Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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Counting

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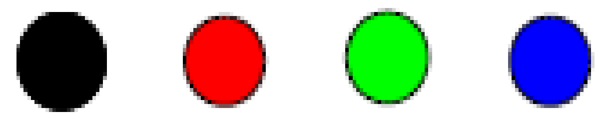
How many different ways are there to choose 2 balls from



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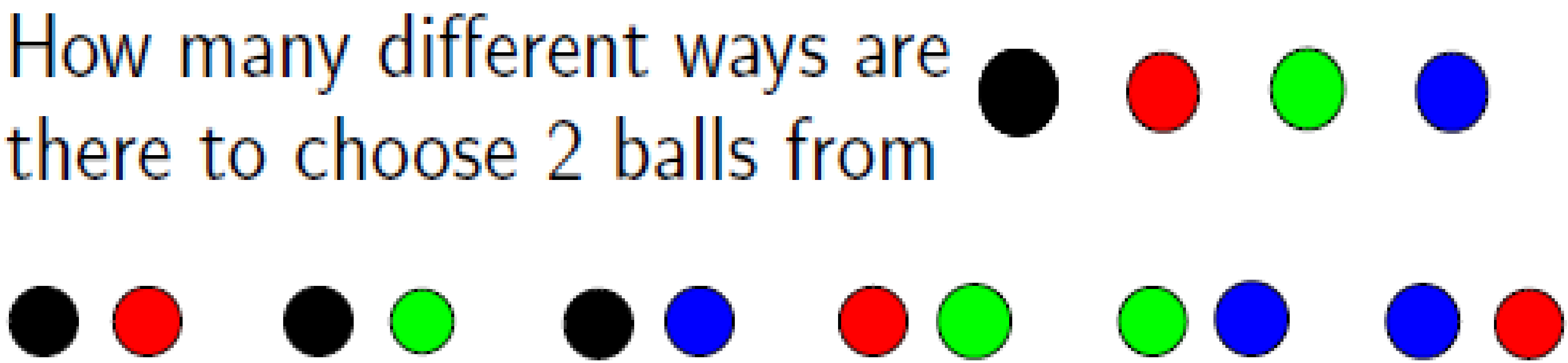

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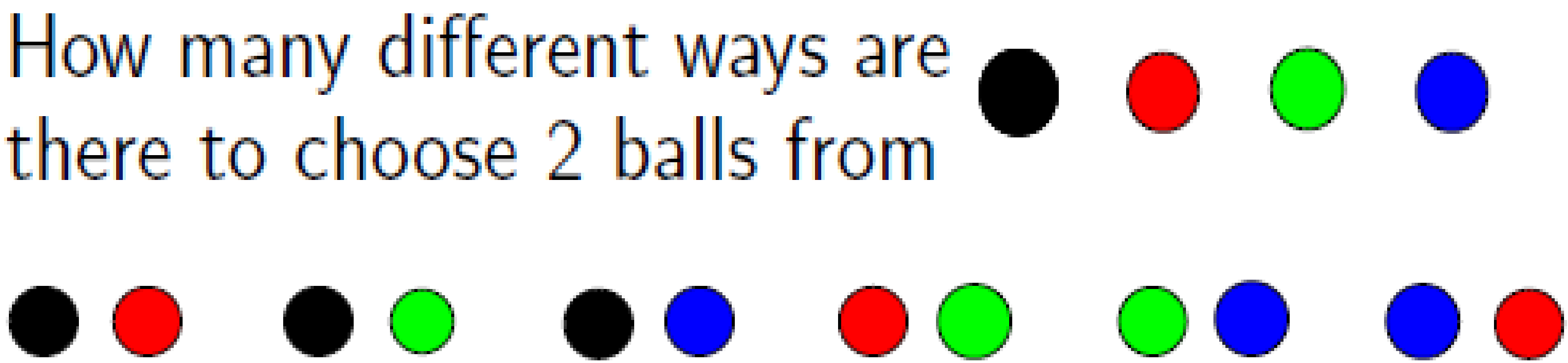

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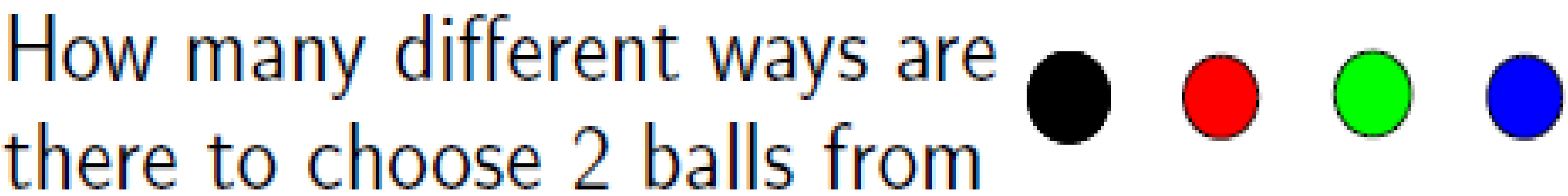

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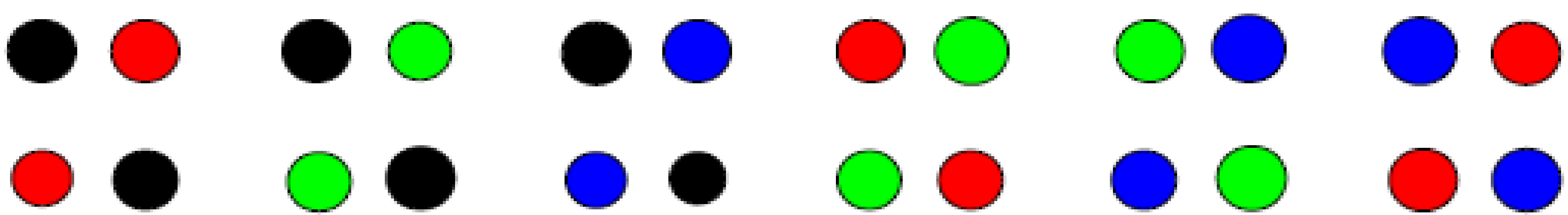

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  – simplify the solution by decomposing the problem

Basic Counting Rules

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Example

In an auditorium, the seats are labeled by a letter and numbers in between 1 to 50 (e.g., $A23$). What is the total number of seats?

