Rings of Differential Operators

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1 Lecture 1

1.1 Motivation

Fix a domain $\Omega \subseteq \mathbb{C}$ and let $\mathcal{O}$ be the ring of holomorphic functions on Ω; it is naturally a $\mathbb{C}$-algebra. Then a *linear ordinary differential equation* (ODE) is an expression of the form

$$g_n(z)\frac{d^n f}{dz^n} + g_{n-1}(z)\frac{d^{n-1} f}{dz^{n-1}} + \cdots + g_1(z)\frac{df}{dz} + g_0(z) f = 0$$

where the $g(z)$ are holomorphic functions on Ω. If you like algebra then you are certainly tempted to let $\partial = d/dz$ and write this as

$$\left(g_n(z)\partial^n + \cdots + g_1(z)\partial + g_0(z)\right) f = 0,$$

but can one make this precise? That is, we want to be able to forget about f and think of

$$P := g_n(z)\partial^n + \cdots + g_1(z)\partial + g_0(z) \tag{1}$$

as an object in its own right.

The key is to think of expressions of the above form as "operators". That is, we think of P as the assignment

$$f \mapsto g_n(z)\frac{d^n f}{dz^n} + g_{n-1}(z)\frac{d^{n-1} f}{dz^{n-1}} + \cdots + g_1(z)\frac{df}{dz} + g_0(z) f.$$

Observe that this assignment is $\mathbb{C}$-linear, and thus we may think of P as an element of $\mathrm{End}_{\mathbb{C}}(\mathcal{O})$.

Definition 1.1. A differential operator on Ω (or $\mathcal{O}$) is an element $P \in \mathrm{End}_{\mathbb{C}}(\mathcal{O})$ of the form (1).

Our goal will be to study the algebraic structure of *collections* of all differential operators. However, it is not a priori clear this should have any nice algebraic structure.

Let D be the collection of all differential operators on Ω. By definition, $D \subseteq \mathrm{End}_{\mathbb{C}}(\mathcal{O})$, so we will start the discussion of the algebraic structure of D with $\mathrm{End}_{\mathbb{C}}(\mathcal{O})$.

Recall that $\mathrm{End}_{\mathbb{C}}(\mathcal{O})$ is a (noncommutative!) ring, where addition is given by addition of operators and multiplication is given by composition. It is, however, a huge horrible ring: one needs to pick a basis of $\mathcal{O}$ over $\mathbb{C}$ (a huge set!) and then $\mathrm{End}_{\mathbb{C}}(\mathcal{O})$ is identified with a "matrix ring" with indexes in this huge set. What's more, this rings "forgets" the $\mathbb{C}$-algebra structure of $\mathcal{O}$ – it only sees the $\mathbb{C}$-vector space structure.

This collection of operators turns out to have a much nicer structure, and this is the content of the following exercise.

Exercise 1.2.

(i) Given a function $g \in \mathcal{O}$ we can assign to it the operator $[f \mapsto gf]$. This gives an injective ring homomorphism $\mathcal{O} \to \mathrm{End}_{\mathbb{C}}(\mathcal{O})$ with image $\mathrm{End}_{\mathcal{O}}(\mathcal{O})$. We will identify $\mathcal{O}$ with its image in $\mathrm{End}_{\mathbb{C}}(\mathcal{O})$ – i.e. a function g with the operator that multiplies by g. Observe that $\mathcal{O} \subseteq D$.

(ii) The operator ∂ given by $\partial(f) = df/dz$ is an element of D. Show that $\partial x = x\partial + 1$ in $\mathrm{End}_{\mathbb{C}}(\mathcal{O})$.

(iii) Let D' be the subring of $\mathrm{End}_{\mathbb{C}}(\mathcal{O})$ generated by $\mathcal{O}$ and ∂. Then $D = D'$. In particular, D forms a ring – a subring of $\mathrm{End}_{\mathbb{C}}(\mathcal{O})$.

Following this exercise, it makes sense to call D the *ring of differential operators* on $\mathcal{O}$.

Remark 1.3. We could have considered the ring of *polynomial* functions on Ω. This is the ring $R = \mathbb{C}[z]$. Observe that the operator ∂ preserves this subring of $\mathcal{O}$. It is therefore intuitive that the ring of differential operators on R should be $k\langle R, \partial\rangle \subseteq \mathrm{End}_{\mathbb{C}}(R)$. It will be the subring of operators that can be expressed in form (1), but where now the g_i are polynomials in z.

1.2 Generalizations

We would like to generalize the above construction in two directions: we want to replace $\mathbb{C}$ with an arbitrary base commutative ring k – most often a field – and $\mathcal{O}$ with an arbitrary k-algebra R.

Before we give the definition, let us observe that if $R = k[x]$ then the operator ∂ given by $\partial(f) = df/dz$ still makes sense. Namely, it is just formally defined by $\partial(x^n) = nx^{n-1}$ for $n \geq 1$, and $\partial(\lambda) = 0$ for $\lambda \in k$, and extended by k-linearly. Similarly, ∂_i makes sense on $R = k[x_1, \ldots, x_n]$.

The following definition is unmotivated, but we will prove that whenever k is a field of characteristic zero and $R = k[x_1, \ldots, x_n]$ we will see that the ring of differential operators will be $k\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$.

Again we observe that, in this setting, $\mathrm{End}_k(R)$ is an R-algebra, and we identify R with its image $\mathrm{End}_R(R)$ in $\mathrm{End}_k(R)$.

