DiscretMath Assignment1

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1 Q.1

a. $r \wedge \neg q$

b. $r \to p$

c. $p \wedge \neg q \wedge r$

d. $(p \wedge q) \to r$

e. $r \leftrightarrow (p \vee q)$

2 Q.2

2. a

p	q	$\neg q$	$p \oplus q$	$p \wedge q$	$p \oplus \neg q$	$(p \oplus q) \to (p \wedge q)$	$(p \oplus q) \to (p \oplus \neg q)$
0	0	1	0	0	1	1	1
0	1	0	1	0	0	0	0
1	0	1	1	0	0	0	0
1	1	0	0	1	1	1	1

they are equivalent

2. b

p	q	$p \leftrightarrow q$	$p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
0	0	1	0	1	1
0	1	0	0	0	0
1	0	0	0	0	0
1	1	1	1	0	1

they are equivalent

2. c

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \wedge \neg(p \to q)$
0	0	1	1	1	0
0	1	1	0	1	0
1	0	0	1	0	1
1	1	0	0	1	0

they are not equivalent

2. d

p	q	r	$p \to \neg q$	$p \vee \neg q$	$r \to (p \vee \neg q)$	$(p \to \neg q) \leftrightarrow (r \to (p \vee \neg q))$	$\neg p \wedge \neg r$	$q \vee (\neg p \wedge \neg r)$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	0	0
0	1	0	1	0	1	1	1	1
0	1	1	1	0	0	0	0	1
1	0	0	1	1	1	1	0	0
1	0	1	1	1	1	1	0	0
1	1	0	0	1	1	0	0	1
1	1	1	0	1	1	0	0	1

they are not equivalent

3 Q.3

a.

$$\begin{aligned}
(p \wedge \neg q) \to r &\equiv \neg(p \vee \neg q) \vee r \qquad \text{Useful} \\
&\equiv (\neg p \vee \neg\neg q) \vee r \qquad \text{De Morgan's} \\
&\equiv (\neg p \vee q) \vee r \qquad \text{Double Negation} \\
&\equiv \neg p \vee (q \vee r) \qquad \text{Associative} \\
&\equiv p \to (q \vee r) \qquad \text{Useful}
\end{aligned}$$

b.

$$
\begin{aligned}
&((p \to q) \land (q \to r)) \to (p \to r) \\
&\qquad \equiv \neg((\neg p \lor q) \land (\neg q \lor r)) \lor (\neg p \lor r) \qquad \text{Useful} \\
&\qquad \equiv (\neg(\neg p \lor q) \lor \neg(\neg q \lor r)) \lor (\neg p \lor r) \qquad \text{De Morgan's} \\
&\qquad \equiv (\neg\neg p \land \neg q) \lor (\neg\neg q \land \neg r) \lor (\neg p \lor r) \qquad \text{De Morgan's} \\
&\qquad \equiv (p \land \neg q) \lor (q \land \neg r) \lor (\neg p \lor r) \qquad \text{Double Negation} \\
&\qquad \equiv (p \lor (q \land \neg r) \lor (\neg p \lor r)) \land (\neg q \lor (q \land \neg r) \lor (\neg p \lor r)) \qquad \text{Distributive} \\
&\qquad \equiv (p \lor (\neg p \lor r) \lor (q \land \neg r)) \land ((q \land \neg r) \lor \neg q \lor (\neg p \lor r)) \qquad \text{Commutative} \\
&\qquad \equiv ((p \lor \neg p) \lor r \lor (q \land \neg r)) \land ((q \land \neg r) \lor \neg q \lor (\neg p \lor r)) \qquad \text{Associative} \\
&\qquad \equiv ((p \lor \neg p) \lor r \lor (q \land \neg r)) \land ((q \lor \neg q) \lor (\neg p \lor r)) \land ((\neg r \lor \neg q) \lor (\neg p \lor r) \quad \text{Distributive} \\
&\qquad \equiv ((p \lor \neg p) \lor r \lor (q \land \neg r)) \land ((q \lor \neg q) \lor (\neg p \lor r)) \land ((\neg r \lor r)\neg q \lor \neg p)) \quad \text{Associative} \\
&\qquad \equiv (T \lor r \lor (q \land \neg r)) \land (T \lor (\neg p \lor r)) \land (T \lor \neg q \lor \neg p) \qquad \text{Negation} \\
&\qquad \equiv T \land T \land T \qquad \text{Domination} \\
&\qquad \equiv T \qquad \text{Identity}
\end{aligned}
$$

4 Q.4

p	q	r	$p \to q$	$(p \to q) \to r$	$q \to r$	$p \to (q \to r)$
0	0	0	1	0	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	1
0	1	1	1	1	1	1
1	0	0	0	0	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

From the Truth table, we can get $(p \to q) \to r$ and $p \to (q \to r)$ are **not equivalent**.

