Numerical calculation of a radiation spectrum emitted by a charge and storing it in the form of the Stokes parameters

1 Theory of radiation

For references see [1, §66], and also [2, §14.5]. We use the units of $c = 1$.

- Maxwell equation $\partial_\nu F^{\mu\nu} = -4\pi j^\mu$ upon using the Lorentz gauge $\partial_\nu A^\nu = 0$ takes the form

$$\partial_\nu \partial^\nu A^\mu = 4\pi j^\mu. \tag{1}$$

- Retarded Green's function:

$$D_R(z) = -\frac{1}{4\pi|\mathbf{z}|}\delta(z^0 - |\mathbf{z}|), \quad \partial^2 D_R(x-y) = -\delta^{(4)}(x-y). \tag{2}$$

- $\Rightarrow$ Retarded vector potential is

$$\mathbf{A}(t,\mathbf{r}) = \int d^3r' \, \frac{\mathbf{j}(t-R,\mathbf{r}')}{R}, \quad \mathbf{R} = \mathbf{r} - \mathbf{r}', \quad R = |\mathbf{R}|. \tag{3}$$

- For a *distant* radiation at the point $\mathbf{R}_0$ we have $R = |\mathbf{R}_0 - \mathbf{r}| \approx R_0 - \mathbf{n}\cdot\mathbf{r}$ and

$$\mathbf{A}(t,\mathbf{R}_0) \approx \frac{1}{R_0}\int d^3r \, \mathbf{j}(t - R_0 + \mathbf{n}\cdot\mathbf{r}, \, \mathbf{r}), \quad \mathbf{n} \equiv \frac{\mathbf{R}_0}{R_0}. \tag{4}$$

  $\Rightarrow \mathbf{H} = \boldsymbol{\nabla}\times\mathbf{A} \approx \dot{\mathbf{A}}\times\mathbf{n}$ and in a plane wave $\mathbf{E} = \mathbf{H}\times\mathbf{n}$, so

$$\mathbf{E} \approx \mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{A}}), \quad \dot{\mathbf{A}} \equiv \frac{\partial \mathbf{A}}{\partial t}. \tag{5}$$

- For a Fourier transform $\mathbf{A}(t,\mathbf{R}_0) = \frac{1}{2\pi}\int d\omega \, e^{-i\omega t}\mathbf{A}_\omega(\mathbf{R}_0)$ we have

$$\mathbf{A}_\omega(\mathbf{R}_0) = \frac{e^{i\omega R_0}}{R_0}\int d^3r \, \mathbf{j}_\omega(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{k} \equiv \omega\mathbf{n}, \tag{6}$$

  where $\mathbf{j}_\omega(\mathbf{r}) = \int dt \, e^{i\omega t}\mathbf{j}(t,\mathbf{r})$, and for a point particle $\mathbf{j}(t,\mathbf{r}) = e\mathbf{v}(t)\delta(\mathbf{r} - \mathbf{r}_0(t))$, where $\mathbf{r}_0(t)$ is the particle's trajectory and $\mathbf{v}(t)$ — its velocity, we get

$$\mathbf{j}_\omega(\mathbf{r}) = e\int dt \, \mathbf{v}(t) e^{i\omega t}\delta(\mathbf{r} - \mathbf{r}_0(t)), \tag{7}$$

  so that

$$\mathbf{A}_\omega(\mathbf{R}_0) = e\frac{e^{i\omega R_0}}{R_0}\int dt \, \mathbf{v}(t) e^{i\omega[t - \mathbf{n}\cdot\mathbf{r}_0(t)]}. \tag{8}$$

- Apparently, $\big[\dot{\mathbf{A}}\big]_\omega \equiv \dot{\mathbf{A}}_\omega = -i\omega\mathbf{A}_\omega$, so

$$\mathbf{E}_\omega(\mathbf{R}_0) = -i\omega\mathbf{n}\times\big(\mathbf{n}\times\mathbf{A}_\omega(\mathbf{R}_0)\big) = -i\omega e\frac{e^{i\omega R_0}}{R_0}\int dt\,\mathbf{n}\times\big(\mathbf{n}\times\mathbf{v}(t)\big)e^{i\omega[t-\mathbf{n}\cdot\mathbf{r}_0(t)]}. \tag{9}$$

- Poynting's vector for a plane wave is

$$\mathbf{S} = \frac{1}{4\pi}\mathbf{E}\times\mathbf{H} = \frac{1}{4\pi}E^2\mathbf{n}. \tag{10}$$

