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CPTS 453

Graph Theory

11-20-2019

Homework 6

1.

Let G be a K-regular graph and let D be a degree matrix so that

$$D_{ij} = \begin{cases} \deg(v_i), & i = j \\ 0, & i \neq j \end{cases}$$

Then let D be a scalar matrix with all diagonal entries, k. Then let suppose x is an Eigen vector which corresponds to the Eigen value d of A which is

$$Ax = dx$$

Then suppose we have the Laplacian matrix

$$L = D - A$$

Then we have

$$Lx = (D - A)x$$

$$Lx = Dx - Ax$$

$$Lx = kx - dx$$

$$Lx = (k - d)x$$

Note that

$Ax = dx$ And $Dx = kx$

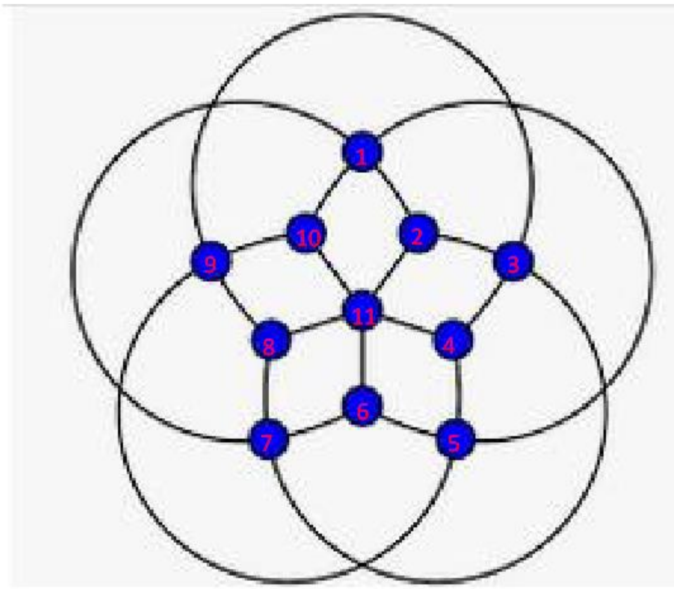
Because we let D be a scalar matrix with all diagonals being equal to k.

Therefore x is an Eigen vector of L that corresponds to the Eigen value $k - d$.

Then we can see that

$$Dx = \begin{bmatrix} k & 0 & . & . & 0 \\ 0 & k & . & . & . \\ . & . & & & . \\ . & . & & & . \\ 0 & . & . & . & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \\ . \\ . \\ kx_n \end{bmatrix} = k \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ . \\ x_n \end{bmatrix} = kx$$

2.




Laplacian Matrix: $L = D - A$

D is the Degree Matrix and A is the Adjacency Matrix

$$D = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$
L = \begin{bmatrix}
4 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\
-1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & -1 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 \\
0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 5
\end{bmatrix}
$$

3.

A)

http://bluebit.gr/matrix-calculator/

**Input matrix:**

```
 2.000 -1.000  0.000 -1.000  0.000  0.000  0.000
-1.000  2.000 -1.000  0.000  0.000  0.000  0.000
 0.000 -1.000  3.000 -1.000  0.000  0.000 -1.000
-1.000  0.000 -1.000  3.000 -1.000  0.000  0.000
 0.000  0.000  0.000 -1.000  2.000 -1.000  0.000
 0.000  0.000  0.000  0.000 -1.000  2.000 -1.000
 0.000  0.000 -1.000  0.000  0.000 -1.000  2.000
```

**Eigenvalues Eigenvectors:**

**Eigenvalues:**

```
(4.879,0.000i)
(0.000,0.000i)
(3.802,0.000i)
(0.753,0.000i)
(1.468,0.000i)
(2.445,0.000i)
(2.653,0.000i)
```

