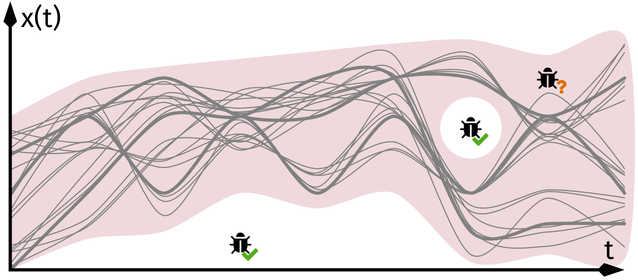

Abstract Interpretation

Software Quality Assurance - Static Code Analysis, II | Florian Sihler | December 11, 2024

Software Engineering
Programming Languages

universität
uulm

1. The Why

The Why

```java
public static void main(String[] args) {
   int a = 1;
   double r = Math.random() * 10;
   if (r > 5) {
      a = 2;
   }
   System.out.println(a);
}
```

The Why

```java
public static void main(String[] args) {
    int a = 1;
    double r = Math.random() * 10;
    if (r > 5) {
        a = 2;
    }
    System.out.println(a);
}
```

The Why

```
public static void main(String[] args) {
    int a = 1;                              ⦃ a ∈ {1} ⦄
    double r = Math.random() * 10;
    if (r > 5) {
      a = 2;
    }
    System.out.println(a);
}
```

The Why

```
public static void main(String[] args) {
    int a = 1;                                ⦃ a ∈ {1} ⦄
    double r = Math.random() * 10;            ⦃ r ∈ [0..10) ⦄
    if (r > 5) {
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    System.out.println(a);
}
```

The Why

```
public static void main(String[] args) {
    int a = 1;                              ⦃ a ∈ {1} ⦄
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    if (r > 5) {
      a = 2;                                ⦃ a ∈ {2} ⦄
    }
    System.out.println(a);
}
```

The Why

```
public static void main(String[] args) {
    int a = 1;                             ≀ a ∈ {1} ≀
    double r = Math.random() * 10;         ≀ r ∈ [0..10) ≀
    if (r > 5) {
      a = 2;                               ≀ a ∈ {2} ≀
    }
    System.out.println(a);                 ≀ a ∈ {1,2} ≀
}
```

The Why

```
public static void main(String[] args) {
    int a = 1;
    double r = Math.random() * 10;
    if (r > 5) {
      a = 2;
    }
    System.out.println(a);
}
```

$\wr a \in \{1\} \wr$

$\wr r \in [0..10) \wr$

$\wr a \in \{2\} \wr$

$\wr a \in \{1, 2\} \wr$ $\longrightarrow$ Valid? Ok? Safe?

The Why

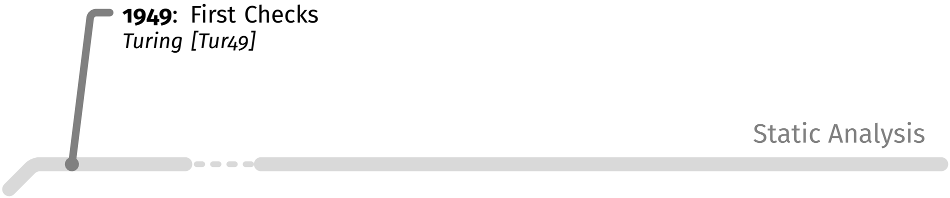
```
public static void main(String[] args) {
    int a = 1;                              ⦃ a ∈ {1} ⦄
    double r = Math.random() * 10;          ⦃ r ∈ [0..10) ⦄
    if (r > 5) {
      a = 2;                                ⦃ a ∈ {2} ⦄
    }
    System.out.println(a);                  ⦃ a ∈ {1,2} ⦄  →  Valid? Ok? Safe?
}
```

- We want to proof, that a program satisfies certain properties

Origins

Based on the amazing "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" by Miné [Min17], https://www.di.ens.fr/~cousot/AI/, and [Bal+18; GR22]

Origins



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Origins

1949: First Checks
Turing [Tur49]

1953: Rice Theorem
Non-trivial Properties are undecidable [Ric53]

Static Analysis

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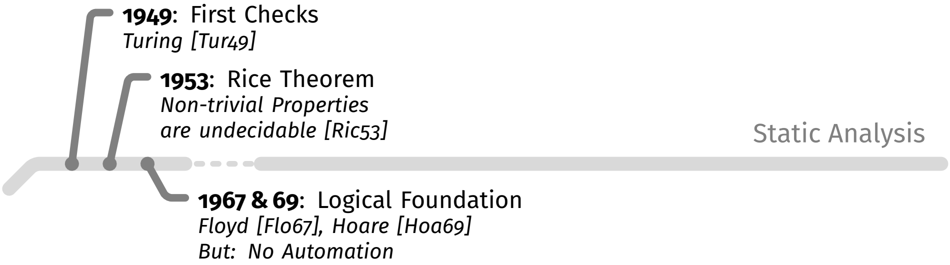
Origins



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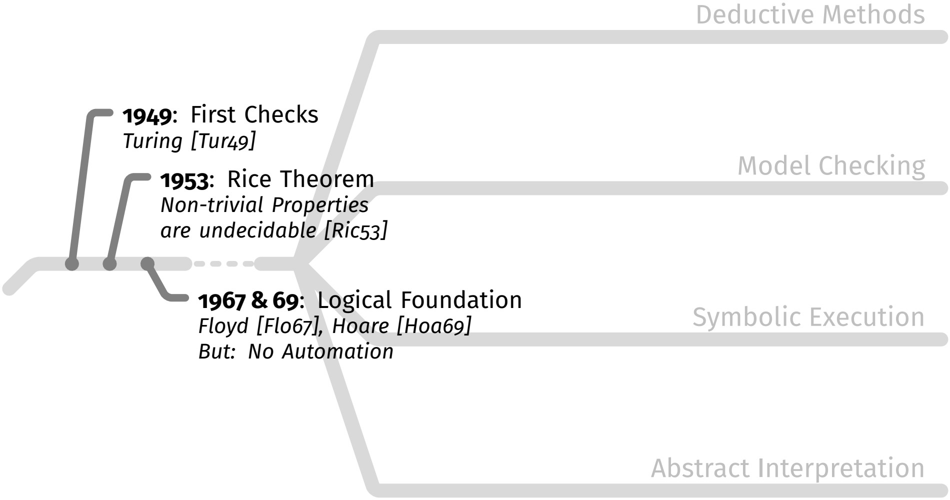

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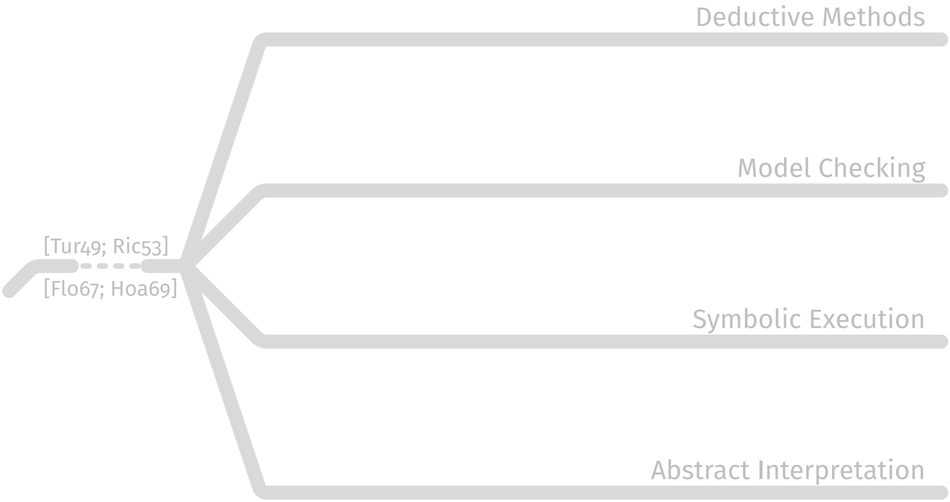

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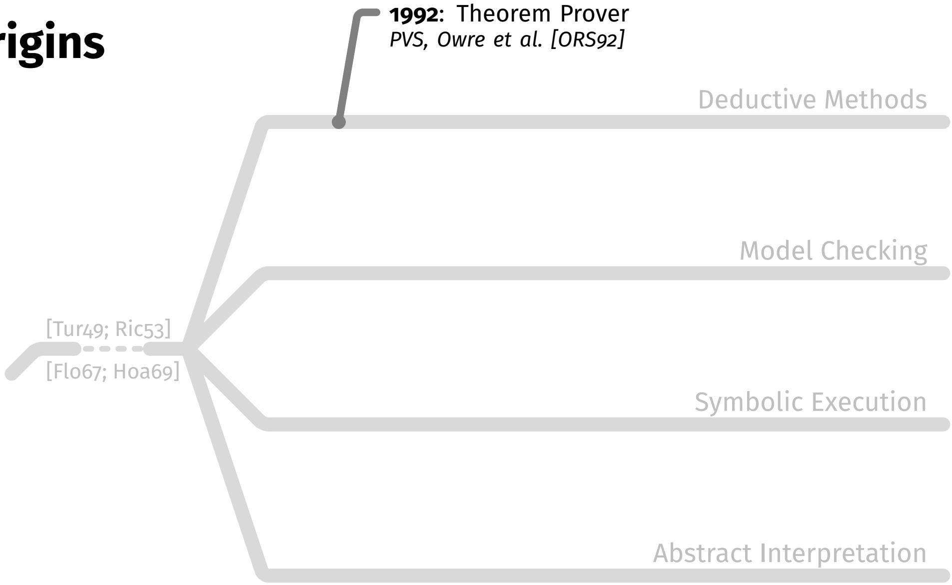

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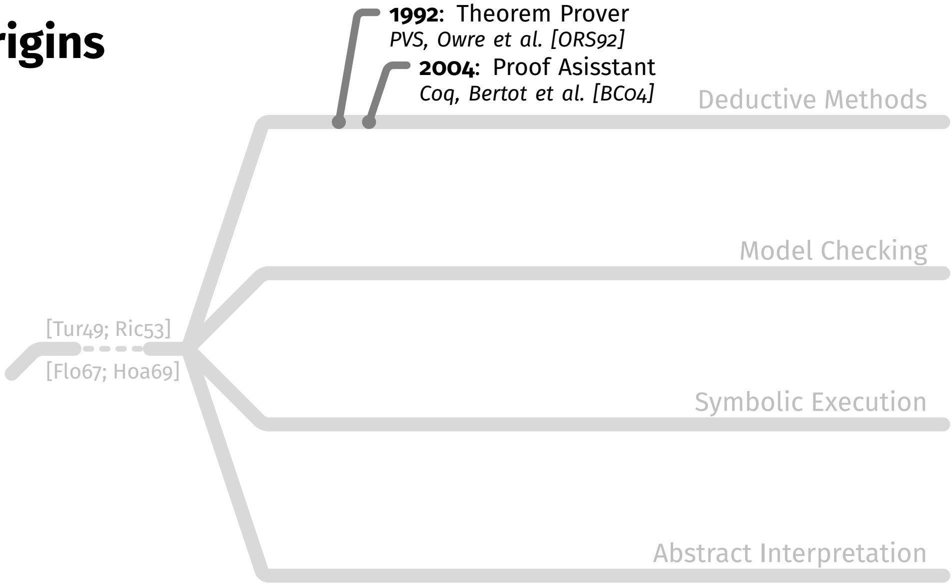