5 Q.5

$$
\begin{aligned}
(q \to (r \lor p)) &\to ((\neg r \lor s) \land \neg s) \\
&\equiv \neg(\neg q \lor (r \lor p)) \lor ((\neg r \lor s) \land \neg s) \qquad \text{Useful} \\
&\equiv (q \land \neg(r \lor p)) \lor ((\neg r \lor s) \land \neg s) \qquad \text{De Morgan's} \\
&\equiv (q \land \neg r \land \neg p) \lor ((\neg r \lor s) \land \neg s) \qquad \text{De Morgan's} \\
&\equiv (q \land \neg r \land \neg p) \lor (\neg r \land \neg s) \lor (s \land \neg s) \qquad \text{Distributive} \\
&\equiv (q \land \neg r \land \neg p) \lor (\neg r \land \neg s) \lor F \qquad \text{Negation} \\
&\equiv (q \land \neg r \land \neg p) \lor (\neg r \land \neg s) \qquad \text{Identity} \\
&\equiv \neg r \land ((q \land \neg p) \lor \neg s) \qquad \text{Distributive} \\
&\to \neg r \qquad \text{Simplication Inference}
\end{aligned}
$$

6 Q.6

a. $\forall x \exists y L(x, y)$

b. $\exists y \forall x L(x, y)$

c. $\neg \exists x \forall y L(x, y)$

d. $\exists y \forall x \neg L(x, y)$

e. $\exists y \forall x (L(x, y) \land \forall z (L(x, z) \to y = z))$

f. $\exists y_1 \exists y_2 (y_1 \neq y_2 \land L(lynn, y_1) \land L(lynn, y_2) \land \forall z (L(lynn, z) \to (z = y_1 \lor z = y_2)))$

g. $\exists x (L(x, x) \land \forall y (L(x, y) \to x = y))$

7 Q.7

1. $\forall x (I(x) \to E(x))$
2. $\forall x \forall y (L(x, y) \to \neg Q(x, y))$
3. $\neg \exists y \forall x L(x, y)$

8 Q.8

1. Domain of the x, y are both $\mathbb{N}, P(x, y) = (x \geq y)$. Thus $\exists x \forall y P(x, y)$ is **false** but $\forall y \exists x P(x, y)$ is **true**
2. NO, as the example in 1.

9 Q.9

1. not equivalent. When $P(x)$ is $x \geq 0$ and $Q(x)$ is $x < 0$. Then i is **false** but ii is **true**
2. equivalent
3. equivalent
4. equivalent

10 Q.10

a. Replace $n^3 + 6n + 5$ *is odd* with $P(n)$ and n *is even* with $Q(n)$.

Thus

$$\begin{aligned}\neg\forall n \in \mathbb{N}(P(n) \to Q(n)) &\equiv \exists n \in \mathbb{N}\neg(P(n) \to Q(n)) \\ &\equiv \exists n \in \mathbb{N}\neg(\neg P(n) \vee Q(n)) \\ &\equiv \exists n \in \mathbb{N}(\neg\neg P(n) \wedge \neg Q(n) \\ &\equiv \exists n \in \mathbb{N}(P(n) \wedge \neg Q(n)\end{aligned}$$

Replace the Predict backward,the negation is:

$$\exists n \in \mathbb{N}(n^3 + 6n + 5 \textit{ is odd} \wedge n \textit{ is odd})$$

b. Assume the negation is true.Then we can get $\exists n \in \mathbb{N} n$ *is odd*. Thus $n^3 + 6n+5$ is even, which lead to the negation false.Because of the contradiction, negation is false.So the original statement is true.

11 Q.11

a. using truth table

p	q	$(q \vee \neg p)$	$\neg(p \leftrightarrow (q \vee \neg p))$	$\neg p \vee \neg q$
0	0	1	1	1
0	1	1	1	1
1	0	0	1	1
1	1	1	0	0

b. Regardless how long P is and logical connectives P uses, the truth table of P have only $2^2 = 4$ rows. It implies that P have totally $2^{2^2} = 16$ truth tables.

Meanwhile, for the situation in the question, it have $4 \times 3 \times 4$ collections. Because the connection $\wedge, \vee, \leftrightarrow$ are all symmetric, we can remove half of them for equivalence by the **Commutative laws**, getting $4 \times 3 \times 4/2 = 24$.

Despite the Commutative laws. Think other situations that $A \square B$ can produce equivalence. For the reason that $A \square B$ is the simplest form of binary connections, $\wedge, \vee, \leftrightarrow$ can't be equivalent in this form **unless** they are **tautology** or **contradiction**.

