Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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Binary Relation

- **Definition**: Let A and B be two sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

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- **Theorem** The number of binary relations on a set A, where $|A| = n$ is 2^{n^2}.

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  Yes. If $a|b$ and $b|c$, then $a|c$.

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No. $(1,2),(2,1) \in R_{\neq}$ but $(1,1) \notin R_{\neq}$.

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  Set operations: union, intersection, difference, etc.

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- **Example**: Let $A = \{1, 2, 3\}$, $B = \{u, v\}$, and
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What is $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$?

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  We may also combine relations by matrix operations.

Composite of Relations

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$R^k =?$ for $k > 3$

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"only if" part: by induction.

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  How many subsets on $n(n-1)$ elements are there?

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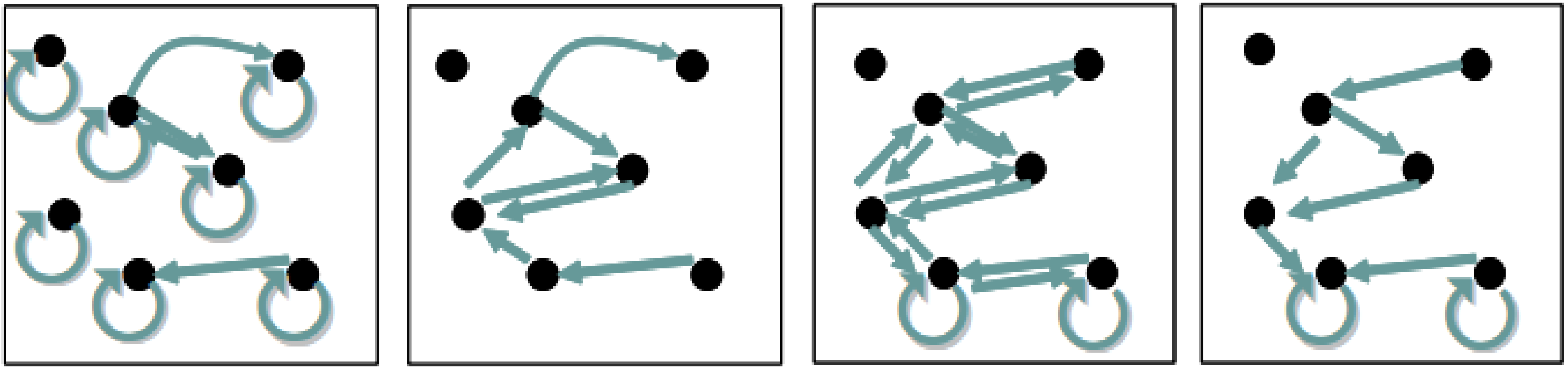

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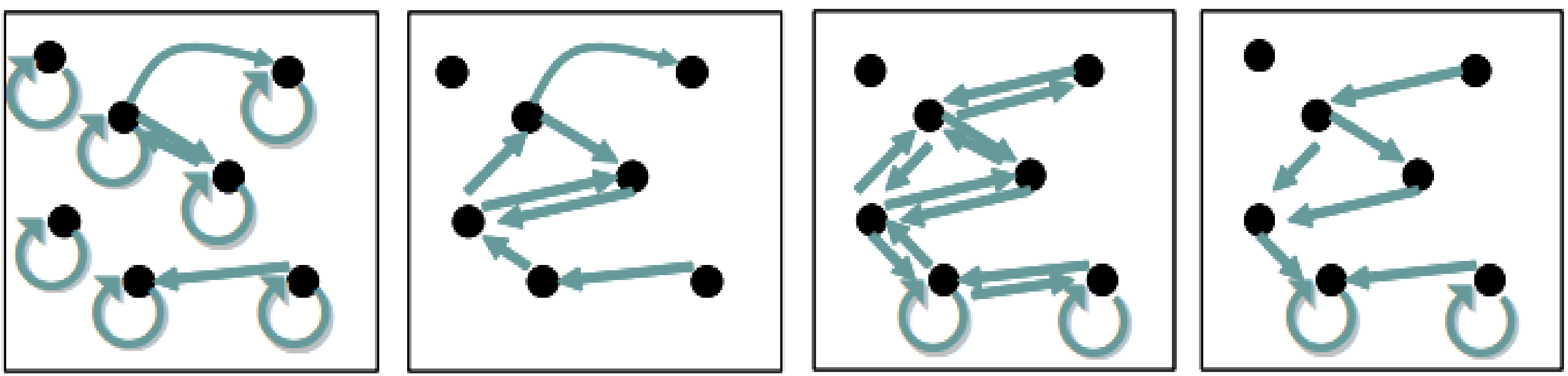