Graph A

**Eigenvectors:**

```
(-0.318, 0.000i) ( 0.378, 0.000i) ( 0.119, 0.000i) ( 0.482, 0.000i) ( 0.207, 0.000i) ( 0.333, 0.000i) (-0.597, 0.000i)
( 0.318, 0.000i) ( 0.378, 0.000i) ( 0.119, 0.000i) ( 0.482, 0.000i) (-0.207, 0.000i) ( 0.333, 0.000i) ( 0.597, 0.000i)
(-0.597, 0.000i) ( 0.378, 0.000i) (-0.333, 0.000i) ( 0.119, 0.000i) (-0.318, 0.000i) (-0.482, 0.000i) ( 0.207, 0.000i)
( 0.597, 0.000i) ( 0.378, 0.000i) (-0.333, 0.000i) ( 0.119, 0.000i) ( 0.318, 0.000i) (-0.482, 0.000i) (-0.207, 0.000i)
(-0.207, 0.000i) ( 0.378, 0.000i) ( 0.482, 0.000i) (-0.333, 0.000i) ( 0.597, 0.000i) (-0.119, 0.000i) ( 0.318, 0.000i)
( 0.000, 0.000i) ( 0.378, 0.000i) (-0.535, 0.000i) (-0.535, 0.000i) ( 0.000, 0.000i) ( 0.535, 0.000i) ( 0.000, 0.000i)
( 0.207, 0.000i) ( 0.378, 0.000i) ( 0.482, 0.000i) (-0.333, 0.000i) (-0.597, 0.000i) (-0.119, 0.000i) (-0.318, 0.000i)
```

Equal

**Input matrix:**

```
 2.000  0.000  0.000  0.000 -1.000  0.000 -1.000
 0.000  3.000 -1.000  0.000 -1.000  0.000 -1.000
 0.000 -1.000  2.000  0.000  0.000 -1.000  0.000
 0.000  0.000  0.000  2.000  0.000 -1.000 -1.000
-1.000 -1.000  0.000  0.000  2.000  0.000  0.000
 0.000  0.000 -1.000 -1.000  0.000  2.000  0.000
-1.000 -1.000  0.000 -1.000  0.000  0.000  3.000
```

**Eigenvalues Eigenvectors:**

**Eigenvalues:**

```
(4.879,0.000i)
(0.000,0.000i)
(3.802,0.000i)
(0.753,0.000i)
(1.468,0.000i)
(2.445,0.000i)
(2.653,0.000i)
```

