

DIGITAL LOGIC(H)

Lecture 1 Number Systems

2024 Fall

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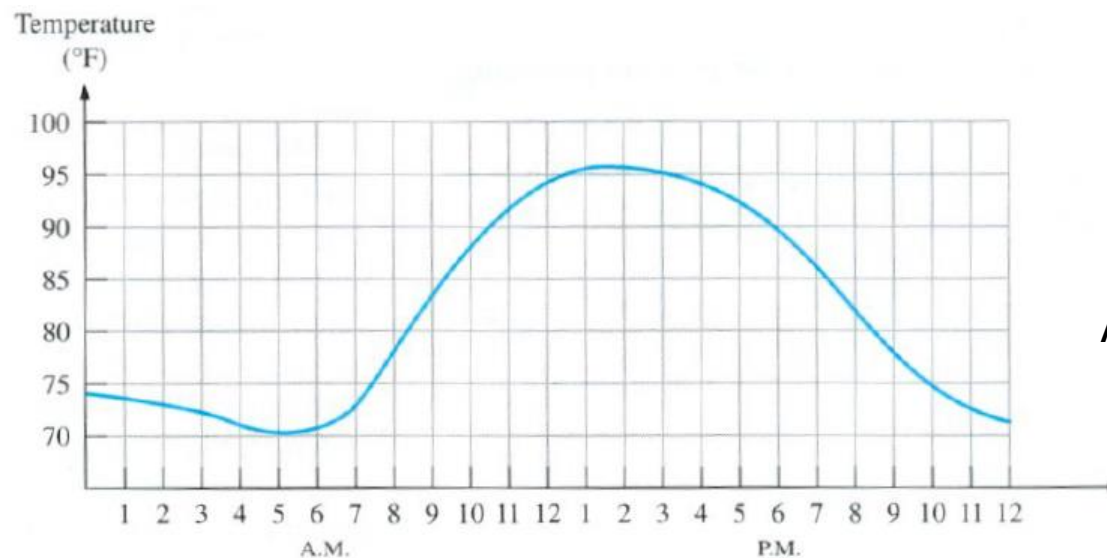


Outline

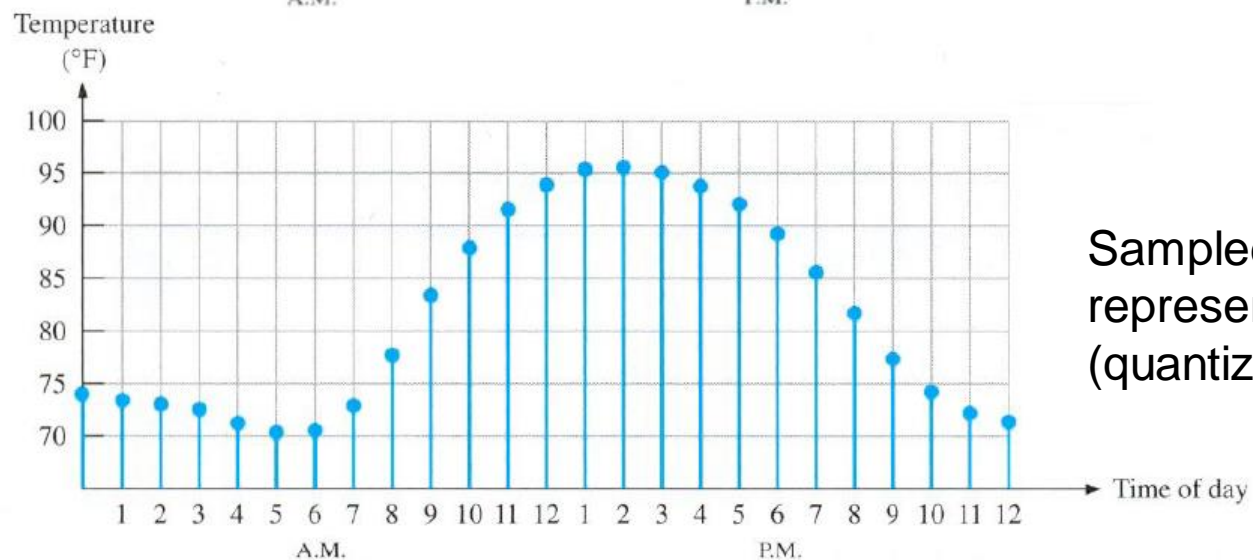
- Digital Number Systems
- Data Representation
- Binary Logic
- Reading: Textbook, Chapter 1

Analog vs. Digital

- Analog vs. digital
- continuous vs. discrete



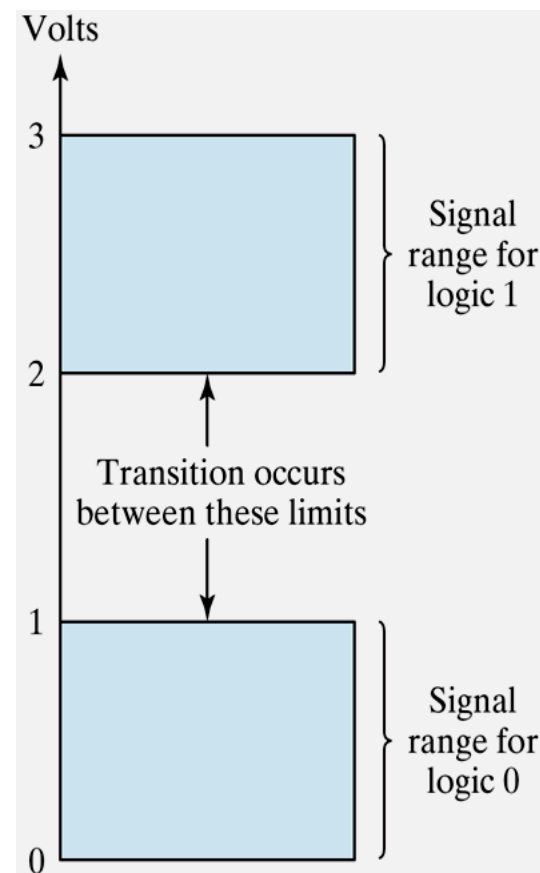
Analog quantity



Sampled-value
representation
(quantization)