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  Add $(2,2)$ and $(3,3)$

  Then $S = \{(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)\} \supseteq R$

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  Is this relation R reflexive?

  No. Why? $(2,2)$ and $(3,3)$ are not in R.

  The question is what is the minimal relation $S \supseteq R$ that is reflexive?

  How to make R reflexive by minimum number of additions?

  Add $(2,2)$ and $(3,3)$

  Then $S = \{(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)\} \supseteq R$

  The minimal set $S \supseteq R$ is called *the reflexive closure of R*.

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- The set S is called *the reflexive closure of* R if it:
  - ◇ contains R
  - ◇ is reflexive
  - ◇ is minimal (is contained in every reflexive relation Q that contains R ($R \subseteq Q$), i.e., $S \subseteq Q$)

Closures on Relations

- Relations can have different properties:
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  We define:
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  - symmetric closures
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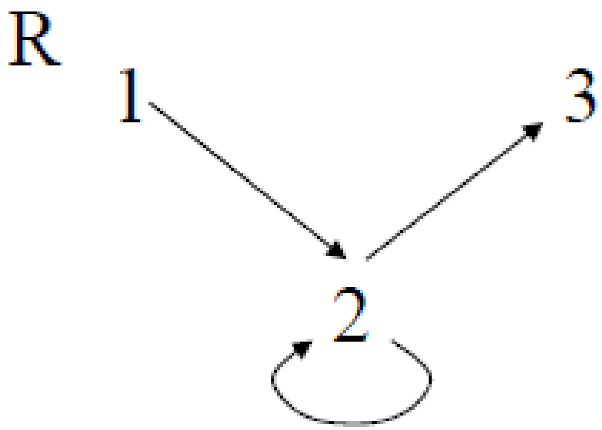

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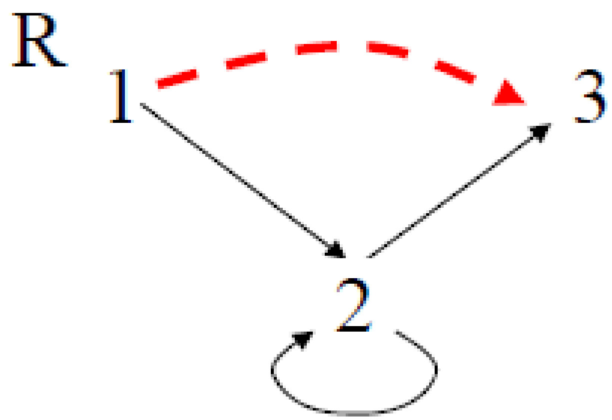

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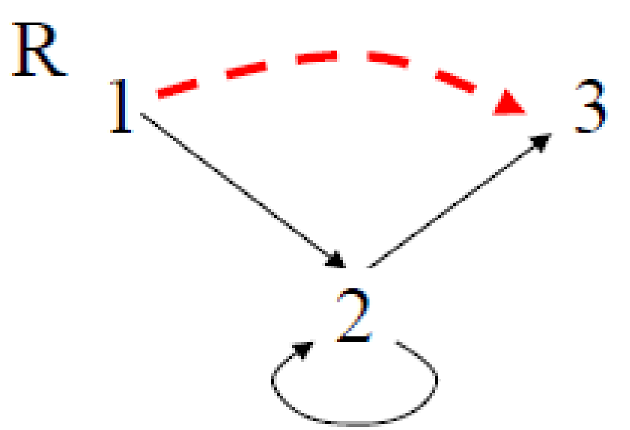