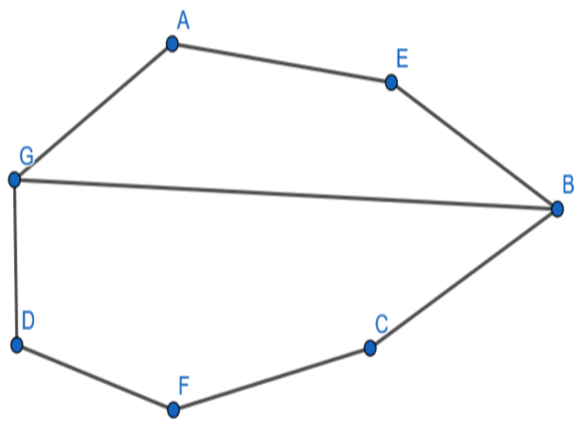
Graph B

**Eigenvectors:**

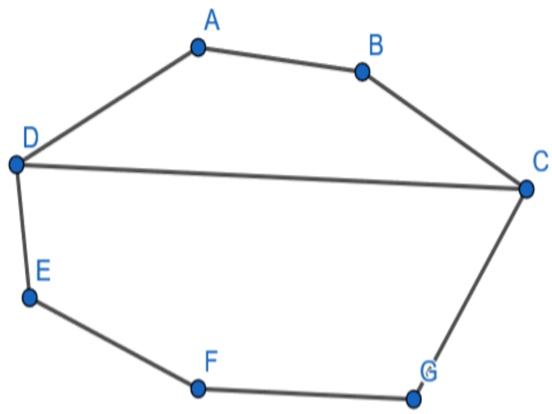
```
(-0.318, 0.000i) (-0.378, 0.000i) (-0.119, 0.000i) ( 0.482, 0.000i) ( 0.207, 0.000i) (-0.333, 0.000i) ( 0.597, 0.000i)
(-0.597, 0.000i) (-0.378, 0.000i) ( 0.333, 0.000i) ( 0.119, 0.000i) (-0.318, 0.000i) ( 0.482, 0.000i) (-0.207, 0.000i)
( 0.207, 0.000i) (-0.378, 0.000i) (-0.482, 0.000i) (-0.333, 0.000i) (-0.597, 0.000i) ( 0.119, 0.000i) ( 0.318, 0.000i)
(-0.207, 0.000i) (-0.378, 0.000i) (-0.482, 0.000i) (-0.333, 0.000i) ( 0.597, 0.000i) ( 0.119, 0.000i) (-0.318, 0.000i)
( 0.318, 0.000i) (-0.378, 0.000i) (-0.119, 0.000i) ( 0.482, 0.000i) (-0.207, 0.000i) (-0.333, 0.000i) (-0.597, 0.000i)
( 0.000, 0.000i) (-0.378, 0.000i) ( 0.535, 0.000i) (-0.535, 0.000i) ( 0.000, 0.000i) (-0.535, 0.000i) ( 0.000, 0.000i)
( 0.597, 0.000i) (-0.378, 0.000i) ( 0.333, 0.000i) ( 0.119, 0.000i) ( 0.318, 0.000i) ( 0.482, 0.000i) ( 0.207, 0.000i)
```

B)

Graph A




Graph B




To check whether or not graph A and graph B are isomorphic we need to check the degree of every vertex on both graphs. Both graphs have 7 vertices.

**Degrees for Graph A**

$$\deg(A_A) = 2 \quad \deg(B_A) = 3 \quad \deg(C_A) = 2 \quad \deg(D_A) = 2$$

$$\deg(E_A) = 2 \quad \deg(F_A) = 2 \quad \deg(G_A) = 3$$

**Degrees for Graph B**

$$\deg(A_B) = 2 \quad \deg(B_B) = 2 \quad \deg(C_B) = 3 \quad \deg(D_B) = 3$$

$$\deg(E_B) = 2 \quad \deg(F_B) = 2 \quad \deg(G_B) = 2$$

**Match Vertices of Graph A and Graph B**

$$A_A = A_B, E_A = B_B, B_A = C_B, C_A = G_B, F_A = F_B, D_A = E_B, G_A = D_B$$

Incidence relation is preserved and therefore graph A and graph B are in fact isomorphic.

4.

Since it's a path we know that the adjacency matrix number of ones in symmetric positions is 2n. Since this is true then it is known that the adjacency matrix of the path $P_n > 2$.

First we have

$$A_n = \begin{bmatrix} 0 & 1 & 0 & \dots & \dots \\ 1 & 0 & 1 & \dots & \dots \\ 0 & 1 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

$$A_n - sI_n = \begin{bmatrix} -s & 1 & 0 & \dots & \dots \\ 1 & -s & 1 & \dots & \dots \\ 0 & 1 & -s & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

$$\begin{aligned} \det(A_n - sI_n) &= \det \begin{bmatrix} -s & 1 & 0 & \dots & \dots \\ 1 & -s & 1 & \dots & \dots \\ 0 & 1 & -s & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}_n \\ &= -s \ \det \begin{bmatrix} -s & 1 & 0 & \dots & \dots \\ 1 & -s & 1 & \dots & \dots \\ 0 & 1 & -s & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{n-1} + \det \begin{bmatrix} 1 & 1 & \dots & \dots & \dots \\ 0 & -s & 1 & \dots & \dots \\ 0 & 1 & -s & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{n-1} \\ &= -s \ \det \begin{bmatrix} -s & 1 & 0 & \dots & \dots \\ 1 & -s & 1 & \dots & \dots \\ 0 & 1 & -s & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{n-1} - \det \begin{bmatrix} 1 & 1 & \dots & \dots & \dots \\ 0 & -s & 1 & \dots & \dots \\ 0 & 1 & -s & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}_{n-1} \\ &= -s \det(A_{n-1} - sI_{n-1}) - \det(A_{n-2} - sI_{n-2}) \end{aligned}$$

This shows that $n \geq 3$ , $f_n(s) = -s\, f_{n-1}(s) - f_{n-2}(s)$.