CS215 DISCRETE MATH

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Some Thoughts on Algorithm Design

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- **Algorithm Design**, is mainly about designing algorithms that have small Big-O running time.
- Being able to do good algorithm design lets you identify the hard parts of your problem and deal with them effectively.
- Too often, programmers try to slove problems using brute force techniques and end up with slow complicated code!
- A few hours of abstract thought devoted to algorithm design could have speeded up the solution substantially and simplified it!

Dealing with Hard Problems

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- What happens if you can't find an efficient algorithm for a given problem?

Blame yourself.

I couldn't find a polynomial-time algorithm.
I guess I am too dumb.

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- What happens if you can't find an efficient algorithm for a given problem?

Show that no-efficient algorithm exists.

I couldn't find a polynomial-time algorithm, because no such algorithm exists.

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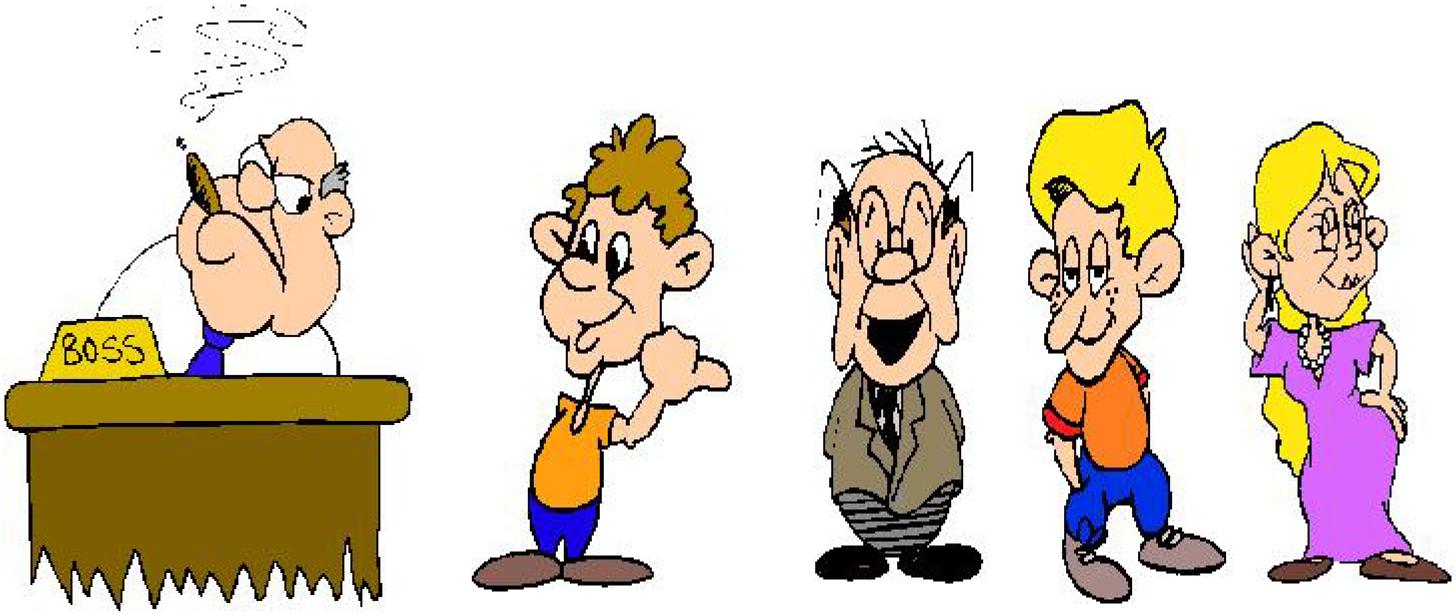
  We will now learn about NP-Complete problems, which provide us with a way to approach this question.

