Discrete Math Ass 4 　　12310401 王子恒

1. Base case: For $n=1$ 　$1^3+2\times 1 = 1+2 = 3$

since $3 \mid 3$ the base case holds

Inductive: Assume the statement is true for k then it also holds for k+1

proof: $(k+1)^3 + 2\times(k+1) = k^3+3k^2+5k+3$

$= k^3+2k+3(k^2+k+1)$

because $3 \mid k^3+2k$ 　thus $3 \mid (k+1)^3+2(k+1)$

By the induction, the statement is true for all positive integers n.
Hence, 3 divides n^3+2n for every positive integers n.

2. Base case: For $n=1$ left $= a-b$, right $= 1\times a^{1-1}\times(a-b) = a-b$.

since $a-b \le a-b$ 　the base case holds

Inductive: Assume the statement is true for k then it also holds for k+1

proof: for $n=k+1$

left: $a^{k+1}-b^{k+1} = a^k\cdot a - a^k\cdot b + a^k\cdot b - b^{k+1}$

$= a^k(a-b) + b(a^k-b^k)$

because $a^k-b^k \le k(a^{k-1})(a-b)$

thus $a^{k+1}-b^{k+1} \le a^k(a-b) + bka^{k-1}(a-b)$

$\le (a+bk)\,a^{k-1}(a-b)$

since $b<a$ 　$\le (a+ak)a^{k-1}(a-b)$

$= (k+1)a^k(a-b)$

Then proof have done.

By the principle of induction, the statements holds for all positive integers n

3. since both 10 and 25 are multiples of 5, we cannot form any amount that is not a multiple of 5. So let's determine for which values of n we can form $5n$ dollars using these gift certificates, the first of which provides 2 copies of \$5, and the second provides 5 copies. we can achieve the following values of n: $2=2$, $4=2+2$, $5=5$ By having considered all the combinations, we know that the gaps in this list cannot be filled. We can form total amounts of the form $5n$ for all $n \geq 4$ using these gift certificates.

To prove by strong induction, let $P(n)$ be the statement that we can form $5n$ dollars in gift certificates using \$10 and \$40. We want to prove that $P(n)$ is true for all $n \geq 4$. From above, we know that $P(n)$ is true for $n = 4, 5$. Assume the inductive hypothesis, that $P(j)$ is true for all j with $4 \leq j \leq k$, where k is a fixed integer greater than or equal to 5. We want to show that $P(k+1)$ is true. Because $k-1 \geq 4$, we know that $P(k-1)$ is true, that we can form $5(k-1)$ dollars. Add one more 10-dollar can form $5(k+1)$ dollars, as disired.

4. The strong induction principle clearly implies ordinary induction, for if $P(k) \to P(k+1)$, then it follows that $[P(1) \wedge \cdots \wedge P(k)] \to P(k+1)$

Conversely, suppose that $P(n)$ is a statement that one can prove using strong induction. Let $Q(n)$ be $P(1) \wedge \cdots \wedge P(n)$. Then we have $\forall n P(n)$ logically equivalent to $\forall n Q(n)$. We show how $\forall n Q(n)$ can be proved using ordinary induction. First, $Q(1)$ is true because $Q(1) = P(1)$ and $P(1)$ is true by the basis step for the proof of $\forall n P(n)$ by strong induction. Now suppose $Q(k)$ is true, which means $P(1) \wedge \cdots \wedge P(k)$ By the proof of $\forall n P(n)$ by strong induction it follow that $P(k+1)$ is true. And $Q(k) \wedge P(k+1)$ is just $Q(k+1)$, Thus we have proved $\forall n Q(n)$ by ordinary induction.

5. function (n, a):

```
function (n, a):
    if n=1    return a^2
    else   return  function(n-1, a)^2.
```

6. (a) $f(4) = 2f(2) + \log_2 4 = 4$

$f(16) = 2f(4) + \log_2 16 = 12$

(b) let $m = \log n \Rightarrow n = 2^m$

let $g(m) = f(2^m)$

Then $f(2^m) = 2f(2^{m/2}) + m$ can be written by

$g(m) = 2g(\frac{m}{2}) + m$

$a=2,\ b=2,\ c=1,\ d=1$

$\Rightarrow g(m) = O(m \log m)$

$\Rightarrow f(n) = O(\log n \cdot \log\log n)$

7. a) when $n > 1$ $\quad n = 2^k$

$S(n) = 9S(\frac{n}{2}) + n^2$

$= 9\left(9S(\frac{n}{4}) + (\frac{n}{2})^2\right) + n^2$

$= 9^2 S(\frac{n}{4}) + 9(\frac{n}{2})^2 + n^2$

$= 9^3 S(\frac{n}{8}) + 9^2(\frac{n}{4})^2 + 9(\frac{n}{2})^2 + n^2$

...

