Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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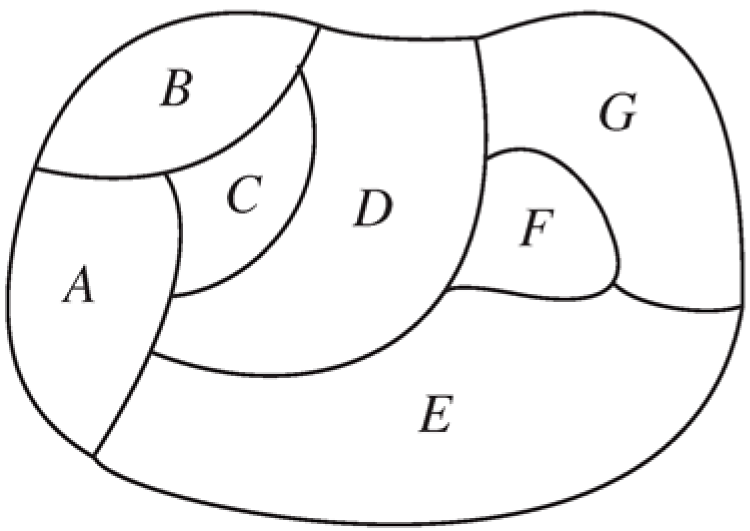
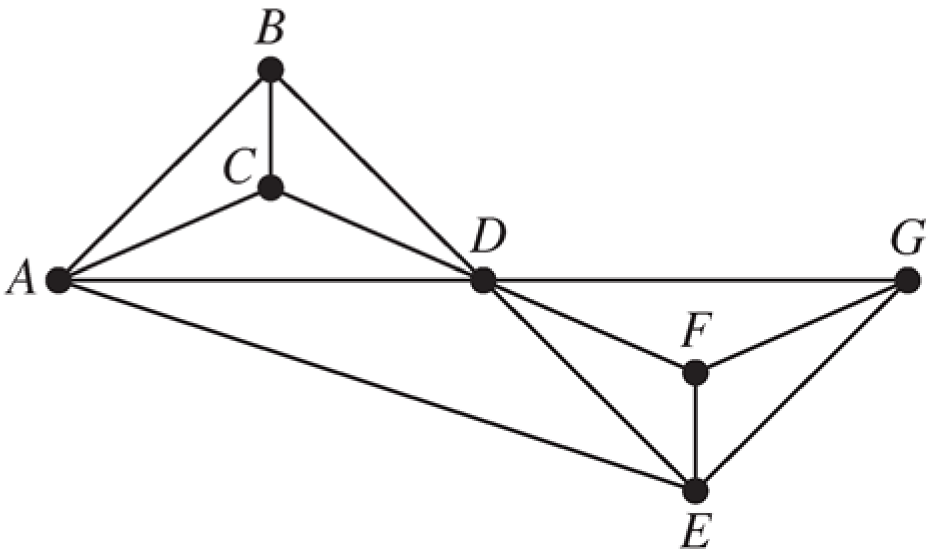
Graph Coloring

- A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

  The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph, denoted by $\chi(G)$.





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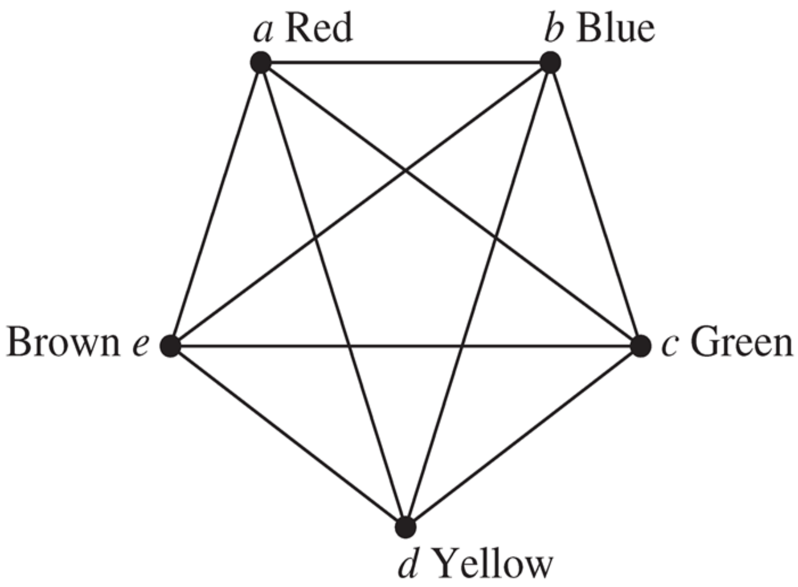

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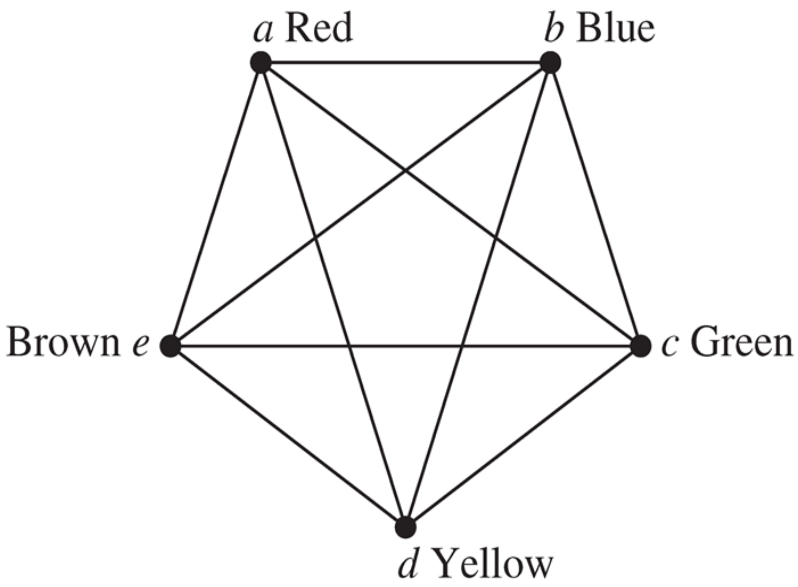

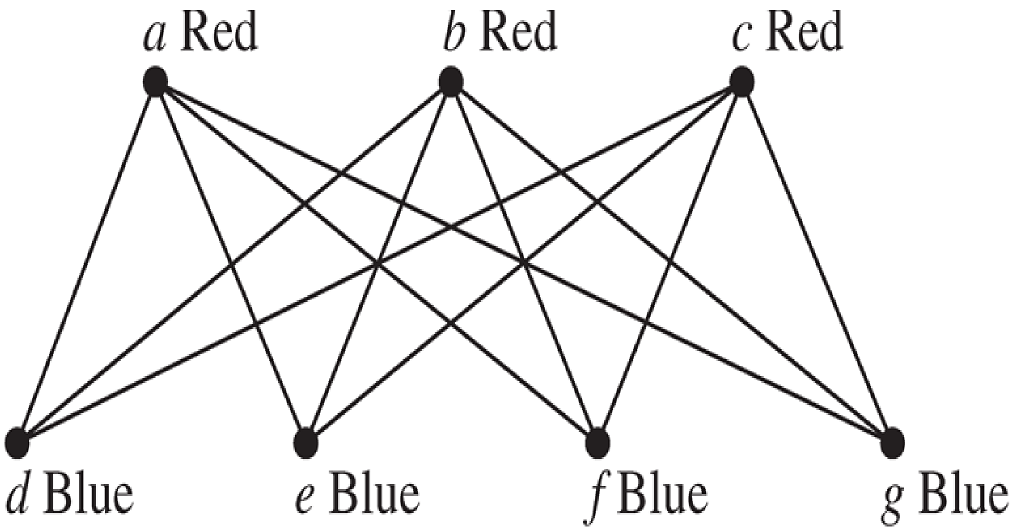

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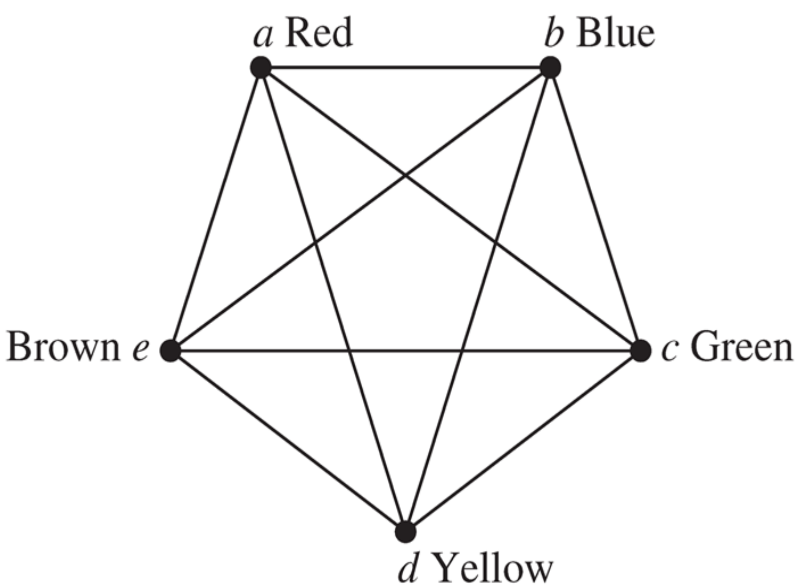

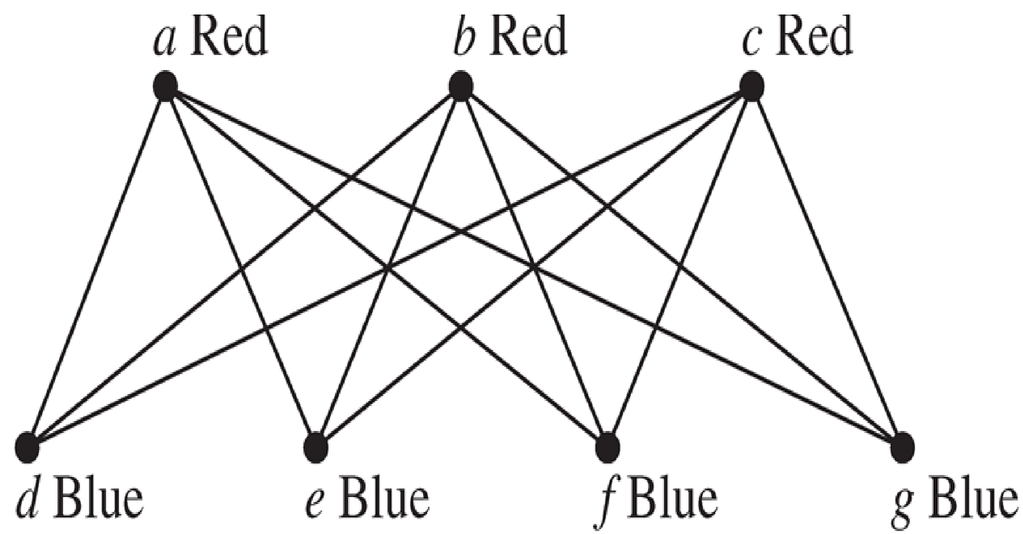

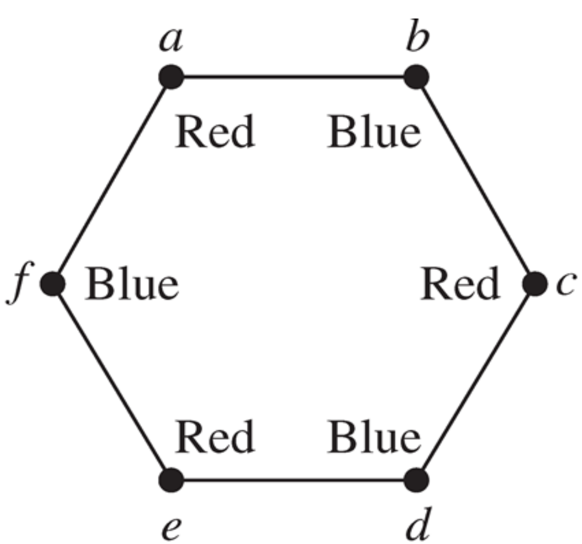

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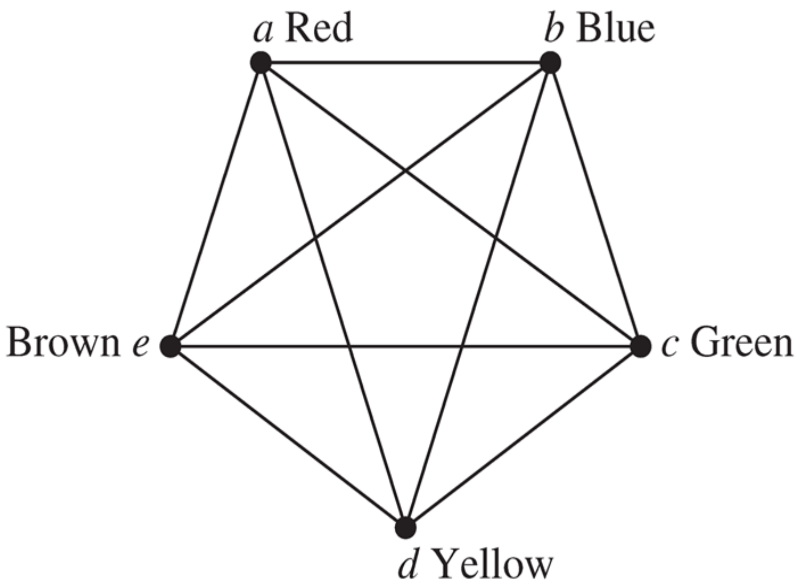
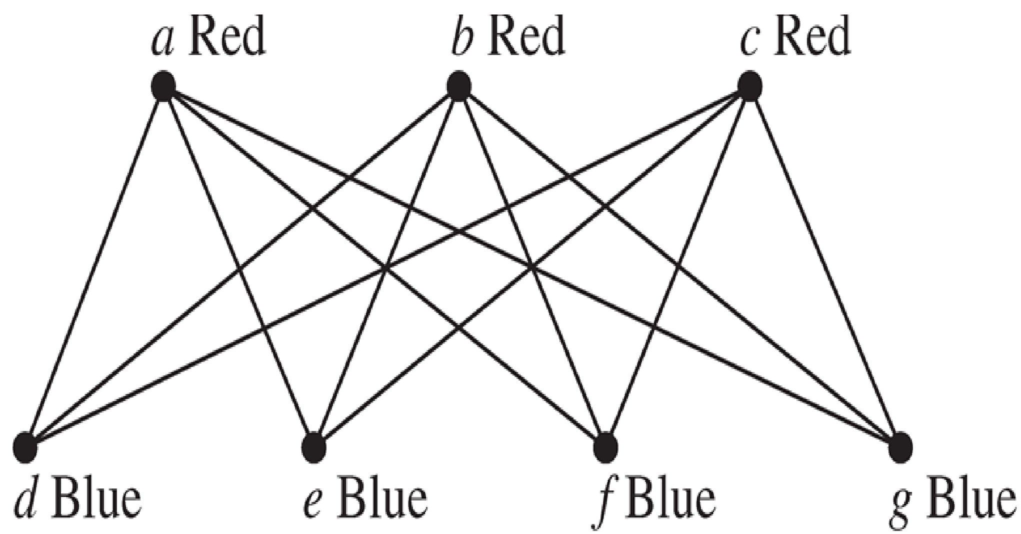
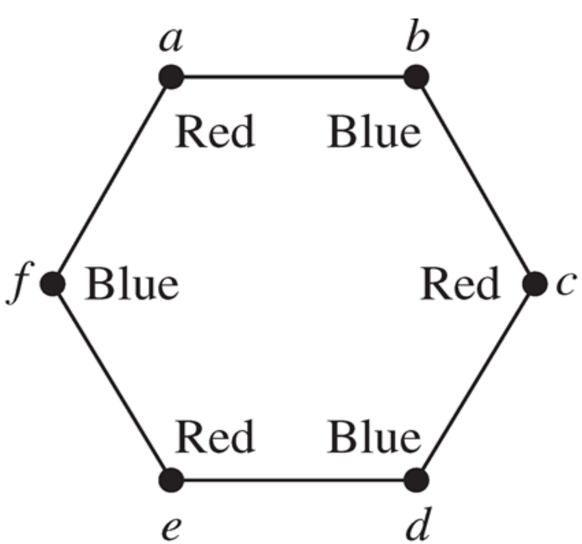
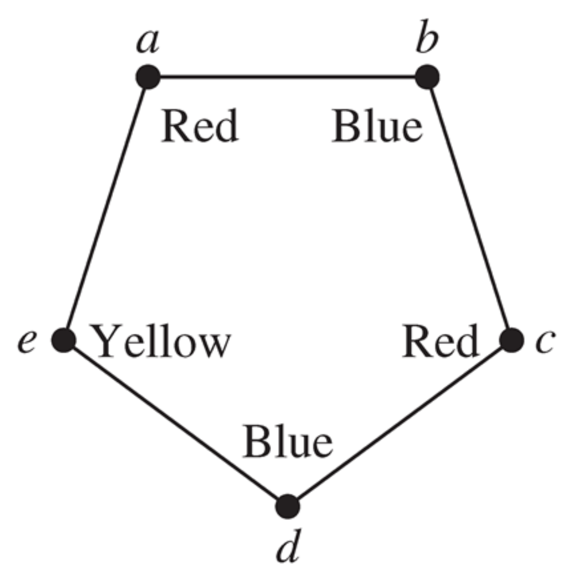




