



CS215 DISCRETE MATH

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Properties of Relations

■ **Reflexive Relation:** A relation R on a set A is called *reflexive* if $(a, a) \in R$ for **every** element $a \in A$.

Irreflexive Relation: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for **every** element $a \in A$.

Symmetric Relation: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for **all** $a, b \in A$.

Antisymmetric Relation: A relation R on a set A is called *antisymmetric* if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for **all** $a, b \in A$.

Transitive Relation: A relation R on a set A is called *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for **all** $a, b, c \in A$.

Connectivity

- **Lemma:** Let A be a set with n elements, and R a relation on A . If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n - 1$.

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Theorem: The transitive closure of a relation R equals the connectivity relation R^* .

Recall Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

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R has the following pairs:

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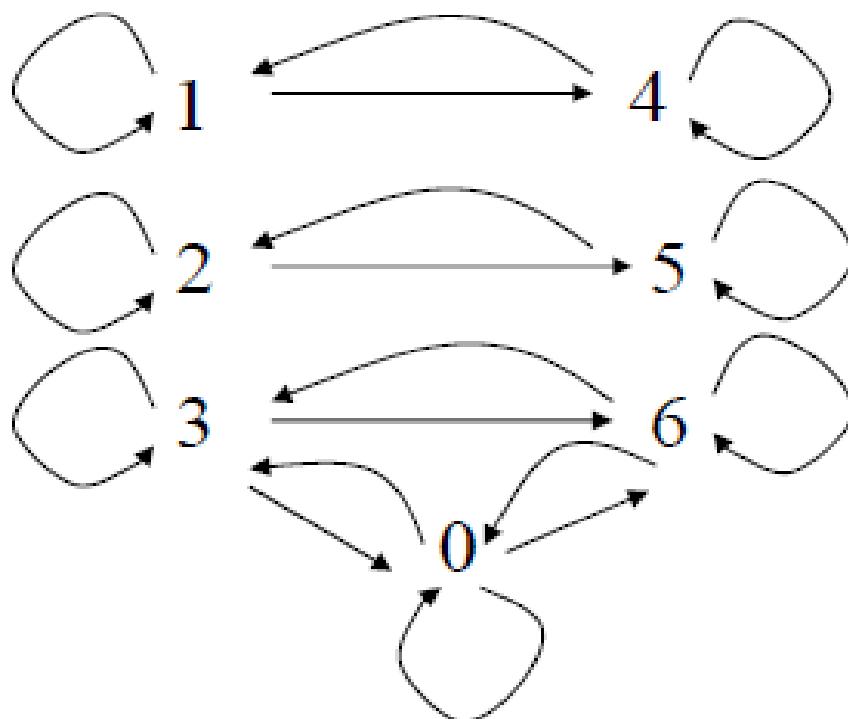
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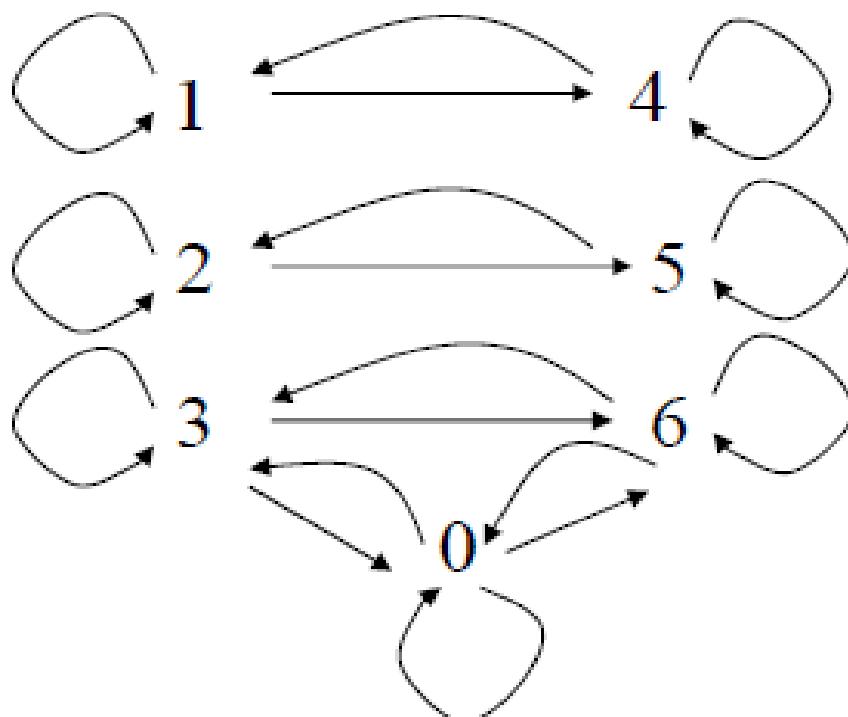
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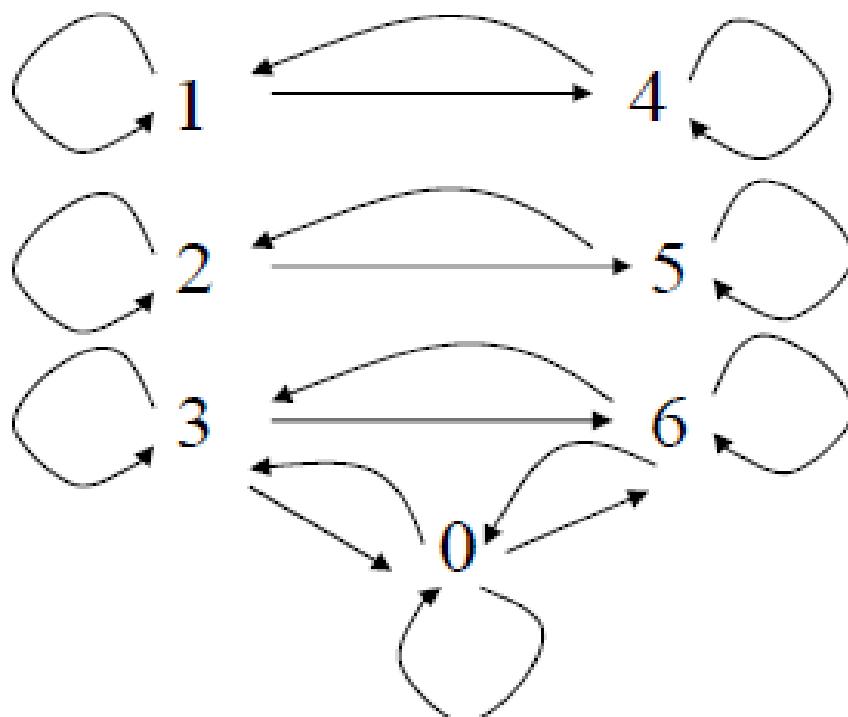


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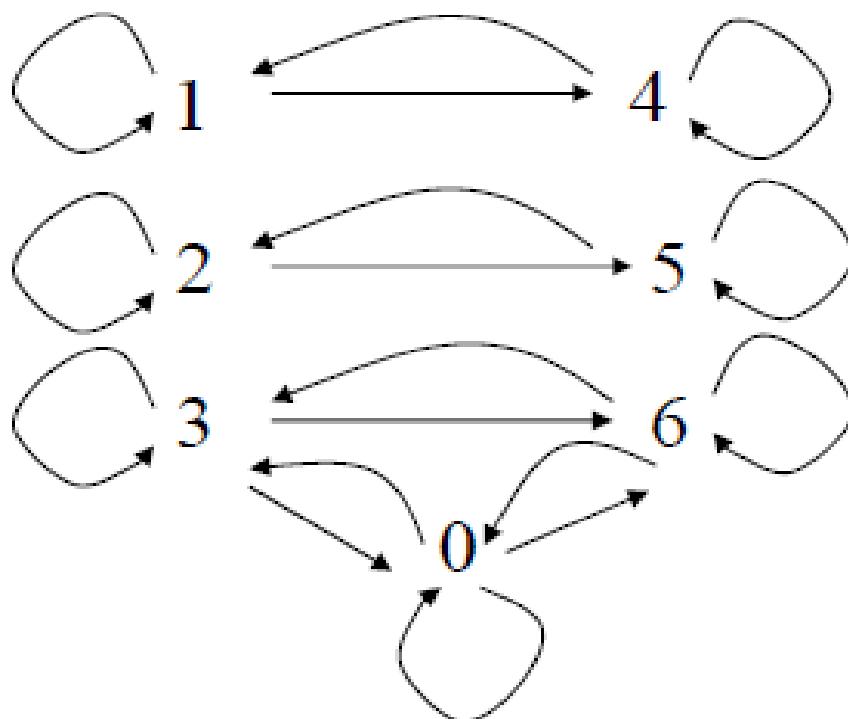


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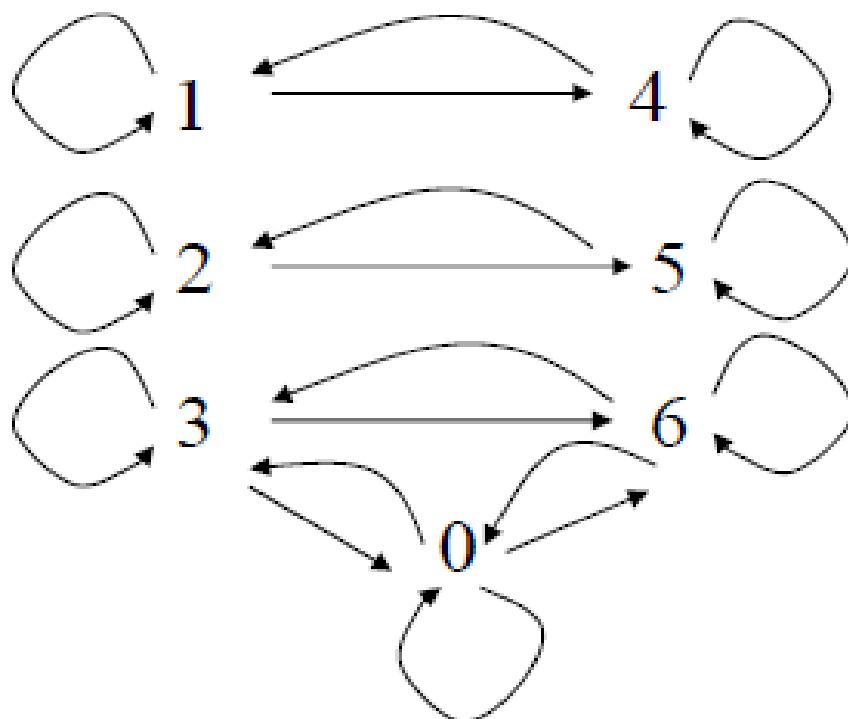
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Is R transitive?



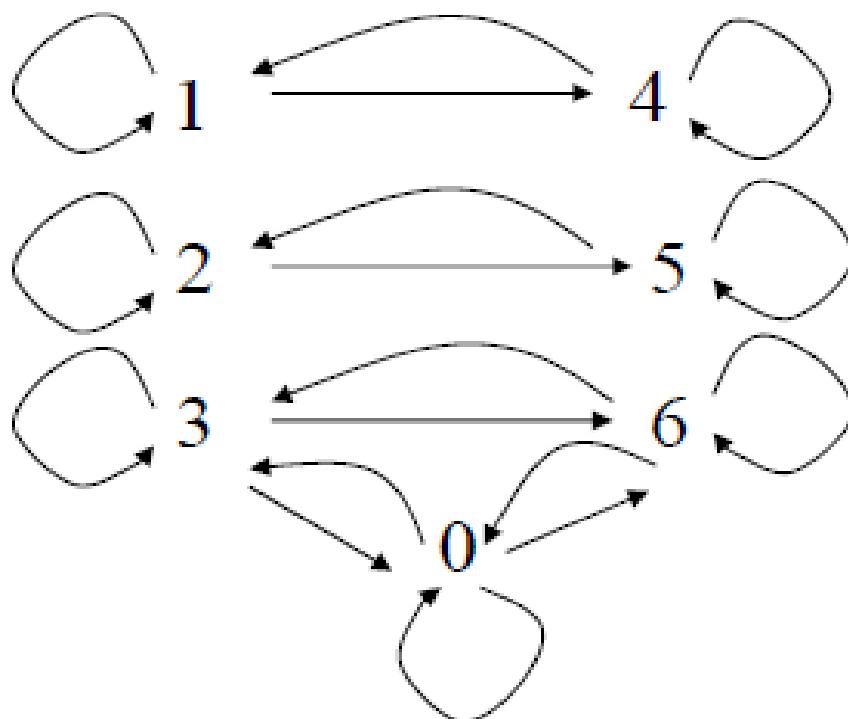
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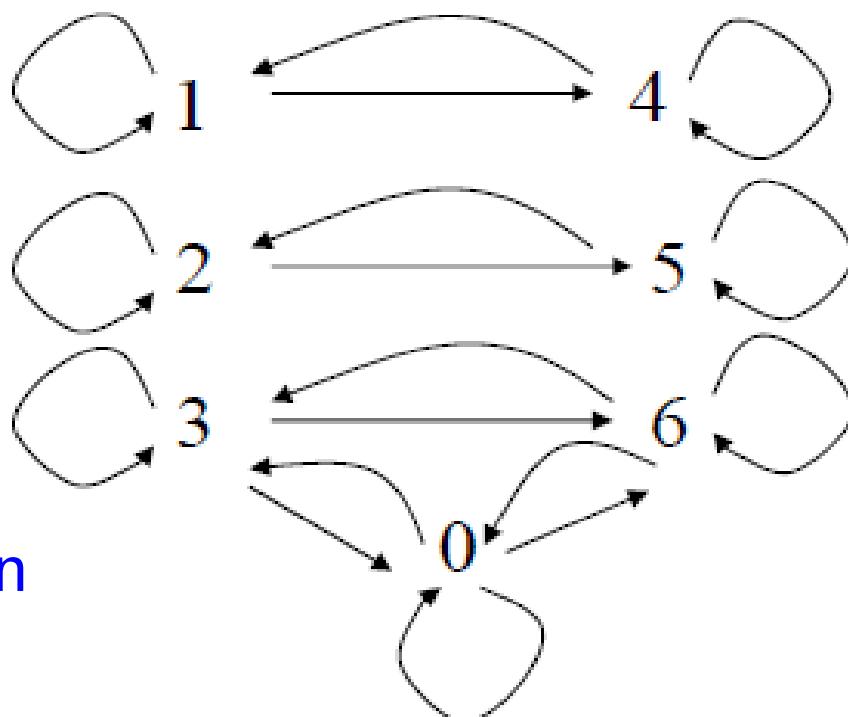
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R is an equivalence relation



Examples of Equivalence Relations

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“Strings a and b have the same length.”

“Integers a and b have the same absolute value.”

“Real numbers a and b have the same fractional part (i.e., $a - b \in \mathbb{Z}$).”

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“The relation \geq between real numbers.”

“has a common factor greater than 1 between natural numbers.”

Equivalence Class

- **Definition** Let R be an equivalence relation on a set A . The set of all elements that are related to an element a of A is called the *equivalence class* of a , denoted by $[a]_R$. When only one relation is considered, we use the notation $[a]$.

