CS215 DISCRETE MATH

Dr. Qi Wang

Department of Computer Science and Engineering
Office: Room413, CoE South Tower
Email: wangqi@sustech.edu.cn

Hamilton Paths and Circuits

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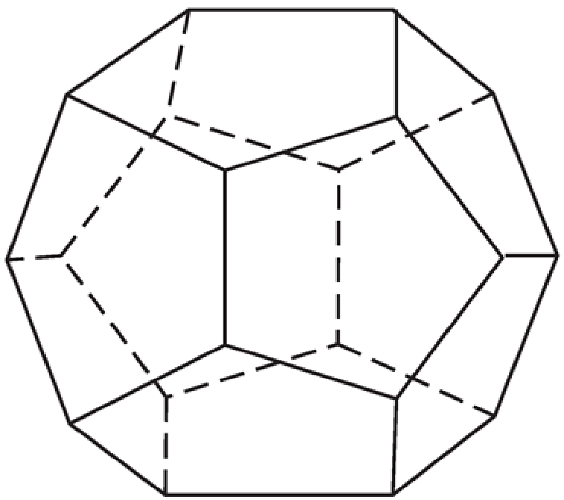
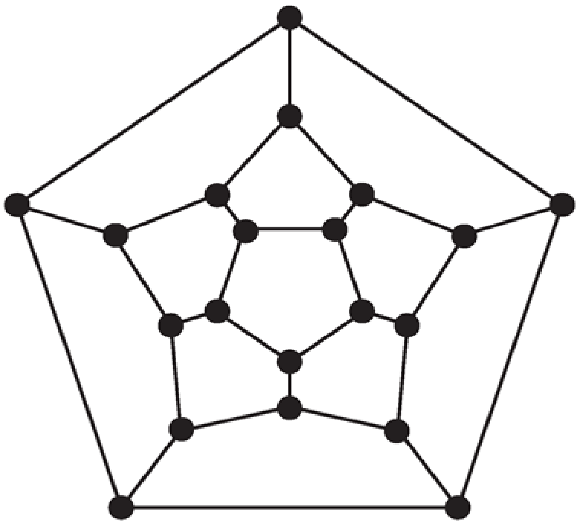


(a)



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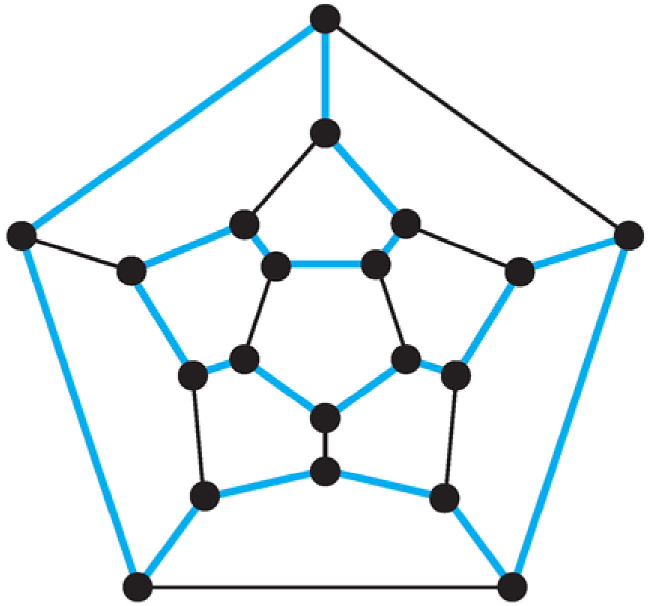

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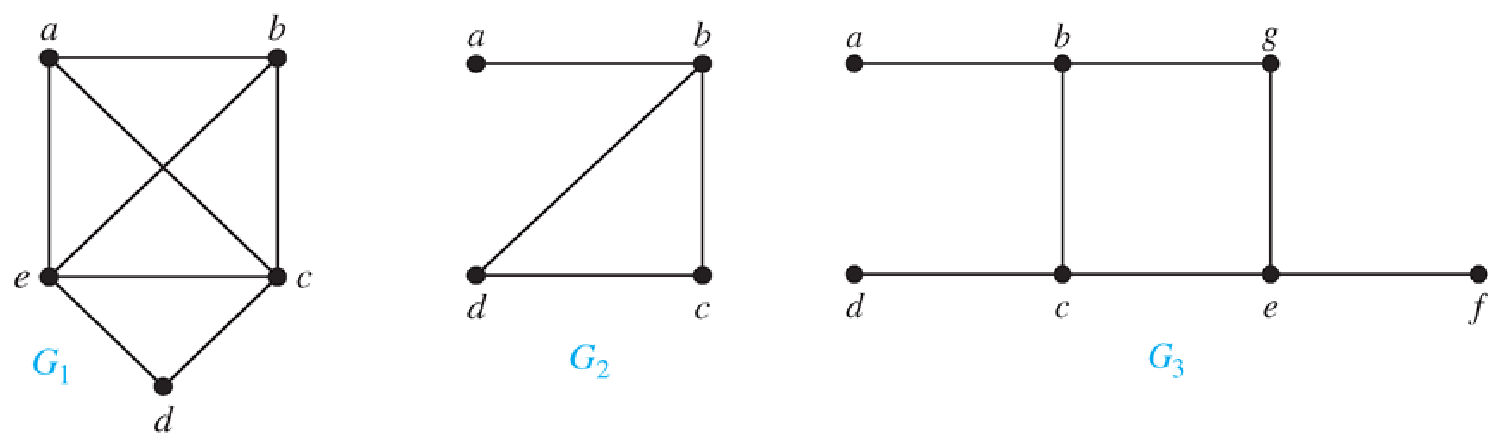
  Example Which of these simple graphs has a Hamilton circuit or, if not, a Hamilton path?



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  Hamilton path problem $\in$ NPC

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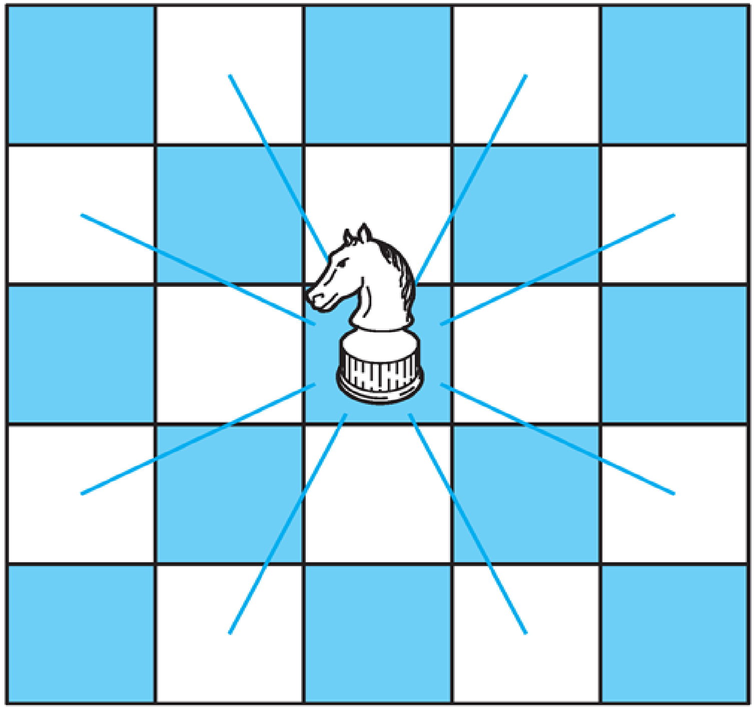
  the decision version of the TSP $\in$ NPC

Applications of Hamilton Paths and Circuits

- Can we traverse every space (and come back) in the 5×5 chessboard?



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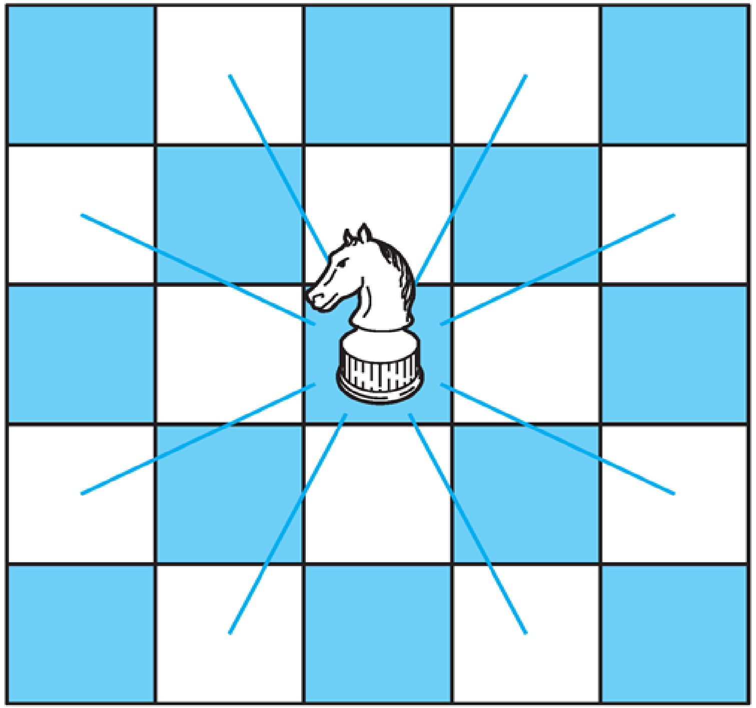

What about in 6×6 chessboard?

Shortest Path Problems

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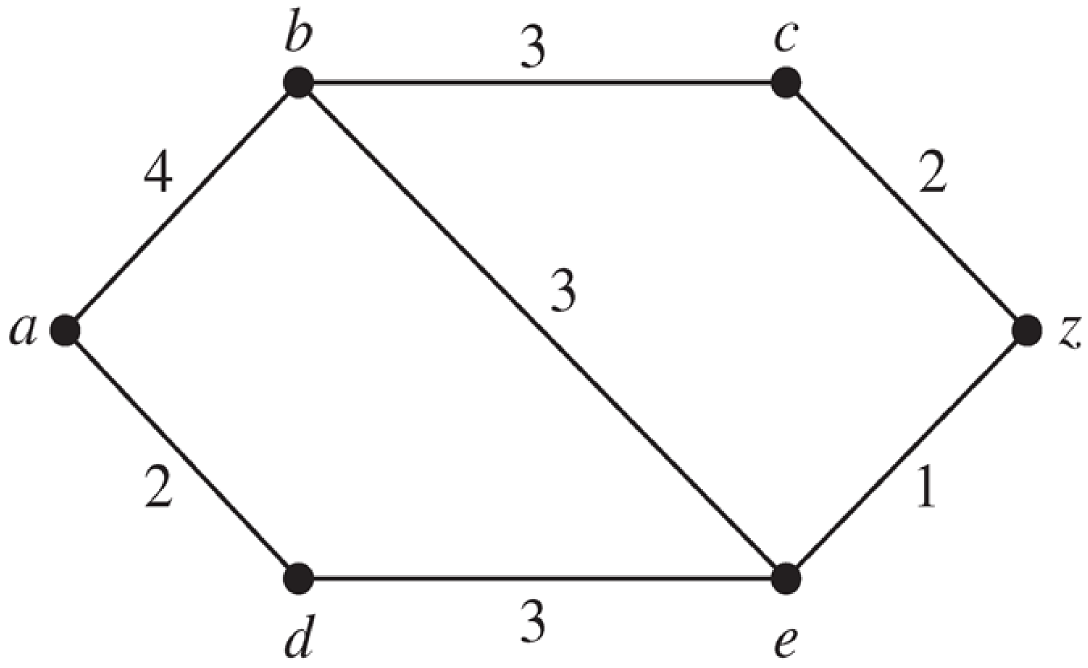

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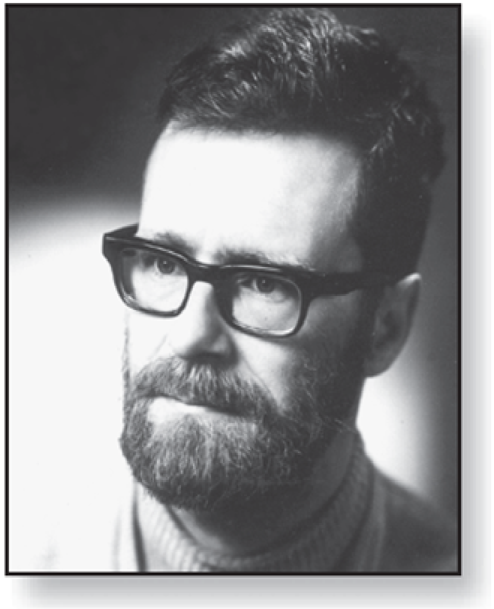

Edsger Wybe Dijkstra

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  let $v \notin S$ be the vertex with the least value $d(v)$,

  $S = S \cup \{v\}$ for each $u \notin S$, replace $d(u)$ by

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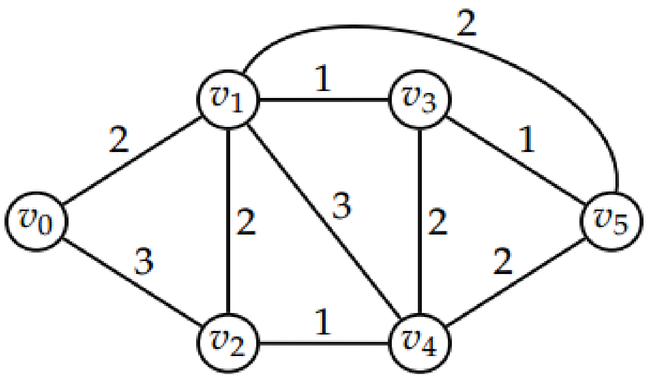