It's the energy per unit area per unit time, so the intensity (energy per unit time) per unit solid angle is

$$dI = \frac{E^2}{4\pi}R_0^2\,d\Omega\,, \tag{11}$$

$\Rightarrow$ radiated energy for a whole time per unit solid angle is

$$d\mathcal{E} = R_0^2\,d\Omega\int\limits_{-\infty}^{+\infty}dt\,\frac{E^2}{4\pi} = R_0^2\,d\Omega\cdot 2\int\limits_0^\infty\frac{d\omega}{2\pi}\frac{|\mathbf{E}_\omega|^2}{4\pi}, \tag{12}$$

where the Parceval's theorem is used. Hence, the radiation spectrum is

$$d\mathcal{E}_{\mathbf{n}\omega} = \frac{|\mathbf{E}_\omega|^2}{2\pi}R_0^2\,d\Omega\,\frac{d\omega}{2\pi}. \tag{13}$$

- So finally, for the radiation spectrum of a charge we have

$$\frac{d\mathcal{E}_{\mathbf{n}\omega}}{d\Omega\,d\omega} = \frac{e^2\omega^2}{4\pi^2}\left|\int dt\,\mathbf{n}\times\big(\mathbf{n}\times\mathbf{v}(t)\big)e^{i\omega[t-\mathbf{n}\cdot\mathbf{r}_0(t)]}\right|^2, \tag{14}$$

where the phase factor $-ie^{i\omega R_0}$ was omitted under the sign of the absolute value.

- If we recall (6) that $\mathbf{k}\equiv\omega\mathbf{n}$, such that $d\Omega\,\omega^2\,d\omega = d^3k$, we can write (14) also in the form

$$\frac{d\mathcal{E}_{\mathbf{n}\omega}}{d^3k} = \frac{e^2}{4\pi^2}\left|\int dt\,\mathbf{n}\times\big(\mathbf{n}\times\mathbf{v}(t)\big)e^{i\omega[t-\mathbf{n}\cdot\mathbf{r}_0(t)]}\right|^2. \tag{15}$$

2 Stokes parameters

- For a monochromatic plane wave of a general polarization

$$\mathbf{E}(t,\mathbf{r}) = (\mathbf{e}_1E_1+\mathbf{e}_2E_2)e^{-i(\omega t-\mathbf{k}\cdot\mathbf{r})}, \tag{16}$$

where $\mathbf{e}_1\perp\mathbf{e}_2$ and $\mathbf{e}_1\times\mathbf{e}_2=\mathbf{n}=\mathbf{k}/\omega$, and E_1 and E_2 are complex in the general case.

- Intensity $I\sim|\mathbf{E}|^2 = |E_1|^2+|E_2|^2$, but we can't restore the information about polarization, phase, etc. from I. However, we can do that from the so called Stokes parameters [2, §7.2]:

$$\begin{aligned}
S_0 &= |E_1|^2+|E_2|^2\sim I,\\
S_1 &= |E_1|^2-|E_2|^2,\\
S_2 &= 2\,\mathrm{Re}[E_1^*E_2],\\
S_3 &= 2\,\mathrm{Im}[E_1^*E_2] = -2\,\mathrm{Im}[E_1E_2^*].
\end{aligned} \tag{17}$$

- Due to the form of S_i, they do not depend on the phase factor $-ie^{i\omega R_0}$ from $\mathbf{E}_\omega$, so we can omit it here too.

- The plane wave is transverse and this is true for a distant radiation, as you can see from (5), as $\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{A}})\perp\mathbf{n}$. Therefore, you can always consider it as a 2D vector $(E_1,\,E_2)$ in a plane transverse to $\mathbf{n}$.

