Department of Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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Recursion

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- A classical example of *recursion* is the **Towers of Hanoi** Problem.

Towers of Hanoi

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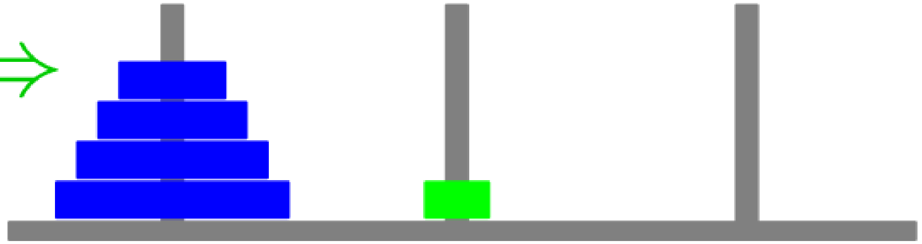
- 3 pegs; n disks of different sizes
- A *legal move* takes a disk from one peg and moves it onto another peg so that it is not on top of a smaller disk
- **Problem**: Find an (efficient) way to move all of the disks from one peg to another

Towers of Hanoi

Towers of Hanoi

legal move

$\Rightarrow$ 

Towers of Hanoi

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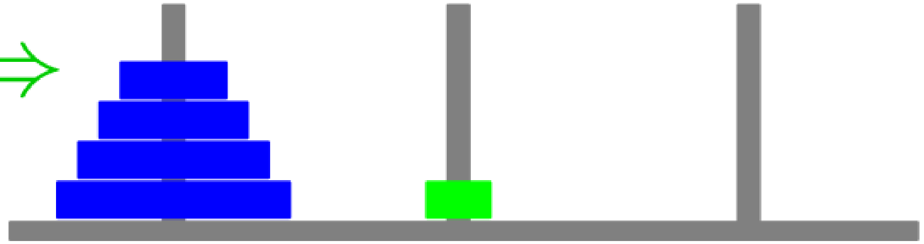

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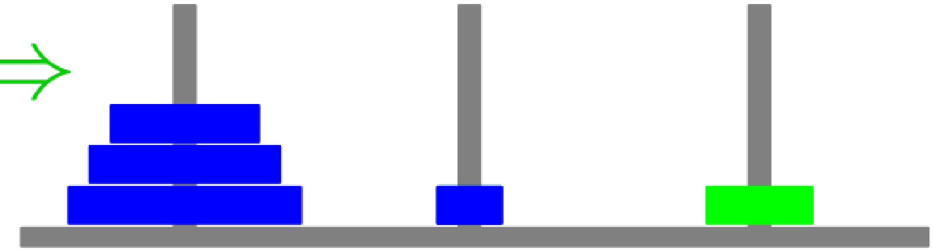

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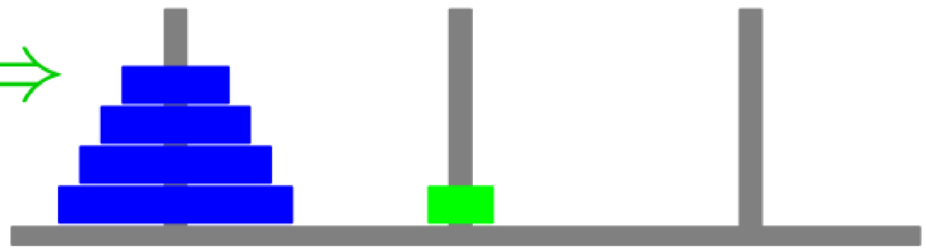

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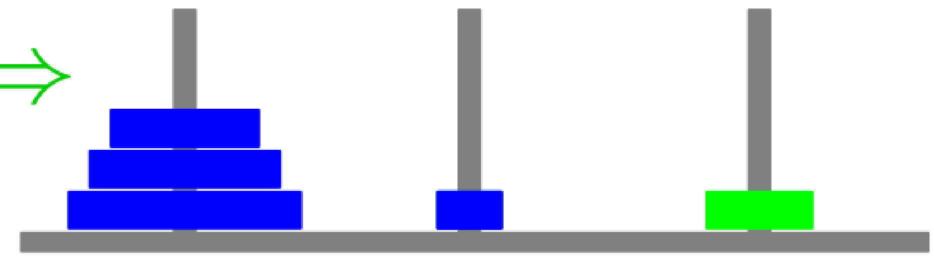

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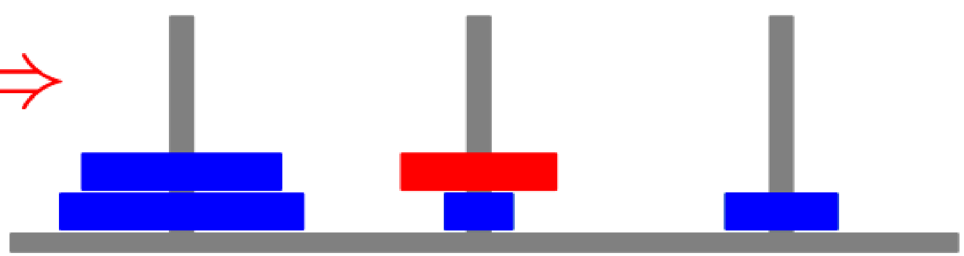

Towers of Hanoi

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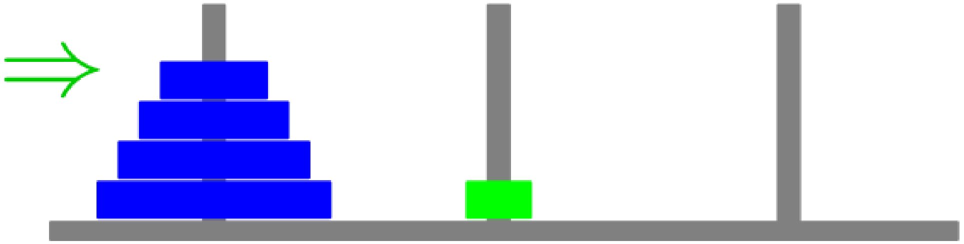

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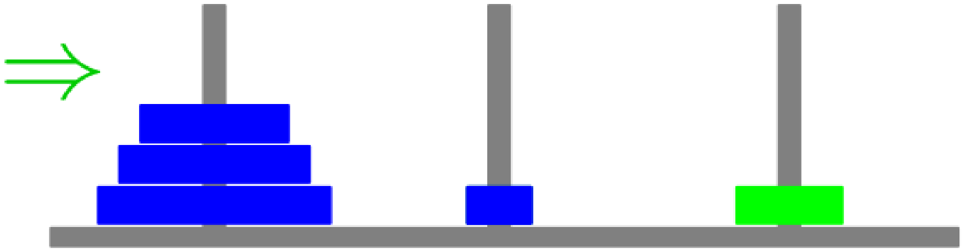

