Discrete Math Ass 2 12310401 王予恒

Q.1 (a) False. counter example: $A=\{1,2,3\}$ $B=\{4,5\}$.

(b) True $A\cap B\cap C = \{x \mid x\in A \wedge x\in B \wedge x\in C)$.

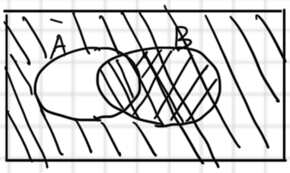
$$\subseteq \{x \mid x\in A \wedge x\in B\}$$

$$\subseteq \{x \mid x\in A \vee x\in B\}$$

$$= A\cup B$$

(c) False

$$\overline{(A-B)} \cap (B-A) = \{x \mid (x\in A \wedge x\notin B)' \wedge (x\in B \wedge x\notin A)\}$$



$$= \{x \mid (x\notin A \vee x\in B) \wedge (x\in B \wedge x\notin A)\}$$

$$= \{x \mid x\notin A \wedge x\in B\}$$

$$= B-A$$

Q.2.

(1) Disproof: conterexample:

$A=\{1,2,3,4\}$ $B=\{2,3,5,6\}$ $C=\{3,4,7,8\}$

$C-(A\cap B) = \{4,7,8\}$

$(C-A)\cap(C-B) = \{7,8\}$.

(2) Proof:

$X\in P(A)\cap P(B)$ means $X\in P(A)$ and $X\in P(B)$, implies that $X\subseteq A$ and $X\subseteq B$, then $X\subseteq A\cap B$, thus $X\in P(A\cap B)$. So $P(A)\cap P(B)\subseteq P(A\cap B)$

$X\in P(A\cap B)$ means that X is a subset of both A and B, which implies $X\in P(A)$ and $X\in P(B)$

Then we get $X\in P(A)\cap P(B)$. So $P(A\cap B)\subseteq P(A)\cap P(B)$

Since two statement above, we have $P(A)\cap P(B) = P(A\cap B)$

(3) False $A=\{1\}$ $B=\{2\}$

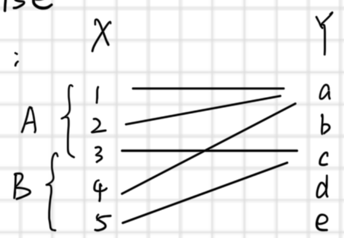
$P(A)\cup P(B) = \{\{1\},\{2\}\}$

$P(A\cup B) = \{\{1\},\{2\},\{1,2\}\}$.

$P(A)\cup P(B) \neq P(A\cup B)$

(4) False

counter :



$f(A \cap B) = \{c\}$.

$f(A) \cap f(B) = \{a, c\}$

Q.3 (a)

A	B	C	$A \oplus B$	$B \oplus C$	$A \oplus (B \oplus C)$	$(A \oplus B) \oplus C$
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	0	0	0
1	0	0	1	0	1	1
1	0	1	1	1	0	0
1	1	0	0	1	0	0
1	1	1	0	0	1	1

From the truth table, we can get $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.

(b) Suppose $A \neq B$, then $\exists x\ x \in A \wedge x \notin B$ or $\exists x\ x \notin A \wedge x \in B$.

when $\exists x, x \in A \wedge x \notin B$ $\quad A \oplus C = 1 \oplus C = \neg C$, $B \oplus C = 0 \oplus C = C$.

$A \oplus C \neq B \oplus C$

when $\exists x, x \notin A \wedge x \in A$ $\quad A \oplus C = 0 \oplus C = C$, $B \oplus C = 1 \oplus C = 0$

$A \oplus C \neq B \oplus C$.

By contradiction, $A = B$

Q.4. (a) finity $\Rightarrow$ countable.

(b) countable

contstructing the list: first list (a,b) with $a+b=0$, then list (a,b) with $a+b=1$, and so on

$(0,0)$ $(1,0)$ $(2,0)$ ...

$(0,1)$ $(1,1)$ ⋮ ....

$(0,2)$ ⋮ ...

⋮

(c) uncountable.

Because for any $a_0 \in N$, it has a subset $\{(a_0, b) \mid b \in R\}$ which is an uncountable set.

Q.5

(a) $A = B = R$

(b) $A = R$ $B = R - N$

(c) $A = R$ $B = (0, 1)$.