$= 9^k S(\frac{n}{2^k}) + 9^{k-1}(\frac{n}{2^{k-1}})^2 + 9^{k-2}(\frac{n}{2^{k-2}})^2 + \cdots + 9^0(\frac{n}{2^0})^2$

$= 9^k S(1) + \left[9^{k-1}(2)^2 + 9^{k-2}4^2 + \cdots + 9^0(2^k)^2\right]$

$= 9^k b + \frac{4}{5}(9^k - 4^k)$

$= 9^{\log_2 n} b + \frac{4}{5}(9^{\log_2 n} - 4^{\log_2 n})$

$= n^{\log_2 9} b + \frac{4}{5}(n^{\log_2 9} - n^2)$

$= n^{\log_2 9}(b + \frac{4}{5}) - \frac{4}{5}n^2$

b) $T(n) = aT(\frac{n}{4}) + n^2)$

$= a\left(aT(\frac{n}{4^2}) + (\frac{n}{4})^2\right) + n^2$

$= a^2 T(\frac{n}{4^2}) + a(\frac{n}{4})^2 + n^2$

$= a^3 T(\frac{n}{4^3}) + a^2(\frac{n}{4^2})^2 + a(\frac{n}{4})^2 + n^2$

...

$= a^k T(\frac{n}{4^k}) + a^{k-1}(\frac{n}{4^{k-1}})^2 + a^{k-2}(\frac{n}{4^{k-2}})^2 + \cdots a^0(\frac{n}{4^0})^2$

$\frac{16}{a}$

$= a^k T(\frac{n}{4^k}) + a^{k-1}(4)^2 + a^{k-2}(4^2)^2 + \cdots + a^0(4^k)^2$

$= a^k T(1) + \frac{a^{k-1}4^2(1-(\frac{16}{a})^k)}{1-\frac{16}{a}}$

$= a^k c + \frac{16a^k - 16^{k+1}}{a-16}$

$= a^{\log_4 n} c + \frac{16a^{\log_4 n} - 16 \cdot 16^{\log_4 n}}{a-16}$

$= n^{\log_4 a} c + \frac{16n^{\log_4 a} - 16n^2}{a-16}$

c) $\log_4 a \leq \log_2 9$

$16 < a \leq 81$

8. $|A| = 2^3 = 8 \quad |B| = 2^4 = 16 \quad |C| = 2^4 = 16$

$|A \cap B| = 0 \quad |A \cap C| = 2 \quad |B \cap C| = 2^2 = 4$

$|A \cap B \cap C| = 0$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 8 + 16 + 16 - 0 - 2 - 4 + 0 = 34$

9. (a) there are 3^n functions

(b) if $n=1$ $\Rightarrow$ 3

if $n=2$ $\Rightarrow$ $3\times 2=6$

if $n=3$ $\Rightarrow$ $3\times 2\times 1=6$

if $n\geqslant 4$ 0 no one to one

(c) divide $\{1 \cdots n\}$ in to 3 none empty set

if $n\leq 2$ 0

if $n\geqslant 3$: $C_{n-1}^{2}=\frac{(n-1)(n-2)}{2}$

10. $N=C_{13}^{2}\times C_{4}^{2}\times C_{4}^{2}\times C_{44}^{2}=2656368$

11. 5 consecutive 0s: $\underline{0}0000XXXXX$ 2^5

$1\underline{00000}XXXX$ $2^4\times 5$

00000

00000

00000

00000

$2^5+2^4\times 5=112$

$|A|=112$

5 consecutive 1s: same as 0: $|B|=112$

$|A\cap B|$: 0000011111

1111100000

$|A\cup B|=|A|+|B|-|A\cap B|=222$

12. (1) $10-(3+0-2+0+0)=9$

0000 0|0|0000

divide 9 balls into 5 parts

$N=C_{9+5-1}^{5-1}=C_{13}^{4}=715$

(2) when $X_1>5$ $C_{4+5-1}^{5-1}=C_8^4$

$C_{10+5-1}^{5-1}-C_8^4=C_{14}^{4}-C_8^4=931$

13. $M_5 = \{0, 1, 2, 3, 4\}$.

Thus we need $5 \times 5 + 1 = 26$ pairs to guarantees

14. $\binom{p}{k} = \frac{p!}{k!(p-k)!}$

Because p is a prime, then p cannot be eliminate by

$\underset{k-1 \cdots}{k} < p$ and $\underset{(p-k-1)\cdots}{p-k} < p$, then p still there.

Thus p can divides $\binom{p}{k}$

15. (a) $\binom{n+r+1}{r}$ counts the number of a sequence of r 0s and $n+1$ 1s by choosing the position of 0s.

From other side, we can find where is the last 1 and then the 0s can be placed before the last 1. If there are $i+1$ terms of the last 1 and $n \le i \le n+r$, then $\sum_{i=n}^{n+r} \binom{i}{i-n} = \sum_{k=0}^{r} \binom{n+k}{k}$

Thus $\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$

(b) $P(r)$ is the induction statement.