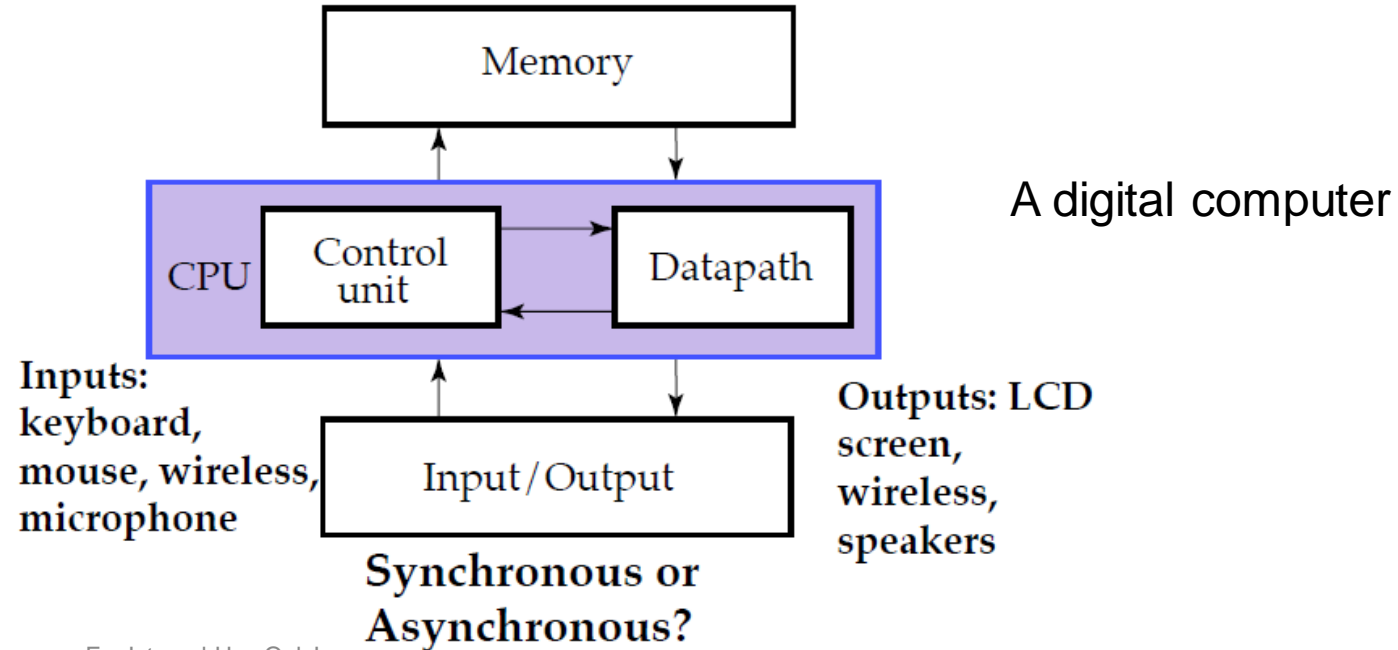
Binary Digits and Logic Levels

- Bit: binary digit
 - 1: HIGH (TRUE)
 - 0: LOW (FALSE)
- Codes: group of bits (combinations of 1s and 0s)
 - Used to represent numbers, letters, symbols, instructions, and anything else required in a given application
- Logic levels



Digital Systems

- A digital system is a system that processes digital signals or data. It operates on discrete values and performs operations such as logic, arithmetic, and data storage in a binary format.
- Digital systems are prevalent in modern electronics, including computers, smartphones, and digital communication devices, due to their reliability and ease of processing.



Common Number Systems

- It is natural for human to use **decimal system**(十进制)
- In a digital world, we think in **binary**(二进制)
- The **octal** (八进制) and **hexadecimal** (十六进制) numbers are shorter forms for representing binary numbers.