Definition 1.4. Let k be a commutative ring and R be a commutative k-algebra. We inductively define R-submodules $D_k^n(R)$ of $\mathrm{End}_k(R)$ by:

$$\begin{aligned} D_k^0(R) &= \mathrm{End}_R(R) = R. \\ D_k^n(R) &= \{\xi \in \mathrm{End}_k(R) : [\xi, f] \in D_k^{n-1}(R) \text{ for all } f \in R\}. \end{aligned}$$

Then the *ring of k-linear differential operators on R* is

$$D_k(R) = \bigcup_{n=0}^{\infty} D_k^n(R).$$

(By exercise 1.5 below, this is an increasing union). Observe that one needs to check that $D_k(R)$ is indeed a ring!

Exercise 1.5.

(i) Show that $D_k^0(R) \subseteq D_k^1(R) \subseteq D_k^2(R) \subseteq \cdots$

(ii) Show that $D_k^1(R) = R + \mathrm{Der}_k(R)$ where $\mathrm{Der}_k(R)$ is the space of k-linear derivations on R – recall a k-linear operator θ is a derivation if $\theta(fg) = f\theta(g) + g\theta(f)$ for all $f, g \in R$.

(iii) Show that $D_k^n(R) \cdot D_k^m(R) \subseteq D_k^{n+m}(R)$.

(iv) Show that $[D_k^n(R), D_k^m(R)] \subseteq D_k^{n+m-1}(R)$.

Observe that Exercise 1.5 tells us that $D_k(R)$ is a ring that is *filtered* by the submodules $D_k^n(R)$. Moreover, part (iv) shows that the associated graded

$$\mathrm{gr}\, D_k(R) = \bigoplus_{i=0}^{\infty} \frac{D_k^i(R)}{D_k^{i-1}(R)}$$

is a *commutative* ring.

As promised, this kind of recovers our intuitive notion of differential operator. The following exercise provides a proof, which is taken from [1]. However, we will prove this in a much more efficient way later.

Exercise 1.6. Suppose k is a field of characteristic zero and $R = k[x_1, \dots, x_n]$. Our goal is to show that

$$D_k(R) = k\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle.$$

Moreover, we will prove that

$$D_k^n(R) = \bigoplus_{|\alpha| \le n} R\partial^\alpha.$$

To start with, let us call $C_n := \bigoplus_{|\alpha|\le n} R\partial^\alpha$, so that our goal is to show $D_k^n(R) = C_n$.

(i) Convince yourself that ∂_i makes sense as an operator on R – i.e. define its action formally. Moreover, prove that $\partial_i \in D_k^1(R)$. By Exercise 1.5(iii) this proves the inclusion $C_n \subseteq D_k^n(R)$.

(ii) Prove that $D_k^1(R) = C_1$.

(iii) Suppose $\xi \in D_k(R)$ is such that $[\xi, x_i] = 0$ for all i. Then $\xi \in R$.

(iv) Show that $C_n = C_{n+1} \cap D_k^n(R)$.

(v) Suppose $\xi_1, \dots, \xi_r \in C_{n-1}$ are such that $[\xi_i, x_j] = [\xi_j, x_i]$ for all i, j. Then there is some $\eta \in C_n$ such that $\xi_i = [\eta, x_i]$. [Hint: induction on r].

(vi) Show that $D_k^n(R) = C_n$. [Hint: induction on n; suppose $\xi \in D_k^n(R)$, then set $\xi_i := [\xi, x_i]$ and use part (v).

One might ask about the characteristic $p > 0$ case. The reason why the above does not work is because for this case one has "divided powers". Namely, set $R = k[x]$ where k is a field of characteristic $p > 0$. Then consider the operator defined by

$$x^n \mapsto \begin{cases} \binom{n}{p} x^{n-p} \text{ if } n \ge p \\ 0 \text{ otherwise.} \end{cases}$$

Then this gives a differential operator of order $\le p$ which, in characteristic 0, is just the operator

$$\frac{1}{p!}\partial^p.$$

But in characteristic $p > 0$ this operator is not given a composition of two operators of smaller degree.

2 Lecture 2

Even though we have only defined the ring of differential operators on a k-algebra R, given two R-modules M and N one can talk about the *module* of differential operators from M to N. This will be useful for us when we want to compare differential operators between two rings. (Given a map of k-algebras $R \to S$ there is no induced map $D_k(R) \to D_k(S)$ in general).

Exercise 2.1.

(i) Define a notion of differential operators $D^n_{R/k}(M,N)$ from M to N of order $\leq n$, and $D_{R/k}(M,N)$. Once again these will be R-submodules of $\mathrm{Hom}_k(M,N)$ defined inductively, with $D^0_{R/k}(M,N) = \mathrm{Hom}_R(M,N)$. The result will no longer be a ring unless $M = N$ – it will, however, retain an R-module structure.

(ii) Given $\xi \in D^n_{R/k}(M,N)$ and $\eta \in D^m_{R/k}(N,P)$, $\eta \circ \xi \in D^{n+m}_{R/k}(M,P)$.

Sadly, computing differential operators is no easy task. So far, the only thing we know is that if k is a field of characteristic zero and $R = k[x_1, \ldots, x_n]$ then

$$D_k(R) = k\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle \subseteq \mathrm{End}_k(R).$$

This was done in the form of an exercise, but at least the inclusion ($\supseteq$) should be clear. It is not hard to find (or at least guess) a presentation for this ring, namely

$$D_k(R) = \frac{k\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle}{\langle [x_i, x_j] = 0, [\partial_i, \partial_j] = 0, [\partial_i, x_j] = \delta_{ij}\rangle}.$$

This ring is called the Weyl algebra in $2n$-generators. We will denote it by $\mathcal{D}$.