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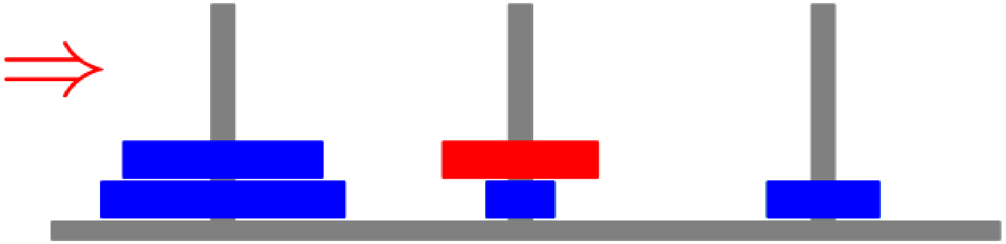

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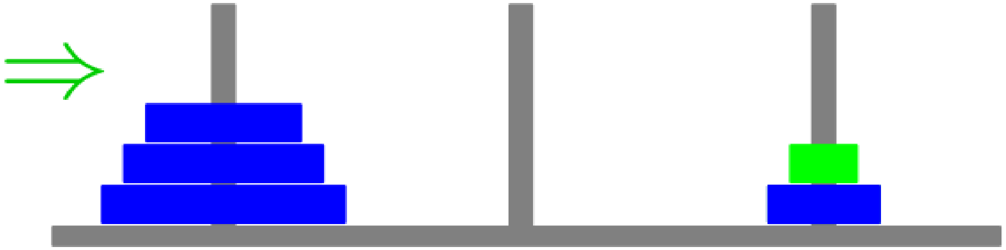

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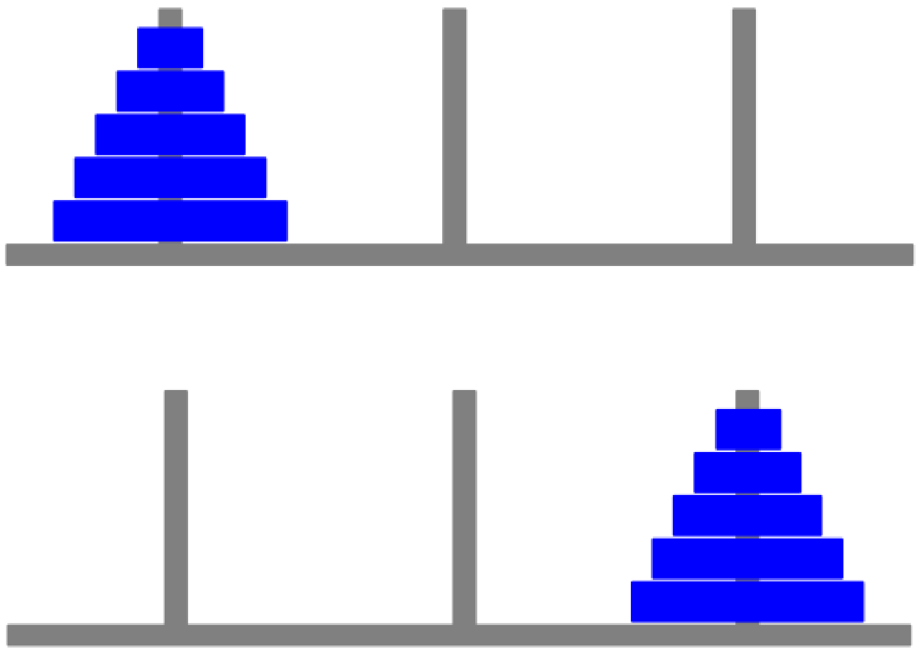

Towers of Hanoi

- **Problem:** Start with n disks on leftmost peg

  using only legal moves

  move all disks to rightmost peg.

Given $i, j \in \{1, 2, 3\}$, let $\overline{\{i, j\}} = \{1, 2, 3\} - \{i\} - \{j\}$, i.e., $\overline{\{1, 2\}} = \{3\}$, $\overline{\{1, 3\}} = \{2\}$, $\overline{\{2, 3\}} = \{1\}$.

Towers of Hanoi

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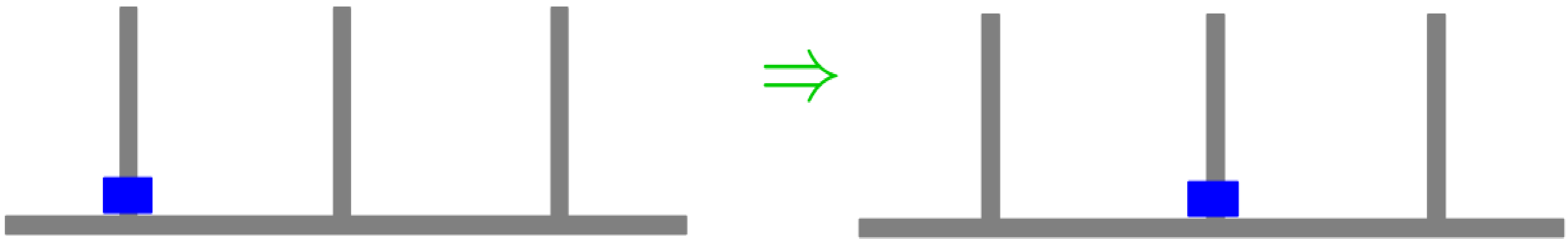

Towers of Hanoi

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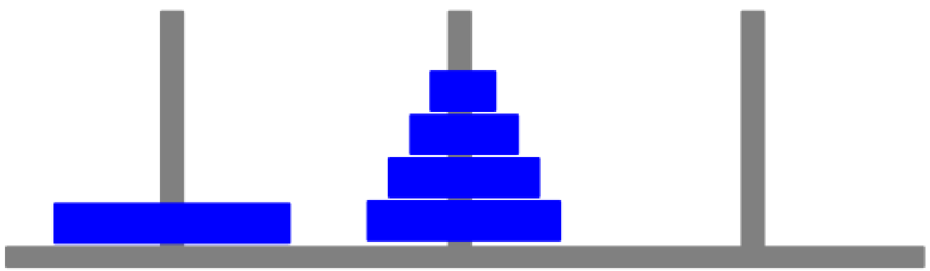

