


Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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The Chinese Remainder Theorem

- About 1500 years ago, the Chinese mathematician Sun-Tsu asked:
  "There are certain things whose number is unknown. When divided by 3, the remainder is 2; when divided by 5, the remainder is 3; when divided by 7, the remainder is 2. What will be the number of things?"

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The Chinese Remainder Theorem

- **Theorem** (*The Chinese Remainder Theorem*) Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime positive integers greater than 1 and $a_1, a_2, \ldots, a_n$ arbitrary integers. Then the system

$$
\begin{aligned}
x &\equiv a_1 \pmod{m_1} \\
x &\equiv a_2 \pmod{m_2} \\
&\ldots \\
x &\equiv a_n \pmod{m_n}
\end{aligned}
$$

has a unique solution modulo $m = m_1 m_2 \cdots m_n$.

The Chinese Remainder Theorem

- **Proof** Let $M_k = m/m_k$ for $k = 1, 2, \ldots, n$ and $m = m_1 m_2 \cdots m_n$. Since $\gcd(m_k, M_k) = 1$, there is an integer y_k, an inverse of M_k modulo m_k such that $M_k y_k \equiv 1 \pmod{m_k}$. Let

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots a_n M_n y_n.$$

It is checked that x is a solution to the n congruences.

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How to prove the uniqueness of the solution modulo m?

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Let $m = 3 \cdot 5 \cdot 7 = 105$, $M_1 = m/3 = 35$, $M_2 = m/5 = 21$, $M_3 = m/7 = 15$.

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$$x = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 \equiv 233 \equiv 23 \pmod{105}$$

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三人同行七十稀，五树梅花廿一枝，
七子团圆正月半，除百零五便得知。
-- 程大位《算法统要》(1593年)

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Fermat's Little Theorem

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$$\{1, 2, \ldots, p-1\} = \{x, 2x, \ldots, x(p-1) \pmod{p}\}$$

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  Example: 3 is a primitive root of $\mathbb{Z}_7$. 2 is not a primitive root of $\mathbb{Z}_7$.

  Theorem * There is a primitive root modulo n if and only if $n = 2, 4, p^e$ or $2p^e$, where p is an odd prime.

  $\mathcal{Q}$: proof? The number of primitive roots? *

Number Theory and Cryptography

- Division, Primes
- Congruence
- Greatest Common Divisor (GCD)
- Euler's Theorem / Fermart's Little Theorem

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  Find the GCD of 286 and 503.

$$
\begin{array}{ll}
\gcd(503, 286) & 503 = 1 \cdot 286 + 217 \\
= \gcd(286, 217) & 286 = 1 \cdot 217 + 69 \\
= \gcd(217, 69) & 217 = 3 \cdot 69 + 10 \\
= \gcd(69, 10) & 69 = 6 \cdot 10 + 9 \\
= \gcd(10, 9) & 10 = 1 \cdot 9 + 1 \\
= 1 & 9 = 9 \cdot 1
\end{array}
\qquad
\begin{array}{rcl}
1 &=& 10 - 1 \cdot 9 \\
1 &=& 7 \cdot 10 - 1 \cdot 69 \\
1 &=& 7 \cdot 217 - 22 \cdot 69 \\
1 &=& 29 \cdot 217 - 22 \cdot 286 \\
1 &=& 29 \cdot 503 - 51 \cdot 286
\end{array}
$$

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  (extended) Euclidean algorithm
  find the modular inverse
  solve linear congruence $ax \equiv b \pmod{m}$ $(\gcd(a, m) = 1)$

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Number Theory Summary

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  $x^{\phi(n)} \equiv 1 \bmod n$ if $\gcd(x, n) = 1$
  $x^{p-1} \equiv 1 \bmod p$ if $x \not\equiv 0 \bmod p$

Modular Arithmetic in CS

- Modular arithmetic and congruencies are used in CS:
  - ◇ Pseudorandom number generators
  - ◇ Hash functions
  - ◇ Cryptography

Pseudorandom Number Generators

- *Linear congruential method*

  We choose four numbers:

  - ◇ the modulus m
  - ◇ multiplier a
  - ◇ increment c
  - ◇ seed x_0

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  - ◇ the modulus m
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  - ◇ increment c
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  We generate a sequence of numbers $x_1, x_2, \ldots, x_n, \ldots$ with $0 \leq x_i < m$ by using the congruence

$$x_{n+1} = (ax_n + c) \pmod{m}$$

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Example:

- Assume : m=9,a=7,c=4, $x_0 = 3$
- $x_1 = 7*3+4 \bmod 9 = 25 \bmod 9 = 7$
- $x_2 = 53 \bmod 9 = 8$
- $x_3 = 60 \bmod 9 = 6$
- $x_4 = 46 \bmod 9 = 1$
- $x_5 = 11 \bmod 9 = 2$
- $x_6 = 18 \bmod 9 = 0$
- ....

Hash Functions

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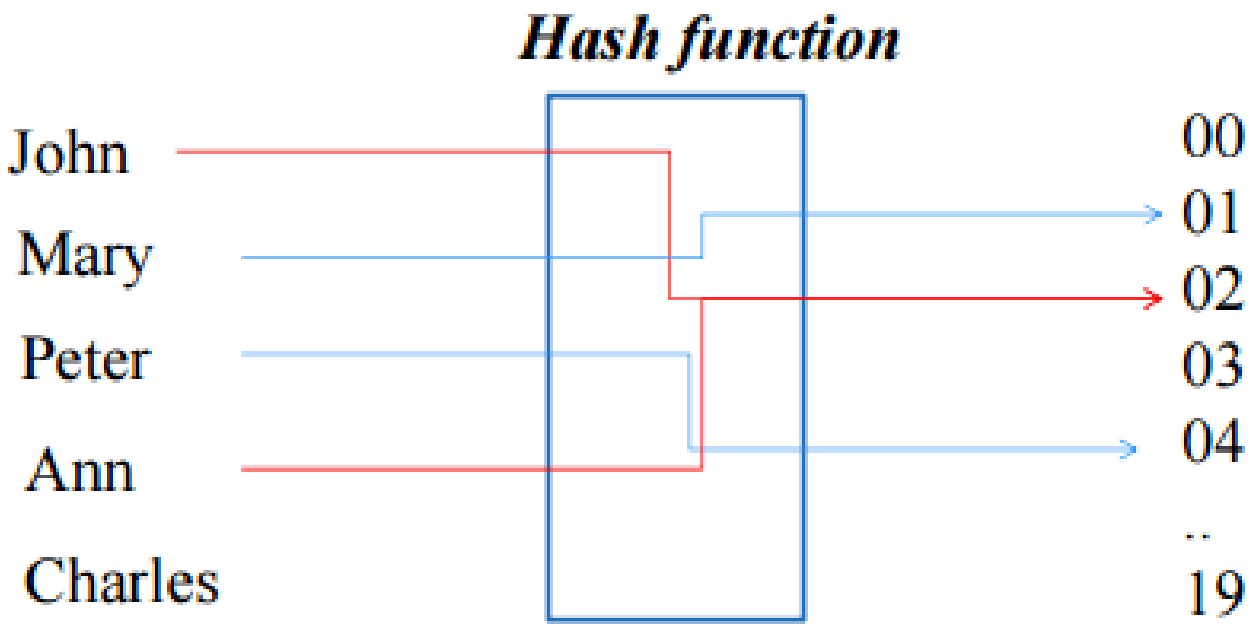
Example:



Hash Functions

- **Problem**: Given a large collection of records, how can we store and find a record quickly?