Our first goal today is to study behaviour under quotients, so that we may begin to give our first examples of things which are not polynomial.

2.1 Behaviour under quotients for polynomial rings

This discussion is taken from [2], Ch. 15.

Fix a field k of characteristic zero and let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a k. Thus $D_k(S) = \mathcal{D}$.

Exercise 2.2.

(i) A differential operator ξ of order $\leq n$ is uniquely determined by its values on x^α where $|\alpha| \leq n$.

(ii) Conversely, given a collection $\{f_\alpha : |\alpha| \leq n\}$ there is a differential operator ξ of order $\leq n$ with $\xi(x^\alpha) = f_\alpha$.

Given an ideal $I \subseteq S$ we define

$$\mathcal{D}[I] = \{\xi \in \mathcal{D} : \xi(I) \subseteq I\}.$$

Observe that $\{\xi \in \mathcal{D} : \xi(R) \subseteq I\}$ is a right ideal of $\mathcal{D}$, and that $\mathcal{D}[I]$ is a subring of $\mathcal{D}$.

Definition 2.3. Given an right ideal J in a (not necessarily commutative) ring A, the idealizer of J in A is the biggest subring of A that contains J as a two-sided ideal.

Exercise 2.4. Show there is a "biggest" subring that contains J as a two-sided ideal.

Proposition 2.5 (ref. McConnell & Robson)**.**

(i) We have $I\mathcal{D} = \{\xi \in \mathcal{D} : \xi(R) \subseteq I\}$*.*

(ii) The subring $\mathcal{D}[I]$ *is the idealizer of* $I\mathcal{D}$ *in* $\mathcal{D}$*.*

Proof.

(i) For this proof set $J := \{\xi \in \mathcal{D} : \xi(R) \subseteq I\}$. Clearly $I\mathcal{D} \subseteq J$. Now suppose $\xi = \sum f_\alpha \partial^\alpha \in J$, and we wish to show that $\xi \in I\mathcal{D}$. We may assume that $f_\alpha \notin I$ for all α. If ξ is then nonzero take an α with minimal $|\alpha|$ amongst the ones with $f_\alpha \neq 0$. Then

$$f_\alpha = (scalar) \cdot \xi(x^\alpha) \in I$$

giving a contradiction.

(ii) For simplicity set $A := \mathbb{I}(I\mathcal{D})$. As $I\mathcal{D}$ is a two-sided ideal of $\mathcal{D}[I]$ (use part (i)), $\mathcal{D}[I] \subseteq A$. Now suppose $\xi \in A$, and we wish to show that $\xi(I) \subseteq I$. Thus let $g \in I$. On the one hand, $\xi g \in I\mathcal{D}$ as ξ is in the idealizer of $I\mathcal{D}$ and $g \in I\mathcal{D}$. But then we conclude that $\xi(g) = \xi g(1) \in I$ by part (i).

□

Proposition 2.6. *Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a commutative ring k and $I \subseteq S$ be an idea. Set $R = S/I$. Then*

$$\mathcal{D}_k(R) \simeq \frac{\mathcal{D}[I]}{I\mathcal{D}}.$$

Moreover, this isomorphism respects the order filtration.

Observe that by the previous proposition $I\mathcal{D}$ is a two-sided ideal of $\mathcal{D}[I]$, and thus the proposition makes sense.

Proof. We have a map $\mathcal{D}[I]/I\mathcal{D} \to \mathcal{D}_k(R)$ which is injective – we will denote the image of $\eta \in \mathcal{D}[I]$ by η_R. To show it is surjective it suffices to show that every differential operator ξ on R can be lifted to a differential operator $\widetilde{\xi}$ on S – the lift will necessarily preserve I. Suppose ξ has order $\leq n$.

For this, consider lifts f_α of $\xi(\overline{x^\alpha})$ for $|\alpha| \leq n$. By Exercise 2.2, there exists a differential operator $\eta \in \mathcal{D}$ with $\eta(x^\alpha) = f_\alpha$. The result then follows from the following claim:

Claim.- Suppose $\xi \in D^n_k(R)$ and $\eta \in D^n_k(S)$ are such that $\xi(\bar{f}) = \overline{\eta(f)}$ for all polynomials $f \in S$ of degree $\leq n$. Then $\xi(\bar{f}) = \overline{\eta(f)}$ for all $f \in S$ – in particular, $\eta \in \mathcal{D}[I]$.

Proof of Claim: We will induct on the order n, with $n = 0$ being clear. We know that, for $i = 1, \ldots, n$,

$$[\xi, \overline{x_i}](f) = \overline{[\eta, x_i](f)}$$

for all polynomials f of degree $\leq n - 1$. As the order has dropped, the inductive hypothesis implies that the above statement holds for all $f \in S$.

We now show that $\xi(\bar{f}) = \overline{\eta(f)}$ for all $f \in S$ by induction on the degree of f, knowing the statement whenever the degree is $\leq n$.