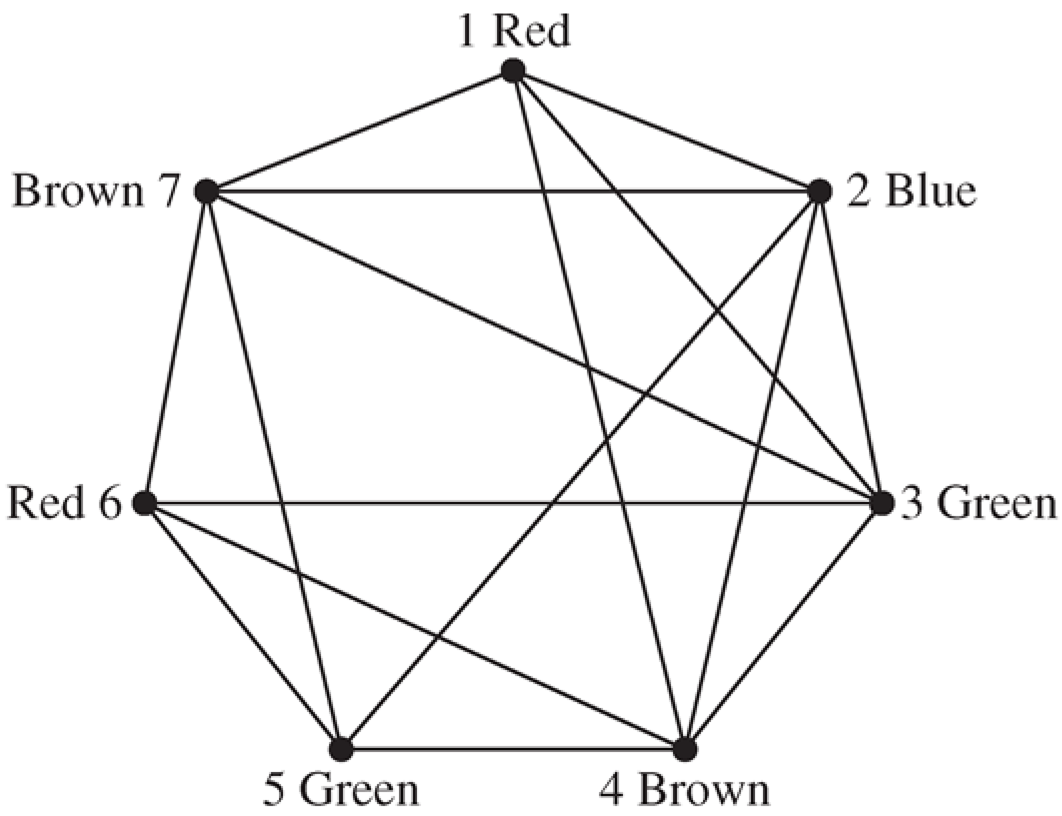




Applications of Graph Coloring

- Scheduling Final Exams

Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

Applications of Graph Coloring

- Channel Assignments

  Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel . How can the assignment of channels be modeled by graph coloring?

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Graph Coloring $\in$ NPC

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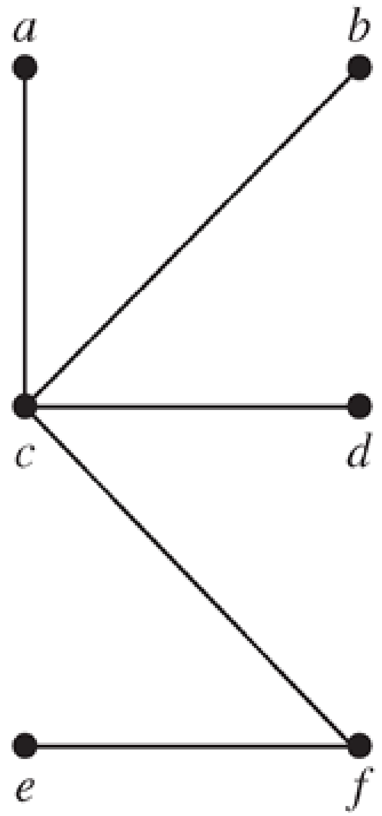

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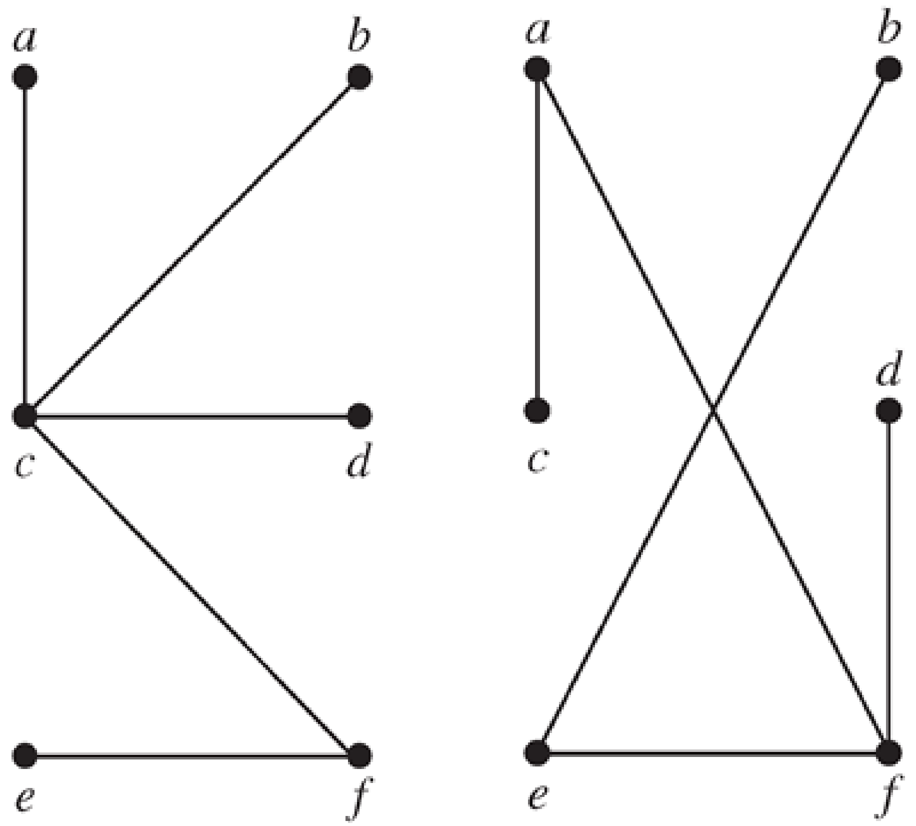

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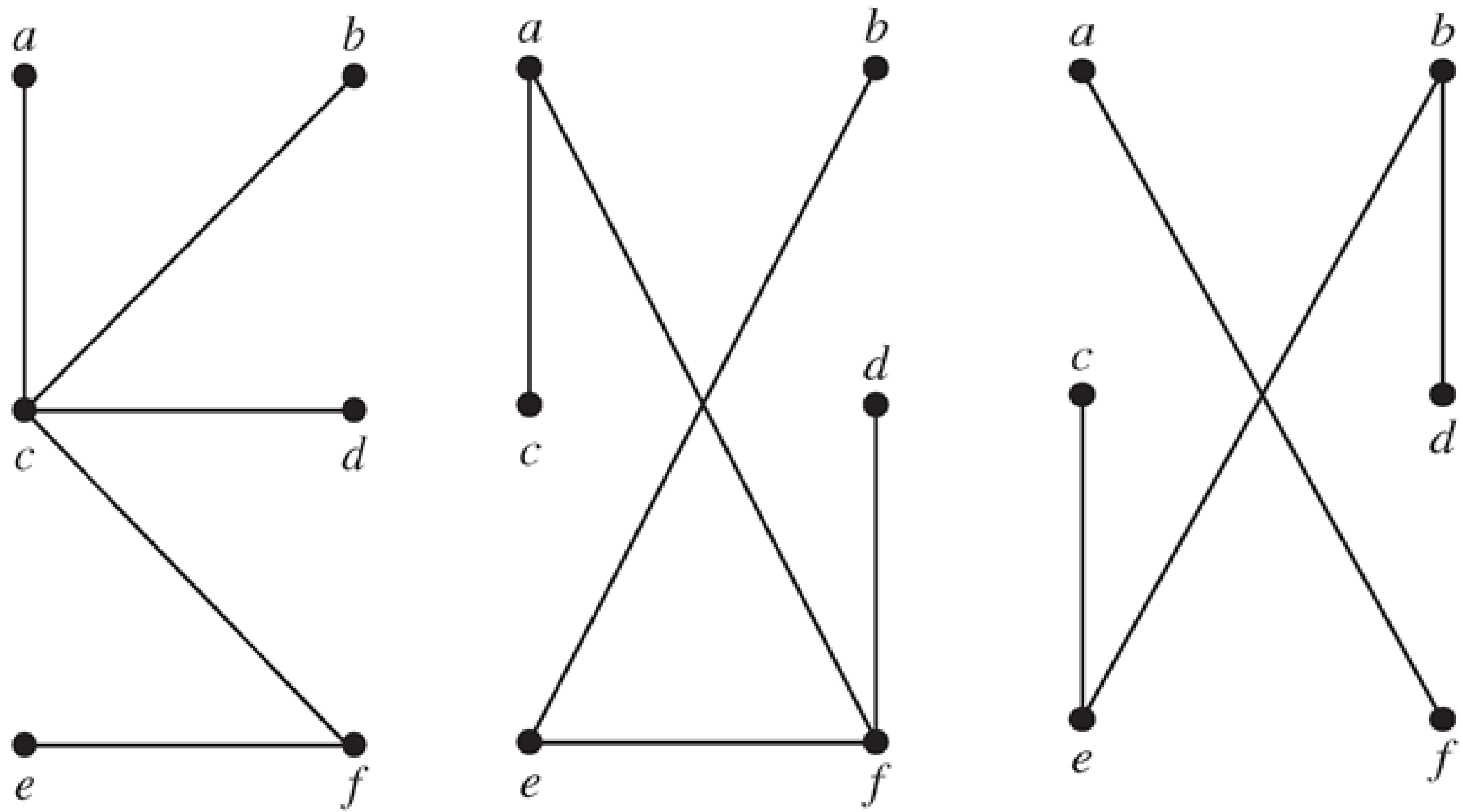

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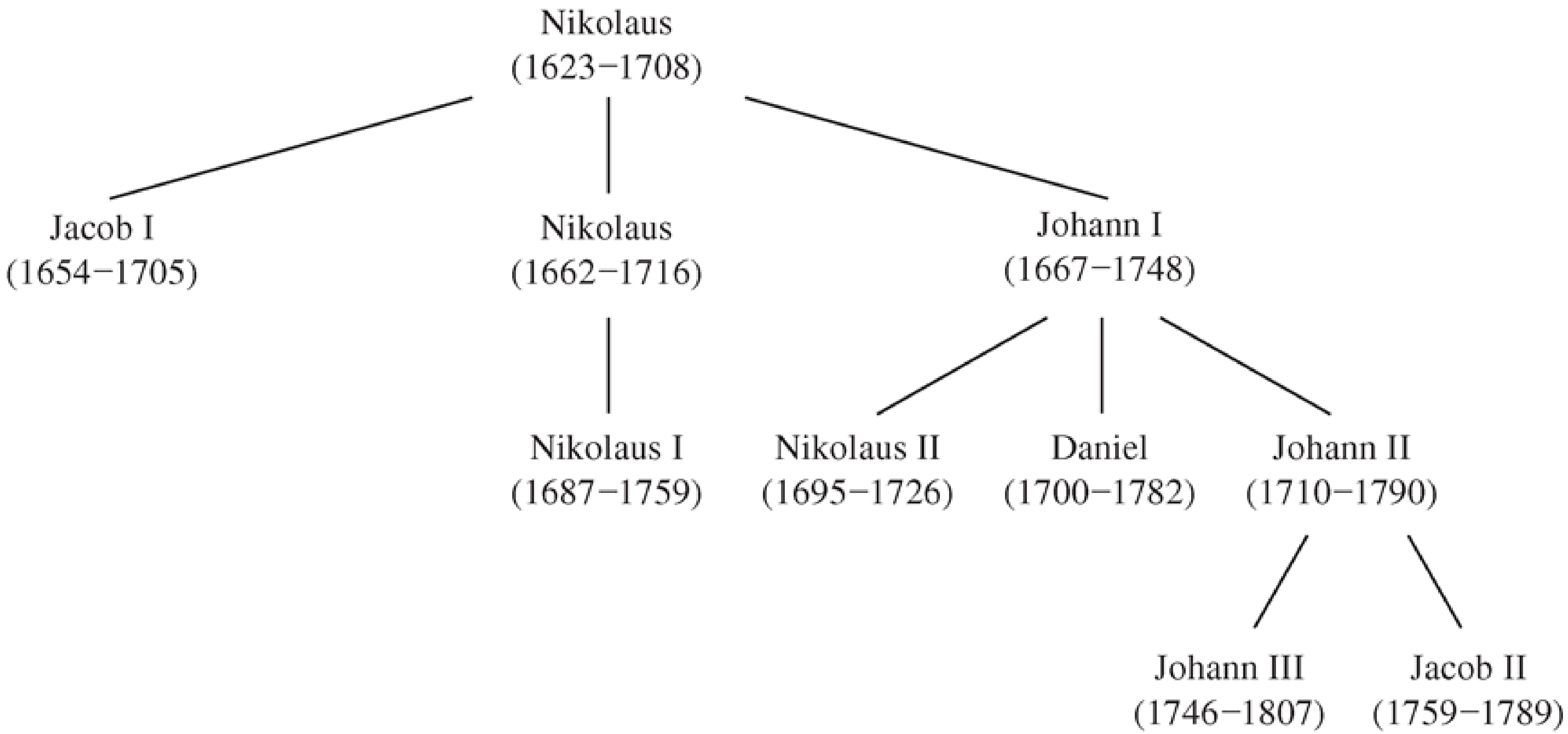

Trees

- **Theorem** An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

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  Two properties of tree: connected, no circuit