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$$\begin{aligned}[0] &= [3] = [6] = \{0, 3, 6\} \\[1] &= [4] = \{1, 4\} \\[2] &= [5] = \{2, 5\}\end{aligned}$$

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“Strings a and b have the same length.”

$[a] =$ the set of all strings of the same length as a

“Integers a and b have the same absolute value.”

$[a] =$ the set $\{a, -a\}$

“Real numbers a and b have the same fractional part (i.e., $a - b \in \mathbf{Z}$).”

$[a] =$ the set $\{\dots, a - 2, a - 1, a, a + 1, a + 2, \dots\}$

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Partition of a Set S

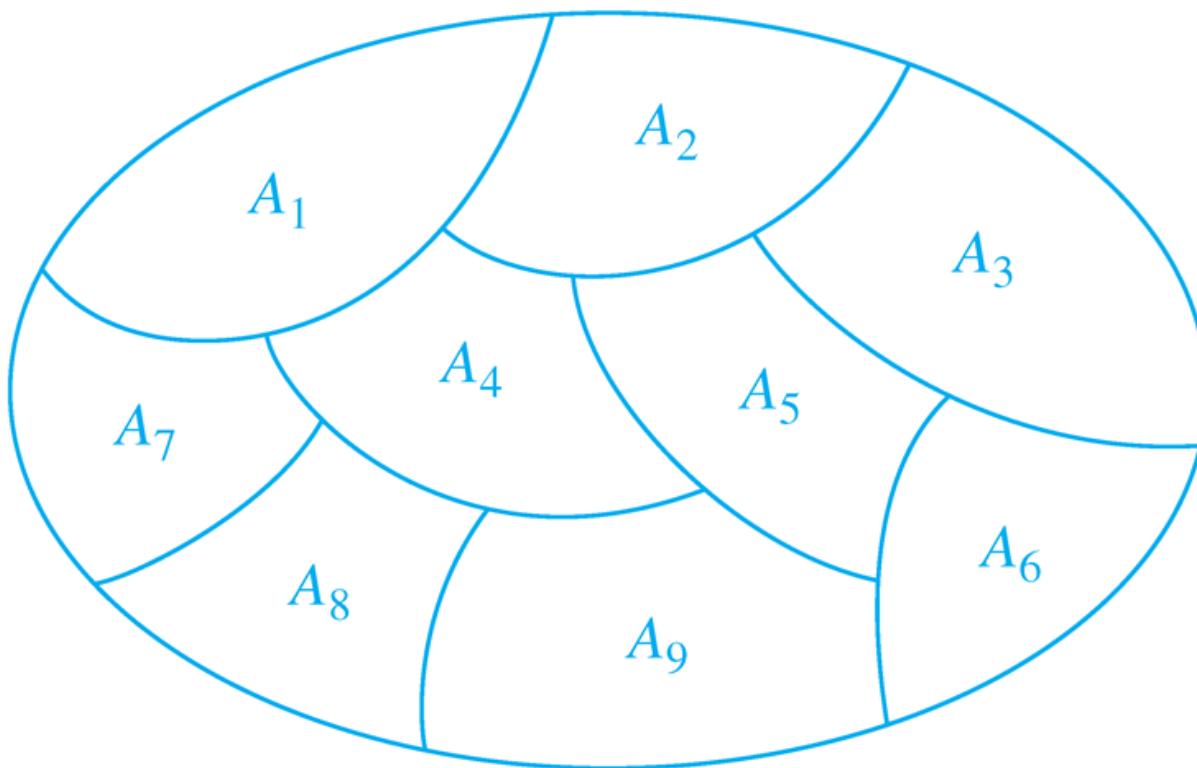
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Is A_1, A_2, A_3 a partition of S ?

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Theorem Let $\{A_1, A_2, \dots, A_i, \dots\}$ be a partition of S . Then there is an equivalence relation R on S , that has the sets A_i as its equivalence classes.

Partial Ordering

- **Definition** A relation R on a set S is called a *partial ordering*, or *partial order*, if it is **reflexive**, **antisymmetric**, and **transitive**. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, denoted by (S, R) . Members of S are called *elements of the poset*.

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2, 4 are comparable, 3, 5 are incomparable.

Total Ordering

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Lexicographic Ordering

- **Definition** Given two posets (A_1, \preccurlyeq_1) and (A_2, \preccurlyeq_2) , the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is **less than** (b_1, b_2) , i.e., $(a_1, a_2) \preccurlyeq (b_1, b_2)$, either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ then $a_2 \preccurlyeq_2 b_2$.

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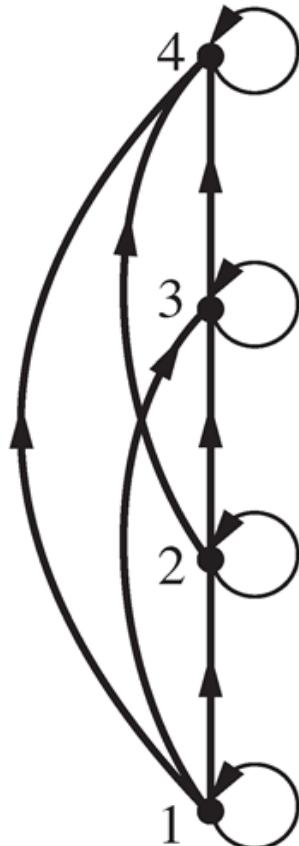
- ◊ *discreet* \prec *discrete*
- ◊ *discreet* \prec *discreteness*

Hasse Diagram

- A Hasse diagram is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.

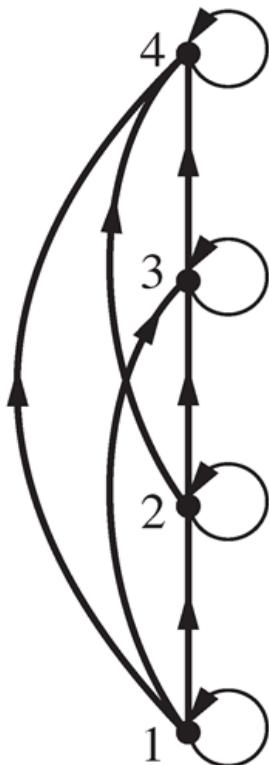
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Hasse Diagram

- (a) A partial ordering. The loops are due to the **reflexive property**
- (b) The edges that must be present due to the **transitive property** are deleted
- (c) The Hasse diagram for the partial ordering (a)



Procedure for Constructing Hasse Diagram

- Start with the directed graph of the relation:

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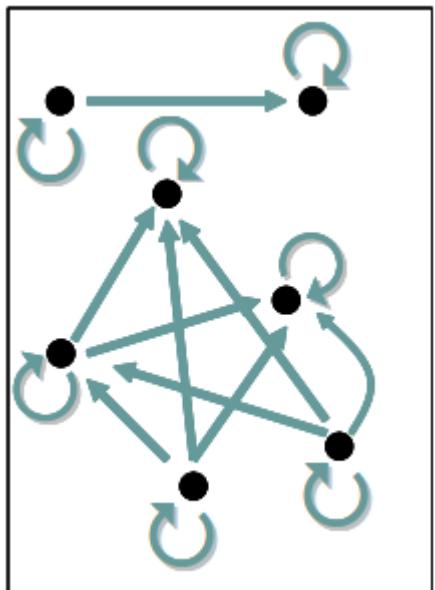
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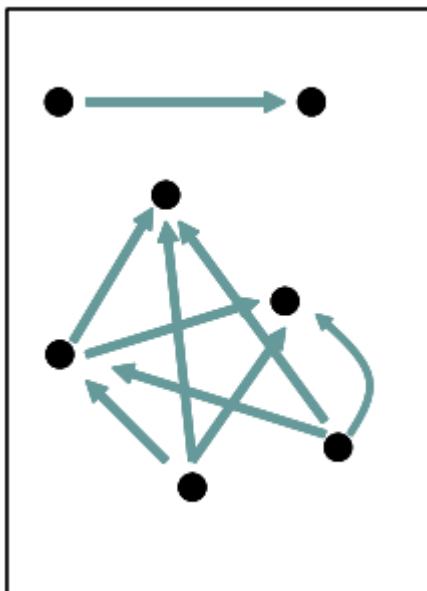
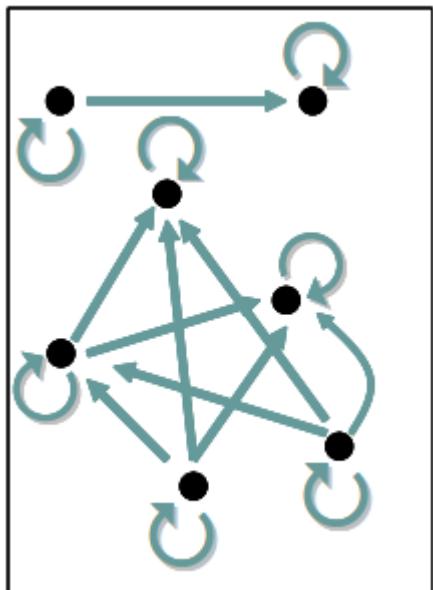
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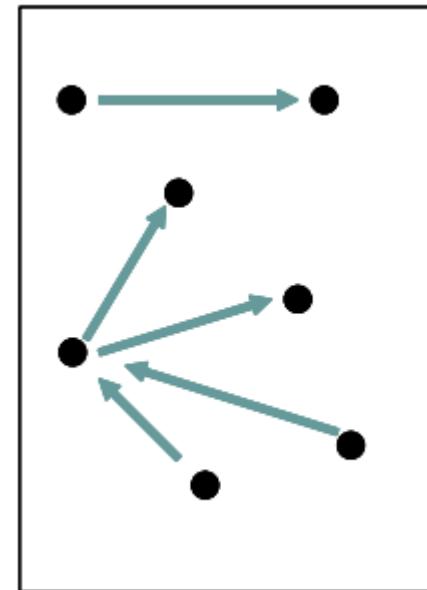
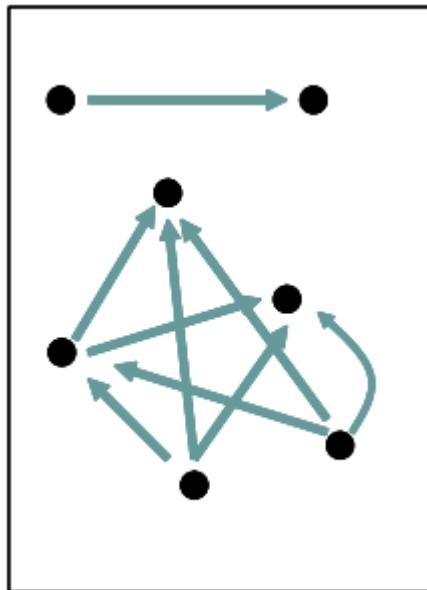
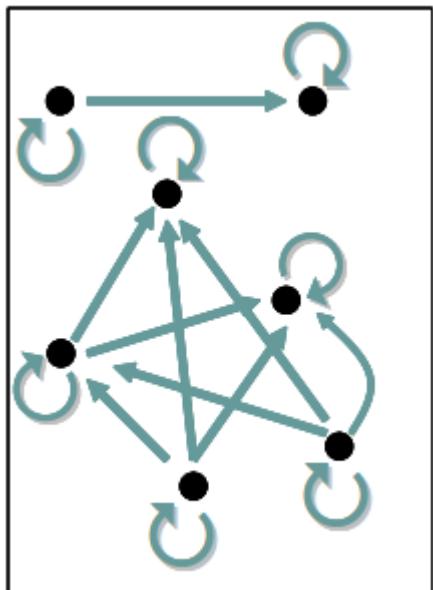
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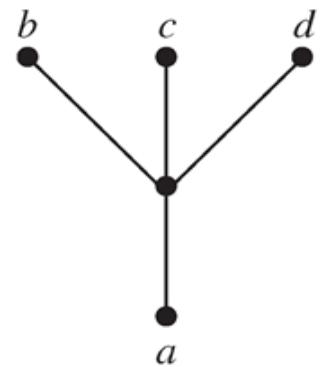
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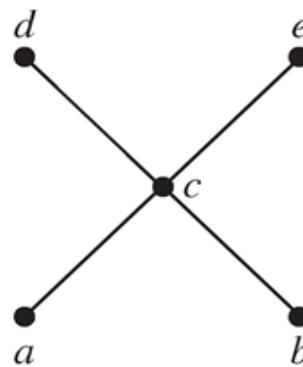
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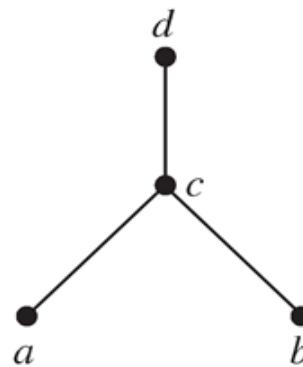
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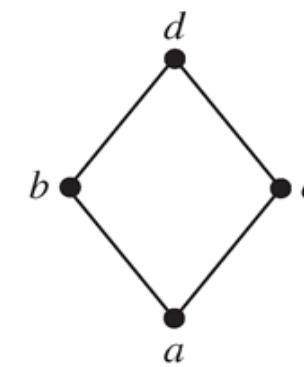
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Example Find the *greatest lower bound* and the *least upper bound* of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbb{Z}^+, |)$.

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- **Definition** (S, \preccurlyeq) is a *well-ordered set* if it is a poset such that \preccurlyeq is a *total ordering* and every nonempty subset of S has a *least element*.

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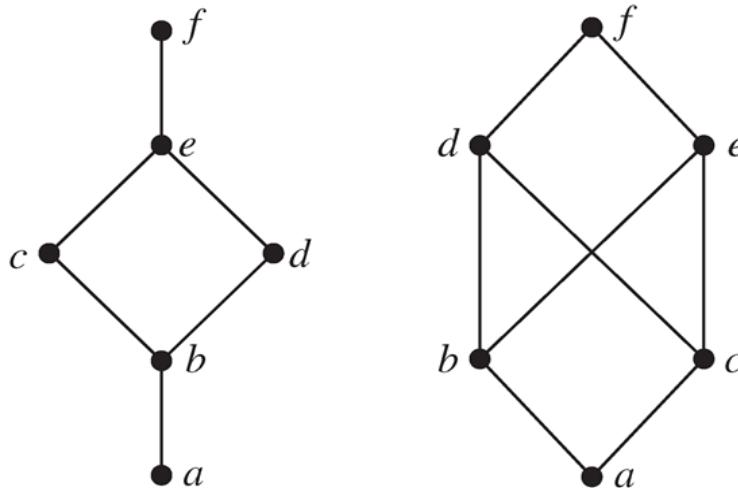
p.620, Theorem 1

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- **Definition** A partial ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a *lattice*.