If x^α is a monomial of positive degree, we may write $x^\alpha = x_i x^\beta$ for some i, β, with x^β having degree one less. Then

$$\xi(x^\alpha) = x_i \xi(x^\beta) + [\xi, x_i](x^\beta).$$

and

$$\eta(x^\alpha) = x_i \eta(x^\beta) + [\eta, x_i](x^\beta).$$

The induction on degree then gives $\xi(x^\alpha) = \eta(x^\alpha)$. □

Example 2.7. Let $R = k[x, y]/(f)$ where $f = x^3 - x + y^2$, and denote $S = k[x, y]$. We will compute $D_k(R)$, but in order to do so we need the following fact:

• If R is a regular k-algebra, where k is a field of characteristic zero, then $D_k(R)$ is generated by R and its derivations.

Because R is regular in our case, it suffices to compute the derivations of R over k. As R is one-dimensional, the module of derivations will be generated by one element, and since

$$\mathrm{Der}_k(R) = \frac{\{\theta \in \mathrm{Der}_k(S) : \theta((f)) \subseteq (f)\}}{(f)\,\mathrm{Der}_k(S)},$$

it suffices to find a derivation θ on S with $\theta((f)) \subseteq (f)$.

The reason why it is easy to compute these derivations is the following:

Exercise 2.8. Let θ be a derivation on S. Then $\theta((f)) \subseteq (f)$ if and only if $\theta(f) \in (f)$. This may fail for higher-order differential operators.

But now we can guess a derivation θ on S with $\theta(f) \in (f)$ – in fact with $\theta(f) = 0$! Set:

$$\begin{aligned}\theta &= \frac{df}{dy}\partial_y - \frac{df}{dx}\partial_x \\ &= 2y\partial_y - (3x^2 - 1)\partial_x.\end{aligned}$$

We conclude that $D_k(R)$ is the subring of $\mathrm{End}_k(R)$ generated by R and $\overline{\theta}$ – the derivation induced by θ on R.

$$D_k(R) = k\langle \bar{x}, \bar{y}, \overline{\theta}\rangle.$$

3 Lecture 3

Last lecture we introduced a theorem that is useful for computing rings of differential operators of regular finite-type k-algebras – especially easy in the case of a hypersurface!

We would like to understand how differential operators of singular k-algebras look like. For example, we are interested in rings like

$$k[x,y]/(x^3-y^2) \text{ or } k[s,t,u]/(su-t^2).$$

While these algebras are singular, they are closely related to regular algebras. The first one we can express as the subring $k[t^2,t^3]$ of $k[t]$. The second one is the subring $k[x^2,xy,y^2]$ of $k[x,y]$.

The machinery we introduce today will allow us to use these "resolutions" to compute the rings of differential operators.

3.1 Modules of principal parts

Fix a k-algebra R. Recall that a k-linear derivation from R to an R-module M is a k-linear map $\theta : R \to M$ with the property $\theta(fg) = f\theta(g) + g\theta(f)$. There is an R-module $\Omega_{R/k}$, called the module of Kähler differentials, that "represents derivations". Namely, there is a derivation $d : R \to \Omega_{R/k}$ such that, given a k-linear derivation $\theta : R \to M$ there is a unique R-linear map $\overline{\theta} : \Omega_{R/k} \to M$ such that the diagram

$$\begin{tikzcd} R \arrow[r, "\theta"] \arrow[dr, "d"'] & M \\ & \Omega_{R/k}. \arrow[u, "\overline{\theta}"'] \end{tikzcd}$$

Moreover, there is a one-to-one correspondence

$$\mathrm{Der}_k(R,M) \simeq \mathrm{Hom}_R(\Omega_{R/k}, M).$$

We are now looking for a similar construction to $\Omega_{R/k}$ for the case of differential operators. That is, an R-module $P^n_{R/k}$ together with a universal differential operator of order $\le n$, $d^n : R \to P^n_{R/k}$, with the property that

$$D^n_{R/k}(R,M) \simeq \mathrm{Hom}_R(P^n_{R/k}, M).$$

Before going into this we need a new way of thinking about differential operators.

Set $T = R \otimes_k R$. There is a multiplication map $\mu : T \to R$, and we denote the kernel by $J = J_{R/k}$.

Exercise 3.1. The ring T has two different R-module structures (one by multiplication on each side), and thus so does J. Show that J is generated by elements of the form $a \otimes 1 - 1 \otimes a$, both as a right and left R-module.

Given now R-modules M and N, observe that T acts on $\mathrm{Hom}_k(M,N)$ by $(r \otimes s) \cdot \phi = r\phi(s\bullet)$.

Proposition 3.2. *With the above notation,*

$$D^n_{R/k}(M,N) = \{\xi \in \mathrm{Hom}_k(M,N) : J^{n+1}\xi = 0\}.$$

Proof. Observe that $\nu_r = r \otimes 1 - 1 \otimes r$ acts on ξ by $\nu_r \cdot \xi = [r,\xi]$. In particular, $J \cdot \xi = 0$ if and only if ξ is R-linear. This gives the statement for $n = 0$. For the inductive step, observe that $J^{n+1}\xi = 0$ if and only if $J^n(\nu_r\xi) = 0$ for all $r \in R$, i.e. if and only if $[r,\xi] \in D^{n-1}_k(R)$ for all $r \in R$. □

Corollary 3.3. *With the notation as above,*

$$D_{R/k}(M,N) = H^0_J(\mathrm{Hom}_k(M,N)).$$

We will also need a stronger form of the tensor-Hom adjunction that most people have usually seen.

