Set 9

(A) Hat Problems In each of the following problems, assume all of the people involved are perfectly logical and know that everyone else is perfectly logical as well. You should also assume that they will avoid punishment (getting an F, etc.) at all costs. For parts 2-5, indicate if the students can guarantee an A. If so, explain a possible strategy, and if not, explain why not.

(1) Three students are standing in a single-file line and is either given a red hat or a yellow hat by a TA. Since they are standing single-file, the student at the back can see both of the other students, the middle student can only see the front student, and the front student can see no one (not even themselves). The instructor says: "I only brought 2 red hats and 3 yellow hats today. If you know your own hat color, let me know by either saying 'red' or 'yellow.' If anyone gets the wrong answer, you all get an F, and if anyone gets the correct answer, you all get an A. You may not communicate with each other in any way." After a few minutes of silent thought, the students get an A because the student in front (who cannot see any hats) gets their own hat color correct. What color was it, and how is this possible?

(2) The instructor, continuing to play with students' grades, asks 3 new students to line up, who can again only see students in front of themselves. This time, each student will get either a white hat or a black hat, completely randomly (there are enough hats for everyone to get either color). The rules are as follows: a TA will walk from the back of the line to the front, asking each student to name their own hat color. If a majority of students (i.e. 2 of the 3 of them) get the correct answer, they all get A's; otherwise they all get F's. Each student can hear the answer that the previous students gave, and the TA will say if they are correct as this happens. Before this process starts, the TA gives the students 1 hour to come up with a plan.

(3) (Difficult) Now, the instructor raises the stakes. She asks the entire class of 20 students to line up single-file and play the same game with white and black hats. They can again strategize beforehand, but this time the instructor says that they will only get an A if 19 out of the 20 students get their own hat color correct.

(4) The students return the next week to see that they have a new instructor. He doesn't like having students line up single-file and instead asks the 20 students to stand in a circle, so that everyone can see everyone else, but still cannot see themselves, as they are given white and black hats at random. This time, the students will all write down their own hat color on a piece of paper and hand it to a TA, so the students don't know each others' answers. If at least 10 of the students get their own hat color correct, the whole class wins. They are allowed to strategize beforehand.

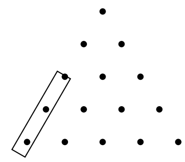
(5) (VERY difficult) In an alternate universe, infinitely many students walk into class and are given either a white hat or a black hat. The students stand in an infinitely large circle so that they can all see each other but not themselves. They will again write down their own hat color on a private piece of paper and hand it to a TA (who has infinitely many arms). If only finitely many students get the wrong answer for their own hat color, the students win; otherwise they lose. Once again, they are allowed to strategize beforehand (on an infinite-bandwidth Zoom call).

(B) One-Player Games

(1) A domino is a 2×1 size rectangle, and a chessboard is (usually) an 8×8 grid of square tiles, but we will also consider different size chessboards in this problem. In each part, determine if you can cover the stated chessboard with dominoes that do not overlap.

(i) A (normal) 8×8 chessboard.
(ii) A 5×5 chessboard.
(iii) A 5×5 chessboard with one corner tile removed.
(iv) A 5×5 chessboard with the center tile removed.
(v) A 5×5 chessboard with a tile adjacent to the center tile removed.
(vi) A 8×8 chessboard with both corners removed.

(2) Consider an equilateral triangle made of dots, with n dots on each side. On each turn, you can remove three dots if they are adjacent and in a straight line (see below).

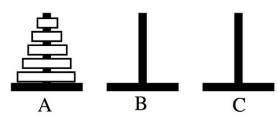


For which values of n can you remove all of the dots?

(3) (***HW***) You are standing in front of a table wearing a blindfold and gloves. You can feel that there are 100 coins on the table in front of you, but cannot tell which are heads-up and which are tails-up. A TA (wandering in from the previous problem) tells you that 20 of them are heads and the other 80 are tails. You are allowed to move the coins around and flip them over at will and need to divide them into two piles so that each pile has the same number of heads. Can you do it? [Hint: the piles don't have to be the same size.]