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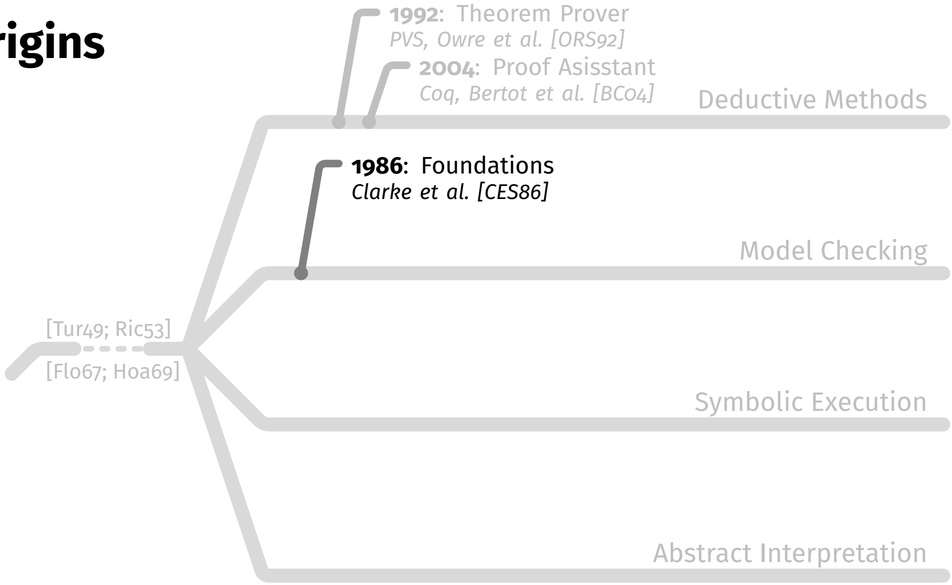

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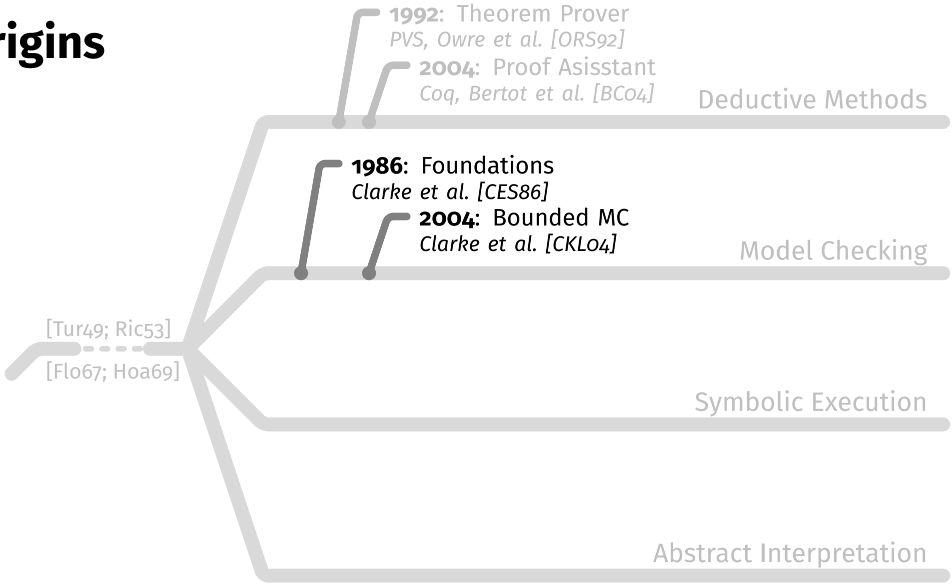

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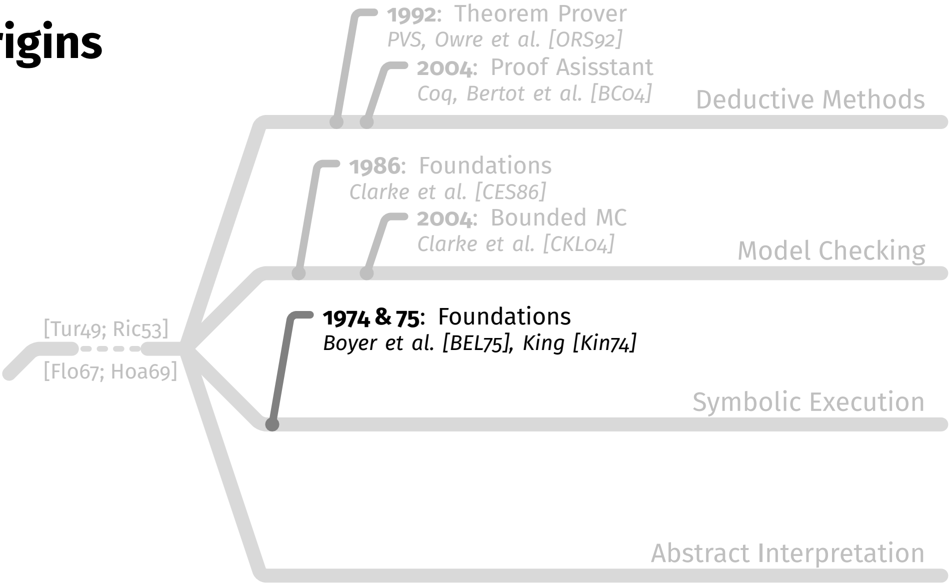

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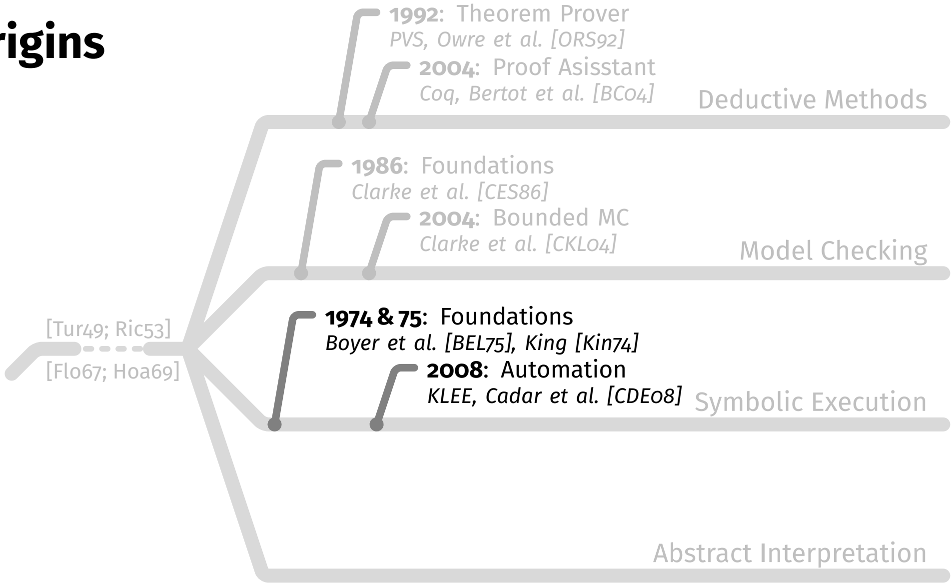

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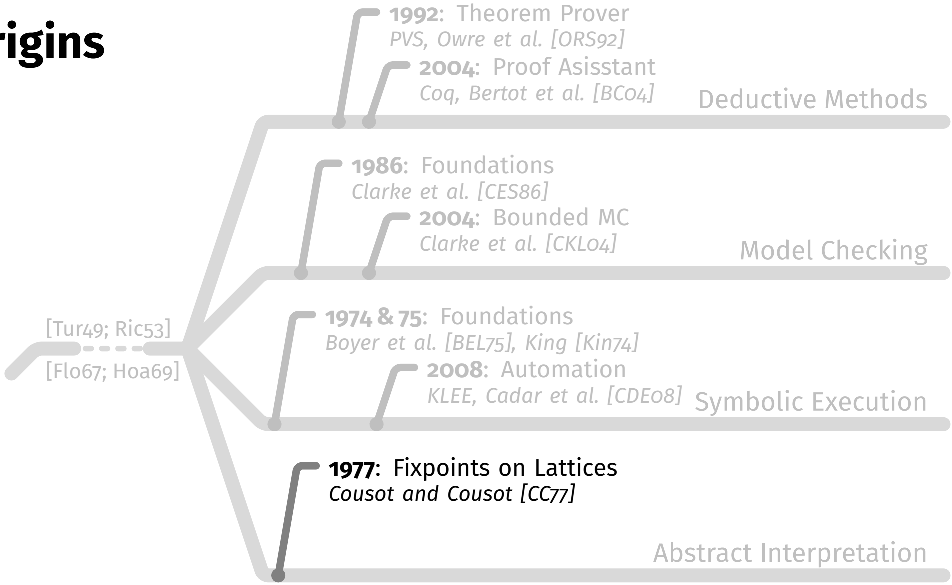

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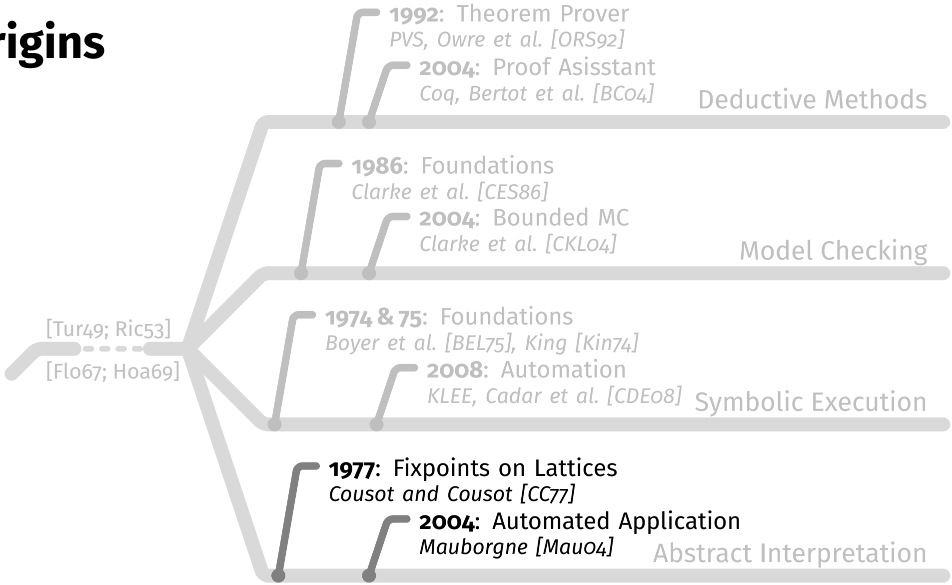