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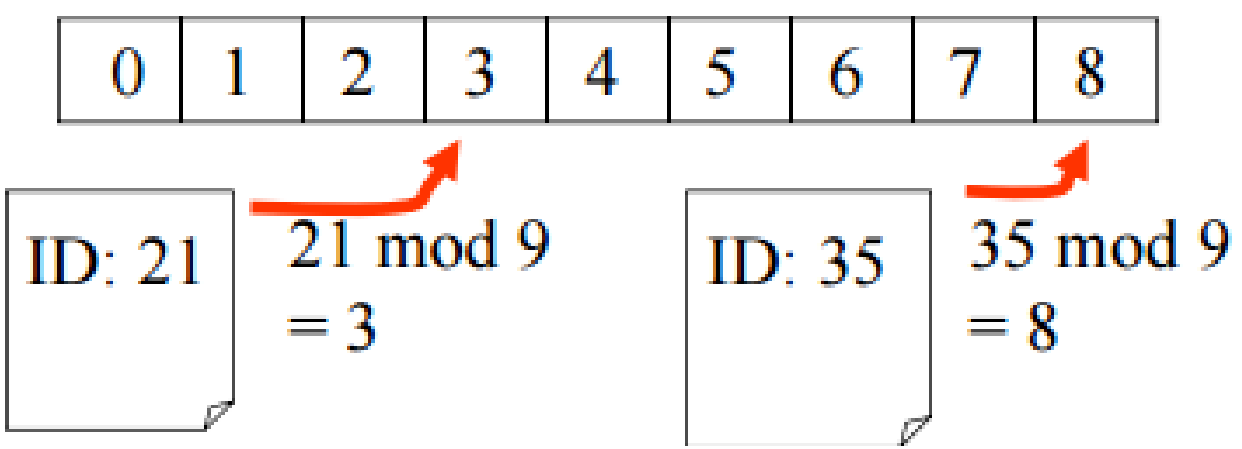
  Solution: Use a hash function, calculate the location of the record based on the record's ID.

  Example: A common hash function is

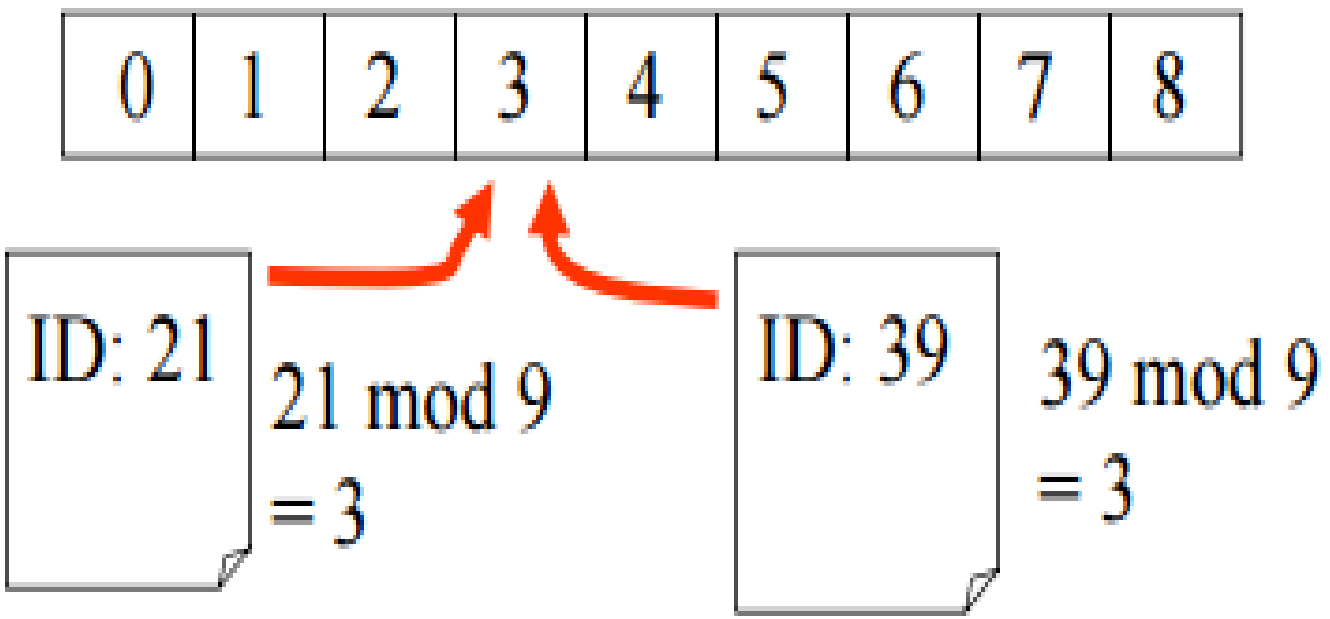
  - $h(k) = k \bmod n$,

  where n is the number of available storage locations.