Decimal (base 10)	Binary (base 2)	Octal (base 8)	Hexadecimal (base 16)
0	0000	00	0x0
1	0001	01	0x1
2	0010	02	0x2
3	0011	03	0x3
4	0100	04	0x4
5	0101	05	0x5
6	0110	06	0x6
7	0111	07	0x7
8	1000	010	0x8
9	1001	011	0x9
10	1010	012	0xA
11	1011	013	0xB
12	1100	014	0xC
13	1101	015	0xD
14	1110	016	0xE
15	1111	017	0xF

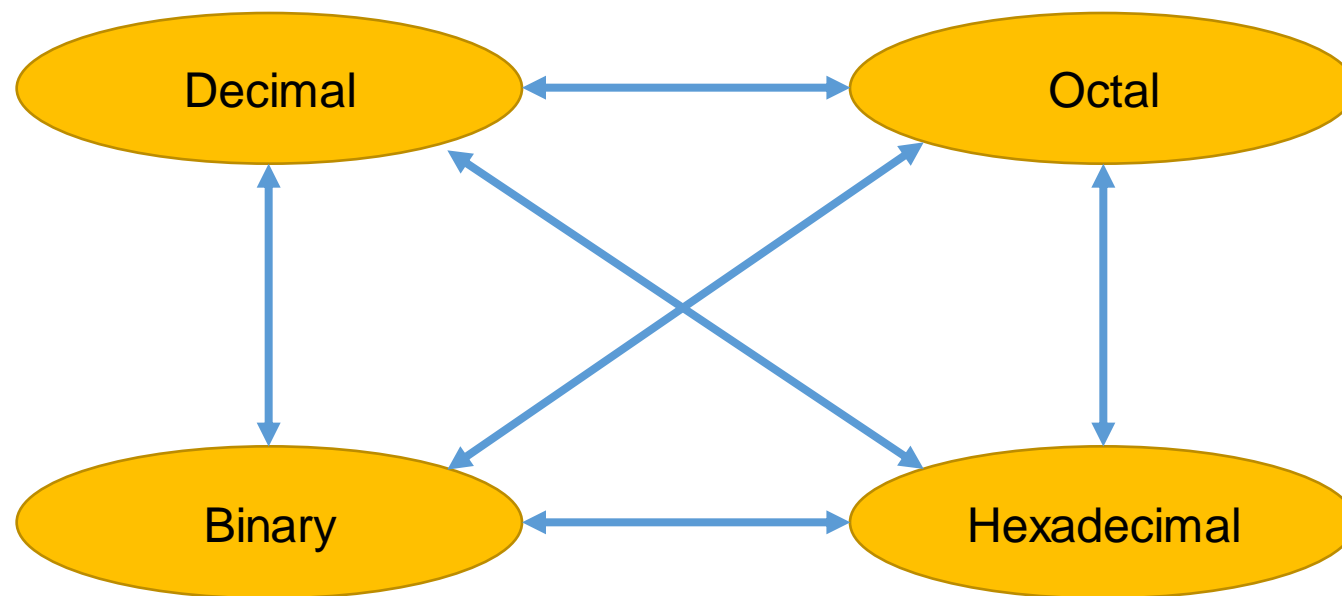
Conversion among Bases

- A quick example:

$$25_{10} = 11001_2 = 31_8 = 19_{16}$$

Base or Radix

- The possibilities:



Radix-r to Decimal Conversion

- We use Positional Number Systems: Let r be the **radix** (or **base**), then the $(n+m)$ -digit number

$$D = d_{n-1}d_{n-2} \dots d_1d_0.d_{-1}d_{-2}\dots d_{-m} \quad 0 \leq d < r$$

has the value

radix point

$$D = d_{n-1}r^{n-1} + d_{n-2}r^{n-2} + \dots + d_1r + d_0 + d_{-1}r^{-1} + d_{-2}r^{-2} + \dots + d_{-m}r^{-m}$$

Most-significant Digit (MSD)

Least-significant Digit (LSD)

$$D = \sum_{i=-m}^{n-1} d_i r^i$$

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Radix-r to Decimal Conversion

- **Decimal** Number System: Base (radix) $r = 10$

2 1 0 -1 -2

5 **1** **2** • **7** **4**

- Coefficients $D=(d_2d_1d_0.d_{-1}d_{-2}) = (512.74)_{10}$
- $(512.74)_{10} = 5 \times 10^2 + 1 \times 10^1 + 2 \times 10^0 + 7 \times 10^{-1} + 4 \times 10^{-2}$

- **Binary** Number System: Base (radix) $r = 2$

2 1 0 -1 -2

1 **0** **1** • **0** **1**

- Coefficients $D=(b_2b_1b_0.b_{-1}b_{-2}) = (101.01)_2$
- $(101.01)_2 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} = (5.25)_{10}$

Exercise:

$$1010.101_2 = ?_{10}$$

$$22.22_4 = ?_{10}$$

$$12.5_8 = ?_{10}$$

$$A.A_{16} = ?_{10}$$

Radix-r to Decimal Conversion

$$D = d_{n-1}r^{n-1} + d_{n-2}r^{n-2} + \dots + d_1r + d_0 + d_{-1}r^{-1} + d_{-2}r^{-2} + \dots + d_{-m}r^{-m}$$

Most-significant Digit (MSD)

Least-significant Digit (LSD)

Exercise:

$$1010.101_2 = 1*2^3 + 0*2^2 + 1*2^1 + 0*2^0 + 1*2^{-1} + 0*2^{-2} + 1*2^{-3} = 10.625_{10}$$

$$22.22_4 = 2*4^1 + 2*4^0 + 2*4^{-1} + 2*4^{-2} = 10.625_{10}$$

$$12.5_8 = 1*8^1 + 2*8^0 + 5*8^{-1} = 10.625_{10}$$

$$A.A_{16} = 10*16^0 + 10*16^{-1} = 10.625_{10}$$

Decimal to Radix-r Conversion

- Integer part: Successive divisions by r and observe the remainders
- Fraction: Successive multiplications by r and observe the integer part

Decimal to Binary Conversion (1)

- For Integer
- Divide the number by the 'Base' (=2)
- Take the remainder (either 0 or 1) as a coefficient
- Take the quotient and repeat the division