Enumerate all the situation that are **tautology** or **contradiction**:

Tautology	Contradiction
$p \vee \neg p$	$p \wedge \neg p$
$q \vee \neg q$	$q \wedge \neg q$
$p \leftrightarrow p$	$p \leftrightarrow \neg p$
$q \leftrightarrow q$	$q \leftrightarrow \neg q$
$\neg p \leftrightarrow \neg p$	
$\neg q \leftrightarrow \neg q$	

Just remain one tautology and one contradiction, we can get the **distinct truth table** $A\Box B$ can produce: $24 - 5 - 3 = 16$. Match the number of P's truth tables.

Thus, we have proven that whatever P's truth table is, we can select one $A\Box B$ that satisfy the situation.

12 Q.12

"All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. (This movie about coal miners is wonderful)"
$\rightarrow$ **Hypothetical syllogism**
"(This movie about coal miners is wonderful) Therefore, there is a wonderful movie about coal miners." $\rightarrow$ **Existential Generalization(EG)**

13 Q.13

1. counterexample1:$a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$,$a^b = 2$ If a is irrational number, then we have proved it. If a is rational number, then we have the $b^b = a$ implies irrational number $\sqrt{2}$ power irrational number $\sqrt{2}$ is rational number $\sqrt{2}^{\sqrt{2}}$

   counterexample2:$a = e$,$b = ln2$,$a^b = 2$

2. counterexample:$a = -\sqrt{2}$, $\sqrt{a} \notin \mathbb{R}$, it's not an irrational number.

   • I guess there miss a condition that $a > 0$. In this condition, proof:

   Suppose a is an irrational number but $\sqrt{a}$ is rational ,Then there exist two integers m and n such that $gcd(m, n) = 1$, and $\sqrt{a} = \frac{m}{n}$. We have then $a = \frac{m^2}{n^2}$. It then shows a is a rational number, which contradicts to the assumption.Thus prove the proposition.

3. For example:$x = 2, y = \frac{1}{\ln 2}$

   proof:$2^{\frac{1}{\ln 2}} = e$ is an irrational number.

   Because $e^{\ln 2} = 2$, thus $2^{\frac{1}{\ln 2}} = e$, which is obviously an irrational number.

14 Q.14

Suppose $r = \sqrt{2}+\sqrt{3}$ is rational, then $r^2 = 5+2\sqrt{6}$ should be rational.It follows that $\frac{r^2-5}{2} = \sqrt{6}$ should be rational. From the theorem we can get that $\sqrt{6}$ is an irrational number, which contradicts to the assumption. Thus $\sqrt{2}+\sqrt{3}$ is irrational.

15 Q.15

Suppose we have rational number x and irrational number y,we can select an irrational number $z = \frac{x+y}{2}$. Then we just need to proof z is irrational number. proof: Suppose $z = \frac{x+y}{2}$ is rational, then $y = 2z - x$ should be rational, which contradicts to the assumption.Then $z = \frac{x+y}{2}$ is an irrational number.

Hense between every rational number and every irrational number there is an irrational number.

16 Q.16

Suppose $a^2 + b^2 = 2n$,then $a^2 - b^2 = 2n - 2b^2$. It follows that $(a+b)(a-b)$ is even. Thus $a+b$ or $a-b$ is even. Meanwhile, $a+b = a-b+2b$, which implies $a+b$ and $a-b$ are both even or both odd. Combine the two condition, $a+b$ is even.

17 Q.17

Assume one real root r of $f(x) = 0$ is neither integral or irrational, which means r can be expressed by $r = \frac{m}{q}$ and $m, q \in \mathbb{Z}$ and $gcd(m,q) = 1$.

Get $f(r) = f(\frac{m}{q}) = a_0 + a_1(\frac{m}{q}) + a_2(\frac{m}{q})^2 + \cdots + a_{n-1}(\frac{m}{q})^{n-1} + (\frac{m}{q})^n = 0$

$\rightarrow a_0q^n + a_1mq^{n-1} + a_2m^2q^{n-2} + \cdots + a_{n-1}m^{n-1}q + m^n = 0$

$\rightarrow a_0q^n + a_1mq^{n-1} + a_2m^2q^{n-2} + \cdots + a_{n-1}m^{n-1}q = -m^n$

$\rightarrow (a_0q^{n-1} + a_1mq^{n-2} + a_2m^2q^{n-3} + \cdots + a_{n-1}m^{n-1})q = -m^n$

Because q is factor of $(a_0q^{n-1} + a_1mq^{n-2} + a_2m^2q^{n-3} + \cdots + a_{n-1}m^{n-1})q$

$\rightarrow$ q is factor of m^n

With the result in the question $\rightarrow q$ is factor of m

Denote that $gcd(m,q) = 1$

$\rightarrow q$ is factor of 1

$\rightarrow q = 1$

$\rightarrow r = m$ $(is\ an\ integer)$

Which is contradicts to the assumption. Thus the real root is either integral or irrational.