CS215: Discrete Math (H)
2024 Fall Semester Written Assignment # 5
Due: Dec. 23rd, 2024, please submit at the beginning of class

Q.1 Show that a subset of an *antisymmetric* relation is also *antisymmetric*.
Solution: Suppose that $R_1 \subseteq R_2$ and that R_2 is antisymmetric. We must show that R_1 is also antisymmetric. Let $(a, b) \in R_1$ and $(b, a) \in R_1$. Since these two pairs are also both in R_2, we know that $a = b$, as desired.

$\square$

Q.2 Define a relation R on $\mathbb{R}$, the set of real numbers, as follows: For all x and y in $\mathbb{R}$, $(x, y) \in R$ if and only if $x - y$ is rational. Answer the followings, and explain your answers.

(1) Is R reflexive?

(2) Is R symmetric?

(3) Is R antisymmetric?

(4) Is R transitive?

Solution:

(1) Yes. Note that for all x we have $x - x = 0$, which is rational.

(2) Yes. Suppose that $(x, y) \in R$. Then $x - y = \frac{m}{n}$ for two integers m and n. Hence $y - x = \frac{-m}{n}$, which is again rational.

(3) No. Let $x = \sqrt{2}$ and $y = \sqrt{2} + 2$. Then we have $(x, y) \in R$ and $(y, x) \in R$, but $x \neq y$.

(4) Yes. Let $(x, y) \in R$ and $(y, z) \in R$. Then by definition both $x - y$ and $y - z$ are rational. Consequently, their sum $(x - y) + (y - z) = x - z$ is also rational. By definition, we have $(x, z) \in R$.

Q.3 How many relations are there on a set with n elements that are

(a) symmetric?

(b) antisymmetric?

(c) irreflexive?

(d) both reflexive and symmetric?

(e) neither reflexive nor irreflexive?

(f) both reflexive and antisymmetric?

(g) symmetric, antisymmetric and transitive?

Solution:

(a) $2^{n(n+1)/2}$

(b) $2^n 3^{n(n-1)/2}$

(c) $2^{n(n-1)}$

(d) $2^{n(n-1)/2}$

(e) $2^{n^2} - 2 \cdot 2^{n(n-1)}$

(f) $3^{n(n-1)/2}$

(g) 2^n

$\square$

Q.4 Suppose that the relation R is symmetric. Show that R^* is symmetric.
Solution: The result follows from

$$(R^*)^{-1} = (\cup_{n=1}^{\infty} R^n)^{-1} = \cup_{n=1}^{\infty} (R^n)^{-1} = \cup_{n=1}^{\infty} R^n = R^*.$$

$\square$

Q.5 Prove or give a counterexample to the following: For a set A and a binary relation R on A, if R is reflexive and symmetric, then R must be transitive as well.
Solution: Counterexample: Consider $A = \{1, 2, 3\}$ and

$$R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}.$$

Then R is symmetric and reflexive, but not transitive.

$\square$

Q.6 Let R be a reflexive relation on a set A. Show that $R \subseteq R^2$.
Solution: Suppose that $(a, b) \in R$. Because $(b, b) \in R$, it then follows that $(a, b) \in R^2$. Thus, R is a subset of R^2.

$\square$

Q.7 Let R and S both be *transitive* relations on a set A. For each of the relations below, either prove that it is transitive, or give a counterexample, showing that it may not be transitive.

(1) $R \cap S$

(2) $R \cup S$

(3) $R \circ S$

Solution:

(1) $R \cap S$ is transitive. Consider $(a, b), (b, c) \in R \cap S$, we have $(a, b), (b, c) \in R$ and $(a, b), (b, c) \in S$. Since both R and S are transitive, it follows that $(a, c) \in R$ and $(a, c) \in S$ and thus $(a, c) \in R \cap S$. Hence, $R \cap S$ is transitive.

(2) $R \cup S$ may not be transitive. Let $A = \{1, 2, 3\}$, and $R = \{(1, 3)\}$, $S = \{3, 1\}$. It is easy to check that both R and S are transitive. However, $R \cup S = \{(1, 3), (3, 1)\}$, which is not transitive.