Example: $(13)_{10}$

	Quotient	Remainder	Coefficient
$13 / 2 =$	6	1	$a_0 = 1$
$6 / 2 =$	3	0	$a_1 = 0$
$3 / 2 =$	1	1	$a_2 = 1$
$1 / 2 =$	0	1	$a_3 = 1$

Answer: $(13)_{10} = (a_3 a_2 a_1 a_0)_2 = (1101)_2$

MSB

LSB

Decimal to Binary Conversion (2)

- For Fraction, the computation is reversed again
- Multiply the number by the 'Base' (=2)
- Take the integer (either 0 or 1) as a coefficient
- Take the resultant fraction and repeat the division

Example: $(0.625)_{10}$

	Integer	Fraction	Coefficient
$0.625 * 2 =$	1	. 25	$a_{-1} = 1$
$0.25 * 2 =$	0	. 5	$a_{-2} = 0$
$0.5 * 2 =$	1	. 0	$a_{-3} = 1$

Answer: $(0.625)_{10} = (0.a_{-1} a_{-2} a_{-3})_2 = (0.101)_2$

↑
MSB

↑
LSB

Decimal to Binary Conversion (3)

- Easier method

Example: $(35.375)_{10}$

2	35		
2	17	...	1
2	8	...	1
2	4	...	0
2	2	...	0
2	1	...	0
	0	...	1

	0.375		
x	2		
	0.750	...	0
x	2		
	1.50	...	1
x	2		
	1.0	...	1

- $(100011.011)_2$

Decimal to Octal Conversion

Example: $(175.3125)_{10}$

	Quotient	Remainder	Coefficient
$175 / 8 =$	21	7	$a_0 = 7$
$21 / 8 =$	2	5	$a_1 = 5$
$2 / 8 =$	0	2	$a_2 = 2$

Integer part: $(175)_{10} = (a_2 a_1 a_0)_8 = (257)_8$

	Integer	Fraction	Coefficient
$0.3125 * 8 =$	2	. 5	$a_{-1} = 2$
$0.5 * 8 =$	4	. 0	$a_{-2} = 4$

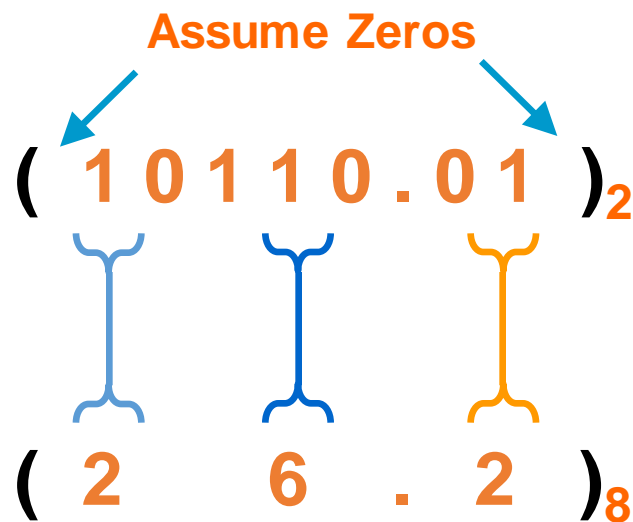
Fraction part: $(0.3125)_{10} = (0.a_{-1} a_{-2} a_{-3})_8 = (0.24)_8$

Answer: $(175.3125)_{10} = (a_2 a_1 a_0 . a_{-1} a_{-2} a_{-3})_8 = (257.24)_8$

Radix-r to Radix-r Conversion

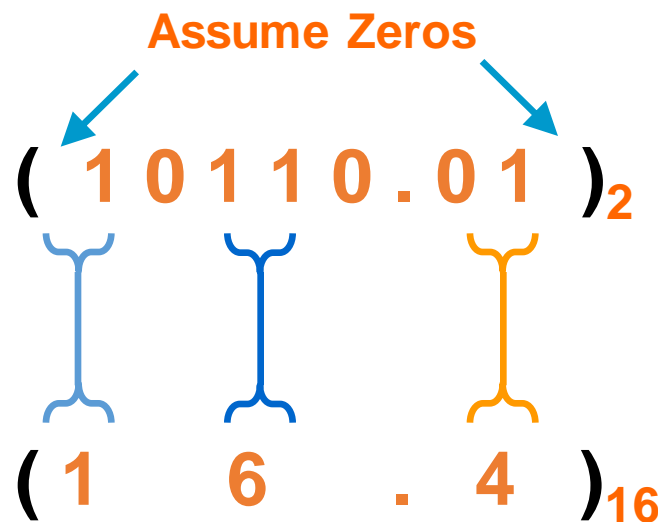
- Binary – Octal

- Each group of 3 bits represents an octal digit starting from radix point
- Works both ways (Binary to Octal & Octal to Binary)