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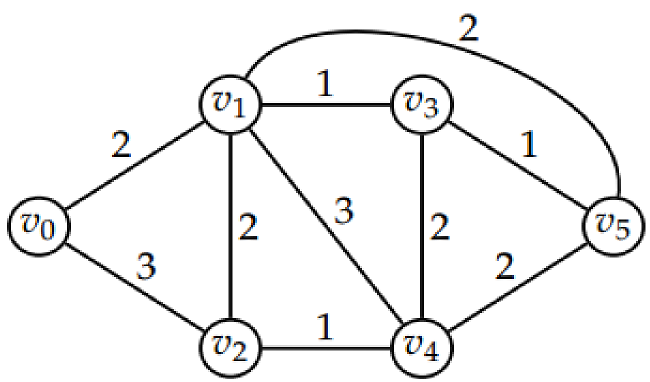

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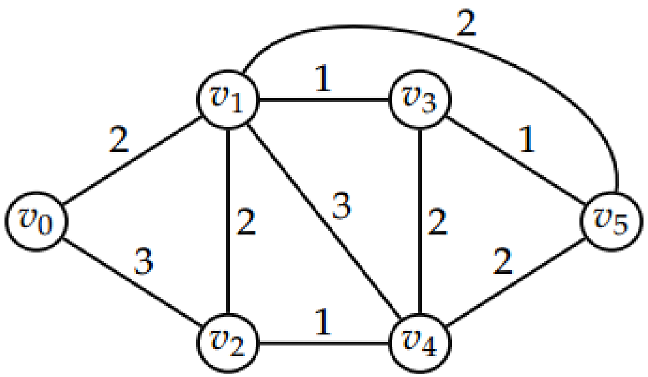

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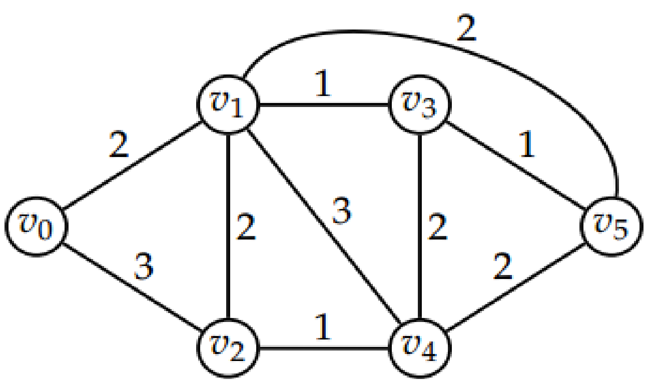

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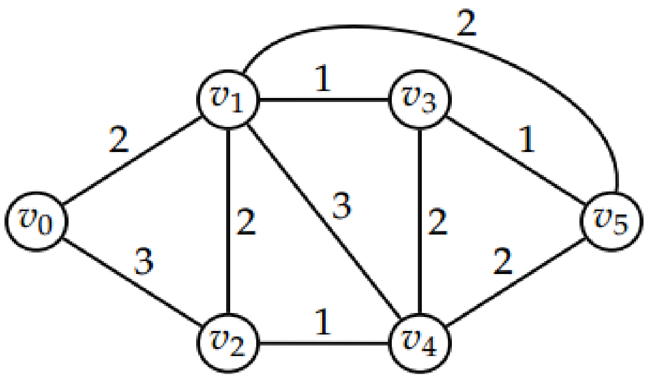

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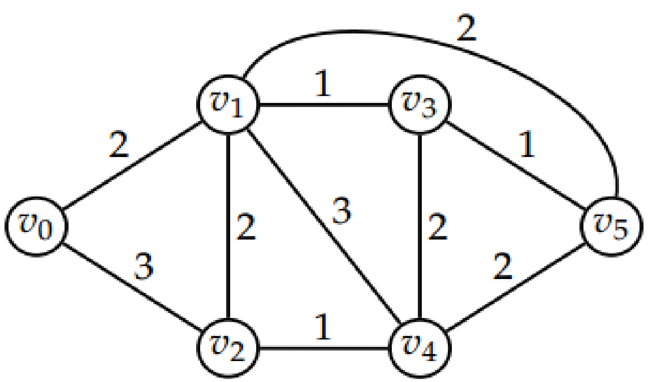

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$d(v_3) = 2 + 1 = 3$, $d(v_4) = 2 + 3 = 5$,

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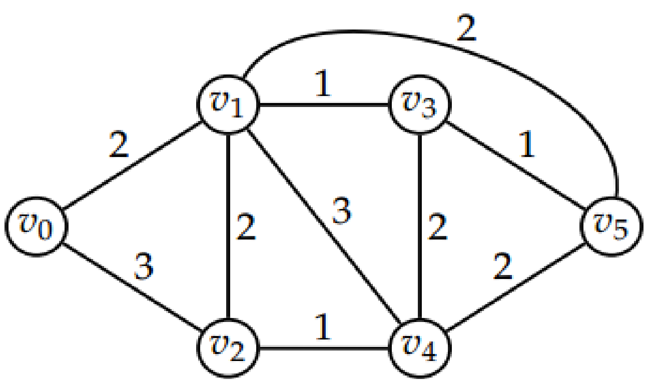

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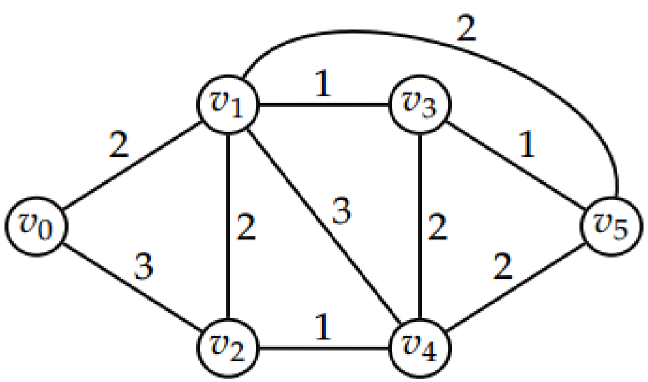

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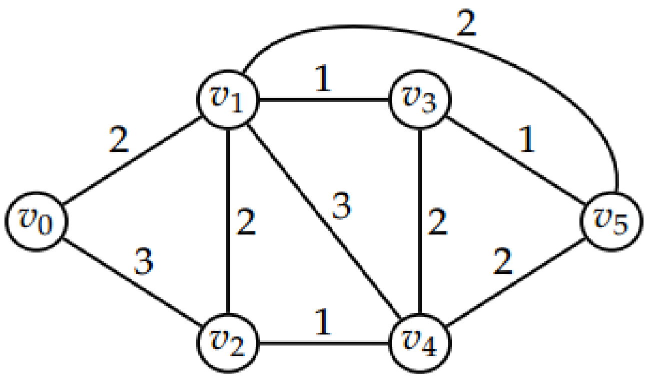

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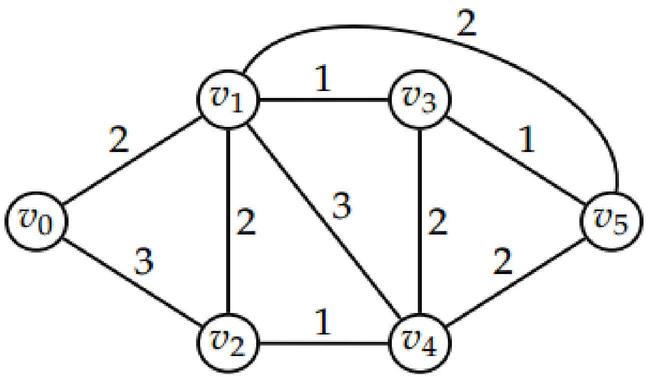

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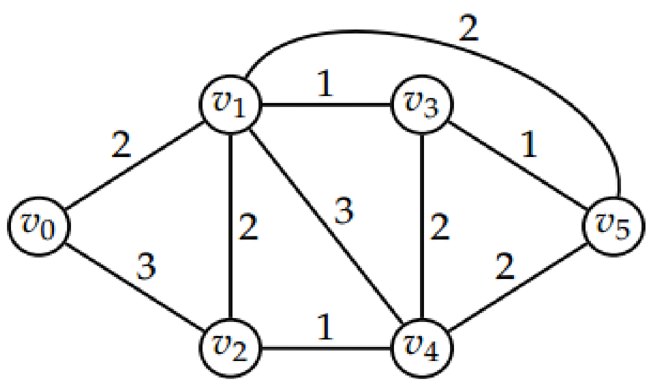

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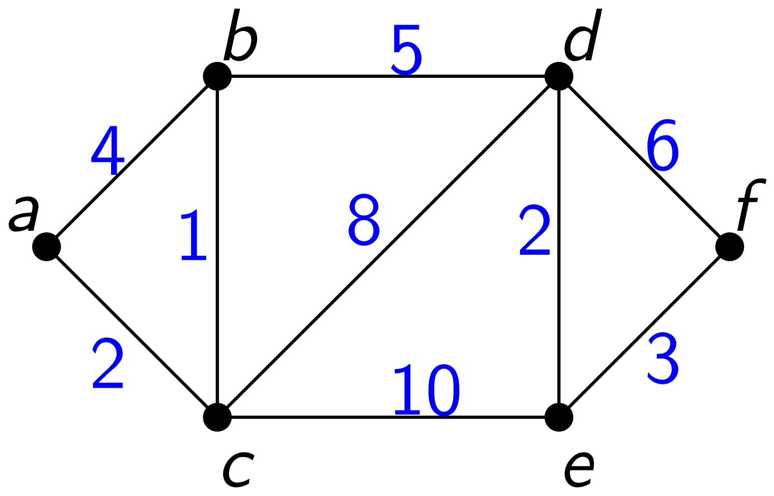
  read the Textbook p.712 – p.714

Another Example



The Single-Source Shortest Paths (SSSP) Problem

- Dijkstra's algorithm

  $O(v^2)$ using an *array* [Dijkstra 1956]

  $O(e + v \log v)$ using a *Fibonacci heap min-priority queue*

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- New result

Negative-Weight Single-Source Shortest Paths in Near-linear Time

Aaron Bernstein* Danupon Nanongkai† Christian Wulff-Nilsen‡

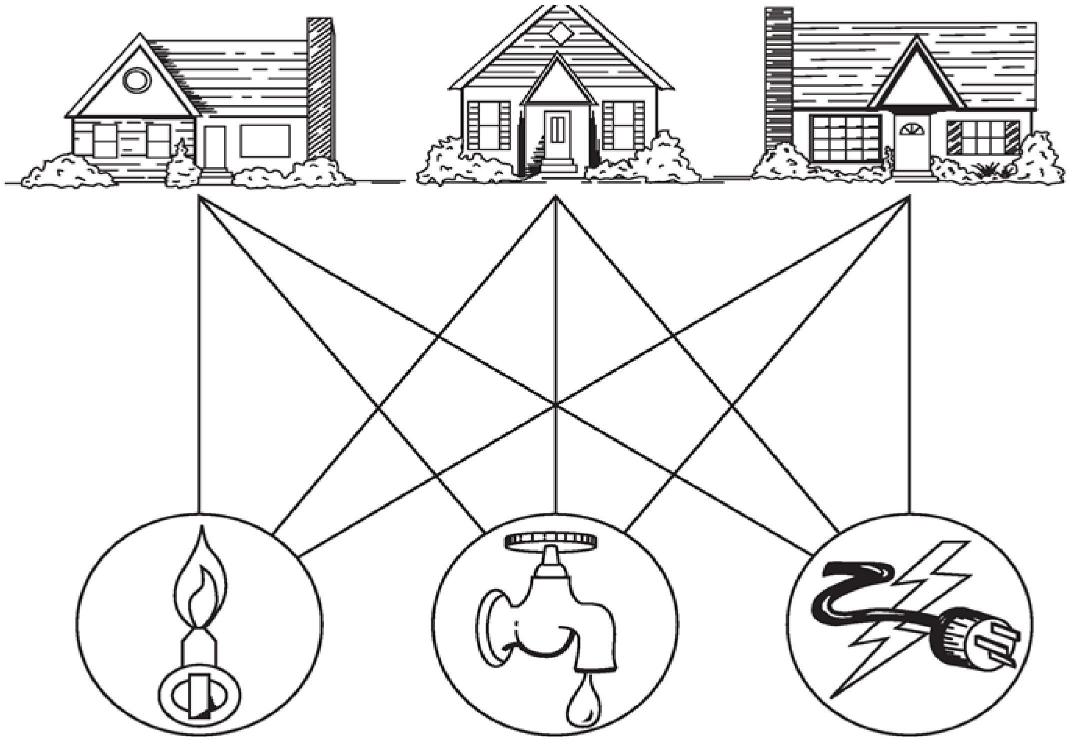
Abstract

We present a randomized algorithm that computes single-source shortest paths (SSSP) in $O(m \log^8(n) \log W)$ time when edge weights are integral and can be negative.[1] This essentially resolves the classic negative-weight SSSP problem. The previous bounds are $\tilde{O}((m+n^{1.5}) \log W)$ [BLNPSSSW FOCS'20] and $m^{4/3+o(1)} \log W$ [AMV FOCS'20]. Near-linear time algorithms were known previously only for the special case of planar directed graphs [Fakcharoenphol and Rao FOCS'01].