Proposition 3.4. *Let A and B be possibly noncommutative rings. Given a bimodule $X = {}_AX_B$ and modules $Y = {}_BY$ and $Z = {}_AZ$ there is a natural isomorphism*

$$\mathrm{Hom}_A(X \otimes_B Y, Z) \simeq \mathrm{Hom}_B(Y, \mathrm{Hom}_A(X, Z)).$$

(Recall that $Z(A)$ is the center of the ring A.)

Exercise 3.5.

(i) The tensor-hom adjunction gives an isomorphism

$$\mathrm{Hom}_k(M, N) \simeq \mathrm{Hom}_R(R \otimes_k M, N)$$

of $R \otimes_k R$-modules, where the structure on the module $\mathrm{Hom}_R(R \otimes_k M, N)$ comes from $((a \otimes b)\psi)(r \otimes u) := \psi(ar \otimes bm)$.

(ii) One has $\mathrm{Hom}_T(P^n, \mathrm{Hom}_k(M, N)) = D^n_{R/k}(M, N)$.

(iii) On the other hand

$$\mathrm{Hom}_T\left(P^n, \mathrm{Hom}_k(M, N)\right) \simeq \mathrm{Hom}_R(P^n \otimes_R M, N),$$

where $P^n \otimes_R M$ means "tensor with the right R-module structure", and the Hom on the right is taken with respect to the left R-module structure on $P^n \otimes_R M$.

Therefore one has

$$D^n_{R/k}(M, N) \simeq \mathrm{Hom}_R(P^n \otimes_R M, N).$$

In particular, when $M = R$ we have $D^n_{R/k}(R, M) \simeq \mathrm{Hom}_R(P^n, N)$ (where the Hom is taken with respect to the left R-module structure on P^n). We have thus an analogue with the Kähler differential situation.

Exercise 3.6. Trace back the isomorphisms one finds that the universal differential operator $d^n : R \to P^n$ is given by $r \mapsto 1 \otimes r$. Why is this a differential operator of order $\leq n$?

3.2 Back to polynomial algebras

Let us compute the module of principal parts for $R = k[x_1, \ldots, x_n]$, where k is now an arbitrary commutative ring.

After identifying $x_i = x_i \otimes 1$ and $y_1 = 1 \otimes x_i$, we have

$$P^n = \frac{k[x_1, \ldots, x_n, y_1, \ldots, y_n]}{\langle y_1 - x_1, \ldots y_n - x_n \rangle^{n+1}}.$$

By making a change of coordinates $dx_i = y_i - x_i = 1 \otimes x_i - x_i \otimes 1$, we get

$$\begin{aligned} P^n &= \frac{k[x_1, \ldots, x_n, dx_1, \ldots, dx_n]}{\langle dx_1, \ldots, dx_n \rangle^{n+1}} \\ &= \bigoplus_{|\alpha| \leq n} R dx^\alpha, \end{aligned}$$

i.e. P^n is a free module over R with basis $\{dx^\alpha : |\alpha| \leq n\}$.

We conclude that

$$D^n_k(R) = \bigoplus_{|\alpha| \leq n} R(dx^\alpha)^*$$

where $(dx_\alpha)^*$ is the R-dual of dx^α. We conclude that $D^n_k(R)$ is, as an R-module, free and generated by the differential operators $(dx^\alpha)^*$.

We now compute what the differential operators of order at most n on R are. It remains to check what the $(dx^\alpha)^*$ actually are as differential operators. Let $D^{(\alpha)}$ be the differential operator corresponding to $(dx^\alpha)^*$, i.e. $D^{(\alpha)} = (dx^\alpha)^* \circ d^n$. Let us describe the action on a monomial x^β.

We claim $D^{(\alpha)}(x^\beta) = \binom{\beta}{\alpha} x^{\beta-\alpha}$. This follows because $d^n(x^\beta) = (d^n x)^\beta = (dx - x)^\beta$ – we are using multi-index notation. By the multi-index binomial theorem,

$$(dx - x)^\beta = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} x^{\beta-\gamma} dx^\gamma,$$

from which the statement follows.

We have thus proven;

Proposition 3.7. *Let $R = k[x_1, \ldots, x_n]$, where k is a commutative ring. Then*

$$D_k^n(R) = \bigoplus_{|\alpha| \leq n} R \cdot D^{(\alpha)},$$

where $D^{(\alpha)}$ is the unique k-linear map satisfying

$$D^{(\alpha)}(x^\beta) = \binom{\beta}{\alpha} x^{\beta-\alpha}.$$

Observe that when k is a field of characteristic zero, $D^{(\alpha)} = (1/\alpha!)\partial^\alpha$.

4 Lecture 4

From now on, we will assume that k is noetherian and that all k-algebras are essentially of finite type. That is, when we say R is a k-algebra, R will be assumed to be a quotient of $k[x_1, \ldots, x_n]$ for some n, or a localization of such a ring, and the map $k \to R$ will be assumed to be the natural map coming from this identification.

Recall that our goal is to understand differential operators of rings like $k[t^2, t^3]$ or $k[x^2, xy, y^2]$. The good thing about these rings is that they admit nice maps to smooth algebras, the differential operators of which we understand well.

To exploit this to our advantage we need to explore how a morphism $f : R \to S$ of k-algebras can help us relate the differential operators of R and S. The first step would be to observe that the most natural hope is not realized.

Exercise 4.1. If $f : R \to S$ is a morphism of k-algebras, f does not in general induce a map between the rings of differential operators of R and S.

