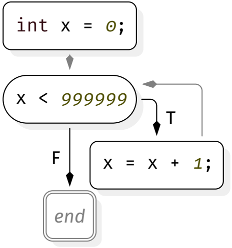

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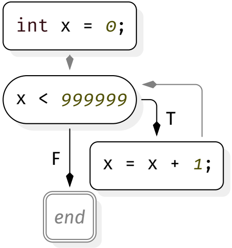

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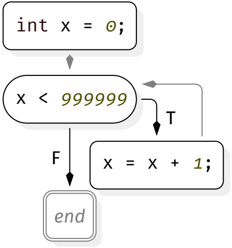

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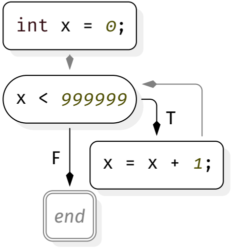

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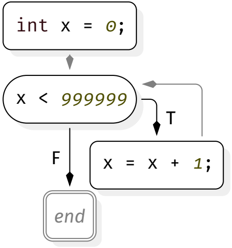

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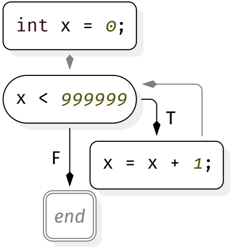

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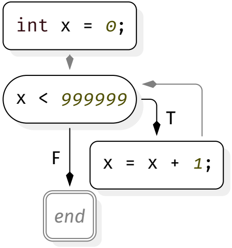

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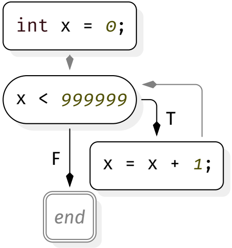

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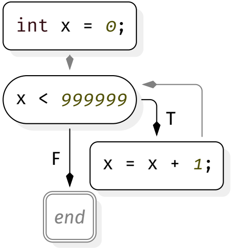

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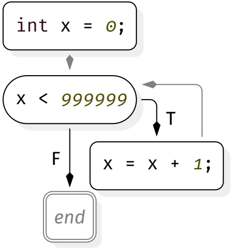

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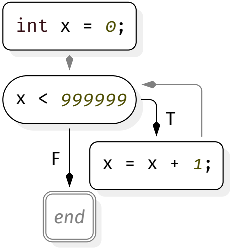

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$$\{\, \text{[post]}\; x_4 \in [999999 \,..\, \infty) \quad ([0 \,..\, \infty) \cap [999999 \,..\, \infty)) \,\}$$

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3. Provide an integer concretization for $\gamma(0)$ and $\gamma(\leq 0)$!

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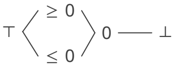

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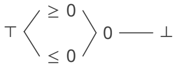

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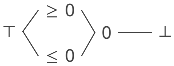

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\begin{tikzcd}[column sep=small, row sep=tiny]
 & \geq 0 \arrow[dl, dash] \arrow[dr, dash] & & \\
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   We do not need widening here as our lattice is finite and our semantics do not introduce new elements. (We will concretize this next week.)

4. Outlook and Comments

This is incredible, I need more!

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- However, we have only looked at single variables and single domains so far!

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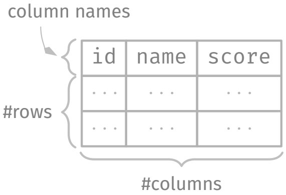

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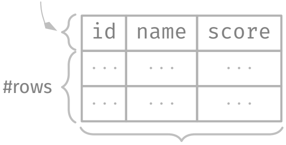

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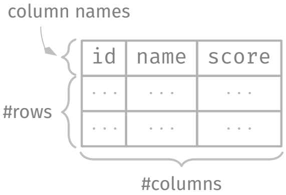

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column names

#rows

id	name	score
...	...	...
...	...	...

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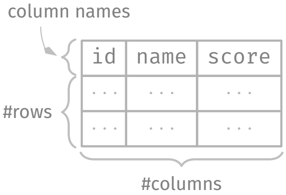

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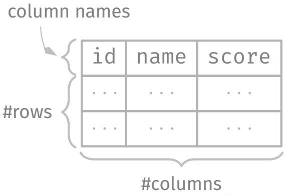

[Ger25] Oliver Gerstl. Tracking the shape of data frames in R programs using abstract interpretation (Ulm University, 2025)

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Domains can also capture relations between variables (e.g. *polyhedra*), their provenance, and much more!