Rooted Trees

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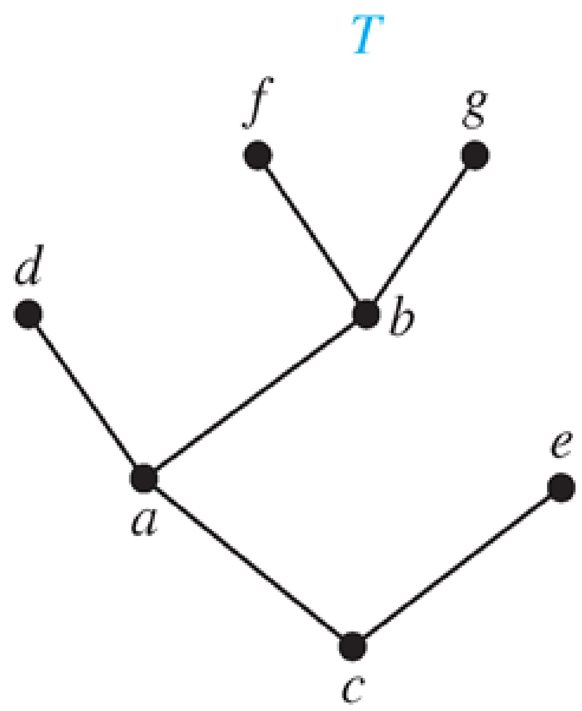

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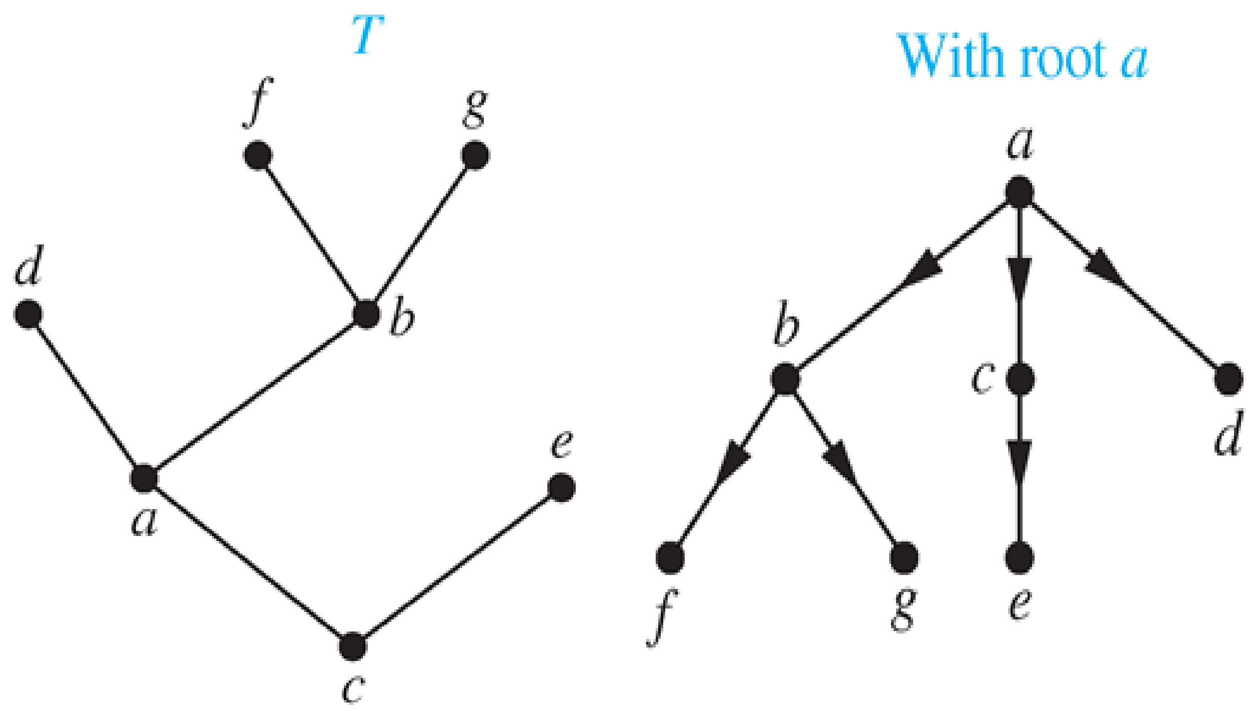

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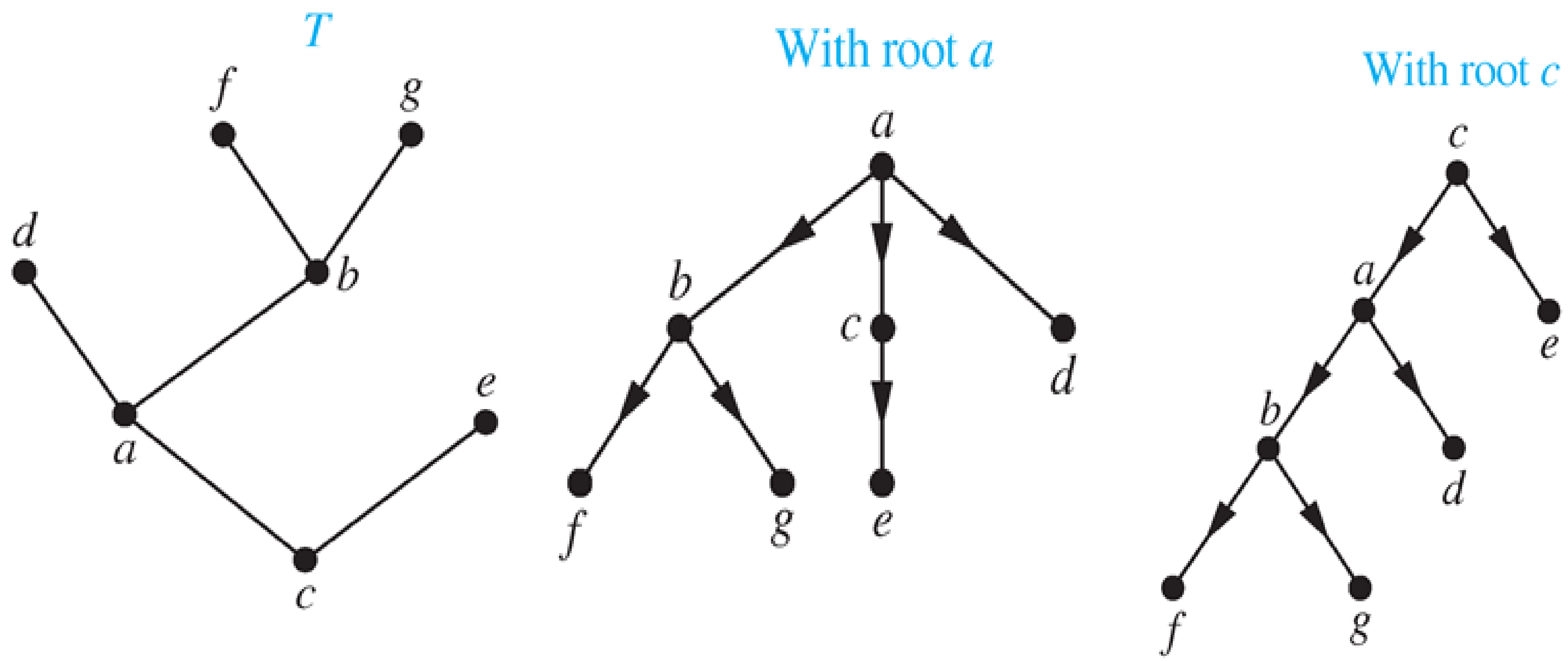

Rooted Trees

- *parent, child, sibling*

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  subtree with a as its root: consists of a and its descendants and all edges incident to these descendants

m-Ary Trees

- **Definition** A rooted tree is called an *m-ary tree* if every internal vertex has no more than m children. The tree is called a *full m-ary tree* if every internal vertex has exactly m children. In particular, an m-ary tree with $m = 2$ is called a *binary tree*.

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Full m-Ary Trees

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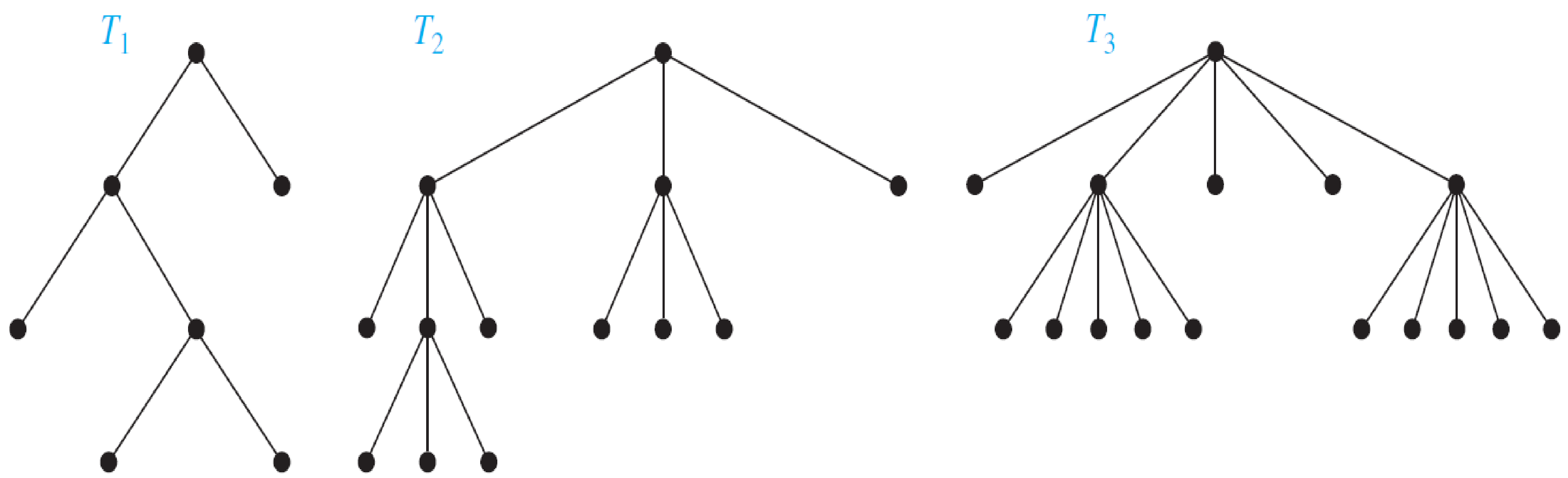

Counting Vertices in a Full m-Ary Trees

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  (i) n vertices, $i = (n-1)/m$, $\ell = [(m-1)n+1)]/m$
  (ii) i internal vertices, $n = mi + 1$, $\ell = (m-1)i + 1$
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  using $n = mi + 1$ and $n = i + \ell$

Level and Height

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  Definition A rooted m-ary tree of height h is *balanced* if all leaves are at levels h or $h-1$. (differ no greater than 1)

The Number of Leaves

- **Theorem** There are at most m^h leaves in an m-ary tree of height h.

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Binary Trees

- **Definition** A *binary tree* is an ordered rooted tree where each internal tree has two children, the first is called the *left child* and the second is the *right child*. The tree rooted at the left child of a vertex is called the *left subtree* of this vertex, and the tree rooted at the right child of a vertex is called the *right subtree* of this vertex.

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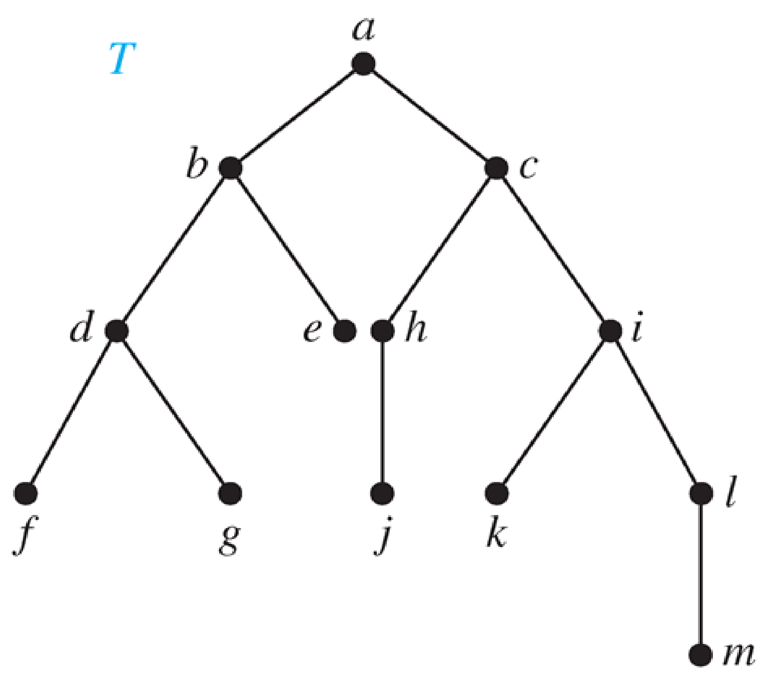

Tree Traversal

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  The three most commonly used traversals are *preorder traversal*, *inorder traversal*, *postorder traversal*.

Preorder Traversal

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *preorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *preorder traversal* begins by visiting r, and continues by traversing T_1 in preorder, then T_2 in preorder, and so on, until T_n is traversed in preorder.