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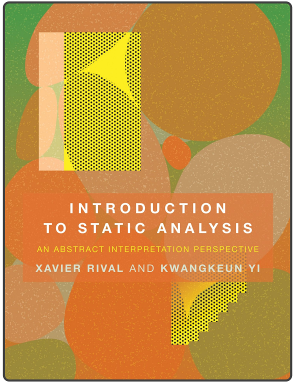
Recommended Resources

And for an overview: “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17]

Recommended Resources

Using Analyses [RY20]



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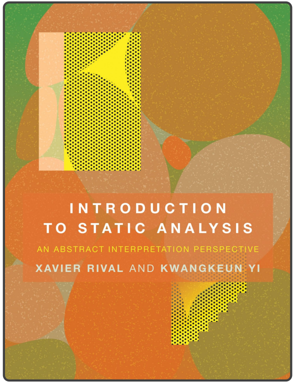
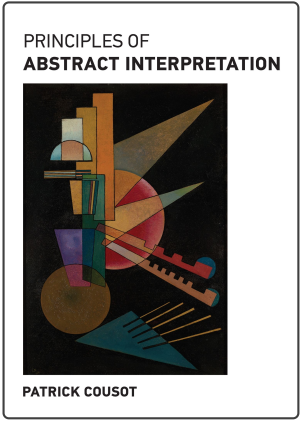


Formal Foundations [Cou21]



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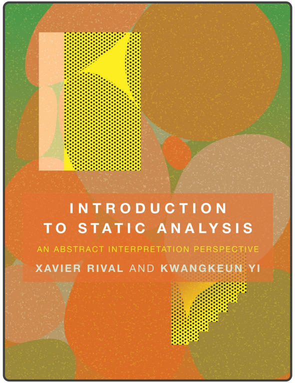

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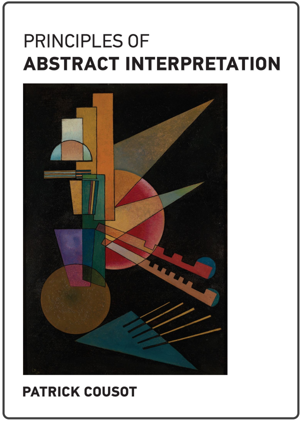
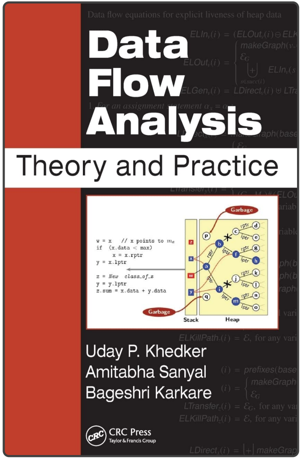


Dataflow Perspective [KSK09]



And for an overview: “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17]

2. The How

Abstract Interpretation

Abstract Interpretation

```
public static void main(String[] args) {
    int a = 1;                              ⦃ a ∈ {1} ⦄
    double r = Math.random() * 10;          ⦃ r ∈ [0..10) ⦄
    if (r > 5) {
        a = 2;                              ⦃ a ∈ {2} ⦄
    }
    System.out.println(a);                  ⦃ a ∈ {1,2} ⦄ → Valid? Ok? Safe?
}
```

“Principles of Abstract Interpretation” [Cou21, p. 722],“History of Abstract Interpretation” [GR22, pp. 37]

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Abstract Interpretation

- We want to proof interesting properties of programs

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- We want to proof interesting properties of programs
  - *Dataflow Properties*
    Liveness, Fainting, Reaching Definitions, …