However, this implies trade-offs which we discuss next time.

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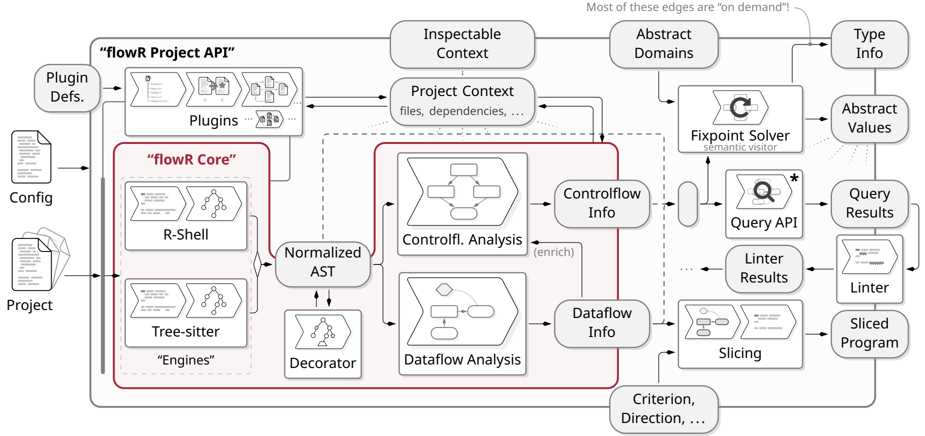

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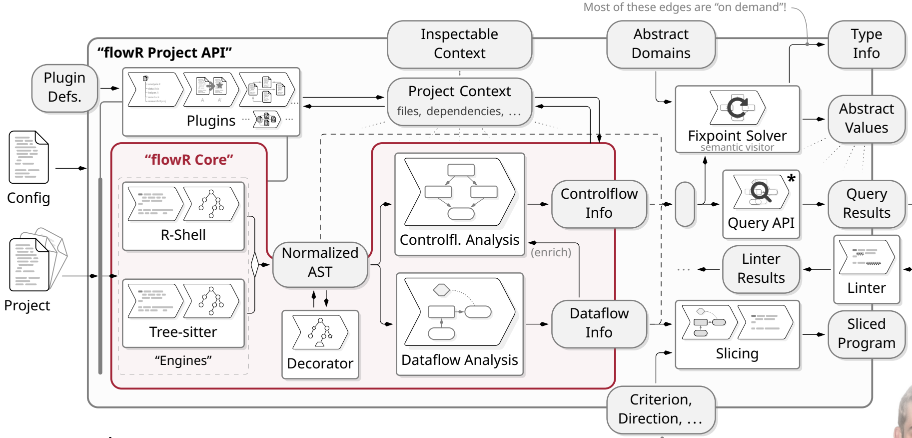

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Bibliography

References I

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A Little Bit of History

Static Analysis

Based on the amazing "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" by Miné [Min17], https://www.di.ens.fr/~cousot/AI/, and [Bal+18; GR22]

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1949: First Checks
Turing [Tur49]

1953: Rice Theorem
Non-trivial properties are undecidable [Ric53]

Static Analysis

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1949: First Checks
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1953: Rice Theorem
Non-trivial properties
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Static Analysis

