CS215: Discrete Math (H)
2024 Fall Semester Written Assignment #1
Due: Oct. 14th, 2024, please submit at the beginning of class

Q.1 Let p, q and r be the following propositions
p: You get an A on the final exam.
q: You do every exercise in this book.
r: You get an A in this class.
Write these propositions using p, q, and r and logical connectives (including negations).

(a) You get an A in this class, but you do not do every exercise in this book.

(b) To get an A in this class, it is necessary for you to get an A on the final.

(c) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

(d) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.

(e) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

Solution:

(a) $r \land \neg q$

(b) $r \to p$

(c) $p \land \neg q \land r$

(d) $(p \land q) \to r$

(e) $r \leftrightarrow (q \lor p)$

□

Q.2 Use truth tables to decide whether or not the following two propositions are equivalent.

(a) $(p \oplus q) \to (p \wedge q)$ and $(p \oplus q) \to (p \oplus \neg q)$

(b) $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$

(c) $(\neg q \wedge \neg(p \to q))$ and $\neg p$

(d) $(p \to \neg q) \leftrightarrow (r \to (p \vee \neg q))$ and $q \vee (\neg p \wedge \neg r)$

Solution:

(a) The combined truth table is

p	q	$(p \oplus q) \to (p \wedge q)$	$(p \oplus q) \to (p \oplus \neg q)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

By comparing the last two columns, we have that they are equivalent.

(b) The combined truth table is:

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \wedge q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
F	F	T	T	T	T	F	T
F	T	F	T	F	F	F	F
T	F	F	F	T	F	F	F
T	T	T	F	F	F	T	T

By comparing the third and last columns, we have that they are equivalent.

(c) The combined truth table is:

p	q	$p \to q$	$\neg q$	$\neg(p \to q)$	$\neg q \wedge \neg(p \to q)$	$\neg p$
F	F	T	T	F	F	T
F	T	T	F	F	F	T
T	F	F	T	T	T	F
T	T	T	F	F	F	F

By comparing the last two columns, we have that they are not equivalent.

(d) The two truth tables are:

p	q	r	$p \to \neg q$	$(p \vee \neg q)$	$r \to (p \vee \neg q)$	$(p \to \neg q) \leftrightarrow (r \to (p \vee \neg q))$
F	F	F	T	T	T	T
F	F	T	T	T	T	T
F	T	F	T	F	T	T
F	T	T	T	F	F	F
T	F	F	T	T	T	T
T	F	T	T	T	T	T
T	T	F	F	T	T	F
T	T	T	F	T	T	F

p	q	r	$\neg p \wedge \neg r$	$q \vee (\neg p \wedge \neg r)$
F	F	F	T	T
F	F	T	F	F
F	T	F	T	T
F	T	T	F	T
T	F	F	F	F
T	F	T	F	F
T	T	F	F	T
T	T	T	F	T

Since the final columns are not the same in both truth tables, we know that these two propositions are not equivalent.

$\square$

Q.3 Use logical equivalences to prove the following statements.

(a) $(p \wedge \neg q) \to r$ and $p \to (q \vee r)$ are equivalent.

(b) $((p \to q) \wedge (q \to r)) \to (p \to r)$ is a tautology.

Solution:

(a) We have

$$
\begin{aligned}
&(p \wedge \neg q) \rightarrow r \\
\equiv\ & \neg(p \wedge \neg q) \vee r \quad \text{Useful} \\
\equiv\ & (\neg p \vee q) \vee r \quad \text{De Morgan} \\
\equiv\ & \neg p \vee (q \vee r) \quad \text{Associative} \\
\equiv\ & p \rightarrow (q \vee r) \quad \text{Useful}
\end{aligned}
$$

Therefore, they are equivalent.