In contrast to all recent developments that rely on sophisticated continuous optimization methods and dynamic algorithms, our algorithm is simple: it requires only a simple graph decomposition and elementary combinatorial tools. In fact, ours is the first combinatorial algorithm for negative-weight SSSP to break through the classic $\tilde{O}(m\sqrt{n} \log W)$ bound from over three decades ago [Gabow and Tarjan SICOMP'89].

Planar Graphs

- Join three houses to each of three seperate utilities.



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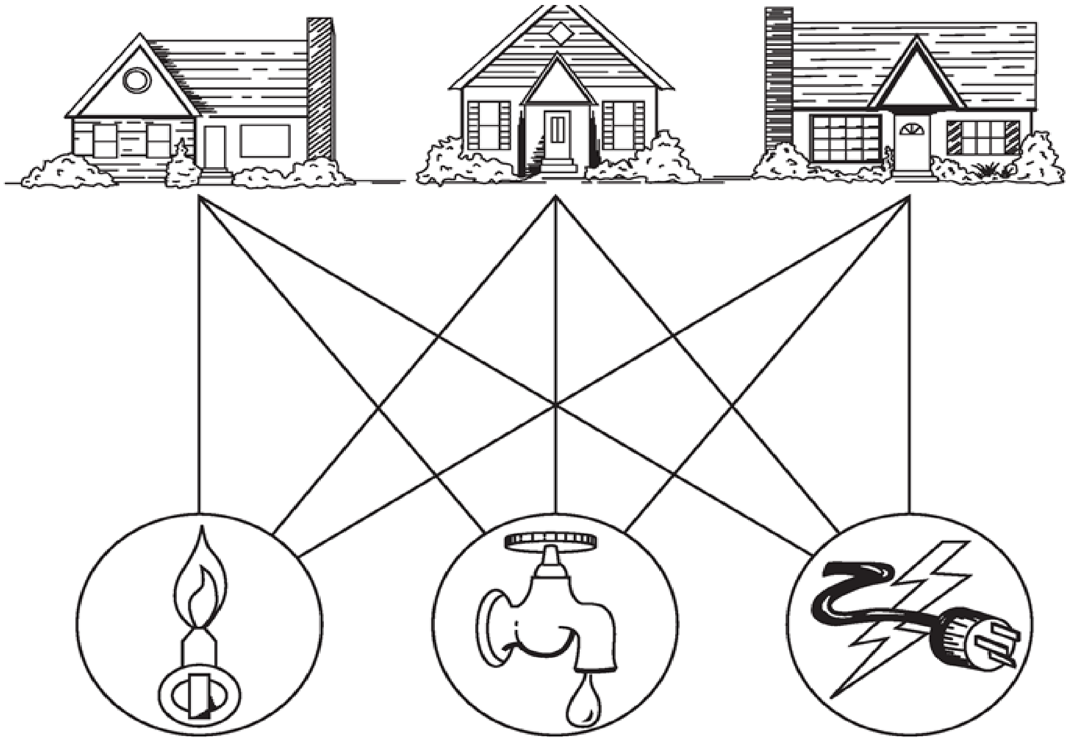

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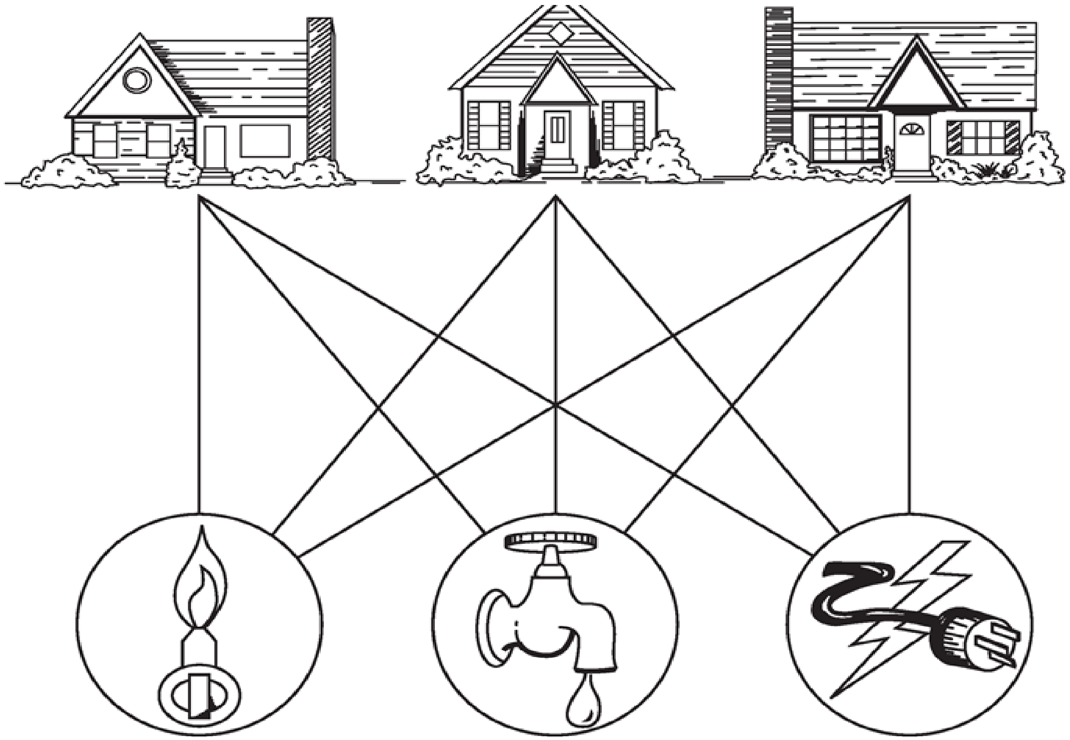

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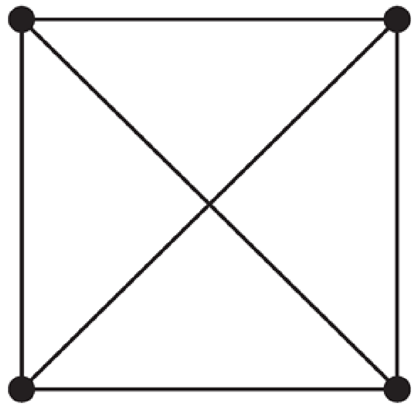

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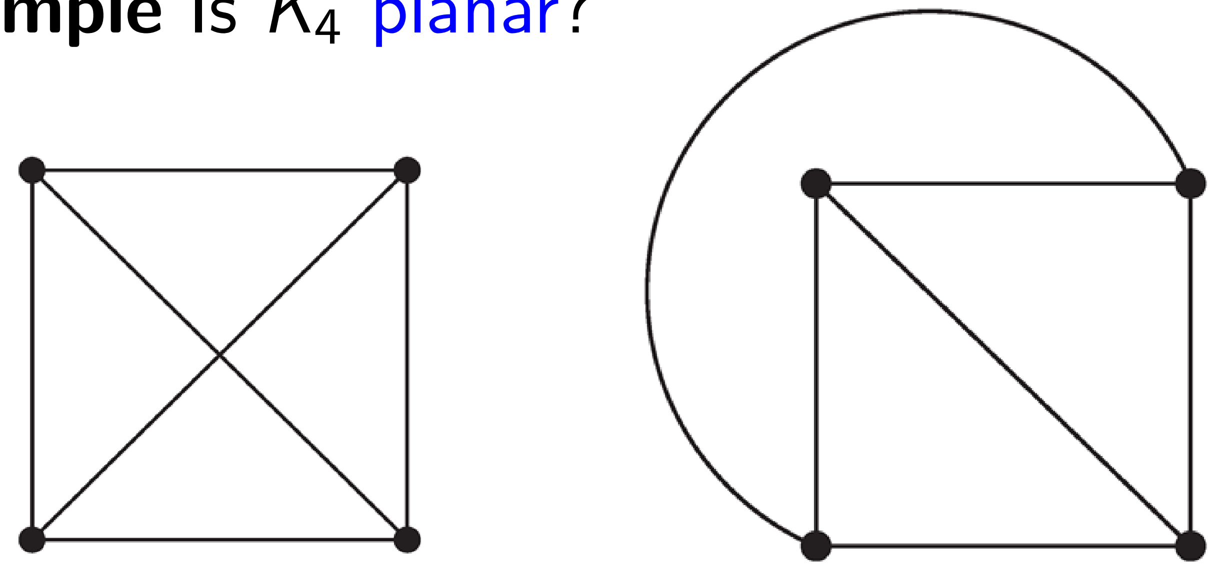
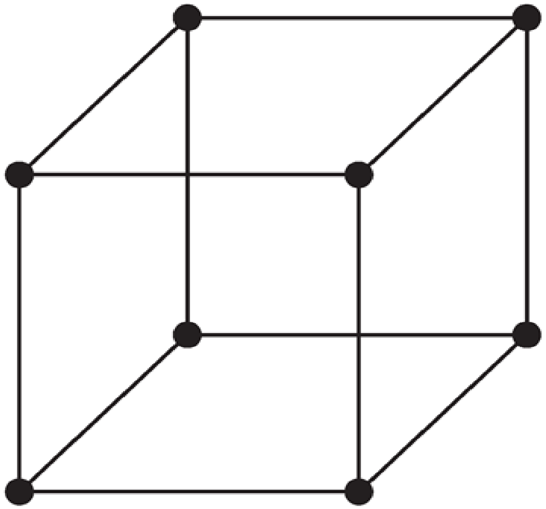
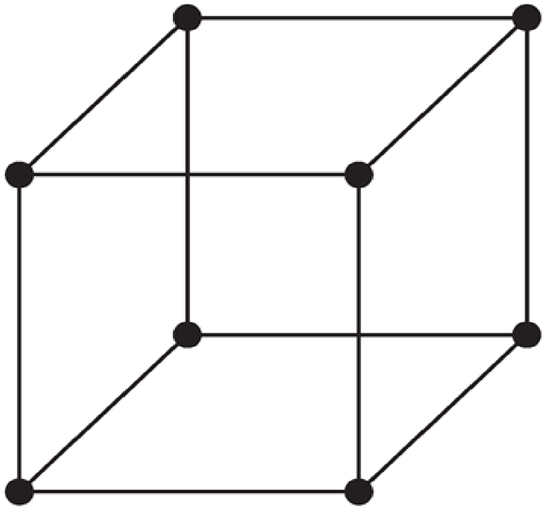


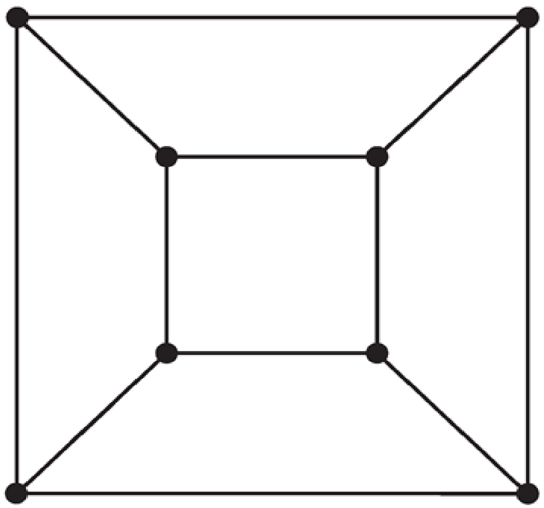
- **Example**



Planar Graphs

- **Example**





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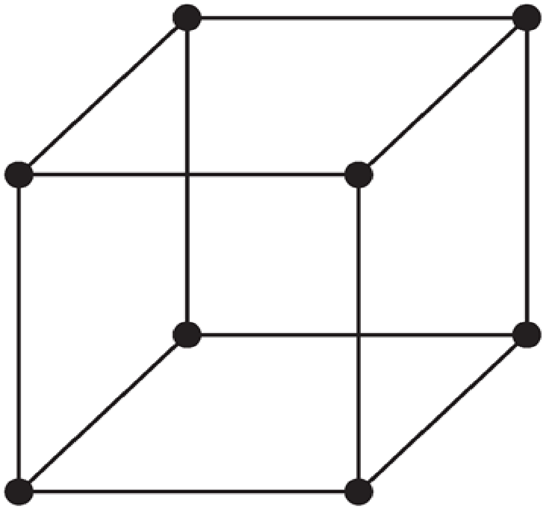

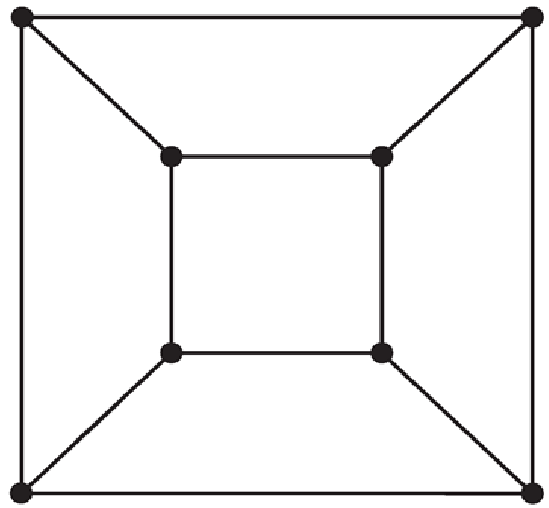

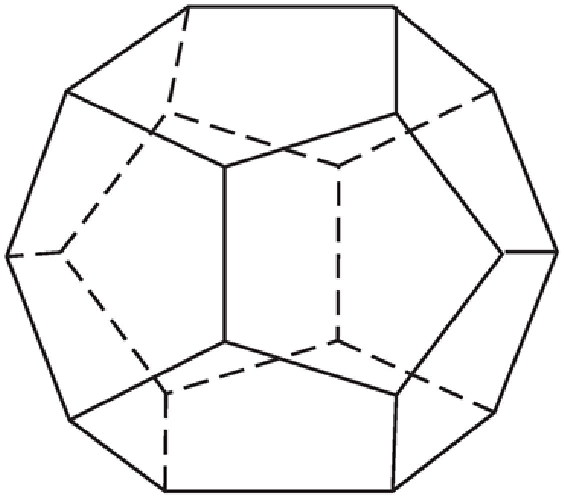

(a)