(3) $R \circ S$ may not be transitive. Let $A = \{(2, 3), (4, 1)\}$ and $S = \{(1, 2), (3, 4)\}$. Then we have $R \circ S = \{(1, 3), (3, 1)\}$, which is not transitive.

Q.8

(1) Give an example to show that the transitive closure of the symmetric closure of a relation is not necessarily the same as the symmetric closure of the transitive closure of this relation.

(2) Show that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

Solution:

(1) Let $R = \{(a,b),(a,c)\}$. The transitive closure of the symmetric closure of R is $\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$ and is different from the symmetric closure of the transitive closure of R, which is $\{(a,b),(a,c),(b,a),(c,a)\}$.

(2) Suppose that (a,b) is in the symmetric closure of the transitive closure of R. We must show that (a,b) is in the transitive closure of the symmetric closure of R. We know that at least one of (a,b) and (b,a) is in the transitive closure of R. Hence, there is either a path from a to b in R or a path from b to a in R (or both). In the former case, there is a path from a to b in the symmetric closure of R. In the latter case, we can form a path from a to b in the symmetric closure of R by reversing the directions of all the edges in a path from b to a, going backward.

Hence, (a,b) is in the transitive closure of the symmetric closure of R.

$\square$

Q.9 Let R be the relation on $\mathbb{Z}$, the set of integers, as follows: For all m and n in $\mathbb{Z}$, $(m,n) \in R$ if and only if 3 divides $(m^2 - n^2)$.

(1) Prove that R is an equivalence relation.

(2) Describe the equivalence classes of R.

Solution:

(1) Since $3|0$, the relation R is obviously reflexive. If $(m,n) \in R$, then $3|(m^2-n^2)$. Hence $3|(n^2-m^2)$. By definition, $(n,m) \in R$. This proves the symmetry. We now prove transitivity. Suppose that $(m,n) \in R$ and $(n,\ell) \in R$, by definition, we then have

$$3x = m^2 - n^2 \text{ and } 3y = n^2 - \ell^2$$

for some integers x and y. It then follows that

$$3(x+y) = m^2 - \ell^2,$$

which means that $3|(m^2 - \ell^2)$. By definition, we have $(m, \ell) \in R$. Hence, R is an equivalence relation on $\mathbb{Z}$.

(2) Every integer $m \in \mathbb{Z}$ can be expressed as $m = 3x + r$, where x is an integer and r is an integer with $0 \leq r \leq 2$.

Let $m = 3x + r$ and $n = 3y + s$, where $0 \leq r \leq 2$ and $0 \leq s \leq 2$. We then have

$$m^2 - n^2 = 9(x^2 - y^2) + 6(xr - ys) + r^2 - s^2.$$

Hence, there are only the following two equivalence classes:

$$\overline{0} = \{a \in \mathbb{Z} : 3|a\} \text{ and } \overline{1} = \{b \in \mathbb{Z} : 3 \nmid b\}.$$

Q.10 Let S be a finite set and T be a subset of S. We define a binary relation R on the power set $\mathcal{P}(S)$ of set S: for subsets A and B of S, $(A, B) \in R$ if and only if $(A \cup B) \setminus (A \cap B) \subseteq T$. Prove that the relation R is an equivalence relation.

Solution: Since $(A \cup A) \setminus (A \cap A) = \emptyset \subseteq T$, we have $(A, A) \in R$ for all $A \subseteq S$. The relation R is *reflexive*.

If $(A, B) \in R$, then $(A \cup B) \setminus (A \cap B) \subseteq T$, but since $\cup$ and $\cap$ are both symmetric, $A \cup B = B \cup A$ and $A \cap B = B \cap A$. So, $(B \cup A) \setminus (B \cap A) \subseteq T$. We then have the relation R is *symmetric*.