Introduction

- A very large class of thousands of practical problems for which it is not known if the problems have "efficient" solutions.
- It is known that if any one of the NP-Complete problems has an efficient solution then all of the NP-Complete problems have efficient solutions.
- Researchers have spent innumberable man-years trying to find efficient solutions to these problems but failed.
- So, NP-Complete problems are very likely to be hard.
- What do you do: prove that your problem is NP-Complete.

What do you actually do:



I couldn't find a polynomial-time algorithm,
but neither could all these other smart people!

Encoding the Inputs of Problems

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- In order to formally discuss how hard a problem is, we need to be much more formal than before about the input size of a problem.

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- The exact input size s, determined by an optimal encoding method, is hard to compute in most cases.

  However, we do not need to determine s exactly.

  For most problems, it is sufficient to choose some natural, and (usually) simple, encoding and use the size s of this encoding.

Input Size Example: Composite

- **Example:**

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  Any integer $n > 0$ can be represented in the binary number system as a string $a_0 a_1 \cdots a_k$ of length $\lceil \log_2(n+1) \rceil$.

  Thus, a natural measure of input size is $\lceil \log_2(n+1) \rceil$ (or just $\log_2 n$)

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  Using fixed length encoding, we write a_i as a binary string of length $m = \lceil \log_2 \max(|a_i| + 1) \rceil$.

  This coding gives an input size nm.

Complexity in terms of Input Size

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  This makes $\Theta(n)$ comparisons, so it might seem linear and very efficient.

  But, note that the input size of this problem is $size(n) = \log_2 n$, so the number of comparisons performed is actually $\Theta(n) = \Theta(2^{size(n)})$, which is exponential.

Input Size of Problems

- **Definition** Two positive functions $f(n)$ and $g(n)$ are of the same type if

$$c_1 g(n^{a_1})^{b_1} \leq f(n) \leq c_2 g(n^{a_2})^{b_2}$$

for all large n, where $a_1, b_1, c_1, a_2, b_2, c_2$ are some positive constants.

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Example:

All polynomials are of the same type, but *polynomials* and *exponentials* are of different types.

Input Size Example: Integer Multiplication

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  The minimum inpute size is

  $$s = \lceil \log_2(a+1) \rceil + \lceil \log_2(b+1) \rceil.$$

  A natural choice is to use $t = \log_2 \max(a, b)$ since $\frac{s}{2} \le t \le s$.

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  If L is the problem, and x is the input, we will often write $x \in L$ to denote a yes answer and $x \notin L$ to denote a no answer.

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  Examples:

  Knapsack vs. Decision Knapsack (DKnapsack)

Knapsack vs. DKnapsack

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  Optimization problem: (Knapsack)

  Find the largest value $\sum_{i \in T} v_i$ of any subset T that fits in the knapsack, i.e., $\sum_{i \in T} w_i \leq W$.

  Decision problem: (DKnapsack)

  Given k, is there a subset of the objects that fits in the knapsack and has total value at least k?

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  First solve the optimization problem, then check the decision problem. If it does, answer yes, otherwise no.

  Thus, if we prove that a given decision problem is hard to solve efficiently, then it is obvious that the optimization problem must be (at least as) hard.

Complexity Classes

- The Theory of Complexity deals with

  ◇ the classification of certain "decision problems" into several classes:

  - ◇ the class of "easy" problems
  - ◇ the class of "hard" problems
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A. Use polynomial-time algorithms.

Polynomial-Time Algorithms

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  Example:

  The standard multiplication algorithm has time $O(m_1 m_2)$, where m_1, m_2 denote the number of digits in the two integers, respectively.

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  Let's return to the Composite problem.

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  ◇ The complete algorithm therefore uses $\Theta(n(\log n)^2)$ time.

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  In terms of the input size, the complexity is $\Theta(2^N N^2)$.

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  In reality, an $O(n^{20})$ algorithm is not really practical.

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  Definition (The Class P) The class P consists of all decision problems that are solvable in polynomial time. That is, there exists an algorithm that will decide in polynomial time if any given input is a yes-input or a no-input.