Hash Functions

- Two records mapped to the same location

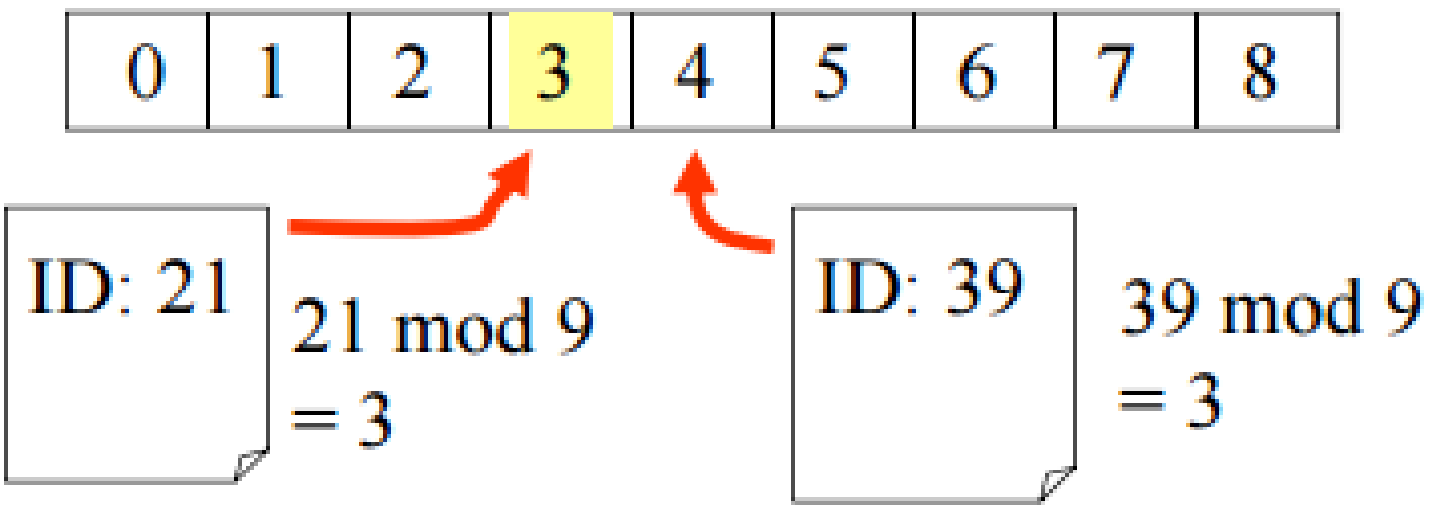


Hash Functions

- **Solution 1**: move to the next available location

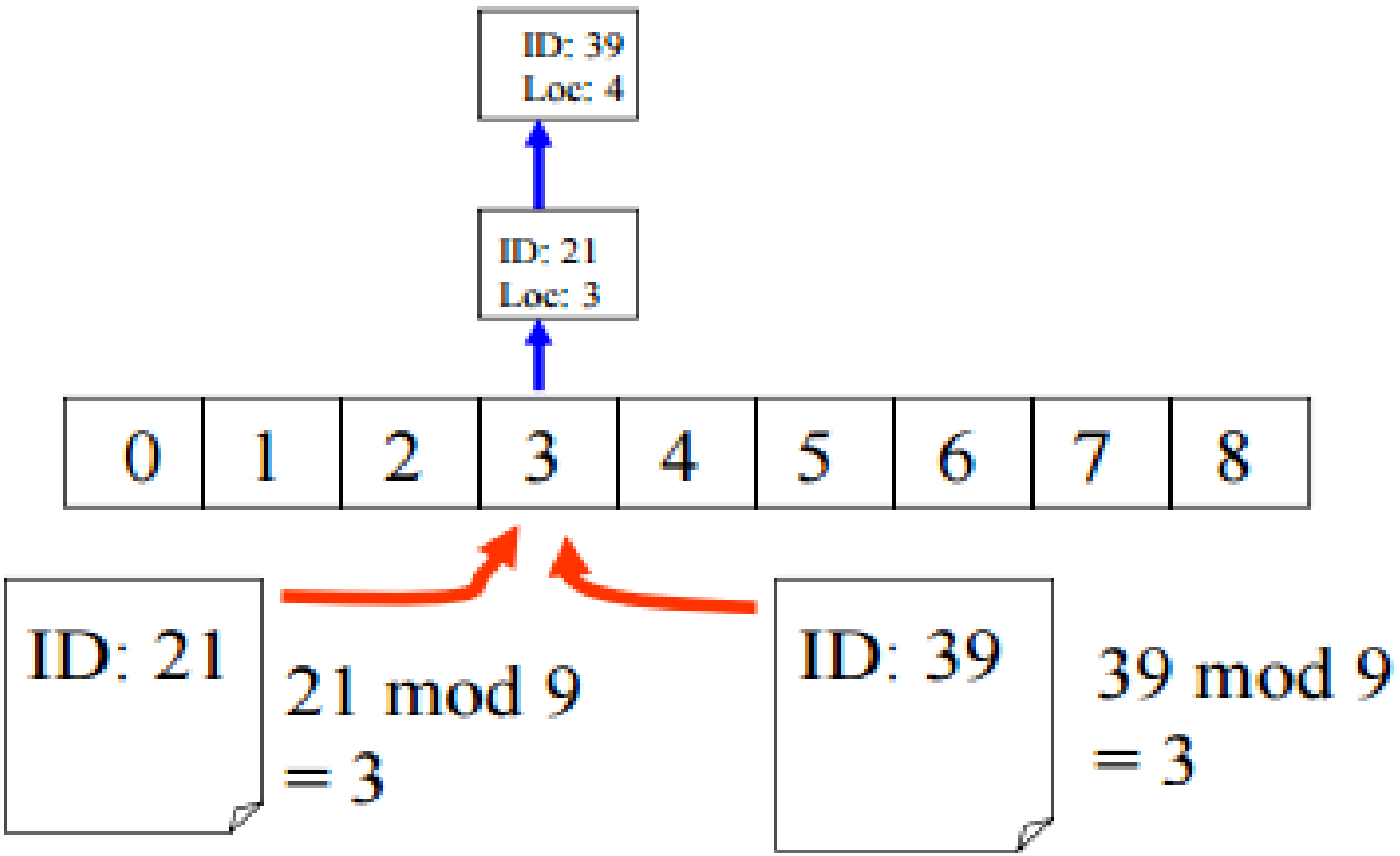
try

$$h_0(k) = k \bmod n$$
$$h_1(k) = (k+1) \bmod n$$
$$\ldots$$
$$h_m(k) = (k+m) \bmod n$$



Hash Functions

- **Solution 2**: remember the exact location in a secondary structure that is searched sequentially



Applications of Number Theory in Cryptography

- Introduction
- Symmetric cryptography
- Asymmetric cryptography
- RSA Cryptosystem
- DLP and El Gamal cryptography
- Diffie-Hellman key exchange protocol
- Crytocurrency, e.g., bitcoin

Cryptography

- History of almost 4000 years (from 1900 B.C.)

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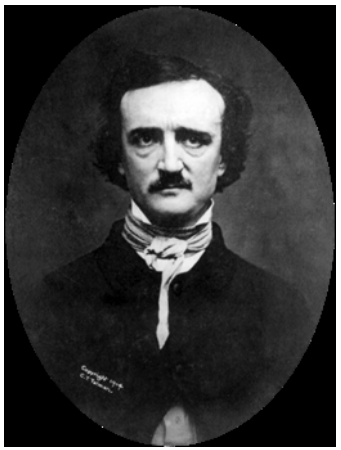
The term was first used in *The Gold-Bug*, by Edgar Allan Poe (1809 - 1849).



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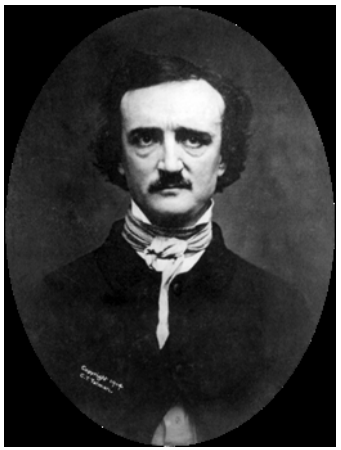


"Human ingenuity cannot concoct a cipher which human ingenuity cannot resolve." – 1941