Assume that $(A, B), (B, C) \in R$. Note that e is an element of $S = (A \cup B) \setminus (A \cap B)$ if and only if it is in exactly one of A and B. So, $(A, B) \in R$ implies that every such element is in T. Similarly, $(B, C) \in R$ means that every element in exactly one of B and C is in T. Now consider an element e in exactly one of A and C. Assume that it is in A, hence not in C. If it is also in B, then it satisfies the condition to be an element of $(B \cup C) \setminus (B \cap C)$ and thus is in T. If e is not in B, then it satisfies the condition to be in $(A \cup B) \setminus (A \cap B)$ and hence is in T. An analogous line of reasoning applies to show that if e is in C but not in A then it is in T. So we have $(A, C) \in R$ and the relation R is *transitive*.

To sum up, the relation R is an equivalence relation.

Q.11 Show that the relation R on $\mathbb{Z} \times \mathbb{Z}$ defined on $(a, b)\mathbb{R}(c, d)$ if and only if $a + d = b + c$ is an *equivalence* relation.

Solution: $((a, b), (a, b)) \in R$ because $a + b = a + b$. Hence R is reflexive.

If $((a, b), (c, d)) \in R$ then $a + d = b + c$, so that $c + b = d + a$. It then follows that $((c, d), (a, b)) \in R$. Hence R is symmetric.

Suppose that $((a, b), (c, d))$ and $((c, d), (e, f))$ belong to R. Then $a + d = b + c$ and $c + f = d + e$. Adding these two equations and subtracting $c + d$ from both sides gives $a + f = b + e$. Hence $((a, b), (e, f))$ belongs to R. Hence, R is transitive.

□

Q.12 Let $\sim$ be a relation defined on $\mathbb{N}$ by the rule that $x \sim y$ if $x = 2^k y$ or $y = 2^k x$ for some $k \in \mathbb{N}$. Show that $\sim$ is an equivalence relation.

Solution: We first show the following lemma.

Lemma For any $x, y \in \mathbb{N}$, $x \sim y$ if and only if there exists some $k \in \mathbb{Z}$ such that $x = 2^k y$ in $\mathbb{Q}$.

Proof. Suppose that $x \sim y$. Then either $x = 2^k y$ for some $k \in \mathbb{N} \subseteq \mathbb{Z}$ and we are done, or $y = 2^{k'} x$ for some $k' \in \mathbb{N}$. In the latter case, solve for $x = 2^{-k'} y$ and let $k = -k'$. In the other direction, if $x = 2^k y$, and $k \geq 0$, then $x = 2^k y$ for some $k \in \mathbb{N}$, giving $x \sim y$. If instead $k < 0$, then $y = 2^{-k}$, again giving $x \sim y$.

To show $\sim$ is an equivalence relation, we show the following three properties.

Reflexive For any $x \in \mathbb{N}$, $x = 2^0 x$ so $x \sim x$.

Symmetric If $x \sim y$, then from **Lemma** there exists $k \in \mathbb{Z}$ such that $x = 2^k y$. But then $y = 2^{-k} x$, so applying the lemma again, gives $y \sim x$.

Transitive If $x \sim y \sim z$, then $x = 2^k y$ and $y = 2^\ell z$ for some $k, \ell \in \mathbb{Z}$ by **Lemma**. Solve to get $x = 2^{k+\ell} z$, which gives $x \sim z$.

□

Q.13 Which of these are posets?

(a) $(\mathbf{Z}, =)$

(b) $(\mathbf{Z}, \neq)$

(c) $(\mathbf{Z}, \geq)$

(d) $(\mathbf{Z}, \nmid)$

Solution:

(a) Yes. The only ordered pairs we will have in this relation is (a, a) for all $a \in \mathbf{Z}$. This would mean that the relation is reflexive, antisymmetric, and transitive.

(b) No. It is not reflexive. The relation is also not antisymmetric, and not transitive.

(c) Yes. For reflexive, we can have the ordered pair (a, a) for all $a \in \mathbf{Z}$. This is also antisymmetric because consider the ordered pair (a, b) and $a \neq b$. This would mean that $a > b$. If this is the case, then $b > a$ is not true and you cannot have (b, a). This is also transitive because if $a > b$, $b > c$, and $a \neq b \neq c$. Then it follows that $a > c$ for all $a, b, c \in \mathbf{Z}$.

(d) No. The relations is not reflexive, not antisymmetric, not transitive.

