Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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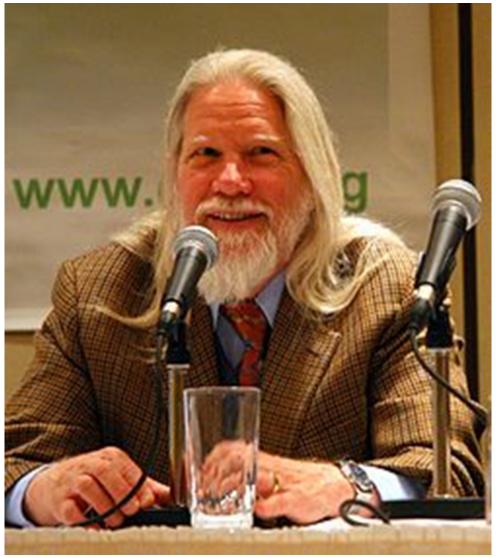
Cryptography History

- History (from 1976)
  - ◇ W. Diffie, M. Hellman, "New direction in cryptography", *IEEE Transactions on Information Theory, vol. 22, pp. 644-654, 1976.*

"We stand today on the brink of a revolution in cryptography."



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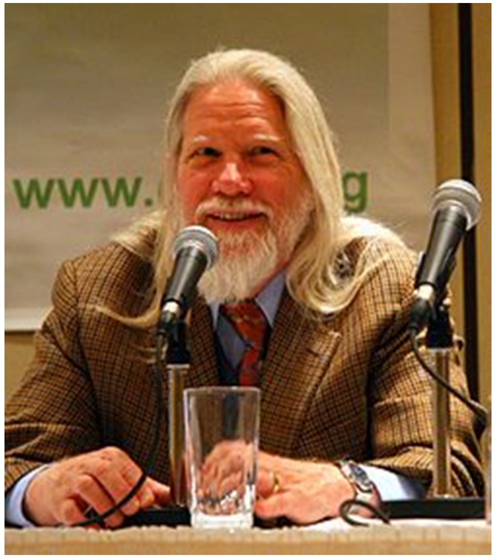
2015 **Turing Award**



Bailey W. Diffie

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2015	Martin E. Hellman Whitfield Diffie	For fundamental contributions to **modern cryptography**. Diffie and Hellman's groundbreaking 1976 paper, "New Directions in Cryptography,"[39] introduced the ideas of public-key cryptography and digital signatures, which are the foundation for most regularly-used security protocols on the internet today.[40]

Public Key Cryptography

- Alice wants to send a message to Bob

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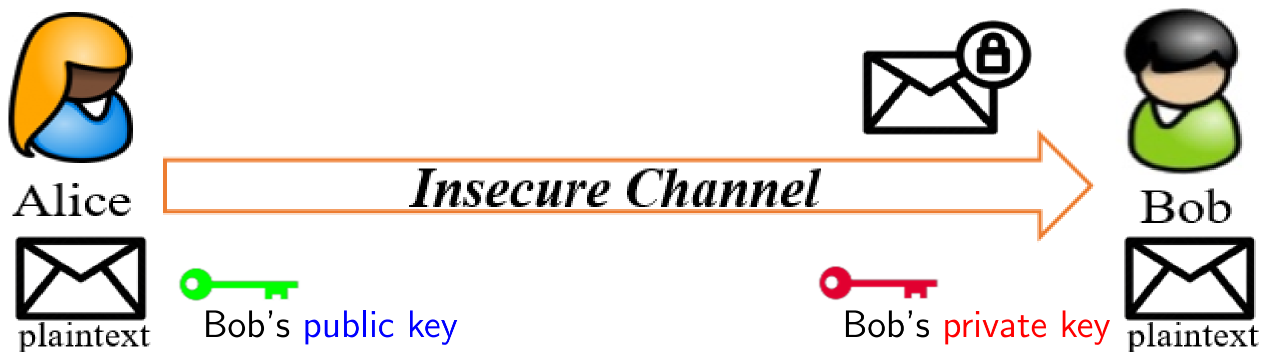

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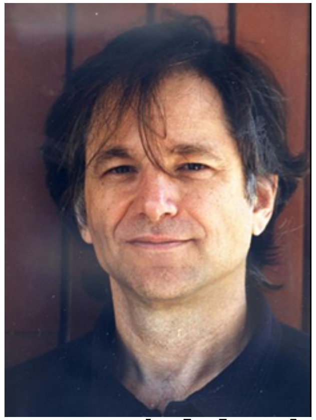
Leonard M. Adleman

R. Rivest, A. Shamir, L. Adleman, “A method for obtaining digital signatures and public-key cryptosystems”, *Communications of the ACM*, vol. 21-2, pages 120-126, 1978.