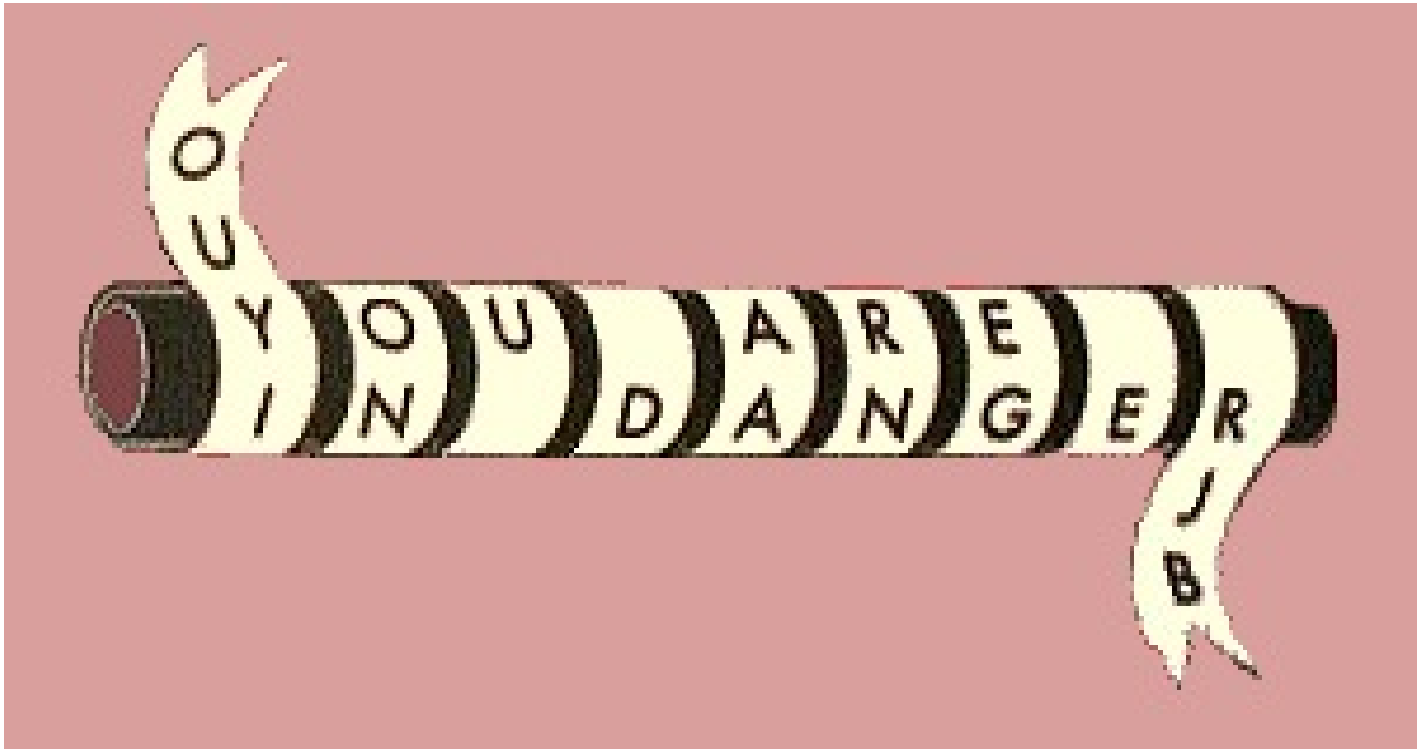
Cryptography

- One-sentence definition:

  "Cryptography is the practice and study of techniques for secure communication in the presence of third parties called *adversaries*." – Ronald L. Rivest

Some Examples

- In 405 BC, the Greek general LYSANDER OF SPARTA was sent a coded message written on the inside of a servant's belt.



Some Examples

- The Greeks also invented a cipher which changed letters to numbers. A form of this code was still being used during *World War I.*

	1	2	3	4	5
1	A	B	C	D	E
2	F	G	H	I/J	K
3	L	M	N	O	P
4	Q	R	S	T	U
5	V	W	X	Y	Z

Some Examples

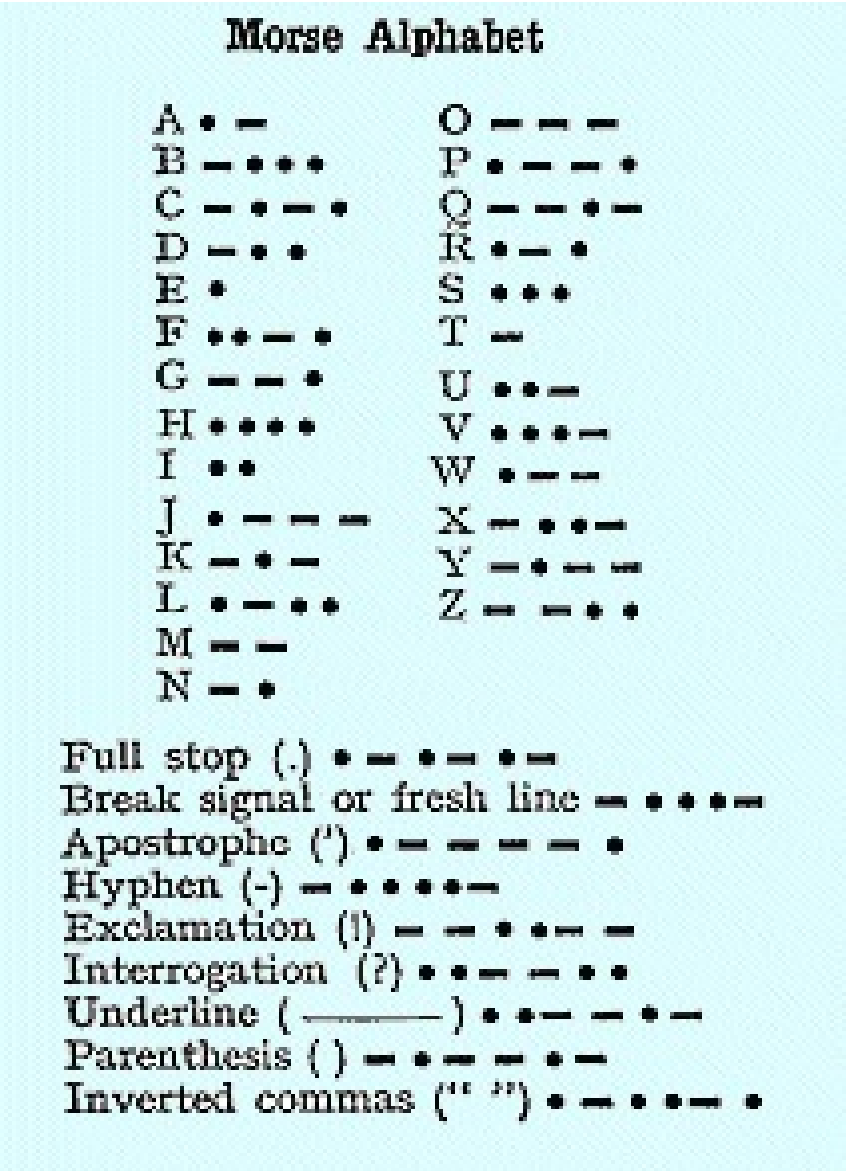
- Caesar Cipher (after the name of JULIUS CAESAR)

VENI, VIDI, VICI

YHQL YLGL YLFL

Some Examples

- Morse Code: created by Samuel Morse in 1838



Morse Alphabet

A • —	O — — —
B — • • •	P • — — •
C — • — •	Q — — • —
D — • •	R • — •
E •	S • • •
F • • — •	T —
G — — •	U • • —
H • • • •	V • • • —
I • •	W • — —
J • — — —	X — • • —
K — • —	Y — • — —
L • — • •	Z — — • •
M — —	
N — •	

Full stop (.) • — • — • —
Break signal or fresh line — • • • —
Apostrophe (') • — — — — •
Hyphen (-) — • • • • —
Exclamation (!) — — • • — —
Interrogation (?) • • — — • •
Underline (_____) • • — — • —
Parenthesis ( ) — • — — • —
Inverted commas (" ") • — • • — •