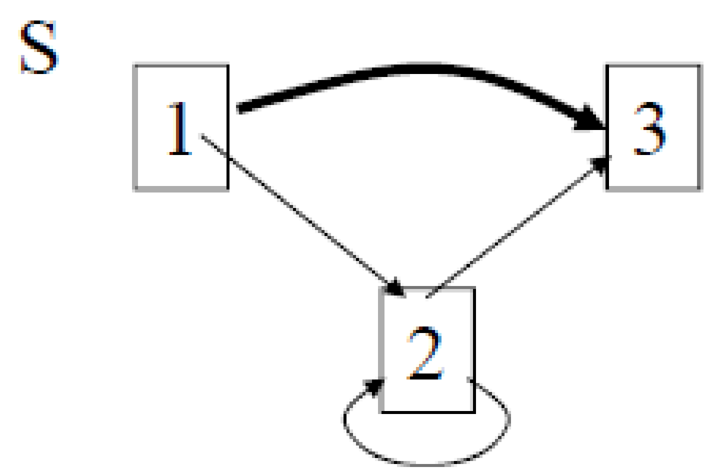

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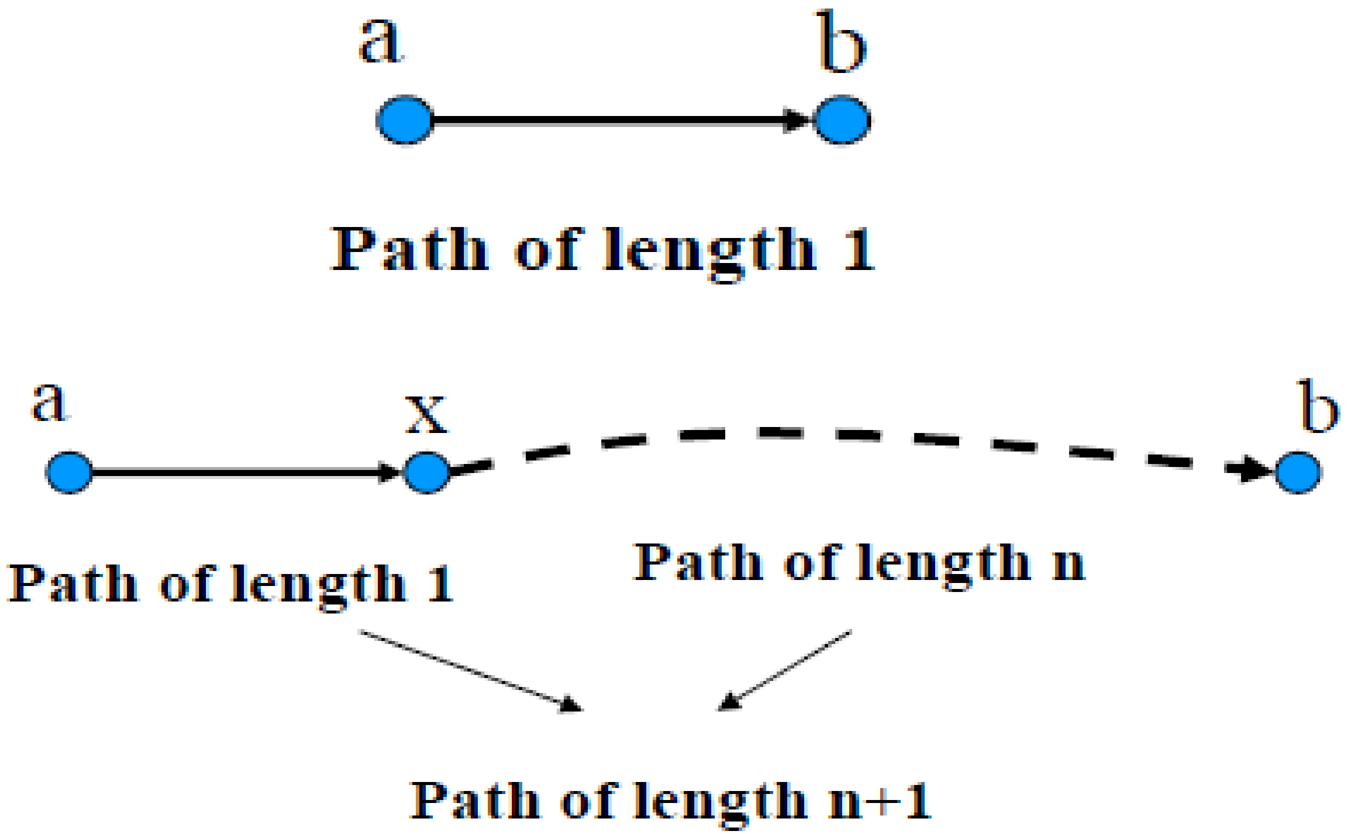

