Set 10

(A) Euler characteristic.

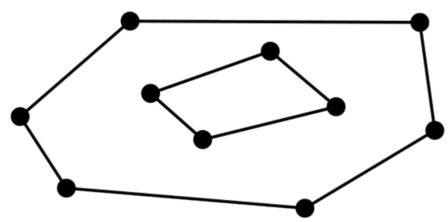
(1) Draw five sufficiently sophisticated doodles (your signature could be one of them!). Then turn them into tilings and compute their Euler characteristic. Any conjectures?

(2) Suppose you have a tiling of the plane. What happens to its Euler characteristic when you perform the following operations?

- (i) Split an edge into two by introducing a vertex in the middle.
- (ii) Take a face and a vertex on its boundary, and draw an edge from the vertex to the interior of the face.
- (iii) Introduce a new edge between two vertices that are on the boundary of the same face.
- (iv) Introduce a new edge between two vertices that are *not* on the boundary of the same face.

(3) Use your answer to (2) to explain your conjecture from (1).

(4) (***HW***) Compute the Euler characteristic of the following shape. Does this contradict your conjecture from (1)? How can you "fix" this tiling?



(B) Platonic solids.

(1) With the people at your table glue together the five platonic solids. For each platonic solid, calculate #Vertices − #Edge + #Faces. Any conjectures?

(2) We say that the Euler characteristic of a space is the Euler characteristic of any tiling of that space—as you have shown above, all tilings give the same answer. Explain why your conjectures from (A1) and (C1) both follow from the fact that The Euler characteristic of the sphere is 2.

(3) If you make n circular holes into a sphere, what is the Euler characteristic of the resulting space?

(4) (***HW***) Suppose n spheres are glued together at points to form a bead collar. What is the Euler characteristic of this space?

(5) Take two spheres, and make a circular hole into each of them. Then glue the two resulting shapes at the holes. What is the Euler characteristic of the resulting shape?

(C) Planar graphs.

(1) A graph is called planar whenever it can be drawn on the plane in such a way that no edges intersect. For each integer $n \geq 1$, let K_n denote the complete graph in n vertices. Which K_n do you believe are planar? For now a conjecture is just fine, but do prove that if K_n is not planar then K_{n+1} is not planar.

(2) For a pair of integers $n, m \geq 1$, let $K_{n,m}$ be the complete bipartite graph in sets of n and m vertices. Which $K_{n,m}$ do you believe are planar? Again, a conjecture will do, but prove that if $K_{n,m}$ is not planar, then $K_{n+1,m}$ and $K_{n,m+1}$ cannot be planar.

(3) Suppose that $K_{3,3}$ was planar, and let V, E, and F denote the number of vertices, edges, and faces respectively.

   (i) Show that every face would be bounded by at least four edges.

   (ii) Note that every edge lies in the boundary of at most two faces. Let E_2 denote the number of edges that lie in *exactly* two faces, and show that $E \geq 4F - E_2$.

   (iii) Note that $E_2 \leq E$, and use this to prove your conjecture from (C2).

(4) Prove your conjecture from (C1).

(D) Yasuda's "Euler Getter" game. With a peer, play five matches of Yasuda's game. Who has a winning strategy?