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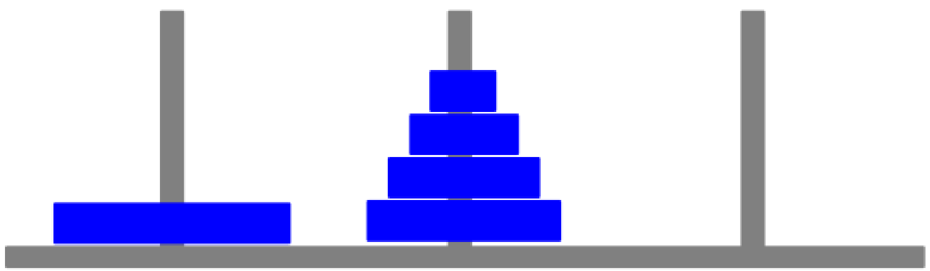

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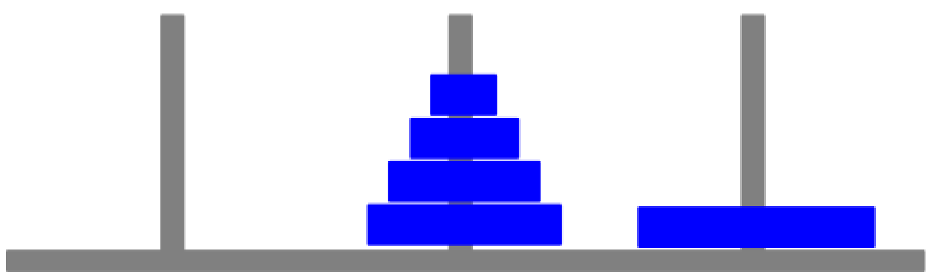

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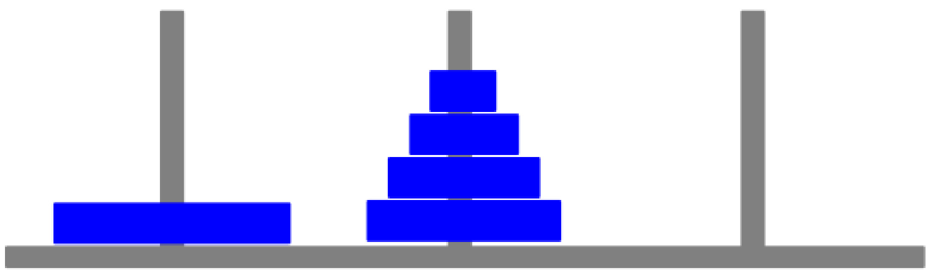

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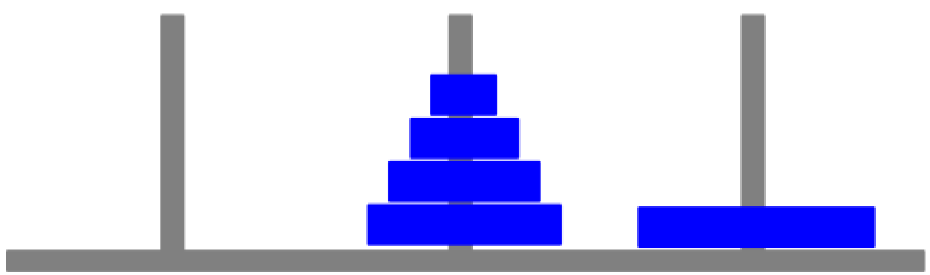

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3)

move top $n-1$ disks from $\overline{\{i,j\}}$ to j

Towers of Hanoi

```java
public class Hanoi
{

    public void move(int n, char a, char b, char c)
    {
        if (n == 1)
            System.out.println("plate " + n + " from " + a + " to " + c);
        else
        {
            move(n-1,a,c,b);
            System.out.println("plate " + n + " from " + a + " to " + c);
            move(n-1,b,a,c);
        }

    }

```

Towers of Hanoi

To move n disks from i to j

i) move top $n-1$ disks from i to $\overline{\{i,j\}}$

ii) move largest disk from i to j

iii) move top $n-1$ disks from $\overline{\{i,j\}}$ to j

Towers of Hanoi

- To prove correctness of solution, we are implicitly using induction

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- $p(1)$ is statement that algorithm works for $n = 1$ disks, which is obviously true
- $p(n-1) \to p(n)$ is *recursion* statement that

  if our algorithm works for $n-1$ disks, then we can build a correct solution for n disks

Towers of Hanoi

- Running time

  $M(n)$ is number of disk moves needed for n disks

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$M(1) = 1$

if $n > 1$, then $M(n) = 2M(n-1) + 1$

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  Later, we'll also see how to solve without guessing

Towers of Hanoi

- Formally, given

$$M(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2M(n-1) + 1 & \text{otherwise} \end{cases}$$

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The base case $n = 1$ is true, since $2^1 - 1 = 1$.

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Then $M(n) = 2M(n-1) + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 1$

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$$M(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2M(n-1) + 1 & \text{otherwise} \end{cases}$$

- The second time was to derive the closed form solution $M(n) = 2^n - 1$ of the recurrence.

Recurrences

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Towers of Hanoi

Fibonacci Sequence

$$F(n) = \begin{cases} 1 & \text{if } n = 0, 1 \\ F(n-1)+F(n-2) & \text{otherwise} \end{cases}$$

- **Example 2**: Let $S(n)$ be the number of subsets of a set of size n. What is the formula for $S(n)$?

  The empty set, of size $n = 0$ has only one subset (itself), so $S(0) = 1$.

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  We "guess" that $S(n) = 2^n$. But, in order to prove formula, we'll need to think recursively.

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  This suggests that the recurrence for the number of subsets of an n-element set $\{1,2,\ldots,n\}$ is

  $$S(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2S(n-1) & \text{if } n \geq 1 \end{cases}$$

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  Proof by induction is easy.

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  Can we generalize this to find a closed-form solution?

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T(n) &= rT(n-1) + a \\
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&= r^2T(n-2) + ra + a \\
&= r^2(rT(n-3) + a) + ra + a \\
&= r^3T(n-3) + r^2a + ra + a \\
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&= r^3(rT(n-4) + a) + r^2 a + ra + a \\
&= r^4 T(n-4) + r^3 a + r^2 a + ra + a.
\end{aligned}$$

Guess $T(n) = r^n T(0) + a \sum_{i=0}^{n-1} r^i$

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  $$T(1) = rT(0) + a = rb + a$$
  $$T(2) = rT(1) + a = r(rb + a) + a = r^2 b + ra + a$$
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\end{aligned}
$$

  This would lead to the same guess

$$T(n) = r^n b + a \sum_{i=0}^{n-1} r^i.$$

Formula of Recurrences

- **Theorem** If $T(n) = rT(n-1) + a$, $T(0) = b$, and $r \neq 1$, then

$$T(n) = r^n b + a\frac{1-r^n}{1-r}$$

for all nonnegative integers n.