(b) We have

$$
\begin{aligned}
&((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r) \\
\equiv\ & \neg((p \rightarrow q) \wedge (q \rightarrow r)) \vee (p \rightarrow r) \quad \text{Useful} \\
\equiv\ & \neg(p \rightarrow q) \vee \neg(q \rightarrow r) \vee (p \rightarrow r) \quad \text{De Morgan} \\
\equiv\ & \neg(\neg p \vee q) \vee \neg(\neg q \vee r) \vee (p \rightarrow r) \quad \text{Useful} \\
\equiv\ & (\neg\neg p \wedge \neg q) \vee (\neg\neg q \wedge \neg r) \vee (p \rightarrow r) \quad \text{De Morgan} \\
\equiv\ & (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r) \quad \text{Double negation, Useful} \\
\equiv\ & ((p \wedge \neg q) \vee q) \wedge ((p \wedge \neg q) \vee \neg r) \vee (\neg p \vee r) \quad \text{Distributive} \\
\equiv\ & ((p \vee q) \wedge (\neg q \vee q)) \wedge ((p \vee \neg r) \wedge (\neg q \vee \neg r)) \vee (\neg p \vee r) \quad \text{Distributive} \\
\equiv\ & ((p \vee q) \wedge (p \vee \neg r) \wedge (\neg q \vee \neg r)) \vee (\neg p \vee r) \quad \text{Negation} \\
\equiv\ & ((p \vee q) \vee (\neg p \vee r)) \wedge ((p \vee \neg r) \vee (\neg p \vee r)) \wedge ((\neg q \vee \neg r) \vee (\neg p \vee r)) \quad \text{Distributive} \\
\equiv\ & (p \vee q \vee \neg p \vee r) \wedge (p \vee \neg r \vee \neg p \vee r) \wedge (\neg q \vee \neg r \vee \neg p \vee r) \quad \text{Associative} \\
\equiv\ & T \wedge T \wedge T \quad \text{Negation} \\
\equiv\ & T.
\end{aligned}
$$

Thus, it is a tautology.

□

Q.4 Show that $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ are not logically equivalent.
Solution: It suffices to give a counterexample. When p, q and r are all false, $(p \rightarrow q) \rightarrow r$ is false, but $p \rightarrow (q \rightarrow r)$ is true.

□

Q.5 Suppose that p, q, r, s are all propositions. You are given the following statement

$$(q \to (r \lor p)) \to ((\neg r \lor s) \land \neg s).$$

Prove that this implies $\neg r$ using logical equivalences and inference rules.

Solution:
We have the following

$$\begin{aligned}
&(q \to (r \lor p)) \to ((\neg r \lor s) \land \neg s) \\
&\quad \equiv \quad \neg(q \to (r \lor p)) \lor ((\neg r \lor s) \land \neg s) \quad \text{Useful} \\
&\quad \equiv \quad \neg(q \to (r \lor p)) \lor (\neg r \land \neg s) \lor (s \land \neg s) \quad \text{Distributive} \\
&\quad \equiv \quad \neg(q \to (r \lor p)) \lor (\neg r \land \neg s) \lor F \quad \text{Negation} \\
&\quad \equiv \quad \neg(q \to (r \lor p)) \lor (\neg r \land \neg s) \quad \text{Identity} \\
&\quad \equiv \quad \neg(\neg q \lor (r \lor p)) \lor (\neg r \land \neg s) \quad \text{Useful} \\
&\quad \equiv \quad (\neg\neg q \land \neg(r \lor p)) \lor (\neg r \land \neg s) \quad \text{De Morgan} \\
&\quad \equiv \quad (q \land (\neg r \land \neg p)) \lor (\neg r \land \neg s) \quad \text{Double negation} \\
&\quad \equiv \quad ((q \land \neg p) \land \neg r) \lor (\neg s \land \neg r) \quad \text{Commutative, Associative} \\
&\quad \equiv \quad (((q \land \neg p) \lor \neg s) \land \neg r) \quad \text{Distributive} \\
&\quad \to \quad \neg r \quad \text{Simplication}
\end{aligned}$$

$\square$

Q.6 Let $L(x, y)$ be the statement "x loves y", where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statement.

(a) Everybody loves somebody.

(b) There is somebody whom everybody loves.

(c) Nobody loves everybody.

(d) There is somebody whom no one loves.

(e) There is exactly one person whom every body loves.

(f) There are exactly two people whom Lynn loves.

(g) There is someone who loves no one besides himself or herself.