The Class P

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  A. You need to prove that there is no polynomial-time algorithm for this problem. (much much harder)

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  Verifying a certificate: Given a presumed yes-input and its corresponding certificate, by making use of the given certificate, we verify that the input is actually a yes-input.

The Class NP

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NP – "nondeterministic polynomial-time"

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  Just being able to verify a certificate in polynomial time does not necessarily mean we can tell whether an input is a yes-input or a no-input in polynomial time.

  However, we are still no closer to solving it and do not know the answer. The search for a solution, though, has provided us with deep insights into what distinguishes an "easy" problem from a "hard" one.

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- If **Q** can be reduced to **Q'**, then **Q** is "no harder to solve" than **Q'**.

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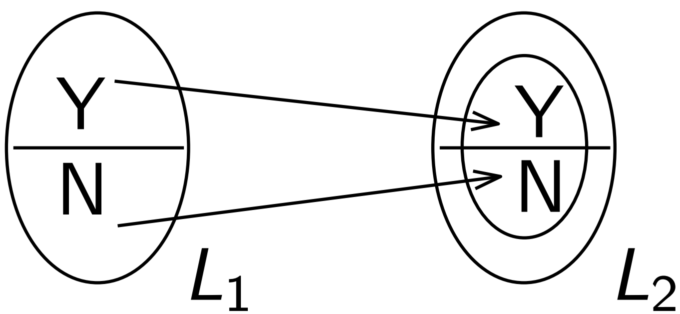
- A *polynomial-time reduction* from L_1 to L_2 is a transformation f with the following two properties:

  (1) f transforms an input x for L_1 into an input $f(x)$ for L_2 s.t.

  a yes-input of L_1 maps to a yes-input of L_2, and a no-input of L_1 maps to a no-input of L_2

  (2) f is computable in *polynomial time* in size(x)



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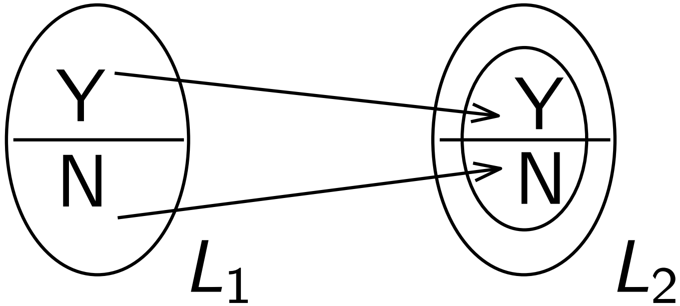

If such an f exists, we say that L_1 is *polynomial-time reducible* to L_2, and write $L_1 \leq_P L_2$.

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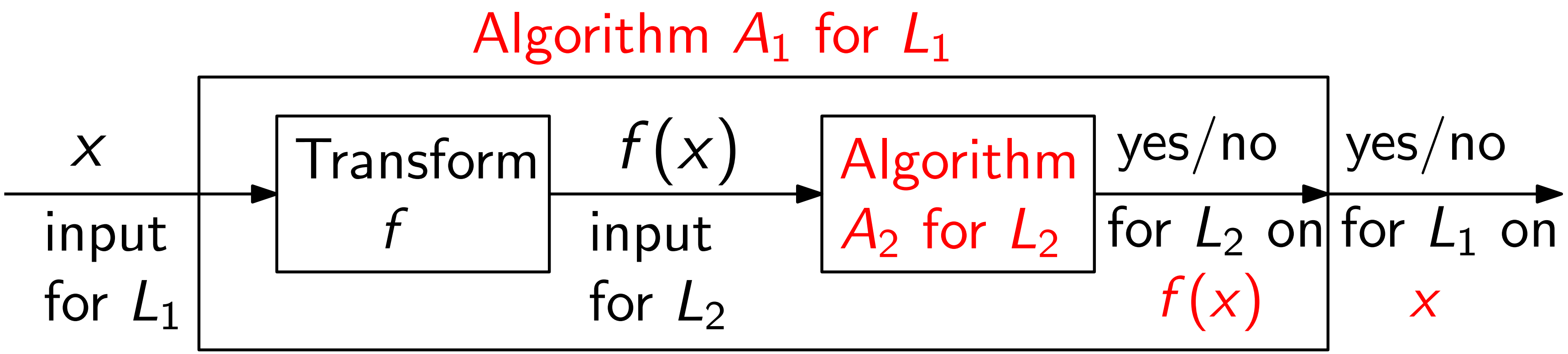

