Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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Cardinality of Sets

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- The sets A and B have *the same cardinality* if there is a one-to-one correspondence between elements in A and B.

- A set that is **either finite** or **has the same cardinality as the set of positive integers $\mathbf{Z}^+$** is called *countable*. A set that is **not countable** is called *uncountable*.

  Why are these called **countable**?

  ◇ The elements of the set can be **enumerated and listed**.

Uncountable Sets

- **Theorem**

  The set $\mathcal{P}(\mathbb{N})$ is uncountable.

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Proof by contradiction:

Assume that $\mathcal{P}(\mathbb{N})$ is countable. This implies that the elements of this set can be listed as $S_0, S_1, S_2, \ldots$, where $S_i \subseteq \mathbb{N}$, and each S_i can be represented uniquely by the bit string $b_{i0}b_{i1}b_{i2}\ldots$, where $b_{ij} = 1$ if $j \in S_i$ and $b_{ij} = 0$ if $j \notin S_i$

– $S_0 = b_{00}b_{01}b_{02}b_{03}\cdots$

– $S_1 = b_{10}b_{11}b_{12}b_{13}\cdots$

– $S_2 = b_{20}b_{21}b_{22}b_{23}\cdots$

$\vdots$

all $b_{ij} \in \{0, 1\}$.

Schröder-Bernstein Theorem

- **Theorem**

If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there are one-to-one functions f from A to B and g from B to A, respectively, then there is a one-to-one correspondence between A and B.

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  Example

  Show that $|(0,1)| = |(0,1]|$.

  $f(x) = x$; $g(x) = x/2$

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  Show that $|(0,1)| = |\mathbb{R}|$.

  $f(x) = x$; $g(x) = (2\arctan(x)/\pi + 1)/2$

Computable vs Uncomputable

- **Definition**

  We say that a function is *computable* if there is a computer program in some programming language that finds the values of this function. If a function is not computable, we say it is *uncomputable*.

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The set of functions from $\mathbf{Z}^+$ to the set $\{0, 1, 2, \ldots, 9\}$ is *uncountable*. Proof?

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There is a bijective function f from S to $\mathcal{P}(S)$.

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Now f is bijective, and T is a subset of S, so there is an element $s_0 \in S$ s.t. $f(s_0) = T$.

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$\mathcal{Q}$: Is $s_0 \in T$?

Algorithms

- An *algorithm* is a finite sequence of **precise instructions** for performing a computation or for solving a problem.