RSA Public Key Cryptosystem

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$\mathcal{Q}$: How to prove this?

RSA Public Key Cryptosystem: Example

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Encryption: $M = 28, C = M^7 \bmod 55 = 52$

Decryption: $M = C^{23} \bmod 55 = 28$

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$\mathcal{Q}$: Consider the RSA system, where $n = pq$ is the modulus. Let (e, d) be a key pair for the RSA. Define

$$\lambda(n) = \mathrm{lcm}(p-1, q-1)$$

and compute $d' = e^{-1} \bmod \lambda(n)$. Will decryption using d' instead of d still work?

Applications of RSA

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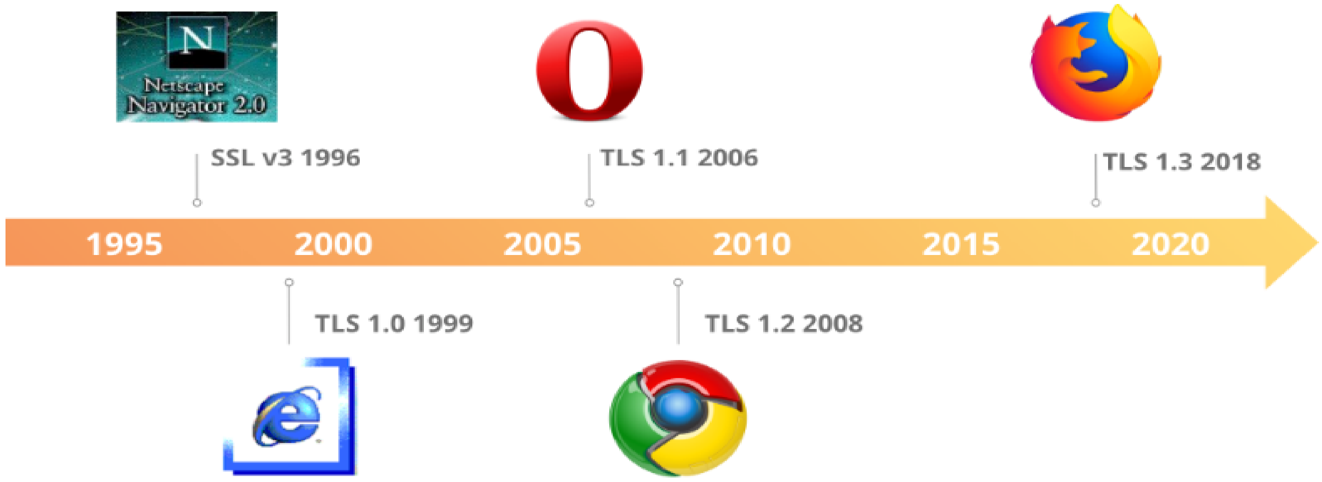

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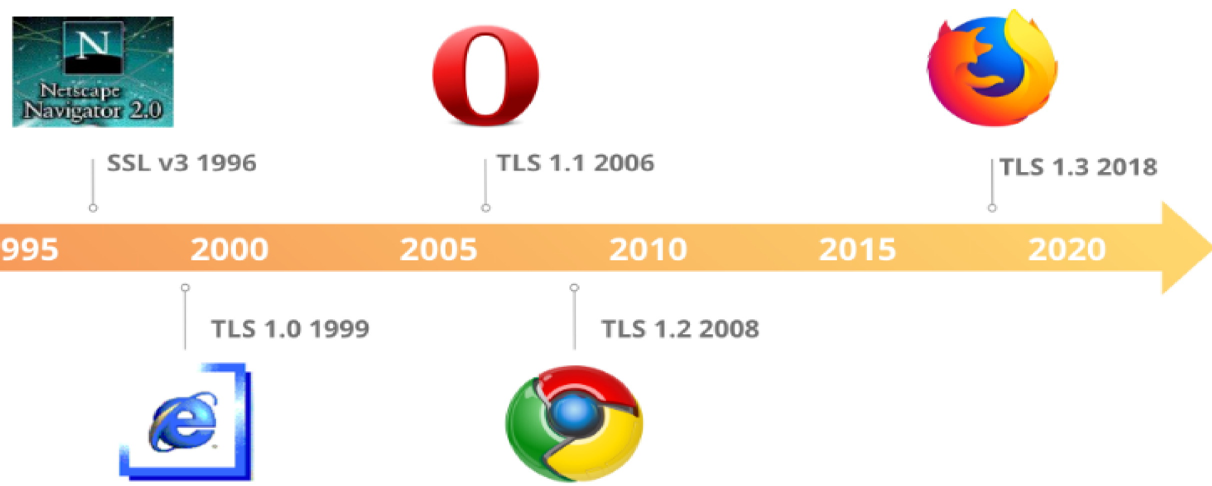