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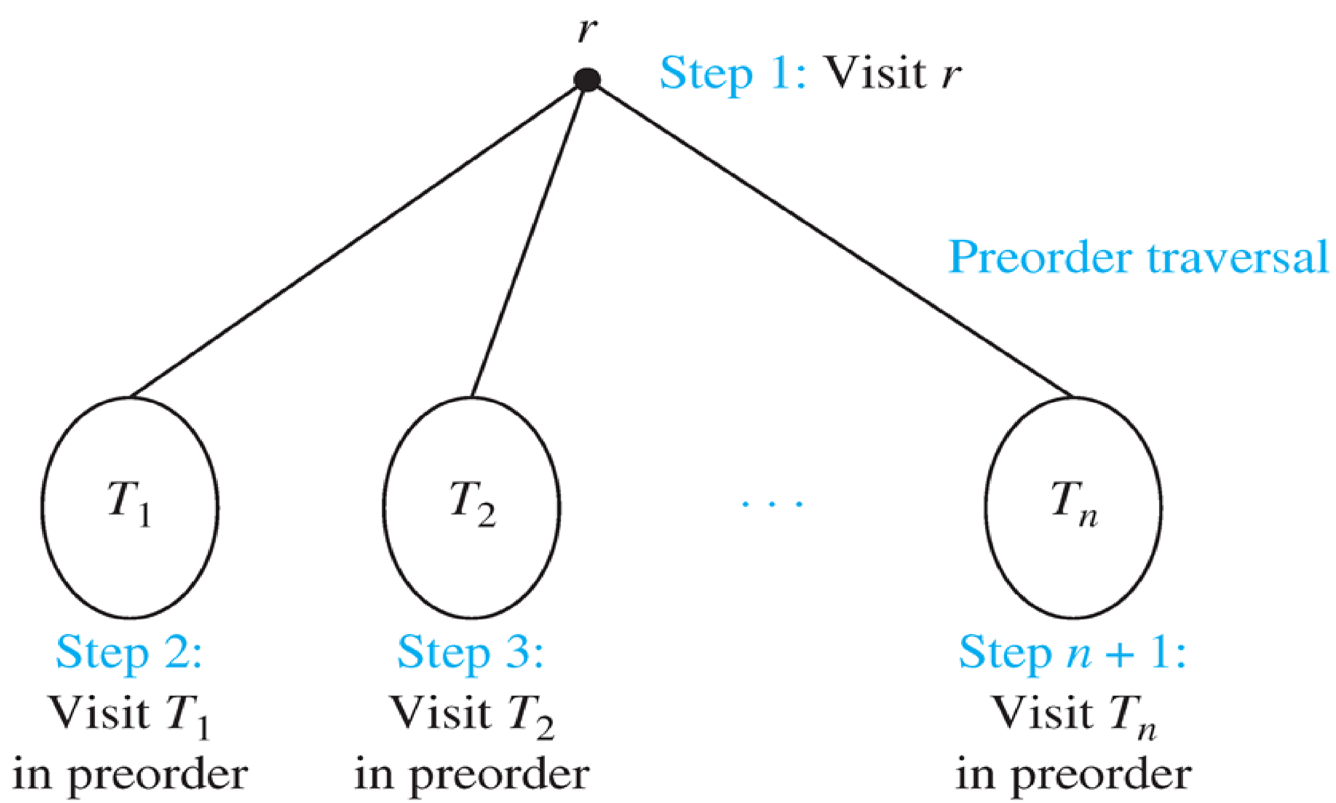
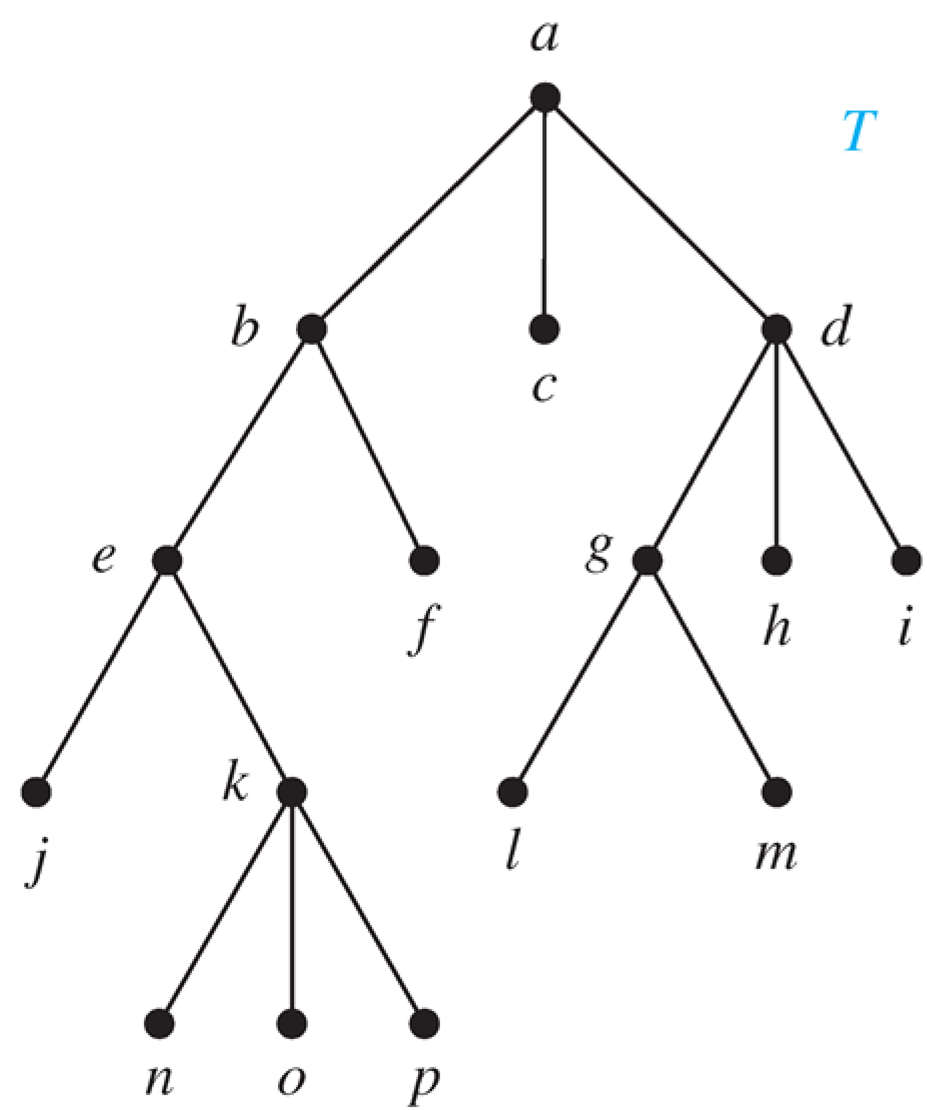


- **Example**



Preorder Traversal

procedure *preorder* (T: ordered rooted tree)
r := root of T
list r
for each child c of r from left to right
    $T(c)$:= subtree with c as root
    preorder($T(c)$)

Inorder Traversal

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *inorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *inorder traversal* begins by traversing T_1 in inorder, then visiting r, and continues by traversing T_2 in inorder, and so on, until T_n is traversed in inorder.

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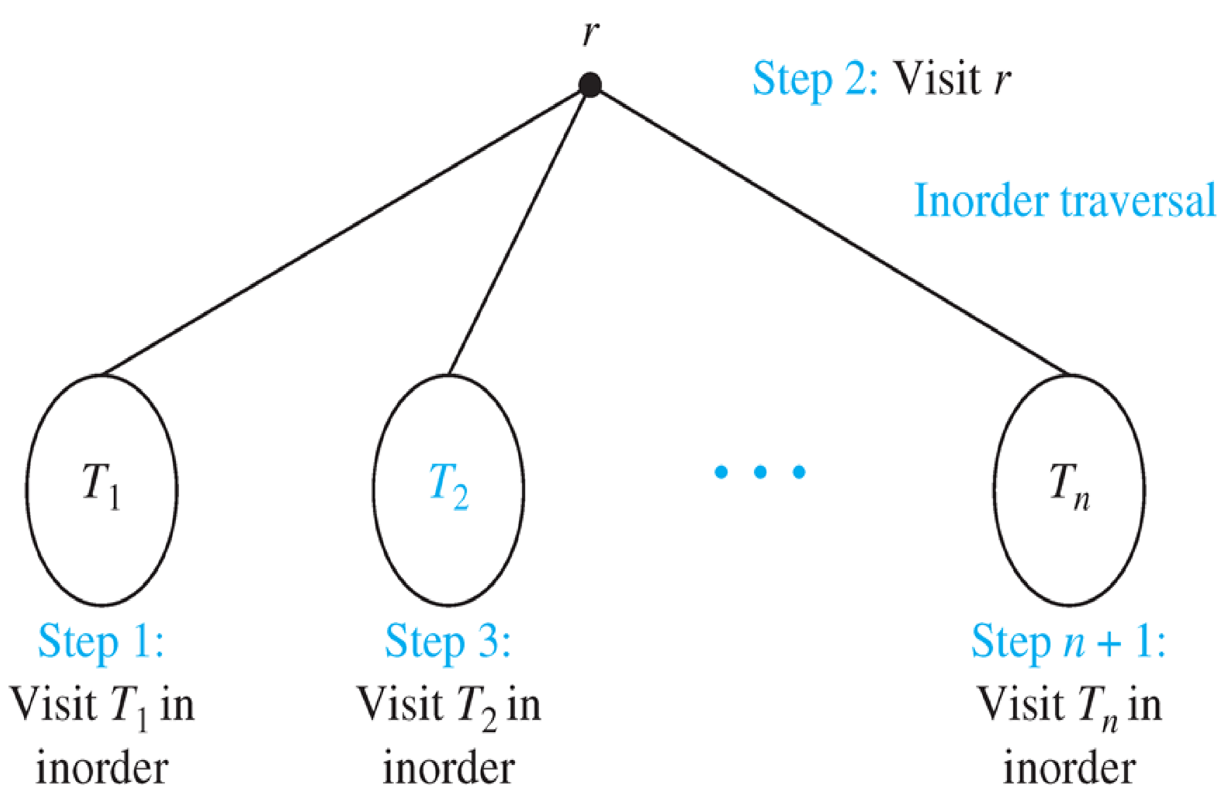

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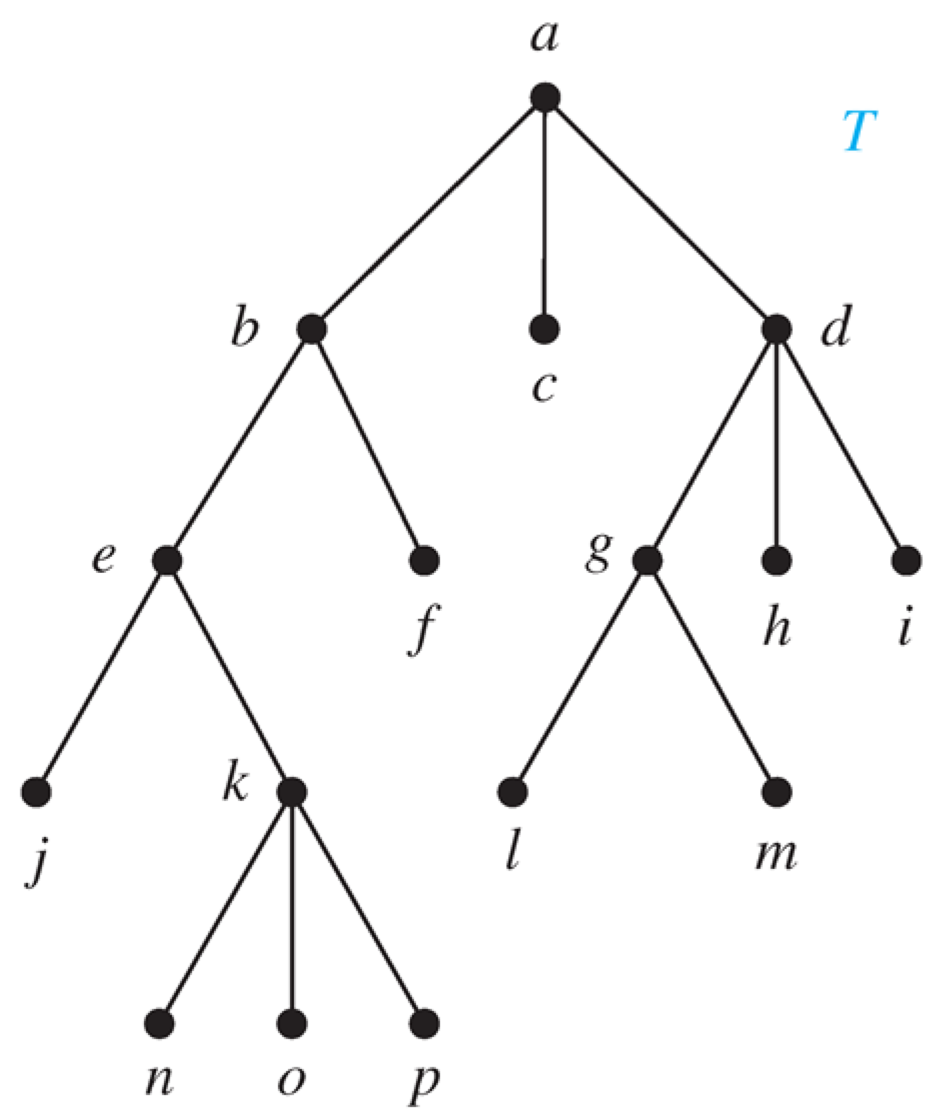

Inorder Traversal

procedure *inorder* (T: ordered rooted tree)
r := root of T
if r is a leaf **then** list r
else
    l := first child of r from left to right
    $T(l)$:= subtree with l as its root
    inorder$(T(l))$
    list(r)
    for each child c of r from left to right
        $T(c)$:= subtree with c as root
        inorder$(T(c))$

Postorder Traversal

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *postorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *postorder traversal* begins by traversing T_1 in postorder, then T_2 in postorder, and so on, after T_n is traversed in postorder, r is visited.

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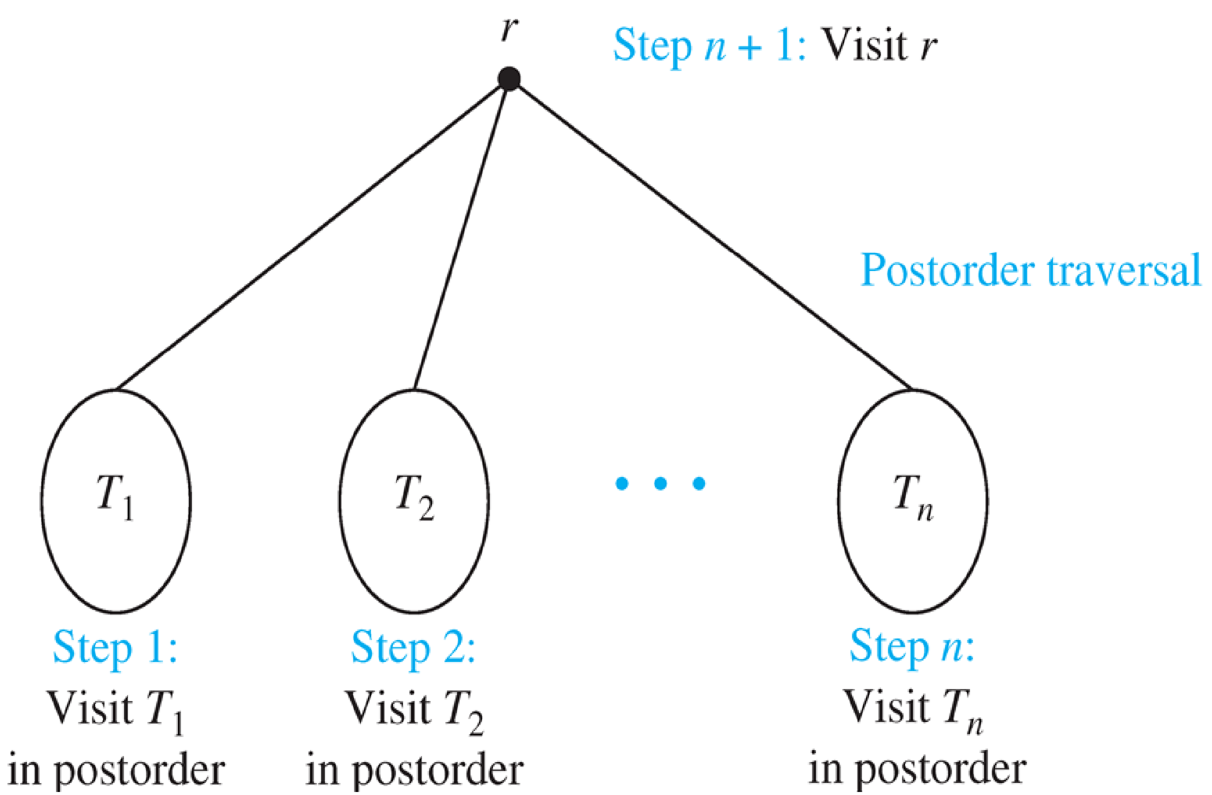