1967 & 69: Logical Foundation
Floyd [Flo67], Hoare [Hoa69]
But: No Automation

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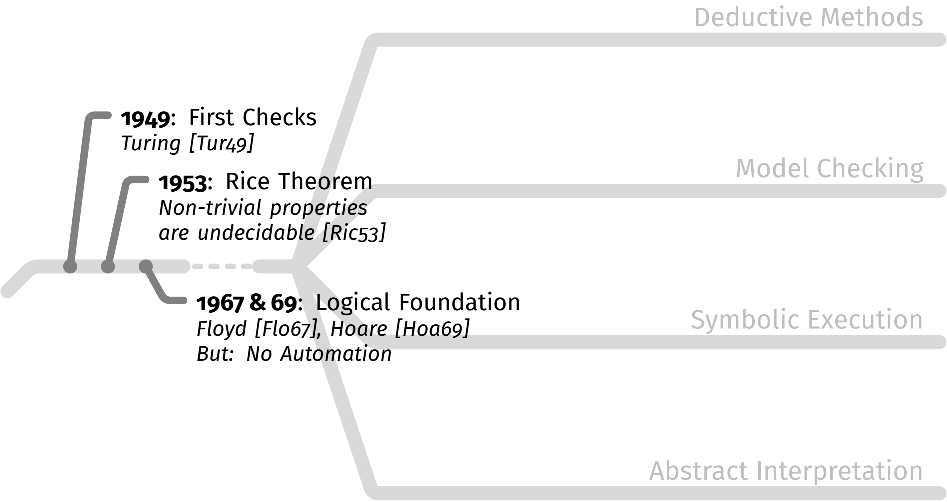

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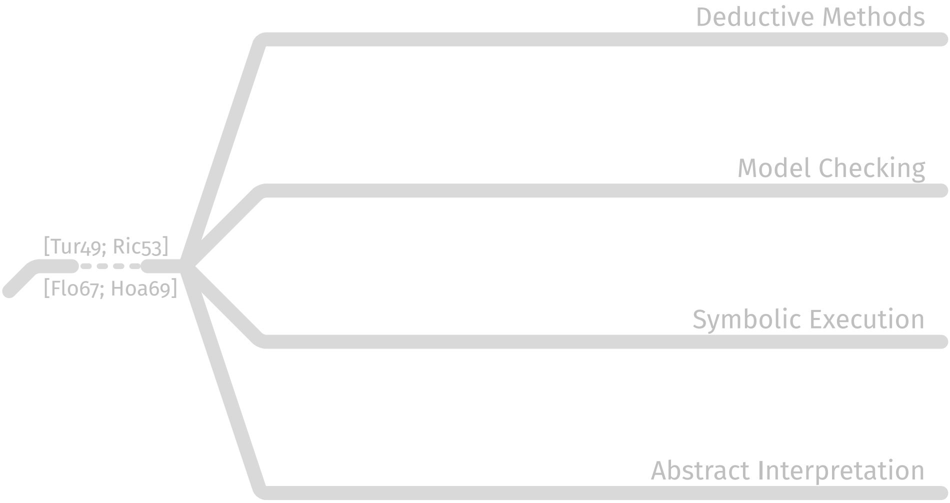

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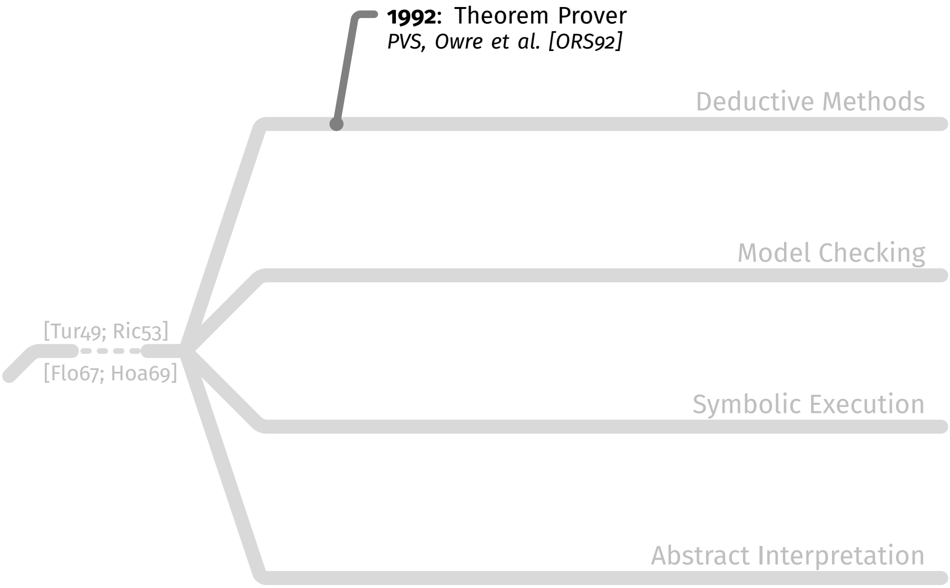