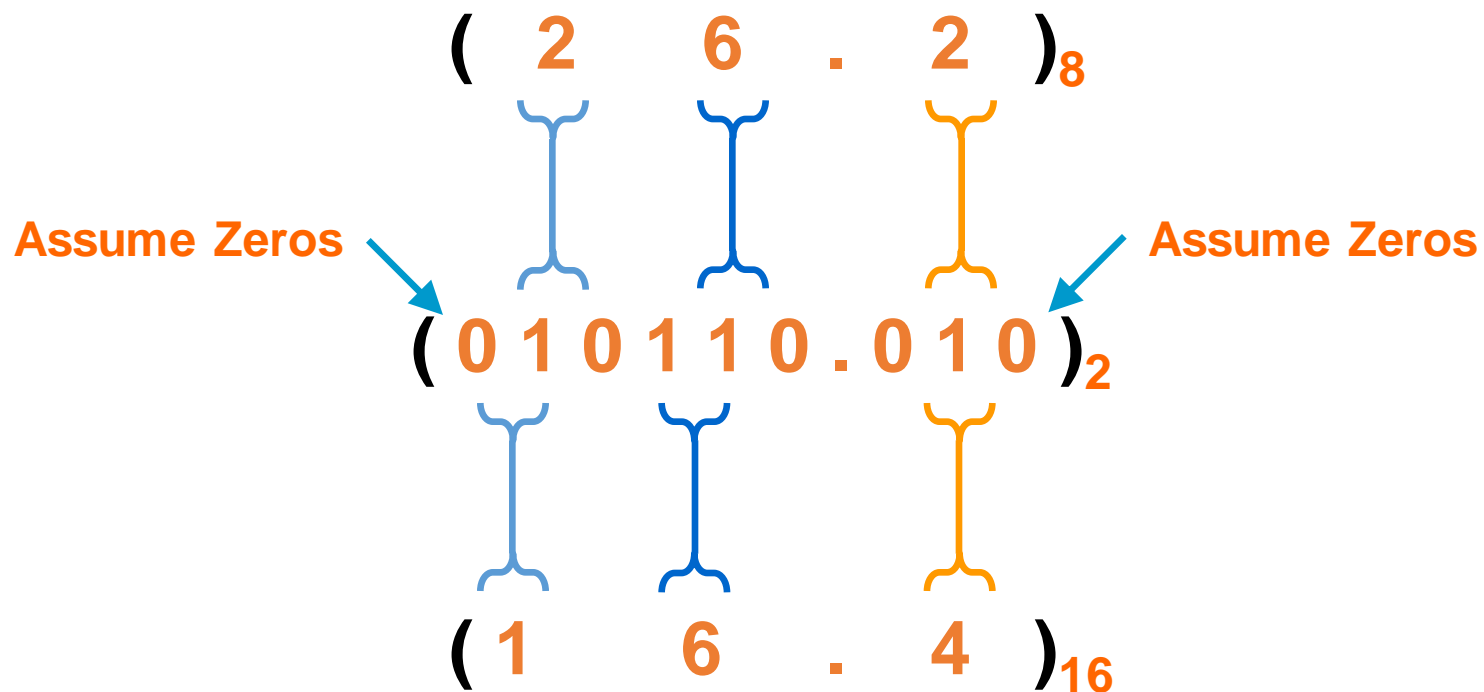
- Binary – Hexadecimal

- Each group of 4 bits represents a hexadecimal digit starting from radix point
- Works both ways (Octal to Hex & Hex to Octal)



Radix-r to Radix-r Conversion (2)

- Octal – Hexadecimal
- Convert to **Binary** as an intermediate step



Common Notions

- Bits(位/位元)

10010110

most significant bit least significant bit

- Bytes(字节)

- 1 byte = 8 bits

byte

10010110

- Bytes

- two hexadecimal digit represent one byte value

CEBF9AD7

most significant byte least significant byte

Power	Meaning	Prefix	Symbol
2^{10}	1024	Kilo	K
2^{20}	1024^2	Mega	M
2^{30}	1024^3	Giga	G
2^{40}	1024^4	Tera	T
2^{50}	1024^5	Peta	P
2^{60}	1024^6	Exa	E
2^{70}	1024^7	Zetta	Z

e.g. 1MB = 1024KB

Binary Addition

- Same rules as for decimal numbers
- Column Addition

						carry	
	1	1	1	1	1	1	$(10)_2 = (2)_{10}$
		1	1	1	1	0	1 = 61 augend
+			1	0	1	1	1 = 23 addend
<hr/>							
	1	0	1	0	1	0	0 = 84 sum

Binary Subtraction

- Same rules as for decimal numbers
- Borrow a “Base” when needed

				borrow				
		1		2			= (10) ₂	
	0	2	2	0	0	2		
	1	0	0	1	1	0	1	= 77 minuend
-			1	0	1	1	1	= 23 subtrahend
<hr/>								
	0	1	1	0	1	1	0	= 54 difference

Complements

- When human do subtraction, we use “borrow” concept to borrow a 1 from a higher significant position.
- It is hard for circuits to design “borrow”. So complements are used to implement subtraction.
 - Simplify the subtraction operation.
 - Simpler, less expensive circuits.
- Two types for radix-r system
 - Diminished radix complement (反码) $((r-1)$'s-complement)
 - Radix complement (补码) $(r$'s-complement)
- Examples:
 - For a binary system: 1's complement and 2's complement.
 - For a decimal system: 9's complement and 10's complement.

Complements for decimal system

- Diminished radix complement
 - 9's-complement of 540 = $999 - 540 = 459$
 - 9's-complement of 12 = $999 - 012 = 987$ (is it equal to 87?)
- Radix complement
 - 10's-complement of 540 = $1000 - 540 = 460$
 - 10's-complement of 12 = $1000 - 012 = 988$ (is it equal to 88?)
 - **Easier method 1:** Calculate the diminished radix complement, then plus one
 - 10's-complement of 540 = $999 - 540 + 1 = 460$
 - **Easier method 2:** use r minus the least significant non-zero digit, and $r - 1$ minus digits on the left
 - The least significant non-zero digit of 540 is 4: $10 - 4 = 6$;
 - Digits on the left is 5: $9 - 5 = 4$;
 - The 10's complement of 540 is 460.