Abu Ja'far Mohammed ibn Musa al-Khowarizmi

Big-O Notation

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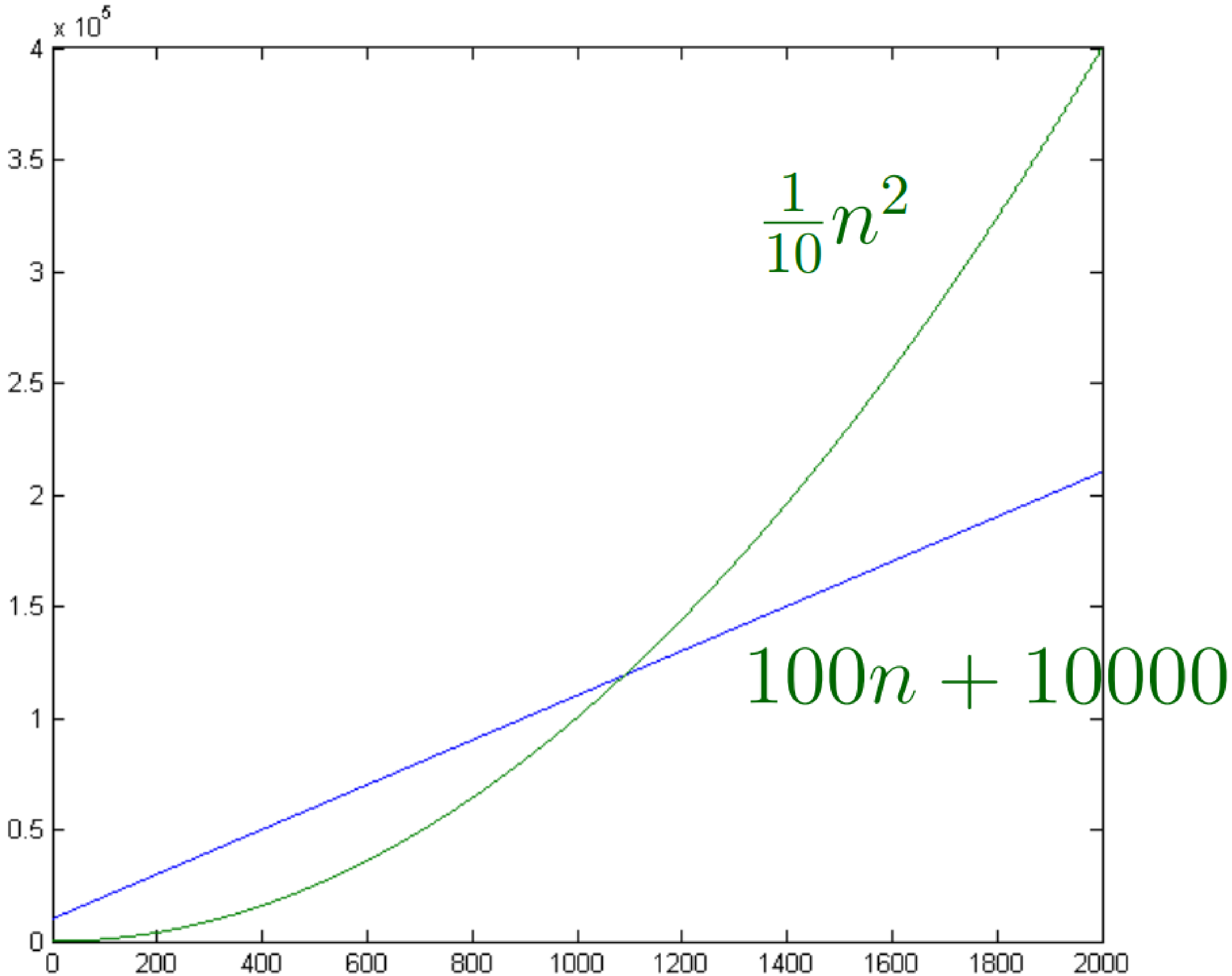
  In Computer Science, we are usually interested in what happens when our problem input size gets large.

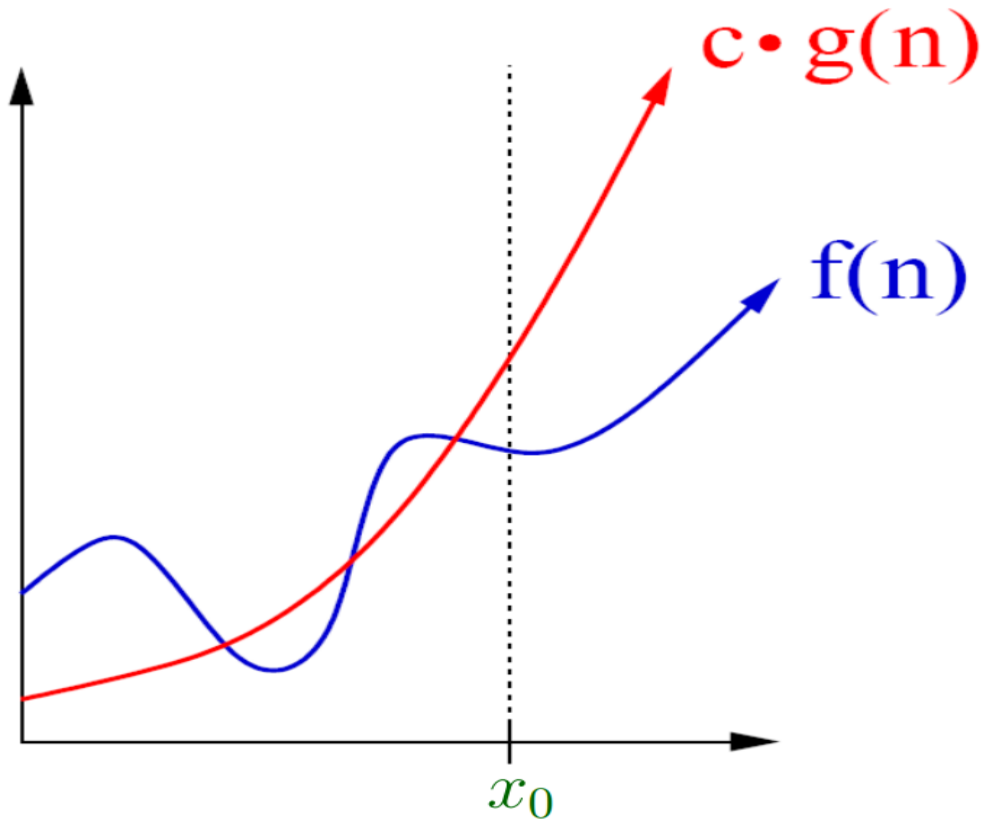
  Notice that when n is "large enough", $\frac{1}{10}n^2$ gets much bigger than $100n + 10000$ and stays larger.