(4) You have 12 black tokens, 20 red tokens, and 35 green tokens. On one turn, you can trade two tokens of different colors for two tokens of the third color. Can you end up with all tokens the same color? What if you begin with 14 black, 33 red, and 4 green tokens?

(5) **Tower of Hanoi** You begin with 3 wooden pegs, the first of which holds a pyramid of 5 disks.



You'd like to move all of the disks from the first peg to the last peg. In one step, you may lift the top disk off of any peg and place it on any other peg, but you may never place a larger disk atop a smaller one. Can you move all of the disks from the first peg to the last peg? If so, what is the fewest number of steps needed to move all of the disks? What if there were 100 disks?

(C) Two-Player Games Each of the following games is played between two players. Player A will always go first and player B will go second. The last player with a valid move wins. If both players play optimally, who will win?

(1) There are three piles of 4 rocks each. On each turn, a player must choose one of the piles and removes some (at least 1) rocks from that pile.

(2) Same as above, but there are 100 rocks in each pile, initially.

(3) (***HW***) There is a single pile of rocks, but each player can only remove either a single rock or two rocks. Who wins if there are 3 rocks initially? What if there are 4 rocks? 5 rocks? Make a conjecture and try to prove it!

(4) There is a large table in the shape of a perfect circle. On a player's turn, they must place a penny on the table without moving or overlapping any of the previous pennies.

(5) Start with a regular octagon. Each turn, a valid move consists of drawing a line segment connecting two nonadjacent vertices, as long as that segment doesn't intersect any previous segment (except possibly at the endpoints). Who wins? What if the shape had 100 sides?

(D) Knights and Knaves You visit a town where each inhabitant is either a knight, who always tells the truth, or a knave, who always lies.

(1) You encounter three people: A, B, and C. You ask A: "Are you a knight or a knave?" but A speaks so softly that you cannot hear them. You ask B: "What did A say?" and B replies: "He said he is a knave." Finally, C speaks up and says: "Don't believe B, he is a knave!" Is C a knight or a knave?

(2) Again, with three people, you ask A: "How many of you are knaves?" You again cannot hear the response, so you ask B: "What did A say?" and B replies: "He said there are two knaves." Again, C speaks up and says: "Don't believe B, he is a knave!" Is C a knight or a knave?

(3) (***HW***) The next day, you meet two new people. A tells you: "Both of us are knaves." Is B a knight or a knave?

(4) Again, with two people, A tells you: "At least one of us is a knave." Is B a knight or a knave?

(5) Again, with two people, A tells you: "B and I are the same type of person, i.e. both knights or both knaves." Is B a knight or a knave?

(6) Again, with two people, you ask A: "Is B a knight?" You also ask B: "Is A a knight?" Do you get the same answer, different answers, or does it depend?

(7) You're unsure if this town has a library. You are passing an inhabitant of the town who you can ask for help. What question can you ask to determine if the town has a library?

(8) You know that the three front-desk employees at the library are the two knaves Larry and Lisa and the knight Tiana, but you don't know what they look like. A mutual friend suggested that you talk to Lisa, so you first ask one question; all three of them answer and their responses tell you who Lisa is. What question can you ask? Can you find a question with fewer than 4 words?

(9) You know that the two twin sisters Maribel and Ramona are not the same—one is a knight and the other is a knave—but you don't know who is who. You run into one of them and want to know if you're talking to Maribel or Ramona. What question can you ask to determine their identity? Can you find a question with fewer than 4 words?

(10) You're in the same situation as above, but you're not interested in who you are talking to in the moment; instead, can you ask a single question to determine which is the knight and which is the knave?

(11) One day, another visitor (V) comes to town. Like you, V is neither a knight nor a knave and can lie and tell the truth at their convenience. You're in a room with three people: A, B, and C. A different knight has told you that the people in the room are a knight, a knave, and V, but you don't know which is which. Can you (eventually) figure out who V is? How many yes/no questions (each directed at a single person) do you need to find V?