Complements for binary system

- 1's Complement (*Diminished Radix Complement*) for binary

- All '0's become '1's
- All '1's become '0's
- just like a bitwise not(按位取反)

Example $(10110000)_2$

$\Rightarrow (01001111)_2$

- 2's Complement: 1's complement, then plus one:

- Another way: leave the first non-zero LSB unchanged, and then replacing 1's with 0's and 0's with 1's in the other MSBs:

$$\begin{array}{r} 10110000 \\ 01001111 \\ + 1 \\ \hline 01010000 \end{array}$$

10110000

first non-zero LSB

01010000

Subtraction with Complements

- subtraction of two k -digit unsigned numbers $M - N$
 - Replace subtraction with addition
 - M and N **both** have k -digit
- $M - N = M + r$'s complement of N
 - If $M \geq N$, the sum will produce an end **carry** r^k which is **discarded**, and what is left is the result $M - N$
 - If $M < N$, the sum does not produce an end carry. It is equal to the r 's complement of $(M - N)$. The correct answer is generated by
 1. taking the r 's **complement** of the answer
 2. then adding a **negative sign** to the front
- Pay attention to align the number of digits for two operands

Subtraction with 10's Complement

- Example with $M \geq N$
 - Using 10's complement, subtract $72532 - 3250$.
- Example with $M < N$
 - Using 10's complement, subtract $3250 - 72532$.

$$\begin{array}{r} M = 72532 \\ 10\text{'s complement of } N = +96750 \\ \hline \text{Sum} = 169282 \\ \text{Discard end carry } 10^5 = -100000 \\ \hline \text{Answer} = 69282 \end{array}$$

$$\begin{array}{r} M = 03250 \\ 10\text{'s complement of } N = +27468 \\ \hline \text{Sum} = 30718 \end{array}$$

There is **no end carry**.
Answer = $-(10\text{'s complement of } 30718)$
= -69282 .

Note: When working with paper and pencil, we can change the answer to a signed negative number in order to put it in a familiar form.

Subtraction with 2's Complement

- Example:

- Given the two binary numbers $X = 1010100$ and $Y = 1000011$ ($X > Y$), perform the subtraction (a) $X - Y$; and (b) $Y - X$, by using 2's complement.

(a)	$X =$	1010100
	2's complement of $Y =$	<u>+0111101</u>
	Sum =	10010001
	Discard end carry $2^7 =$	<u>-10000000</u>
	Answer. $X - Y =$	0010001

(b)	$Y =$	1000011
	2's complement of $X =$	<u>+ 0101100</u>
	Sum =	1101111



There is no end carry.

Answer. $Y - X$
 $= -$ (2's complement of 1101111)
 $= -$ 0010001.

Signed Binary Numbers

- In real life one may have to face a situation where both positive and negative numbers may arise.
 - We have + and -.
 - Digital systems represent everything with binary digits.
- Three types of representations of signed binary numbers:
 - Sign-magnitude representation
 - Signed-1's complement representation
 - Signed-2's complement representation
- In Signed binary system, the convention is to make the **sign bit (MSB) 0** for positive and **1** for negative.

Signed Binary Numbers

- Example, assume 9-bits number representation:
- $(105)_{10}$?
- $105_{10} = 1101001_2$, represent in 9 bits
 - Signed-magnitude representation of 105: **0**01101001
 - Signed-1's-complement representation of 105: **0**01101001
 - Signed-2's-complement representation of 105: **0**01101001
- $(-105)_{10}$?
- Magnitude of -105 is 1101001, represent in 9 bits
 - Signed-magnitude representation of -105: **1**01101001
 - Signed-1's-complement representation of -105: **1**10010110
 - Signed-2's-complement representation of -105: **1**10010111

S	7	6	5	4	3	2	1	0
0	0	1	1	0	1	0	0	1
0	0	1	1	0	1	0	0	1
0	0	1	1	0	1	0	0	1

S	7	6	5	4	3	2	1	0
1	0	1	1	0	1	0	0	1
1	1	0	0	1	0	1	1	0
1	1	0	0	1	0	1	1	1

Signed Binary Numbers

- All possible 4-bit signed binary numbers in the three representations.
- Which one is the best? Why?

Decimal	Signed-2's Complement	Signed-1's Complement	Signed Magnitude
+7	0111	0111	0111
+6	0110	0110	0110
+5	0101	0101	0101
+4	0100	0100	0100
+3	0011	0011	0011
+2	0010	0010	0010
+1	0001	0001	0001
+0	0000	0000	0000
-0	—	1111	1000
-1	1111	1110	1001
-2	1110	1101	1010
-3	1101	1100	1011
-4	1100	1011	1100
-5	1011	1010	1101
-6	1010	1001	1110
-7	1001	1000	1111
-8	1000	—	—