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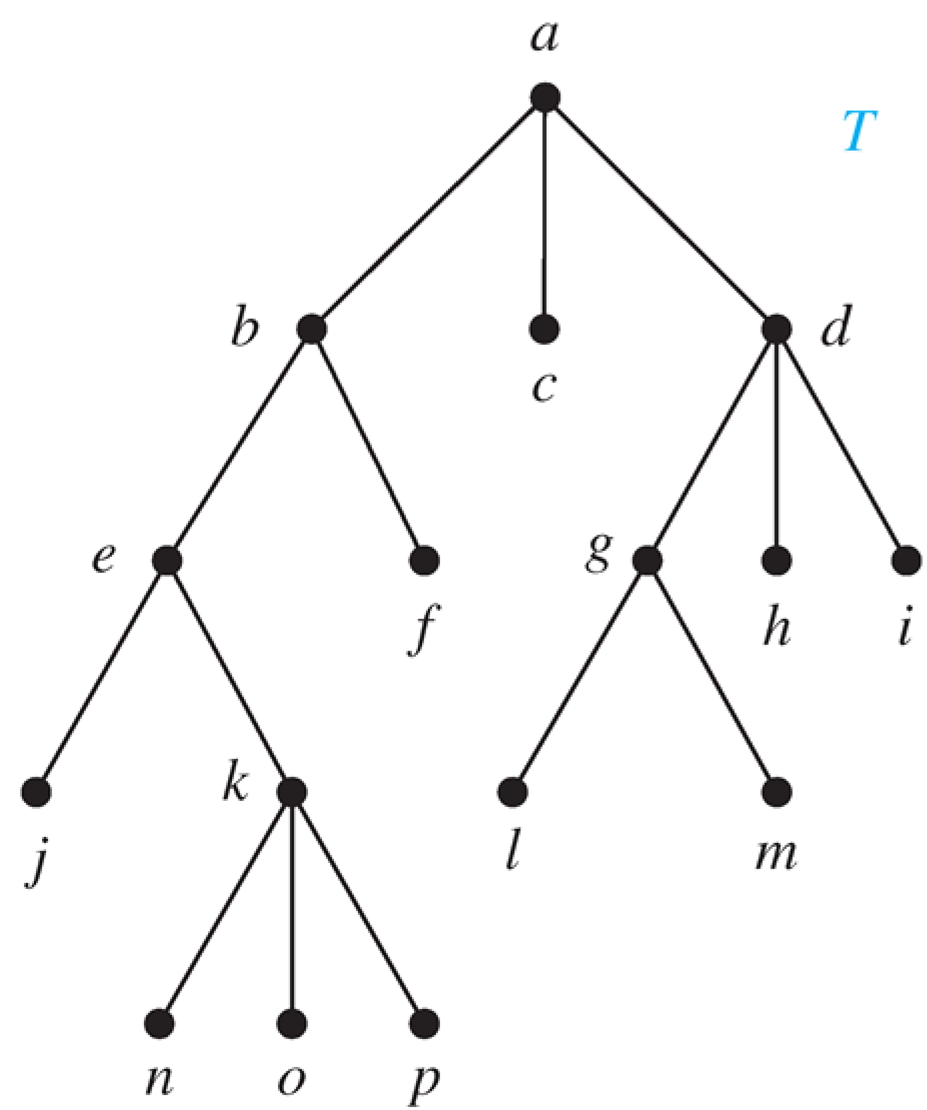
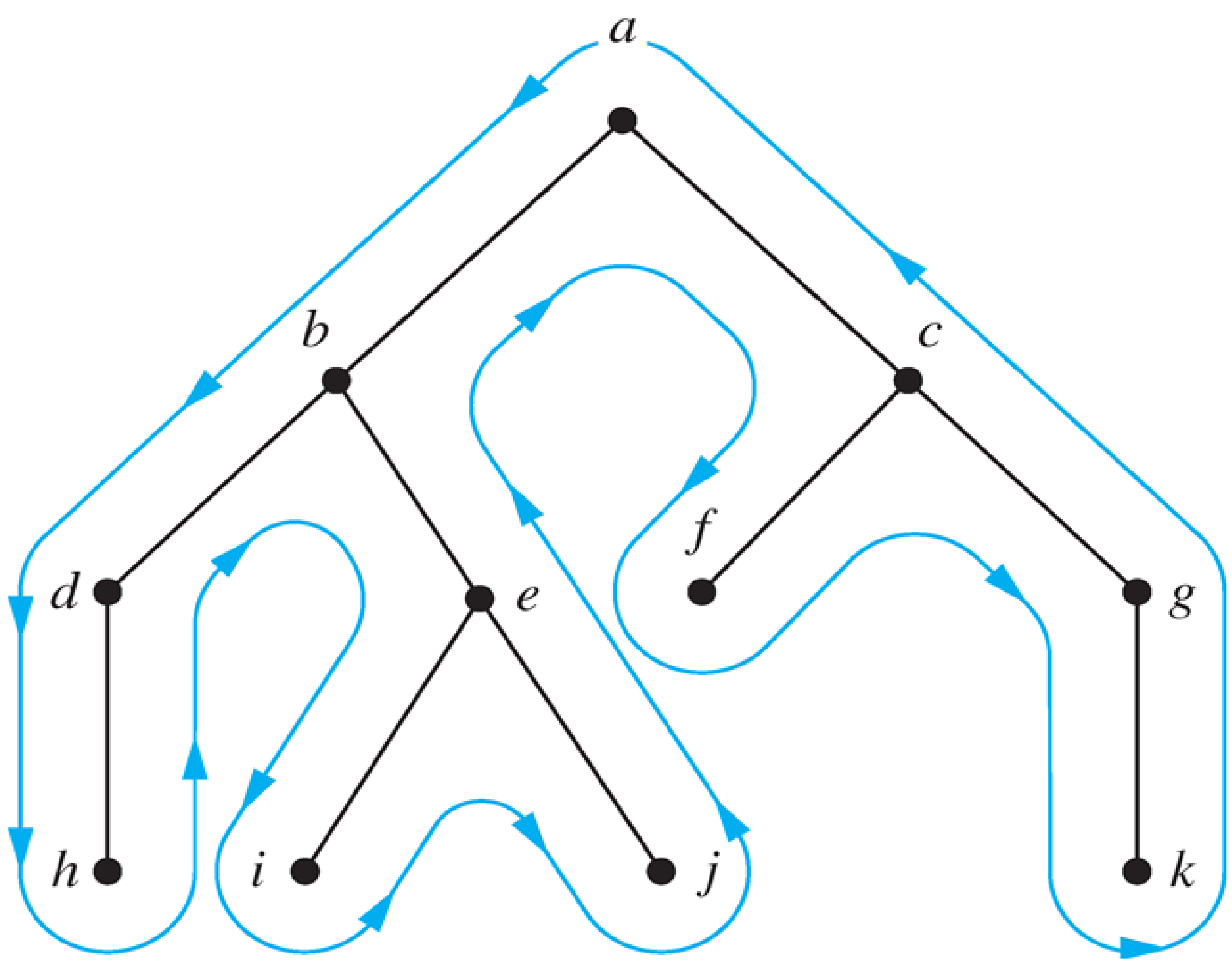


Postorder Traversal

procedure *postordered* (T: ordered rooted tree)
r := root of T
for each child c of r from left to right
    $T(c)$:= subtree with c as root
    postorder($T(c)$)
list r

Preorder, Inorder, Postorder Traversal



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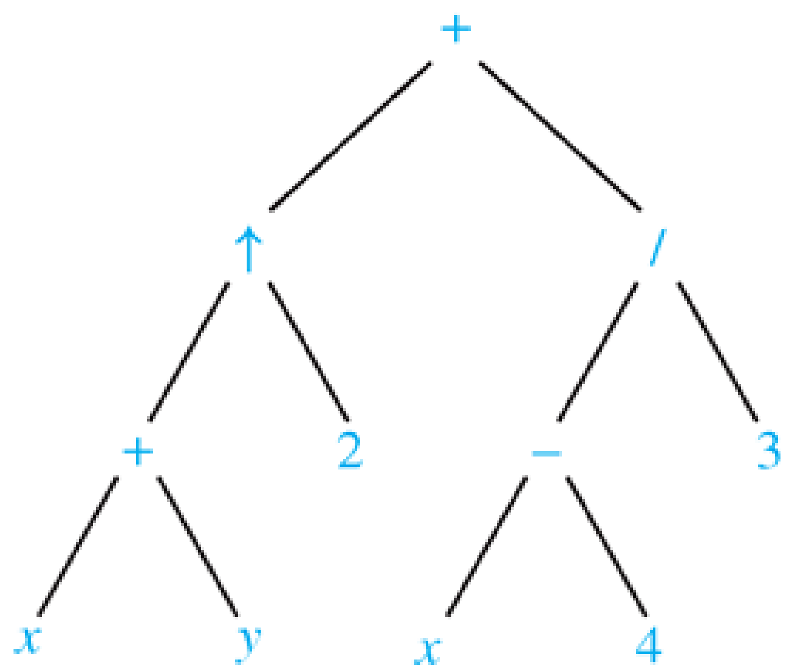

Infix Notation

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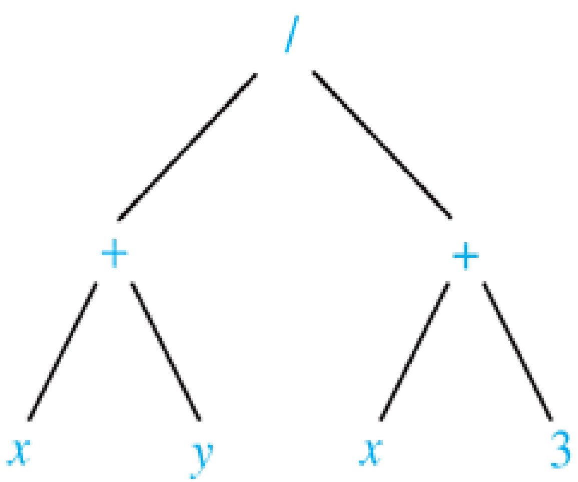

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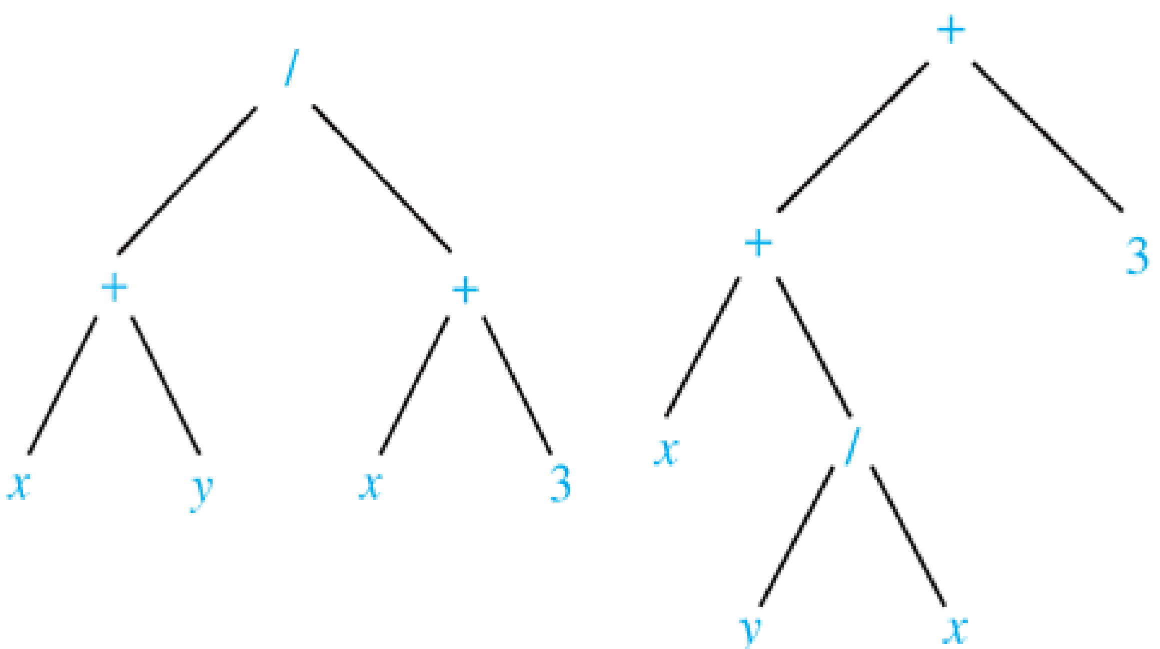

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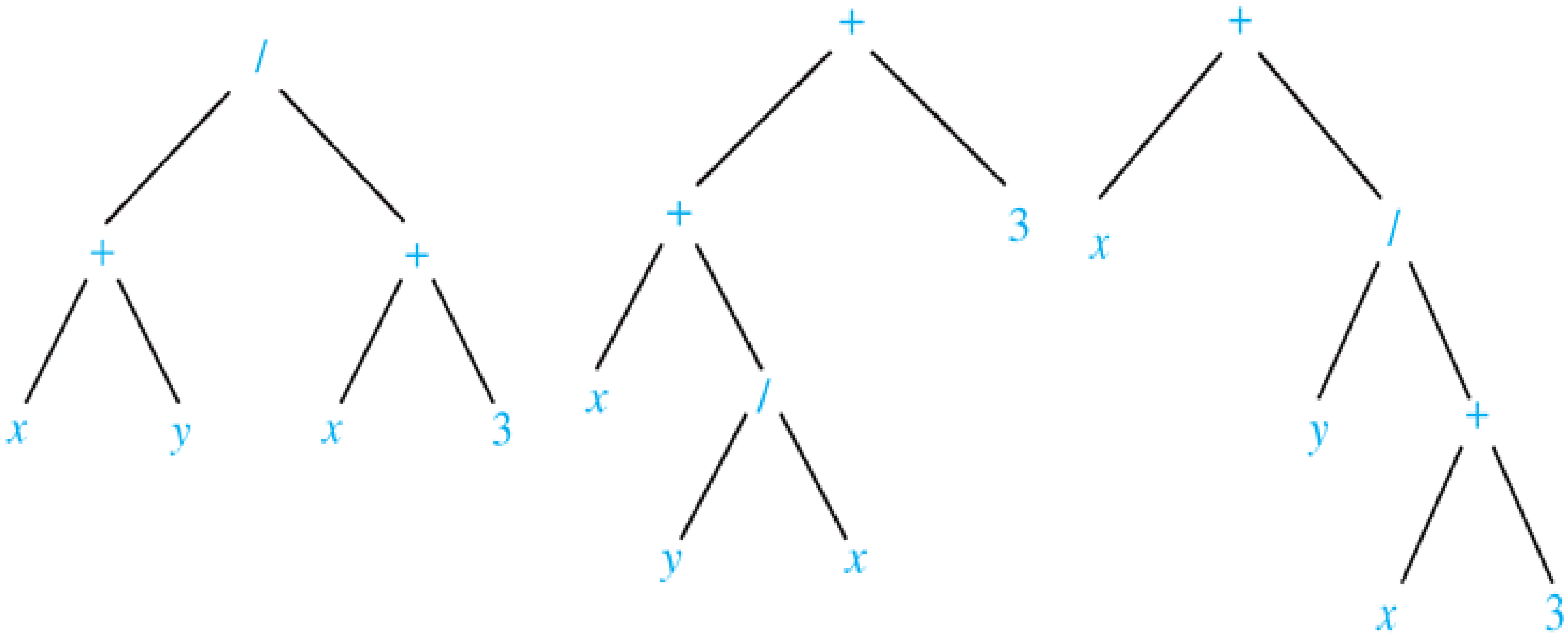

Prefix Notation

- The preorder traversal of expression trees leads to the *prefix form* of the expression (*Polish notation*).

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  Operators precede their operands in the prefix notation. Parentheses are not needed as the representation is unambiguous.

  Prefix expressions are evaluated by working from right to left. When we encounter an operator, we perform the operation with the two operands to the right.

Prefix Notation

- **Example**

  $$+ \; - \; * \, 2 \; 3 \; 5 \; / \; \uparrow \; 2 \; 3 \; 4$$

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  $+ \ - \ * \ 2 \ 3 \ 5 \ / \ \uparrow \ 2 \ 3 \ 4$

$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad / \quad \underbrace{\uparrow \quad 2 \quad 3}_{2 \uparrow 3 = 8} \quad 4$$

$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad \underbrace{/ \quad 8 \quad 4}_{8 / 4 = 2}$$

$$+ \quad - \quad \underbrace{* \quad 2 \quad 3}_{2 * 3 = 6} \quad 5 \quad 2$$

$$+ \quad \underbrace{- \quad 6 \quad 5}_{6 - 5 = 1} \quad 2$$

$$\underbrace{+ \quad 1 \quad 2}_{1 + 2 = 3}$$

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  Postfix expressions are evaluated by working from left to right. When we encounter an operator, we perform the operation with the two operands to the left.