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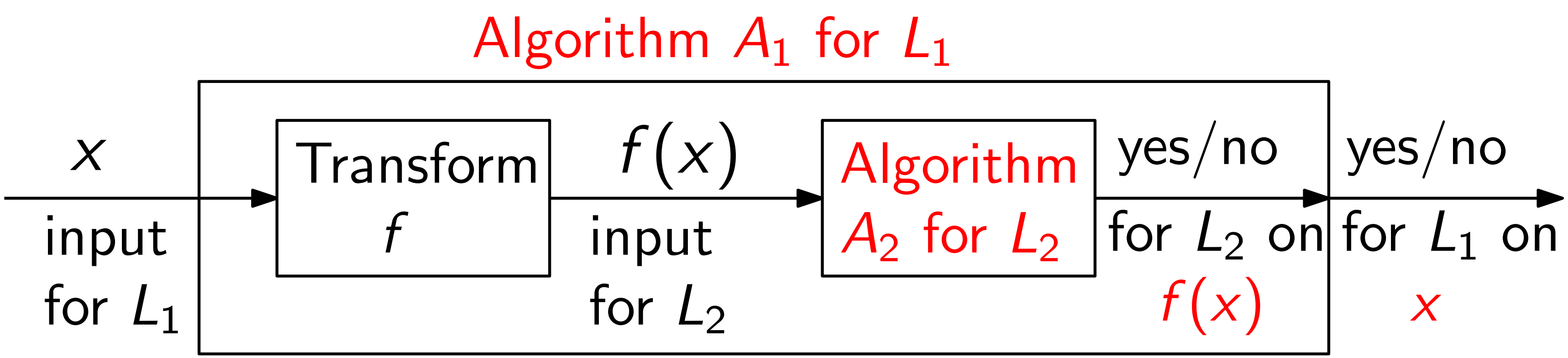

- If A_2 is polynomial-time algorithm, so is A_1

Polynomial-Time Reduction $f : L_1 \to L_2$

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  Proof. $L_2 \in P$ means we have a polynomial-time algorithm A_2 for L_2. Since $L_1 \leq_P L_2$, we have a polynomial-time transformation f mapping input x for L_1 to an input for L_2.

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  Combining these, we get the following polynomial-time algorithm for solving L_1:
  (1) take input x for L_1 and compute $f(x)$
  (2) run algorithm A_2 on input $f(x)$, and return the ans. (for L_2 on $f(x)$) as the ans. for L_1 on x

Polynomial-Time Reduction $f : L_1 \to L_2$

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  Proof. $L_2 \in P$ means we have a polynomial-time algorithm A_2 for L_2. Since $L_1 \leq_P L_2$, we have a polynomial-time transformation f mapping input x for L_1 to an input for L_2.

  Combining these, we get the following polynomial-time algorithm for solving L_1:
  (1) take input x for L_1 and compute $f(x)$
  (2) run algorithm A_2 on input $f(x)$, and return the ans. (for L_2 on $f(x)$) as the ans. for L_1 on x

  Both steps take polynomial time. So the combined algorithm takes polynomial time. Hence, $L_1 \in P$.

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  Note: The converse (if $L_1 \leq_P L_2$ and $L_1 \in P$, then $L_2 \in P$) is not true.

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  Intuitively, *NPC* consists of all the hardest problems in NP.

- **Theorem** Let L be any problem in NPC.