In this level of generality, the most we can observe is that we have the following natural maps, all of which preserve the relevant order filtrations.

$$S \otimes_R D_k(R) \longrightarrow D_{R/k}(R, S) \longleftarrow D_k(S).$$

The arrow on the left is given $s \otimes \xi \mapsto [r \mapsto sf(\xi(r))]$. The arrow on the left is given by $\eta \mapsto \eta \circ f$. Because these preserve the filtrations, one can think of all these maps at the level of modules of principal parts:

$$S \otimes_R \operatorname{Hom}_R(P^n_R, R) \longrightarrow \operatorname{Hom}_R(P^n_R, S) \longleftarrow \operatorname{Hom}_S(P^n_S, S).$$

Now observe that $\operatorname{Hom}_R(P^n_R, S) \simeq \operatorname{Hom}_S(S \otimes_R P^n_R, S)$.

Exercise 4.2. Show that P^n_R is finitely-generated as a left R-module. [Hint: One needs the assumption that k is noetherian and R is essentially of finite-type over k]

This exercise, together with the observations above, already shows that

Lemma 4.3. *Suppose $f : R \to S$ is a flat map of k-algebras. Then*

$$S \otimes_k D_k(R) \simeq D_{R/k}(R, S).$$

The more interesting question is when do we have the map $\operatorname{Hom}_S(P^n_S, S) \to \operatorname{Hom}_S(S \otimes_R P^n_R, S)$ being an isomorphism. A nice case where this happens is when the natural map $S \otimes_R P^n_R \to P^n_S$ is an isomorphism.

Exercise 4.4. Show that there is a natural map $S \otimes_R P^n_R \to P^n_S$.

We will prove this map is an isomorphism whenever the morphism $f : R \to S$ is étale.

4.1 Behaviour under étale extensions

This material is taken from [3]

Definition 4.5. A map of rings $f : R \to S$, essentially of finite type, is étale if for all R-algebras T, all nilpotent ideals $J \subseteq T$ and all R-algebra maps $S \to T/J$ there exists a unique R-algebra map $S \to T$ such that

$$\begin{tikzcd} & T \arrow[d] \\ S \arrow[ur, dashed] \arrow[r] & T/J. \end{tikzcd}$$

commutes.

By requiring just existence, or just uniqueness, one gets the notions of smoothness and unramified-ness respectively.

Remark 4.6. An étale map is flat. In particular, Lemma 4.3 holds for an étale map $f : R \to S$.

Exercise 4.7.

(i) A localization $f : R \to W^{-1}R$ is étale. [Hint: a unit plus a nilpotent is a unit].

(ii) Suppose that n is invertible in k. Then "adjoining n-th roots is an étale operation". To be concrete, show that

$$k[x] \to \frac{k[x,y]_y}{(y^n - x)}$$

is étale.

We now begin our proof of the already-mentioned statement. Namely:

Theorem 4.8. *Let $R \to S$ be an étale morphism of k-algebras. Then the natural map*

$$S \otimes_R P^n_R \to P^n_S$$

is an isomorphism.

We will prove this by constructing an inverse.

Let $R \to S$ be an etale homomorphism of k-algebras (thus flat). Tensoring the exact sequence

$$0 \to J_R/J_R^{n+1} \to P^n_R \xrightarrow{\mu} R \to 0$$

with S on the left gives

$$0 \to S \otimes_R (J_R/J_R^{n+1}) \to S \otimes_R P^n_R \xrightarrow{S\otimes\mu} S \to 0.$$

In particular, the map $S \otimes_R P^n_R \xrightarrow{S\otimes\mu} S$ has nilpotent kernel.

Now regard $S \otimes_R P^n_R$ as an R-algebra via its right R-module structure $d^n : R \to P^n_R$. By étaleness, the dashed arrow can be filled as a map of R-algebras.

$$\begin{tikzcd}
S \arrow[r, "="] \arrow[dr, dashed, "\phi"] & S \\
R \arrow[u] \arrow[r, "1\otimes d^n"'] & S \otimes_R P^n_R. \arrow[u, "S\otimes\mu"']
\end{tikzcd}$$

Lemma 4.9. *The map ϕ above is a differential operator of order $\leq n$.*

Proof. As ϕ is an algebra map, $(1 \otimes a - a \otimes 1) \cdot \phi = (\phi(a) - a \otimes 1)\phi(\bullet)$ for all $a \in S$. By the commutativity of the diagram above, $\phi(a) - a \otimes 1 \in \ker(S \otimes \mu) = S \otimes_R (J_R/J_R^{n+1})$. □

Consequently, there is a unique S-linear map $\bar{\phi}$ such that

$$\begin{tikzcd}
P^n_S \arrow[r, "\bar{\phi}"] & S \otimes_R P^n_R \\
S \arrow[u, "d^n_S"] \arrow[ur, "\phi"] &
\end{tikzcd}$$

commutes. We claim this $\bar{\phi}$ provides an inverse for the natural map. We leave this as an exercise.

Exercise 4.10.

(i) Observe that so far we have only used smoothness.

(ii) Denote by $\psi : S \otimes_R P^n_R \to P^n_S$ the natural map. Show that $\bar{\phi} \circ \psi = \mathrm{id}$.

(iii) Show that $\psi \circ \bar{\phi} = \mathrm{id}$. [Hint: both $\psi \circ \phi$ and d^n_S make a certain diagram commute, and étaleness implies $\psi \circ \phi = d^n_S$].

We have thus proven Theorem 4.8

Corollary 4.11. *With R, S as above,*

$$D^n_k(S) \cong S \otimes_R D^n_k(R)$$

as left S-modules.