Postfix Notation

- **Example**

  $7\ 2\ 3\ *\ -\ 4\ \uparrow\ 9\ 3\ /\ +$

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  $7\ 2\ 3\ *\ -\ 4\ \uparrow\ 9\ 3\ /\ +$

$$7\ \underbrace{2\quad 3\quad *}_{2*3=6}\quad -\quad 4\quad \uparrow\quad 9\quad 3\quad /\quad +$$

$$\underbrace{7\quad 6\quad -}_{7-6=1}\quad 4\quad \uparrow\quad 9\quad 3\quad /\quad +$$

$$\underbrace{1\quad 4\quad \uparrow}_{1^4=1}\quad 9\quad 3\quad /\quad +$$

$$1\quad \underbrace{9\quad 3\quad /}_{9/3=3}\quad +$$

$$\underbrace{1\quad 3\quad +}_{1+3=4}$$

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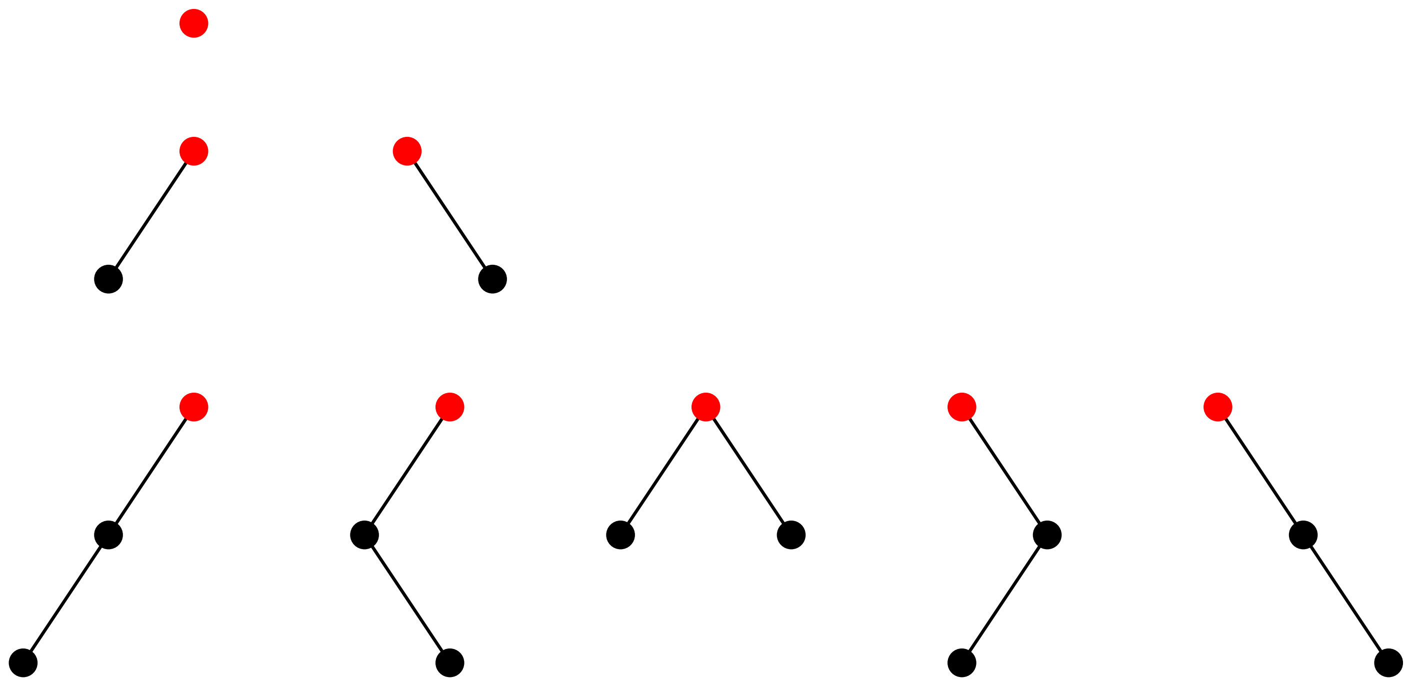

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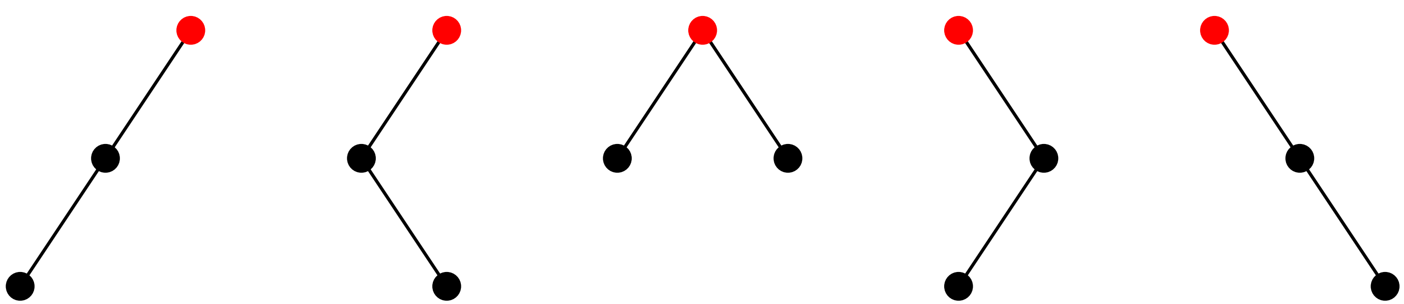

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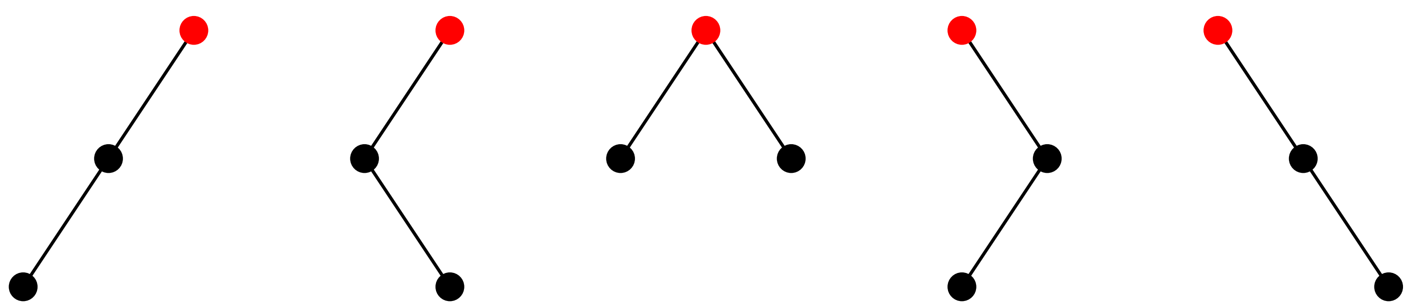

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How to find a formula for C_n?

Catalan Numbers

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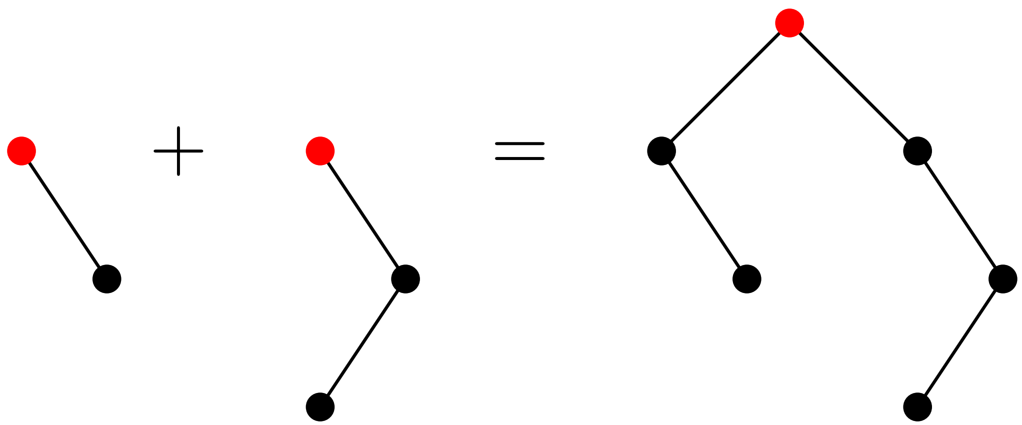

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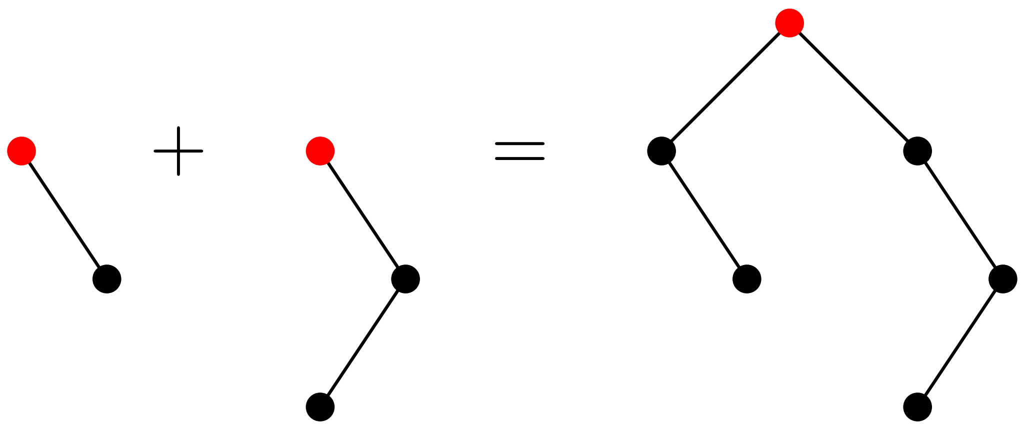

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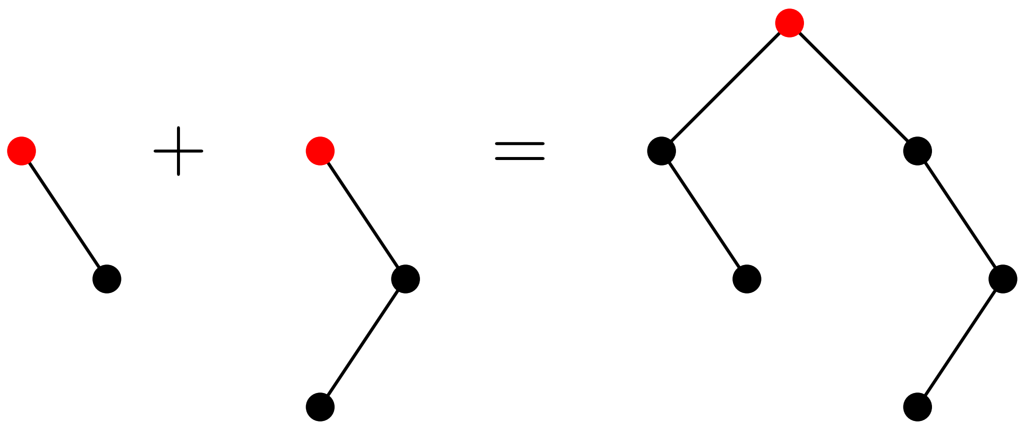

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For example, $C_3 = C_0 C_2 + C_1 C_1 + C_2 C_0 = 1 * 2 + 1 * 1 + 2 * 1 = 5.$

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Catalan Numbers

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Then we have $C_n = \frac{1}{n+1}\binom{2n}{n}$.

This is called the n-th *Catalan number*.

- **Theorem** The number of sequences $a_1, \ldots, a_{2n}$ of $2n$ terms that can be formed using exactly n $+1$'s and exactly n -1's whose partial sums are always nonnegative, i.e., $a_1 + a_2 + \cdots + a_k \geq 0$ for any $1 \leq k \leq 2n$, equals the n-th Catalan number C_n.

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R. Stanley, *Catalan Numbers*, Cambridge University Press, 2015. Includes 214 combinatorial interpretations of C_n, and 68 additional problems!

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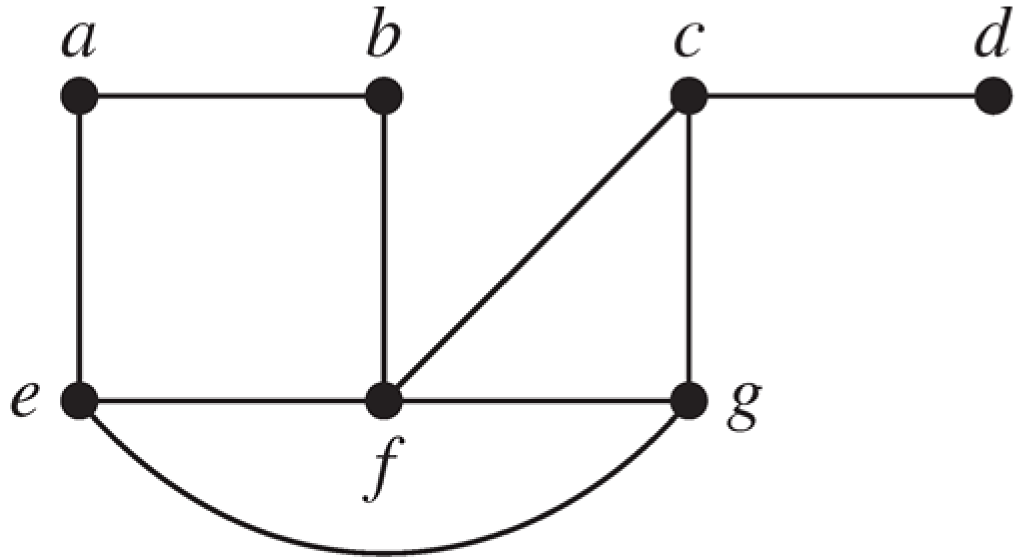

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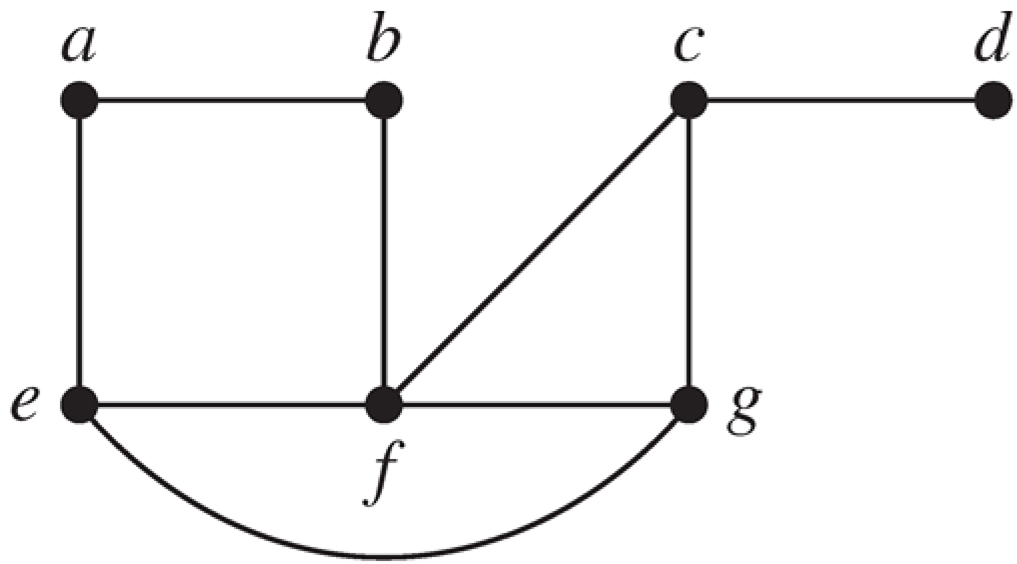

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  ◇ Otherwise, move back to some vertex to repeat this procedure (*backtracking*)