Connectivity Relation

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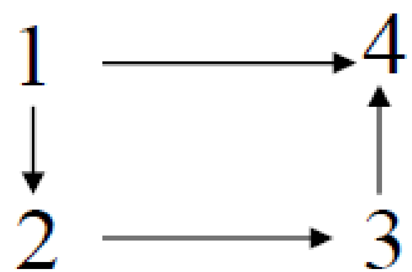

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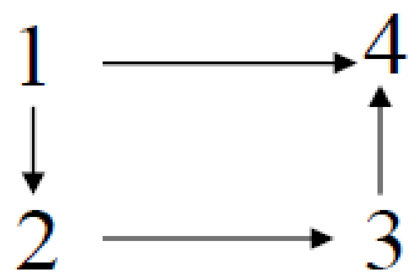

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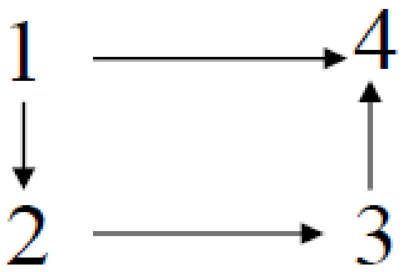

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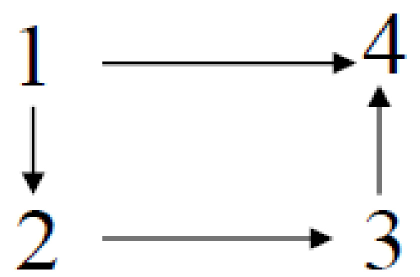

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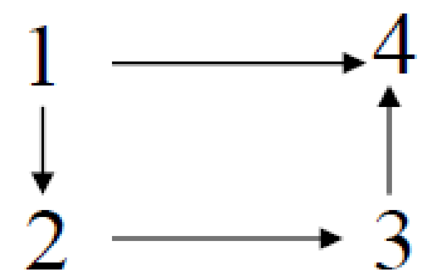

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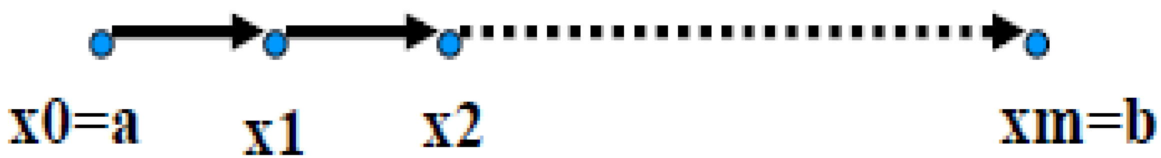

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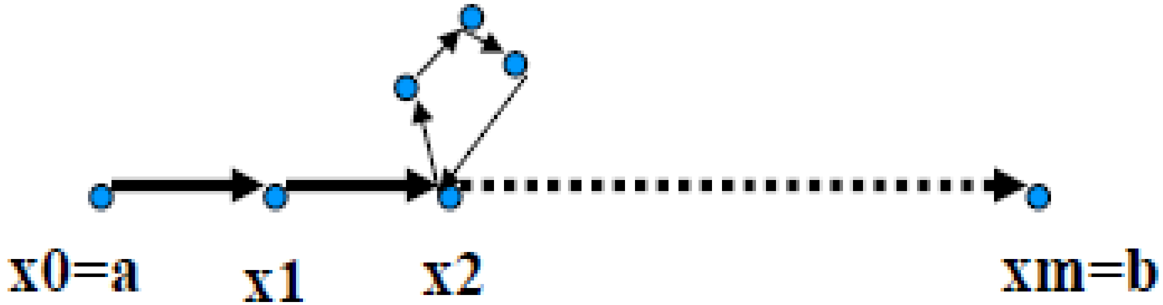