Big-O Notation



Big-O Notation

- Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(n) = O(g(n))$ (reads: $f(n)$ is O of $g(n)$), if there exist some positive constants C and x_0 such that $|f(n)| \leq C|g(n)|$, whenever $n > x_0$.



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(Proof by contradiction)

Examples

$4n^2$
$8n^2 + 2n - 3$
$n^2/5 + \sqrt{n} - 10\log n$
$n(n-3)$

are all $O(n^2)$

Big-O Estimates for Polynomials

- Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_{n-1}$ are real numbers. Then $f(x) = O(x^n)$.

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  Proof:

  Assuming $x > 1$, we have

$$
\begin{aligned}
|f(x)| &= |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0| \\
&\leq |a_n|x^n + |a_{n-1}|x^{n-1} + \cdots + |a_1|x + |a_0| \\
&= x^n(|a_n| + |a_{n-1}|/x + \cdots + |a_1|/x^{n-1} + |a_0|/x^n) \\
&\leq x^n(|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|).
\end{aligned}
$$

Big-O Estimates for Polynomials

- Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_{n-1}$ are real numbers. Then $f(x) = O(x^n)$.

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Assuming $x > 1$, we have

$$\begin{aligned} |f(x)| &= |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0| \\ &\leq |a_n|x^n + |a_{n-1}|x^{n-1} + \cdots + |a_1|x + |a_0| \\ &= x^n(|a_n| + |a_{n-1}|/x + \cdots + |a_1|/x^{n-1} + |a_0|/x^n) \\ &\leq x^n(|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|). \end{aligned}$$

The leading term $a_n x^n$ of a polynomial dominates its growth.