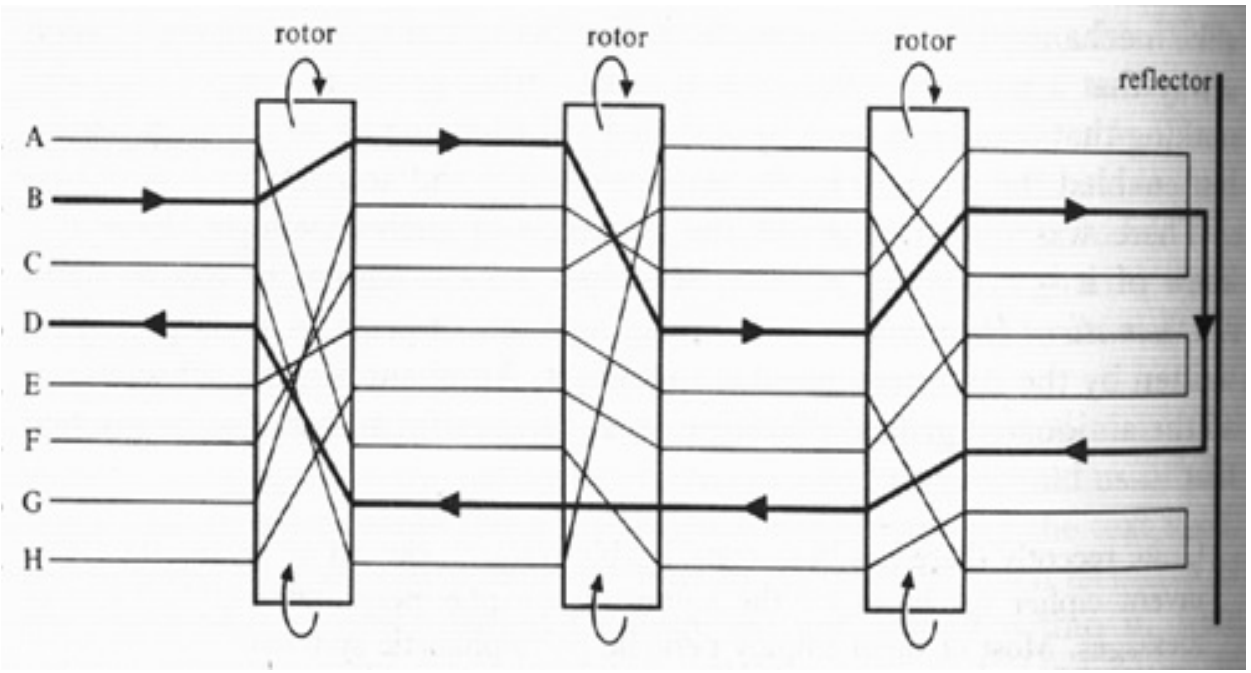
Some Examples

- Crytograms from the Chinese gold bars

http://www.iacr.org/misc/china/china.html

Some Examples

- Enigma, Germany coding machine in *World War II.*



Some Examples

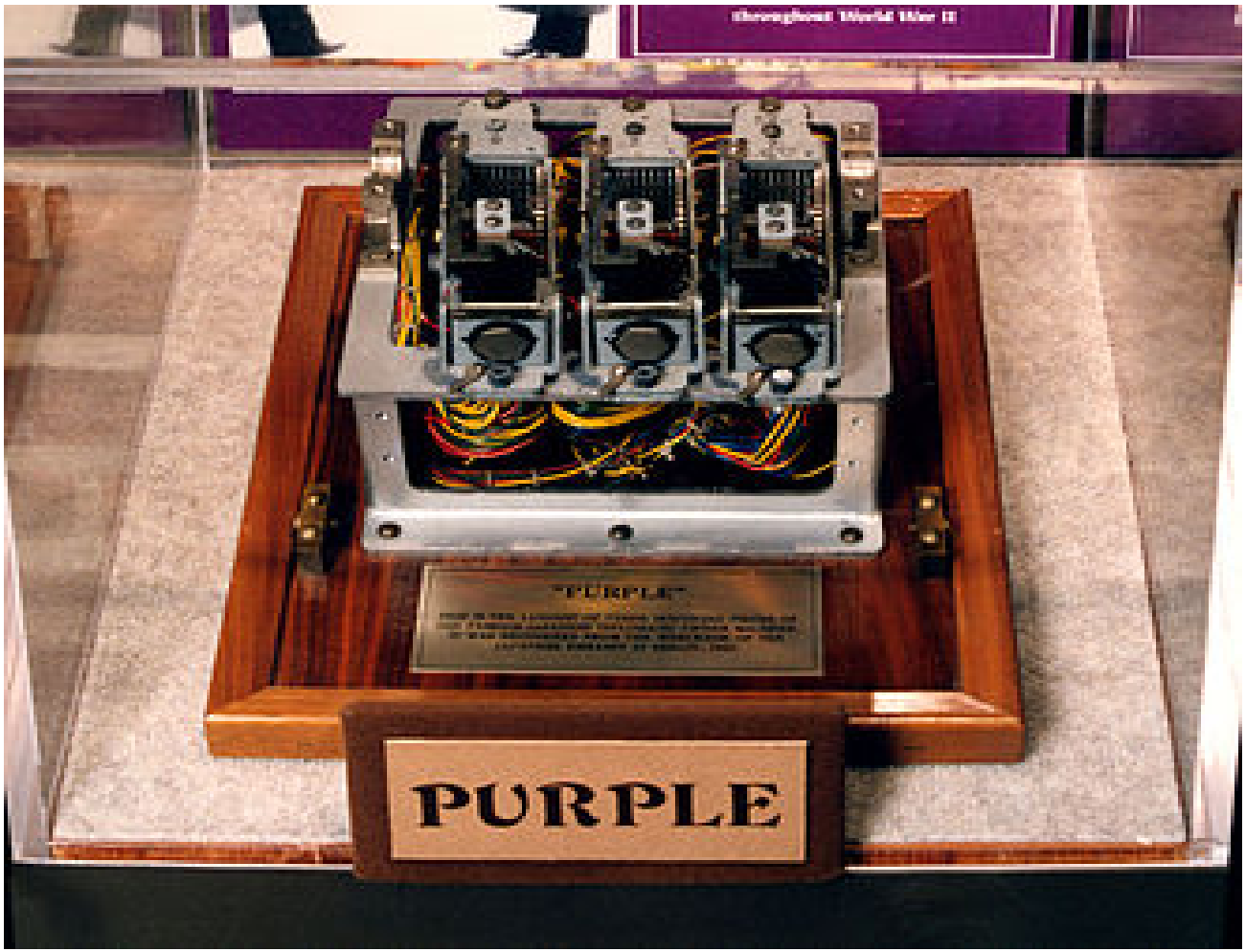
- Sigaba, used by U.S. during *World War II.*

Some Examples

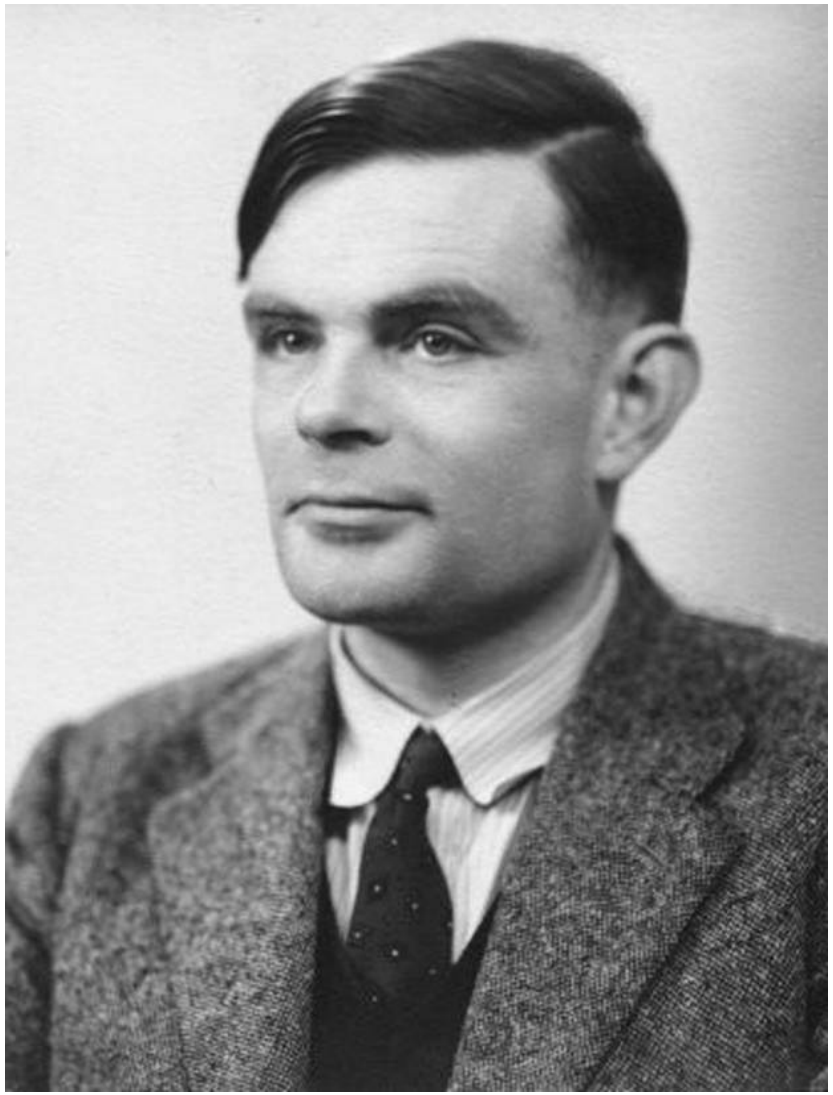
- Japanese "Enigma" Rotor Cipher Machine

Some Examples

- Japanese Purple Machine (97-shiki obun inji-ki)



People Working in Breaking Codes



Alan Turing
(1912-1954)

Claude E. Shannon
(1916-2001)

Cryptography History

- History (until 1970's)

  "*Symmetric*" cryptography

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$\mathcal{Q}$: How can they do this?