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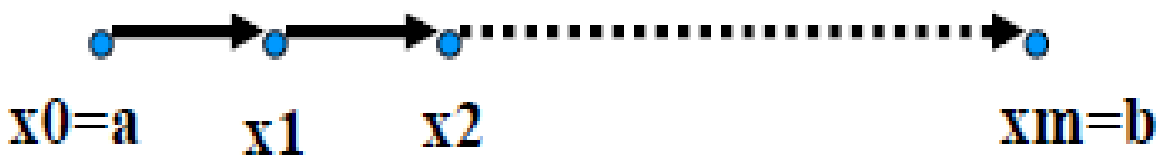

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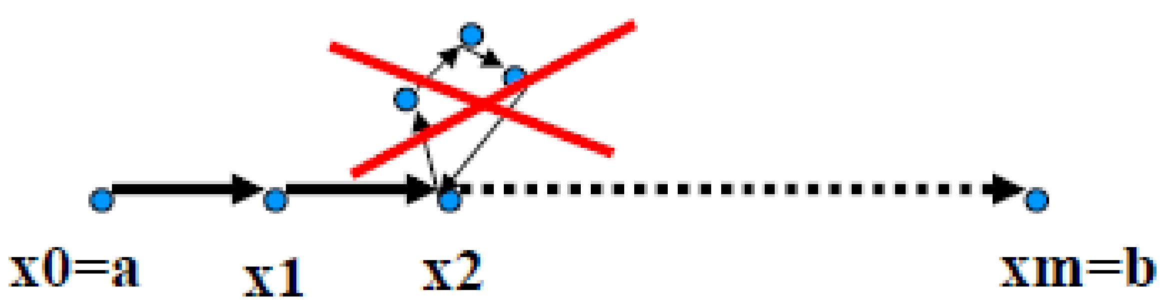

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1. If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c in R. Thus, there is a path from a to c in R. This means that $(a, c) \in R^*$.

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We have $S^* \subseteq S$. Thus, $R^* \subseteq S^* \subseteq S$

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Example

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$\mathbf{M}_{R^*} = ?$

Simple Transitive Closure Algorithm

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n - 1$.

procedure transClosure ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$A := B := \mathbf{M}_R$;
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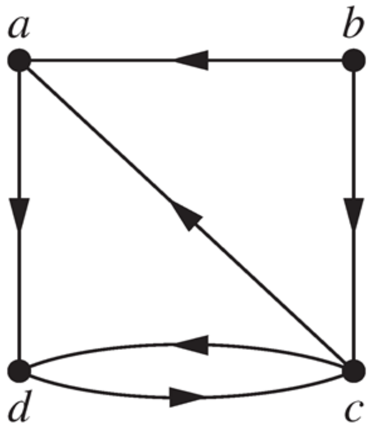
Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



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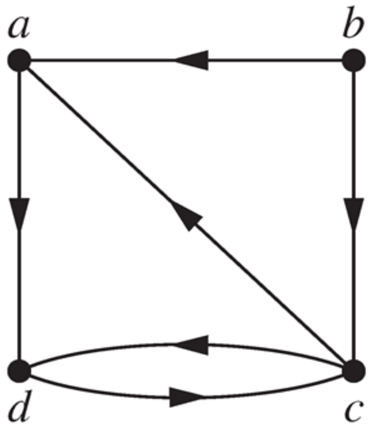

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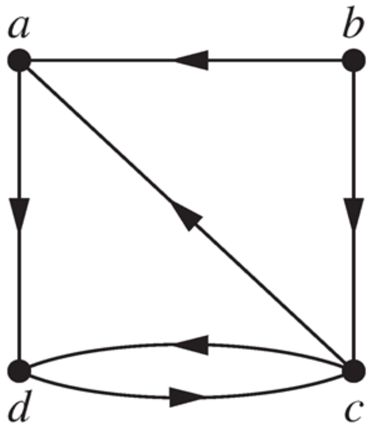