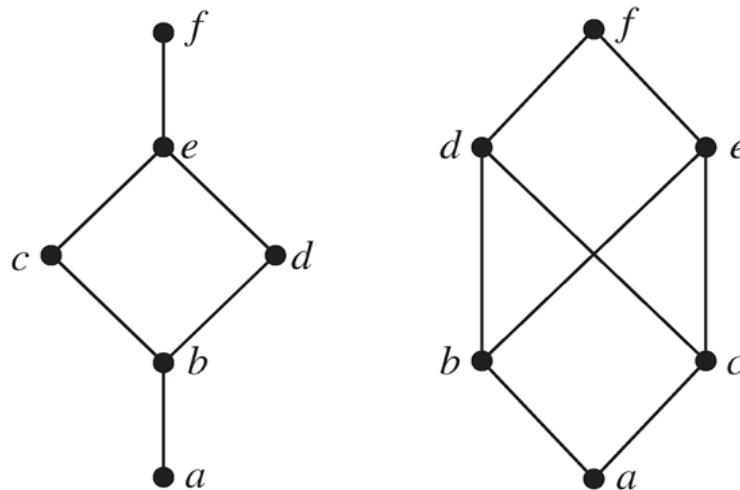
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Example Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

Topological Sorting

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Topological sorting: Given a **partial ordering** R , find a **total ordering** \preccurlyeq such that $a \preccurlyeq b$ whenever $a R b$. \preccurlyeq is said **compatible with** R .

Topological Sorting for Finite Posets

procedure topological_sort (S : finite poset)

$k := 1$;

while $S \neq \emptyset$

$a_k :=$ a minimal element of S

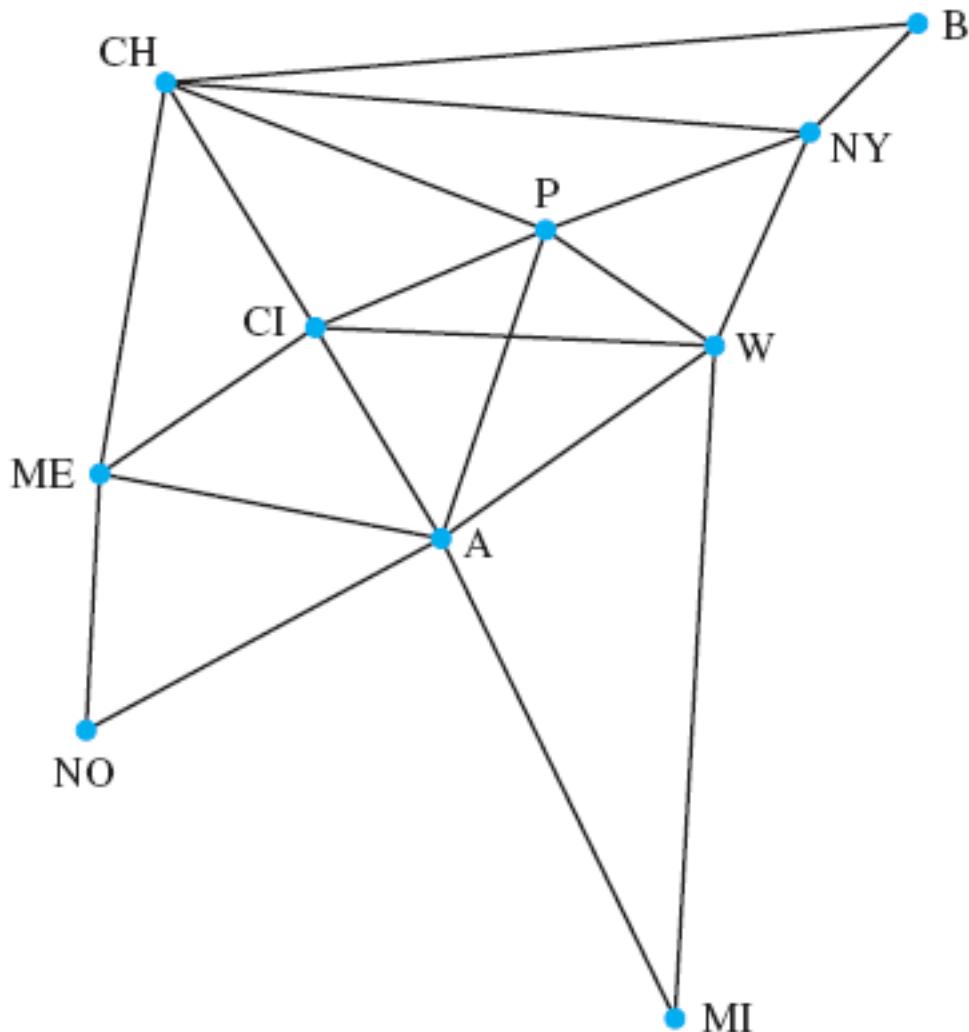
$S := S \setminus \{a_k\}$

$k := k + 1$

end while

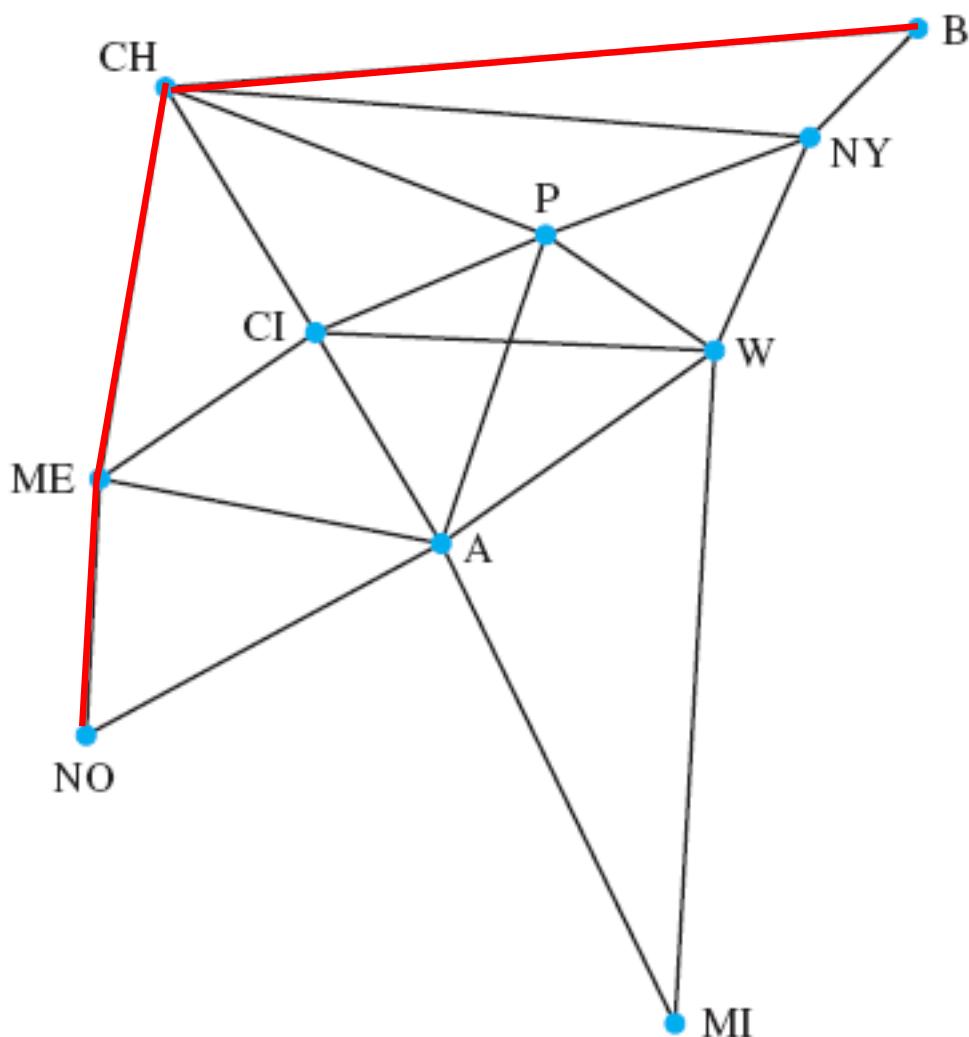
// $\{a_1, a_2, \dots, a_n\}$ is a compatible total ordering of S

Example



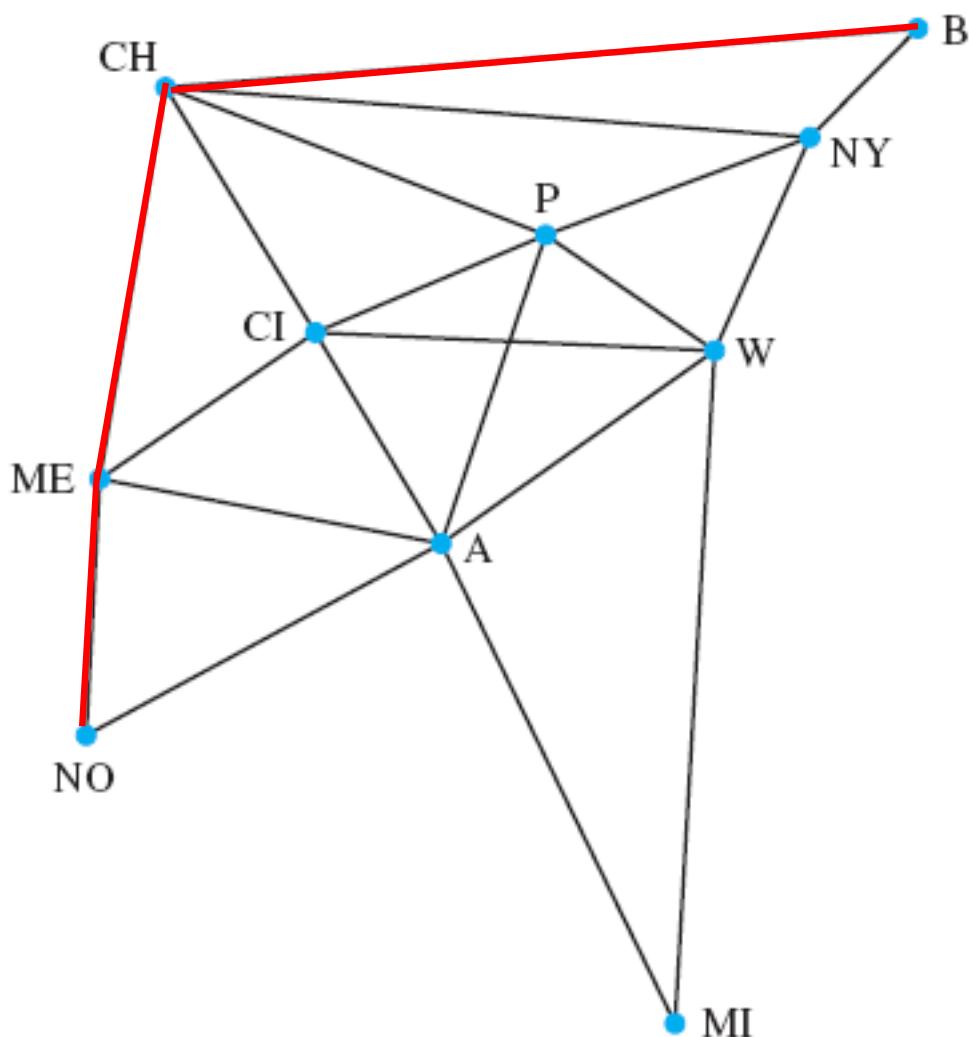
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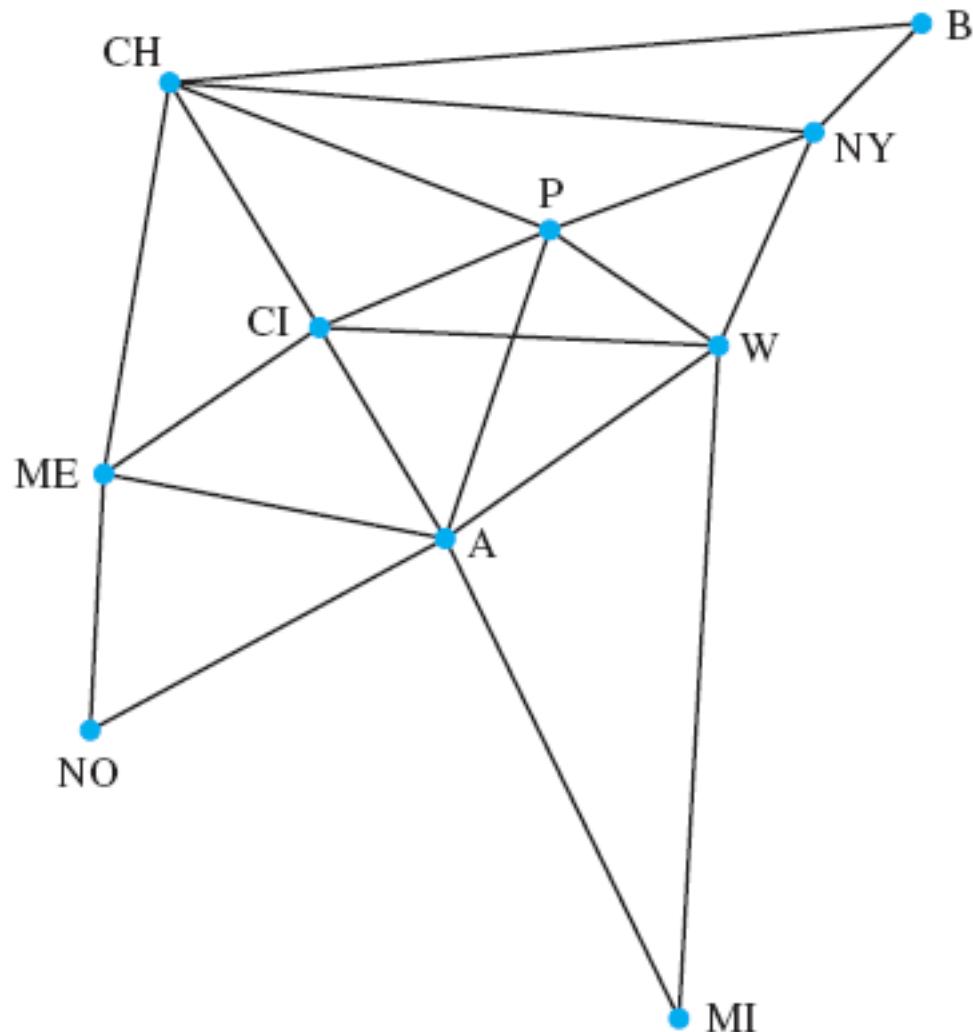
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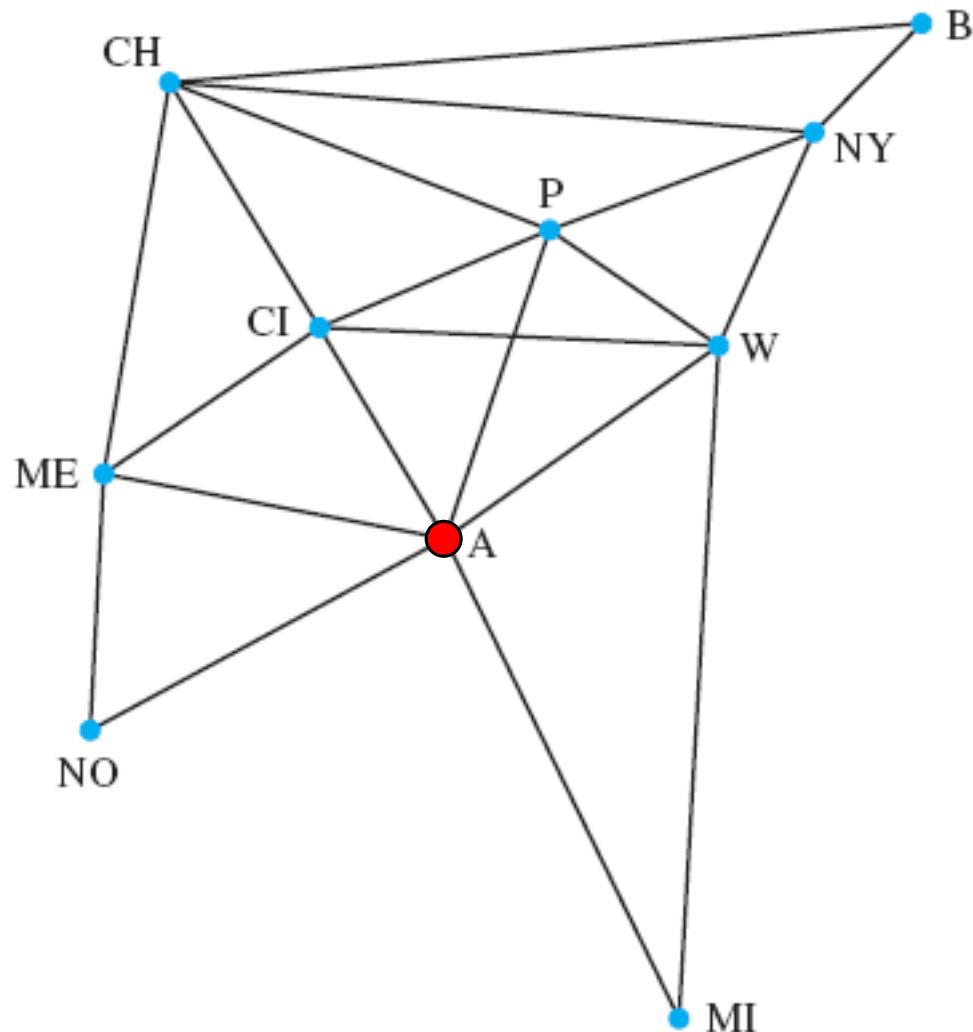