Proof. See the discussion at the beginning of the lecture, as well as Remark 4.6 □

Corollary 4.12. *Let R be a k-algebra and $W \subseteq R$ be a multiplicatively closed subset. Then*

$$D_k(W^{-1}R) = W^{-1}D_k(R).$$

Corollary 4.13. *Let $R \subseteq S$ be an étale inclusion of k-algebras (e.g. an injective localization). Then*

(i) Every differential operator on R lifts to a unique differential operator on S.

(ii) One has

$$D_k(R) \simeq \{\xi \in D_k(S) : \xi(R) \subseteq R\}$$

as rings.

Example 4.14. If $R = k[t^{\pm 1}]$, where k is a field of characteristic zero, then $D_k(R)$ is the k-subalgebra of $\mathrm{End}_k(R)$ generated by t, t^{-1} and ∂_t. In other words,

$$R = k[t^{\pm 1}] = k\langle t^{\pm 1}, \partial_t\rangle.$$

Exercise 4.15. Identifying the ring of Example 4.14 with $k[t, u]/(ut - 1)$, verify the statement of Example 4.14 by using the behaviour of differential operators under quotients.

Corollary 4.16. *Let $R \subseteq S$ be an étale inclusion of k-algebras (e.g. an injective localization). Then*

(i) Every differential operator on R lifts to a unique differential operator on S.

(ii) One has

$$D_k(R) \simeq \{\xi \in D_k(S) : \xi(R) \subseteq R\}$$

as rings.

Exercise 4.17. Even when the map $R \to S$ is not injective there is always a map

$$D_k(R) \to \{\xi \in D_k(S) : \xi(\mathrm{im}\, R) \subseteq \mathrm{im}\, R\}.$$

One might ask what goes wrong if we don't assume that the étale map $R \to S$ is an inclusion. Consider the ring $R = k[x, y]/(xy)$, let $S = R_y$ and consider the localization map $R \to S$ – non-injective localizations are still étale. Show that the map above is not an isomorphism.

Exercise 4.18. Let $\mathcal{D}$ be the Weyl algebra and N be a positive integer. Suppose that some $\xi \in \mathcal{D}$ satisfies $\xi(f) = 0$ for all f of degree $\geq N$. Then $\xi = 0$. [Hint: If S is the polynomial ring, consider the subring R generated by elements of degree $\geq N$. Then R and S have a common localization].

5 Lecture 5

Throughout today, k will be a field of characteristic zero.

We now want to use the material from yesterday to compute the rings of differential operators of our favorite examples. Let us begin with an observation that will be useful.

Exercise 5.1.

(i) Suppose R is a $\mathbb{Z}$ or $\mathbb{N}$ graded k-algebra. Then $D_k(R)$ is $\mathbb{Z}$-graded:

$$D_k(R) = \bigoplus_{d\in\mathbb{Z}} D_k^{(d)}(R)$$

where

$$D_k^{(d)}(R) = \{\xi \in D_k(R) : \xi(R_i) \subseteq R_{d+i} \text{ for all } i\}.$$

(ii) Let k be a field of characteristic zero and $R = k[x_1, \ldots, x_n]$ be standard graded. Then $D_k(R) = k\langle x, \partial\rangle$ is the Weyl algebra $\mathcal{D}$ and the grading inherited from R is given by $\deg x_i = 1$, $\deg \partial_i = -1$.

5.1 The cuspidal cubic

Fix a field k of characteristic zero and let $R = k[t^2, t^3]$ and $S = k[t]$. Then $R \subseteq S$ and the map $R \to S_t = k[t, t^{-1}]$ is an injective localization. We denote by $\mathcal{D}_t := k\langle t, t^{-1}, \partial_t\rangle$. We conclude that

$$D_k(R) = \{\xi \in \mathcal{D}_t : \xi(R) \subseteq R\}.$$

In particular, $t^2\mathcal{D} \subseteq D_k(R)$. We will compute $D_k^{(d)}(R)$. First observe that $D_k^{(d)}(R) = \mathcal{D}^{(d)}$ for $d \geq 2$ and $d = 0$. Moreover, $D_k^{(d)}(R) \supseteq t^2\mathcal{D}^{(d-2)}$ and therefore it suffices to compute the space $A^{(d)} := D_k^{(d)}/t^2\mathcal{D}^{(d-2)}$ for $d = 1$ and $d < 0$.

- For starters, $\mathcal{D}_t^{(1)}/t^2\mathcal{D}^{(-1)}$ is just spanned by the monomial t, and $\lambda t(R) \subseteq R$ if and only if $\lambda = 0$. Thus $A^{(1)} = 0$.
- $\mathcal{D}_t^{(-1)}/t^2\mathcal{D}^{(-1)}$ is spanned by the monomials t^{-1}, ∂ and $t\partial^2$. Plugging in 1 and t^2 we see that $A^{(-1)} = k(\partial - t\partial^2)$. (Everything of degree $\geq$ will automatically map into R).
- Similarly, $A^{(-2)}$ is spanned by the operator $2t^{-1}\partial - \partial^2$.
- Now let $d > 2$, and we want to compute $A^{(-d)}$. Observe that $\mathcal{D}_t^{(-d)}/t^2\mathcal{D}^{(-d-2)}$ is given by

  $$\{a_0t^{-d} + a_1t^{-d+1}\partial + \cdots + a_{d+1}t\partial^{d+1} : a \in k^{d+2}\}.$$

  Let us call $\xi_{d,a} = a_0t^{-d} + a_1t^{-d+1}\partial + \cdots + a_{d+1}t\partial^{d+1}$. The question is: for which values of a does $\xi_{d,a}$ preserve R. That is, for which a do we get $\xi_{d,a}(x^m) \in R$ for $m = 0$ or $m \geq 2$? It is easy that we get $m \geq d+2$ for free, as well as $m = d$. Therefore, there are d equations for a, which has $d+2$ parameters. These equations are linearly independent (exercise) and therefore $A^{(-d)}$ is a vector space of dimension 2 for $d \geq 3$.