- If we denote
$$\mathbf{J}\equiv\frac{e\omega}{2\pi}\int dt\,\mathbf{v}(t)e^{i\omega[t-\mathbf{n}\cdot\mathbf{r}(t)]},\tag{18}$$
which is proportional to $\mathbf{E}_\omega$, see (9), then $\mathbf{n}\times(\mathbf{n}\times\mathbf{J})\perp\mathbf{n}$ and we can describe it as a 2D vector $\mathbf{J}_\perp\equiv(J_1,\,J_2)$ in a plane transverse to $\mathbf{n}$. We can treat $|J_1|^2+|J_2|^2$, which is exactly the RHS of (14), as the first Stokes parameter S_0, because it has the meaning of intensity of an ω-harmonics. Therefore, we can use the following Stokes parameters instead of (17):
$$\begin{aligned}\widetilde{S}_0&=|J_1|^2+|J_2|^2=\frac{d\mathcal{E}_{\mathbf{n}\omega}}{d\Omega\,d\omega},\\ \widetilde{S}_1&=|J_1|^2-|J_2|^2,\\ \widetilde{S}_2&=2\,\mathrm{Re}[J_1^*J_2],\\ \widetilde{S}_3&=2\,\mathrm{Im}[J_1^*J_2]=-2\,\mathrm{Im}[J_1J_2^*].\end{aligned}\tag{19}$$

- Now we connect J_1 and J_2, and ultimately the Stokes parameters (19), with the components of $\mathbf{J}$ and the viewing angles θ, φ.

- $\mathbf{n}\times(\mathbf{n}\times\mathbf{J})=\mathbf{n}(\mathbf{n}\cdot\mathbf{J})-\mathbf{J}=-(\mathbf{J}-J_\parallel\mathbf{n})=-\mathbf{J}_\perp$, where $J_\parallel\equiv\mathbf{n}\cdot\mathbf{J}$, hence
$$\mathbf{J}_\perp=-\mathbf{n}\times(\mathbf{n}\times\mathbf{J}).\tag{20}$$

- We choose $(\mathbf{e}_1,\,\mathbf{e}_2)$ such that $\mathbf{e}_2\perp\mathbf{e}_z$ (in other words, $\mathbf{e}_2$ is in the xy-plane), where z is some preferential direction.

- $\mathbf{n}=\big(\sin\theta\cos\varphi,\,\sin\theta\sin\varphi,\,\cos\theta\big)$,
$$\begin{aligned}&\Rightarrow\mathbf{e}_2=\frac{\mathbf{n}\times\mathbf{e}_z}{\sin\theta}=\frac{1}{\cancel{\sin\theta}}\big(\cancel{\sin\theta}\sin\varphi,\,-\cancel{\sin\theta}\cos\varphi,\,0\big)=\big(\sin\varphi,\,-\cos\varphi,\,0\big),\\ &\Rightarrow\mathbf{e}_1=\mathbf{e}_2\times\mathbf{n}=\big(-\cos\theta\cos\varphi,\,-\cos\theta\sin\varphi,\,\sin\theta\big).\end{aligned}$$
It's easy to explicitly check that indeed $\mathbf{n}\cdot\mathbf{e}_1=\mathbf{n}\cdot\mathbf{e}_2=\mathbf{e}_1\cdot\mathbf{e}_2=0$ and $\mathbf{n}=\mathbf{e}_1\times\mathbf{e}_2$.

- $\Rightarrow J_1=\mathbf{J}_\perp\cdot\mathbf{e}_1=(\mathbf{J}-\cancelto{0}{J_\parallel\mathbf{n}})\cdot\mathbf{e}_1=\mathbf{J}\cdot\mathbf{e}_1$ and similarly $J_2=\mathbf{J}\cdot\mathbf{e}_2$.

- So finally
$$\begin{aligned}J_1&=\mathbf{J}\cdot\mathbf{e}_1=-(J_x\cos\varphi+J_y\sin\varphi)\cos\theta+J_z\sin\theta,\\ J_2&=\mathbf{J}\cdot\mathbf{e}_2=J_x\sin\varphi-J_y\cos\varphi.\end{aligned}\tag{21}$$

3 Numerical calculation of J

- Each component of (18) has the form

$$\mathcal{J} = \omega \int\limits_a^b dt\, f(t) e^{i\omega\xi(t)}, \quad \xi(t) \equiv t - \mathbf{n} \cdot \mathbf{r}(t). \tag{22}$$

In $\mathbf{J}$ the limits $(a,\, b) = (-\infty,\, +\infty)$, but for numerical calculation we divide the whole interval into small steps, and so here $\mathcal{J}$ is an integral for such a step. However, if the integrand highly oscillates, the usual formula

$$\mathcal{J} \approx \omega f(\bar{t}) e^{i\omega\xi(\bar{t})} \Delta t, \quad \Delta t \equiv b - a, \quad \bar{t} \equiv \frac{a+b}{2} \tag{23}$$

is very inaccurate, unless Δt is small enough such that integrand does not vary rapidly on $(a,\, b)$. But it may be impractical to have such a small step of integration for high frequencies of oscillation because usually $f(t)$ and $\xi(t)$ demand a much less step for calculating them with good enough precision (i. e. for solving numerically the equation of motion). So we may use linear approximation for $f(t)$ and $\xi(t)$ on the interval $(a,\, b)$ and calculate the integral analytically. Moreover, without further assumptions the linear approximation looks the most reasonable because usually we know only the initial and final values f_a and f_b for each step of integration.