Planar Graphs

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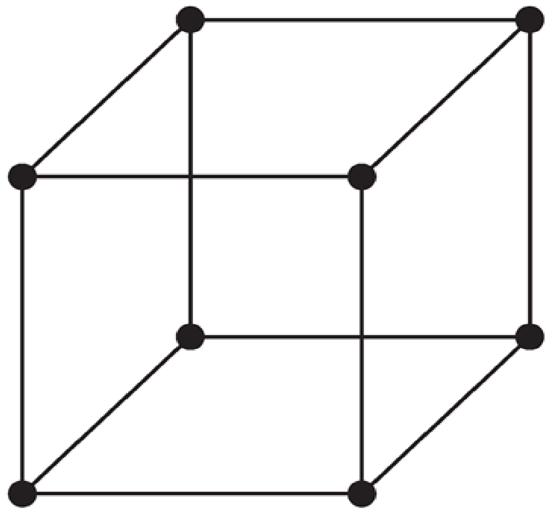

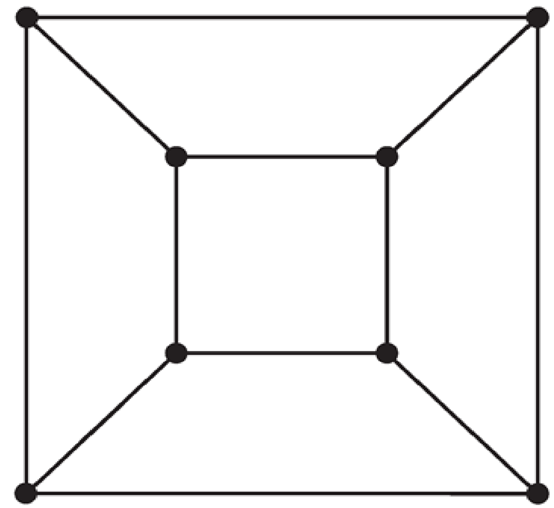

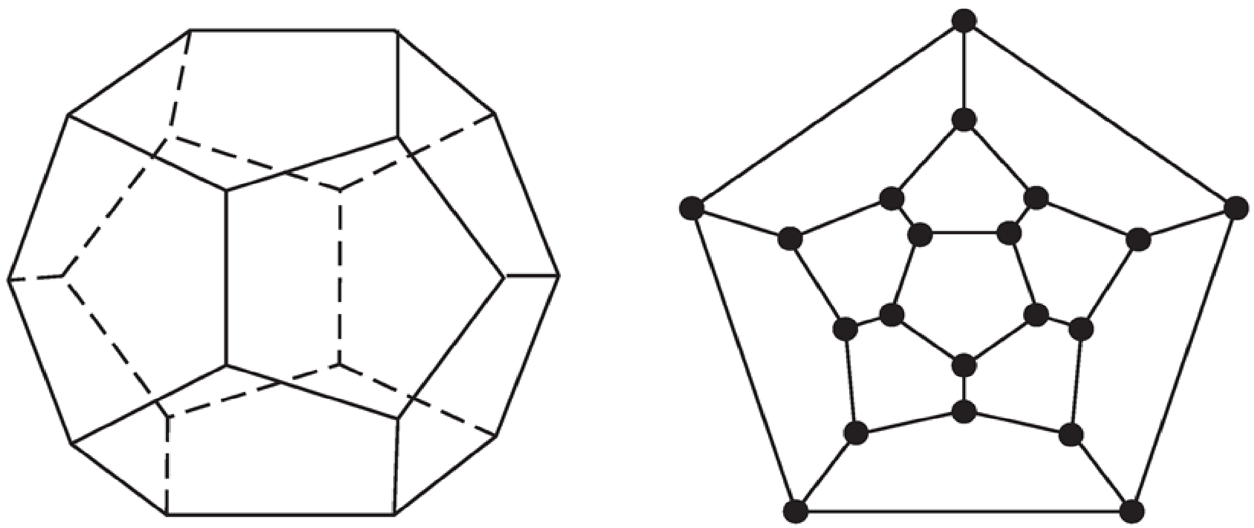

(a) (b)

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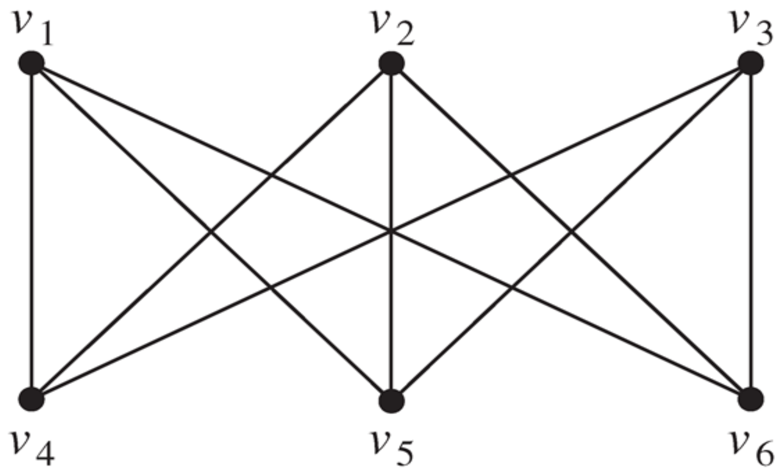

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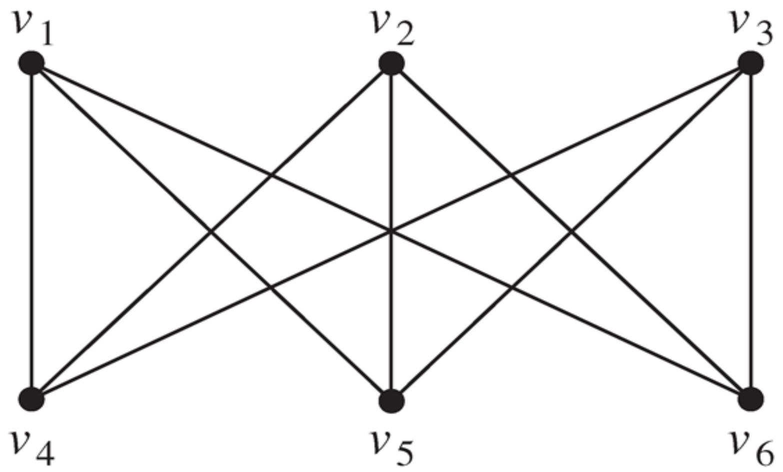

Applications

- ◇ IC design
- ◇ design of road networks

Euler's Formula

- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

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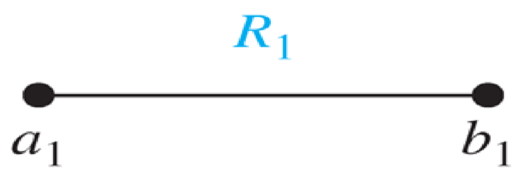

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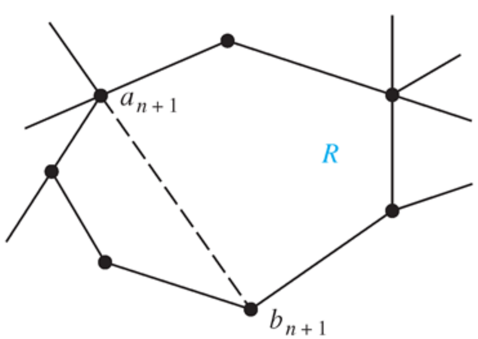

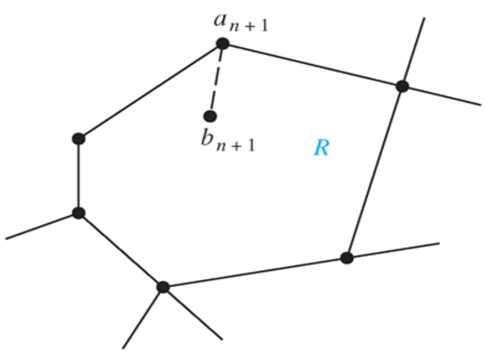

The Degree of Regions

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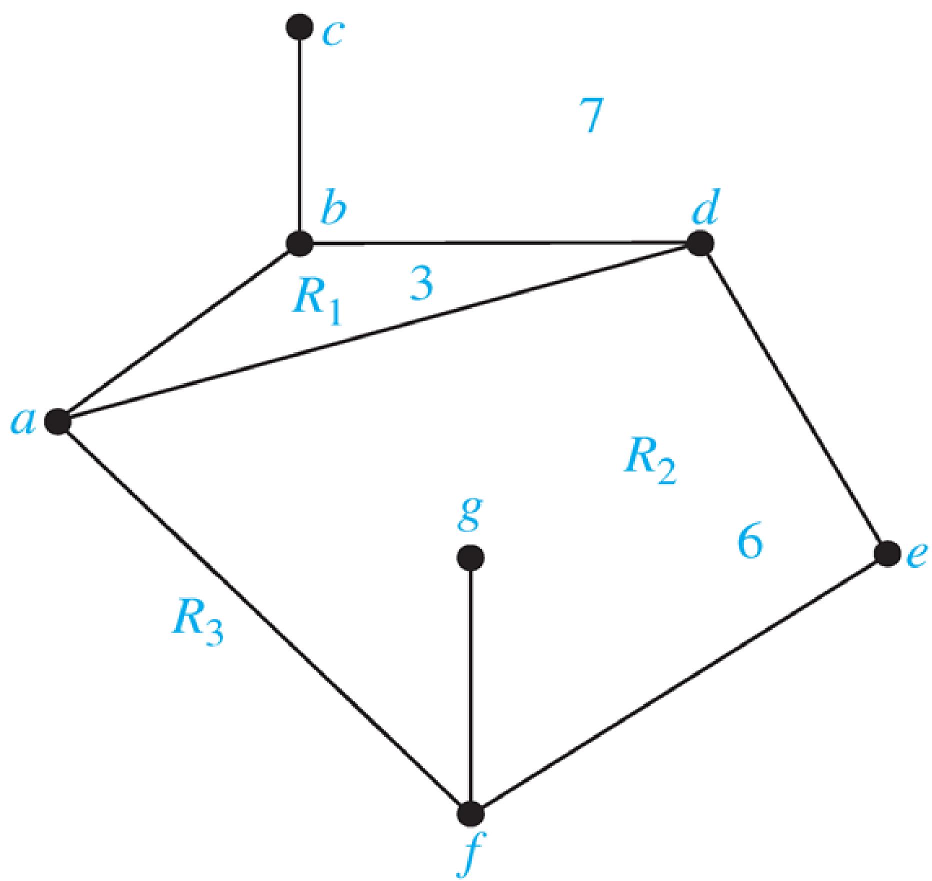

Corollaries

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  By Euler's formula, the proof is completed.

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  Proof similar to that of Corollary 1.

Examples

- Show that K_5 is nonplanar.

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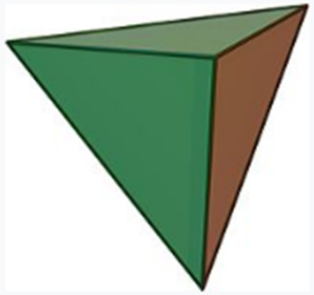
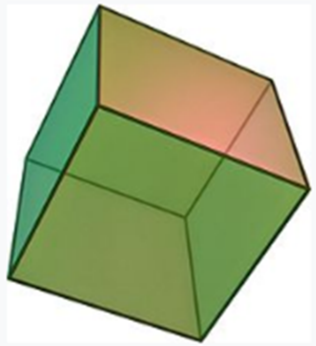
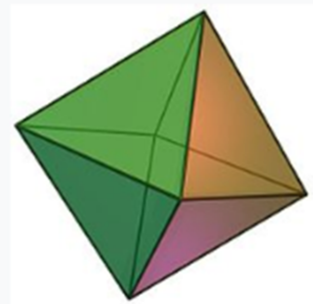
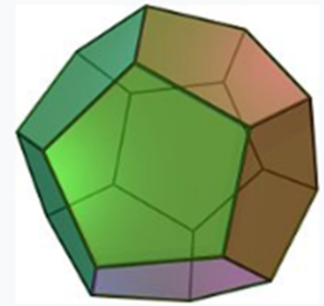
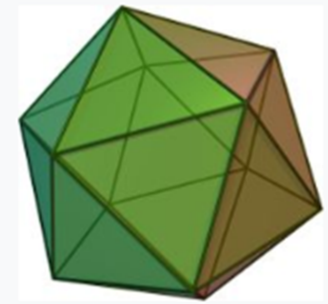
  Corollary 2 is used in the proof of Five Color Theorem.