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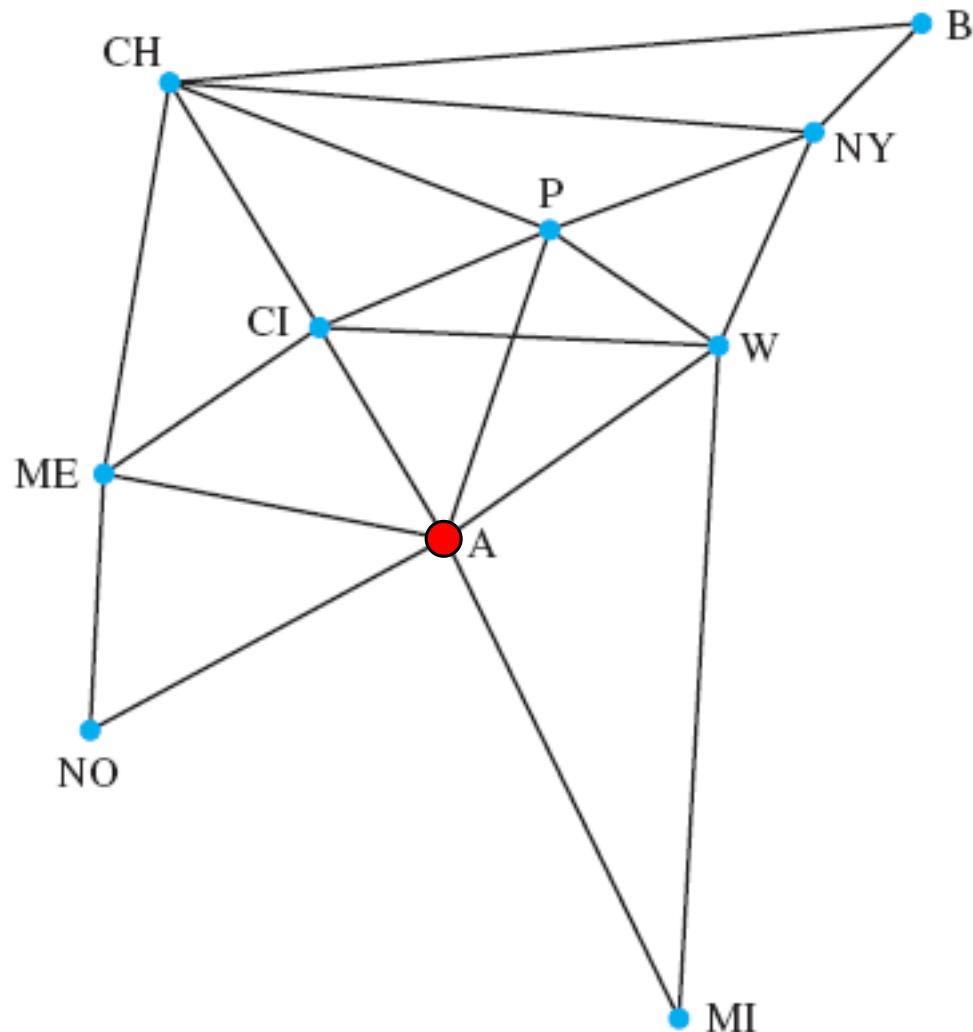


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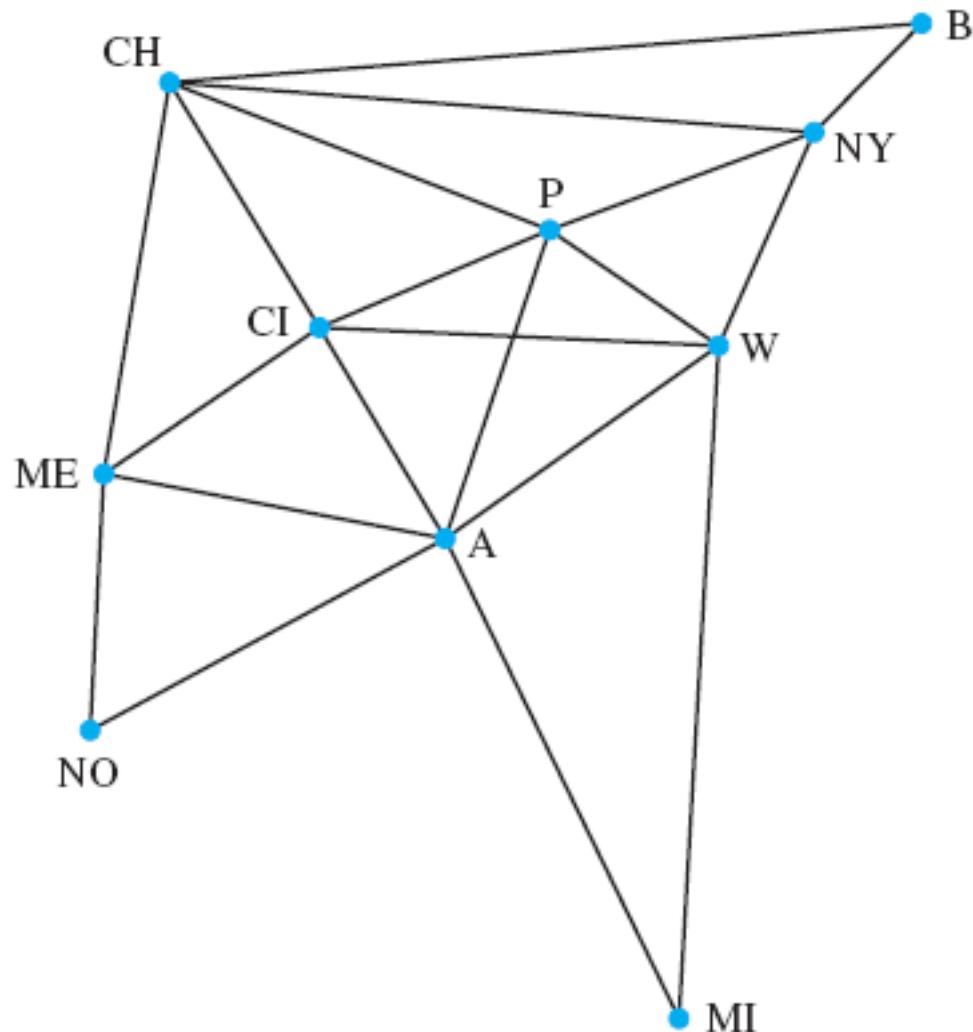
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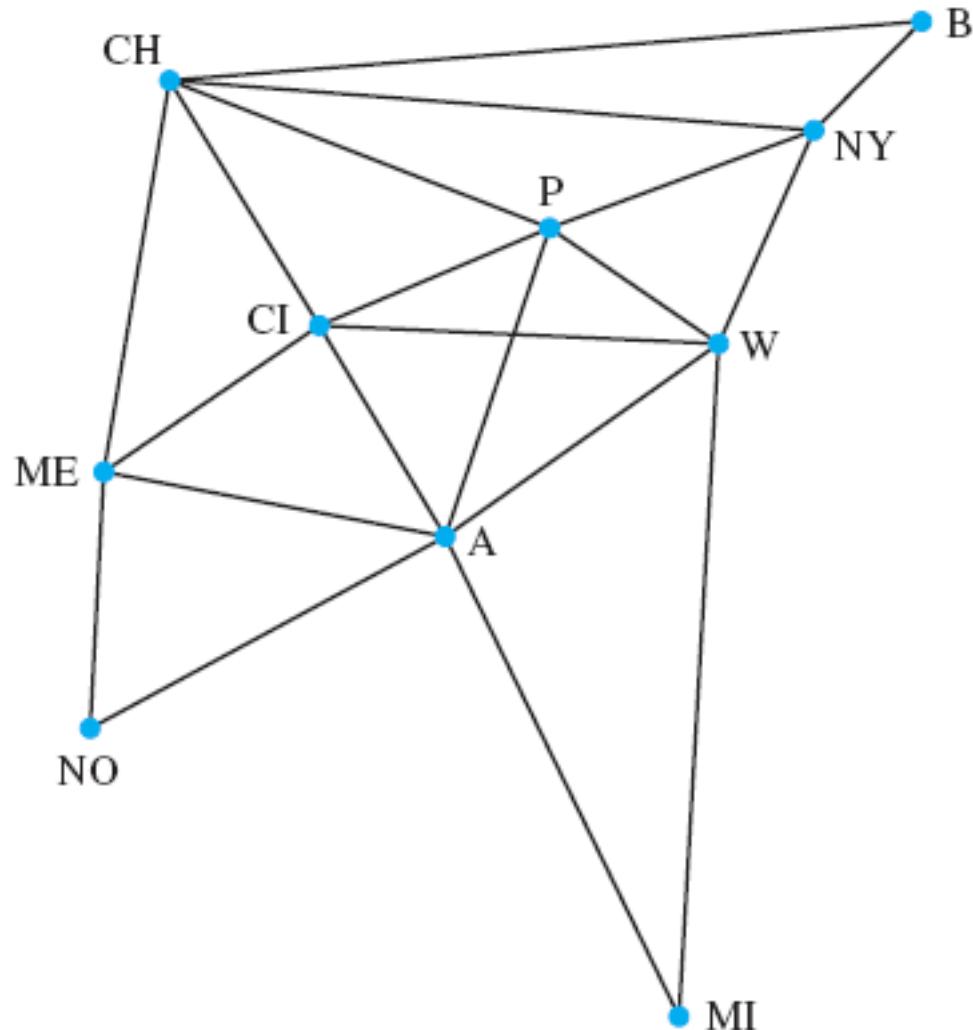
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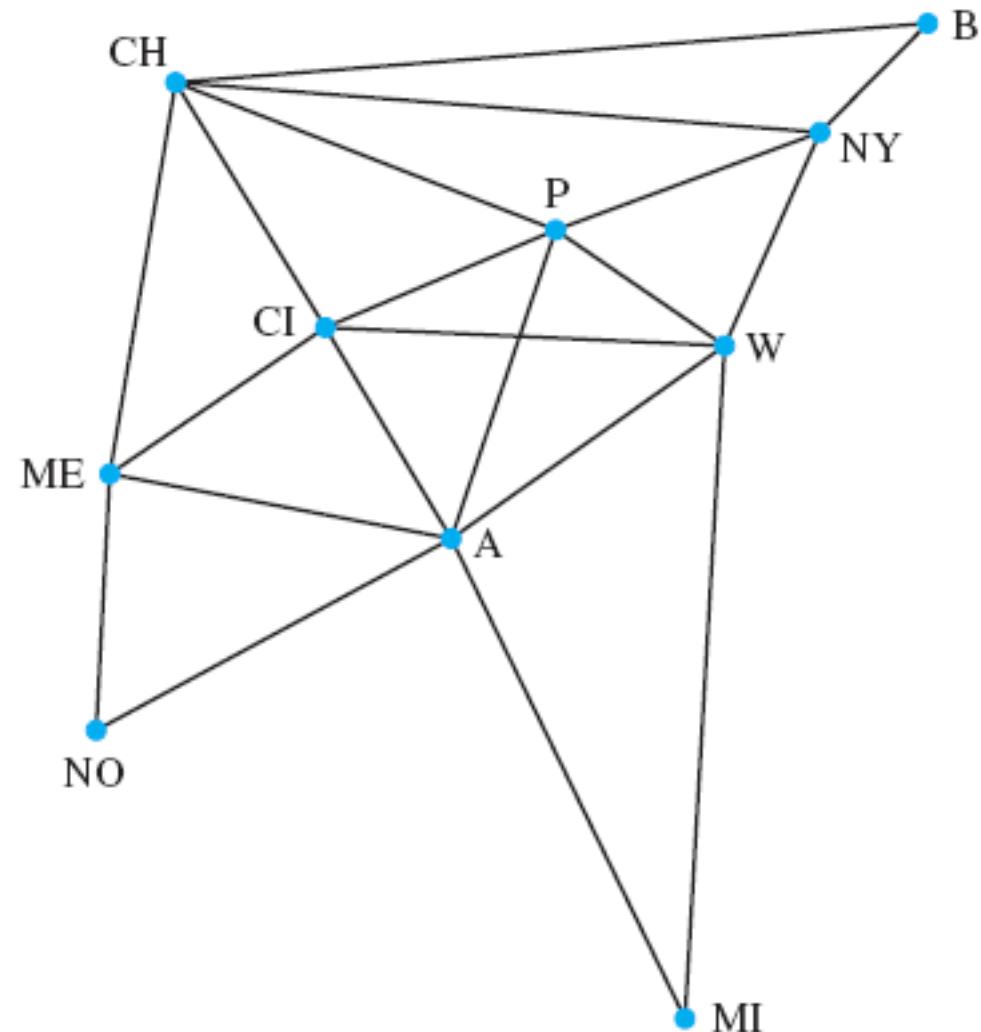
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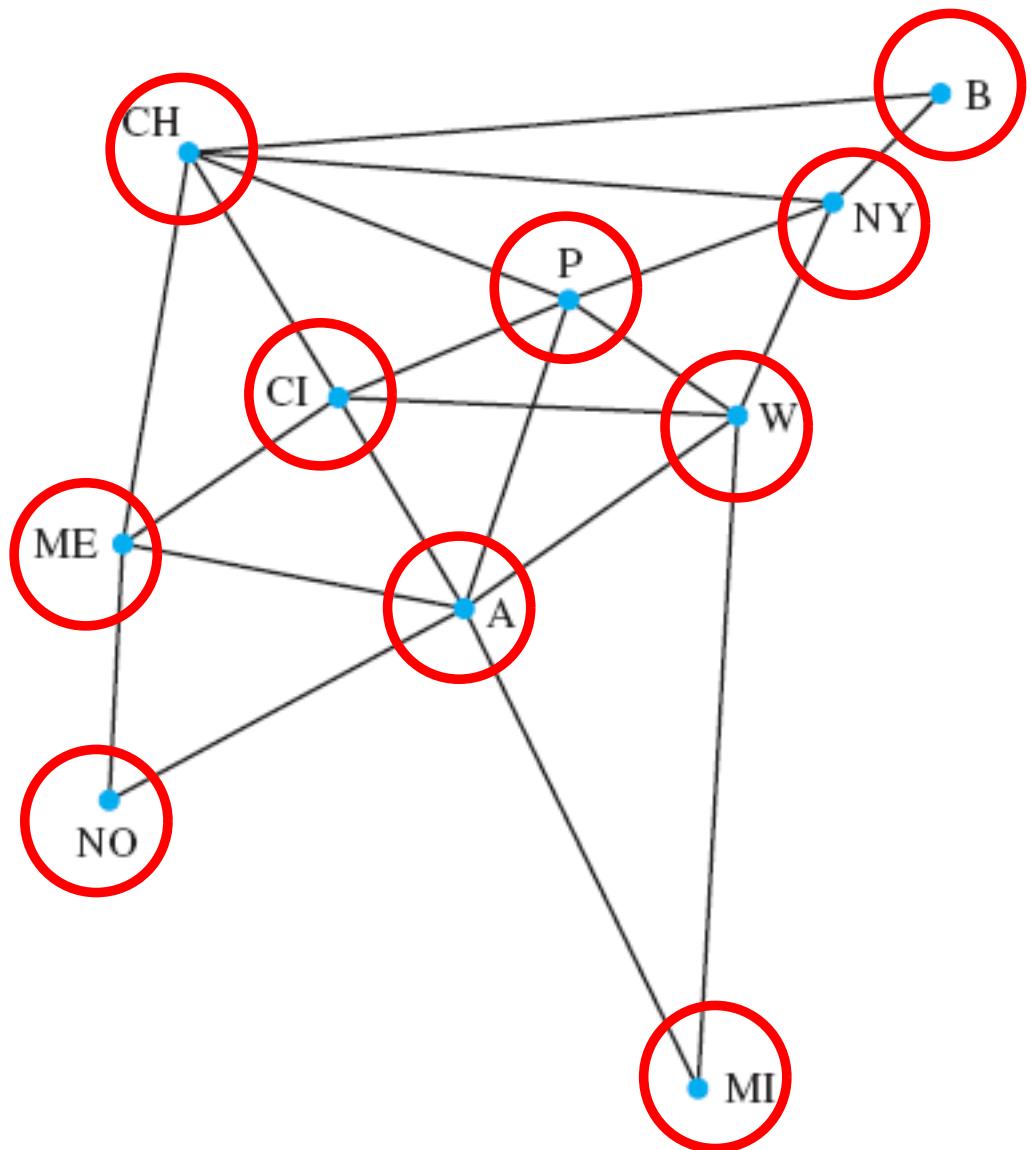
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20 links