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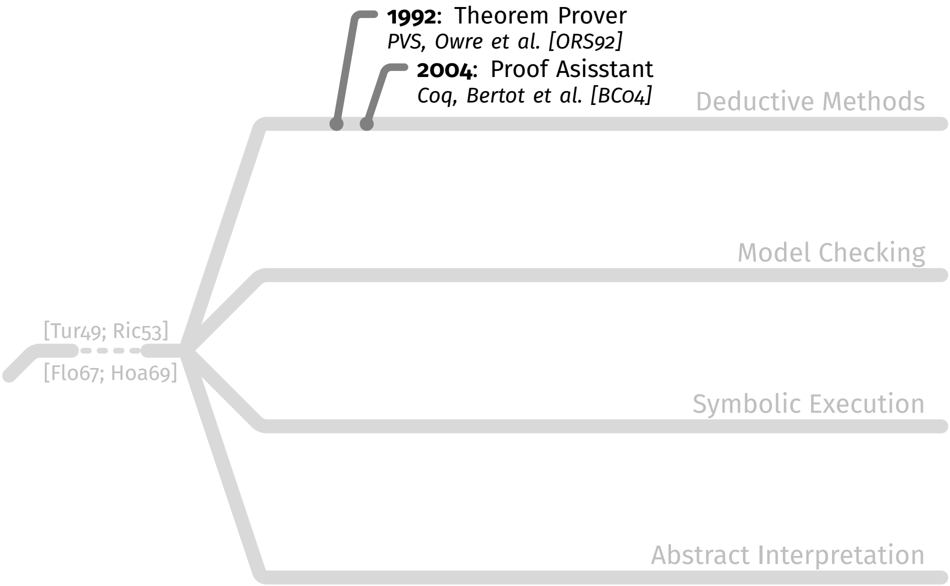

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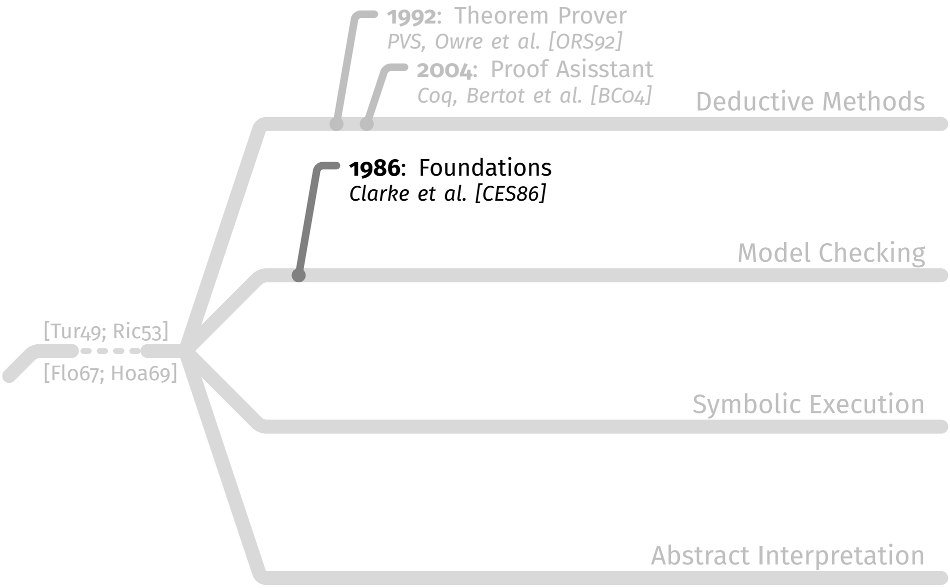

