Set 1

Note: We obtained many of the problems listed here from Stefan Patrikis, who used to run the SHSP at the University of Utah. We are grateful that he allowed us to use his problems. In Patrikis' original problem set, he also mentioned "These problem sets are derived from those assigned at the Ross Mathematics Program every summer since 1957; it is with gratitude and admiration that we follow the example of the extraordinary mathematics educator Arnold E. Ross (1906-2022)."

(A) Ice-breaker:

(B) Have each of the members in your group answer this question, in order:

(1) What is your name?

(2) What school do you attend?

(3) Do you have any hobbies?

(4) Do you have a favorite number and, if so, which one? Why?

(C) Warm-up problems:

(1) What is an irrational number? Show that sometimes the sum of two irrational numbers can be rational, and that the product of two irrational numbers can be irrational. For us, this means that the set of irrational numbers does not form a number system.

(2) It is now 10am. What time will it be in 25 hours? In 75 hours? In 1000 hours?

(3) It is now Monday. What day will it be after 25 days? How about 75 days? How about 1000 days?

(4) Suppose you have mysterious big positive integers A and B. All that you know about them is that the last two digits of A are 26, and the last two digits of B are 45. What are the last two digits of $A + B$ and $A \times B$? What is the last digit of A^2? How about A^3?

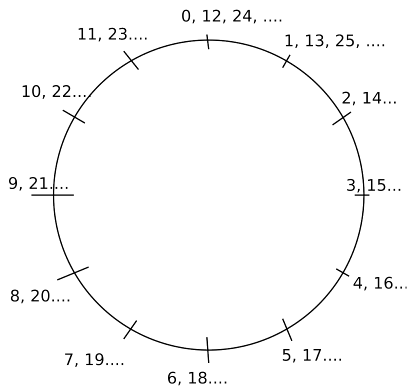
(5) What is the last digit of 98713^{4003}?

(6) Recall that the Fibonacci sequence (f_n) is obtained by setting $f_0 = f_1 = 1$, and by declaring that $f_n = f_{n-1} + f_{n-2}$ for all integers $n \geq 2$. What can you say about the last digits of the Fibonacci sequence?

(7) (***HW***) If we wrote the Fibonacci sequence in base 7, what would the sequence of last digits look like?

(D) Clock arithmetic

(1) Let's think of $\mathbb{Z}/12$ as a clock, where each hour is a different congruence class, as follows:



(2) Where should the negative numbers $-1, -2, -3, \ldots$ appear on the picture?

(3) Investigate where the primes distinct from 2 or 3 appear on the clock. Can you think about why that would be the case? What is special about 2 and 3 in this situation?

(4) (***HW***) Where do the primes distinct from 2 and 5 appear in a clock with 10 or 11 hours? Can you explain why?

(E) You may have encountered at some point the following simple test for whether an integer is divisible by 3: simply add the digits, and check whether the sum is divisible by 3 (for instance, 1014 has digits adding to $1 + 0 + 1 + 4 = 6$, which is divisible by 3, and indeed $1014 = 3 \cdot 338$; on the other hand, 1015 has digits summing to 7, which is not divisble by 3, and indeed $1015 = 3 \cdot 338 + 1$ has remainder 1 when divided by 3).

(1) Is the integer 123455678910111213141516171819 divisible by 3?

(2) We can often spot mathematical patterns by combining two kinds of inductive reasoning: accumulating special cases (e.g. numerical examples), and using analogies with previously-observed patterns. Throughout these problems sets, you will have to combine these methods of investigation. Can you devise simple tests, analogous to the one for divisibility by 3, for whether an integer is divisible by 9? By 11?

(3) In all of these cases (divisibility tests for division by 3, 9, 11), at what point do you find it "safe" to extrapolate from experimental observation to the belief that a general pattern holds? Why?

(4) (For enthusiasts) Observe that $1001 = 7 \cdot 11 \cdot 13$, and that 1001 is very close to 10^3. Can you generalize the sort of reasoning used in the previous two problems to devise divisibility tests for divisibility by $7, 11, 13$?

(F) Consider the following number systems: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}/3, \mathbb{Z}/6, \mathbb{Z}/8, \mathbb{Z}/11$. You are familiar with the binary operations addition and multiplication on $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

(1) How do you define precisely addition and multiplication in the modular systems $\mathbb{Z}/n$? Does your definition satisfy familiar properties of integer arithmetic such as the commutative, associative, and distributive properties?

(2) Granted that each of the listed number systems has two operations, addition and multiplication, consider some basic algebraic properties and determine for which of these systems those properties hold. Some properties to consider are:

(i) If $a^2 = 1$, then $a = 1$ or $a = -1$.

(ii) If $bc = 0$ then either $b = 0$ or $c = 0$.

(iii) If $d \neq 0$, then the system also contains an element e such that $d \cdot e = 1$ (so we are entitled to label e as d^{-1}).

(iv) Make your own!

Make a list of several such basic properties, noting which systems have which properties. Which of these systems would you consider to be the "most similar" algebraically? Why?

(3) The rational number $1/2$ simply refers to the unique rational number that when multiplied by 2 yields 1, i.e. it is a solution to the equation $2x = 1$ in $\mathbb{Q}$. We can similarly ask whether this equation is solvable in modular arithmetic. Can you solve the equation $2x = 1$ in $\mathbb{Z}/5$? In $\mathbb{Z}/6$? $\mathbb{Z}/10$? $\mathbb{Z}/11$? $\mathbb{Z}/13$? In each case, if there is a unique solution, we are entitled to call it the element "1/2" in the number system under consideration.

(4) Find the following other elements in $\mathbb{Z}/5$: $2/3, 3/4, \sqrt{-1}$. Much of your task here is to understand precisely what is being asked! Compare with the previous problem.

(5) (***HW***) Which of these elements $2/3, 3/4, \sqrt{-1}$ can you find in $\mathbb{Z}/6$? $\mathbb{Z}/10$? $\mathbb{Z}/11$?, $\mathbb{Z}/13$?

(6) You know that 2, 3, 5, 7, 11 and 13 are all prime in $\mathbb{Z}$. In $\mathbb{Z}[i]$, 5 factors as $5 = (2 + i)(2 - i)$. Which of these other primes factor in $\mathbb{Z}[i]$? Which ones "remain prime"?

(7) Which integers can be expressed in the form $2x + 3y$ where x and y are integers? What about in the form $4x + 6y$?