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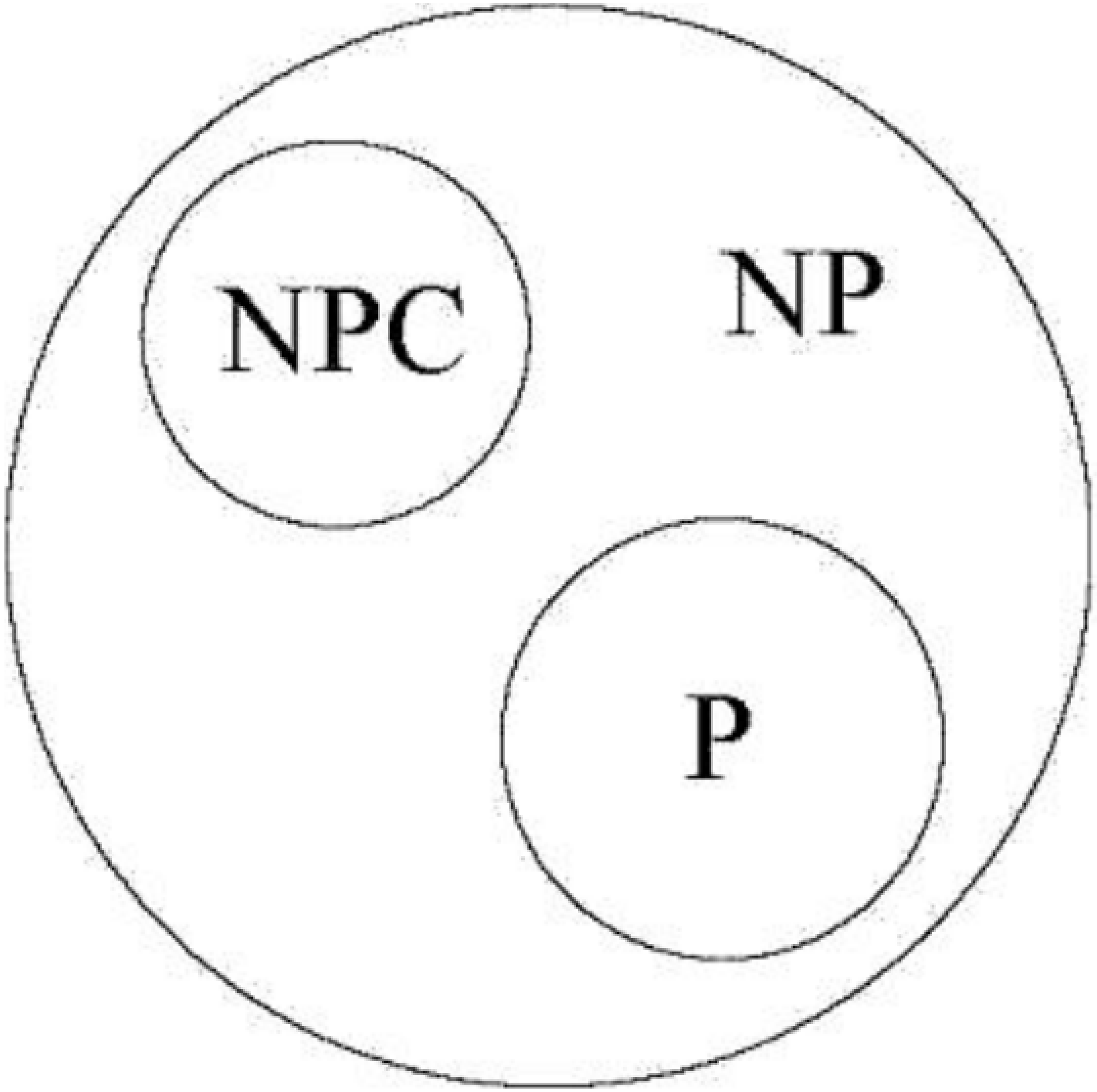
  This is the major reason why we are interested in NP-Completeness.

The Classes P, NP, and NPC



Next Lecture

- number theory ...