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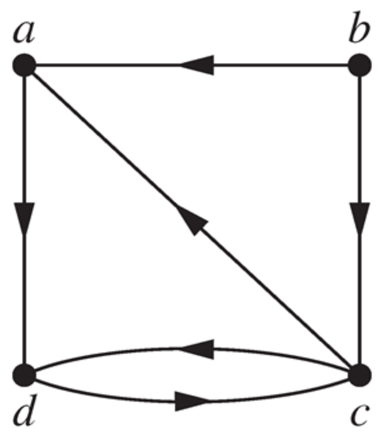

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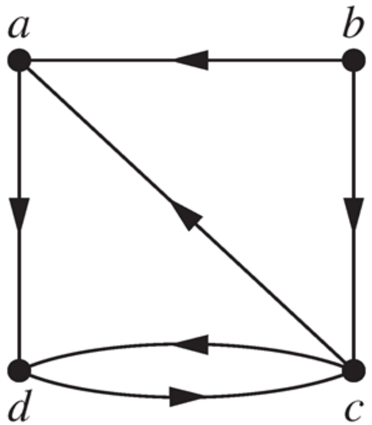
Example

Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



Let $v_1 = a$, $v_2 = b$, $v_3 = c$, $v_4 = d$.

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_2 = W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & \color{red}{1} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & \color{red}{1} \\ 1 & 0 & 0 & 1 \\ \color{red}{1} & 0 & 1 & \color{red}{1} \end{bmatrix} \qquad W_4 = \begin{bmatrix} \color{red}{1} & 0 & \color{red}{1} & 1 \\ 1 & 0 & 1 & \color{red}{1} \\ 1 & 0 & \color{red}{1} & 1 \\ \color{red}{1} & 0 & 1 & \color{red}{1} \end{bmatrix}$$

n-ary Relations

- **Definition** An n-ary relation R on sets $A_1, \ldots, A_n$, written as $R : A_1, \ldots, A_n$, is a subset $R \subseteq A_1 \times \cdots \times A_n$.

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  – R is *functional* in domain A_i if it contains at most one n-tuple $(\cdots, a_i, \cdots)$ for any value a_i within domain A_i.

Relational Databases

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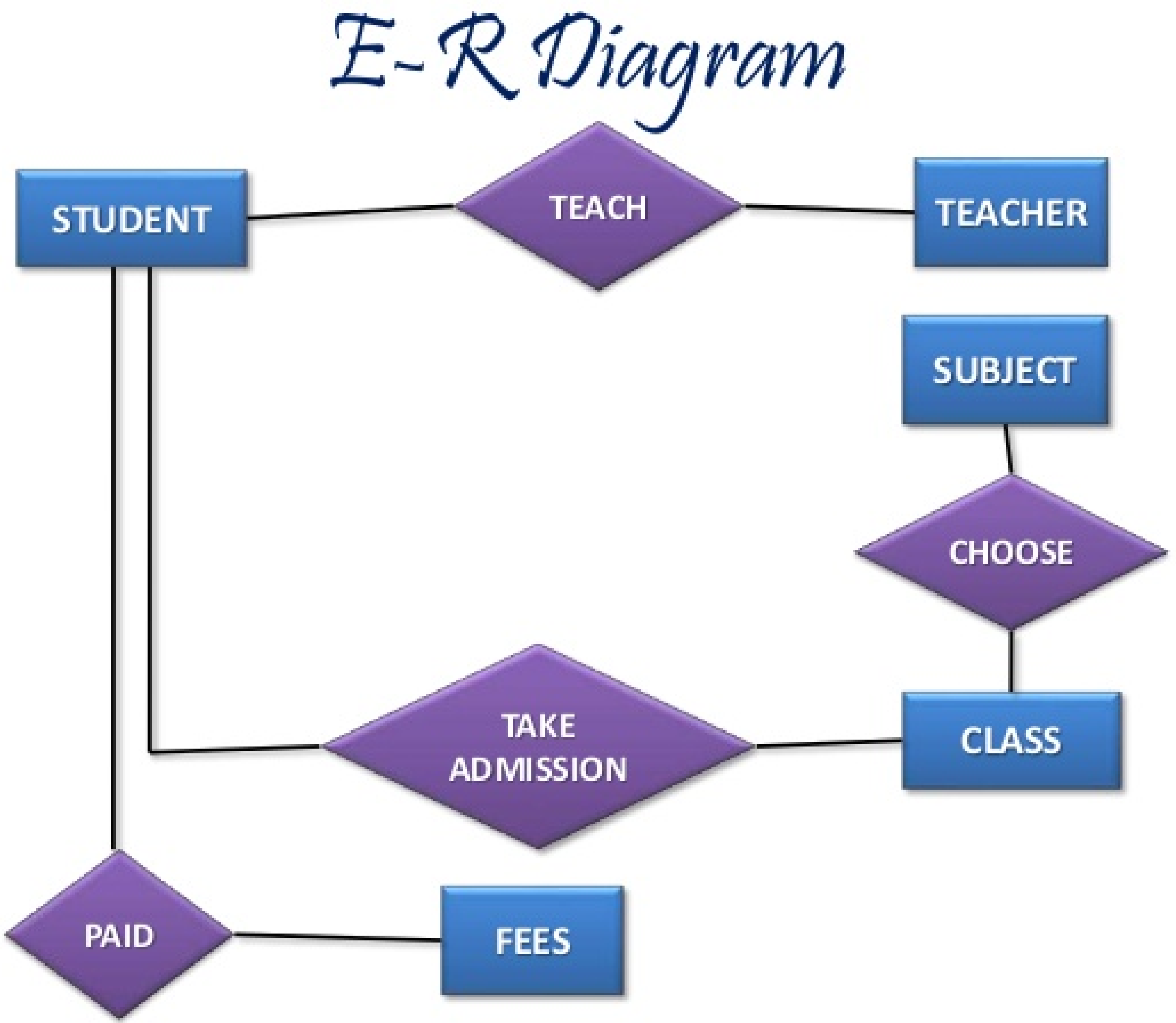
- A domain A_i is a *primary key* for the database if the relation R is functional in A_i.