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Abstract Interpretation

- We want to proof interesting properties of programs
  - *Dataflow Properties*
    Liveness, Fainting, Reaching Definitions, ...
  - *Safety Properties*
    No Null Dereference, No Division by Zero, ...

```
public static void main(String[] args) {
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  - *Numerical Properties*
    Signs, Intervals, Octagons, Polyhedra, ...

"Principles of Abstract Interpretation" [Cou21, p. 722],"History of Abstract Interpretation" [GR22, pp. 37]

Abstract Interpretation

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  - …

“Principles of Abstract Interpretation” [Cou21, p. 722],“History of Abstract Interpretation” [GR22, pp. 37]

Abstract Interpretation

See “A casual introduction to Abstract Interpretation” [Cou12]

Concrete Interpretation

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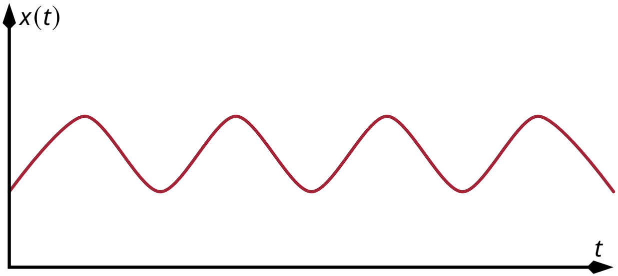
Concrete Interpretation



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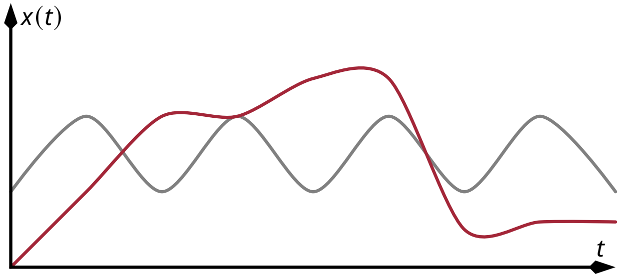

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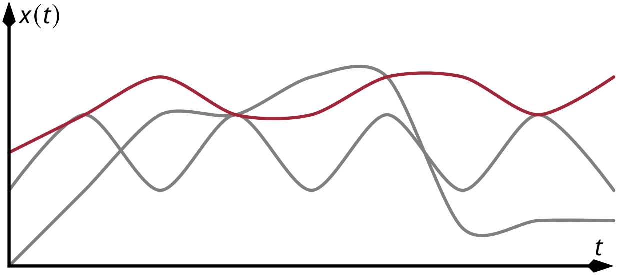

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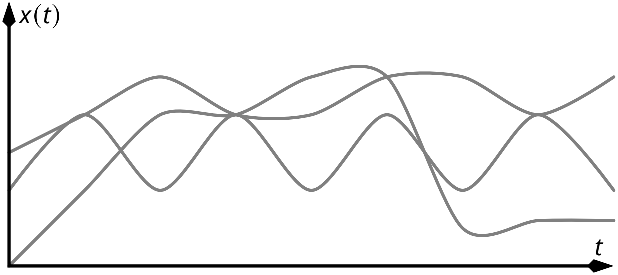

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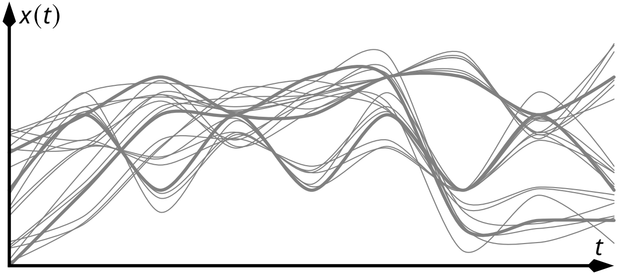

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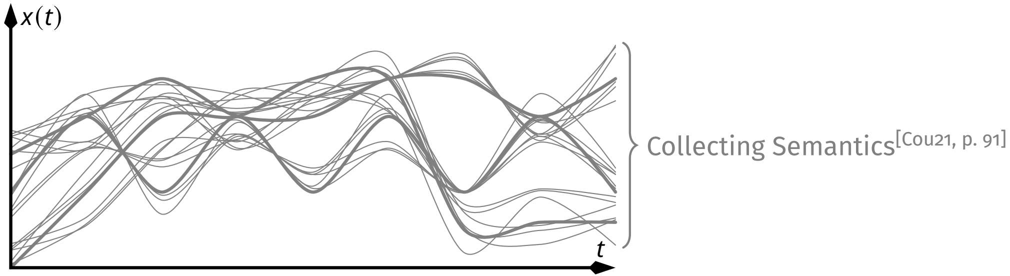

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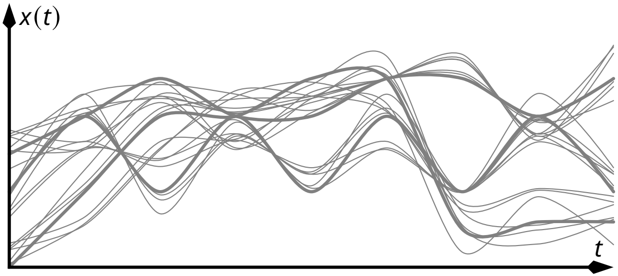

Collecting Semantics[Cou21, p. 91]

- Maybe impossible to compute statically
- … or very expensive (▸ *dynamic*)

▸ Abstract Interpretation to the rescue

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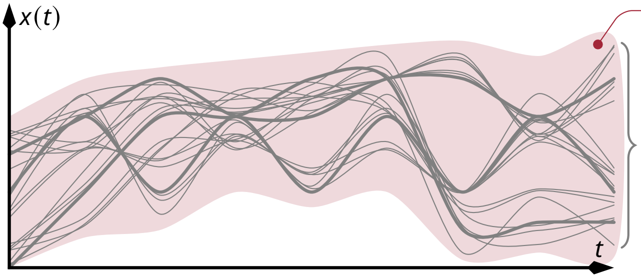
Abstract Interpretation



(Trace) Abstraction[Cou21, p. 92]
just one of many

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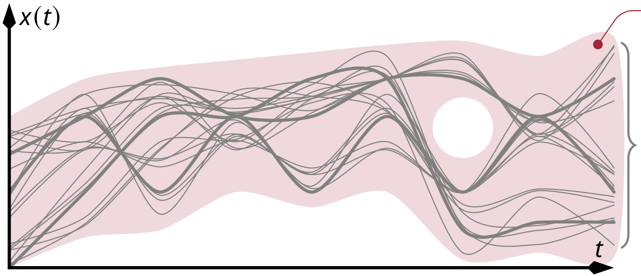

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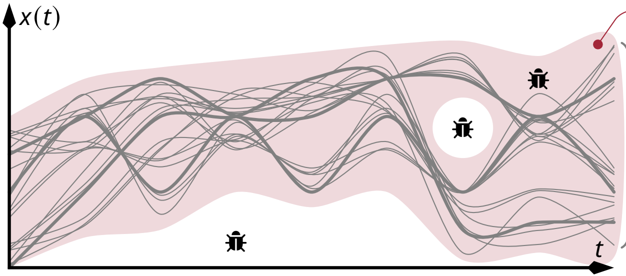

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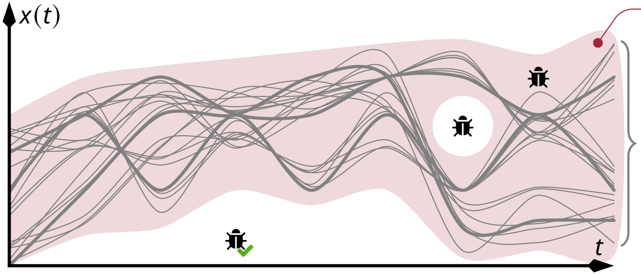

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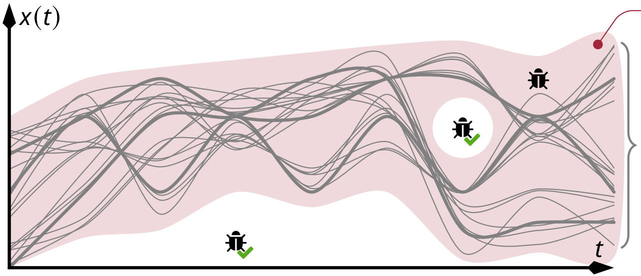

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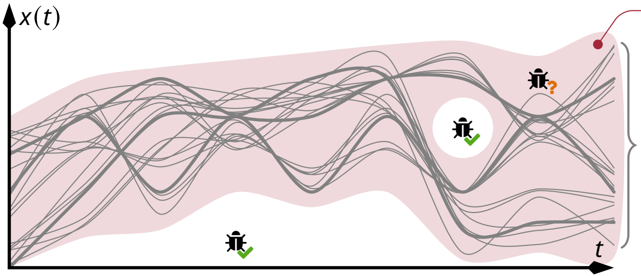

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Terminology

- **Property**

“Principles of Abstract Interpretation” [Cou21, p. 15],“Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 18]

Terminology

- **Property** — Set of states/traces that satisfy that property

“Principles of Abstract Interpretation” [Cou21, p. 15],“Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 18]

Terminology

- **Property** — Set of states/traces that satisfy that property
  Even integers: $P = \{ z \in \mathbb{Z} \mid \exists k \in \mathbb{Z} : z = 2k \} = \{ 0, 2, 4, 6, \ldots \} \subseteq \mathcal{P}(\mathbb{Z})$

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  Even integers: $\mathrm{P} = \{\, z \in \mathbb{Z} \mid \exists k \in \mathbb{Z} : z = 2k \,\} = \{\, 0, 2, 4, 6, \ldots \,\} \subseteq \mathcal{P}(\underbrace{\mathbb{Z}}_{\text{universe } (\mathbb{U})})$

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$$\varnothing \subseteq P_1 \subseteq P_2 \subseteq \mathbb{U}$$

"Principles of Abstract Interpretation" [Cou21, p. 15],"Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 18]

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$$\underbrace{\varnothing}_{\text{strongest}} \subseteq \mathrm{P}_1 \subseteq \mathrm{P}_2 \subseteq \mathbb{U}$$

“Principles of Abstract Interpretation” [Cou21, p. 15],“Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 18]

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strongest
stronger
weaker
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strongest · stronger · weaker · weakest

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  strongest stronger weaker weakest

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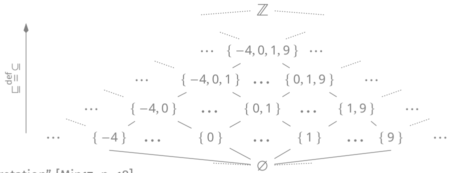

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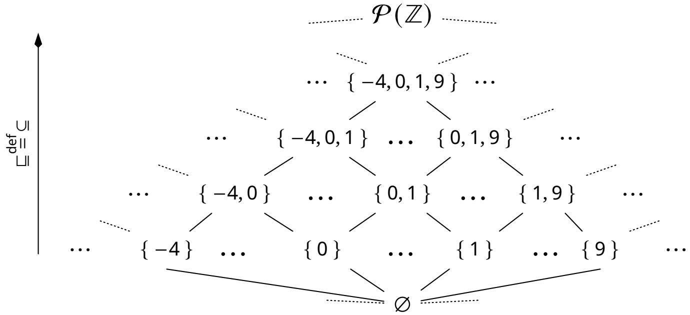
Chains and Lattices



"Lattice theory" [Bir67], see also sublattices [Min17, p. 25]

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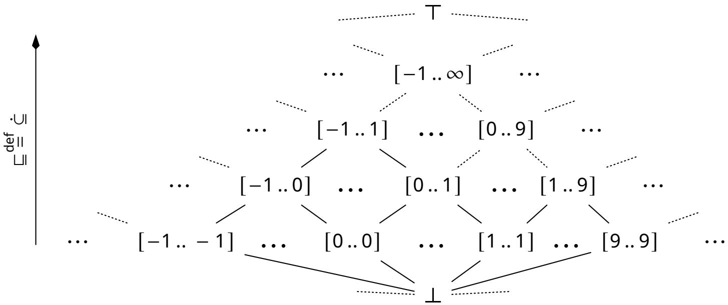

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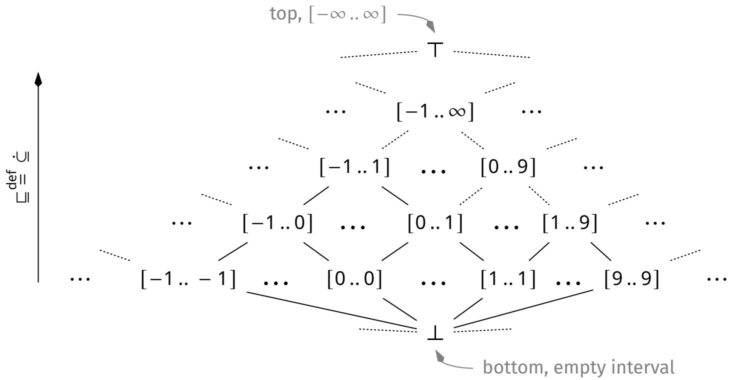

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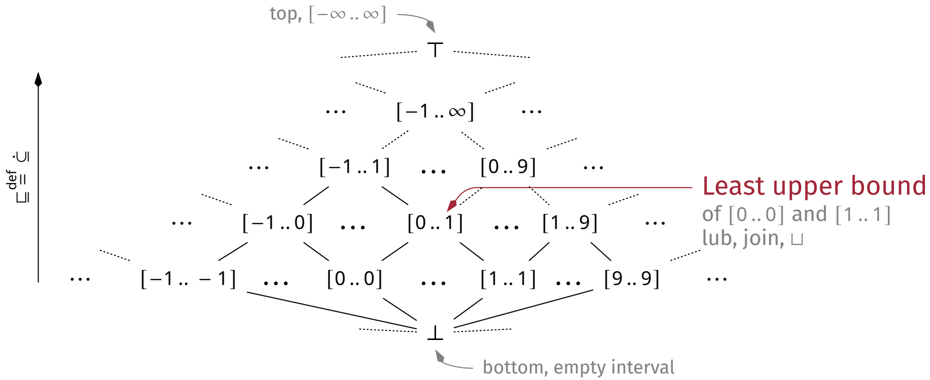

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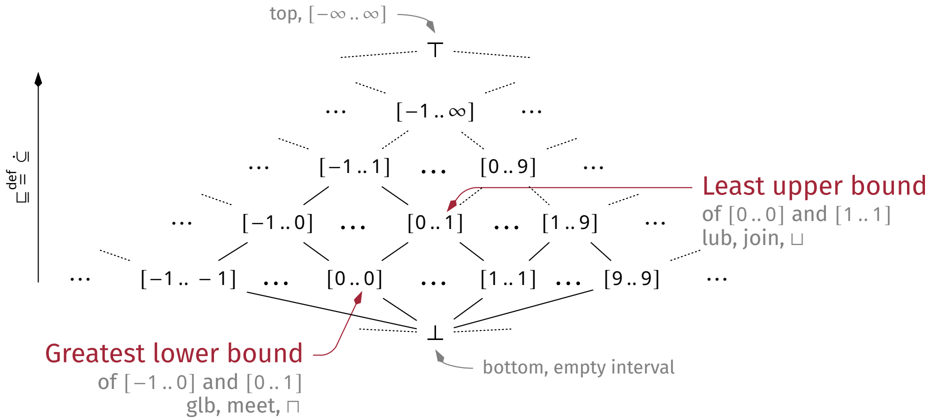

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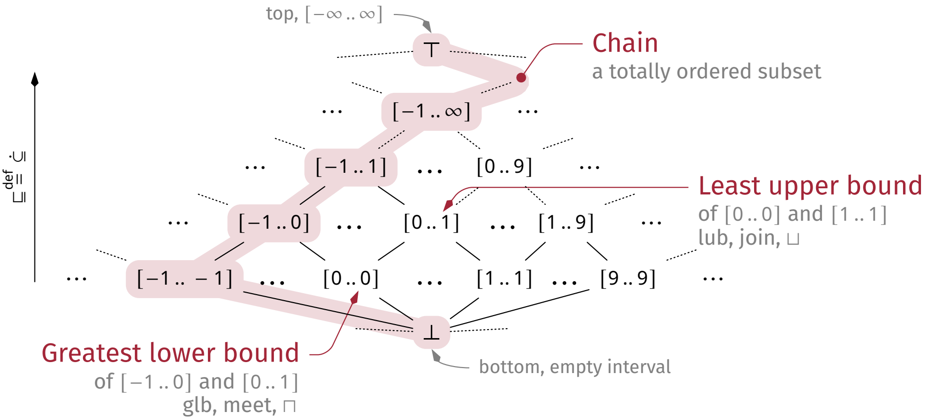

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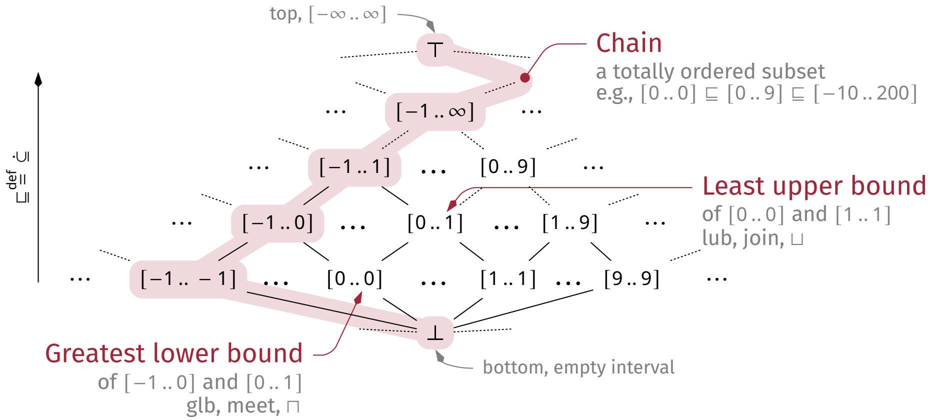

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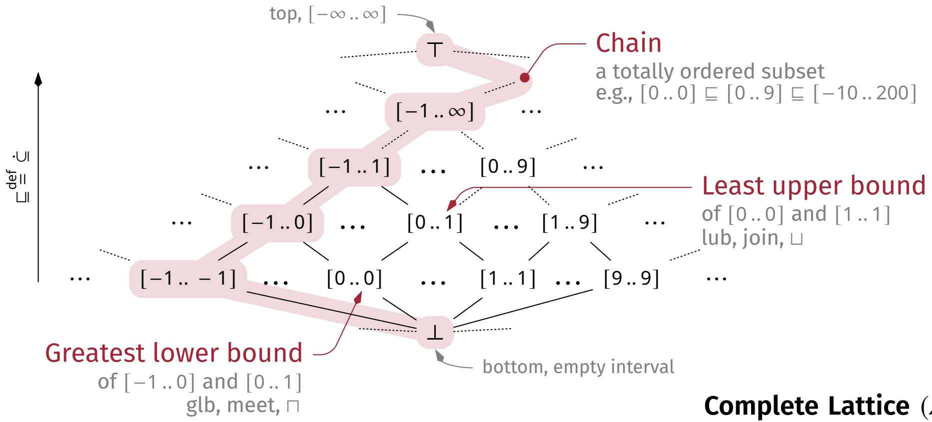
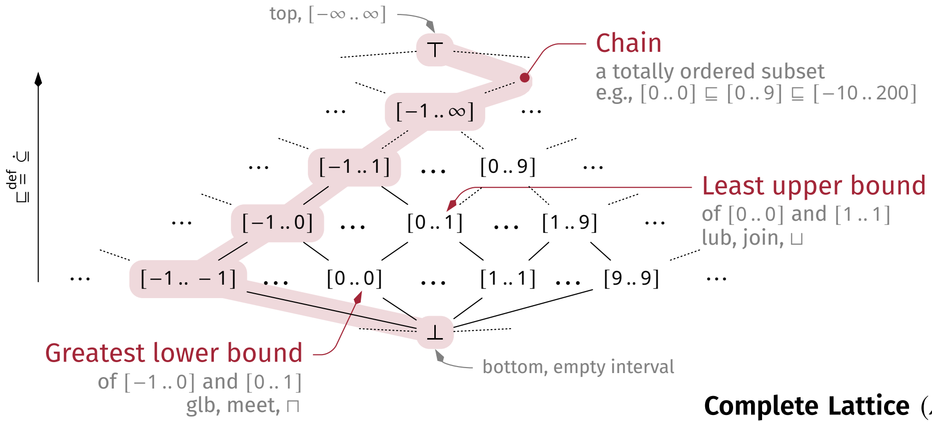


Complete Lattice $(X, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$

“Lattice theory” [Bir67], see also sublattices [Min17, p. 25]

Chains and Lattices



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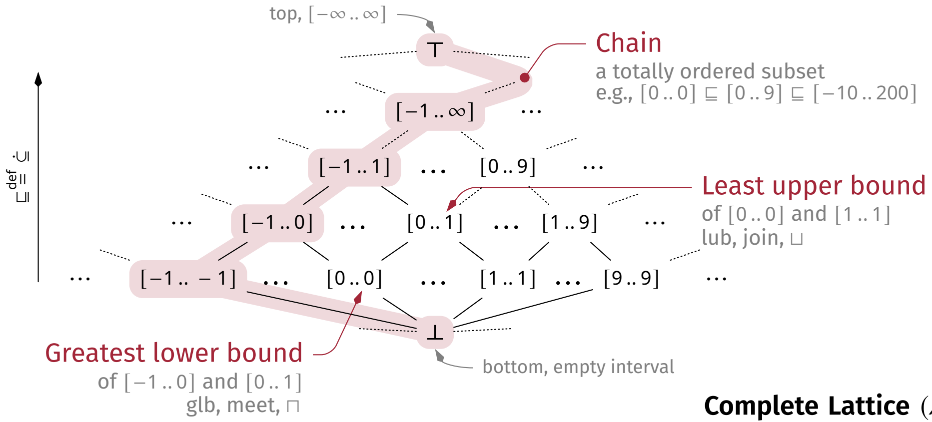

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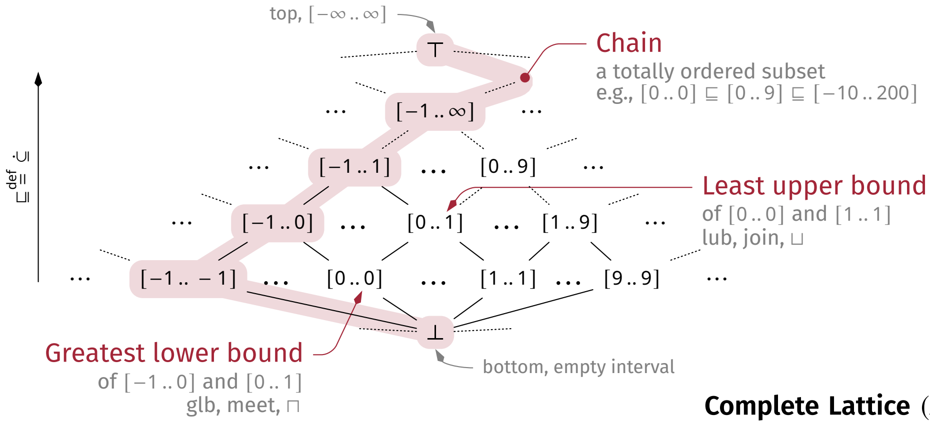

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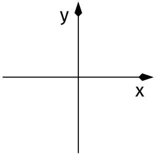
Abstract Domains Numerical



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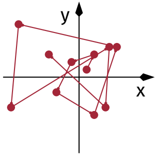

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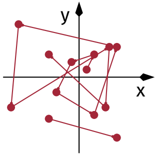

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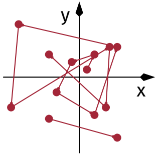
Abstract Domains — Numerical



Collecting Semantics

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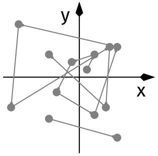

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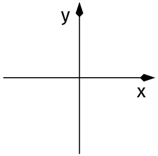

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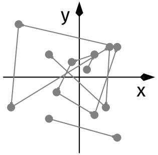

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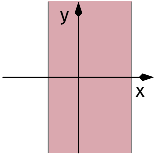

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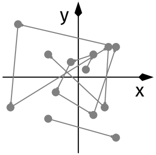

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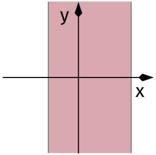

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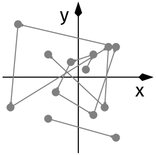

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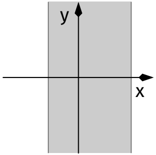

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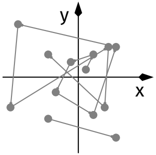

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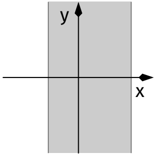

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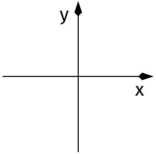

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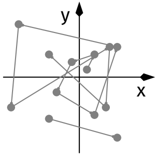

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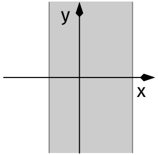

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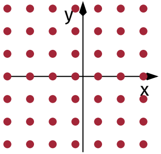

Simple Congruences

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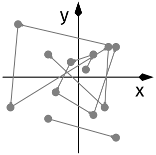

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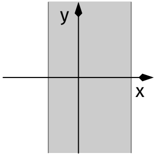

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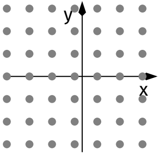

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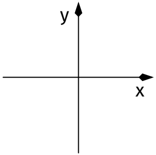

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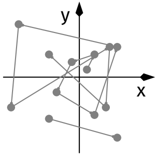

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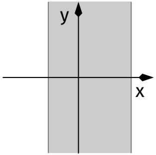

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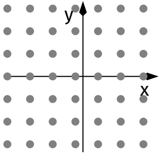

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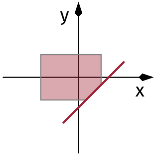
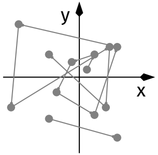


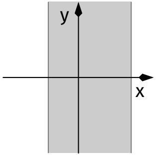
Pentagons

“Principles of Abstract Interpretation” [Cou21], “Pentagons: a weakly relational abstract domain for the efficient validation of array accesses” [LF08, p. 25]

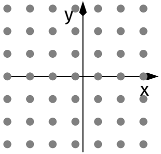
Abstract Domains Numerical



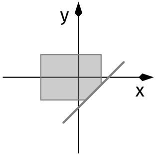
Collecting Semantics



Intervals $x \in [a \mathinner{\ldotp\ldotp} b]$



Simple Congruences



Pentagons

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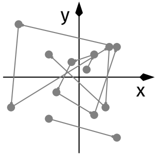

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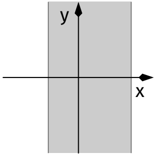

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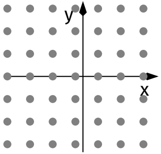

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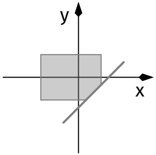

Pentagons

- Octagons

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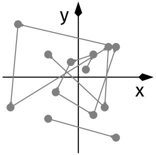

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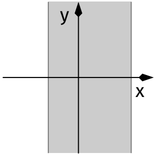

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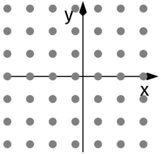

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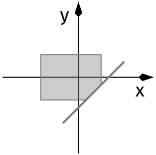

Pentagons

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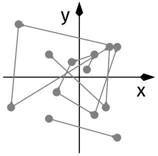

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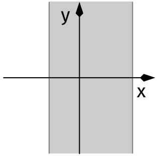

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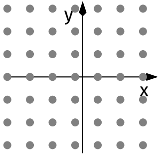

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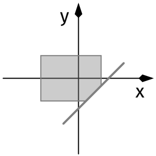

Pentagons

- Octagons
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“Principles of Abstract Interpretation” [Cou21], “Pentagons: a weakly relational abstract domain for the efficient validation of array accesses” [LF08, p. 25]

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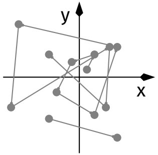

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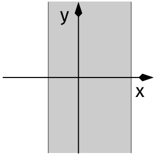

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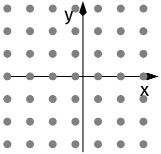

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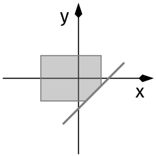


Pentagons

- Octagons
- Ellipses
- Exponentials
- Signs

"Principles of Abstract Interpretation" [Cou21], "Pentagons: a weakly relational abstract domain for the efficient validation of array accesses" [LF08, p. 25]

Sign Analysis

Sign Analysis

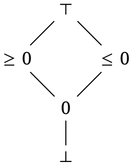
Simple Sign Domain

$$\begin{tikzcd}[column sep=small]
 & \top \arrow[dl, dash] \arrow[dr, dash] & \\
\geq 0 \arrow[dr, dash] & & \leq 0 \arrow[dl, dash] \\
 & 0 \arrow[d, dash] & \\
 & \bot &
\end{tikzcd}$$

Sign Analysis

- We still have no program semantics, but we can try...