Graph G

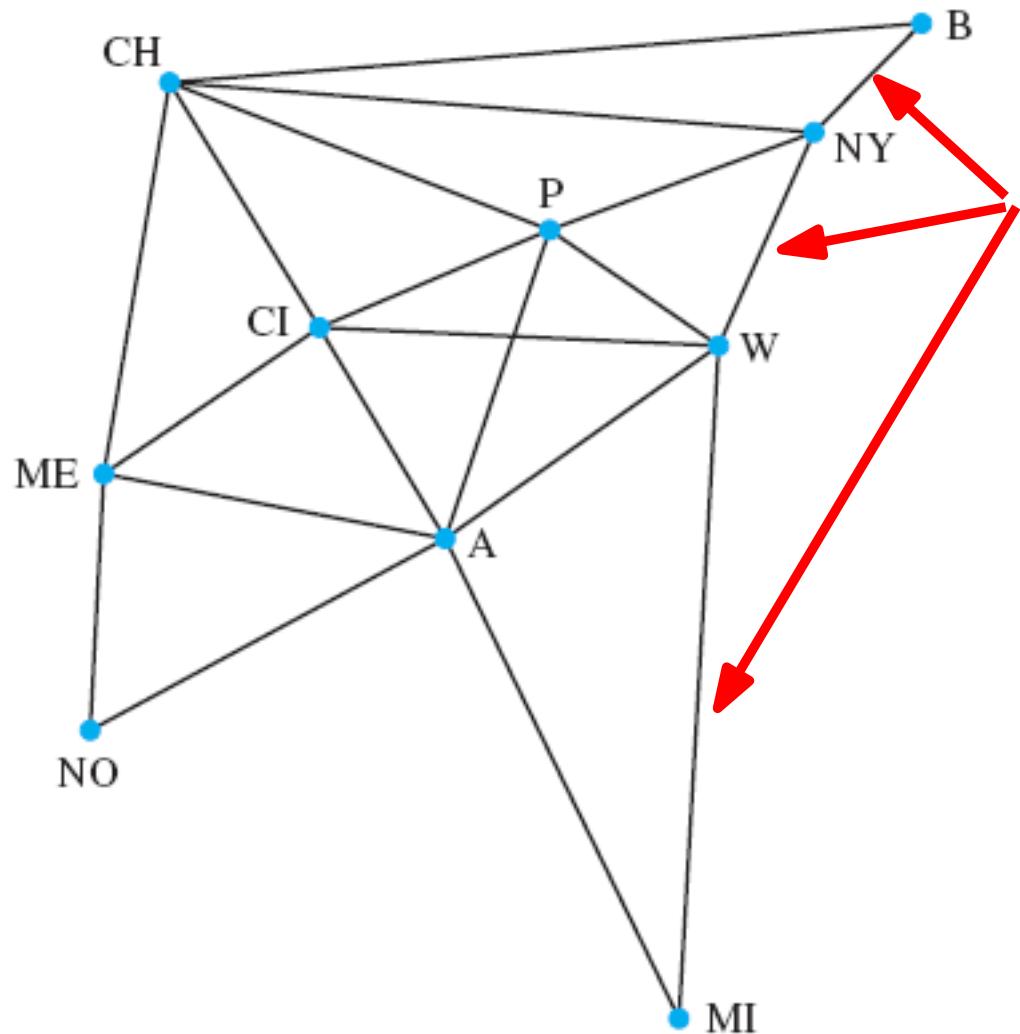


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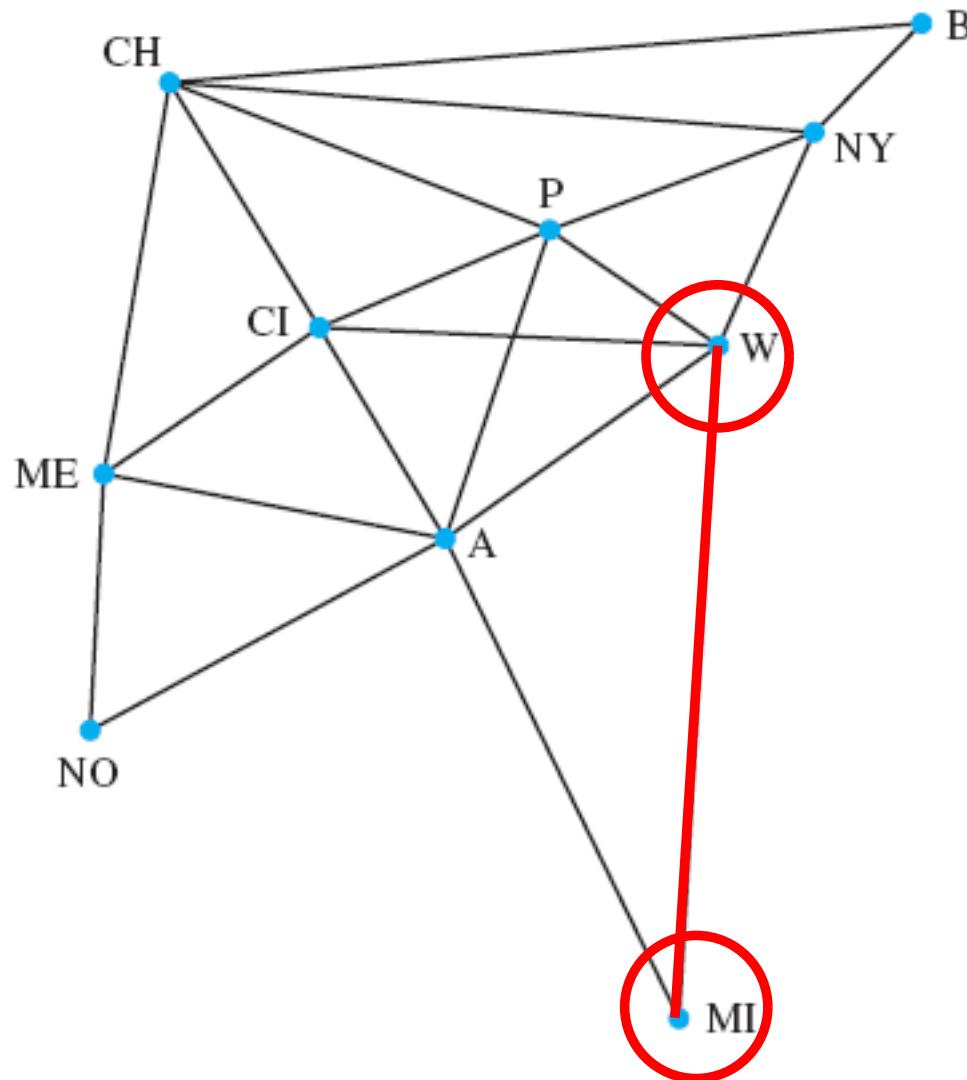
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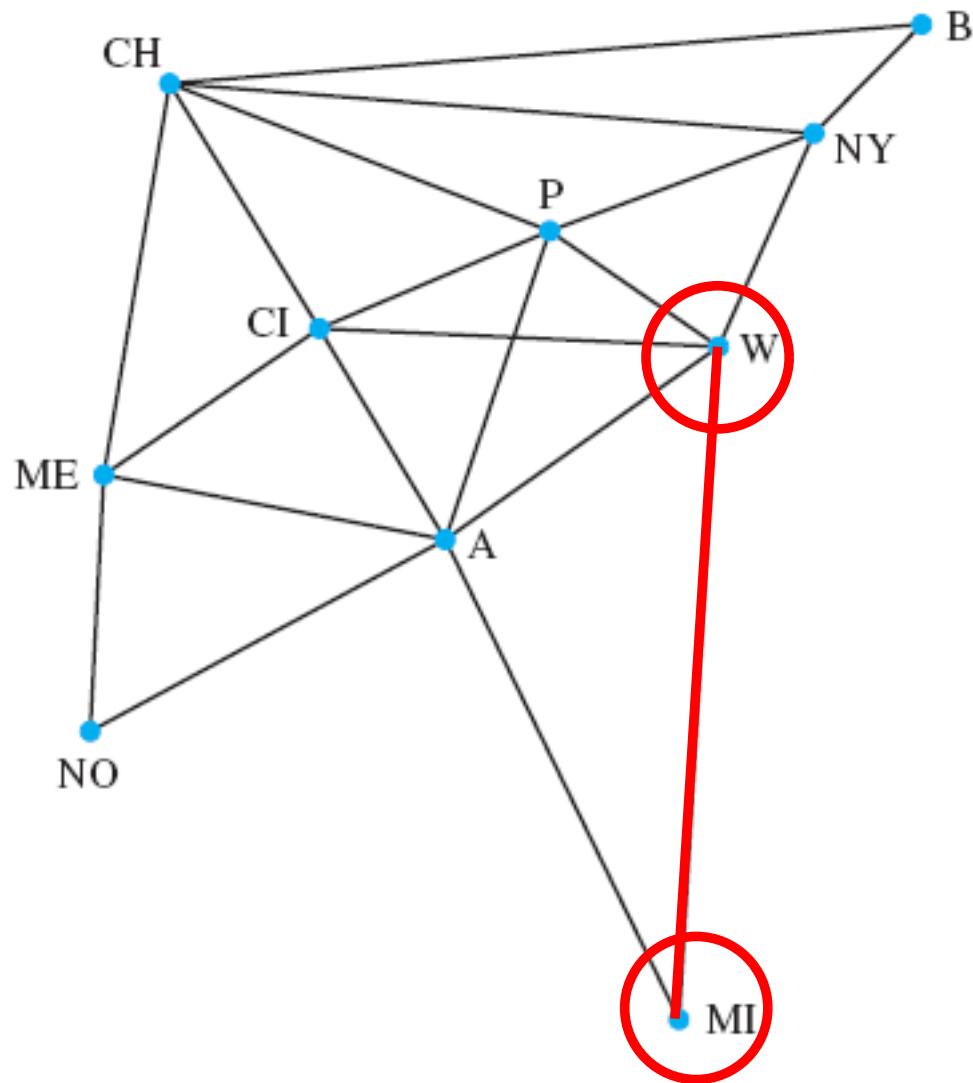