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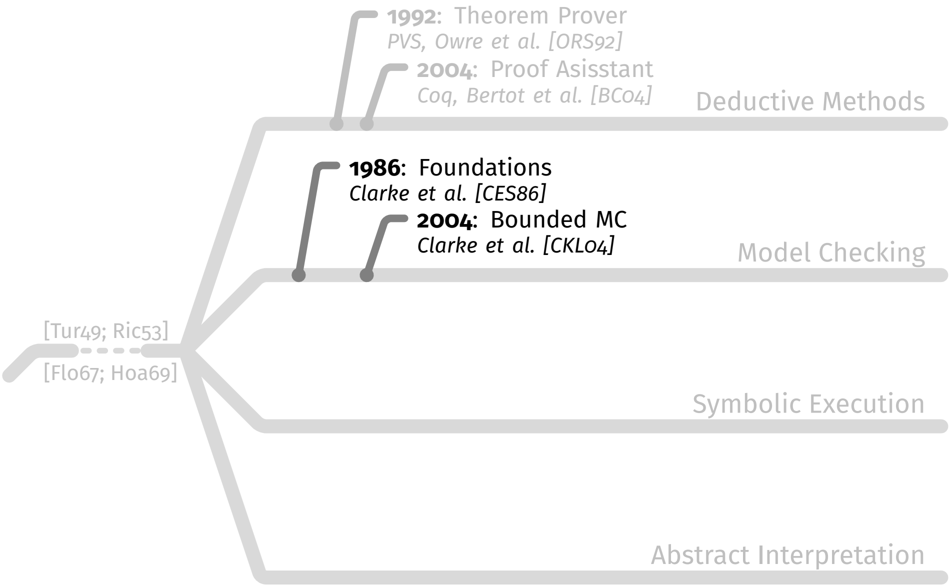

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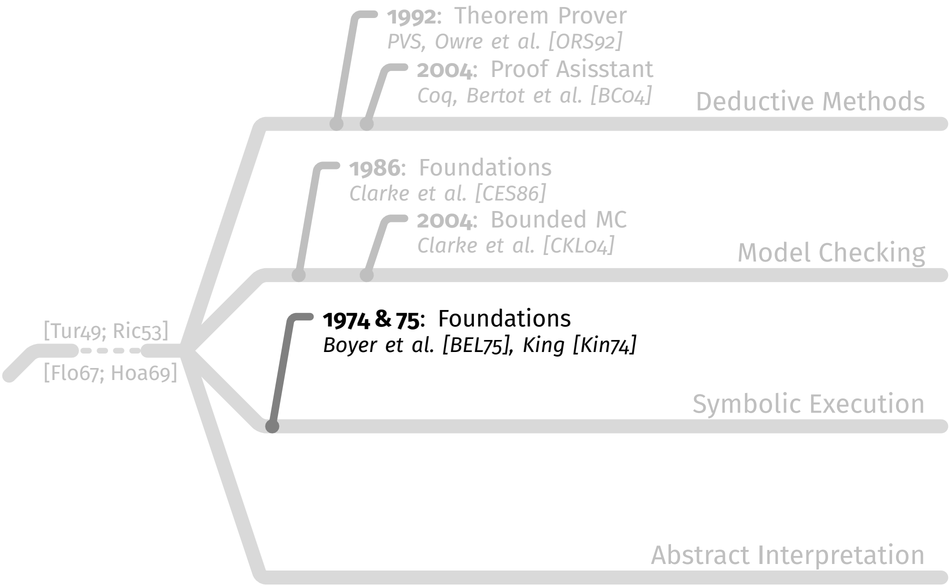

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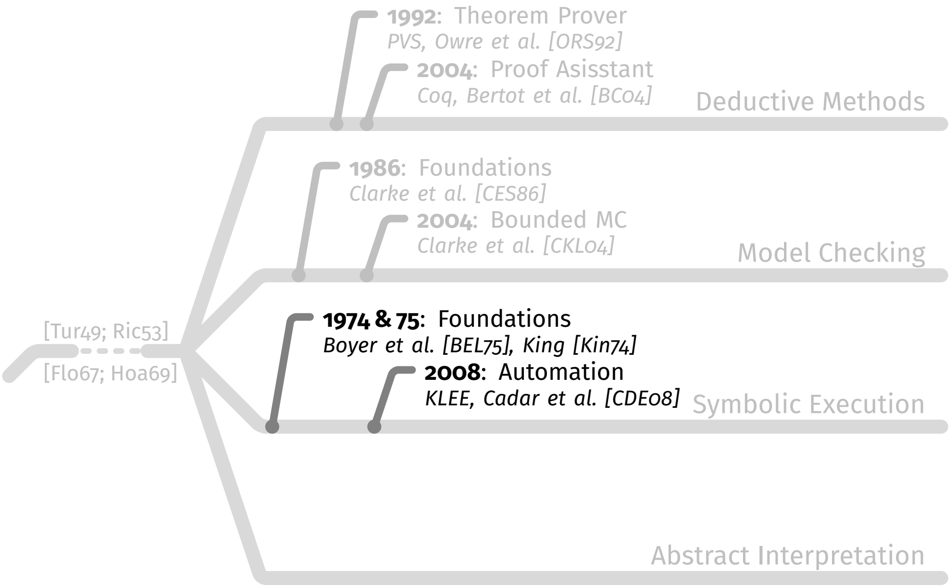