Solution:

(a) $\forall x \exists y\ L(x, y)$

(b) $\exists y \forall x\ L(x, y)$

(c) $\forall x \exists y \neg L(x, y)$

(d) $\exists x \forall y \neg L(y, x)$

(e) $\exists x (\forall y L(y, x) \wedge \forall z ((\forall w L(w, z)) \rightarrow z = x))$

(f) $\exists x \exists y (x \neq y \wedge L(Lynn, x) \wedge L(Lynn, y) \wedge (\forall z (L(Lynn, z) \rightarrow (z = x \vee z = y)))$

(g) $\exists x \forall y (L(x, y) \leftrightarrow x = y)$

$\square$

Q.7 Suppose that variables x and y represent real numbers, and $L(x, y)$: $x < y$, $Q(x, y)$: $x = y$, $E(x)$: x is even, $I(x)$: x is an integer. Write the following statements using these predicates and any needed quantifiers.

(1) Every integer is even.

(2) If $x < y$, then x is not equal to y.

(3) There is no largest real number.

Solution:

(1) $\forall x (I(x) \rightarrow E(x))$

(2) $\forall x \forall y (L(x, y) \rightarrow \neg Q(x, y))$

(3) $\forall x \exists y L(x, y)$

$\square$

Q.8 For the predicate $P(x, y)$ with two variables x, y, answer the following two questions.

(1) Give an example of a predicate $P(x, y)$ such that $\exists x \forall y P(x, y)$ and $\forall y \exists x P(x, y)$ have *different* truth values.

(2) If $\forall y \exists x P(x, y)$ is true, does it necessarily follow that $\exists x \forall y P(x, y)$ is true?

Solution:

(1) Let the domain be the set of natural numbers $\mathbb{N}$, and the predicate $P(x, y)$ denote $x > y$. Then, $\exists x \forall y P(x, y)$ is false, but $\forall y \exists x P(x, y)$ is true.

(2) No. From (1), we know that if $\forall y \exists x P(x, y)$ is true, $\exists x \forall y P(x, y)$ is not necessarily true but false.

$\square$

Q.9 Each of the two below contains a pair of statements, (i) and (ii). For each pair, say whether (i) is equivalent to (ii), i.e., for all $P(x)$ and $Q(x)$, (i) is true if and only if (ii) is true. Here $\mathbb{R}$ denotes the set of all *real numbers.*

If they are equivalent, *all you have to do is to say that they are equivalent.* If they are not equivalent, give a counterexample. A counterexample should involve a specification of $P(x)$ and $Q(x)$ and an explanation as to why the resulting statement is false.

(1) (i) $(\forall x \in \mathbb{R}\ P(x)) \vee (\forall x \in \mathbb{R}\ Q(x))$
(ii) $\forall x \in \mathbb{R}\,(P(x) \vee Q(x))$

(2) (i) $(\forall x \in \mathbb{R}\ P(x)) \wedge (\forall x \in \mathbb{R}\ Q(x))$
(ii) $\forall x \in \mathbb{R}\,(P(x) \wedge Q(x))$

(3) (i) $(\forall x \in \mathbb{R}\ P(x)) \wedge (\exists y \in \mathbb{R}\ Q(y))$
(ii) $\forall x \in \mathbb{R}\,(\exists y \in \mathbb{R}\ (P(x) \wedge Q(y)))$

(4) (i) $(\forall x \in \mathbb{R}\ P(x)) \vee (\exists y \in \mathbb{R}\ Q(y))$
(ii) $\forall x \in \mathbb{R}\,(\exists y \in \mathbb{R}\ (P(x) \vee Q(y)))$

Solution:

(1) Not equivalent. Let $P(x)$ be "$x \geq 0$" and $Q(x)$ be "$x < 0$". (i) is false but (ii) is true.

(2) Equivalent.

(3) Equivalent.

(4) Equivalent.

$\square$

Q.10

(a) Give the negation of the statement

$$\forall n \in \mathbb{N}\ (n^3 + 6n + 5 \text{ is odd} \rightarrow n \text{ is even}).$$

(b) Either the original statement in (a) or its negation is true. Which one is it and explain why?

Solution:

(a) The negation is

$$\exists x \in \mathbb{N}\ (n^3 + 6n + 5 \text{ is odd} \wedge n \text{ is odd}).$$

(b) If n is odd then $n^3 + 6n + 5$ is even because n^3 is then odd and $6n$ is then even. Therefore, the original statement is true.