$\mathcal{Q}$: What if Bob could send Alice a "special key" useful only for encryption but no help for decryption?

Caesar Cipher

- Key: $k = 0, 1, \ldots, 25$

  Encryption: encode i as $(i + k) \bmod 26$

  Decryption: decode j as $(j - k) \bmod 26$

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  Kerchoff's Principle (1883): System should be secure even if algorithms are known, as long as key is secret.

Substitution Cipher

- Key: table mapping each letter to another letter

A	B	C	...	Z
V	R	E	...	D

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Key observation: can recover plaintext using *statistics* on *letter frequencies.*

Substitution Cipher

- Table 1: Relative frequencies of the letters of the English language

Letter	Relative Frequency (%)	Letter	Relative Frequency (%)
a	8.167	n	6.749
b	1.492	o	7.507
c	2.782	p	1.929
d	4.253	q	0.095
e	12.702	r	5.987
f	2.228	s	6.327
g	2.015	t	9.056
h	6.094	u	2.758
i	6.966	v	0.978
j	0.153	w	2.360
k	0.772	x	0.150
l	4.025	y	1.974
m	2.406	z	0.074

Substitution Cipher

Table 2: Number of Diagraphs Expected in 2,000 Letters of English Text

th	-	50	at	-	25	st	-	20
er	-	40	en	-	25	io	-	18
on	-	39	es	-	25	le	-	18
an	-	38	of	-	25	is	-	17
re	-	36	or	-	25	ou	-	17
he	-	33	nt	-	24	ar	-	16
in	-	31	ea	-	22	as	-	16
ed	-	30	ti	-	22	de	-	16
ne	-	30	to	-	22	rt	-	16
ha	-	26	it	-	20	ve	-	16

Table 3: The 15 Most Common Trigraphs in the English Language

1	-	the	6	-	tio	11	-	edt
2	-	and	7	-	for	12	-	tis
3	-	tha	8	-	nde	13	-	oft
4	-	ent	9	-	has	14	-	sth
5	-	ion	10	-	nce	15	-	men

Substitution Cipher

- LIVITCSWPIYVEWHEVSRIQMXLEYVEOIEWHRXEXIPFE
  MVEWHKVSTYLXZIXLIKIIXPIJVSZEYPERRGERIMWQL
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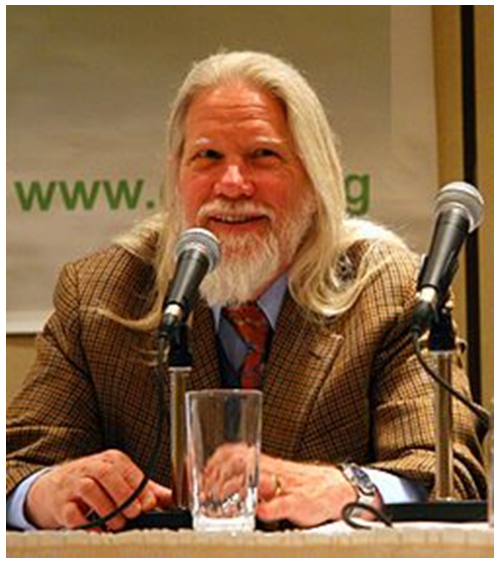
HereUpOnLeGrandAroseWithAGraveAndStatelyAirAndBroug
MeTheBeetleFromAGlassCaseInWhichItWasEnclosedIt-
WasABe

Cryptography History

- History (from 1976)
  - ◇ W. Diffie, M. Hellman, "New direction in cryptography", *IEEE Transactions on Information Theory, vol. 22, pp. 644-654, 1976.*

"We stand today on the brink of a revolution in cryptography."



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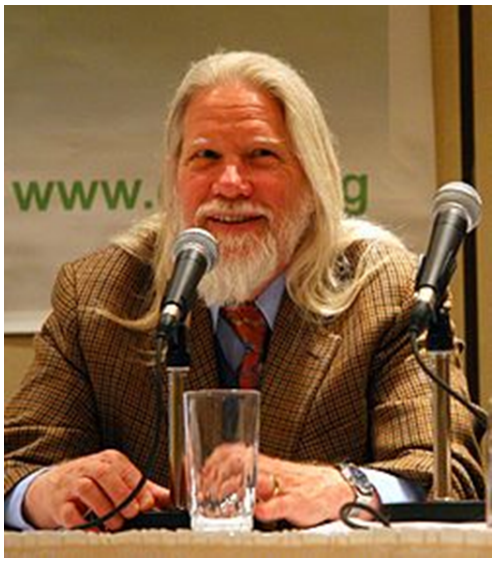

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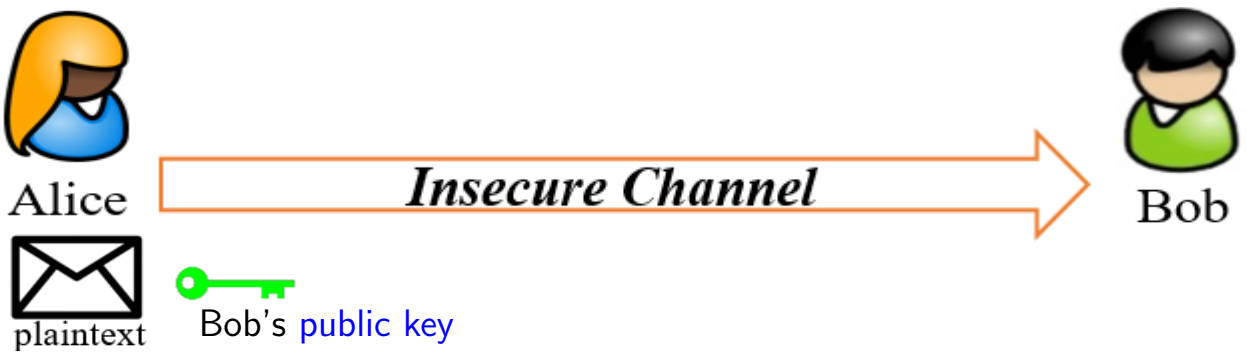
2015 **Turing Award**

2015	Martin E. Hellman Whitfield Diffie	For fundamental contributions to **modern cryptography**. Diffie and Hellman's groundbreaking 1976 paper, "New Directions in Cryptography,"[39] introduced the ideas of public-key cryptography and digital signatures, which are the foundation for most regularly-used security protocols on the internet today.[40]