$\square$

Q.14 Consider a relation $\propto$ on the set of functions from $\mathbb{N}^+$ to $\mathbb{R}$, such that $f \propto g$ if and only if $f = O(g)$.

(a) Is $\propto$ an equivalence relation?

(b) Is $\propto$ a partial ordering?

(c) Is $\propto$ a total ordering?

Solution:

(a) No. $\propto$ is not symmetric. Let $f(n) = n$ and $g(n) = n^2$. Here $f = O(g)$ but $g \neq O(f)$.

(b) No. $\propto$ is not antisymmetric. Let $f(n) = n$ and $g(n) = 2n$. Then $f = O(g)$ and $g = O(f)$, but $f \neq g$.

(c) No. It is not partial ordering, then not a total ordering.

□

Q.15 The relation R on the set $X = \{(a, b, c) : \; a, b, c \in \mathbb{N}\}$ with $(a_1, b_1, c_1)R(a_2, b_2, c_2)$ if and only if $2^{a_1}3^{b_1}5^{c_1} \le 2^{a_2}3^{b_2}5^{c_2}$.

(1) Prove that R is a partial ordering.

(2) Write two comparable and two incomparable elements if they exist.

(3) Find the least upper bound and the greatest lower bound of the two elements $(5, 0, 1)$ and $(1, 1, 2)$.

(4) List a minimal and a maximal element if they exist.

Solution:

(1) Reflexive: Consider $(a, b, c) \in X$. Note that $2^a3^b5^c \le 2^a3^b5^c$ by definition of $\le$ (equals). Thus, the relation is reflexive.

Antisymmetric: Consider $(a_1, b_1, c_1), (a_2, b_2, c_2) \in X$ such that $(a_1, b_1, c_1)R(a_2, b_2, c_2)$ and $(a_2, b_2, c_2)R(a_1, b_1, c_1)$. By definition of the relation, we have

$$\begin{aligned} 2^{a_1}3^{b_1}5^{c_1} &\le 2^{a_2}3^{b_2}5^{c_2}, \\ 2^{a_2}3^{b_2}5^{c_2} &\le 2^{a_1}3^{b_1}5^{c_1}, \\ 2^{a_1}3^{b_1}5^{c_1} &= 2^{a_2}3^{b_2}5^{c_2}, \\ a_1 &= a_2, \\ b_1 &= b_2, \\ c_1 &= c_2. \end{aligned}$$

Transitive: Consider $(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) \in X$ such that $(a_1, b_1, c_1)R(a_2, b_2, c_2)$ and $(a_2, b_2, c_2)R(a_3, b_3, c_3)$. By definition of the relation, we have

$$\begin{aligned} 2^{a_1}3^{b_1}5^{c_1} &\le 2^{a_2}3^{b_2}5^{c_2}, \\ 2^{a_2}3^{b_2}5^{c_2} &\le 2^{a_3}3^{b_3}5^{c_3}, \\ 2^{a_1}3^{b_1}5^{c_1} &\le 2^{a_3}3^{b_3}5^{c_3}. \end{aligned}$$

The latter is by transitivity of $\le$. Thus, the relation is transitive.

(2) $(1,2,3)$ and $(4,5,6)$ are comparable. No pairs are incomparable. Every pair of integers has a lesser integer.

(3) Since $2^5 3^0 5^1 = 160$ and $2^1 3^1 5^2 = 150$. Thus, the least upper bound is $(5,0,1)$ and the greatest lower bound is $(1,1,2)$.

(4) The minimal element is $(0,0,0)$ because $2^0 3^0 5^0 = 1$ which is the smallest nonzero, nonnegative integer. There is no maximal element, because there is always a bigger integer.

Q.16 Define the relation $\preceq$ on $\mathbb{Z} \times \mathbb{Z}$ according to

$$(a,b) \preceq (c,d) \Leftrightarrow (a,b) = (c,d) \text{ or } a^2 + b^2 < c^2 + d^2.$$

Show that $(\mathbb{Z}\times\mathbb{Z}, \preceq)$ is a poset; Construct the Hasse diagram for the subposet $(B, \preceq)$, where $B = \{0,1,2\} \times \{0,1,2\}$.