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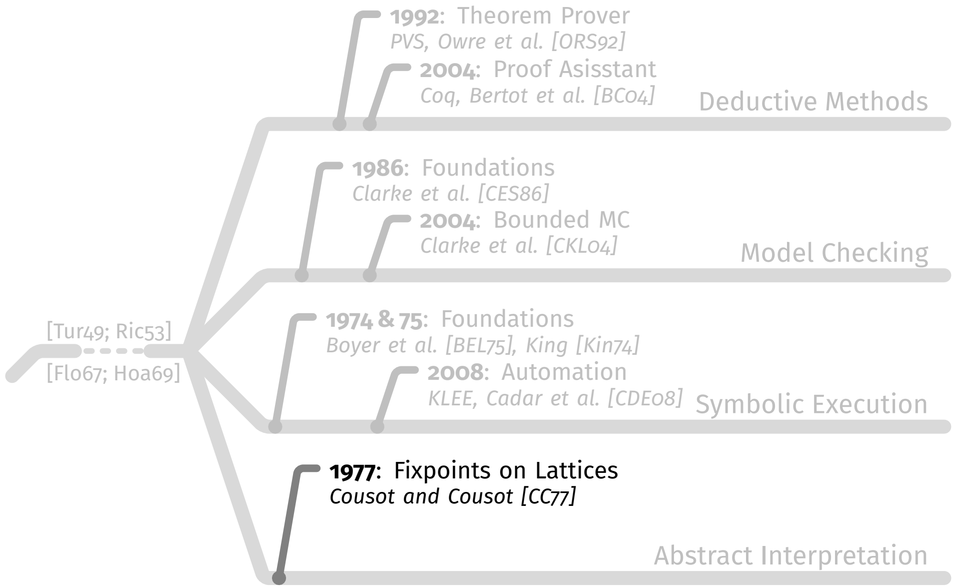

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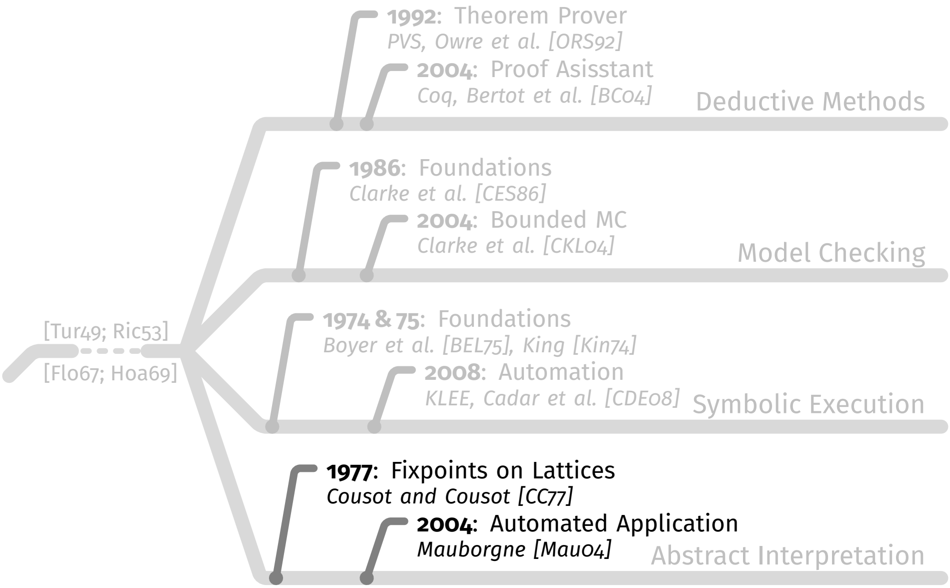

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