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Proof by induction

The base case:

$$T(0) = r^0 b + a\frac{1 - r^0}{1 - r} = b.$$

So the formula is true when $n = 0$.

Now assume that $n > 0$ and

$$T(n-1) = r^{n-1} b + a\frac{1 - r^{n-1}}{1 - r}.$$

- **Proof by induction**

$$
\begin{aligned}
T(n) &= rT(n-1) + a \\
&= r\left(r^{n-1}b + a\frac{1-r^{n-1}}{1-r}\right) + a \\
&= r^n b + \frac{ar - ar^n}{1-r} + a \\
&= r^n b + \frac{ar - ar^n + a - ar}{1-r} \\
&= r^n b + a\frac{1-r^n}{1-r}.
\end{aligned}
$$

- **Theorem** If $T(n) = rT(n-1) + a$, $T(0) = b$, and $r \neq 1$, then

$$T(n) = r^n b + a\frac{1 - r^n}{1 - r}$$

for all nonnegative integers n.

Formula of Recurrences

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Example:
$T(n) = 3T(n-1) + 2$ with $T(0) = 5$

Plugging $r = 3, a = 2, b = 5$ in the formula, gives

$$T(n) = 3^n \cdot 5 + 2\frac{1 - 3^n}{1 - 3} = 3^n \cdot 6 - 1$$

First-Order Linear Recurrences

- A recurrence of the form $T(n) = f(n)T(n-1) + g(n)$ is called a *first-order linear recurrence*.

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  Something like $T(n) = (T(n-1))^2 + 3$ would be a non-linear first-order recurrence relation.

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- $T(n) = f(n)T(n-1) + g(n)$

  When $f(n)$ is a constant, say r, the general solution is almost as easy as we derived before. Iterating the recurrence gives

$$
\begin{aligned}
T(n) &= rT(n-1) + g(n) \\
&= r(rT(n-2) + g(n-1)) + g(n) \\
&= r^2 T(n-2) + rg(n-1) + g(n) \\
&= r^3 T(n-3) + r^2 g(n-2) + rg(n-1) + g(n) \\
&\vdots \\
&= r^n T(0) + \sum_{i=0}^{n-1} r^i g(n-i)
\end{aligned}
$$

First-Order Linear Recurrences

- **Theorem** For any positive constants a and r, and any function g defined on nonnegative integers, the solution to the first-order linear recurrence

$$T(n) = \begin{cases} rT(n-1) + g(n) & \text{if } n > 0 \\ a & \text{if } n = 0 \end{cases}$$

is

$$T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i).$$

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$$
\begin{aligned}
T(n) &= 6 \cdot 4^n + \sum_{i=1}^{n} 4^{n-i} \cdot 2^i \\
&= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} 4^{-i} \cdot 2^i \\
&= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} (\frac{1}{2})^i \\
&= 6 \cdot 4^n + (1 - \frac{1}{2^n}) \cdot 4^n \\
&= 7 \cdot 4^n - 2^n.
\end{aligned}
$$

Examples

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Examples

- Solve $T(n) = 3T(n-1) + n$ with $T(0) = 10$

$$\begin{aligned} T(n) &= 10 \cdot 3^n + \sum_{i=1}^{n} 3^{n-i} \cdot i \\ &= 10 \cdot 3^n + 3^n \sum_{i=1}^{n} i \cdot 3^{-i} \end{aligned}$$

Theorem. For any real number $x \neq 1$,

$$\sum_{i=1}^{n} ix^i = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(1-x)^2}.$$

Examples

- Solve $T(n) = 3T(n-1) + n$ with $T(0) = 10$

$$
\begin{aligned}
T(n) &= 10 \cdot 3^n + \sum_{i=1}^{n} 3^{n-i} \cdot i \\
&= 10 \cdot 3^n + 3^n \sum_{i=1}^{n} i \cdot 3^{-i} \\
&= 10 \cdot 3^n + 3^n \left(-\frac{3}{2}(n+1)3^{-(n+1)} - \frac{3}{4}3^{-(n+1)} + \frac{3}{4} \right. \\
&= \frac{43}{4}3^n - \frac{n+1}{2} - \frac{1}{4}.
\end{aligned}
$$

Growth Rates of Solutions to Recurrences

- Divide and conquer algorithms

- Iterating recurrences

- Three different behaviors

Divide and conquer algorithms

- We just analyzed recurrences of the form

$$T(n) = \begin{cases} b & \text{if } n = 0 \\ r \cdot T(n-1) + a & \text{if } n > 0 \end{cases}$$

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These corresponded to the analysis of recursive algorithms in which a problem of size n is solved by recursively solving a problem of size $n-1$.

$$T(n) = r^n b + a\frac{1-r^n}{1-r}$$

We will now look at recurrences of the form

$$T(n) = \begin{cases} \text{something given} & \text{if } n \le n_0 \\ r \cdot T(n/m) + a & \text{if } n > n_0 \end{cases}$$

Binary Search

- Someone has chosen a number x between 1 and n. We need to discover x.

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  We need to discover x.

  We are only allowed to ask two types of questions:

  - ◇ Is x greater than k?
  - ◇ Is x equal to k?

  Our strategy will be to always ask greater than questions, at each step halving our search range, until the range only contains one number, when we ask a final equal to question.