Big-O Estimates for Some Functions

- $1 + 2 + \cdots + n = O(n^2)$

  $n! = O(n^n)$

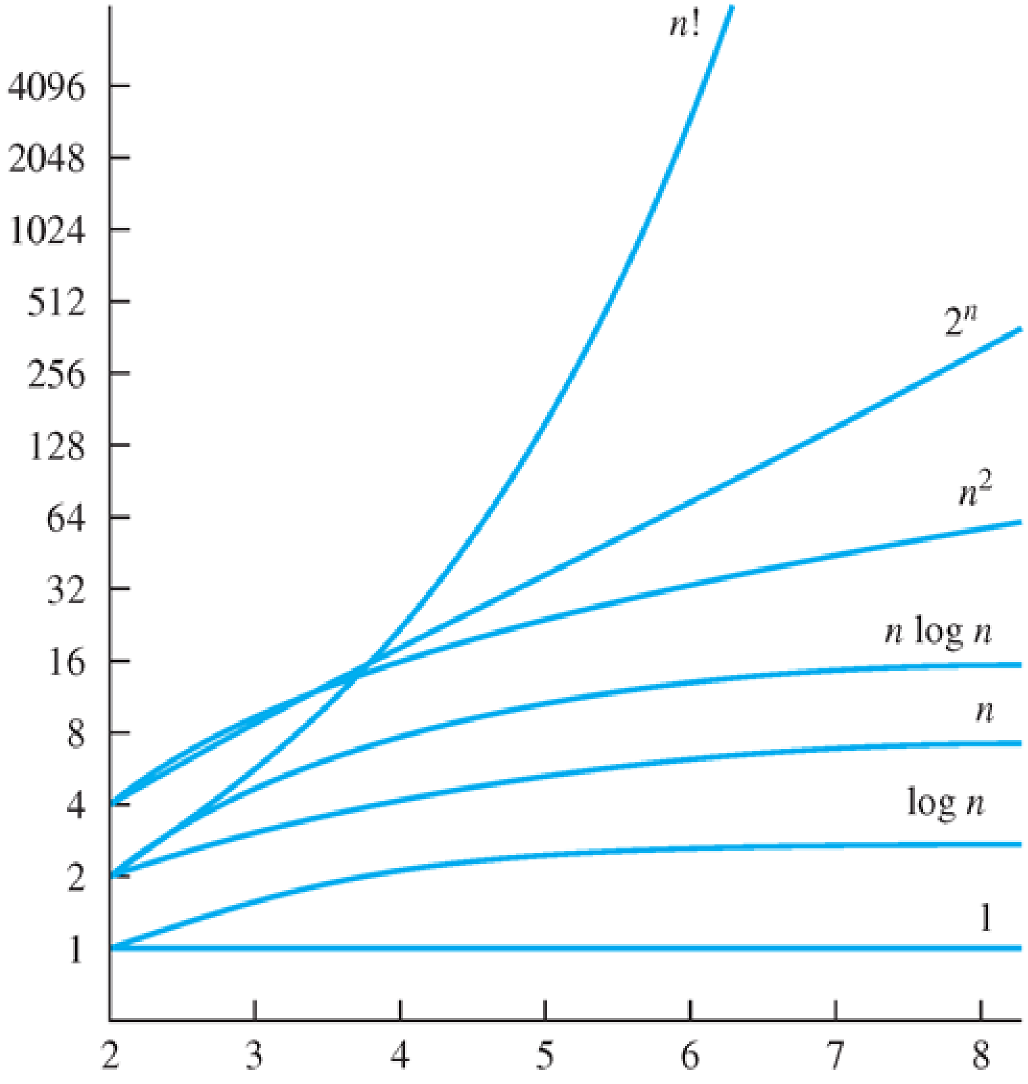
  $\log n! = O(n \log n)$

  $\log_a n = O(n)$ for an integer $a \geq 2$

  $n^a = O(n^b)$ for integers $a \leq b$

  $n^a = O(2^n)$ for an integer a

Display of Growth of Functions



Combinations of Functions

- If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$ then
  $(f_1 + f_2)(x) = O(\max(|g_1(x)|, |g_2(x)|))$

Proof:

By definition, there exist constants C_1, C_2, k_1, k_2 such that
$|f_1(x)| \leq C_1|g_1(x)|$ when $x > k_1$ and
$|f_2(x)| \leq C_2|g_2(x)|$ when $x > k_2$. Then

$$\begin{aligned}
|(f_1 + f_2)(x)| &= |f_1(x) + f_2(x)| \\
&\leq |f_1(x)| + |f_2(x)| \\
&\leq C_1|g_1(x)| + C_2|g_2(x)| \\
&\leq C_1|g(x)| + C_2|g(x)| \\
&= (C_1 + C_2)|g(x)| \\
&= C|g(x)|,
\end{aligned}$$

where $g(x) = \max(|g_1(x)|, |g_2(x)|)$ and $C = C_1 + C_2$.

Combinations of Functions

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Proof:

When $x > \max(k_1, k_2)$,

$$
\begin{aligned}
|(f_1 f_2)(x)| &= |f_1(x)||f_2(x)| \\
&\leq C_1|g_1(x)|C_2|g_2(x)| \\
&\leq C_1 C_2|(g_1 g_2)(x)| \\
&\leq C|(g_1 g_2)(x)|,
\end{aligned}
$$

where $C = C_1 C_2$.

Ordering Functions by Order of Growth

- $f_1(n) = (1.5)^n$
  $f_2(n) = 8n^3 + 17n^2 + 111$
  $f_3(n) = (\log n)^2$
  $f_4(n) = 2^n$
  $f_5(n) = \log(\log n)$
  $f_6(n) = n^2(\log n)^3$
  $f_7(n) = 2^n(n^2 + 1)$
  $f_8(n) = n^3 + n(\log n)^2$
  $f_9(n) = 100000$
  $f_{10}(n) = n!$

Big-Omega Notation

- Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(n) = \Omega(g(n))$ (reads: $f(n)$ is Ω of $g(n)$), if there exist some positive constants C and x_0 such that $|f(n)| \geq C|g(n)|$, whenever $n > x_0$.