Only 5 Platonic Solids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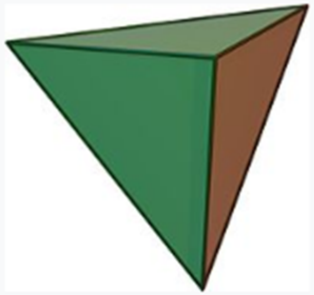	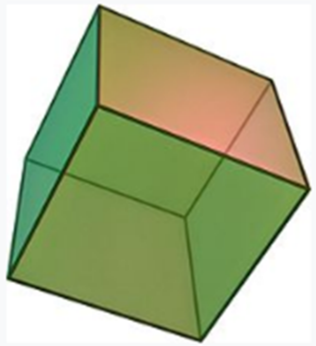	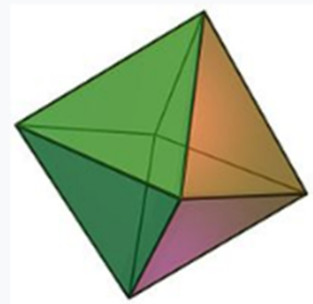	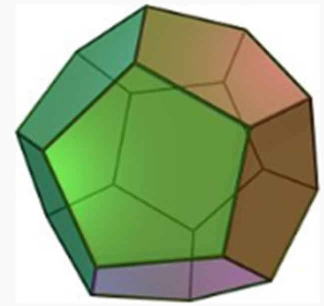	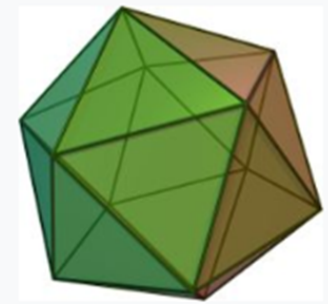
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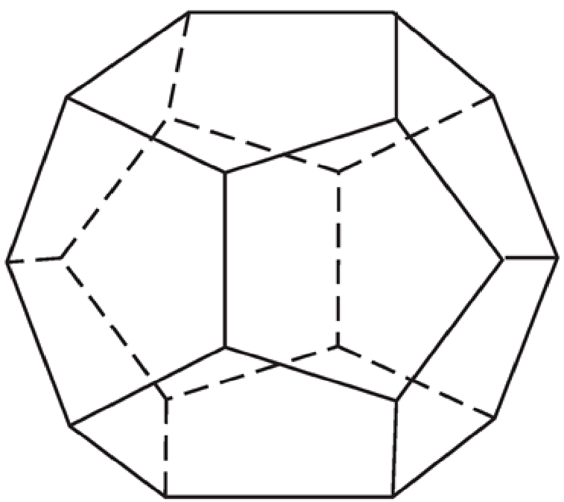
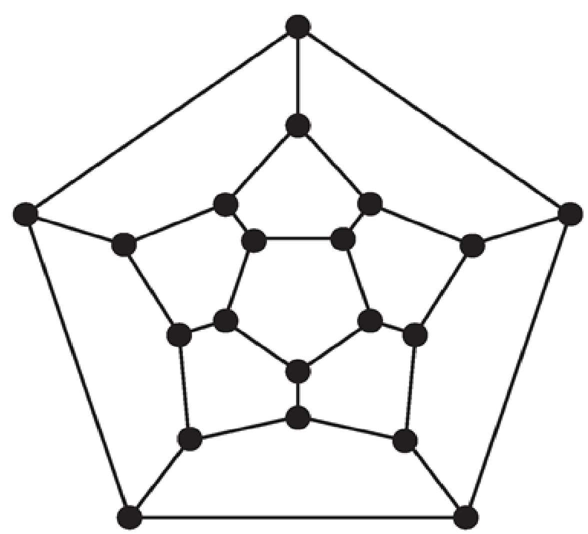


(a)



(b)

- **Definition** If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an *elementary subdivision*. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *homomorphic* if they can be obtained from the same graph by a sequence of elementary subdivisions.

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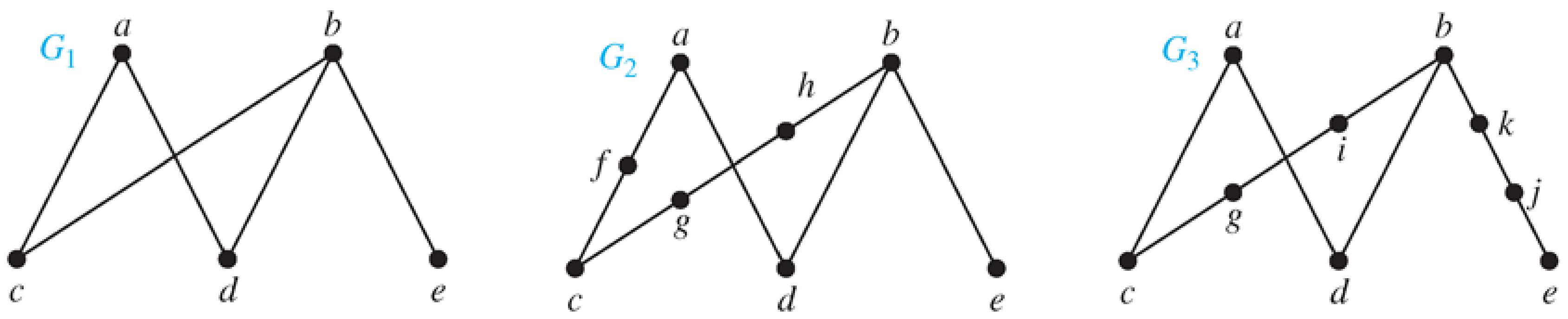

Kuratowski's Theorem

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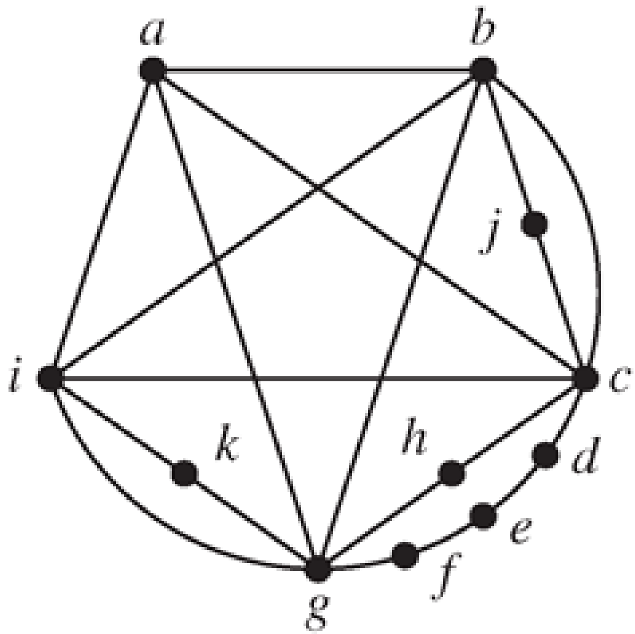
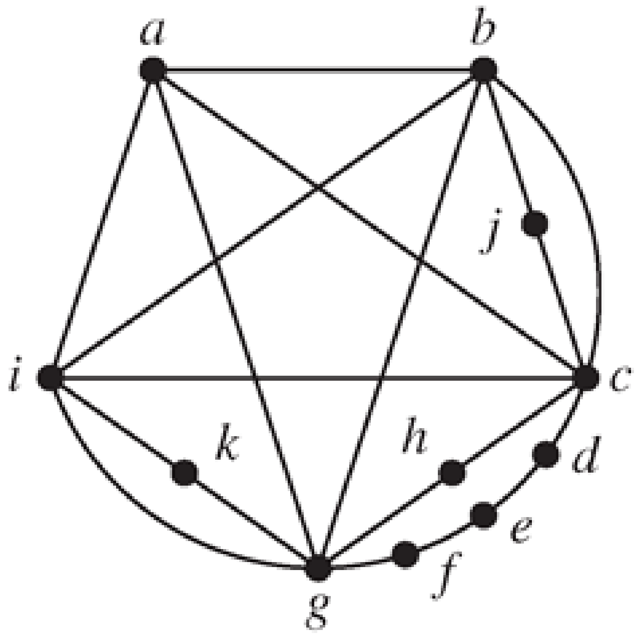
  Theorem A graph is nonplanar if and only if it contains a subgraph homomorphic to $K_{3,3}$ or K_5.