- Expanding $f(t)$ and $\xi(t)$ to the linear term, we obtain:

$$\mathcal{J} = \omega \int\limits_0^{\Delta t} dt\, f(a+t) e^{i\omega\xi(a+t)} \approx e^{i\omega\xi(a)} \omega \int\limits_0^{\Delta t} dt\, \big(f(a) + f'(a)t\big) e^{i\omega\xi'(a)t} =$$

$$= e^{i\omega\bar{\xi}} \frac{1}{\xi'(a)} \left\{ 2\overline{f} \sin\frac{\widetilde{\omega}\Delta t}{2} + i f'(a)\Delta t \left[\frac{\sin\frac{\widetilde{\omega}\Delta t}{2}}{\frac{\widetilde{\omega}\Delta t}{2}} - \cos\frac{\widetilde{\omega}\Delta t}{2} \right] \right\}, \tag{24}$$

where $\widetilde{\omega} \equiv \omega\xi'(a) = \omega(1 - \mathbf{n} \cdot \mathbf{v}(a))$ is the frequency of oscillation of the approximated integrand, $\overline{f} \equiv f(a) + f'(a)\Delta t/2$ is the mean value of linearly approximated $f(t)$ on the integration interval, and $\bar{\xi}$ has the same meaning as $\overline{f}$.

- For derivatives and mean values of $f(t)$ and $\xi(t)$ we can use the formulae

$$f'(a) = \frac{f_b - f_a}{b - a} \equiv \frac{\Delta f}{\Delta t}, \quad \overline{f} = f_a + \frac{\Delta f}{\cancel{\Delta t}} \frac{\cancel{\Delta t}}{2} = \frac{f_a + f_b}{2}. \tag{25}$$

- Using (25) we have $\widetilde{\omega}\Delta t = \omega\Delta\xi$, and we can write (24) in the form

$$\mathcal{J} \approx e^{i\omega\bar{\xi}} \frac{\Delta t}{\Delta\xi} \left\{ 2\overline{f} \sin\frac{\omega\Delta\xi}{2} + i\Delta f \left[\frac{\sin\frac{\omega\Delta\xi}{2}}{\frac{\omega\Delta\xi}{2}} - \cos\frac{\omega\Delta\xi}{2} \right] \right\}. \tag{26}$$

- Note that if $\widetilde{\omega}(b-a) = \omega\Delta\xi \ll 1$, i. e. if $(a,\, b)$ is much less than one period of oscillation of the integrand, then (26) becomes the usual formula for integrating "smooth" functions:

$$\mathcal{J} \approx \omega \overline{f} e^{i\omega\bar{\xi}} \Delta t, \tag{27}$$

where instead of $f(\bar{t})$ and $\xi(\bar{t})$, as in (23), we have $\overline{f}$ and $\bar{\xi}$.

- For high frequencies $\omega\Delta\xi \gg 1$ (26) works well because it's in nature an asymptotic expansion with respect to $1/\omega\Delta\xi \ll 1$ (terms $\sim 1/(\omega\Delta\xi)^2$ and higher do not appear because of the linear approximation of $f(t)$ and $\xi(t)$).

- Finally, we can write that for each step of integration of $\mathbf{J}$ we have

$$e^{i\omega\bar{\xi}}\frac{\Delta t}{\Delta\xi}\left\{2\overline{\mathbf{v}}\sin\frac{\omega\Delta\xi}{2}+i\Delta\mathbf{v}\left[\frac{\sin\frac{\omega\Delta\xi}{2}}{\frac{\omega\Delta\xi}{2}}-\cos\frac{\omega\Delta\xi}{2}\right]\right\}, \tag{28}$$

where $\overline{\mathbf{v}} \equiv (\mathbf{v}_a+\mathbf{v}_b)/2,\ \Delta\mathbf{v} \equiv \mathbf{v}_b-\mathbf{v}_a$.