Base step: $\sum_{k=0}^{1} \binom{n+k}{k} = \underset{\Rightarrow \binom{n}{1}}{\binom{n}{0}} + \binom{n+1}{1} = \binom{n+2}{1}$

Inductive step:

$$\sum_{k=0}^{r+1} \binom{n+k}{k} = \sum_{k=0}^{r} \binom{n+k}{k} + \binom{n+r+1}{r+1}$$

$$= \binom{n+r+1}{r} + \binom{n+r+1}{r+1}$$

$$= \binom{n+r+2}{r+1}$$

16. $\sum_{r=k}^{n}\binom{n}{r}\binom{r}{k} = \sum_{r=k}^{n} C_n^r C_r^k$

$= \sum_{r=k}^{n} \frac{n!\ r!}{(n-r)!\,r!\cdot k!\,(r-k)!}$

$= \sum_{r=k}^{n} \frac{n!}{(n-r)!\,k!\,(r-k)!}$

$= \sum_{r=k}^{n} \frac{n!}{k!\,(n-k)!} \cdot \frac{(n-k)!}{(n-r)!\,(r-k)!}$

$= C_n^k \sum_{r=k}^{n} \frac{(n-k)!}{(n-r)!\,(r-k)!}$

$= C_n^k \sum_{r=k}^{n} C_{n-k}^{r-k}$

$= C_n^k \cdot 2^{n-k}$.

17. $\lambda^3 - 2\lambda^2 - \lambda + 2$

$= \lambda^2(\lambda-1) - \lambda(\lambda-1) - 2(\lambda-1)$

$= (\lambda^2-\lambda-2)(\lambda-1)$

$= (\lambda+1)(\lambda-2)(\lambda-1) = 0$

$\lambda_1 = -1$, $\lambda_2 = 1$ $\lambda_3 = 2$

$a_n = C_1\cdot(-1)^n + C_2\cdot 1^n + C_3\cdot 2^n$

$\begin{cases} C_1 + C_2 + C_3 = 3 \\ -C_1 + C_2 + 2C_3 = 6 \\ C_1 + C_2 + 4C_3 = 0 \end{cases}$ $\begin{cases} C_1 = -2 \\ C_2 = 6 \\ C_3 = -1 \end{cases}$

$a_n = -2\cdot(-1)^n + 6 - 2^n$

18. let $a_n = V(n)$, $b_n = 10^n - a_n$

then $a_{n+2} = 9a_{n+1} + b_{n+1}$

$= 9a_{n+1} + a_n + 9b_n$

$= 9a_{n+1} + a_n + 9(a_{n+1} - 9a_n)$

$= 18a_{n+1} - 80a_n$

$x^2 = 18x - 80$

$(x-8)(x-10) = 0$ $x_1 = 8$ $x_2 = 10$

$V(n) = C_1\cdot 8^n + C_2\cdot 10^n$ $\begin{cases} 8C_1 + 10C_2 = 1 \\ 64C_1 + 100C_2 = 18 \end{cases} \Rightarrow \begin{cases} C_1 = -\frac{1}{2} \\ C_2 = \frac{1}{2} \end{cases}$

$V(n) = -\frac{1}{2}8^n + \frac{1}{2}\cdot 10^n$.

19. $$A_n = \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix}$$

$d_n = 2d_{n-1} - 1 \times d_{n-2}$ $\qquad x^2 = 2x - 1 \qquad x = 1$

$A_1 = [2] \qquad d_1 = 2$

$A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad d_2 = 4 - 1 = 3$

$$\begin{aligned} d_n &= a_1 \cdot 1^n + a_2 n \cdot 1^n \\ &= a_1 + a_2 n \end{aligned}$$

$$\begin{cases} a_1 + a_2 = 2 \\ a_1 + 2a_2 = 3 \end{cases} \qquad \begin{cases} a_1 = 1 \\ a_2 = 1 \end{cases}$$

$\Rightarrow d_n = n + 1$

20. According to the hint $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$

$$\begin{aligned} \sum_{r=0}^{n} C(n,r)x^r &= \sum_{r=0}^{n-1} C(n-1,r)\,x^r + \sum_{r=0}^{n-1} C(n-1,r)\,x^{r+1} \\ &= \sum_{r=0}^{n-1} C(n-1,r)x^r + \sum_{r=0}^{n} C(n-1,r-1)\,x^r \end{aligned}$$

Compare the coefficient, we can see that

$$C(n,r) = C(n-1,r) + C(n-1,r-1)$$

21. According to the binomial theorem to $(1+x)^{m+n} = (1+x)^m (1+x)^n$

$$\begin{aligned} \sum_{r=0}^{m+n} \underline{C(m+n,r)}\, \underline{\underline{x}}^r &= \sum_{r=0}^{m} C(m,r)\,x^r \cdot \sum_{r=0}^{n} C(n,r)\,x^r \\ &= \sum_{r=0}^{m+n} \left[\underline{\sum_{k=0}^{r} C(m,r-k)\,C(n,k)} \right] \underline{\underline{x}}^r \end{aligned}$$

Thus $C(m+n,r) = \sum_{k=0}^{r} C(m,r-k)\,C(n,k)$