Binary Search Example

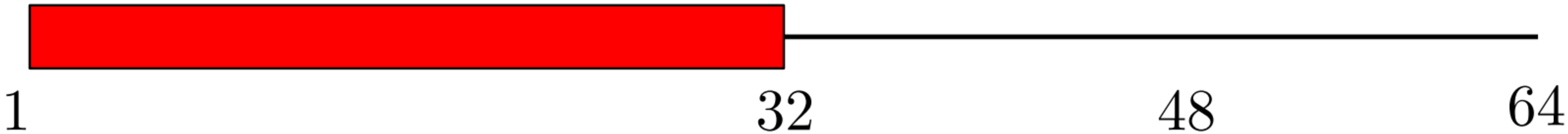
1 32 48 64

Binary Search Example

1 32 48 64

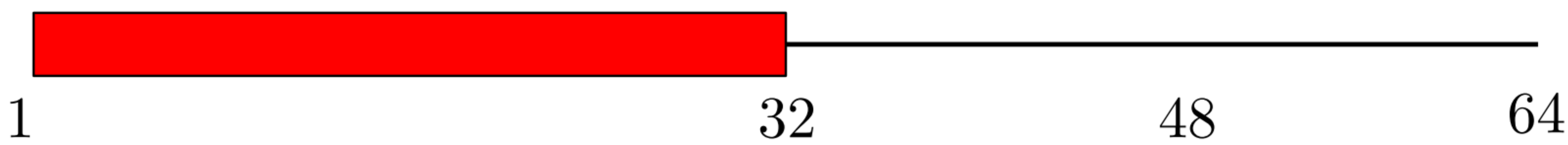
Is $x > 32$?

Binary Search Example



Is $x > 32$? Answer: Yes

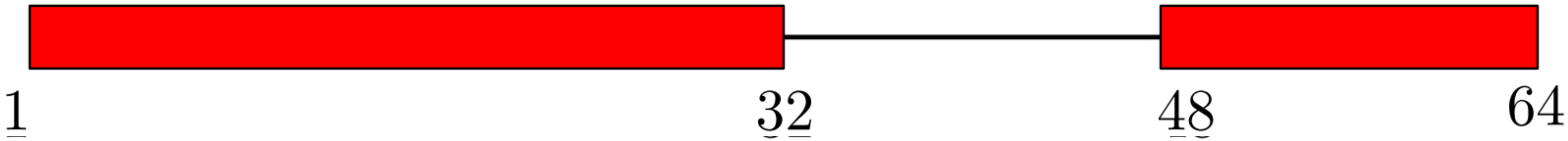
Binary Search Example



Is $x > 32$?   Answer: Yes

Is $x > 48$?

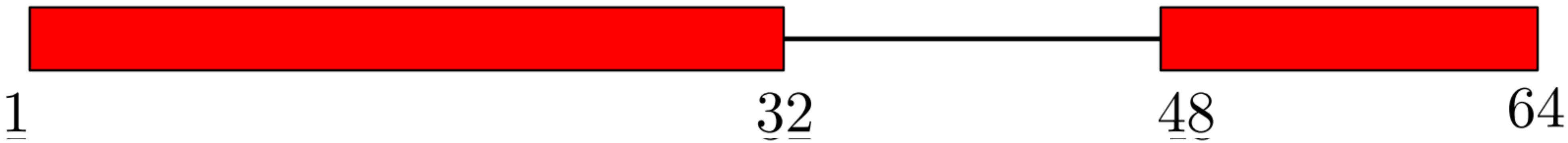
Binary Search Example



Is $x > 32$? Answer: Yes

Is $x > 48$? Answer: No

Binary Search Example

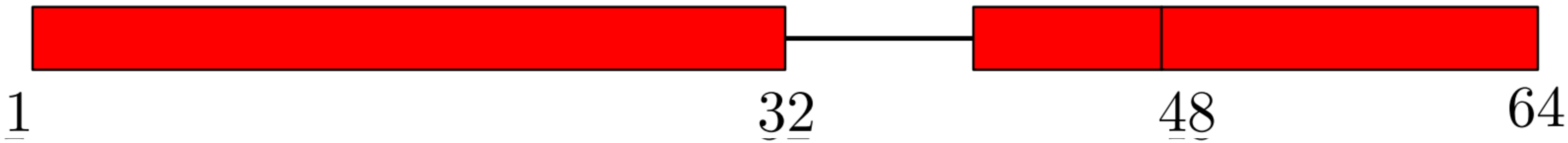


Is $x > 32?$ Answer: Yes

Is $x > 48?$ Answer: No

Is $x > 40?$

Binary Search Example

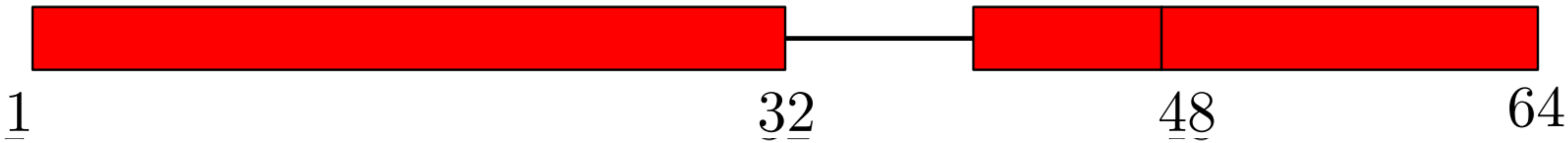


Is $x > 32$? Answer: Yes

Is $x > 48$? Answer: No

Is $x > 40$? Answer: No

Binary Search Example



Is $x > 32?$ Answer: Yes

Is $x > 48?$ Answer: No

Is $x > 40?$ Answer: No

Is $x > 36?$

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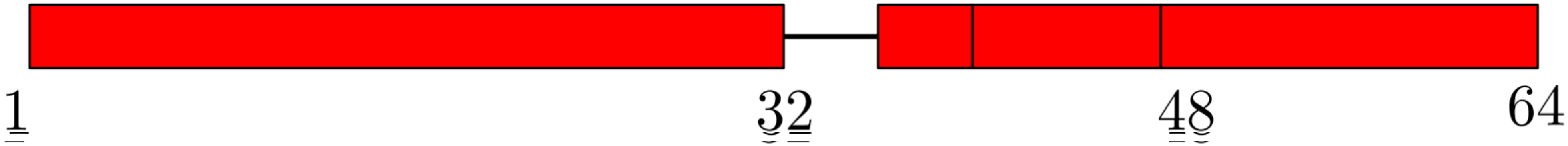

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Is $x > 36$? Answer: No

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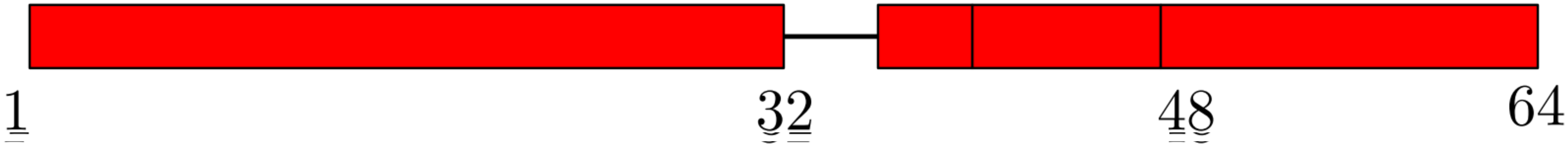

Is $x > 32$? Answer: Yes

Is $x > 48$? Answer: No

Is $x > 40$? Answer: No

Is $x > 36$? Answer: No

Is $x > 34$?