Examples

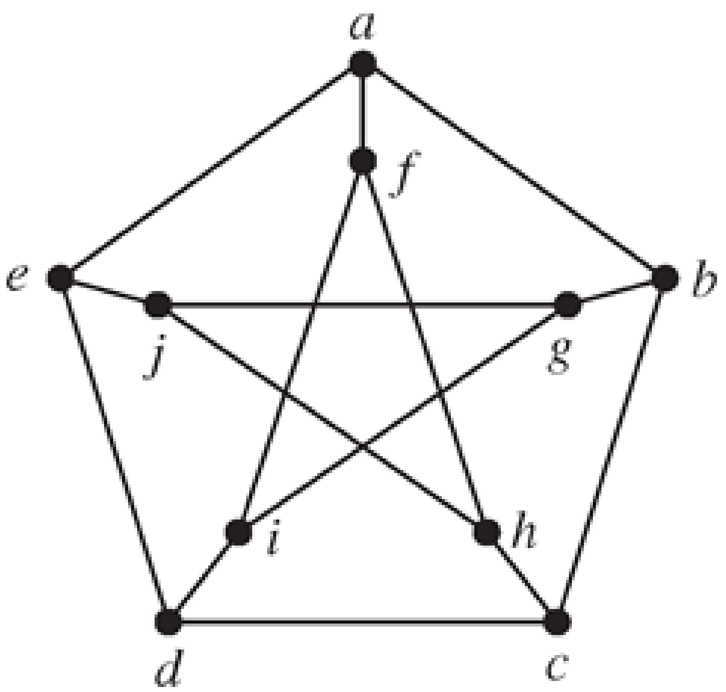


G

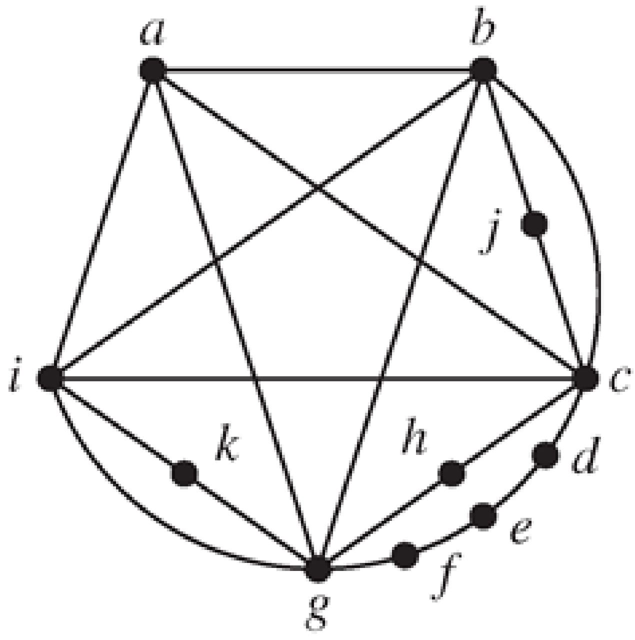
Examples



G



Examples



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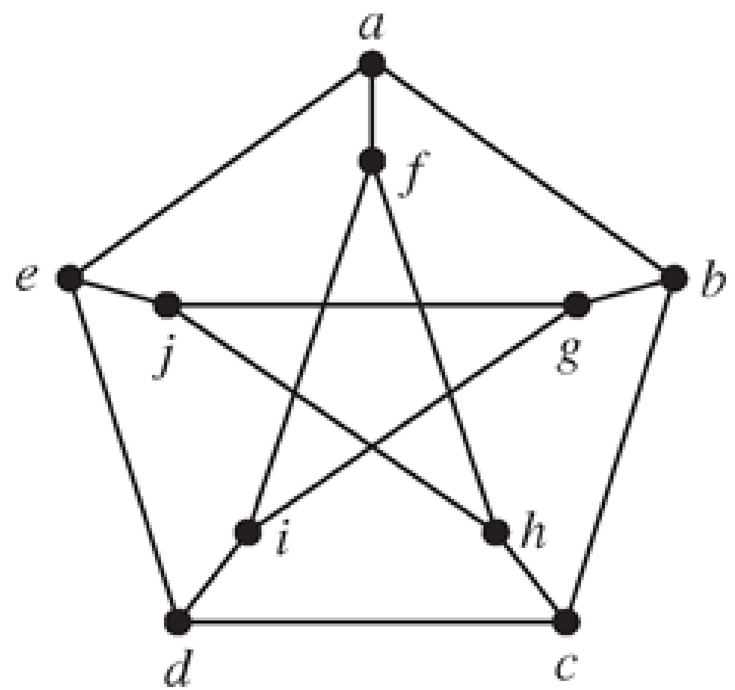
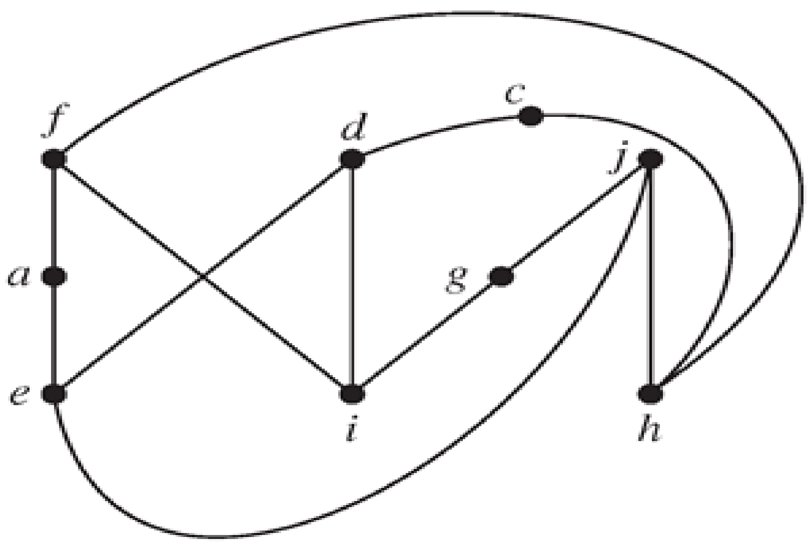
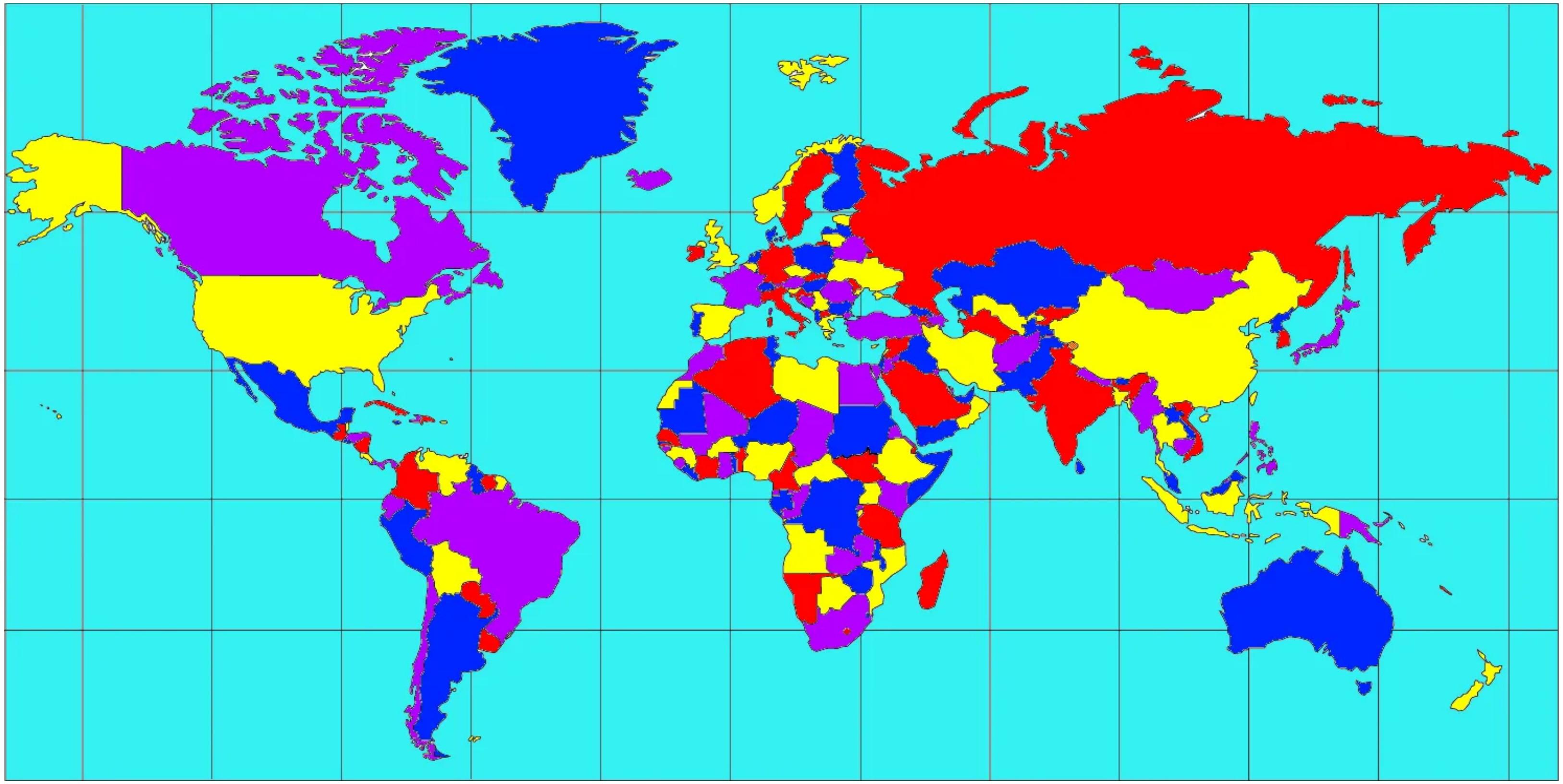




Graph Coloring

- **Four-color theorem** Given any separation of a plane into contiguous regions, producing a figure called a *map*, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.



- **Four-color theorem**
  - ◇ first proposed by Francis Guthrie in 1852
  - ◇ his brother Frederick Guthrie told Augustus De Morgan
  - ◇ De Morgan wrote to William Hamilton
  - ◇ Alfred Kempe proved it incorrectly in 1879
  - ◇ Percy Heawood found an error in 1890 and proved the *five-color theorem*
  - ◇ Finally, Kenneth Appel and Wolfgang Haken proved it with case by case analysis by computer in 1976 (*the first computer-aided proof*)
  - ◇ Kempe's incorrect proof serves as a basis

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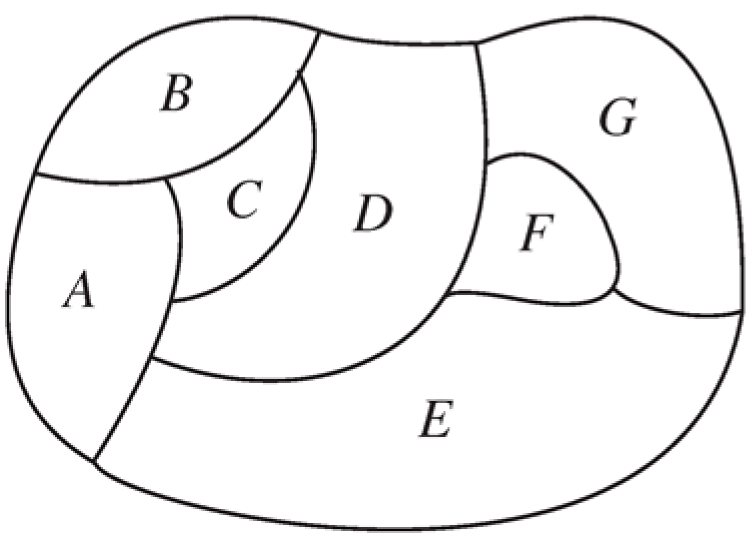

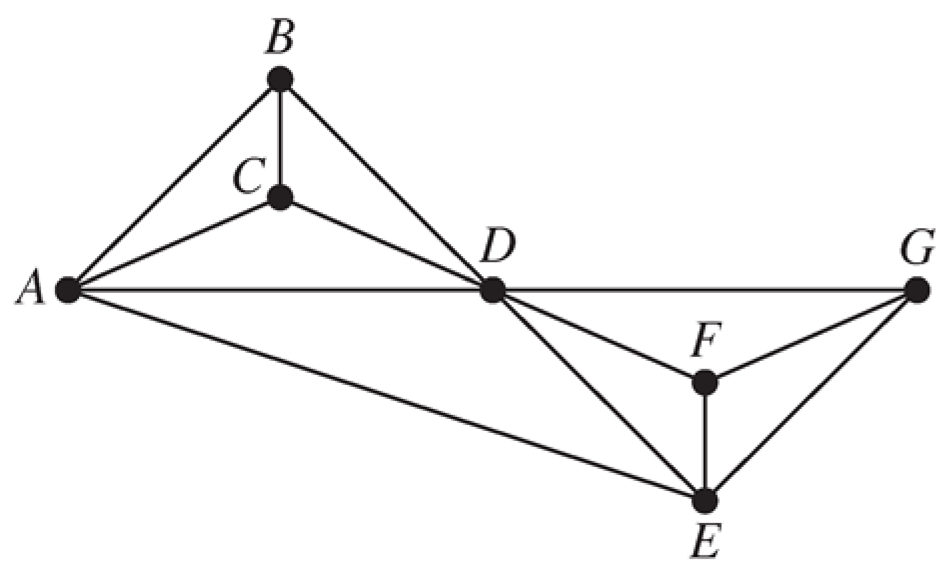

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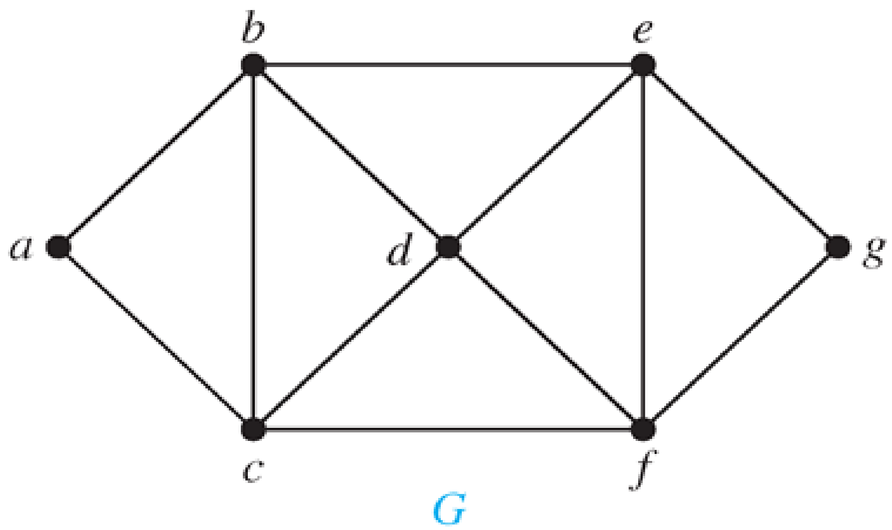

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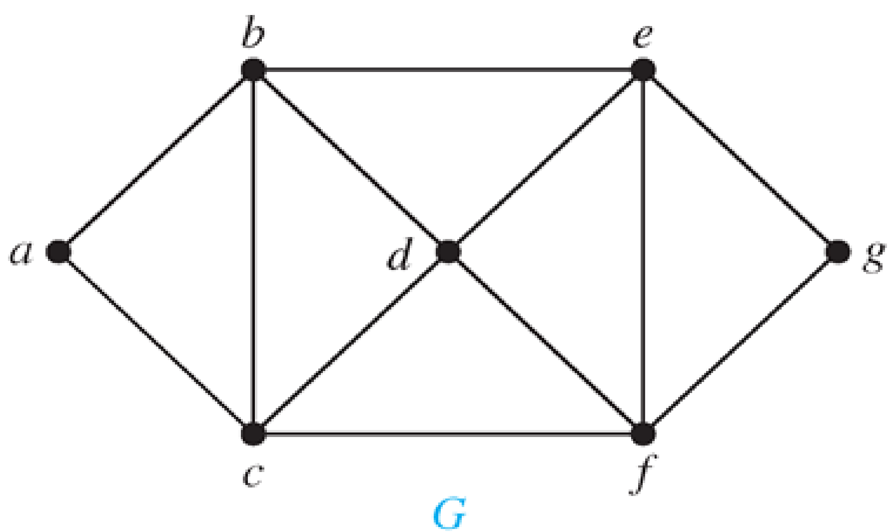

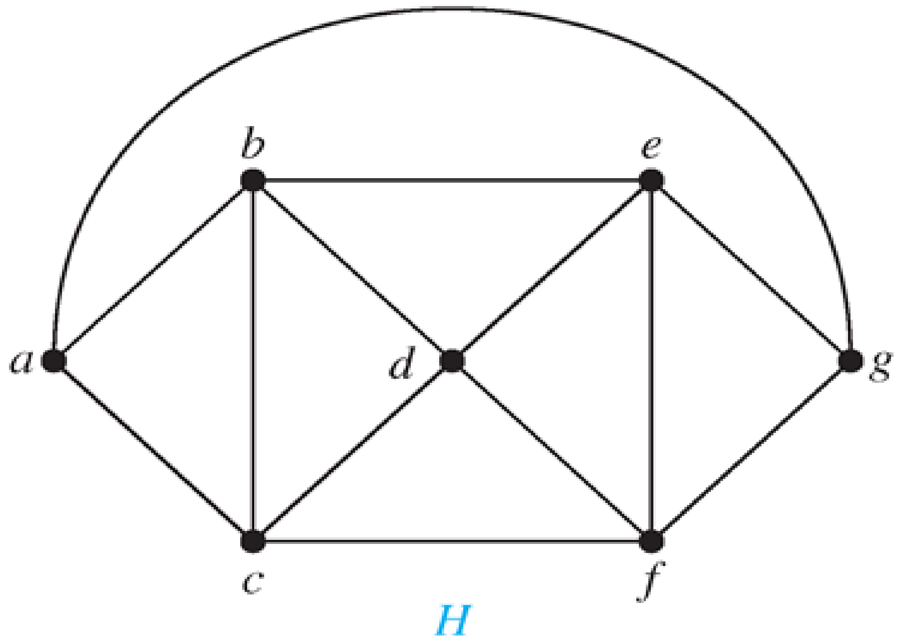

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  w.l.o.g., assume that the graph is connected.

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  Inductive step: Consider a planar graph with $k+1$ vertices. Recall Corollary 2 (the graph has a vertex of degree 5 or fewer). Remove this vertex, by i.h., we can color the remaining graph with 6 colors. Put the vertex back in. Since there are at most 5 colors adjacent, so we have at least one color left.