Key exchange/agreement and authentication

Algorithm	SSL 2.0	SSL 3.0	TLS 1.0	TLS 1.1	TLS 1.2	TLS 1.3
RSA	Yes	Yes	Yes	Yes	Yes	No
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DHE-RSA (forward secrecy)	No	Yes	Yes	Yes	Yes	Yes
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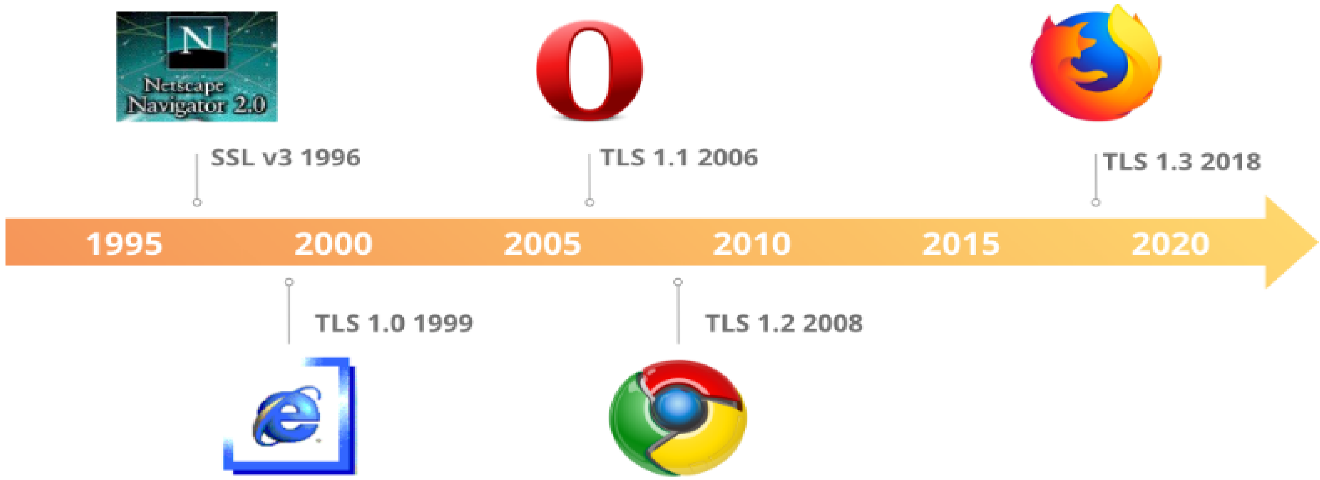

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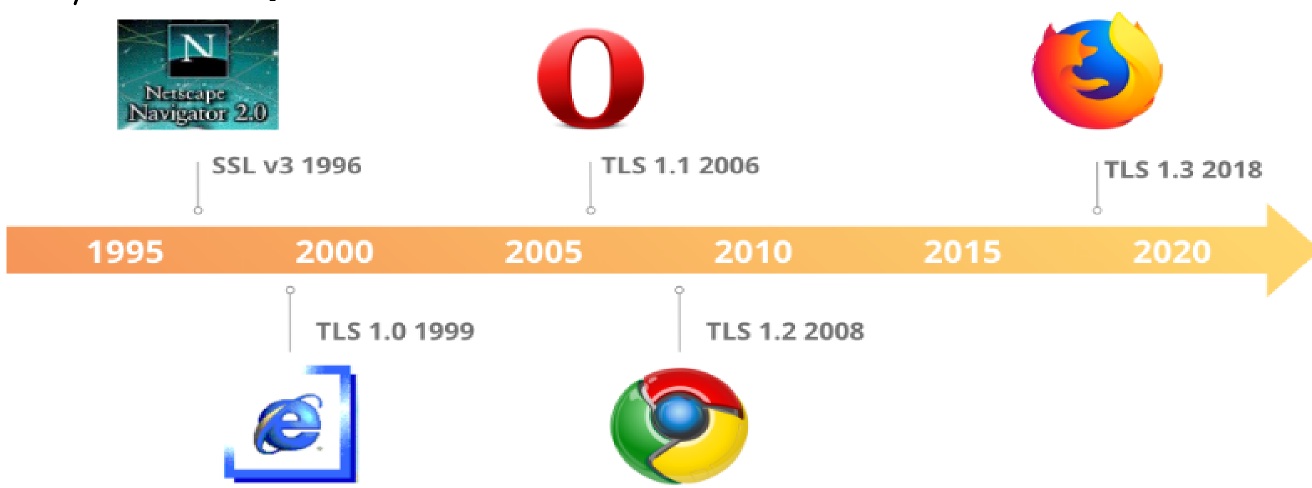