Public Key Cryptography

- Alice wants to send a message to Bob



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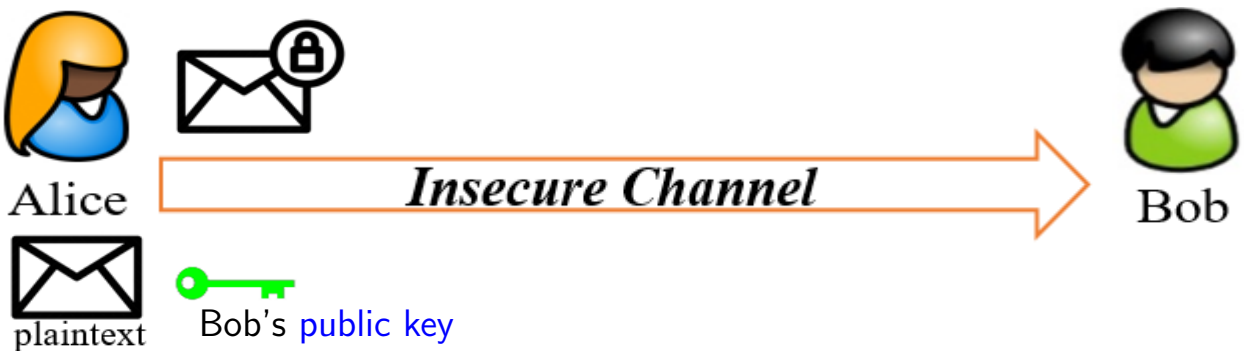

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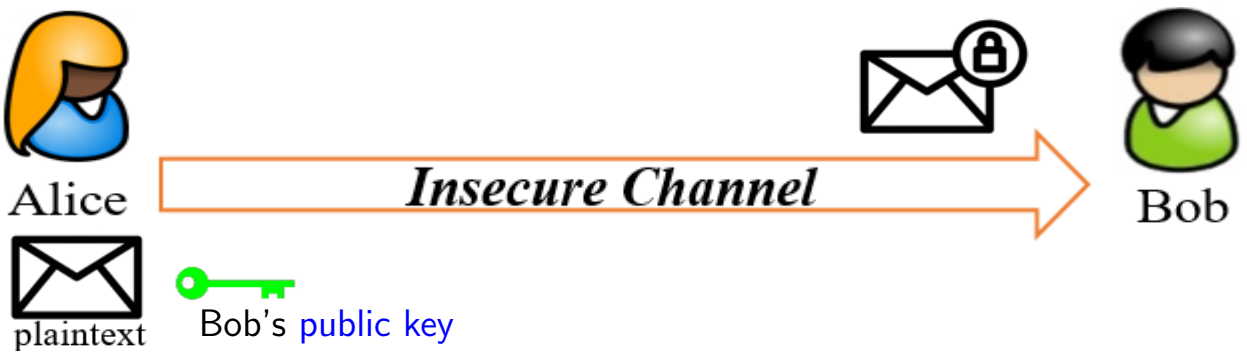

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R. Rivest, A. Shamir, L. Adleman, "A method for obtaining digital signatures and public-key cryptosystems", *Communications of the ACM*, vol. 21-2, pages 120-126, 1978.

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Pick two large primes, p and q. Let $n = pq$, then $\phi(n) = (p-1)(q-1)$. Encryption and decryption keys e and d are selected such that

- $\gcd(e, \phi(n)) = 1$
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Theorem (*Correctness*) : Let p and q be two odd primes, and define $n = pq$. Let e be relatively prime to $\phi(n)$ and let d be the multiplicative inverse of e modulo $\phi(n)$. For each integer x such that $0 \leq x < n$,

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$\mathcal{Q}$: How to prove this?

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Encryption: $M = 28, C = M^7 \bmod 55 = 52$

Decryption: $M = C^{23} \bmod 55 = 28$

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$\mathcal{Q}$: Consider the RSA system, where $n = pq$ is the modulus. Let (e, d) be a key pair for the RSA. Define

$$\lambda(n) = \text{lcm}(p-1, q-1)$$

and compute $d' = e^{-1} \bmod \lambda(n)$. Will decryption using d' instead of d still work?

Applications of RSA

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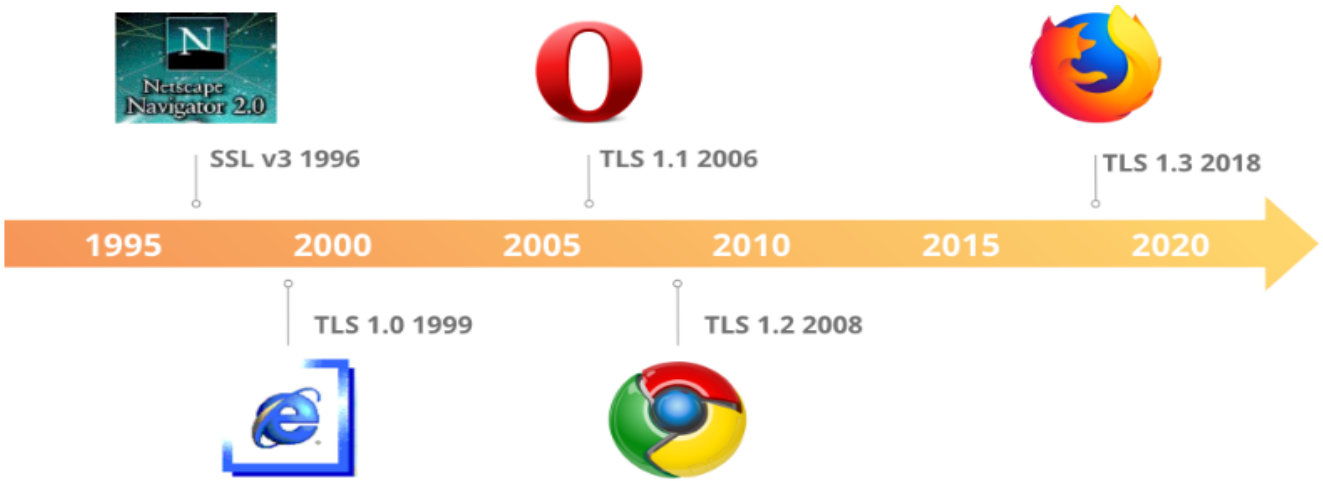

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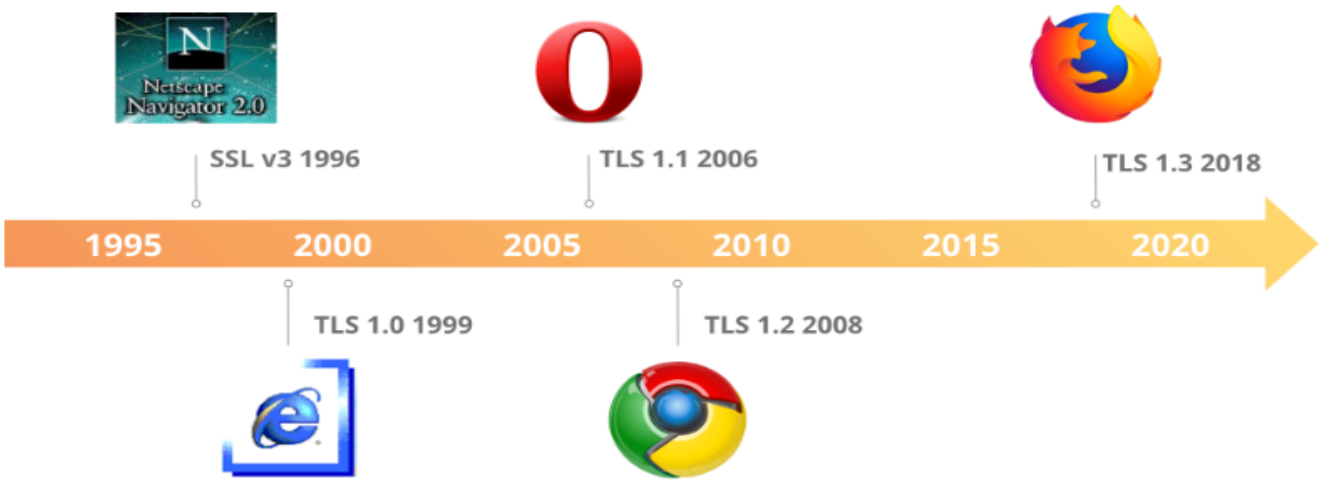