4 Units of measure

- If we have some characteristic frequency (say, a laser frequency) ω_L or characteristic time t_L (a laser period), we can express time and frequency it terms of t_L and ω_L:

$$t = t_L\cdot\frac{t}{t_L} \equiv t_L\widetilde{t}, \quad \omega = \omega_L\cdot\frac{\omega}{\omega_L} \equiv \omega_L\widetilde{\omega}, \tag{29}$$

where $\widetilde{t}$ and $\widetilde{\omega}$ are dimensionless time and frequency.

- In order to avoid the introduction of 2π factors from $t_L = 2\pi/\omega_L$ we may use ordinary frequency ν_L instead of the angular one, such that $t_L\nu_L = 1$. Or we may use $\tau_L = 1/\omega_L = t_L/2\pi$, which is probably preferable in order to express $\widetilde{\omega}$ in units of ω_L, and not ν_L.

- So $t = \tau_L\cdot t/\tau_L \equiv \tau_L\widetilde{t} = (t_L/2\pi)\cdot\widetilde{t} = \widetilde{t}/\omega_L$, therefore

$$\widetilde{t} = \omega_L t, \quad \widetilde{\omega} = \frac{\omega}{\omega_L}. \tag{30}$$

- Due to $c = 1$, length is measured in the same units as time, so for example $x = \widetilde{x}/\omega_L$, or

$$\widetilde{x} = \omega_L x. \tag{31}$$

- So $\omega\xi = \omega(t-\mathbf{n}\cdot\mathbf{r}) = \widetilde{\omega}(\widetilde{t}-\mathbf{n}\cdot\widetilde{\mathbf{r}}) = \widetilde{\omega}\widetilde{\xi}$ doesn't change, and so does the following:

$$\frac{\Delta t}{\Delta\xi} = \frac{\Delta\widetilde{t}}{\Delta\widetilde{\xi}}, \quad \omega\Delta\xi = \widetilde{\omega}\Delta\widetilde{\xi}, \quad \omega\bar{\xi} = \widetilde{\omega}\bar{\widetilde{\xi}}. \tag{32}$$

Hence we do not need to change (28) at all in order to perform integration with respect to dimensionless variables and parameters $\widetilde{t}$, $\widetilde{\omega}$, etc.

- We may measure charges in the units of the elementary charge e (i. e. the absolute value of the electron charge $-e$). And if we plan to have charges q of different values, then we can write $q = e\widetilde{q}$, where $\widetilde{q}$ may be $\pm1, \pm2$, etc.

- From applying (30) to (14) we see that radiated energy $d\mathcal{E}_{\mathbf{n}\omega}$ is measured in units of $e^2\omega_L$.

- Similarly, the unit of $\mathbf{E}$ for radiation is $e\omega_L^2$, and the unit of $\mathbf{E}_\omega$ is $e\omega_L$ (due to $E \sim E_\omega \omega$).

- A charge dynamics includes the mass of a charge m, so me may measure it in the units of the electron mass m_e: $m = m_e \widetilde{m}$. The momentum $\mathbf{p} = m\mathbf{u} = m_e \widetilde{m}\mathbf{u}$, and we can write the equation for the spatial part of 4-velocity $\mathbf{u}$ in the form

$$\frac{d\mathbf{u}}{d\widetilde{t}} = \frac{e}{m_e \omega_L} \cdot \frac{\widetilde{q}}{\widetilde{m}} (\mathbf{E} + \mathbf{v} \times \mathbf{H}). \tag{33}$$

Hence $m_e \omega_L / e$ is a unit of measurement for the external field, which drives the charge dynamics (that's why there is m_e here, as opposed to the radiation field). This implies the usual definition of dimensionless amplitude of the wave field $a_0 \equiv eE/m_e \omega_L$. So

$$\frac{d\mathbf{u}}{d\widetilde{t}} = \frac{\widetilde{q}}{\widetilde{m}} (\widetilde{\mathbf{E}} + \mathbf{v} \times \widetilde{\mathbf{H}}), \quad \widetilde{\mathbf{E}} = \frac{e\mathbf{E}}{m_e \omega_L}, \quad \widetilde{\mathbf{H}} = \frac{e\mathbf{H}}{m_e \omega_L}. \tag{34}$$

If we introduce the dimensionless momentum $\widetilde{\mathbf{p}} = \widetilde{m}\mathbf{u}$ which accounts for the value of the charge mass, we can write (34) in the form

$$\frac{d\widetilde{\mathbf{p}}}{d\widetilde{t}} = \widetilde{q} (\widetilde{\mathbf{E}} + \mathbf{v} \times \widetilde{\mathbf{H}}). \tag{35}$$