5.2 Veronese subrings

Recall that our second example was $k[x^2, xy, y^2] \subseteq k[x, y]$, which consists of polynomials whose homogeneous components all have even degree. This fits into a more general setting.

Definition 5.2. Let $S = k[x_1, \dots, x_n]$ be a polynomial ring in $n \geq 2$ variables over k, and let $d \geq$ be a positive integer. The *Veronese subring* of degree d of S is

$$S^{(d)} := \{f \in S : f_i = 0 \text{ unless } d|i\},$$

where f_i is the component of f of degree i.

It follows that $S^{(d)}$ is generated, as a k-algebra, by the monomials of degree d in $x_1, \dots, x_n$. We recover our example by $d = 2$, $n = 2$.

The geometric picture to have in mind is as follows.

Set $X = \operatorname{Spec} S \simeq \mathbb{A}^n$ and $Y = \operatorname{Spec} S^{(d)}$. Then we have a diagram

$$\begin{tikzcd} X & X \setminus \{0\} \arrow[l, hook', "i"'] \arrow[d, "f"] \\ Y & Y \setminus \{0\} \arrow[l, hook', "j"'] \end{tikzcd}$$

where f is étale.

Recall that there is a map $D_k(S) \to D_{R/k}(R, S)$ that just restricts operators. At order $\leq n$ this map is just $\operatorname{Hom}_S(P^n_S, S) \to \operatorname{Hom}_R(P^n_R, S)$. We have a diagram

$$\begin{tikzcd} & \operatorname{Hom}_S(S \otimes_R P^n_R, S) \arrow[d, equal] \\ \operatorname{Hom}_S(P^n_S, S) \arrow[ur] \arrow[r] & \operatorname{Hom}_R(P^n_R, S). \end{tikzcd}$$

Now observe that the map $\operatorname{Hom}_S(P^n_S, S) \to \operatorname{Hom}_S(S \otimes_R P^n_R, S)$ is an isomorphism away from the origin. But they are both S_2-modules on $k[x_1, \dots, x_n]$. It follows that the map is actually an isomorphism (see [4] for details, in particular Corollary 2.11).

We conclude:

Proposition 5.3. *Let $S = k[x_1, \dots, x_n]$ and $R = S^{(d)}$. Then $D_k(S) \simeq D_{R/k}(R, S) \simeq S \otimes_R D_k(R)$. In particular, every differential operator on R extends to a unique differential operator on S and, moreover,*

$$D_k(R) \simeq \{\xi \in D_k(S) : \xi(R) \subseteq R\}.$$

Corollary 5.4. *The ring of differential operators on R is given by*

$$D_k(R) = \mathcal{D}^{(d)},$$

where $\mathcal{D} = D_k(S)$.

Proof. Suffices to show that if $\xi(R) \subseteq R$ then ξ has all homogeneous terms with degree divisible by d. Subtract of terms with degree divisible by d to obtain an operator $\eta = \sum \eta_i$ where η_i has degree i not divisible by d. Then $\eta(x^{d\alpha}) = \sum \eta_i(x^{d\alpha})$ and since $\eta_i(x^{d\alpha})$ has degree $d|\alpha| + i$ and must be in R it follows that $\eta(R) = 0$. But then η corresponds to the zero operator under the isomorphism $D_k(S) \simeq D_{R/k}(R, S)$. □

Example 5.5. For $R = k[x^2, xy, y^2] \subseteq k[x, y]$ we have

$$D_k(R) = k\langle x^2, xy, y^2, x\partial_x, x\partial_y, y\partial_x, y\partial_y, \partial_x^2, \partial_y^2\rangle \subseteq \mathcal{D}.$$

5.3 Morita equivalences for curves

To finish, I would like to discuss a result of Smith and Stafford.

Definition 5.6. Recall that two (not necessarily commutative) rings A and B are Morita equivalent if their categories of left modules are equivalent.

Miraculously, this is equivalent to their categories of right modules being equivalent.

Theorem 5.7. *(S.P. Smith and J.T. Stafford [5]) Let R be a k-algebra of dimension 1 and $\widetilde{R}$ be its normalization. Let $\pi : \operatorname{Spec}(\widetilde{R}) \to \operatorname{Spec}(R)$ be the induced map. Then the following are equivalent:*

- *(i) The map π is injective.*
- *(ii) The ring $D_k(R)$ Morita equivalent to $D_k(\widetilde{R})$.*

(Smith and Stafford provide further equivalences in [5]).

Ben-Zvi and Nevins proved in [6] that one can take the bimodule inducing this equivalence to be $D_{R/k}(R, \widetilde{R})$.

Corollary 5.8. *Let $R = k[t^2, t^3]$. Then $D_k(R)$ is Morita-equivalent to the Weyl algebra in 2 generators. In particular, it is left and right noetherian and simple.*

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