- A *composite key* for the database is a set of domains $\{A_i, A_j, \cdots\}$ such that R contains at most 1 n-tuple $(\cdots, a_i, \cdots, a_j, \cdots)$ for each composite value $(a_i, a_j, \cdots) \in A_i \times A_j \times \cdots$.

Relational Databases



Selection Operators

- Let A be any *n-ary domain* $A = A_1 \times \cdots \times A_n$, and let $C : A \to \{T, F\}$ be any *condition* (predicate) on elements (n-tuples) of A.

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  – $\forall R \subseteq A$,

$$
\begin{aligned}
s_C(R) &= R \cap \{a \in A \mid s_C(a) = T\} \\
&= \{a \in R \mid s_C(a) = T\}.
\end{aligned}
$$

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- Suppose that we have a domain

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$$\begin{aligned} &UpperLevel(name,\ standing,\ ssn) \\ &\quad :\equiv \quad [(standing = junior) \vee (standing = senior)] \end{aligned}$$

- Then, $s_{UpperLevel}$ is the selection operator that takes any relation R on A (database of students) and produces a relation consisting of just the upper-level classes (juniors and seniors).

Projection Operators

- Let $A = A_1 \times \cdots \times A_n$ be any n-ary domain, and let $\{i_k\} = (i_1, \ldots, i_m)$ be a sequence of indices all falling in the range 1 to n.

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- Then the *projection operator* on n-tuples

  $$P_{\{i_k\}} : A \to A_{i_1} \times \cdots \times A_{i_m}$$

  is defined by

  $$P_{\{i_k\}}(a_1, \cdots, a_n) = (a_{i_1}, \cdots, a_{i_m})$$

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  $$(a_{i_1}, a_{i_2}) = (a_1, a_3) = (model, color)$$

- This operator can be usefully applied to a whole relation $R \subseteq Cars$ (database of cars) to obtain a list of $model/color$ combinations available.

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- A, B, C can also be sequences of elements rather that single elements.

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- Suppose that R_1 is a teaching assignment table, relating *Professors* to *Courses*.

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- Suppose that R_2 is a room assignment table relating *Courses* to *Rooms* and *Times*.

- Then $J(R_1, R_2)$ is like your class schedule, listing $(professor, course, room, time)$.

Next Lecture

- relation III ...