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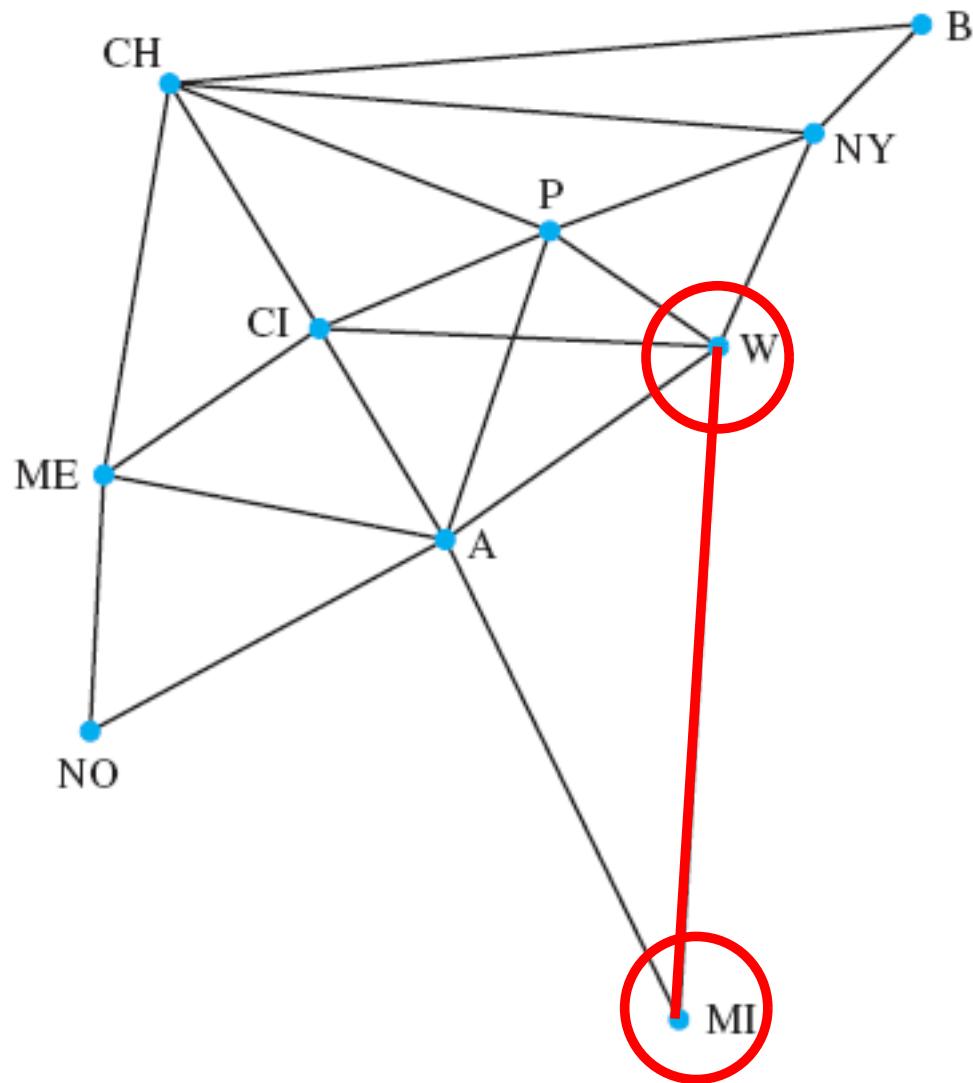
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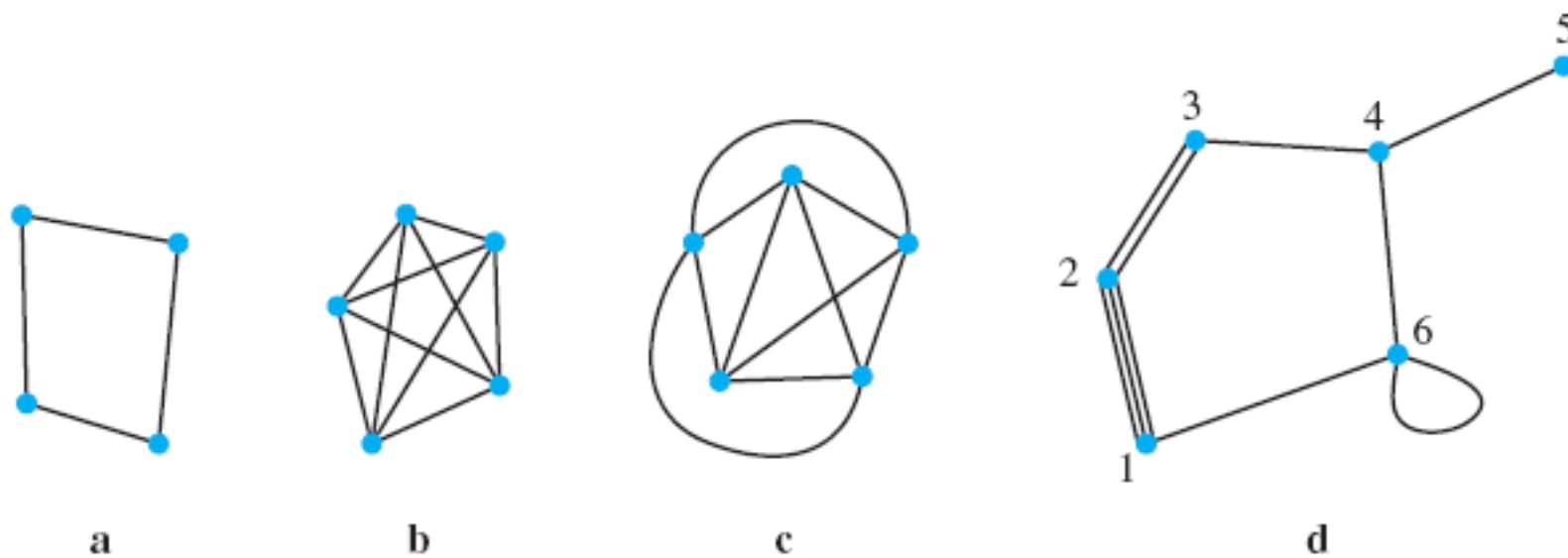
When a vertex is an
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A graph with n vertices that has an edge between **each pair** of vertices

Graphs

- **Graphs** and **graph theory** can be used to model:
 - ◊ Computer networks
 - ◊ Social networks
 - ◊ Communication networks
 - ◊ Information networks
 - ◊ Software design
 - ◊ Transportation networks
 - ◊ Biological networks

Graph Models

- Computer Networks

Vertices: computers

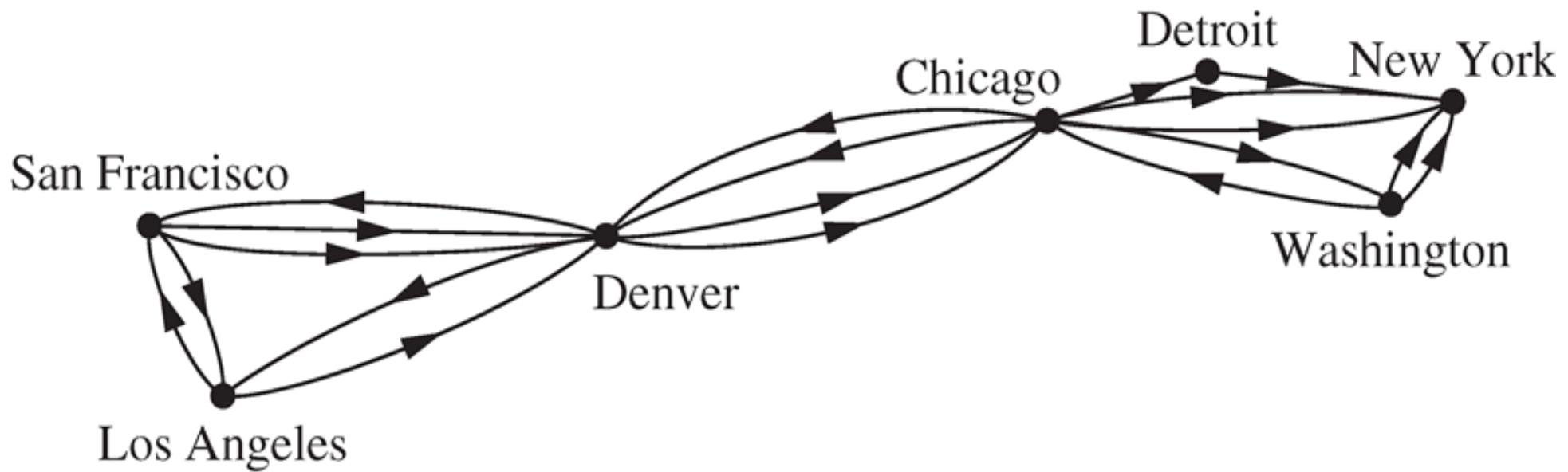
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Graph Models

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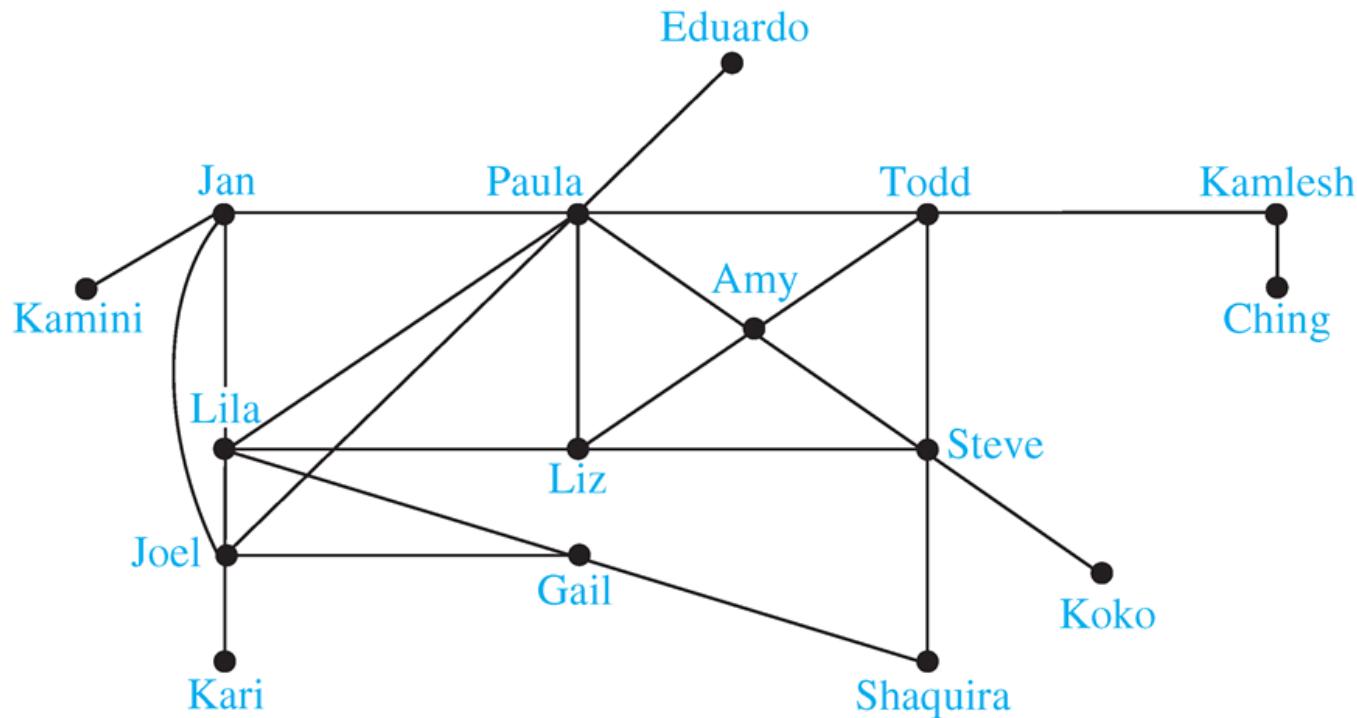
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Example

- the Hollywood graph

- the Erdős number

The Erdős Number

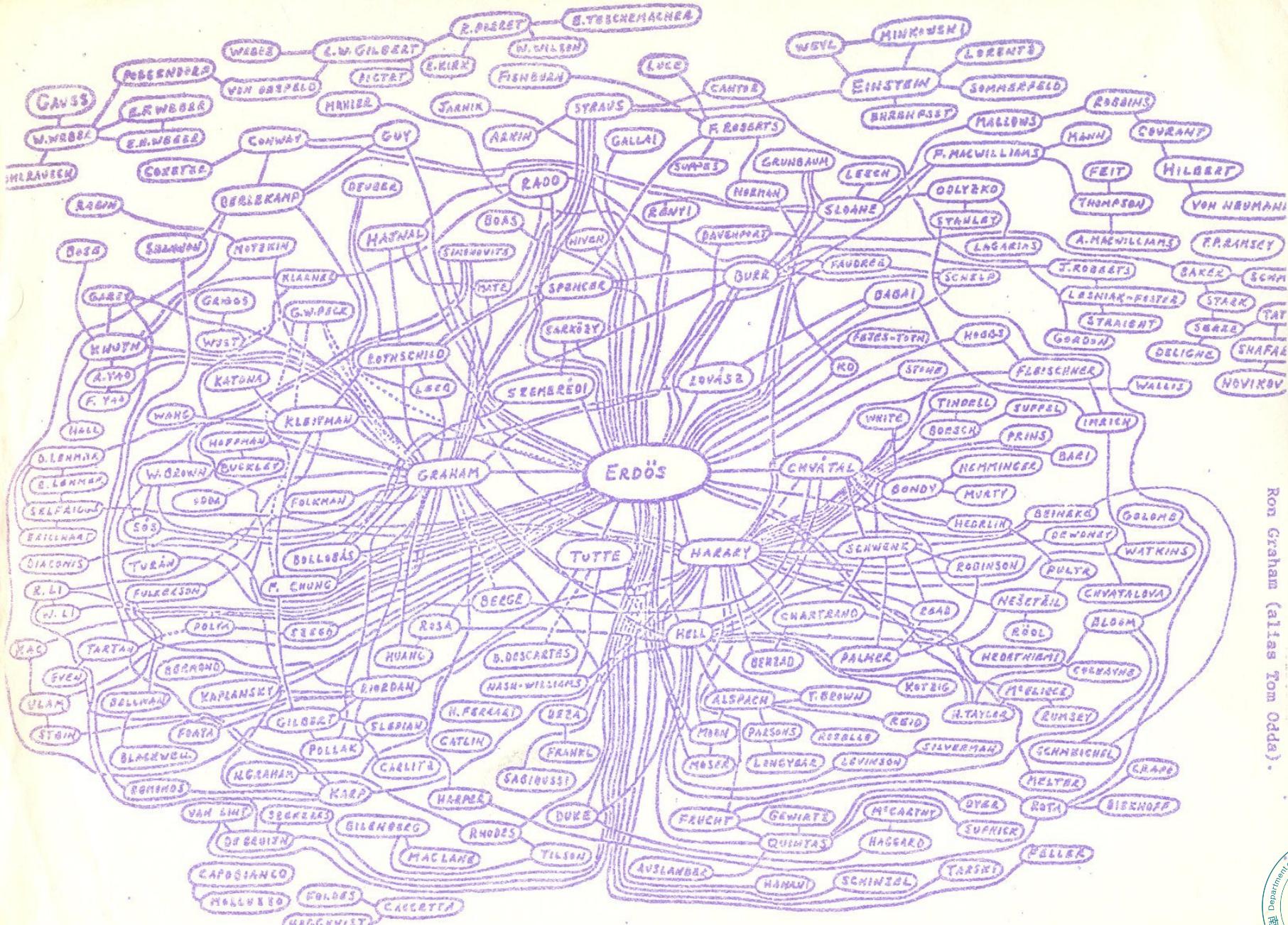
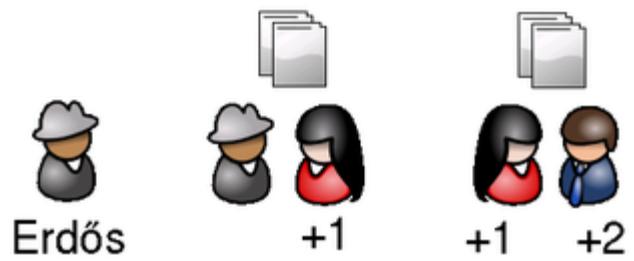


Figure 2

To appear in Topics in Graph Theory (F. Harary, ed.), New York Academy of Sciences (1979).