```
int a = 0;
int b = 12;
int c = a + b;
int d = c - b;
```



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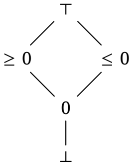

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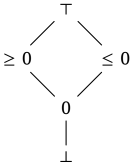
$\{\, b \geq 0 \,\}$



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$\{\!\{\, c \geq 0 \quad (= 0 \; + \; \geq 0) \,\}\!\}$

$$
\begin{tikzcd}[arrows=-]
 & \top & \\
\geq 0 \arrow[ur] & & \leq 0 \arrow[ul] \\
 & 0 \arrow[ul] \arrow[ur] & \\
 & \bot \arrow[u] &
\end{tikzcd}
$$

Sign Analysis — Simple Sign Domain

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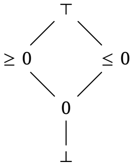
$\{\, d = \top \quad (\geq 0\ -\ \geq 0) \,\}$



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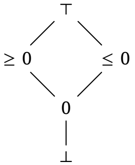
- But how to handle control flow? Loops? Recursion?



Fixpoints

"A lattice-theoretical fixpoint theorem and its applications." [Tar55], "Introduction to metamathematics" [Kle52], "Principles of Abstract Interpretation" [Cou21, p. 165]

Fixpoints

- For operators $f : X \longrightarrow X$ a **fixpoint** is a $x \in X$ such that $f(x) = x$

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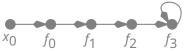
  - 

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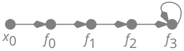
  - 

reach a fixpoint, $f^p = f(f^p)$

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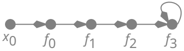
  - 

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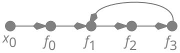
  - 

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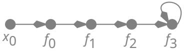
  - 

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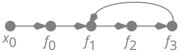
  - 

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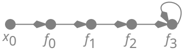
  - 

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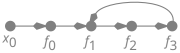
  - 

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$f : \mathbb{N} \longrightarrow \mathbb{N}, f(x) = x + 1$

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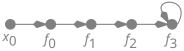
  - 

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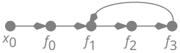
  - 

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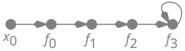
  - 

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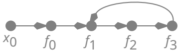
  - 

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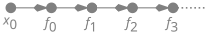
  - 

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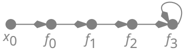
  - 

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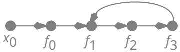
  - 

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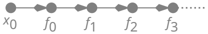
  - 

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for complete, nonempty lattices
Tarski's Theorem

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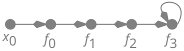
  - 

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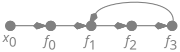
  - 

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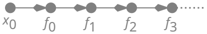
  - 

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- Analyzing, e.g. loops, we "go up" the lattice until we reach a least fixpoint

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Interval Analysis, I

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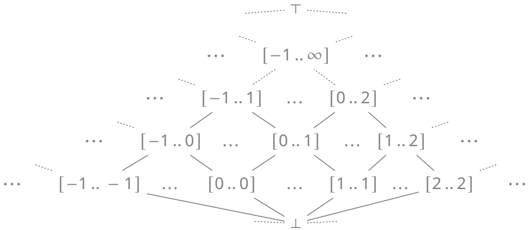

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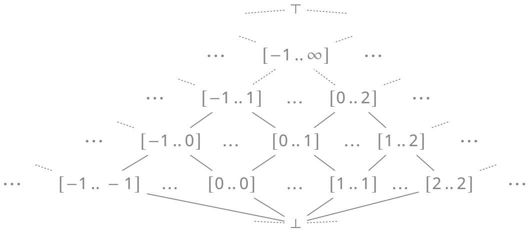

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```
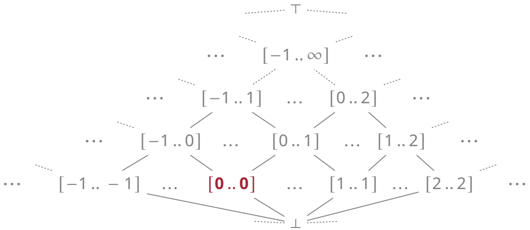

$\wr\, x_0 \in [0\,..\,0] \,\wr$

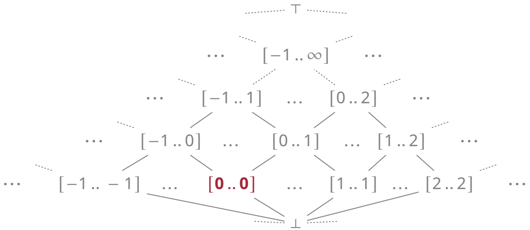


Interval Analysis, I