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Using RSA for Digital Signature

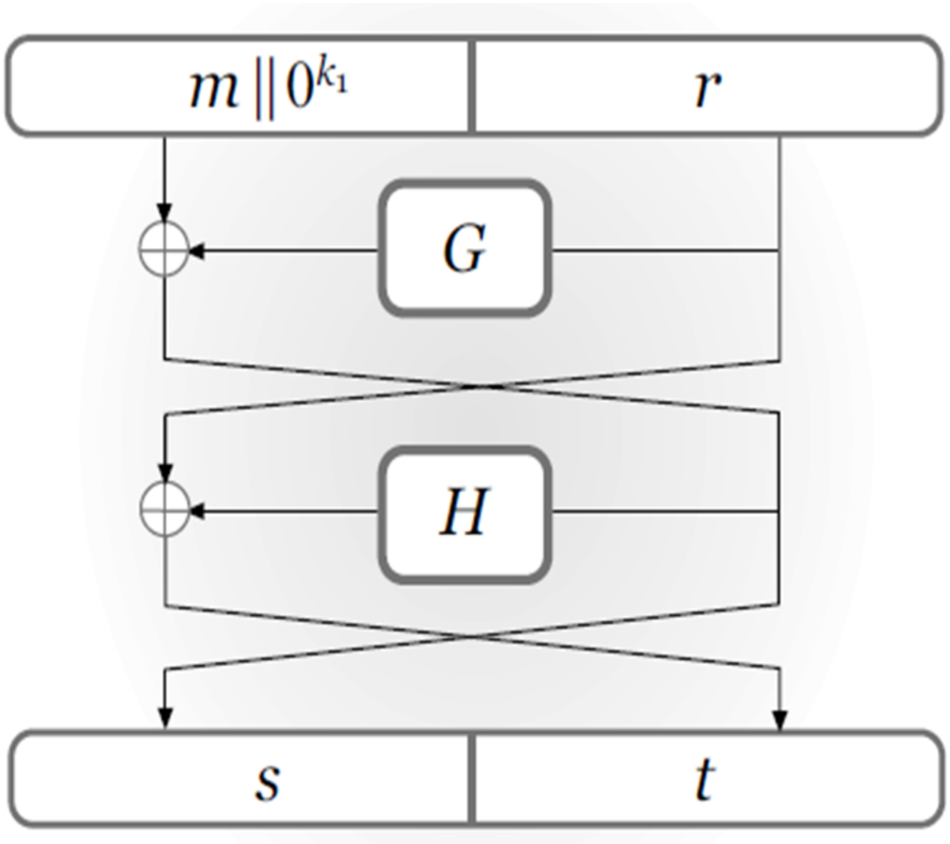
$S = M^d \bmod n$ (RSA **signature)**

$M = S^e \bmod n$ (RSA **verification)**

Why?

RSA-OAEP Standard

- RSA-OAEP (Optimal Asymmetric Encryption Padding) is *IND-CCA2 secure*.
- PKCS#1 V2, RFC2437 Standard



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This is very hard!

El Gamal Encryption

- **Setup** Let p be a prime, and g be a generator of $\mathbb{Z}_p$. The private key x is an integer with $1 < x < p - 2$. Let $y = g^x \bmod p$. The public key for *El Gamal encryption* is (p, g, y).