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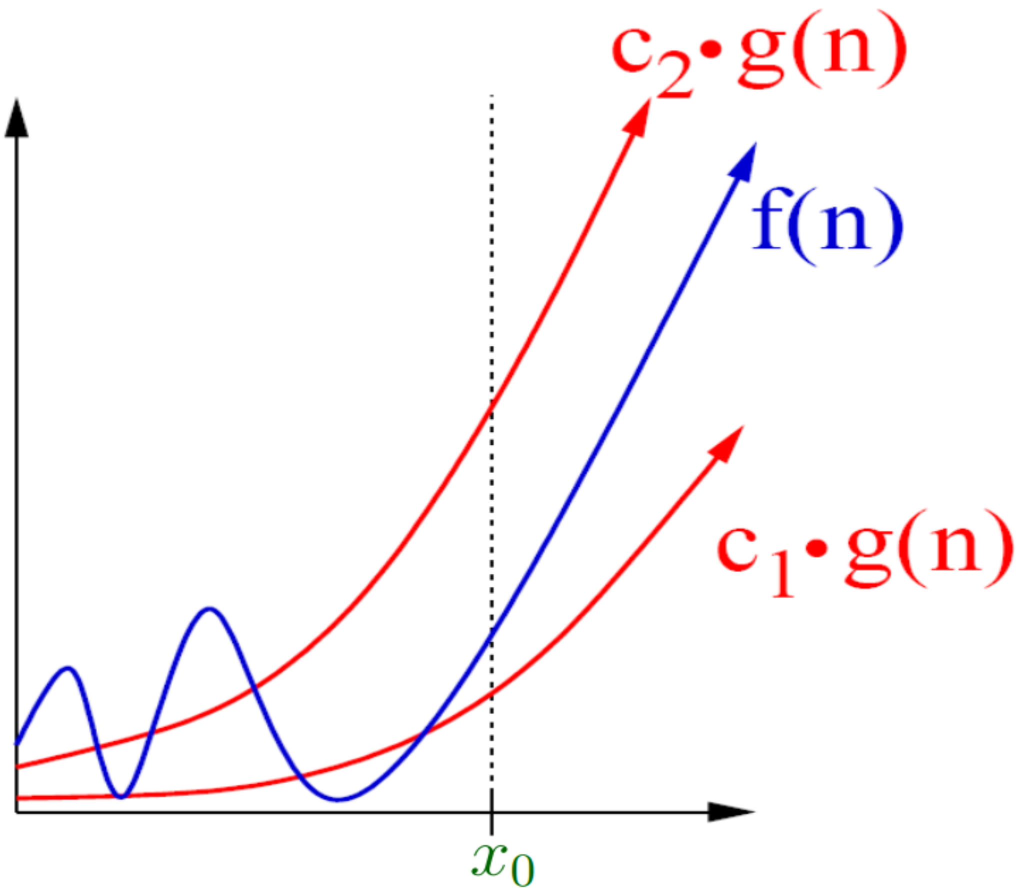
Big-O gives an upper bound on the growth of a function, while Big-Ω gives a lower bound. Big-Ω tells us that a function grows at least as fast as another.

Note: $f(x)$ is $\Omega(g(x))$ if and only if $g(x)$ is $O(f(x))$.

Big-Theta Notation (Big-O & Big-Omega)

- Two functions $f(n)$, $g(n)$ have the same order growth if $f(n) = O(g(n))$ and $g(n) = O(f(n))$. In this case, we say that $f(n) = \Theta(g(n))$, which is the same as $g(n) = \Theta(f(n))$.



Examples ($f(n) = \Theta(g(n))$)

- $3n^2 + 4n = \Theta(n)$?
  $3n^2 + 4n = \Theta(n^2)$?
  $3n^2 + 4n = \Theta(n^3)$?
  $n/5 + 10n \log n = \Theta(n^2)$?
  $n^2/5 + 10n \log n = \Theta(n \log n)$?
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Examples ($f(n) = \Theta(g(n))$)

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Algorithms

- An *algorithm* is a finite sequence of **precise instructions** for performing a computation or for solving a problem.

  A *computational problem* is a specification of the desired input-output relationship.

  Example (Computational Problem and Algorithm)

  The following procedure is an algorithm for calculating the sum of n given numbers $a_1, a_2, \ldots, a_n$.

  Step 1: set $S = 0$

  Step 2: for $i = 1$ to n, replace S by $S + a_i$

  Step 3: output S

Instance

- An *instance* of a problem is all the inputs needed to compute a solution to the problem.