Q.6. only if: Because $A \subseteq P(A)$, $P(A) \subseteq P(B)$, we have $A \subseteq P(B)$.
which means A is a subset of B, Thus $A \subseteq B$

if: Let $X \in P(A)$ which means $X \subseteq A$. Since $X \subseteq A$, $A \subseteq B$
Therefore $X \subseteq B$. Thus $X \in P(B)$
Since for all element X in $P(A)$ we can show that $X \in P(B)$
Hence, $P(A) \subseteq P(B)$

Q.7.

f is one-to-one $\Rightarrow$ Proof every $f(x) = f(y)$ implies $x = y$ for $x, y \in A$.

Proof: Assume $f(x) = f(y)$, $x, y \in A$

since $g \circ f = 1A$, we have $g \circ f(x) = g(f(x)) = x$

$$g \circ f(y) = g(f(y)) = y$$

since $f(x) = f(y)$, thus $g(f(x)) = g(f(y))$

Hence, $x = y$. So f is one-to-one.

g is onto: for every $a \in A$, there exists an $b \in B$ that $g(b) = a$

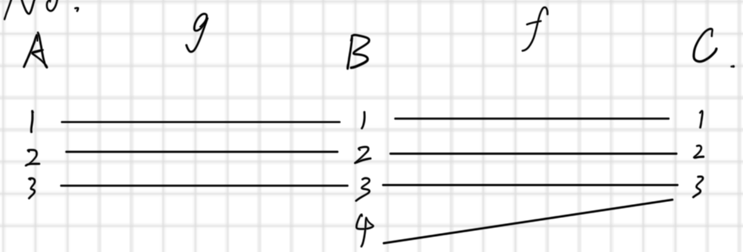
since $g \circ f(a) = g(f(a)) = a$

Thus for every $a \in A$, there exists $f(a) \in B$ that $g(f(a)) = a$

Hence g is onto.

Q.8. (a) No.

counter example:



(b) g must be one-to-one.

Suppose g is not one-to-one, then there exists $a_1 \neq a_2 \in A$ and $g(a_1) = g(a_2)$. Thus $f(g(a_1)) = f(g(a_2))$ which means that $\exists a_1 \neq a_2 \in A$ $f \circ g(a_1) = f \circ g(a_2)$. Contradict to $f \circ g$ is one-to-one. Hence g must be one-to-one.

(c) g must be one-to-one.

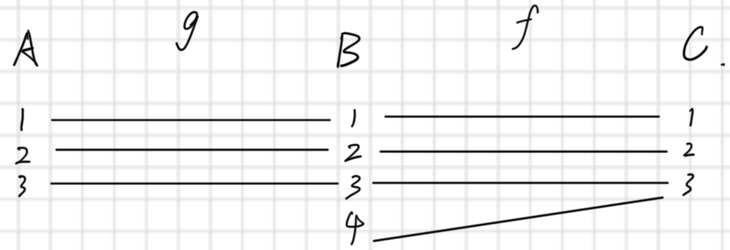
Let $a_1, a_2 \in A$ that $g(a_1) = g(a_2)$ then $f(g(a_1)) = f(g(a_2)$. Because $f \circ g$ is one-to-one, thus $a_1 = a_2$. Hence g is one-to one.

(d) Yes.

If $f \circ g$ is onto, then for every $c \in C$, there exist an $a \in A$ that $f \circ g(a) = c$. Since for every a, we have a $b \in B$ that $b \in B$. Thus for every $c \in C$, we have a $b \in g(b)$ let $f(b) = c$,

Hence f is onto

(e) No.



$$(n-1)^3 = n^3 - 3n^2 + 3n - 1$$

Q.9.

$$n^3 - (n-1)^3 = 3n^2 - 3n + 1$$
$$(n-1)^3 - (n-2)^2 = 3(n-1)^2 - 3(n-1) + 1$$
$$\vdots$$
$$2^3 - 1^3 = 3\times 2^2 - 3\times 2 + 1$$
$$1^3 - 0^3 = 3\times 1^2 - 3\times 1 + 1$$

$$n^3 = 3(1^2 + 2^2 + \cdots + n^2) - 3\times(1+2+\cdots+n) + n$$
$$= 3\sum_{k=1}^{n} k^2 - 3\times\frac{n(n+1)}{2} + n$$
$$\sum_{k=1}^{n} = \frac{n^3}{3} + \frac{n(n+1)}{2} - \frac{n}{3}$$
$$= \frac{2n^3+3n^2+n}{6}$$
$$= \frac{n(2n^2+3n+1)}{6} = \frac{n(2n+1)(n+1)}{6}$$

Q.10.