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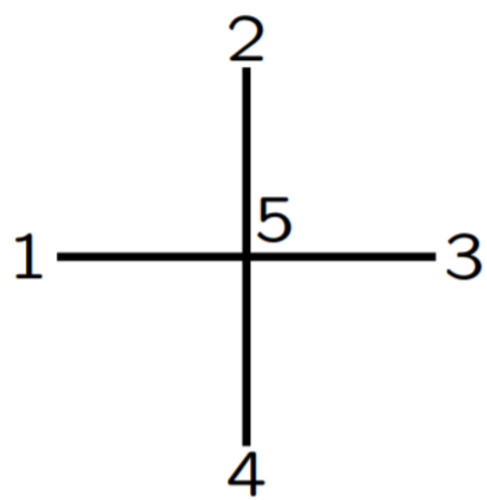
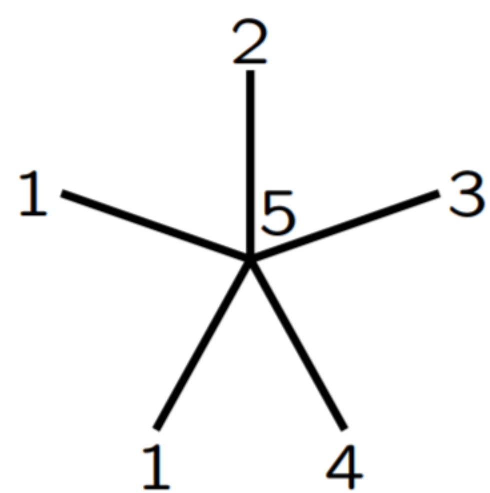
  Inductive hypothesis: Assume that every planar graph with $k \geq 1$ or fewer vertices can be 5-colored.
  Inductive step: Consider a planar graph with $k+1$ vertices. Recall Corollary 2 (the graph has a vertex of degree 5 or fewer). Remove this vertex, by i.h., we can color the remaining graph with 5 colors. Put the vertex back in.

- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

  Proof (by induction on the number of vertices)
  w.l.o.g., assume that the graph is connected.

  If the vertex has degree less than 5, or if it has degree 5 and only ≤ 4 colors are used for vertices connected to it, we can pick an available color for it.





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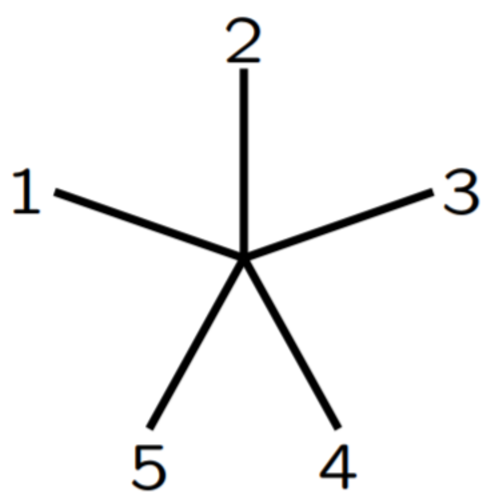
  Proof (by induction on the number of vertices)

  If the vertex has degree 5, and all 5 colors are connected to it, we label the vertices adjacent to the "special" vertex (degree 5) 1 to 5 (in order).



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  We make a subgraph out of all the vertices colored 1 or 3. If the adjacent vertex colored 1 and the adjacent vertex colored 3 are not connected by a path in the subgraph.

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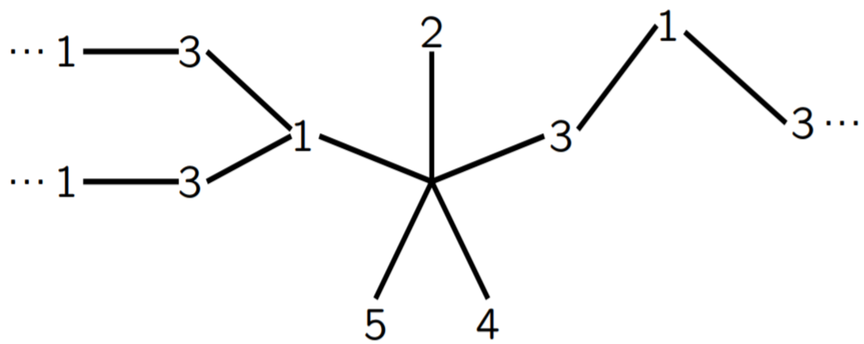

Graph Coloring

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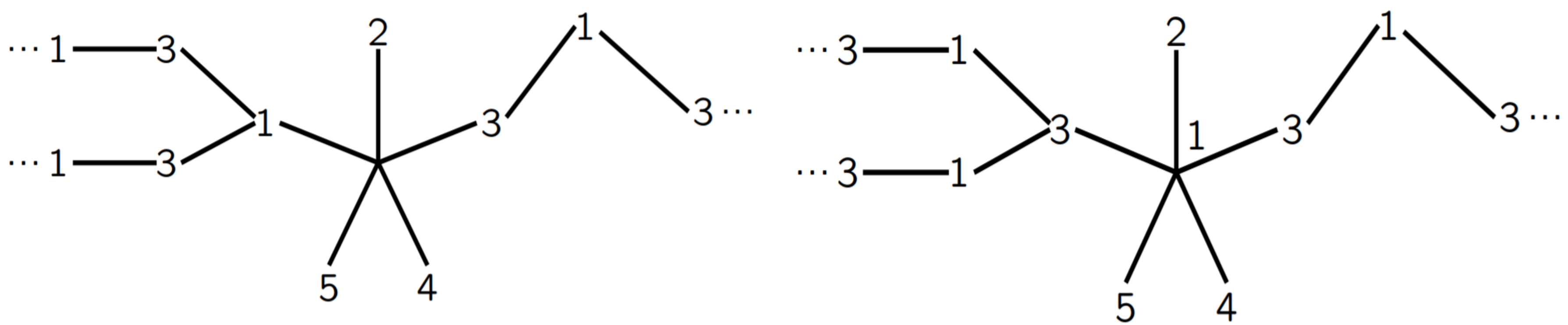

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  On the other hand, if the vertices colored 1 and 3 are connected via a path in the subgraph, we do the the same for the vertices colored 2 and 4. Note that this will be a disconnected pair of subgraphs, separated by a path connecting the vertices colored 1 and 3 (Why?)

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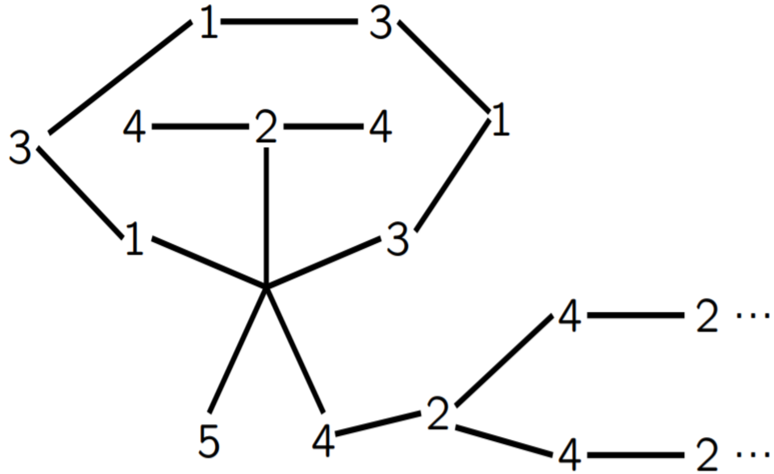

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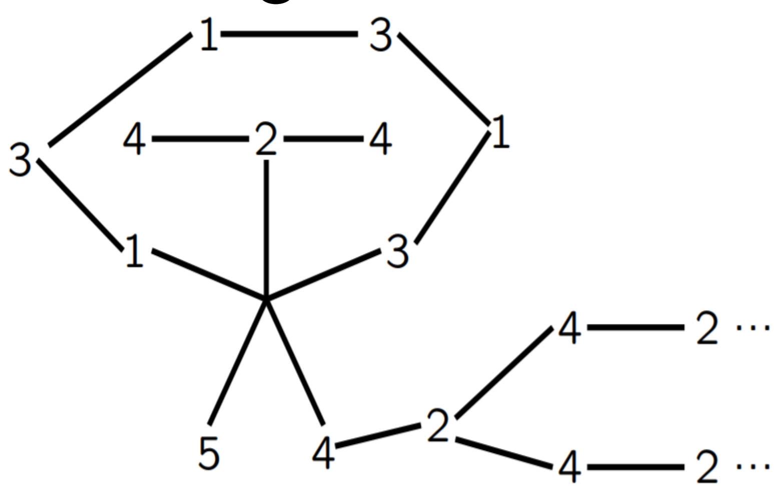

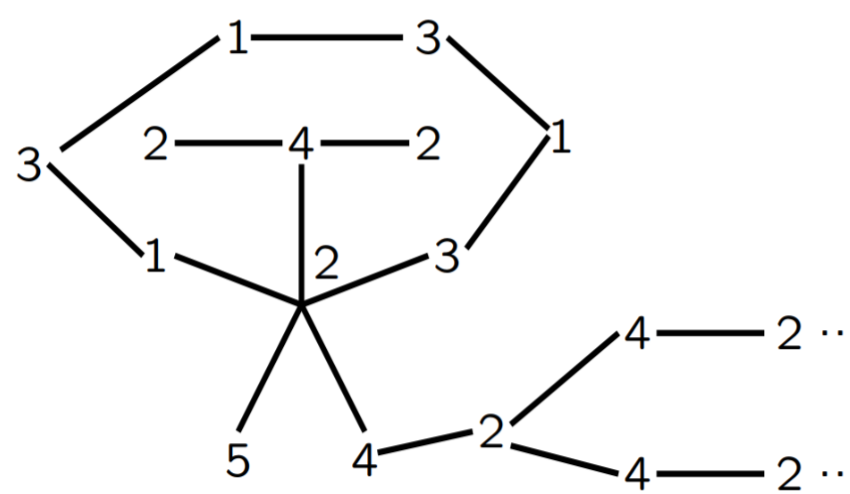

Examples

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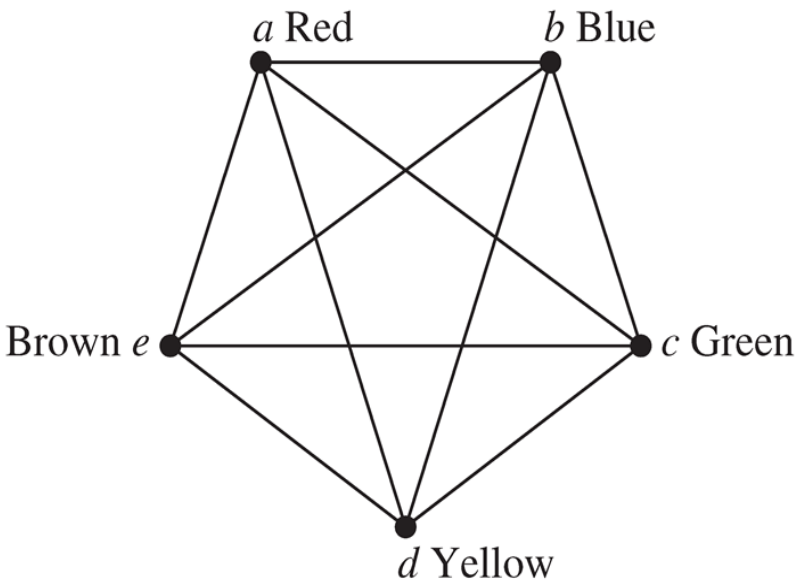

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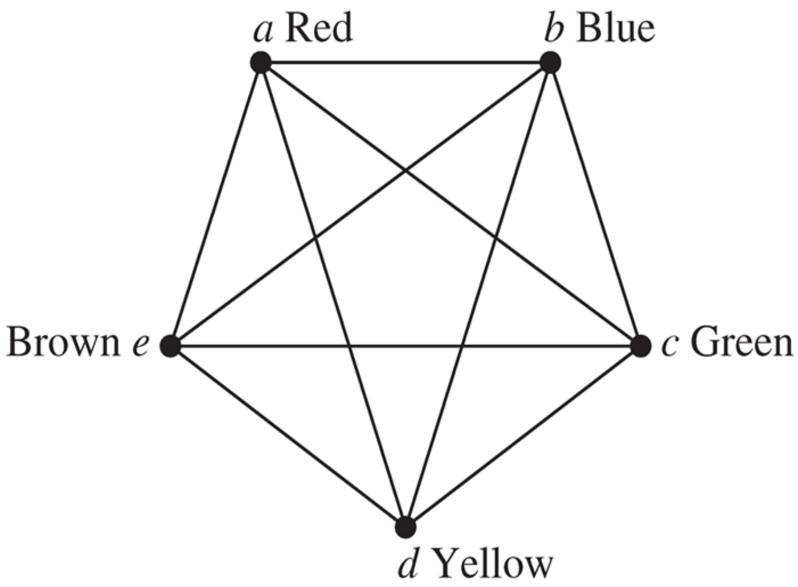