5 Test analytical solutions of the equation of motion

Charge in a uniform constant magnetic field

- In the case of a uniform constant magnetic field (35) takes the form

$$\frac{d\widetilde{\mathbf{p}}}{d\widetilde{t}} = \widetilde{q} \mathbf{v} \times \widetilde{\mathbf{H}}, \quad \widetilde{\mathbf{H}} = (0,\, 0,\, \widetilde{H}). \tag{36}$$

- As $\mathbf{H}$ is constant, we don't have a characteristic frequency of a field. But we know that in such a field a charge moves in a helix with a frequency of transverse circular rotations $\omega_L = eH/m_e\gamma$, where γ is the Lorentz factor of the charge. So we may choose it as a characteristic frequency. Hence

$$\widetilde{H} = \frac{eH}{m_e \omega_L} = \gamma, \tag{37}$$

that is, in such units the amplitude of a dimensionless magnetic field $\widetilde{H}$ is the gamma-factor γ. On the other hand, $eH/m_e\omega_L = a_0$, therefore we put $a_0 = \gamma$ and we can't specify a_0 independently.

- Now, the solution:

$$\begin{aligned}
\widetilde{x} &= -\frac{v_{0\perp}\widetilde{m}}{\widetilde{q}} \cos \widetilde{q}\,\widetilde{t}, & v_x &= v_{0\perp} \sin \frac{\widetilde{q}\,\widetilde{t}}{\widetilde{m}}, & \widetilde{p}_x &= \widetilde{p}_{0\perp} \sin \frac{\widetilde{q}\,\widetilde{t}}{\widetilde{m}}, \\
\widetilde{y} &= \frac{v_{0\perp}\widetilde{m}}{\widetilde{q}} \sin \widetilde{q}\,\widetilde{t}, & v_y &= v_{0\perp} \cos \frac{\widetilde{q}\,\widetilde{t}}{\widetilde{m}}, & \widetilde{p}_y &= \widetilde{p}_{0\perp} \cos \frac{\widetilde{q}\,\widetilde{t}}{\widetilde{m}}, \\
\widetilde{z} &= \widetilde{z}_0 + v_{z0}\widetilde{t}, & v_z &= v_{z0}, & \widetilde{p}_x &= \widetilde{m}\gamma v_{z0},
\end{aligned} \tag{38}$$

where $v_{0\perp}$ is the (constant) velocity of transverse rotations and $\widetilde{p}_{0\perp} = \widetilde{m}\gamma v_{0\perp}$.

Charge in a uniform oscillating electric field

$$E_z(t) = E_0 \cos \omega_L t \;\Rightarrow\; \widetilde{E}_z = a_0 \cos \widetilde{t}, \quad a_0 = \frac{eE_0}{m_e \omega_L},$$

$$\widetilde{p}_x = \widetilde{p}_y = 0, \quad \frac{d\widetilde{p}_z}{d\widetilde{t}} = \widetilde{q} a_0 \cos \widetilde{t} \;\Rightarrow\; \widetilde{p}_z = \widetilde{q} a_0 \sin \widetilde{t},$$

$$\gamma = \sqrt{1 + \frac{\widetilde{q}^{\,2} a_0^2}{\widetilde{m}^2} \sin^2 \widetilde{t}} \;\Rightarrow\; v_z = \frac{\widetilde{p}_z}{\widetilde{m}\gamma} = \frac{\widetilde{q} a_0 \sin \widetilde{t}}{\sqrt{\widetilde{m}^2 + \widetilde{q}^{\,2} a_0^2 \sin^2 \widetilde{t}}},$$

$$d\widetilde{z} = -\frac{\widetilde{q} a_0 \, d \cos \widetilde{t}}{\sqrt{\widetilde{m}^2 + \widetilde{q}^{\,2} a_0^2 - \widetilde{q}^{\,2} a_0^2 \cos^2 \widetilde{t}}} \;\Rightarrow\; \widetilde{z} = -\arcsin\left(\frac{\widetilde{q} a_0}{\sqrt{\widetilde{m}^2 + \widetilde{q}^{\,2} a_0^2}} \cos \widetilde{t} \right).$$

References

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[2] J. D. Jackson. *Classical Electrodynamics*. John Wiley & Sons, New York, 3rd edition, 1999.