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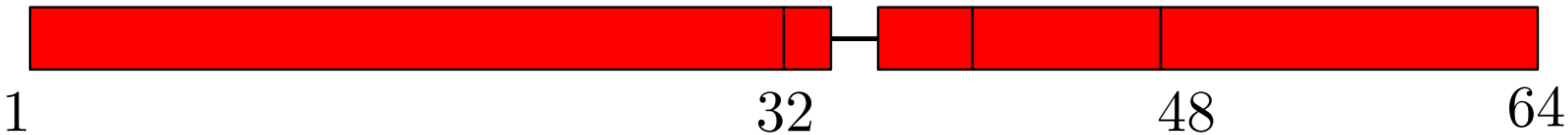

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Is $x > 36$? Answer: No

Is $x > 34$? Answer: Yes

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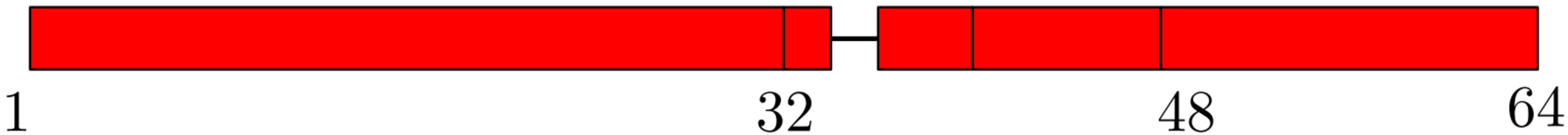

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Is $x > 40$? Answer: No

Is $x > 36$? Answer: No

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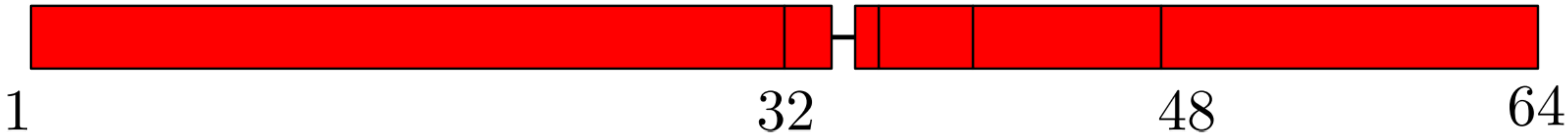


Is $x > 32$? Answer: Yes

Is $x > 48$? Answer: No

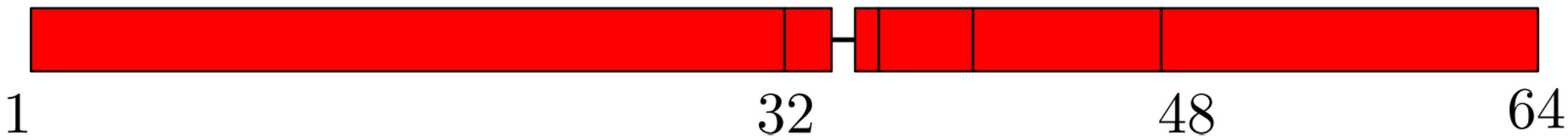
Is $x > 40$? Answer: No

Is $x > 36$? Answer: No

Is $x > 34$? Answer: Yes

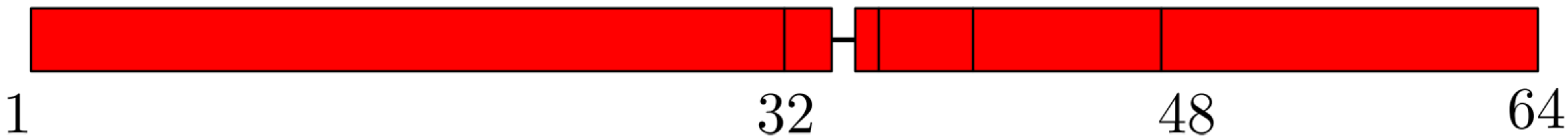
Is $x > 35$? Answer: No

Binary Search Example



Is $x > 32$?	Answer: Yes
Is $x > 48$?	Answer: No
Is $x > 40$?	Answer: No
Is $x > 36$?	Answer: No
Is $x > 34$?	Answer: Yes
Is $x > 35$?	Answer: No
Is $x = 35$?	

Binary Search Example



Is $x > 32$? Answer: Yes

Is $x > 48$? Answer: No

Is $x > 40$? Answer: No

Is $x > 36$? Answer: No

Is $x > 34$? Answer: Yes

Is $x > 35$? Answer: No

Is $x = 35$? Answer: BINGO!

Binary Search Example

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  Note: When n is a power of 2, $T(n)$, the number of questions in a binary search on $[1, n]$, satisfies

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

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  This can also be proved inductively, similar to the tower of Hanoi recurrence.

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  first step

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  time to perform binary search on the remaining $n/2$ items

  Base case (1 item): $T(1) = 1$ to ask: "Is the number k?"

Binary Search Example

- $$(*) \qquad T(n) = \begin{cases} C_1 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + C_2 & \text{if } n \geq 2 \end{cases}$$

  For simplicity, we will (usually) assume that n is a power of 2 (or sometimes 3 or 4) and also often that constants such as C_1, C_2 are 1. This will let us replace a recurrence such as (*) by one such as (**).

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  $$(**) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

  In practice, the solution of (*) will be very close to that of (**) (this can be proved mathematically). Hence, we can restrict attention to (**).

Growth Rates of Solutions to Recurrences

- Divide and conquer algorithms
- Iterating recurrences
- Three different behaviors

Iterating Recurrences: Example 1

- $$(*) \qquad T(n) = \begin{cases} T(1) & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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This corresponds to solving a problem of size n, by

(i) solving 2 subproblems of size $n/2$ and
(ii) doing n units of additional work

or using $T(1)$ work for "bottom" case of $n = 1$

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  We now see how to solve (*) by algebraically iterating the recurrence.