Signed Binary Numbers

	Addition	Representation of 0	Range
Sign-magnitude	Doesn't work $\rightarrow -6+6$ 1110 <u>+ 0110</u> 10100 (wrong!)	Two representations 0000 +0 1000 -0	$[-(2^{N-1}-1), 2^{N-1}-1]$
Signed-1's complement	Doesn't work $\rightarrow -3+6$ 1100 <u>+ 0110</u> 10010 (wrong!)	Two representations 0000 +0 1111 -0	$[-(2^{N-1}-1), 2^{N-1}-1]$
Signed-2's complement	Works $\rightarrow -3+6$ 1101 <u>+ 0110</u> 10011 (correct!)	Only one 0000 ± 0 1000 is -8	$[-2^{N-1}, 2^{N-1}-1]$

BCD Codes

- BCD Code
 - Four bits are required to code each decimal number.
 - Decimal 396 is represented in BCD with 12bits as 0011 1001 0110, with each group of 4 bits representing one decimal digit.
 - Also known as 8-4-2-1 code, as 8, 4, 2, and 1 are the weights of the four bits of BCD.
 - The binary combinations 1010 through 1111 are not used and have no meaning in BCD.

Decimal Symbol	BCD Digit
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001

BCD Addition

- First add the two numbers using normal rules for binary addition.
- If the 4-bit sum is equal to or less than 9, it becomes a **valid** BCD number.
- If the 4-bit sum is greater than 9, or if a carry-out of the group is generated, it is an **invalid** result.
 - In such a case, add $(0110)_2$ or $(6)_{10}$ to the 4-bit sum in order to skip the six invalid states and return the code to BCD. If a carry results when 6 is added, add the carry to the next 4-bit group.
- Example: Consider the addition of $184 + 576 = 760$ in BCD:

BCD	1	1		
	0001	1000	0100	184
	<u>+ 0101</u>	<u>0111</u>	<u>0110</u>	<u>+576</u>
Binary sum	0111	10000	1010	
Add 6	<u> </u>	<u>0110</u>	<u>0110</u>	<u> </u>
BCD sum	0111	0110	0000	760

BCD Subtraction

- Same as in the binary case:
- Take the 10's complement of the subtrahend and add it to the minuend.
- Example: Consider the subtraction of $109 - 132 = -23$ in BCD:
 - Take 10's comp of 132 = 868
 - Convert difference into 10's complement

Subtraction		1		
	0001	0000	1001	109
	+1000	0110	1000	+868
Binary sum	1001	0111	10001	
Add 6	0000	0000	0110	
Difference	1001	0111	0111	977
10's complement				-23

Gray Code

- Gray Code(格雷码)

- Minimum change code: A number changes by only one bit as it proceeds from one number to the next.
 - Error detection.
 - Representation of analog data.
 - Low power design.

Gray Code	Decimal Equivalent
0000	0
0001	1
0011	2
0010	3
0110	4
0111	5
0101	6
0100	7
1100	8
1101	9
1111	10
1110	11
1010	12
1011	13
1001	14
1000	15

ASCII Codes

- American Standard Code for Information Interchange (ASCII) Character Code
 - Many applications of the computer require not only handling of numbers, but also of letters.
 - To represent letters it is necessary to have a binary code for the alphabet.
 - Seven bits to code 128 characters.

$b_4b_3b_2b_1$	$b_7b_6b_5$							
	000	001	010	011	100	101	110	111
0000	NUL	DLE	SP	0	@	P	`	p
0001	SOH	DC1	!	1	A	Q	a	q
0010	STX	DC2	“	2	B	R	b	r
0011	ETX	DC3	#	3	C	S	c	s
0100	EOT	DC4	\$	4	D	T	d	t
0101	ENQ	NAK	%	5	E	U	e	u
0110	ACK	SYN	&	6	F	V	f	v
0111	BEL	ETB	‘	7	G	W	g	w
1000	BS	CAN	(8	H	X	h	x
1001	HT	EM)	9	I	Y	i	y
1010	LF	SUB	*	:	J	Z	j	z
1011	VT	ESC	+	;	K	[k	{
1100	FF	FS	,	<	L	\	l	
1101	CR	GS	—	=	M]	m	}
1110	SO	RS	.	>	N	^	n	~
1111	SI	US	/	?	O	—	o	DEL

Error-Detecting Code

- Error-Detecting Code

- To detect errors in data communication and processing, an eighth bit is sometimes added to the ASCII character to indicate its parity.
- A **parity bit** (校验位) is an extra bit included with a message to make the total number of 1's either even or odd.

- Example:

	With even parity	With odd parity
ASCII A = 1000001	0 1000001	1 1000001
ASCII T = 1010100	1 1010100	0 1010100

- Suppose we use even parity

Original code	With even parity	sender	receiver	Parity check Passed?
1000001	01000001	01000001	01000001	yes
1000001	01000001	0100 0 001	0100 1 001	No
1000001	01000001	0100 0 001	0100 1 101	Yes but fails for double errors

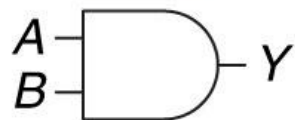
Binary Logic

- Binary logic deals with binary variables(e.g. can have two values,“0”and “1”)
- Binary variables can undergo three basic logical operators AND, OR and NOT
 - **AND** is denoted by a dot (\cdot) $z = x \cdot y$ or $z = xy$.
 - **OR** is denoted by a plus ($+$) $z = x + y$.
 - **NOT** is denoted by a single quote mark ($'$) after the variable, or an overbar ($\bar{}$) above the variable.
 - $x'y$ is pronounced as "x ~~prime~~ y" or "x complement y".
- Binary logic resembles binary arithmetic.
 - However, binary logic should not be confused with binary arithmetic.
 - An arithmetic variable designates a number that may consist of many digits.
 - A logic variable is always either 0 or 1.