Key exchange/agreement and authentication

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DH-RSA	No	Yes	Yes	Yes	Yes	No
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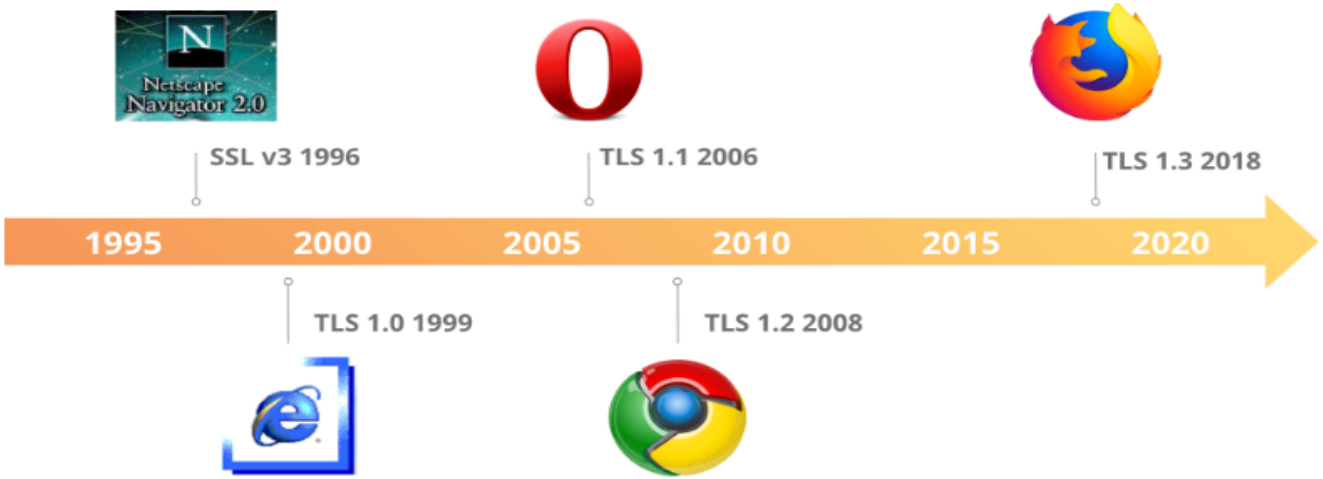

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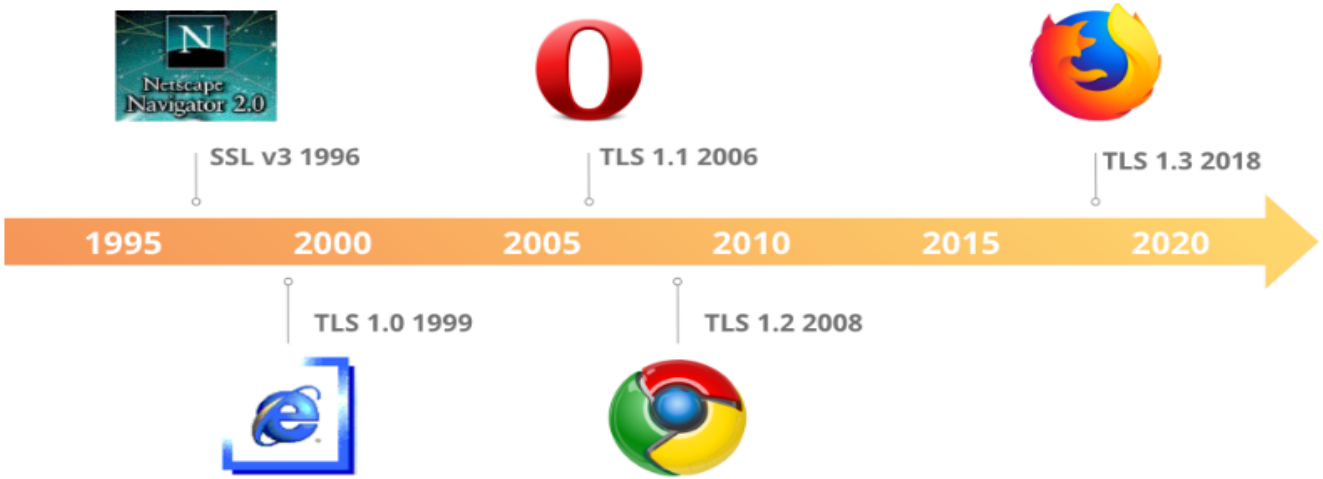

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CS 305 – Computer Networks
CS 403 – Cryptography and Network Security

Using RSA for Digital Signature

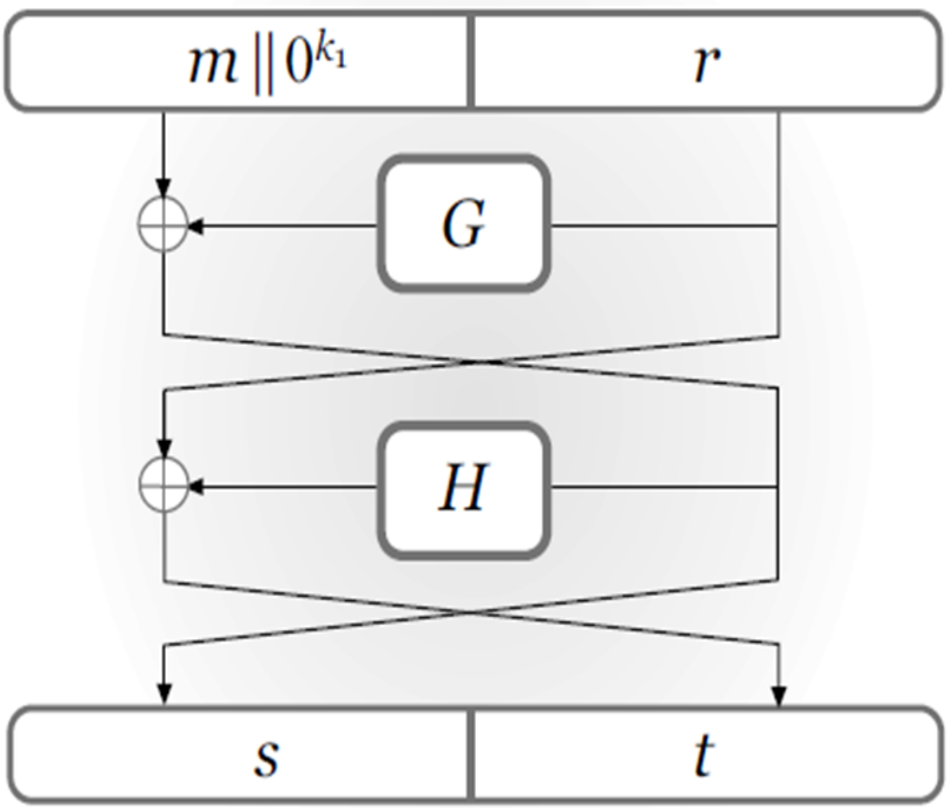
$S = M^d \bmod n$ (RSA **signature)**

$M = S^e \bmod n$ (RSA **verification)**

Why?

RSA-OAEP Standard

- RSA-OAEP (Optimal Asymmetric Encryption Padding) is *IND-CCA2 secure*.
- PKCS#1 V2, RFC2437 Standard



Next Lecture

- cryptography ...