```
int x = 0;
while(x < 2) {
   x = x + 1;
}
```

$\wr\; x_0 \in [0\,..\,0] \;\int$

$\wr\;$ [pre] $x_1 \in [0\,..\,0] \;\int$



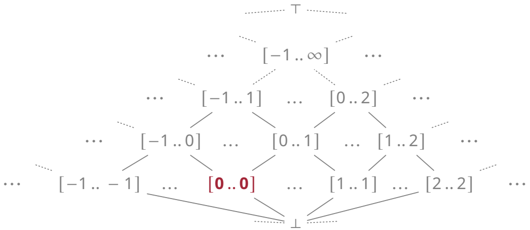
Interval Analysis, I

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$\wr\, x_0 \in [0\,..\,0] \,\wr$

$\wr$ [pre] $x_1 \in [0\,..\,0] \,\wr$

$\wr$ [in] $x_2 \in [0\,..\,0] \quad ([0\,..\,0] \cap (-\infty\,..\,1]) \,\wr$



Interval Analysis, I

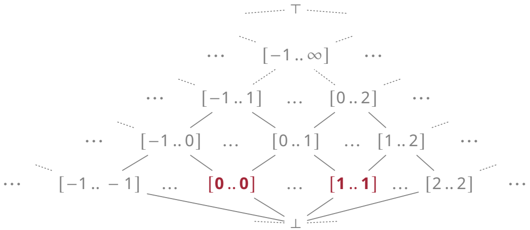
```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$\{\!\{\ x_0 \in [0\,..\,0]\ \}\!\}$

$\{\!\{$ [pre] $x_1 \in [0\,..\,0]\ \}\!\}$

$\{\!\{$ [in] $x_2 \in [0\,..\,0] \quad ([0\,..\,0] \cap (-\infty\,..\,1])\ \}\!\}$

$\{\!\{\ x_3 \in [1\,..\,1] \quad ([0\,..\,0] \oplus [1\,..\,1])\ \}\!\}$



Interval Analysis, I

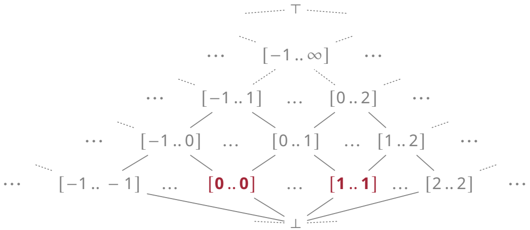
```
int x = 0;
while(x < 2) {
   x = x + 1;
}
```

$\{\!\{\ x_0 \in [0\,..\,0]\ \}\!\}$

$\{\!\{\ \text{[pre]}\ \boldsymbol{x_1 \in [0\,..\,0]}\ \}\!\}$

$\{\!\{\ \text{[in]}\ x_2 \in [0\,..\,0] \quad ([0\,..\,0] \cap (-\infty\,..\,1])\ \}\!\}$

$\{\!\{\ x_3 \in [1\,..\,1] \quad ([0\,..\,0] \oplus [1\,..\,1])\ \}\!\}$



Interval Analysis, I

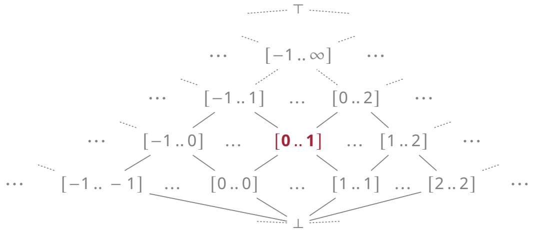
```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$\{\!|\ x_0 \in [0 \mathinner{..} 0]\ |\!\}$

$\{\!|\ \text{[pre]}\ \boldsymbol{x_1 \in [0 \mathinner{..} 1] \quad ([0 \mathinner{..} 0] \cup [1 \mathinner{..} 1])}\ |\!\}$

$\{\!|\ \text{[in]}\ x_2 \in [0 \mathinner{..} 0] \quad ([0 \mathinner{..} 0] \cap (-\infty \mathinner{..} 1])\ |\!\}$

$\{\!|\ x_3 \in [1 \mathinner{..} 1] \quad ([0 \mathinner{..} 0] \oplus [1 \mathinner{..} 1])\ |\!\}$



Interval Analysis, I

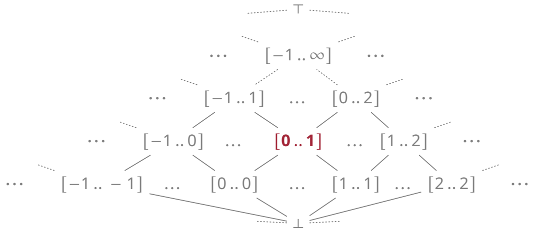
```
int x = 0;
while(x < 2) {
   x = x + 1;
}
```

$\{\!\!\{\, x_0 \in [0 \mathrel{..} 0] \,\}\!\!\}$

$\{\!\!\{\, \text{[pre]}\; x_1 \in [0 \mathrel{..} 1] \quad ([0 \mathrel{..} 0] \cup [1 \mathrel{..} 1]) \,\}\!\!\}$

$\{\!\!\{\, \text{[in]}\; \boldsymbol{x_2 \in [0 \mathrel{..} 1] \quad ([0 \mathrel{..} 1] \cap (-\infty \mathrel{..} 1])} \,\}\!\!\}$

$\{\!\!\{\, x_3 \in [1 \mathrel{..} 1] \quad ([0 \mathrel{..} 0] \oplus [1 \mathrel{..} 1]) \,\}\!\!\}$



Interval Analysis, I

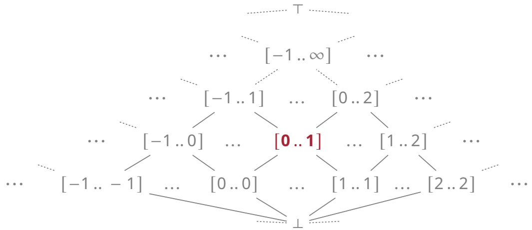
```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$\wr\ x_0 \in [0\,..\,0]\ \int$