$$n^4 - (n-1)^4 = (n^2+(n-1)^2)(n^2-(n-1)^2).$$
$$= (2n^2-2n+1)(2n-1).$$
$$= 4n^3 - 6n^2 + 4n - 1$$
$$(n-1)^4 - (n-2)^4 = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1$$
$$\vdots$$
$$1^4 - 0^4 = 4\times 1^3 - 6\times 1^2 + 4\times 1 - 1$$
$$n^4 = 4\sum_{k=1}^{n} k^3 - 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k - n$$
$$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^4 - \frac{1}{4}n(2n+1)(n+1) + \frac{n(n+1)}{2} - \frac{n}{4}$$
$$= \frac{n^4 + 2n^3 + 3n^2 + n - 2n^2 - 2n + n}{4}$$
$$= \frac{n^4 + 2n^3 + n^2}{4} = \left[\frac{n(n+1)}{2}\right]^2.$$

Q.11. we have $(n+1)^2-n^2$ numbers of n

thus if $\lfloor\sqrt{m}\rfloor = n+1$ $\qquad n=\lfloor\sqrt{m}\rfloor-1$

$$\sum_{k=1}^{m}\lfloor k\rfloor = \sum_{i=1}^{n} i\cdot\left[(i+1)^2-i^2\right] + \sum_{k=(n+1)^2}^{m}\lfloor k\rfloor.$$

$$= \sum_{i=1}^{n} i\cdot(2i+1) + \left[m-(n+1)^2+1\right](n+1).$$

$$= \frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2} + (n+1)\left(m-(n+1)^2+1\right)$$

$$= \frac{4n^3+9n^2+5n}{6} + (m+1)(n+1) - (n+1)^3$$

$$= \frac{n(n+1)(4(n+1)+1)}{6} + (m+1)(n+1) - (n+1)^3$$

$$= \frac{(\lfloor\sqrt{m}\rfloor-1)\lfloor\sqrt{m}\rfloor(4\lfloor\sqrt{m}\rfloor+1)}{6} + (m+1)\lfloor\sqrt{m}\rfloor - (\lfloor\sqrt{m}\rfloor)^3$$

Q.12. $|A\times C| = |A|\times|C|$

$|B\times D| = |B|\times|D|$

Since $|A|=|B|$, $|C|=|D|$

we have $|A|\times|C| = |B|\times|D|$ $\qquad$ Hence $|A\times C| = |B\times D|$

⋆Q.13 If $|A|=|B|$ we can find a bijction $f: A\to B$

Thus we have an injection $f: A\to B$, so $|A|\le|B|$.

Meanwhile we have another injection $f^{-1}: B\to A$. so $|B|\le|A|$.

Q14. We proof a lemma first: If f and g are both onto function, then $f\circ g$ is also an onto function.

Proof: Assume $g: A\to B$ $\qquad f: B\to C$.

Let $y\in C$. Because f is onto, $y=f(b)$ for some $b\in B$.

And Because g is onto $\quad b=g(a)$ for some $a\in A$.

Hence $y=f(b)=f(g(a))=f\circ g(a)$. which means $f\circ g$ is onto.

Solution:

If A is finite, then $|A|=k$. And because there is an onto function from A to B, then $|B|\le|A|=k$ B is finite.

Hence B is countable.

If A is countably infinite, Let f be the onto function from A to B and g be the $\mathbb{Z}^+ \to A$ onto function. Thus we can find an onto function $f \circ g : \mathbb{Z}^+ \to B$.
Hence B is countable.

Q.15. If we fix $m+n = x$, then the value of $f(m,n)$ is $\left[\frac{(x-2)(x-1)}{2}+1, \frac{(x-2)(x-1)}{2}+x-1 \right]$ when $\begin{cases} m=1 \\ n=x-1 \end{cases}$ and $\begin{cases} m=x-1 \\ n=1 \end{cases}$ get "="

Thus the images of fixed $m+n$ are continues integers.
Because $g(x) = \frac{(x-2)(x-1)}{2}$ covers $\mathbb{Z}^+$, to prove f is one-to-one and onto only need to show the values of x and $x+1$ is precisely connected. Which means show $\frac{(x-2)(x-1)}{2} + x-1+1 = \frac{(x+1-2)(x+1-1)}{2}+1$

left: $\frac{(x-2)(x-1)}{2} + x-1+1 = \frac{x^2-3x+2+2x}{2} = \frac{x^2-x+2}{2}$

right: $\frac{(x+1-2)(x+1-1)}{2} + 1 = \frac{(x-1)x+2}{2} = \frac{x^2-x+2}{2}$.

left = right.