Depth-First Search

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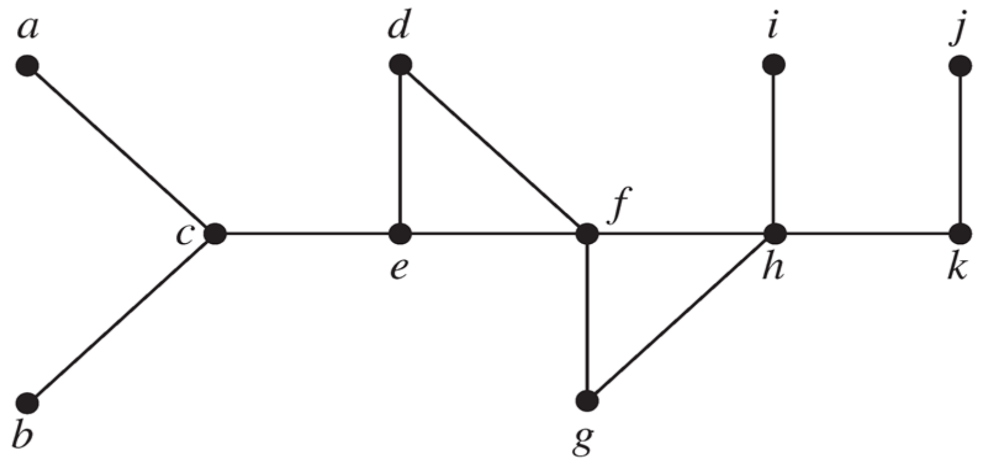

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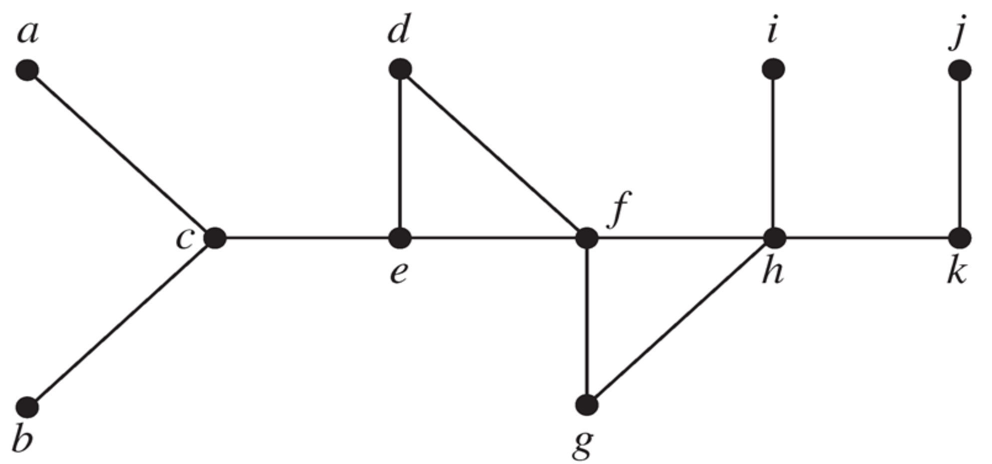

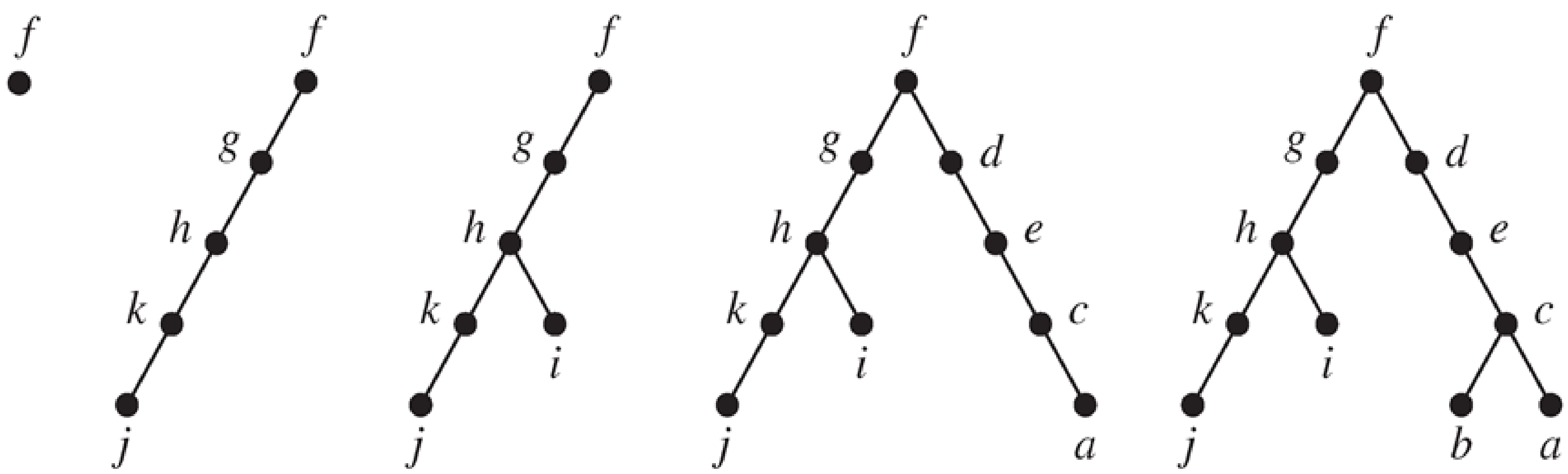

Depth-First Search Algorithm

procedure *DFS*(G: connected graph with vertices $v_1, v_2, \ldots, v_n$)
T := tree consisting only of the vertex v_1
visit(v_1)

procedure *visit*(v: vertex of G)
for each vertex w adjacent to v and not yet in T
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time complexity: $O(e)$

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  ◇ Continue in this manner until all vertices have been added.

Breadth-First Search

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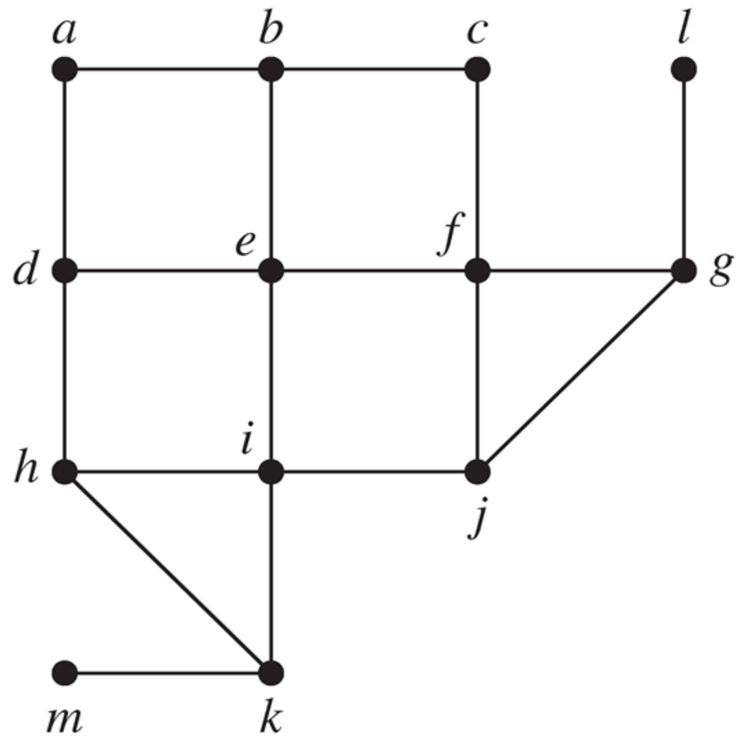

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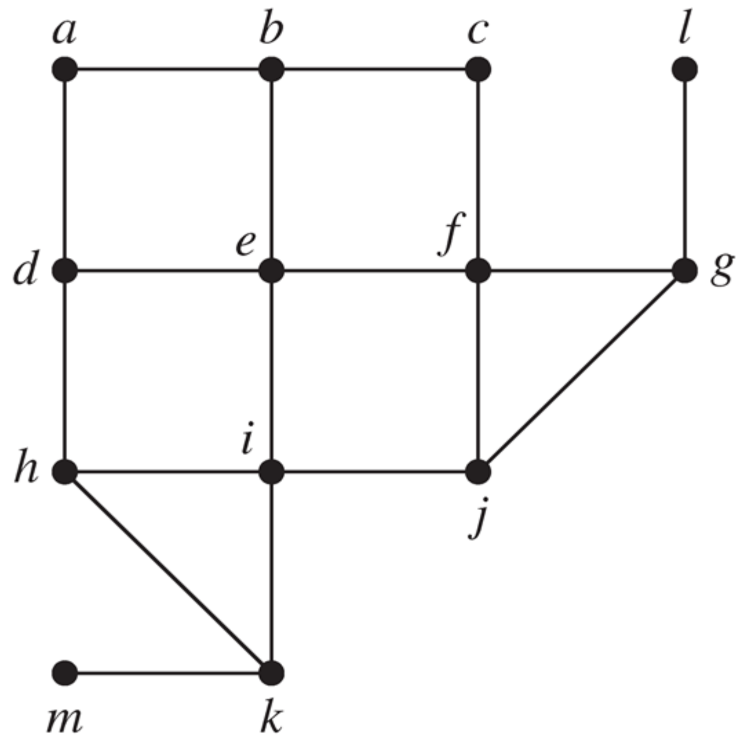

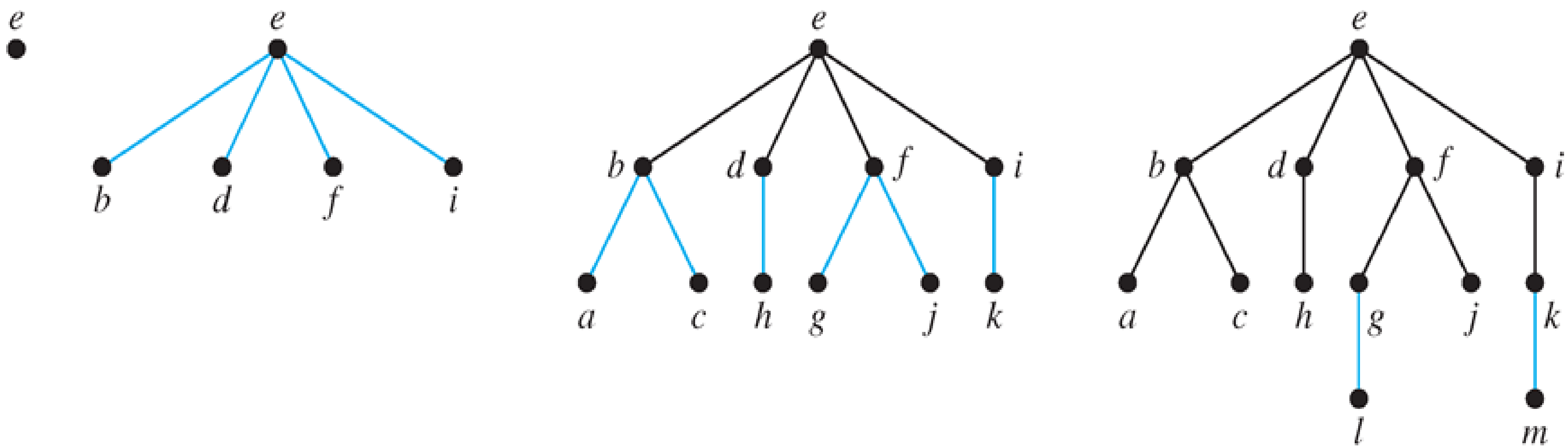

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put v_1 in the list L of unprocessed vertices
while L is not empty
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  graph coloring, sums of subsets, ...

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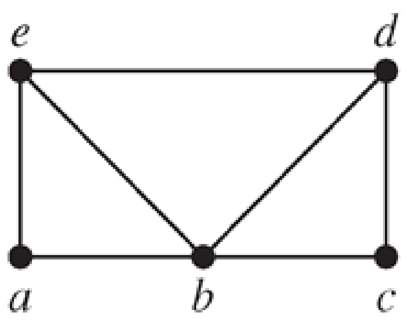