$\wr$ [pre] $x_1 \in [0\,..\,1] \quad ([0\,..\,0] \cup [1\,..\,1])\ \int$

$\wr$ [in] $x_2 \in [0\,..\,1] \quad ([0\,..\,1] \cap (-\infty\,..\,1])\ \int$

$\wr\ \boldsymbol{x_3 \in [1\,..\,2] \quad ([0\,..\,1] \oplus [1\,..\,1])}\ \int$



Interval Analysis, I

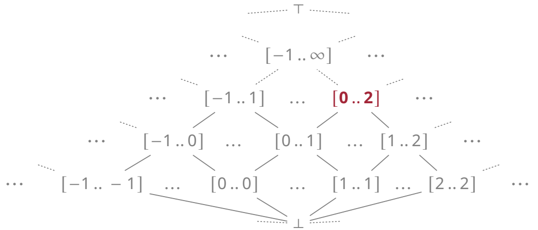
```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$\wr\, x_0 \in [0\,..\,0] \,\int$

$\wr$ [pre] $\boldsymbol{x_1 \in [0\,..\,2]} \quad \boldsymbol{([0\,..\,1] \cup [1\,..\,2])} \,\int$

$\wr$ [in] $x_2 \in [0\,..\,1] \quad ([0\,..\,1] \cap (-\infty\,..\,1]) \,\int$

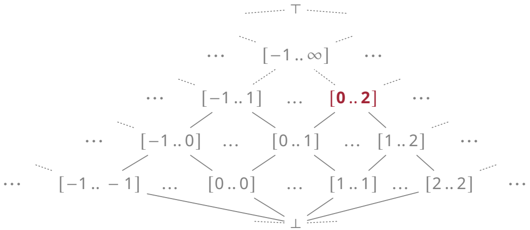
$\wr\, x_3 \in [1\,..\,2] \quad ([0\,..\,1] \oplus [1\,..\,1]) \,\int$



Interval Analysis, I

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

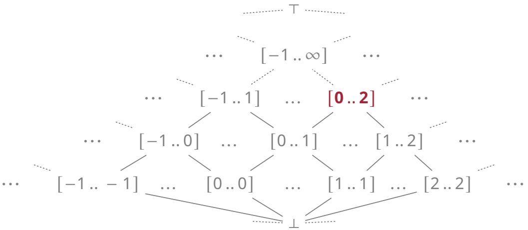
$\wr\, x_0 \in [0\,..\,0] \,\wr$
$\wr$ [pre] $x_1 \in [0\,..\,2] \quad ([0\,..\,1] \cup [1\,..\,2]) \,\wr$
$\wr$ [in] $x_2 \in [0\,..\,1] \quad ([0\,..\,1] \cap (-\infty\,..\,1]) \,\wr$
$\wr\, x_3 \in [1\,..\,2] \quad ([0\,..\,1] \oplus [1\,..\,1]) \,\wr$
$\wr$ [post] $x_4 \in [2\,..\,2] \quad ([0\,..\,2] \cap [2\,..\,\infty)) \,\wr$



Interval Analysis, I

```
int x = 0;
while(x < 2) {
   x = x + 1;
}
```

Intervals	Signs
$\wr\, x_0 \in [0\,..\,0] \,\int$	$\wr\, x_0 = 0 \,\int$
$\wr\, \text{[pre]}\ x_1 \in [0\,..\,2] \quad ([0\,..\,1] \cup [1\,..\,2]) \,\int$	$\wr\, \text{[pre]}\ x_1 \geq 0 \,\int$
$\wr\, \text{[in]}\ x_2 \in [0\,..\,1] \quad ([0\,..\,1] \cap (-\infty\,..\,1]) \,\int$	$\wr\, \text{[in]}\ x_2 \geq 0 \,\int$
$\wr\, x_3 \in [1\,..\,2] \quad ([0\,..\,1] \oplus [1\,..\,1]) \,\int$	$\wr\, x_3 \geq 0 \,\int$
$\wr\, \text{[post]}\ x_4 \in [2\,..\,2] \quad ([0\,..\,2] \cap [2\,..\,\infty)) \,\int$	$\wr\, \text{[post]}\ x_4 \geq 0 \,\int$



Interval Analysis, I

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

Intervals

⦃ $x_0 \in [0 .. 0]$ ⦄

⦃ [pre] $x_1 \in [0 .. 2] \quad ([0 .. 1] \cup [1 .. 2])$ ⦄

⦃ [in] $x_2 \in [0 .. 1] \quad ([0 .. 1] \cap (-\infty .. 1])$ ⦄

⦃ $x_3 \in [1 .. 2] \quad ([0 .. 1] \oplus [1 .. 1])$ ⦄

⦃ [post] $x_4 \in [2 .. 2] \quad ([0 .. 2] \cap [2 .. \infty))$ ⦄

Signs

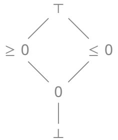
⦃ $x_0 = 0$ ⦄

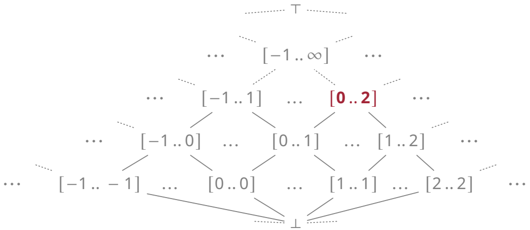
⦃ [pre] $x_1 \geq 0$ ⦄

⦃ [in] $x_2 \geq 0$ ⦄

⦃ $x_3 \geq 0$ ⦄

⦃ [post] $x_4 \geq 0$ ⦄





3. Semantics

Semantics

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

Variable $v \in \mathbb{V}$

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

Variable $v \in \mathbb{V}$

Assignment

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

Variable $v \in \mathbb{V}$

Assignment

```
int x = 0;
```

Numeric Constant $c \in \mathbb{I}$

```
while(x < 2) {

  x = x + 1;

}
```

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

Variable $v \in \mathbb{V}$

Assignment

Sequence

Numeric Constant $c \in \mathbb{I}$

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

Variable $v \in \mathbb{V}$

Assignment

```
int x = 0;
```

Sequence

Loop

Numeric Constant $c \in$ ▯

```
while(x < 2) {

  x = x + 1;

}
```

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

```
int x = 0;
while(x < 2) {
   x = x + 1;
}
```

Variable $v \in \mathbb{V}$

Assignment

Sequence

Numeric Constant $c \in \mathbb{I}$

Loop

Comparison $\bowtie \in \{\leq, <, \ldots\}$

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

```
int x = 0;
while(x < 2) {
   x = x + 1;
}
```

Variable $v \in \mathbb{V}$

Assignment

Sequence

Numeric Constant $c \in \mathbb{Z}$

Loop

Comparison $\bowtie \in \{\leq, <, \ldots\}$

Binary Expression

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Program Syntax (simplified)