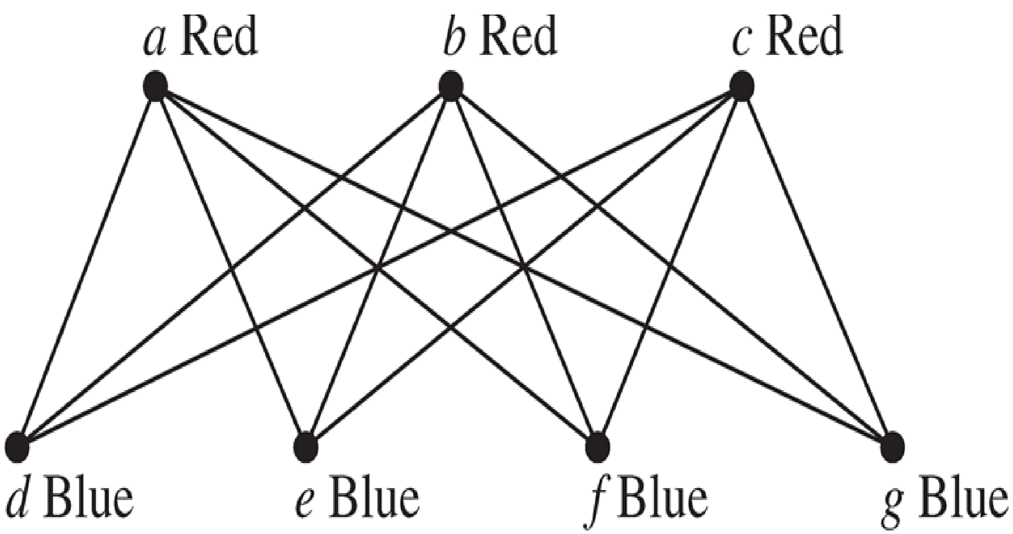

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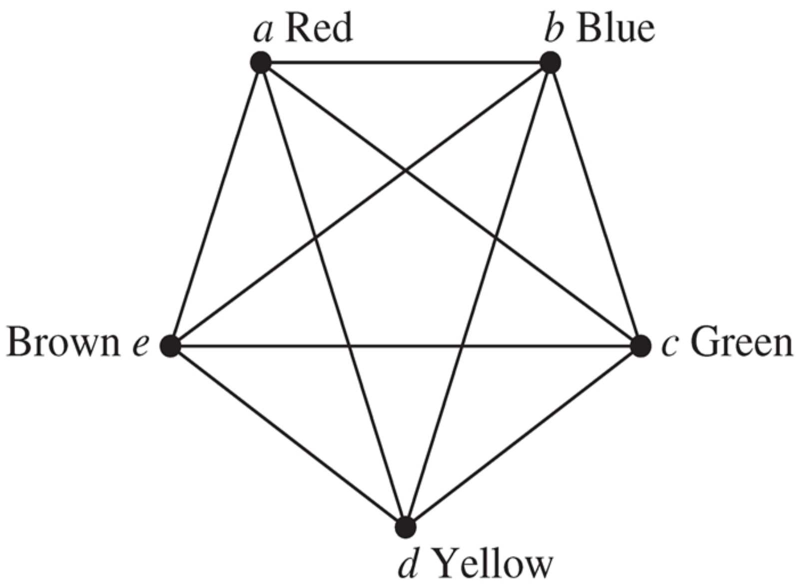

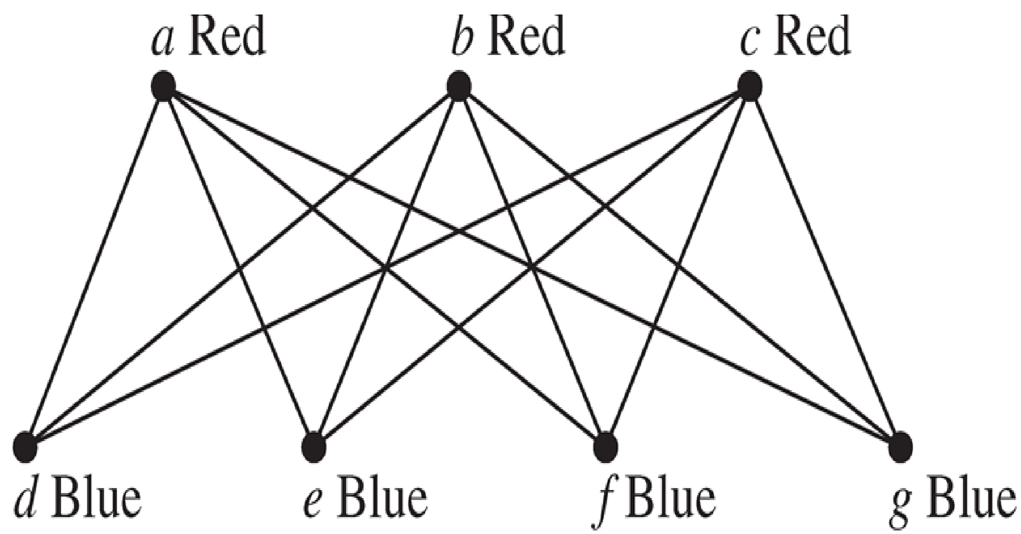

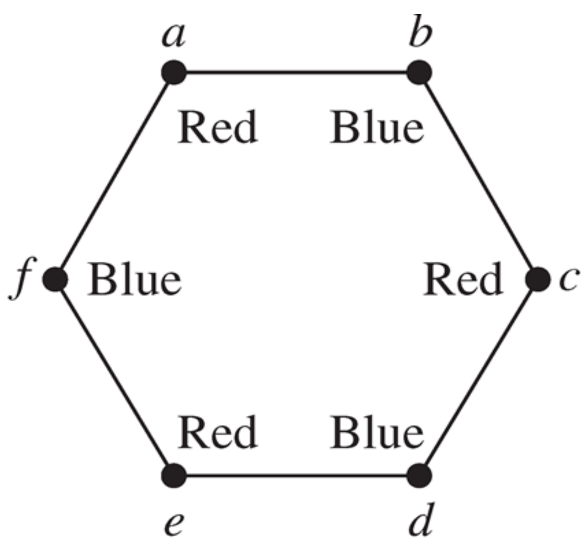

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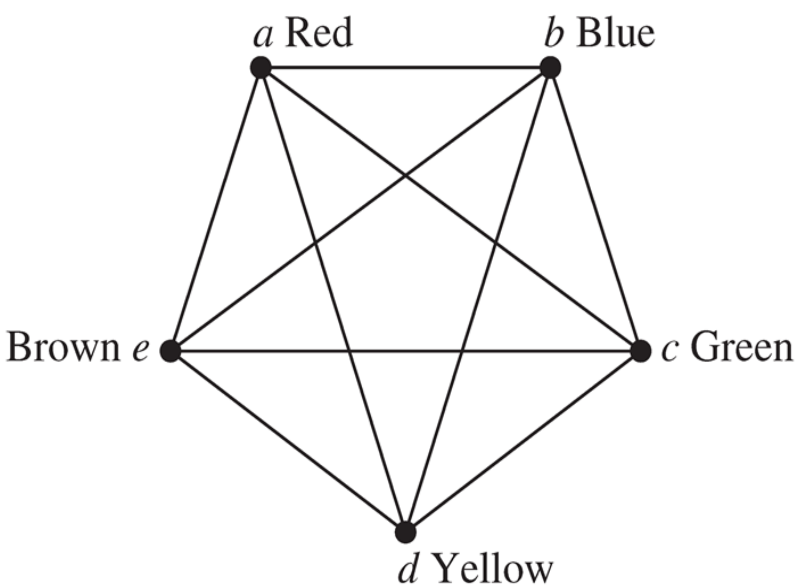
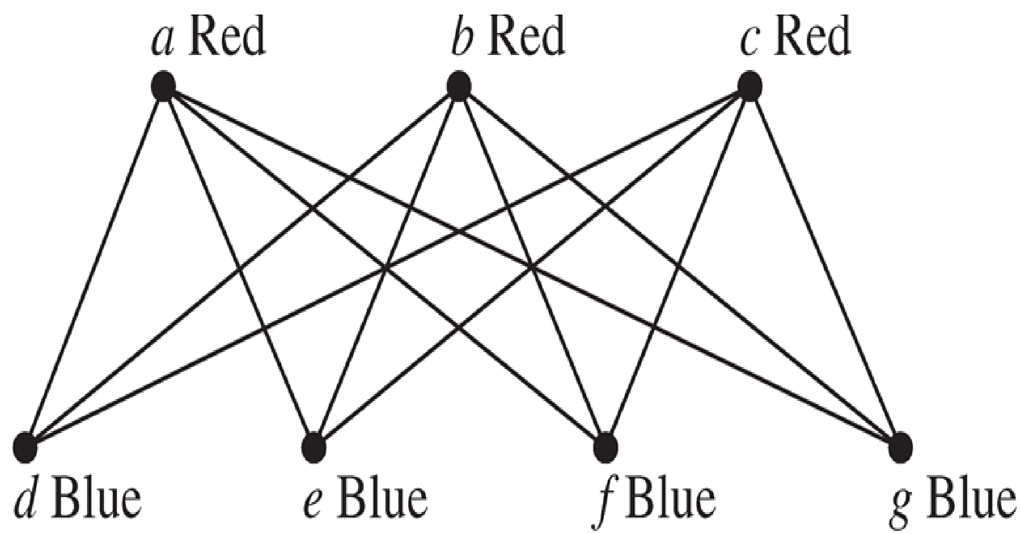
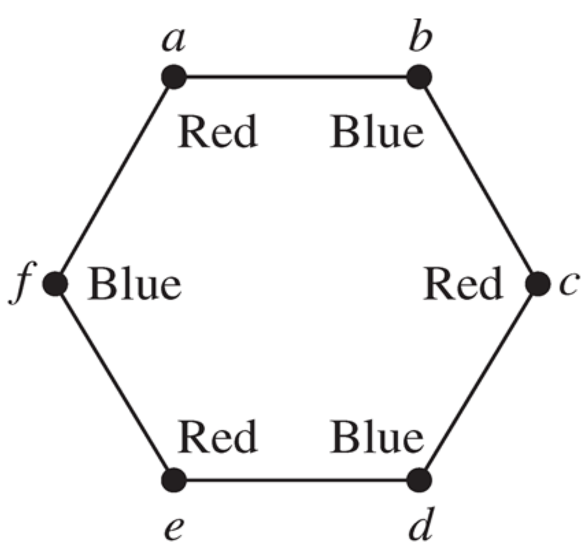
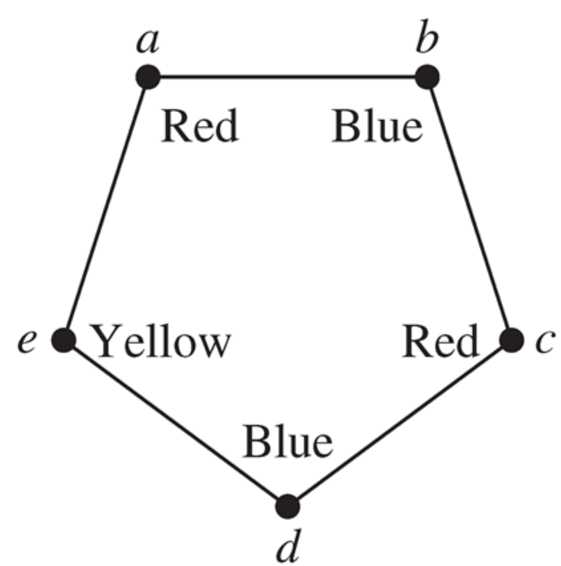




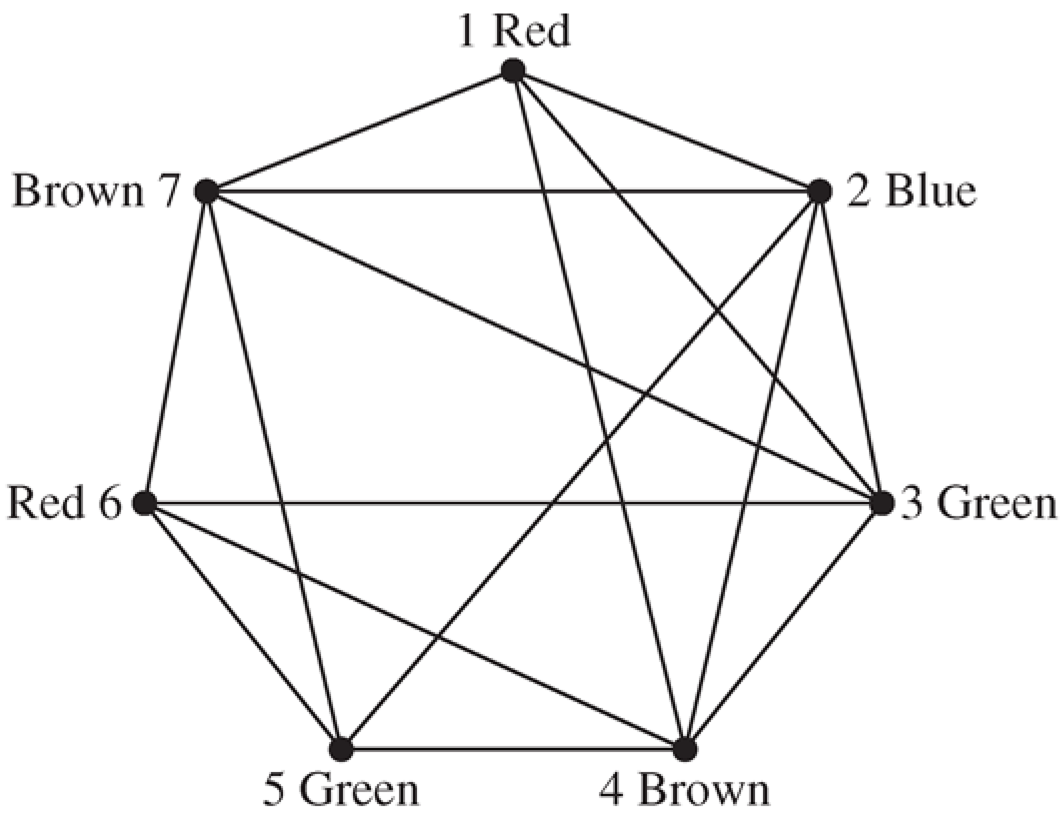




Applications of Graph Coloring

Scheduling Final Exams

Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

Applications of Graph Coloring

- Channel Assignments

  Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel . How can the assignment of channels be modeled by graph coloring?

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Graph Coloring $\in$ NPC

Next Lecture

- tree ...