Binary Logic

- Truth Tables, Boolean Expressions, and Logic Gates

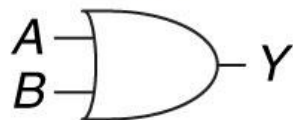
AND



$$Y = AB$$

A	B	Y
0	0	0
0	1	0
1	0	0
1	1	1

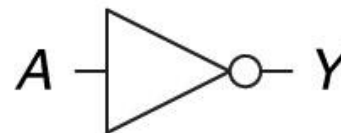
OR



$$Y = A + B$$

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1

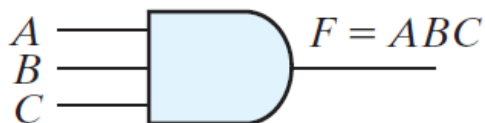
NOT



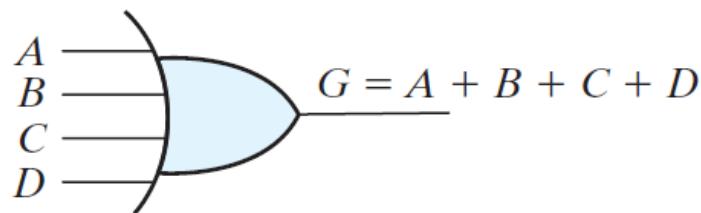
$$Y = \bar{A}$$

A	Y
0	1
1	0

- It is fine to have more than two inputs for AND/OR



$$F = ABC$$

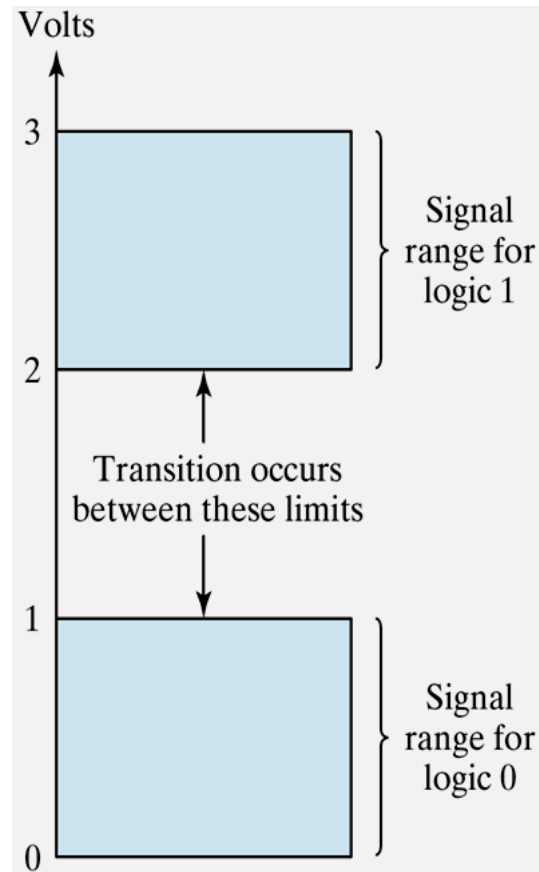


$$G = A + B + C + D$$

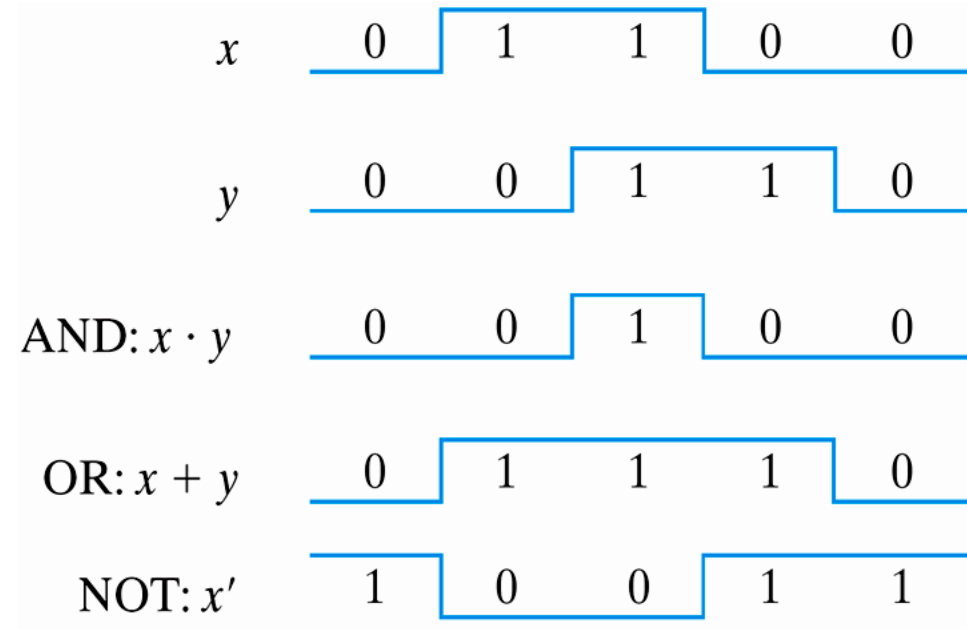
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Binary Logic

- Voltage-operated, though on a range, interpreted to be either of the two values



Signal levels for binary logic values



Input–output signals for gates