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El Gamal Encryption: Pick a random integer k from $\mathbb{Z}_{p-1}$,

$$a = g^k \bmod p$$
$$b = My^k \bmod p$$

The ciphertext C consists of the pair (a, b).

El Gamal Decryption:

$$M = b(a^x)^{-1} \bmod p$$

Using El Gamal for Digital Signature

$$a = g^k \bmod p$$
$$b = k^{-1}(M - xa) \bmod (p-1)$$

(El Gamal **signature**)

$$y^a a^b \equiv g^M \pmod{p}$$

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$\mathcal{Q}$: How to verify it?

An Example

Choose $p = 2579$, $g = 2$, and $x = 765$. Hence $y = 2^{765} \bmod 2579 = 949$.

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Encryption: Let $M = 1299$ and choose a random $k = 853$,

$$\begin{aligned}(a, b) &= (g^k \bmod p, My^k \bmod p) \\ &= (2^{853} \bmod 2579, 1299 \cdot 949^{853} \bmod 2579) \\ &= (435, 2396).\end{aligned}$$

Decryption:

$$M = b(a^x)^{-1} \bmod p = 2396 \times (435^{765})^{-1} \bmod 2579 = 1299.$$

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Security of the El Gamal Cryptosystem

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Attack 1: Use $M = by^{-k}$. However, k is randomly picked.

Attack 2: Use $M = b(a^x)^{-1} \bmod p$, but x is secret.

Diffie-Hellman Key Exchange Protocol

User A		User B
Generate random $X_A < p$		Generate random $X_B < p$
calculate $Y_A = \alpha^{X_A} \bmod p$	$\xrightarrow{Y_A}$	Calculate $Y_B = \alpha^{X_B} \bmod p$
	$\xleftarrow{Y_B}$	
Calculate $k = (Y_B)^{X_A} \bmod p$		Calculate $k = (Y_A)^{X_B} \bmod p$

Cryptography Wonders

- *Digital Signatures*. Electronically sign documents

  Zero-knowledge Proofs. Alice proves to Bob that she earns $< \$50k$ without Bob learning her income.

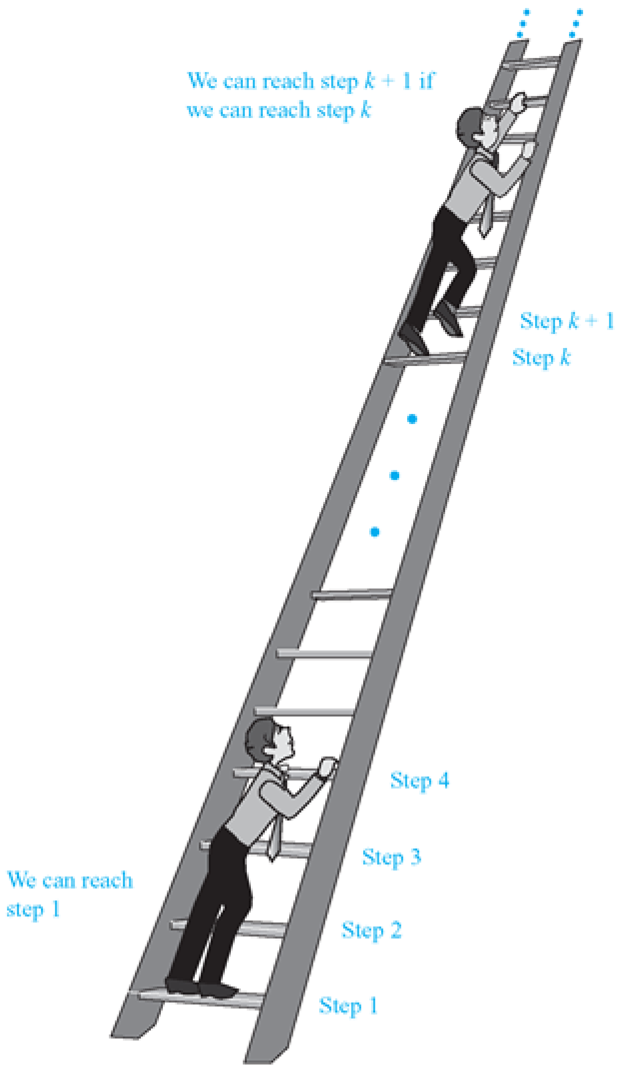
  Privacy-perserving data mining. Bob holds DB. Alice gets answer to one query, without Bob knowing what she asked.