Solution: We now prove that $\preceq$ on the set $\mathbb{Z} \times \mathbb{Z}$ is a partial ordering. Obviously, $(a,b) \preceq (a,b)$, and we have $\preceq$ is reflexive; Suppose that $(a,b) \preceq (c,d)$ and $(c,d) \preceq (a,b)$, then the only possibility is that $(a,b) = (c,d)$. Then $\preceq$ is antisymmetric; Suppose that $(a,b) \preceq (c,d)$ and $(c,d) \preceq (e,f)$, then we have four possible cases: $(a,b) = (c,d)$ and $c^2 + d^2 < e^2 + f^2$; $(a,b) = (c,d)$ and $(c,d) = (e,f)$; $a^2 + b^2 < c^2 + d^2$ and $(c,d) = (e,f)$; $a^2 + b^2 < c^2 + d^2$ and $c^2 + d^2 < e^2 + f^2$. For each of the four cases above, we have $(a,b) \preceq (e,f)$ and thereby the relation $\preceq$ is transitive.

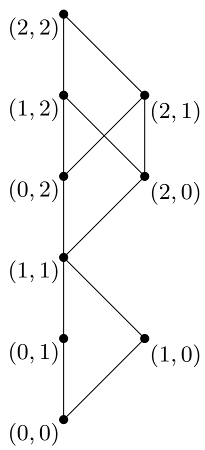

Figure 1: Q.16

$\square$

Q.17 We consider partially ordered sets whose elements are sets of natural numbers, and for which the ordering is given by $\subseteq$. For each such partially ordered set, we can ask if it has a minimal or maximal element. For example, the set $\{\{0\}, \{0,1\}, \{2\}\}$, has minimal elements $\{0\}, \{2\}$, and maximal elements $\{0,1\}, \{2\}$.

(a) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no maximal element.

(b) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no minimal element.

(c) Prove or disprove: there exists a nonempty $T \subseteq \mathcal{P}(\mathbb{N})$ that has neither minimal nor maximal elements.

Solution:

(a) There are many choices here. One is to let $R = \{A_0, A_1, A_2, \ldots\}$ where $A_i = \{j \in \mathbb{N} | j < i\}$. Then R has no maximal element, because for any $A_i \in R$, we have $A_i \subsetneq A_{i+1} \in R$.

(b) For this we will do the same thing as above in reverse. Let $S = \{B_0, B_1, B_2, \ldots\}$ where $B_i = \{j \in \mathbb{N} | j \geq i\}$. Then S has no minimal element, because for any $B_i \in S$, we have $B_i \supsetneq B_{i+1}$.

(c) Here we can combine the previous two results. Let $T = \{C_{ij} | i \in \mathbb{N},\ j \in \mathbb{N}\}$ where each $x \in \mathbb{N}$ is in C_{ij} if and only if $x = 2k$ and $k < i$, or $x = 2k+1$ and $K \geq j$. Now T has no minimal or maximal elements, because for any $C_{ij} \in T$, $C_{i,j+1} \subsetneq C_{ij} \subsetneq C_{i+1,j}$.

$\square$

Q.18 Answer these questions for the poset $(\{3, 5, 9, 15, 24, 45\}, |)$.

(1) Find the maximal elements.

(2) Find the minimal elements.

(3) Is there a greatest element?

(4) Is there a least element?

(5) Find all upper bounds of $\{3, 5\}$.

(6) Find the least upper bound of $\{3, 5\}$, if it exists.

(7) Find all lower bounds of $\{15, 45\}$.

(8) Find the greatest lower bound of $\{15, 45\}$, if it exists.

Solution:

(1) By drawing the Hasse diagram, our maximal elements are 24 and 45.

(2) The minimal elements are 3 and 5.

(3) There is no greatest element because this element would have to be a number that all other elements divide. Since our maximal elements are 24 and 45, and they do not divide each other, we do not have a greatest element.

(4) There is no least element because this element would be a number that can divide all other elements. Since our minimal elements are 3 and 5, and they do not divide each other, we do not have a least element.

(5) 15 and 45.

(6) 15.

(7) 3, 5, and 15.

(8) 15.

$\square$