Ron Graham (alias Tom Gada) -

The Erdős Number



The Erdős Number



Erdős number 0	---	1 person
Erdős number 1	---	504 people
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Statistics on Mathematical Collaboration, 1903-2016

◆	#Laureates ◆	#Erdős ◆	%Erdős ◆	Min ◆	Max ◆	Average ◆	Median ◆
Fields Medal	56	56	100.0%	2	6	3.36	3
Nobel Economics	76	47	61.84%	2	8	4.11	4
Nobel Chemistry	172	42	24.42%	3	10	5.48	5
Nobel Medicine	210	58	27.62%	3	12	5.50	5
Nobel Physics	200	159	79.50%	2	12	5.63	5

Undirected Graphs

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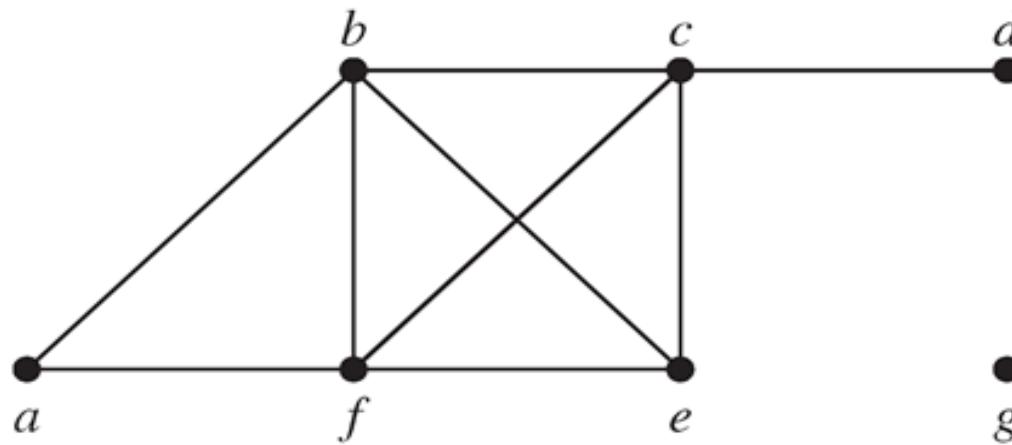
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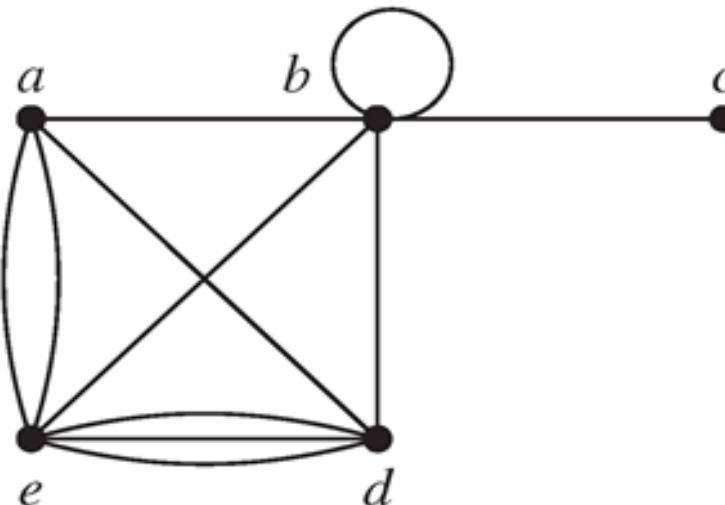
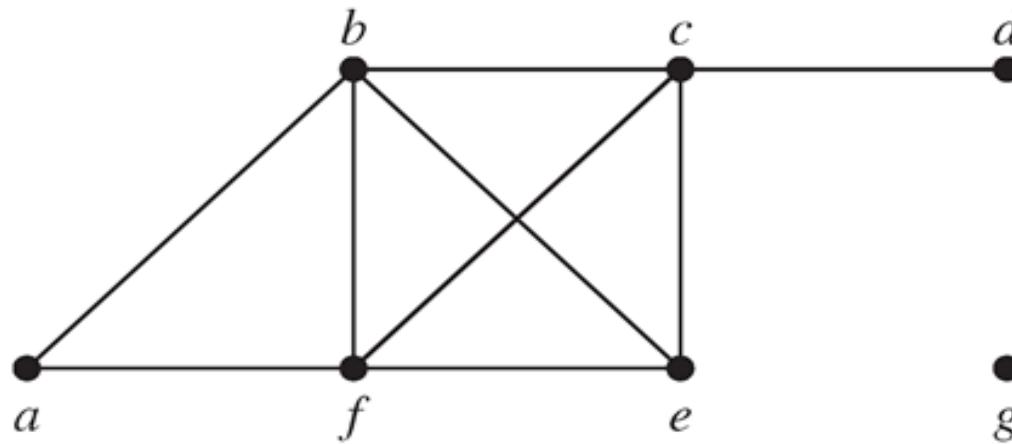
Undirected Graphs

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Undirected Graphs

- **Theorem 1 (Handshaking Theorem)** If $G = (V, E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} \deg(v)$$

Proof

Undirected Graphs

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Directed Graphs

- **Definition** An *directed graph* $G = (V, E)$ consists of V , a nonempty set of vertices, and E , a set of directed edges. Each edge is an **ordered** pair of vertices. The directed edge (u, v) is said to **start at u and end at v** .

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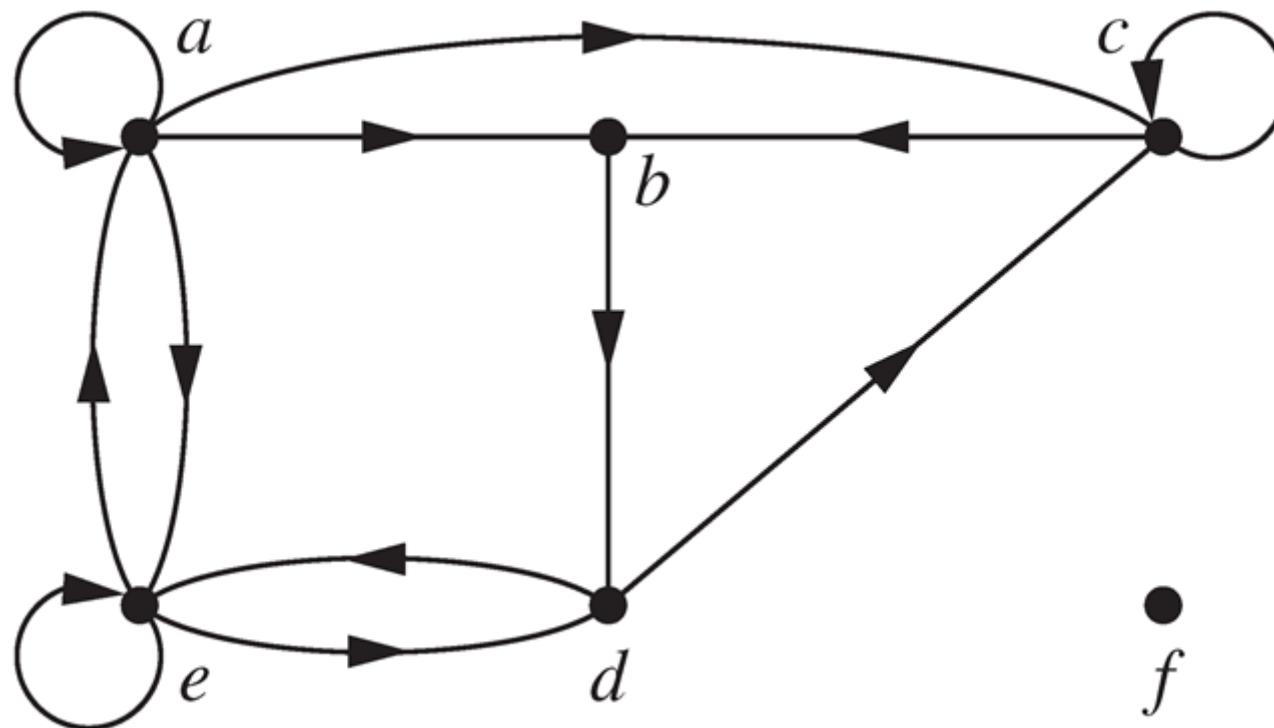
Definition Let (u, v) be an edge in G . Then u is the *initial vertex* of the edge and is *adjacent to v* and v is the *terminal vertex* of this edge and is *adjacent from u* . The initial and terminal vertices of a loop are the same.

Directed Graphs

- **Definition** The *in-degree* of a vertex v , denoted by $\deg^-(v)$, is the number of edges which terminate at v . The *out-degree* of v , denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. Note that a **loop** at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

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Directed Graphs

- **Theorem 3** Let $G = (V, E)$ be a graph with directed edges. Then

$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v)$$

Proof

Complete Graphs

- A *complete graph* on n vertices, denoted by K_n , is the simple graph that contains exactly one edge between **each pair** of distinct vertices.

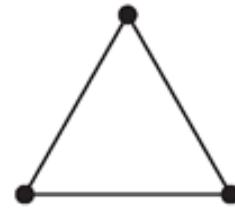
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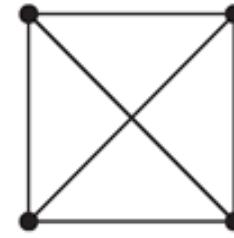
K_1



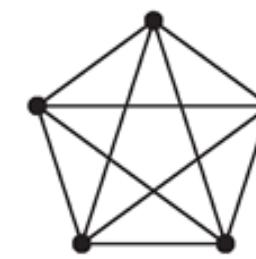
K_2



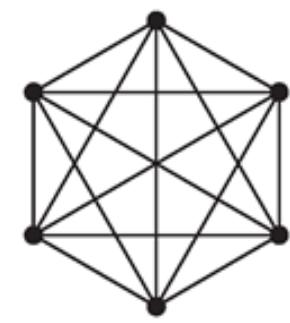
K_3



K_4



K_5



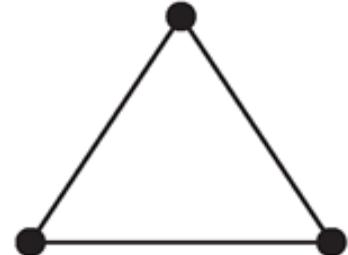
K_6

Cycles

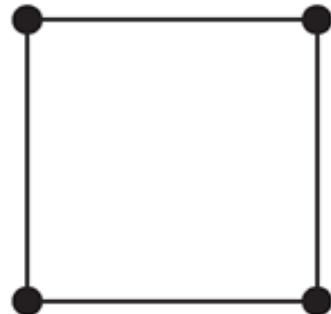
- A *cycle* C_n for $n \geq 3$ consists of n vertices v_1, v_2, \dots, v_n , and edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

Cycles

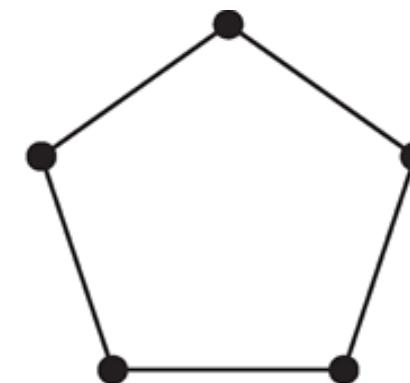
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C_3



C_4



C_5



C_6

Wheels

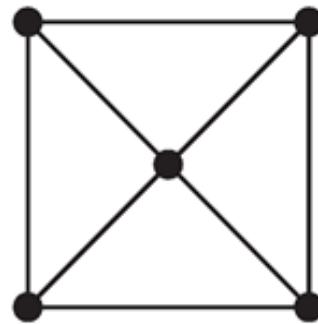
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Wheels

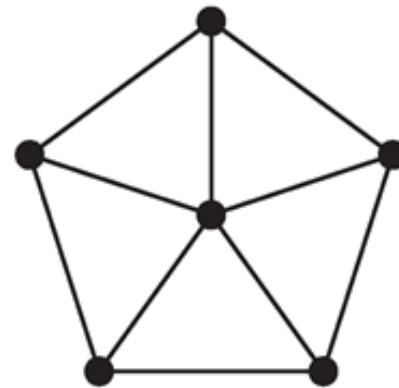
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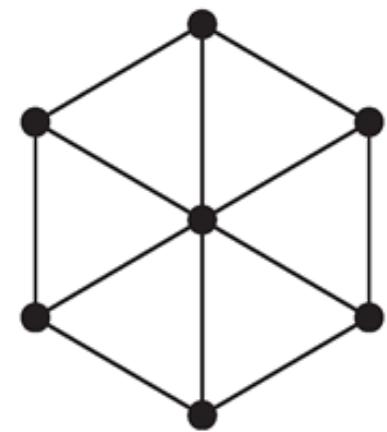
W_3



W_4



W_5



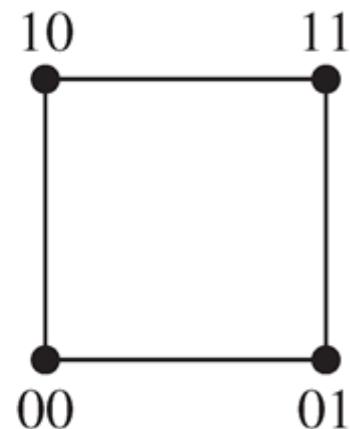
W_6

N -dimensional Hypercube

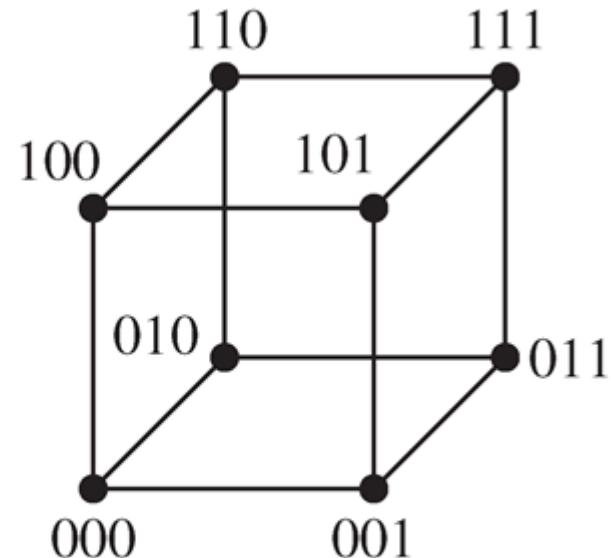
- An *n-dimensional hypercube*, or *n-cube*, Q_n is a graph with 2^n vertices representing all bit strings of length n , where there is an edge between two vertices that differ in exactly one bit position.