  Playing poker over the net. Alice, Bob, Carol and David can play Poker over the net without trusting each other or any central server. (*E-Voting*)

  Electronic Auctions. Can run auctions s.t. no one (even not seller) learns anything other than winning party and bid.

  Fully Homomorphic Encryption. Encrypt $E(m)$ in a way that allows to compute $E(f(m))$.

Mathematical Induction



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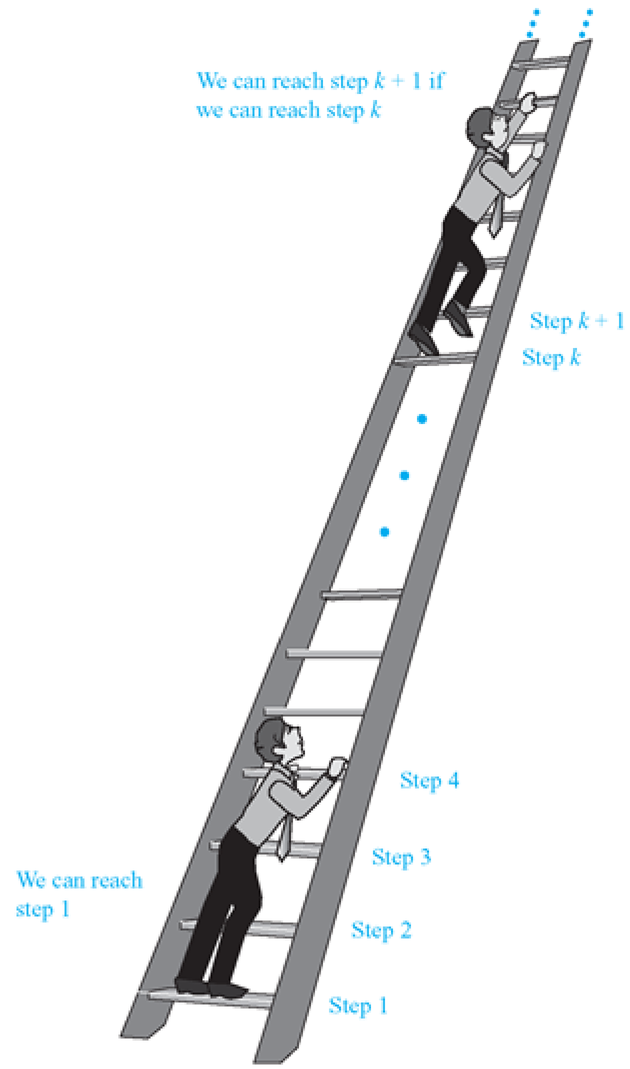