Iterating Recurrences: Example 1

- Algebraically iterating the recurrence

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Iterating Recurrences: Example 1

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  Assume that n is a power of 2

$$T(n) = 2T\left(\frac{n}{2}\right) + n \qquad = 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$$

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$$T(n) = 2T\left(\frac{n}{2}\right) + n \qquad = 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$$

$$= 4T\left(\frac{n}{4}\right) + 2n \qquad = 4\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + 2n$$

Iterating Recurrences: Example 1

- Algebraically iterating the recurrence

  Assume that n is a power of 2

$$
\begin{aligned}
{\color{blue} T(n) = 2T\left(\tfrac{n}{2}\right) + n} \quad &= 2\left(2T\left(\tfrac{n}{4}\right) + \tfrac{n}{2}\right) + n \\
= 4T\left(\tfrac{n}{4}\right) + 2n \quad &= 4\left(2T\left(\tfrac{n}{8}\right) + \tfrac{n}{4}\right) + 2n \\
= 8T\left(\tfrac{n}{8}\right) + 3n &
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$$

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\begin{aligned}
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= 4T\left(\frac{n}{4}\right) + 2n \quad & = 4\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + 2n \\
= 8T\left(\frac{n}{8}\right) + 3n & \\
\vdots \qquad \vdots & \\
= 2^i T\left(\frac{n}{2^i}\right) + in &
\end{aligned}
$$

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  Assume that n is a power of 2

$$
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\vdots \qquad \vdots & \\
= 2^i T\left(\tfrac{n}{2^i}\right) + in & \\
\vdots \qquad \vdots & \\
= 2^{\log_2 n} T\left(\tfrac{n}{2^{\log_2 n}}\right) + (\log_2 n)n &
\end{aligned}
$$

End when $i = \log_2 n$

Iterating Recurrences: Example 1

- Algebraically iterating the recurrence

  Assume that n is a power of 2

$$T(n) = 2T\left(\frac{n}{2}\right) + n \qquad = 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$$

$$= 4T\left(\frac{n}{4}\right) + 2n \qquad = 4\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + 2n$$

$$= 8T\left(\frac{n}{8}\right) + 3n$$

$$\vdots \qquad \vdots$$

$$= 2^i T\left(\frac{n}{2^i}\right) + in$$

End when $i = \log_2 n$

$$\vdots \qquad \vdots$$

$$= 2^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + (\log_2 n)n$$

$$= nT(1) + n\log_2 n$$

Iterating Recurrences: Example 1

- We just iterated the recurrence to derive that the solution to

$$(*) \qquad T(n) = \begin{cases} T(1) & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

is $nT(1) + n\log_2 n$.

Iterating Recurrences: Example 1

- We just iterated the recurrence to derive that the solution to

$$(*) \qquad T(n) = \begin{cases} T(1) & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

is $nT(1) + n \log_2 n$.

Note: Technically, we still need to use **induction** to prove that our solution is correct. Practically, we never explicitly perform this step, since it is obvious how the induction would work.

Iterating Recurrences: Example 2

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

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$$T(n) = T\left(\frac{n}{2}\right) + 1$$

Iterating Recurrences: Example 2

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

$$T(n) = T\left(\frac{n}{2}\right) + 1 \qquad = \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1$$

Iterating Recurrences: Example 2

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

$$T(n) = T\left(\frac{n}{2}\right) + 1 \qquad = \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1$$

$$= T\left(\frac{n}{2^2}\right) + 2$$

Iterating Recurrences: Example 2

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + 1 & &= \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1 \\ &= T\left(\frac{n}{2^2}\right) + 2 & &= \left(T\left(\frac{n}{2^3}\right) + 1\right) + 2 \end{aligned}$$

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$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + 1 & &= \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1 \\ &= T\left(\frac{n}{2^2}\right) + 2 & &= \left(T\left(\frac{n}{2^3}\right) + 1\right) + 2 \\ &= T\left(\frac{n}{2^3}\right) + 3 & & \end{aligned}$$

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- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

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T(n) &= T\left(\tfrac{n}{2}\right) + 1 & &= \left(T\left(\tfrac{n}{2^2}\right) + 1\right) + 1 \\
&= T\left(\tfrac{n}{2^2}\right) + 2 & &= \left(T\left(\tfrac{n}{2^3}\right) + 1\right) + 2 \\
&= T\left(\tfrac{n}{2^3}\right) + 3 \\
&\ \ \vdots \qquad \vdots \\
&= T\left(\tfrac{n}{2^i}\right) + i \\
&\ \ \vdots \qquad \vdots \\
&= T\left(\tfrac{n}{2^{\log_2 n}}\right) + \log_2 n
\end{aligned}$$

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- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned}
T(n) &= T\left(\frac{n}{2}\right) + 1 &&= \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1 \\
&= T\left(\frac{n}{2^2}\right) + 2 &&= \left(T\left(\frac{n}{2^3}\right) + 1\right) + 2 \\
&= T\left(\frac{n}{2^3}\right) + 3 \\
&\ \vdots \qquad \vdots \\
&= T\left(\frac{n}{2^i}\right) + i \\
&\ \vdots \qquad \vdots \\
&= T\left(\frac{n}{2^{\log_2 n}}\right) + \log_2 n &&= 1 + \log_2 n
\end{aligned}$$

Iterating Recurrences: Example 3

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + n \\ &= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n \end{aligned}$$

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Iterating Recurrences: Example 3

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + n \\ &= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n \\ &= T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n \\ &\ \vdots \qquad \vdots \\ &= T\left(\frac{n}{2^i}\right) + \frac{n}{2^{i-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\ &\ \vdots \qquad \vdots \\ &= T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{n}{2^{\log_2 n - 1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \end{aligned}$$

Iterating Recurrences: Example 3

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$
\begin{aligned}
T(n) &= T\left(\frac{n}{2}\right) + n \\
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&= T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n \\
&\ \vdots \qquad \vdots \\
&= T\left(\frac{n}{2^i}\right) + \frac{n}{2^{i-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
&\ \vdots \qquad \vdots \\
&= T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{n}{2^{\log_2 n - 1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
&= 1 + 2 + 2^2 + \cdots + \frac{n}{2^2} + \frac{n}{2} + n
\end{aligned}
$$

Iterating Recurrences: Example 3

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned}
T(n) &= T\left(\frac{n}{2}\right) + n \\
&= T\left(\frac{n}{2^2}\right) + \frac{n}{2} + n \\
&= T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n \\
&\ \ \vdots \qquad\quad \vdots \\
&= T\left(\frac{n}{2^i}\right) + \frac{n}{2^{i-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
&\ \ \vdots \qquad\quad \vdots \\
&= T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{n}{2^{\log_2 n - 1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
&= 1 + 2 + 2^2 + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \qquad = \Theta(n)
\end{aligned}$$