Example (Instance of Problem)

$$< 8, 3, 6, 7, 1, 2, 9 >$$

- A *correct algorithm* halts with the correct output for **every input instance**. We can then say that the algorithm solves the problem.

Time and Space Complexity

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Step 3: output S

Step 1 and Step 3 take one operation. Step 2 takes $2n$ operations. Therefore, altogether this algorithm takes $2n + 2$ operations. The time complexity is $O(n)$.

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  Consider the evaluation of $f(x) = 1 + 2x + 3x^2 + 4x^3$.
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Step 1: set $S = a_n$

Step 2: for $i = 1$ to n, replace S by $a_{n-i} + Sx$

Step 3: output S

Horner's Algorithm and Its Complexity

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The final value of S output at Step 3 is the desired value of $a_0 + a_1 x + \cdots + a_n x^n$. The number of operations needed in this algorithm is $1 + 3n + 1 = 3n + 2$. So the time complexity of this algorithm is $O(n)$.

Time Complexity

- Determine the time complexity of the following algorithm:

  for $i := 1$ to n

  for $j := 1$ to n

  $a := 2 * n + i * j;$

  end for

  end for

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  end for

  In the second loop, computing a takes 4 operations (two multiplications, one addition, and one replacement). For each i, it takes $4n$ operations to complete the second loop. So it takes $n \times 4n = 4n^2$ operations to complete the two loops. The time complexity of this algorithm is $O(n^2)$.

Time Complexity

- Determine the time complexity of the following algorithm:

  $S := 0$

  for $i := 1$ to n

      for $j := 1$ to i

          $S := S + i * j;$

      end for

  end for

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- Determine the time complexity of the following algorithm:

$S := 0$

for $i := 1$ to n

for $j := 1$ to i

$S := S + i * j;$

end for

end for

Computing S takes 3 operations. For each i, completing the second loop takes $3i$ operations. So altogether it takes

$$1 + \sum_{i=1}^{n} 3i = 1 + 3\frac{n(n+1)}{2}$$

operations. So the complexity of this algorithm is $O(n^2)$.

More on Time Complexity

- **Example:** (Insertion Sort)

  Input: $A[1 \dots n]$ is an array of numbers

  for $j := 2$ to n

  $\quad key = A[j];$

  $\quad i = j - 1;$

  $\quad$ while $i \geq 1$ and $A[i] > key$ do

  $\quad\quad A[i+1] = A[i];$

  $\quad\quad i - -;$

  $\quad$ end while

  $\quad A[i+1] = key;$

  end for

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          $A[i + 1] = A[i]$;

          $i - -$;

      end while

      $A[i + 1] = key$;

  end for

	key	
Sorted		Unsorted

Where in the sorted part to put "key"?

Three Cases of Analysis: I

- Best Case: constraints on the input, other than size, resulting in the fastest possible running time for the given size.

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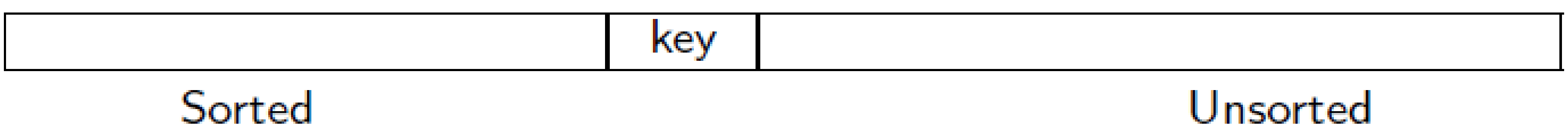
Example: (Insertion Sort)

$$A[1] \leq A[2] \leq A[3] \leq \cdots \leq A[n]$$

The number of comparisons needed is

$$\underbrace{1+1+1+\cdots+1}_{n-1} = n-1 = \Theta(n)$$



"key" is compared to only the element right before it.

Three Cases of Analysis: II

- Worst Case: constraints on the input, other than size, resulting in the slowest possible running time for the given size.