$\square$

Q.11

(a) Let P be a proposition in atomic propositions p and q. If $P = \neg(p \leftrightarrow (q \vee \neg p))$, prove that P is equivalent to $\neg p \vee \neg q$.

(b) If P is of any length, using any of the logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, prove that P is logically equivalent to a proposition of the from

$$A \Box B,$$

where $\Box$ is one of $\wedge$, $\vee$, $\leftrightarrow$, and A and B are chosen from $\{p, \neg p, q, \neg q\}$.

Solution:

(a) This can be proved by truth table.

Alternatively, we prove it using logical equivalences as follows.

$$\begin{aligned}
P &= \neg(p \leftrightarrow (q \vee \neg p)) \\
&\equiv \neg((p \rightarrow (q \vee \neg p)) \wedge ((q \vee \neg p) \rightarrow p)) \quad \text{Definition} \\
&\equiv \neg((\neg p \vee (q \vee \neg p)) \wedge (\neg(q \vee \neg p) \vee p)) \quad \text{Useful} \\
&\equiv \neg((\neg p \vee q) \wedge (\neg(q \vee \neg p) \vee p)) \quad \text{Idempotent}
\end{aligned}$$

For simplicity, let $r = \neg p \vee q$, then we have

$$\begin{aligned}
P &\equiv \neg(r \wedge (\neg r \vee p)) \\
&\equiv \neg r \vee \neg(\neg r \vee p) \quad \text{De Morgan} \\
&\equiv \neg r \vee (r \wedge \neg p) \quad \text{De Morgan and double negation} \\
&\equiv (\neg r \vee r) \wedge (\neg r \vee \neg p) \quad \text{Distributive} \\
&\equiv T \wedge (\neg r \vee \neg p) \quad \text{Negation} \\
&\equiv \neg r \vee \neg p \quad \text{Identity} \\
&\equiv (p \wedge \neg q) \vee \neg p \quad \text{De Morgan} \\
&\equiv (p \vee \neg p) \wedge (\neg q \vee \neg p) \quad \text{Distributive} \\
&\equiv T \wedge (\neg q \vee \neg p) \quad \text{Negation} \\
&\equiv \neg p \vee \neg q \quad \text{Identity.}
\end{aligned}$$

(b) For the proposition P, since the two involved atomic propositions p and q can have at most 4 combinations of truth tables, P has at most 2^4 different forms in terms of truth tables up to logical equivalence. It then suffices to prove that a proposition of the form $A \Box B$ has also 2^4 different forms in terms of truth tables up to logical equivalence.

If $A \in \{p, \neg p\}$, $B \in \{q, \neg q\}$, and $\Box \in \{\wedge, \vee\}$, then $A \Box B$ has $2 \times 2 \times 2 = 8$ different possible forms. If $A \in \{p, \neg p\}$, $B \in \{q, \neg q\}$ and $\Box = \leftrightarrow$, then there are two extra different possibilities: $p \leftrightarrow q$ and $p \leftrightarrow \neg q$. Together with $p \vee p \equiv p$, $p \vee \neg p \equiv T$, $q \vee q \equiv q$, $p \wedge \neg p \equiv F$ and similarly $\neg p$, $\neg q$, we will have the $2^4 = 16$ different forms by $A \Box B$. This proves the statement.

$\Box$

Q.12 For the following argument, explain which rules of inference are used for each step.

"All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners."

Solution:

Let $s(x)$ be "x is a movie produced by Sayles," let $c(x)$ be "x is a movie about coal miners", and let $w(x)$ be "movie x is wonderful". we are given premises $\forall x(s(x) \rightarrow w(x))$ and $\exists x(s(x) \wedge c(x))$, and we want to conclude $\exists x(c(x) \wedge w(x))$.

step	**reason**
1. $\exists x(s(x) \wedge c(x))$	hypothesis
2. $s(y) \wedge c(y)$	existential instantiation using 1.
3. $s(y)$	simplification using 2.
4. $\forall x(s(x) \rightarrow w(x))$	hypothesis
5. $s(y) \rightarrow w(y)$	universal instantiation using 4.
6. $w(y)$	modus ponens using 3. and 5.
7. $c(y)$	simplification using 2.
8. $w(y) \wedge c(y)$	conjunction using 6 and 7.
9. $\exists x(c(x) \wedge w(x))$	existential generalization using 8.