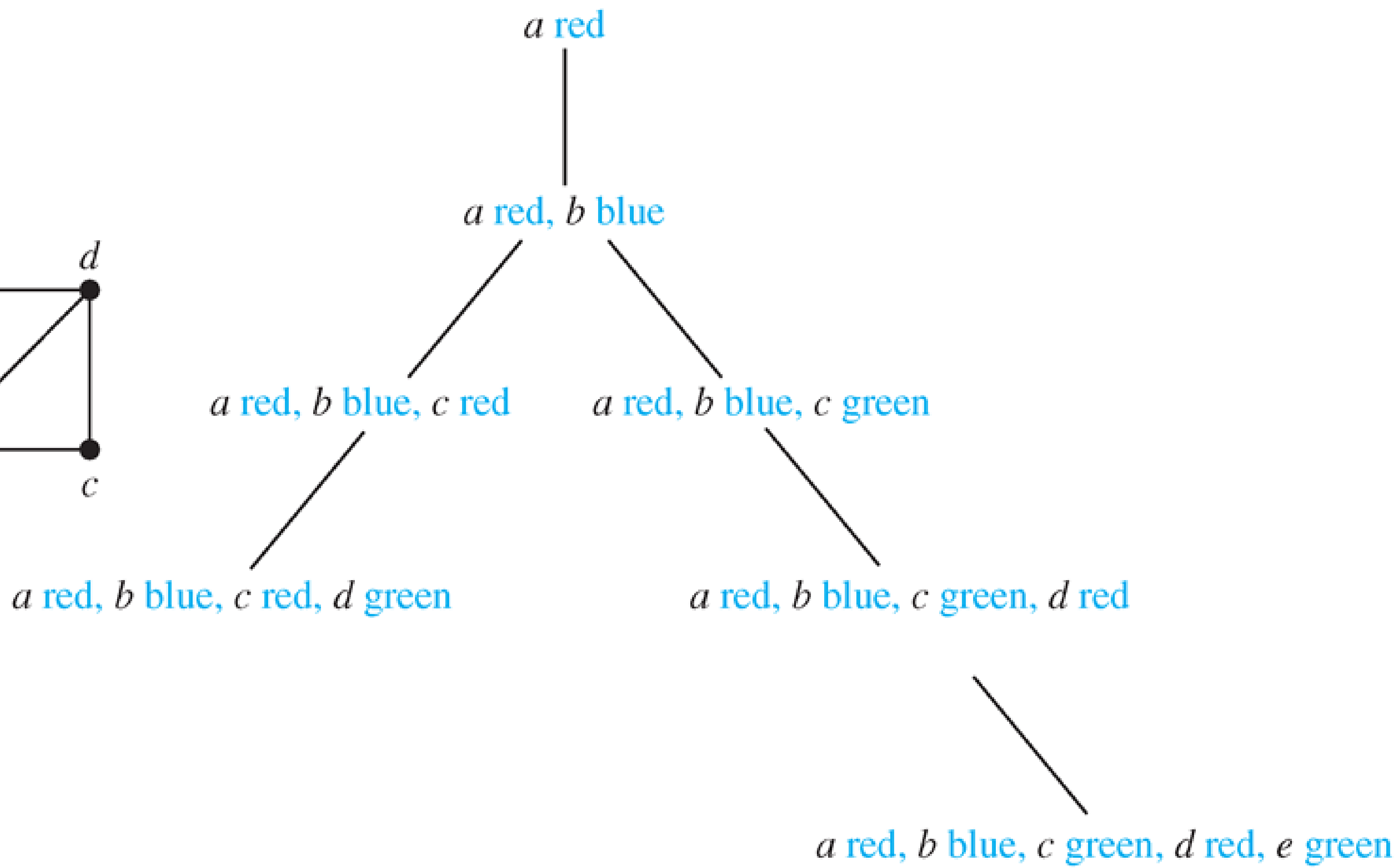

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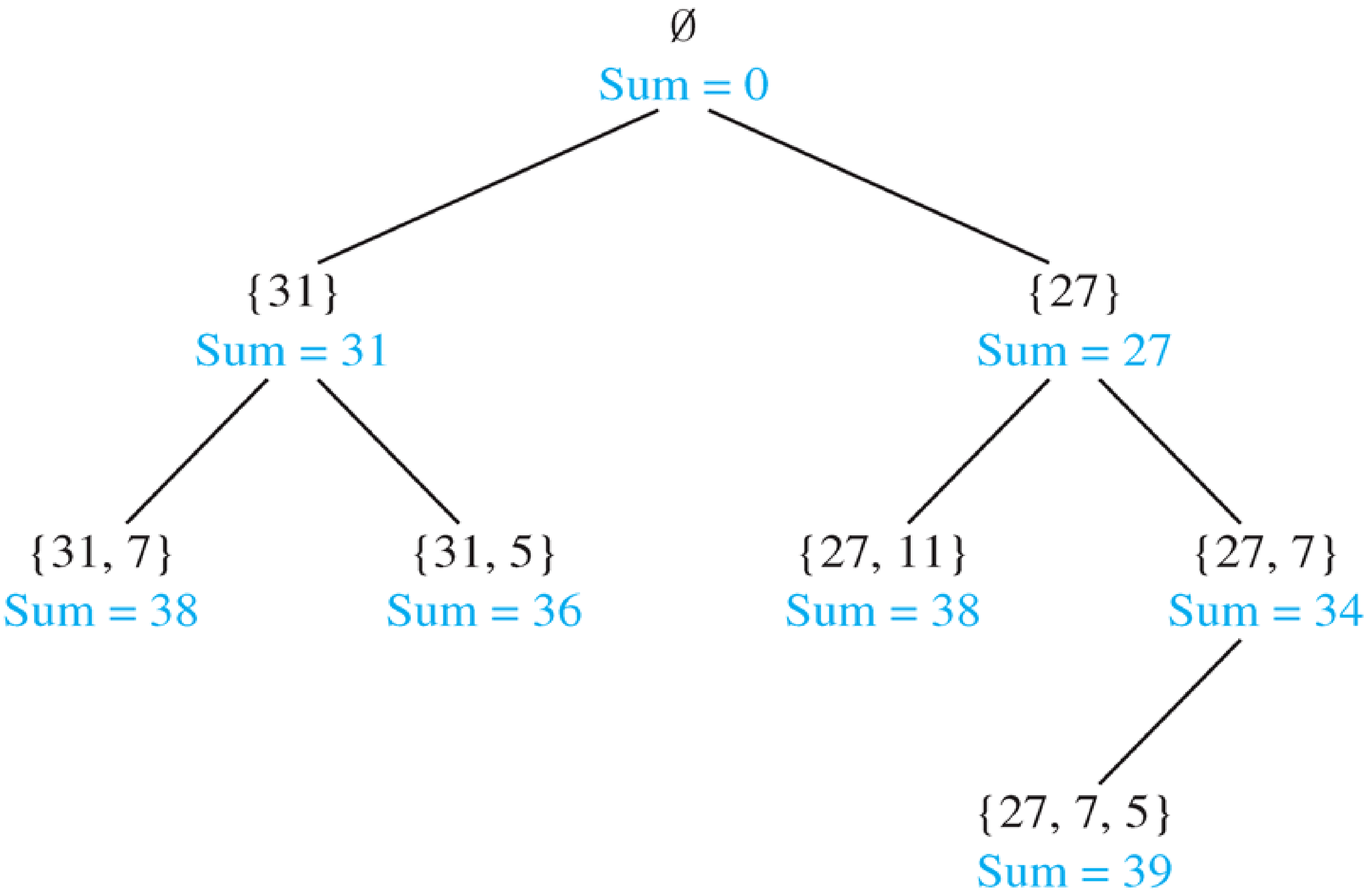
  find

  grap



find a subset of $\{31, 27, 15, 11, 7, 5\}$ with the sum 39

Minimum Spanning Trees

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two greedy algorithms:
Prim's Algorithm, Kruscal's Algorithm

Prim's Algorithm

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procedure *Prim*(G: weighted connected undirected graph with n vertices)
T := a minimum-weight edge
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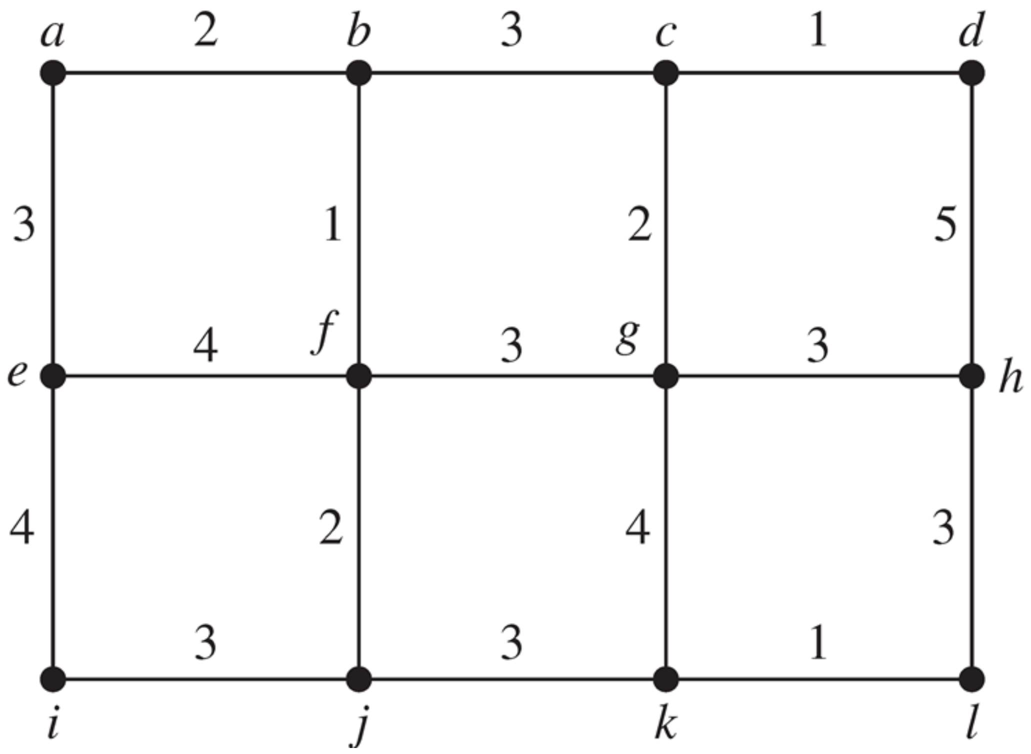
We can maintain a *heap* of all the edges with at least one endpoint in T, and in each iteration, we do Extract-Mins until we see an edge that has one endpoint in T and one endpoint not in T.

time complexity: $e \log v$

Prim's Algorithm

- **Example**



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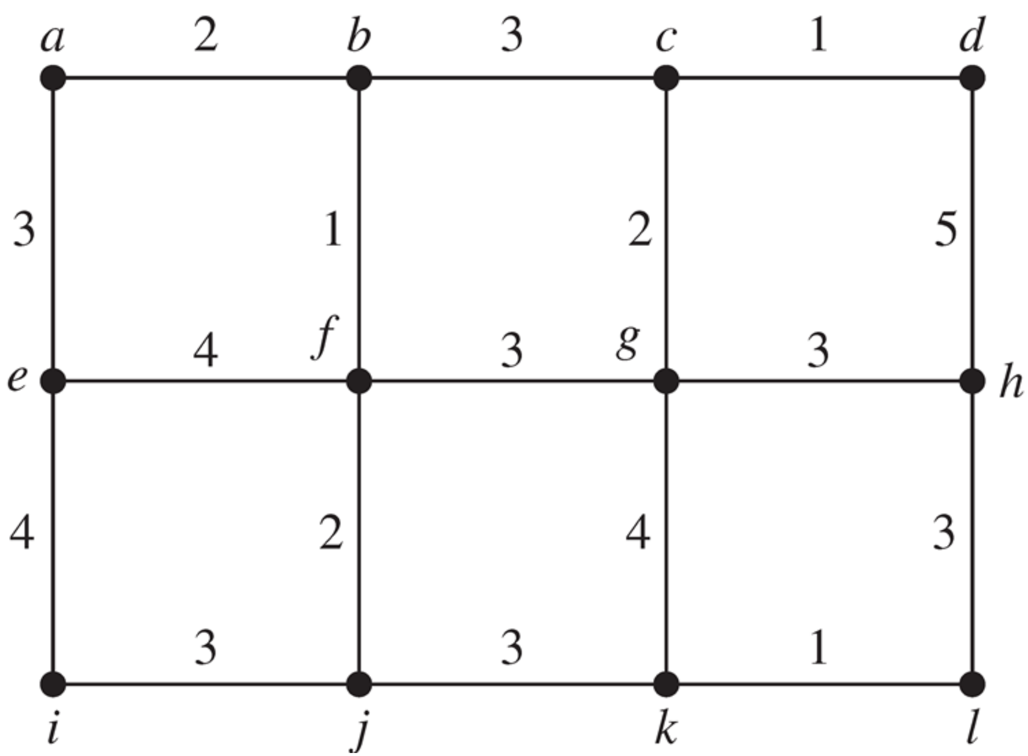

Choice	Edge	Weight
1	$\{b, f\}$	1
2	$\{a, b\}$	2
3	$\{f, j\}$	2
4	$\{a, e\}$	3
5	$\{i, j\}$	3
6	$\{f, g\}$	3
7	$\{c, g\}$	2
8	$\{c, d\}$	1
9	$\{g, h\}$	3
10	$\{h, l\}$	3
11	$\{k, l\}$	1
	Total:	24

Prim's Algorithm: Correctness

- **Proof** by *induction*.

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  Since Prim's algorithm has chosen to add e, we have $w(e) \leq w(e')$. So if we add e to M and remove e' from M, we will have a new tree M' whose total weight $\leq$ that of M, and $T \cup \{e\} \subset M'$.

Kruscal's Algorithm

ALGORITHM 2 Kruskal's Algorithm.

procedure *Kruskal*(G: weighted connected undirected graph with n vertices)
T := empty graph
for $i := 1$ **to** $n - 1$
    e := any edge in G with smallest weight that does not form a simple circuit
        when added to T
    $T := T$ with e added
return T {T is a minimum spanning tree of G}

ALGORITHM 2 Kruskal's Algorithm.

procedure *Kruskal*(G: weighted connected undirected graph with n vertices)
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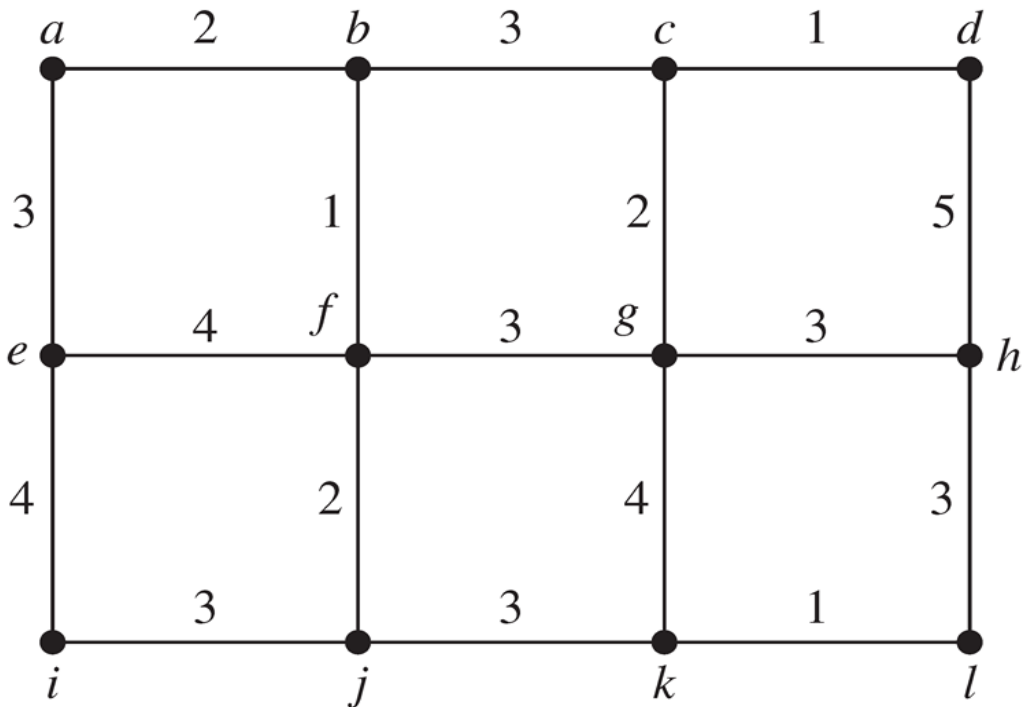
see *CLRS* / *Algorithm Design*, J. Kleinberg, E. Tardos

- **Example**



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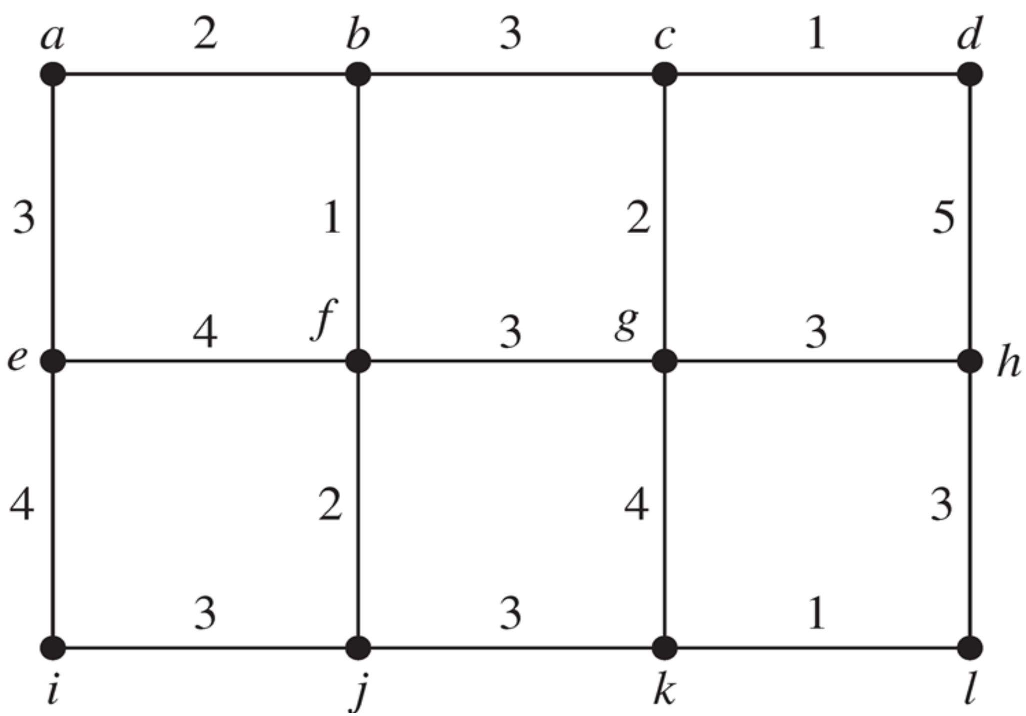

Choice	Edge	Weight
1	$\{c, d\}$	1
2	$\{k, l\}$	1
3	$\{b, f\}$	1
4	$\{c, g\}$	2
5	$\{a, b\}$	2
6	$\{f, j\}$	2
7	$\{b, c\}$	3
8	$\{j, k\}$	3
9	$\{g, h\}$	3
10	$\{i, j\}$	3
11	$\{a, e\}$	3
	Total:	24

Kruscal's Algorithm: Correctness

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  Theorem Let $(S, \bar{S})$ be an arbitrary cut, and let e be an edge across the cut (one endpoint in S, the other in $\bar{S}$) that has the smallest weight of all edges cross the cut. Then there must be an MST T containing e.

  Theorem Let $(S, \bar{S})$ be an arbitrary cut, and let E' be the set of edges across the cut of minimum weight ($w(e) = w(e')$ for any two edges $e, e' \in E'$ and $w(e) < w(e')$ for any $e \in E'$ and $e' \notin E'$). Let T be an arbitrary MST. Then T must contain some edge in E'.

Next Lecture

- reduction, review ...