Iterating Recurrences: Example 4

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

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$$T(n) = 3T\left(\frac{n}{3}\right) + n \qquad = 3\left(3T\left(\frac{n}{3^2}\right) + \frac{n}{3}\right) + n$$

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$$= 3^2 T\left(\frac{n}{3^2}\right) + 2n$$

Iterating Recurrences: Example 4

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

$$\begin{aligned} T(n) = 3T\left(\tfrac{n}{3}\right) + n \quad & = 3\left(3T\left(\tfrac{n}{3^2}\right) + \tfrac{n}{3}\right) + n \\ = 3^2 T\left(\tfrac{n}{3^2}\right) + 2n \quad & = 3^2\left(3T\left(\tfrac{n}{3^3}\right) + \tfrac{n}{3^2}\right) + 2n \end{aligned}$$

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T(n) &= 3T\left(\frac{n}{3}\right) + n & &= 3\left(3T\left(\frac{n}{3^2}\right) + \frac{n}{3}\right) + n \\
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&= 3^3T\left(\frac{n}{3^3}\right) + 3n \\
&\ \ \vdots \quad \vdots \\
&= 3^iT\left(\frac{n}{3^i}\right) + in \\
&\ \ \vdots \quad \vdots \\
&= 3^{\log_3 n}T\left(\frac{n}{3^{\log_3 n}}\right) + n\log_3 n & &= n + n\log_3 n
\end{aligned}$$

Iterating Recurrences: Example 5

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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$$= 4^3T\left(\frac{n}{2^3}\right) + \frac{4^2}{2^2}n + \frac{4}{2}n + n$$

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$$= 4^3 T\left(\frac{n}{2^3}\right) + \frac{4^2}{2^2}n + \frac{4}{2}n + n$$

$$\vdots \qquad \vdots$$

$$= 4^i T\left(\frac{n}{2^i}\right) + \frac{4^{i-1}}{2^{i-1}}n + \cdots + \frac{4^2}{2^2}n + n$$

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$$\vdots \qquad \vdots$$

$$= 4^{\log_2 n}T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{4^{\log_2 n-1}}{2^{\log_2 n-1}}n + \cdots + \frac{4}{2}n + n$$

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$$\vdots \qquad \vdots$$

$$= 4^i T\left(\frac{n}{2^i}\right) + \frac{4^{i-1}}{2^{i-1}}n + \cdots + \frac{4^2}{2^2}n + n$$

$$\vdots \qquad \vdots$$

$$= 4^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{4^{\log_2 n - 1}}{2^{\log_2 n - 1}}n + \cdots + \frac{4}{2}n + n$$

$$= 2n^2 - n$$

Three Different Behaviors

- Compare the iteration for the recurrences

$$T(n) = 2T(n/2) + n$$

$$T(n) = T(n/2) + n$$

$$T(n) = 4T(n/2) + n$$

Three Different Behaviors

- Compare the iteration for the recurrences

$$T(n) = 2T(n/2) + n$$

$$T(n) = T(n/2) + n$$

$$T(n) = 4T(n/2) + n$$

◇ all three recurrences iterate $\log_2 n$ times

◇ in each case, size of subproblem in next iteration is half the size in the preceding iteration level

Three Different Behaviors

- **Theorem** Suppose that we have a recurrence of the form
$$T(n) = aT(n/2) + n,$$
where a is a positive integer and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

    1. If $a < 2$, then $T(n) = \Theta(n)$.
    2. If $a = 2$, then $T(n) = \Theta(n \log n)$.
    3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$

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Proof

We already proved Case 1 when $a = 1$ in Example 3.
(will not prove it for $1 < a < 2$)
We already proved Case 2 in Example 1.
We will now prove Case 3.

Iterating Recurrences

- $T(n) = aT(n/2) + n$, where $a > 2$. Assume that $n = 2^i$.

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  Iterating as in Example 5 gives

  $$T(n) = a^i T\left(\frac{n}{2^i}\right) + \left(\frac{a^{i-1}}{2^{i-1}} + \frac{a^{i-2}}{2^{i-2}} + \cdots \frac{a}{2} + 1\right) n$$

Iterating Recurrences

- $T(n) = aT(n/2) + n$, where $a > 2$. Assume that $n = 2^i$.

  Iterating as in Example 5 gives

$$T(n) = a^i T\left(\frac{n}{2^i}\right) + \left(\frac{a^{i-1}}{2^{i-1}} + \frac{a^{i-2}}{2^{i-2}} + \cdots \frac{a}{2} + 1\right) n$$

$$T(n) = a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$

Work at "bottom"

Iterated Work

Total work

- The total work is

$$T(n) = a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$

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- The total work is

$$T(n) = a^{\log_2 n} T(1) + n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i$$

Since $a > 2$, the geometric series is Θ of the largest term.

$$n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i = n \frac{1-(a/2)^{\log_2 n}}{1-a/2} = n\Theta((a/2)^{\log_2 n - 1})$$

Total work

- n times the largest term in the geometric series is

$$n\left(\frac{a}{2}\right)^{\log_2 n-1} = \frac{2}{a}\cdot\frac{n\cdot a^{\log_2 n}}{2^{\log_2 n}} = \frac{2}{a}\cdot\frac{n\cdot a^{\log_2 n}}{n} = \frac{2}{a}\cdot a^{\log_2 n}$$

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$$\Theta\left(n^{\log_2 a}\right) \qquad \Theta\left(n^{\log_2 a}\right)$$

Example 5 Recap

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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$$T(n) = \Theta\left(n^{\log_2 a}\right) = \Theta\left(n^{\log_2 4}\right) = \Theta(n^2)$$

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This matches with the exact answer of $2n^2 - n$.

Three Different Behaviors

- **Theorem** Suppose that we have a recurrence of the form
$$T(n) = aT(n/2) + n,$$
where a is a positive integer and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

  1. If $a < 2$, then $T(n) = \Theta(n)$.
  2. If $a = 2$, then $T(n) = \Theta(n \log n)$.
  3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$

The Master Theorem

- **Theorem** Suppose that we have a recurrence of the form
$$T(n) = aT(n/b) + cn^d,$$
where a is a positive integer, $b \geq 1$, c, d are real numbers with c positive and d nonnegative, and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

1. If $a < b^d$, then $T(n) = \Theta(n^d)$.
2. If $a = b^d$, then $T(n) = \Theta(n^d \log n)$.
3. If $a > b^d$, then $T(n) = \Theta(n^{\log_b a})$

Next Lecture

- counting ...