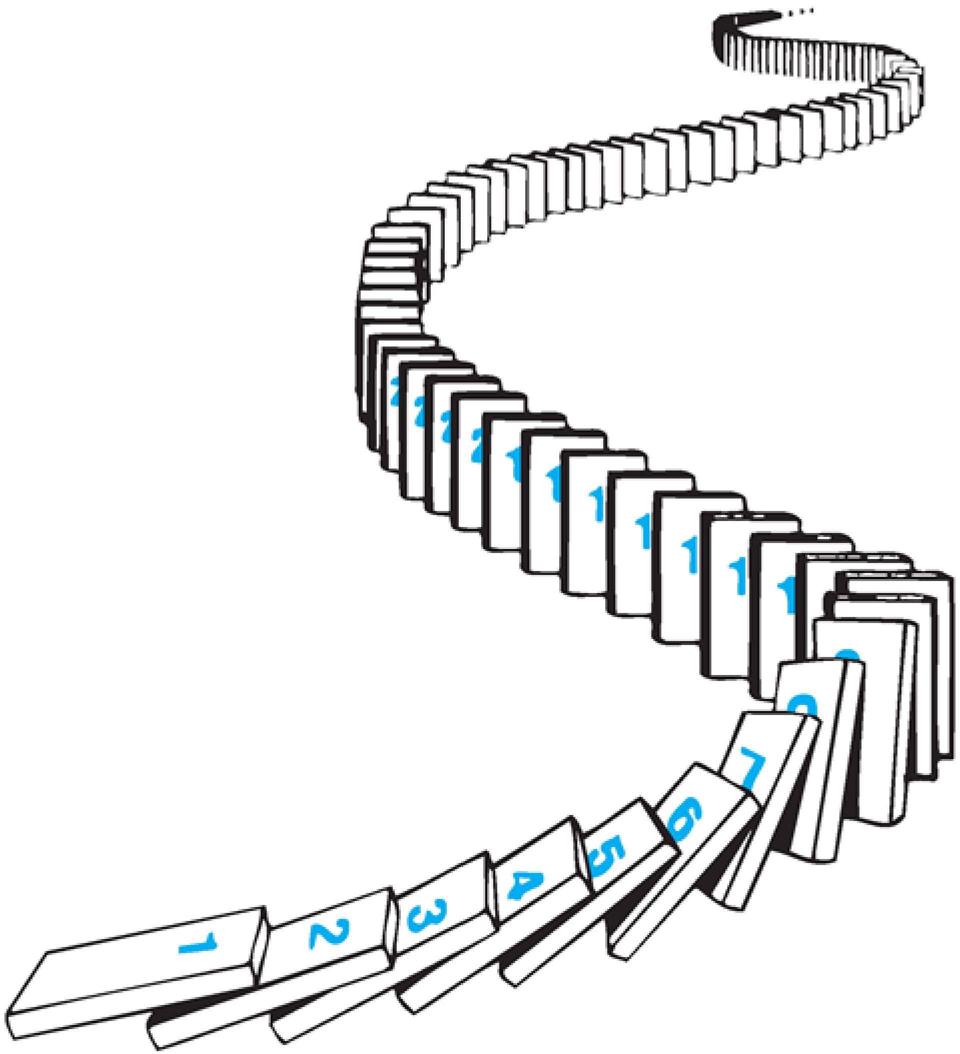

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- We conclude by distinguishing between the *weak principle* of mathematical induction and the *strong principle* of mathematical induction.

  The *strong principle* can actually be derived from the *weak principle*.

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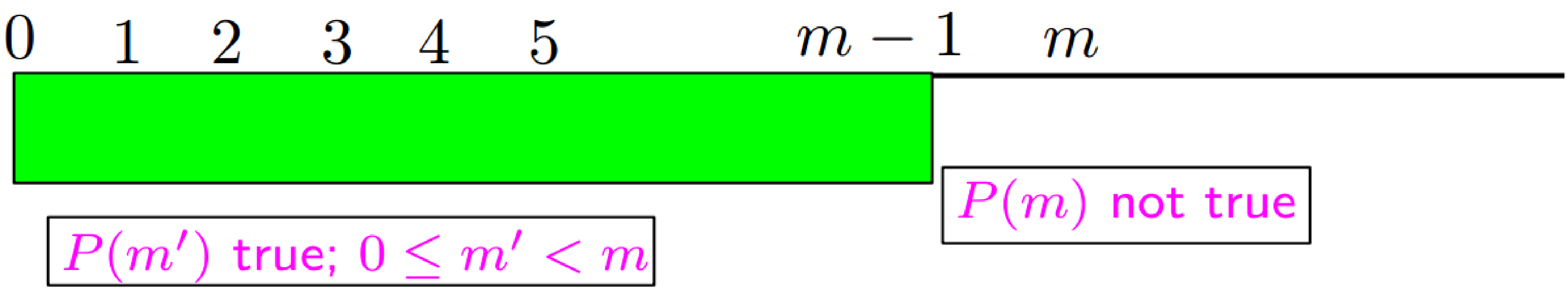

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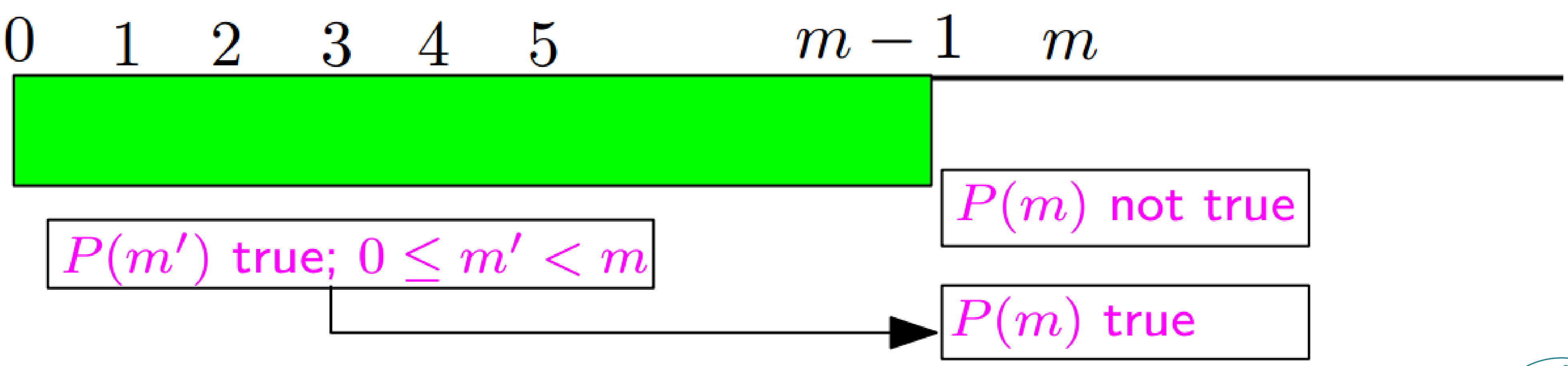