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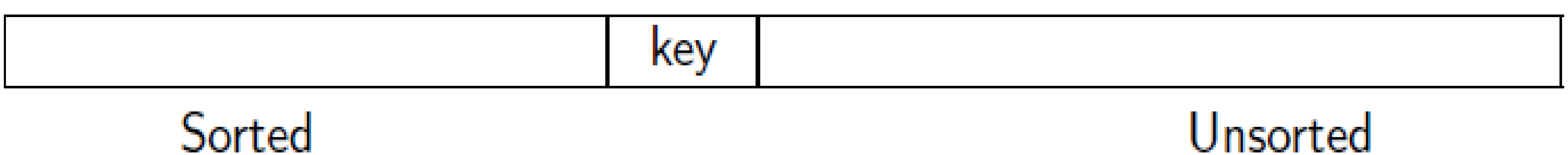
  Example: (Insertion Sort)

$$A[1] \geq A[2] \geq A[3] \geq \cdots \geq A[n]$$

The number of comparisons needed is

$$1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2} = \Theta(n^2)$$



"key" is compared to everything element before it.

Three Cases of Analysis: III

- Average Case: average running time over every possible type of input for the given size (usually involve probabilities of different types of input)

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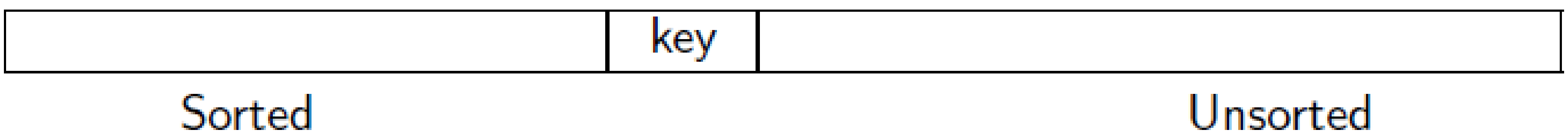
Example: (Insertion Sort)

$\Theta(n^2)$ assuming that each of the $n!$ instances are equally likely



On average, "key" is compared to half of the elements before it.

Some Thoughts on Algorithm Design

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Some Thoughts on Algorithm Design

- **Algorithm Design**, is mainly about designing algorithms that have small Big-O running time.
- Being able to do good algorithm design lets you identify the hard parts of your problem and deal with them effectively.
- Too often, programmers try to slove problems using brute force techniques and end up with slow complicated code!
- A few hours of abstract thought devoted to algorithm design could have speeded up the solution substantially and simplified it!

Dealing with Hard Problems

- What happens if you can't find an efficient algorithm for a given problem?

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- What happens if you can't find an efficient algorithm for a given problem?

Blame yourself.

I couldn't find a polynomial-time algorithm.
I guess I am too dumb.

Dealing with Hard Problems

- What happens if you can't find an efficient algorithm for a given problem?

Show that no-efficient algorithm exists.

I couldn't find a polynomial-time algorithm,
because no such algorithm exists.

Dealing with Hard Problems

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- Proving that no efficient algorithm exists for a particular problem is difficult:

  How can we prove the non-existence of something?

  We will now learn about NP-Complete problems, which provide us with a way to approach this question.

