Set 5

(A)

(1) How many elements do the groups S_3, S_4, and D_3 have?

(2) (***HW***) How many elements does D_4 have? Justify your answer.

(3) Write the group table for the groups S_2, S_3, $(\mathbb{Z}/2)$, D_3, and D_4. Which of them are abelian?

(4) For the groups S_2, S_3, $(\mathbb{Z}/2)$, D_3, and D_4, find the order of each element.

(5) Do you observe any similarities between the group tables of S_2 and $(\mathbb{Z}/2)$? How about S_3 and D_3? Show that S_4 and D_4 could not have "similar" group tables.

(B) Consider a rectangular mattress, which can be used on both sides.

(1) Show that the mattress can be used in four different configurations.

(2) The manufacturer recommends that, once a month, the user change the configuration of the mattress, so that one rotates evenly through all four possible configurations. Is there an operation (e.g. rotate counterclockwise, flip, then rotate clockwise) that, when performed repeatedly, rotates through all four possible configurations?

(3) Show that a cyclic group must be abelian.

(4) How many possible configurations would there be if the mattress was a cube? Is there an operation that rotates through all possible configurations in this case?

(5) Find two transformations X, Y in D_4 such that every other transformation can be obtained by composing X and Y.

(C) Consider the following elements of S_6:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix} \qquad \mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 3 & 1 & 6 \end{pmatrix}$$

(1) Compute the following products: $\tau\sigma$, $\tau^2\sigma$, $\mu\sigma^2$,

(2) (***HW***) Compute $\sigma^{-2}\tau$ and $\sigma^{-1}\tau\sigma$.

(3) Find the orders of σ, τ, and μ. Then compute σ^{100}, τ^{100}, and μ^{100}.

(4) Find the orbits of τ, and write τ as a product of disjoint cycles. Then do the same for μ. Could you have guessed their orders from these representations?

(5) For each $x \in \{1, \ldots, 6\}$ write down the size of its τ-orbit, and which powers of τ stabilize x. Then do the same for μ. Any conjectures? Does your conjecture hold for σ?

(6) Is every permutation in S_n a product of disjoint cycles? For enthusiasts: prove your claim.

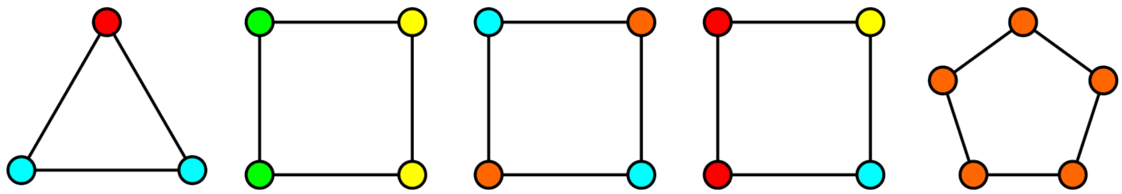
(D)

(1) Among the groups $(\mathbb{Z}/2)$, $(\mathbb{Z}/6)$, $(\mathbb{Z}/12)$, S_2, S_3, S_4, D_3, and D_4, which are isomorphic?

(2) Recall that $\mathbf{U}_n$ is the group of units in $(\mathbb{Z}/n)$. (*Warning:* $\mathbf{U}_n$ is a group under multiplication, whereas $\mathbb{Z}/n$ is a group under addition.) Show that $\mathbf{U}_5$ is isomorphic to $(\mathbb{Z}/4)$, and that $\mathbf{U}_7$ is isomorphic to $(\mathbb{Z}/6)$. Find groups that are isomorphic to $\mathbf{U}_6$ and $\mathbf{U}_8$.

(3) Show that, up to isomorphism, there is only one group with two elements. How many are there with three or four elements?

(E)



(1) (***HW***) For each of the four colored figures above, how many shapes are in the D_n-orbit? How many elements are in the stabilizer? Any conjectures?

(2) What is the fewest number of colors you need in order for a colored n-gon to have trivial stabilizer?

(F) If $(G, \star)$ is a group, a subset $H \subseteq G$ is called *closed under* $\star$ whenever, for all $h_1, h_2 \in H$, we have $h_1 \star h_2 \in H$. We say it is *closed under inverses* whenever, for all $h \in H$, we have $h^{-1} \in H$. A nonempty subset $H \subseteq G$ is called a *subgroup* if it is closed under $\star$ and taking inverses.

(1) Show that if $H \subseteq G$ is a subgroup, then $(H, \star)$ is itself a group.

(2) Show that D_n is (isomorphic to) a subgroup of S_n.

(3) If G is a finite group and $g \in G$ is an element, show that $\{e, g, g^2, g^3, \ldots\}$ is a subgroup of G. You may want to consider the case where G is S_n and g is σ, τ, or μ from the previous exercise.

(4) If G is a finite group, show that G must be isomorphic to a subgroup of S_n for some n. [Hint: write $G = \{g_1, \ldots, g_n\}$, and assign to every $g \in G$ the permutation σ for which $gg_i = g_{\sigma(i)}$.]