```
int x = 0;
while(x < 2) {
    x = x + 1;
}
```

Variable $v \in \mathbb{V}$
Assignment
Sequence
Numeric Constant $c \in \mathbb{I}$
Loop
Comparison $\bowtie \in \{\leq, <, \ldots\}$
Binary Expression

$$
\begin{array}{llll}
\mathit{stm} & ::= & V \leftarrow \mathit{expr} & (\text{assignment}, V \in \mathbb{V}) \\
& | & \mathit{stm}_1 ;\ \mathit{stm}_2 & (\text{sequence}) \\
& | & \textbf{while}(\mathit{cond})\ \{\ \mathit{stm}\ \} & (\text{loop}) \\
\mathit{expr} & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
& | & c & (\text{constant}, c \in \mathbb{I}) \\
& | & \mathit{expr}_1 \diamond \mathit{expr}_2 & (\text{bin. expr.}, \diamond \in \{+, -, \ldots\}) \\
\mathit{cond} & ::= & b & (\text{boolean}, b \in \mathbb{B}) \\
& | & \mathit{expr}_1 \bowtie \mathit{expr}_2 & (\text{comparison}, \bowtie \in \{\leq, <, \ldots\})
\end{array}
$$

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$$
\begin{array}{rcll}
\textit{expr} & ::= & V & \text{(variable, } V \in \mathbb{V}\text{)} \\
 & \mid & c & \text{(constant, } c \in \mathbb{I}\text{)} \\
 & \mid & \textit{expr}_1 \diamond \textit{expr}_2 & \text{(bin. expr., } \diamond \in \{+, -, \ldots\}\text{)}
\end{array}
$$

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```
int x = 0;
while(x < 2) {
  x = x + 1;
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$$
\begin{array}{llll}
\mathit{expr} & ::= & V & \text{(variable, } V \in \mathbb{V}\text{)} \\
& | & c & \text{(constant, } c \in \mathbb{I}\text{)} \\
& | & \mathit{expr}_1 \diamond \mathit{expr}_2 & \text{(bin. expr., } \diamond \in \{+, -, \ldots\}\text{)}
\end{array}
$$

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \mathbb{V} \to \mathbb{I}$ to represent the current program state

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$$
\begin{array}{llll}
\textit{expr} & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
 & | & c & (\text{constant}, c \in ▯) \\
 & | & \textit{expr}_1 \diamond \textit{expr}_2 & (\text{bin. expr.}, \diamond \in \{+, -, \ldots\})
\end{array}
$$

Variable

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \mathbb{V} \to ▯$ to represent the current program state

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$$
\begin{array}{llll}
\mathit{expr} & ::= & V & \text{(variable, } V \in \mathbb{V}\text{)} \\
 & \mid & c & \text{(constant, } c \in \mathbb{I}\text{)} \\
 & \mid & \mathit{expr}_1 \diamond \mathit{expr}_2 & \text{(bin. expr., } \diamond \in \{+, -, \ldots\}\text{)}
\end{array}
$$

- We use an environment $\mathcal{E} \overset{\text{def}}{=} \overset{\text{Variable}}{\mathbb{V}} \to \overset{\text{Integer Values}}{\mathbb{I}}$ to represent the current program state

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

expr	::=	V	(variable, $V \in \mathbb{V}$)
	|	c	(constant, $c \in \mathbb{I}$)
	|	$expr_1 \diamond expr_2$	(bin. expr., $\diamond \in \{+, -, \ldots\}$)

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \underbrace{\mathbb{V}}_{\text{Variable}} \to \underbrace{\mathbb{I}}_{\text{Integer Values}}$ to represent the current program state

$\mathbb{V}$	$\mathbb{I}$
x	0
c	5

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$$
\begin{array}{llll}
\mathit{expr} & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
 & | & c & (\text{constant}, c \in \mathbb{I}) \\
 & | & \mathit{expr}_1 \diamond \mathit{expr}_2 & (\text{bin. expr.}, \diamond \in \{+, -, \dots\})
\end{array}
$$

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \underbrace{\mathbb{V}}_{\text{Variable}} \to \underbrace{\mathbb{I}}_{\text{Integer Values}}$ to represent the current program state
- Now we can define evalExpr$(\mathit{expr}, \mathit{env})$ for an environment $\mathit{env} \in \mathcal{E}$

$\mathbb{V}$	$\mathbb{I}$
x	0
c	5

Shortened form of “Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation” [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$$
\begin{array}{rcll}
expr & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
 & \mid & c & (\text{constant}, c \in \mathbb{I}) \\
 & \mid & expr_1 \diamond expr_2 & (\text{bin. expr.}, \diamond \in \{+, -, \ldots\})
\end{array}
$$

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \underbrace{\mathbb{V}}_{\text{Variable}} \to \underbrace{\mathbb{I}}_{\text{Integer Values}}$ to represent the current program state
- Now we can define $\underbrace{\text{evalExpr}(expr, env)}_{\text{Usually written as } \mathbb{E}[\![expr]\!]\rho}$ for an environment $env \in \mathcal{E}$

$\mathbb{V}$	$\mathbb{I}$
x	0
c	5

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$$
\begin{array}{rcll}
\mathit{expr} & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
 & \mid & c & (\text{constant}, c \in \mathbb{I}) \\
 & \mid & \mathit{expr}_1 \diamond \mathit{expr}_2 & (\text{bin. expr.}, \diamond \in \{+, -, \ldots\})
\end{array}
$$

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \underbrace{\mathbb{V}}_{\text{Variable}} \to \underbrace{\mathbb{I}}_{\text{Integer Values}}$ to represent the current program state

$\mathbb{V}$	$\mathbb{I}$
x	0
c	5

- Now we can define evalExpr(expr, env) for an environment $\mathit{env} \in \mathcal{E}$

Usually written as $\mathbb{E}[\![\, \mathit{expr} \,]\!]\rho$

$$\text{evalExpr}(V,\ \mathit{env}) \stackrel{\text{def}}{=} \mathit{env}(V)$$

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$$
\begin{array}{rcll}
\mathit{expr} & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
 & \mid & c & (\text{constant}, c \in \mathbb{I}) \\
 & \mid & \mathit{expr}_1 \diamond \mathit{expr}_2 & (\text{bin. expr.}, \diamond \in \{+, -, \dots\})
\end{array}
$$

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \underbrace{\mathbb{V}}_{\text{Variable}} \to \underbrace{\mathbb{I}}_{\text{Integer Values}}$ to represent the current program state
- Now we can define evalExpr(expr, env) for an environment $\mathit{env} \in \mathcal{E}$

Usually written as $\mathbb{E}[\![\, \mathit{expr} \,]\!]\rho$

$\mathbb{V}$	$\mathbb{I}$
x	0
c	5

$$\text{evalExpr}(V,\ \mathit{env}) \stackrel{\text{def}}{=} \mathit{env}(V)$$

Value of $V \in \mathbb{V}$ in Environment env

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$$
\begin{array}{rcll}
\mathit{expr} & ::= & V & \text{(variable, } V \in \mathbb{V}\text{)} \\
 & | & c & \text{(constant, } c \in \mathbb{I}\text{)} \\
 & | & \mathit{expr}_1 \diamond \mathit{expr}_2 & \text{(bin. expr., } \diamond \in \{+, -, \ldots\}\text{)}
\end{array}
$$

Variable

Integer Values

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \mathbb{V} \to \mathbb{I}$ to represent the current program state

$\mathbb{V}$	$\mathbb{I}$
x	0
c	5

Usually written as $\mathbb{E}[\![\, \mathit{expr} \,]\!]\rho$

- Now we can define evalExpr(expr, env) for an environment $\mathit{env} \in \mathcal{E}$

$$
\begin{aligned}
\text{evalExpr}(V,\ \mathit{env}) &\stackrel{\text{def}}{=} \mathit{env}(V) \\
\text{evalExpr}(c,\ \mathit{env}) &\stackrel{\text{def}}{=} c
\end{aligned}
$$

Value of $V \in \mathbb{V}$ in Environment env

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$$
\begin{array}{llll}
\textit{expr} & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
 & | & c & (\text{constant}, c \in \mathbb{I}) \\
 & | & \textit{expr}_1 \diamond \textit{expr}_2 & (\text{bin. expr.}, \diamond \in \{+, -, \ldots\})
\end{array}
$$

Variable

Integer Values

- We use an environment $\mathcal{E} \stackrel{\text{def}}{=} \mathbb{V} \to \mathbb{I}$ to represent the current program state

$\mathbb{V}$	$\mathbb{I}$
x	0
c	5

Usually written as $\mathbb{E}[\![\, \textit{expr} \,]\!]\rho$

- Now we can define evalExpr($\textit{expr}$, $\textit{env}$) for an environment $\textit{env} \in \mathcal{E}$

$$
\begin{aligned}
\text{evalExpr}(V,\ \textit{env}) &\stackrel{\text{def}}{=} \textit{env}(V) \\
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Value of $V \in \mathbb{V}$ in Environment $\textit{env}$

Shortened form of "Tutorial on Static Inference of Numeric Invariants by Abstract Interpretation" [Min17, p. 46], craftinginterpreters.com/representing-code.html

Atomic Expression Semantics

```
int x = 0;
while(x < 2) {
  x = x + 1;
}
```

$$
\begin{array}{rcll}
expr & ::= & V & (\text{variable}, V \in \mathbb{V}) \\
 & | & c & (\text{constant}, c \in \mathbb{Z}) \\
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\begin{array}{rcl}
\text{evalExpr}(V,\ env) & \stackrel{\text{def}}{=} & env(V) \\
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Value of $V \in \mathbb{V}$ in Environment $\textit{env}$

- Additionally we can define evalCond($\textit{cond}$, $\textit{envs}$) and evalStm($\textit{stm}$, $\textit{envs}$)

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Value of $V \in \mathbb{V}$ in Environment env

$\mathbb{C}[\![\, cond \,]\!]\mathcal{D}$

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Denotational Semantics while loops

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$\textbf{while}(\mathit{cond})\ \{\ \mathit{stm}\ \}$

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Suppose we start the loop with states *Start*

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iterate to find the least fixpoint [Min17, p. 52]

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We achieve their abstract counterpart using
the same principles but for abstract domains!

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Usually written as $\mathbb{S}^{\#}, \mathbb{C}^{\#}, \mathbb{E}^{\#}, \ldots$

Interval Analysis, II

```
int x = 0;
while(x < 2) {
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}
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“Principles of Abstract Interpretation” [Cou21, p. 393], “Widening and narrowing operators for abstract interpretation” [CZ11]

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int x = 0;
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$\wr$ [post] $x_4 \in [999999\,..\,\infty) \quad ([0\,..\,\infty) \cap [999999\,..\,\infty))\, \int$

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4. Soundness and Completeness

Rice’s Theorem

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- We have to approximate the reality

The Confusion Matrix

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E.g., do we claim there is an error?

Prediction

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Prediction

Pos.

The Confusion Matrix

E.g., do we claim there is an error?

	Prediction	
	Pos.	Neg.

The Confusion Matrix

E.g., do we claim there is an error?

Prediction

Pos. **Neg.**

Actual

E.g., is there really an error?

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		Prediction	
		Pos.	**Neg.**
Actual	**Pos.**		

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The Confusion Matrix

E.g., do we claim there is an error?

		Prediction	
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Actual	Pos.	(TP) True Positive	
	Neg.		

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The Confusion Matrix

E.g., do we claim there is an error?

		Prediction	
		Pos.	**Neg.**
Actual	**Pos.**	**(TP) True Positive**	(FN) False Negative
	Neg.		

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The Confusion Matrix

E.g., do we claim there is an error?

		Prediction	
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	Neg.	(FP) False Positive	

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		Pos.	**Neg.**
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E.g., is there really an error?

- **Precision:** $\mathbf{TP} / (\mathbf{TP} + \mathrm{FP})$ ("how many false alarms")

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E.g., do we claim there is an error?

		Prediction	
		Pos.	**Neg.**
Actual	**Pos.**	**(TP) True Positive**	(FN) False Negative
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E.g., is there really an error?

- **Precision:** $\mathbf{TP}/(\mathbf{TP} + \mathrm{FP})$ ("how many false alarms")
- **Recall:** $\mathbf{TP}/(\mathbf{TP} + \mathrm{FN})$ ("how many errors did we find")

Soundness and Completeness

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Abstract interpretation soundly over-approximates the program semantics

5. Outlook

Outlook

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Outlook

- Domain transformers
  combine abstract domains[Min17, p. 149]
- Galois connections
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- Corresponding to widening, narrowing
  refines approximations[Cou21, p. 395]
- Function calls
  require special handling[MJ12]

Outlook

- Domain transformers
  combine abstract domains[Min17, p. 149]
- Galois connections
  define the relationship between concrete and abstract domains[Cou21, p. 110]
- Corresponding to widening, narrowing
  refines approximations[Cou21, p. 395]
- Function calls
  require special handling[MJ12]
- Existing libraries allow for easy implementation
  LiSA[Fer+21], MOPSA[Jou+19], Apron[JM09]

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