Proof Done.

Q.16. Define $f:(0,1) \to [0,1]$ by $f(x)=x$ thus $|(0,1)| \le |[0,1]|$.
Define $g:[0,1] \to (0,1)$ by $g(x) = \frac{x}{2}+\frac{1}{4}$ thus $|[0,1]| \le |(0,1)|$.
$\therefore |(0,1)| = |[0,1]|$

Q.17. $f(x)$ is $\Theta(g(x)) \to c_1 g(x) \le f(x) \le c_2 g(x)$ $c_1 > 0$.
$g(x)$ is $\Theta(h(x)) \to c_3 h(x) \le g(x) \le c_4 h(x)$ $c_3 > 0$

Then $c_1c_3h(x) \le f(x) \le c_2 c_4 h(x)$ $c_1c_3 > 0$.
Thus $f(x) = \Theta(h(x))$.

Q.18. Counter example: $f_1(x) = x^2+x$ $f_2(x) = x^2$, $g(x) = x^2$
$(f_1 - f_2)x = x = \Theta(x) \ne \Theta(x^2)$.

Q.19.

when $x \geq 1$.

$$|f(x)| = |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0|$$

$$\leq |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x + a_0$$

$$\leq |a_n| x^n + |a_{n-1}| x^n + \cdots + |a_1| x^n + |a_0| x^n$$

Let $M = \max \{|a_n|, |a_{n-1}|, \cdots |a_0|\}$

$$\leq M x^n + M x^n + \cdots + M x^n + M x^n$$

$$= (n+1) M x^n = C_1 x^n.$$

Besides:

$$|f(x)| = |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0|$$

$$= x^n \left| a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x} + \frac{a_0}{x^{n-1}} \right|$$

let $\left| \frac{a_{n-i}}{x^i} + \cdots + \frac{a_0}{x^{n-1}} \right| < \left| \frac{a_n}{2} \right|$.

we should have $\left| \frac{nM}{x} \right| < \left| \frac{a_n}{2} \right|$.

Therefore: when $x > \frac{2nM}{a_n}$.

$$|f(x)| = x^n \left| a_n + \frac{a_{n-i}}{x} + \cdots + \frac{a_1}{x} + \frac{a_0}{x^{n-1}} \right|$$

$$\geq \left| \frac{a_n}{2} \right| x^n = C_2 x^n$$

Hence $f(x) = \Theta(x^n)$.

Q.20. Definition: $C_1 \cdot \log_a n \leq \Theta(\log_a n) \leq C_2 \log_a n \qquad (a>1)$

$$C_1 \cdot \frac{\ln 2}{\ln a} \cdot \frac{\ln n}{\ln a} \cdot \frac{\ln a}{\ln 2} \leq \Theta(\log_a n) \leq C_2 \cdot \frac{\ln 2}{\ln a} \cdot \frac{\ln n}{\ln a} \cdot \frac{\ln a}{\ln 2}$$

$$\underbrace{C_1 \cdot \frac{\ln 2}{\ln a}}_{C_3} \cdot \log_2 n \leq \Theta(\log_a n) \leq \underbrace{C_2 \cdot \frac{\ln 2}{\ln a}}_{C_4} \cdot \log_2 n$$

$$C_3 \log_2 n \leq \Theta(\log_a n) \leq C_4 \log_2 n$$

Thus, $\Theta(\log_a n) = \Theta(\log_2 n)$.

Q21. multiplication : $2n$

addition : n

total : $2n + n = 3n$

Q22. multiplication : n

addition : n

total : $n+n=2n$

Q23.

(1) $2^{(\log n)^{\log(\log n)}} = 2^{\log(\log n)\cdot\log n} = n^{\log(\log n)} \leq 2^{\log(n^n)} = n^n$ when $n>1$

since 2^x is monotonous increasing

Thus $(\log n)^{\log(\log n)} \leq \log(n^n)$.

Hence $(\log n)^{\log(\log n)} = O(\log(n^n))$

(2) $f_1(n) = \log n$

$f_2(n) = (\log n)^{\log\log n}$

$f_3(n) = (\log\log n)^{\log n}$

$f_4(n) = (\log n)^{\log n}$

$f_5(n) = n\log n$

$f_6(n) = (\log n)^n$

$f_7(n) = n^2$

$f_8(n) = n^{\log n}$

$f_9(n) = 3^{\frac{n}{2}}$

Q24. A, C, E.