$\Box$

Q.13 Prove or disprove the following.

(1) For two irrational numbers a and b, a^b is also irrational.

(2) For an irrational number a, $\sqrt{a}$ is also irrational.

(3) There is a rational number x and an irrational number y such that x^y is irrational.

Solution:

(1) The statement is false. Consider $\sqrt{2}^{\sqrt{2}}$, if it is rational, then we have a counterexample that a^b is rational for a, b both irrational; if it is not, then we consider $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$, which equals 2, again a counterexample. Alternatively, consider $\sqrt{2}^{2\log_2 3}$, which equals 3, a counterexample, since both $\sqrt{2}$ and $\log_2 3$ are irrational.

(2) The statement is true. Suppose to the contrary that $\sqrt{a}$ is rational, i.e., $\sqrt{a} = \frac{c}{d}$ for two integers c and d. Then we have $a = \frac{c^2}{d^2}$, which is again rational, contradiction. Thus, $\sqrt{a}$ must be irrational.

(3) Let $x = 2$ and $y = \sqrt{2}$. If $x^y = 2^{\sqrt{2}}$ is irrational, we are done. If not, let $x = 2^{\sqrt{2}}$ and $y = \sqrt{2}/4$. Then $x^y = (2^{\sqrt{2}})^{\sqrt{2}/4} = 2^{\sqrt{2}\cdot(\sqrt{2})/4} = \sqrt{2}$.

$\square$

Q.14 Suppose that we have a theorem: "$\sqrt{n}$ is irrational whenever n is a positive integer that is *not* a perfect square." Use this theorem to prove that $\sqrt{2}+\sqrt{3}$ is irrational.

Solution: We give a proof by contradiction. If $\sqrt{2}+\sqrt{3}$ is a rational number, then its square is also rational, which is $5+2\sqrt{6}$. Subtracting 5 and dividing by 2, we have $\sqrt{6}$ is also rational. However, this contradicts the theorem.

$\square$

Q.15 Prove that between every rational number and every irrational number there is an irrational number.

Solution: The average of two different numbers is certainly always between the two numbers. Furthermore, the average a of rational number x and irrational number y must be irrational, because the equation $a = (x+y)/2$ leads to $y = 2a - x$, which would be rational if a were rational.

$\square$

Q.16 Give a direct proof that: Let a and b be integers. If $a^2 + b^2$ is even, then $a + b$ is even.
Solution: Observe that $a^2 + b^2 = (a + b)^2 - 2ab$. Thus, $(a + b)^2$ has the same parity as $a^2 + b^2$. So $(a + b)^2$ is even. Then $a + b$ is also even.

$\square$

Q.17 Let the coefficients of the polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + x^n$ be integers. Show that any *real* root of the equation $f(x) = 0$ is either integral or irrational. Note that in your proof, you may direct use the following result without a proof. "**Fact.** If a prime p is a factor of some power of an integer, then it is a factor of that integer."

Solution: Let r be a real root of the polynomial, such that

$$a_0 + a_1r + a_2r^2 + \cdots + a_{n-1}r^{n-1} + r^n = 0.$$

There are three cases: either r is an integer, or r is irrational, or $r = s/t$ for integers s and t which have no common factors and such that $t > 1$. We want to eliminate the last case, so assume to the contrary that it holds for some r.

Substituting s/t for r and multiplying both sides of the above equation by t^m yields:

$$\begin{aligned} a_0t^n + a_1st^{n-1} + a_2s^2t^{n-2} + \cdots + a_{n-1}s^{n-1}t + s^n &= 0, \\ a_0t^n + a_1st^{n-1} + a_2s^2t^{n-2} + \cdots + a_{n-1}s^{n-1}t &= -s^n. \end{aligned}$$

Now since $t > 1$, it must have a prime factor p. The prime p therefore divides each term of the left-hand side of the equation above, so p also divides the right-hand side, $-s^n$. This means that p divides s^n. Then by Fact, p is also a factor of s. Thus, we know that p is a common factor of s and t, contradicting the fact that s and t have no common factors.