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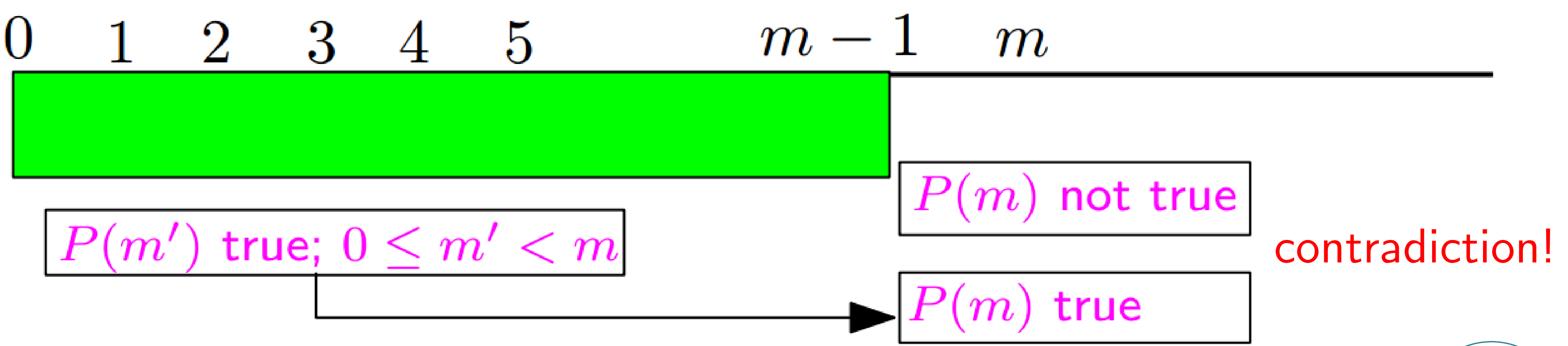

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◇ The smallest counterexample n is larger than 0

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  ◇ Therefore, $(*)$ holds for all positive integers n.

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  The key step was proving that

  $$P(n-1) \rightarrow P(n)$$

  where $P(n)$ is the statement

  $$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

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When a for all quantifier is false, there must be some n for which it is false. Let n be the smallest nonnegative integer for which $2^{n+1} \not\geq n^2 + 2$.

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  Then setting $i = n - 1$ gives
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  Thus, we write

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  Since $P(n-1) \rightarrow P(n)$, we see that

  $P(0)$ implies $P(1)$, $P(1)$ implies $P(2)$, ...

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Inductive Hypothesis

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Inductive Step

Hence, we've just prove that for $n > 2$, $P(n-1) \rightarrow P(n)$.

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  - Iterating gives us a proof of $P(n)$ for all n

Strong Induction

- **Principle** (*The Strong Principle of Mathematical Induction*)

  (a) If the statement $P(b)$ is true

  (b) for all $n > b$, the statement
  $P(b) \wedge P(b+1) \wedge \cdots \wedge P(n-1) \rightarrow P(n)$ is true.

  then $P(n)$ is true for all integers $n \geq b$.

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  ◇ Thus, by the strong principle of mathematical induction, every positive integer is a power of a prime or a product of powers of primes.

Mathematical Induction

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- In reality, they are equivalent to each other in that the weak form is a special case of the strong form, and the strong form can be derived from the weak form.

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  3. We conclude on the basis of the principle of mathematical induction that $P(n)$ is true for all $n \geq b$.

Next Lecture

- recurrence ...