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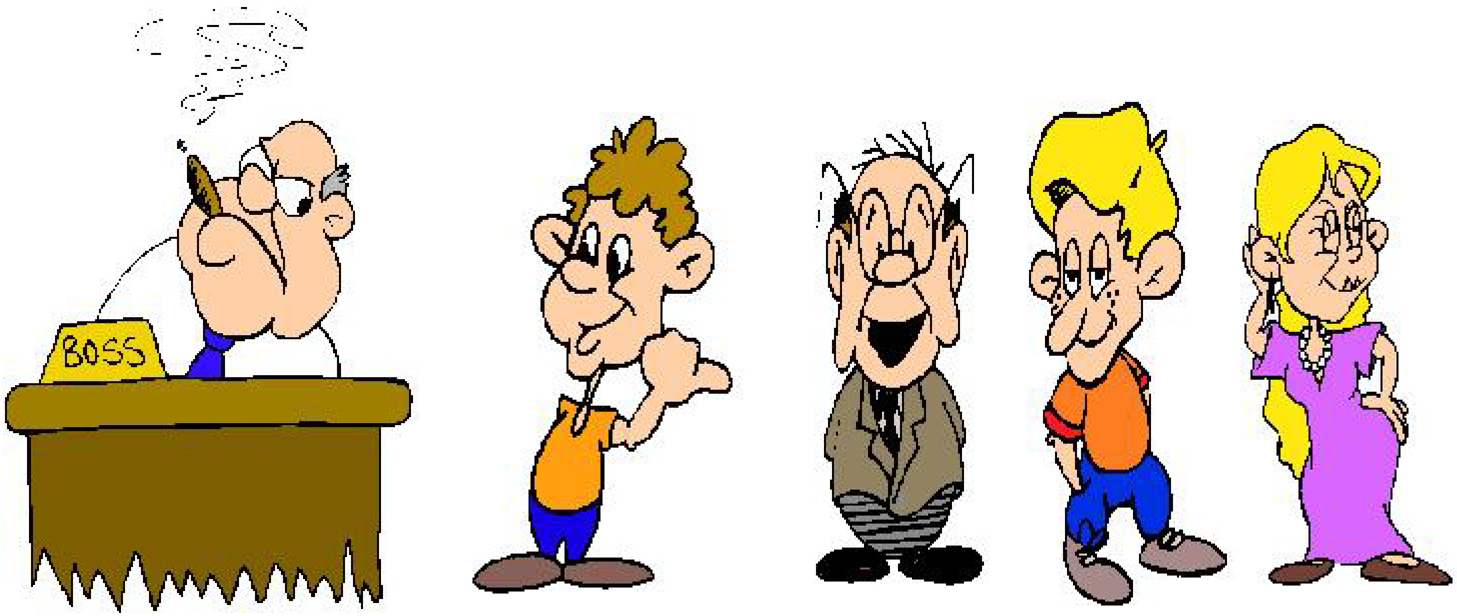
- It is known that if any one of the NP-Complete problems has an efficient solution then all of the NP-Complete problems have efficient solutions.
- Researchers have spent innumberable man-years trying to find efficient solutions to these problems but failed.
- So, NP-Complete problems are very likely to be hard.
- What do you do: prove that your problem is NP-Complete.

What do you actually do:



I couldn't find a polynomial-time algorithm,
but neither could all these other smart people!

Encoding the Inputs of Problems

- Complexity of a problem is measure w.r.t the size of input.

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- In order to formally discuss how hard a problem is, we need to be much more formal than before about the input size of a problem.

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The Input Size of Problems

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  Definition The *input size* of a problem is the minimum number of bits ($\{0,1\}$) needed to encode the input of the problem.

- The exact input size s, determined by an optimal encoding method, is hard to compute in most cases.

  However, we do not need to determine s exactly.

  For most problems, it is sufficient to choose some natural, and (usually) simple, encoding and use the size s of this encoding.

Input Size Example: Composite

- **Example:**

  Given a positive integer n, are there integers $j, k > 1$ such that $n = jk$? (i.e., is n a composite number?)

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  Question:

  What is the input size of this problem?

  Any integer $n > 0$ can be represented in the binary number system as a string $a_0 a_1 \cdots a_k$ of length $\lceil \log_2(n+1) \rceil$.

  Thus, a natural measure of input size is $\lceil \log_2(n+1) \rceil$ (or just $\log_2 n$)

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  Sort n integers $a_1, \ldots, a_n$

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  Using fixed length encoding, we write a_i as a binary string of length $m = \lceil \log_2 \max(|a_i| + 1) \rceil$.

  This coding gives an input size nm.

Complexity in terms of Input Size

- **Example:** (Composite)

  The naive algorithm for determining whether n is composite compares n with the first $n-1$ numbers to see if any of them divides n

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  This makes $\Theta(n)$ comparisons, so it might seem linear and very efficient.

  But, note that the input size of this problem is $size(n) = \log_2 n$, so the number of comparisons performed is actually $\Theta(n) = \Theta(2^{size(n)})$, which is exponential.

Input Size of Problems

- **Definition** Two positive functions $f(n)$ and $g(n)$ are of the same type if

$$c_1 g(n^{a_1})^{b_1} \le f(n) \le c_2 g(n^{a_2})^{b_2}$$

for all large n, where $a_1, b_1, c_1, a_2, b_2, c_2$ are some positive constants.

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Example:

All polynomials are of the same type, but *polynomials* and *exponentials* are of different types.

Input Size Example: Integer Multiplication

- **Example:** (Integer Multiplication problem)

  Compute $a \times b$.

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  The minimum inpute size is

  $$s = \lceil \log_2(a+1) \rceil + \lceil \log_2(b+1) \rceil.$$

  A natural choice is to use $t = \log_2 \max(a, b)$ since $\frac{s}{2} \le t \le s$.

Next Lecture

- P vs NP, number theory ...