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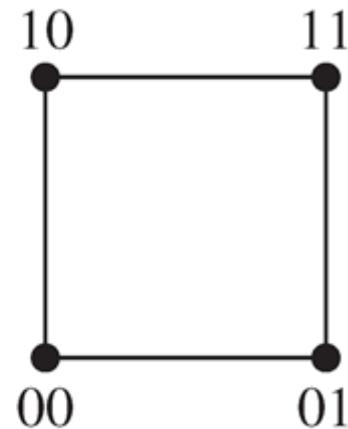
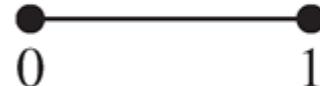
Q_1



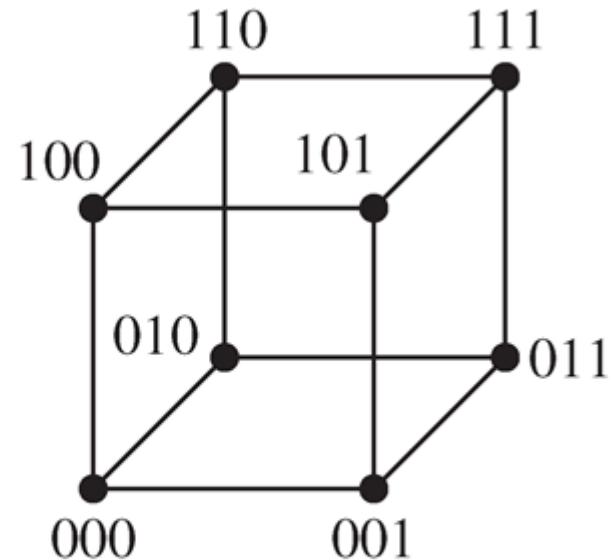
Q_3

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Q_1



Q_3

How many vertices? How many edges?

Bipartite Graphs

- **Definition** A simple graph G is *bipartite* if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2 .

Bipartite Graphs

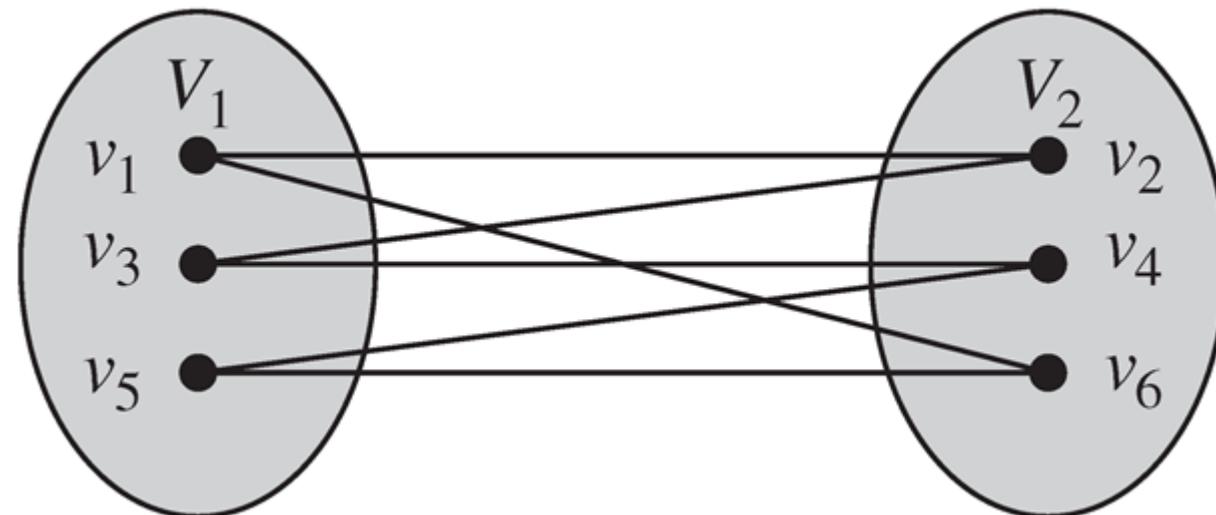
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An equivalent definition of a *bipartite graph* is a graph where it is possible to color the vertices **red** or **blue** so that no two adjacent vertices are of the same color.

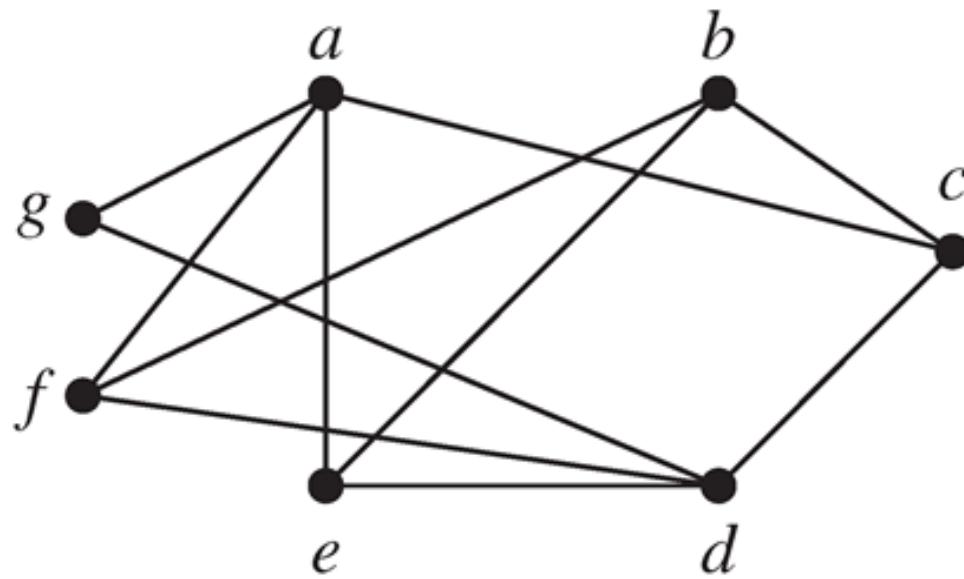
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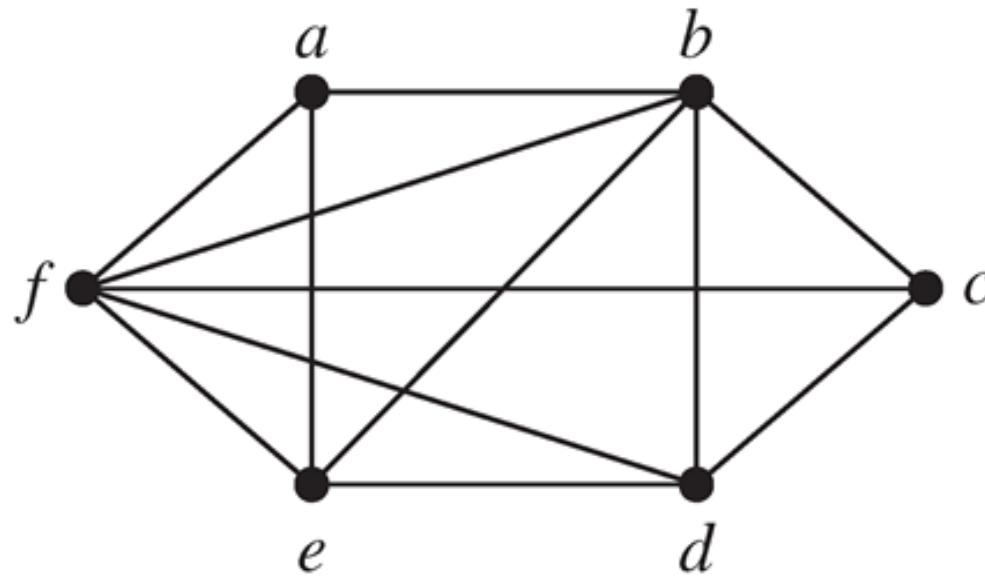
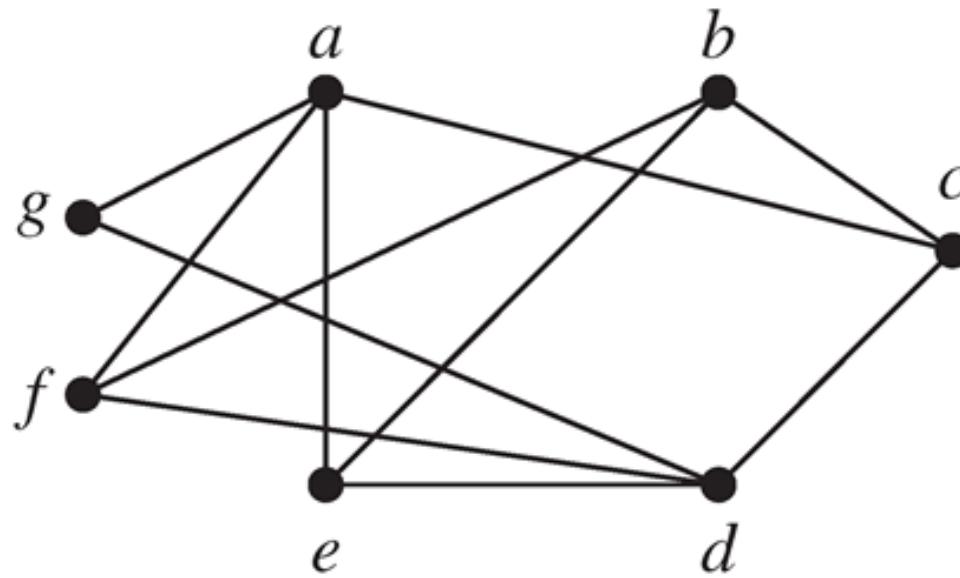
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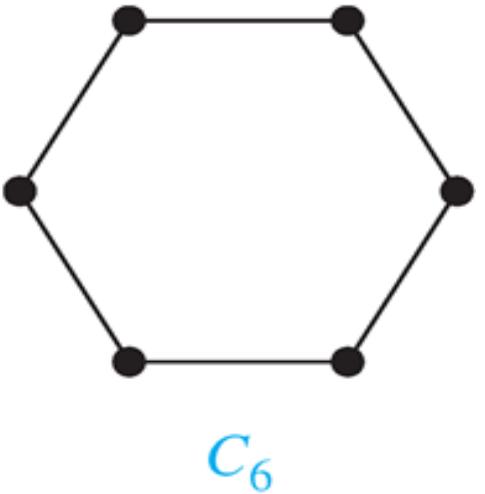


Bipartite Graphs



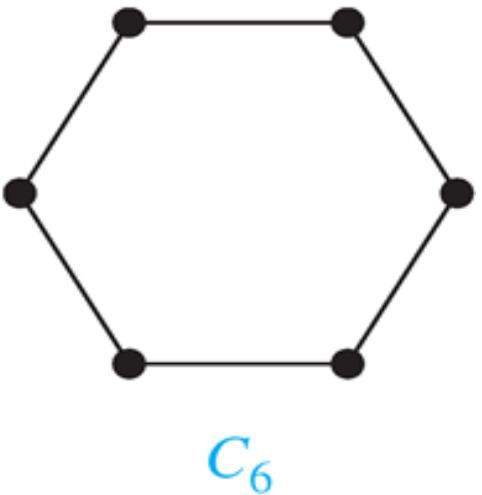
Bipartite Graphs

- **Example** Show that C_6 is bipartite.

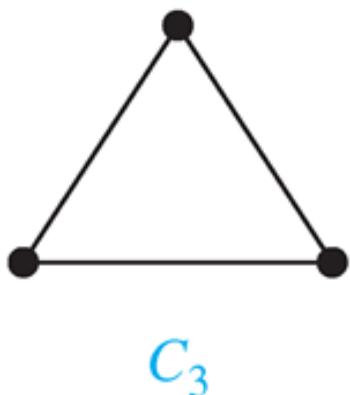


Bipartite Graphs

- **Example** Show that C_6 is bipartite.



- **Example** Show that C_3 is not bipartite.

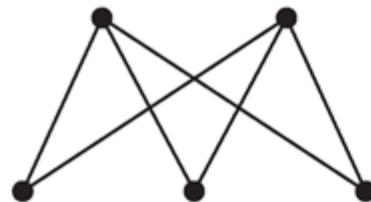


Complete Bipartite Graphs

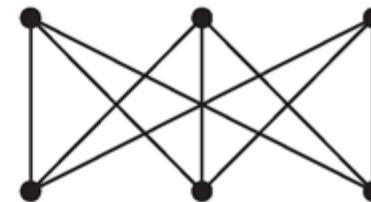
- **Definition** A *complete bipartite graph* $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2 .

Complete Bipartite Graphs

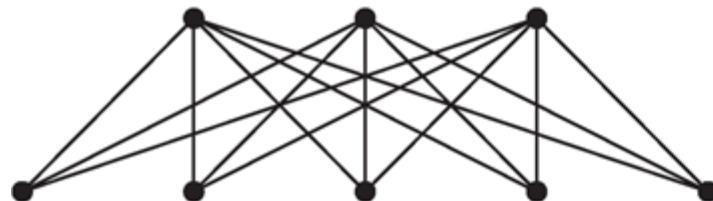
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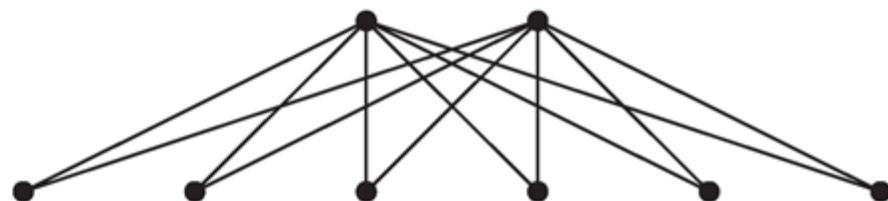
$K_{2,3}$



$K_{3,3}$



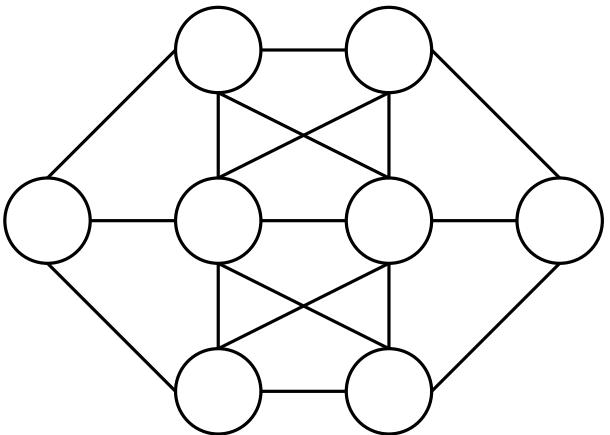
$K_{3,5}$



$K_{2,6}$

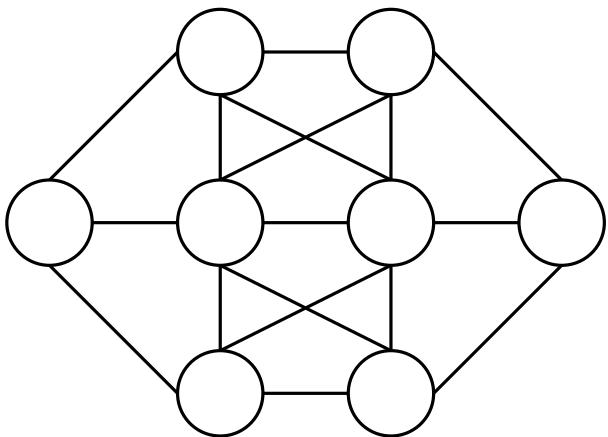
Puzzles using Graphs

- **The eight-circles problem** Place the letters A, B, C, D, E, F, G, H into the eight circles in the figure, in such a way that **no** letter is adjacent to a letter that is next to it in the alphabet.



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- **Six people at a party** Show that, in any gathering of six people, there are either three people who all know each other, or three people none of which knows either of the other two.

Next Lecture

- graph ...