We may either list all or use the product rule.

$$26 \times 50 = 1300$$

The Product Rule

- **Product Rule**: If a count of elements can be broken down into a sequence of dependent counts where the first count yields n_1 elements, the second n_2 elements, and kth count n_k elements, then the total number of elements is

$$n = n_1 \cdot n_2 \cdot \cdots \cdot n_k$$

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How many different bit strings of length 7 are there?

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How many one-to-one functions are there from a set with m elements to a set with n elements?

How many onto functions?

The Product Rule

- The following loop is a part of program computing the product of two matrices.

```
(1) for i = 1 to r
(2)    for j = 1 to m
(3)       S = 0
(4)       for k = 1 to n
(5)          S = S + A[i,k] * B[k,j]
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How many multiplications (in terms of r, m, n) does this program carry out in total among all iterations of line 5?

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Example

You need to travel from city A to B. You may either fly, take a train, or a bus. There are 12 different flights, 5 different trains and 10 buses. How many options do you have to get from A to B?

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We may use the sum rule.

$$12 + 5 + 10$$

The Sum Rule

- **Sum Rule**: If a count of elements can be broken down into a set of independent counts where the first count yields n_1 elements, the second n_2 elements, and kth count n_k elements, then the total number of elements is
$$n = n_1 + n_2 + \cdots + n_k$$

The Sum Rule

- The following loop is from selection sort.

```
(1) for i = 1 to n-1
(2)     for j = i+1 to n
(3)         if (A[i] > A[j])
(4)             exchange A[i] and A[j]
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How many comparisons (in terms of n) does this program carry out in total among all iterations of line 3?

More Complex Counting

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Example

Each password is 6 to 8 characters long, where each character is a lowercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

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$$P = P_6 + P_7 + P_8$$

Tree Diagrams

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  Example

  What is the number of bit strings of length 4 that do not have two consecutive 1's?

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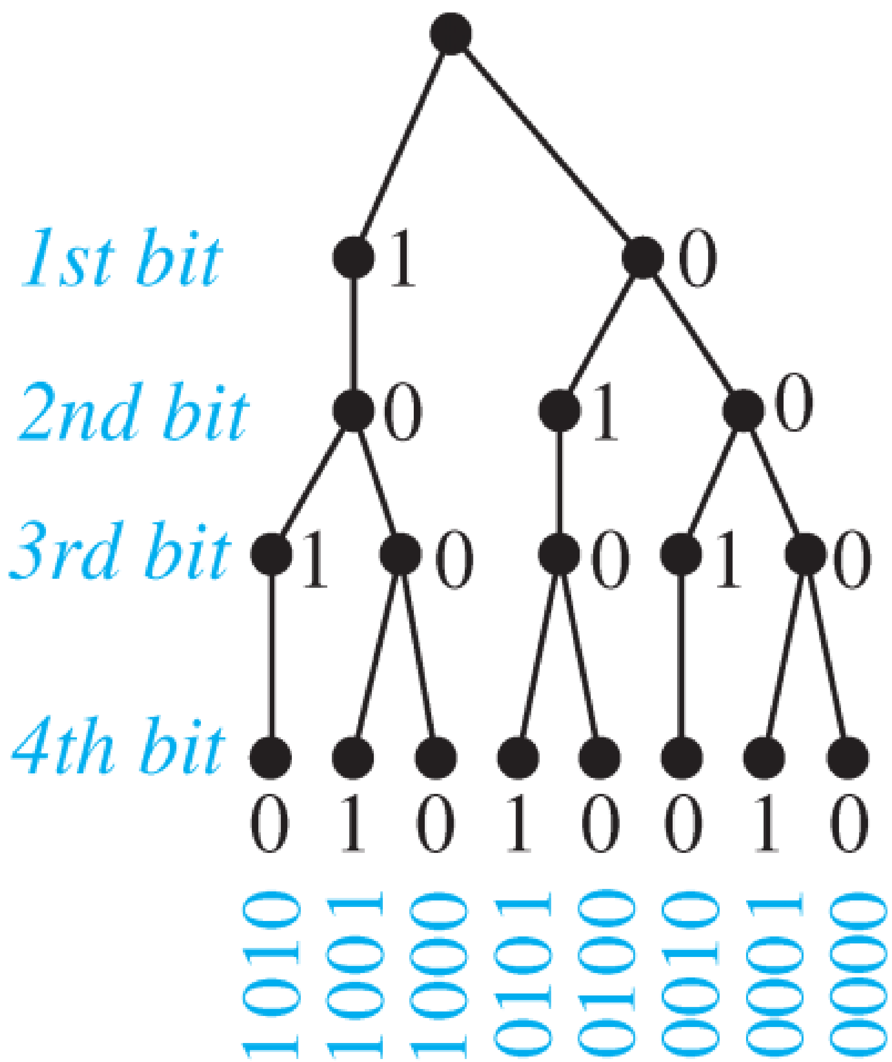

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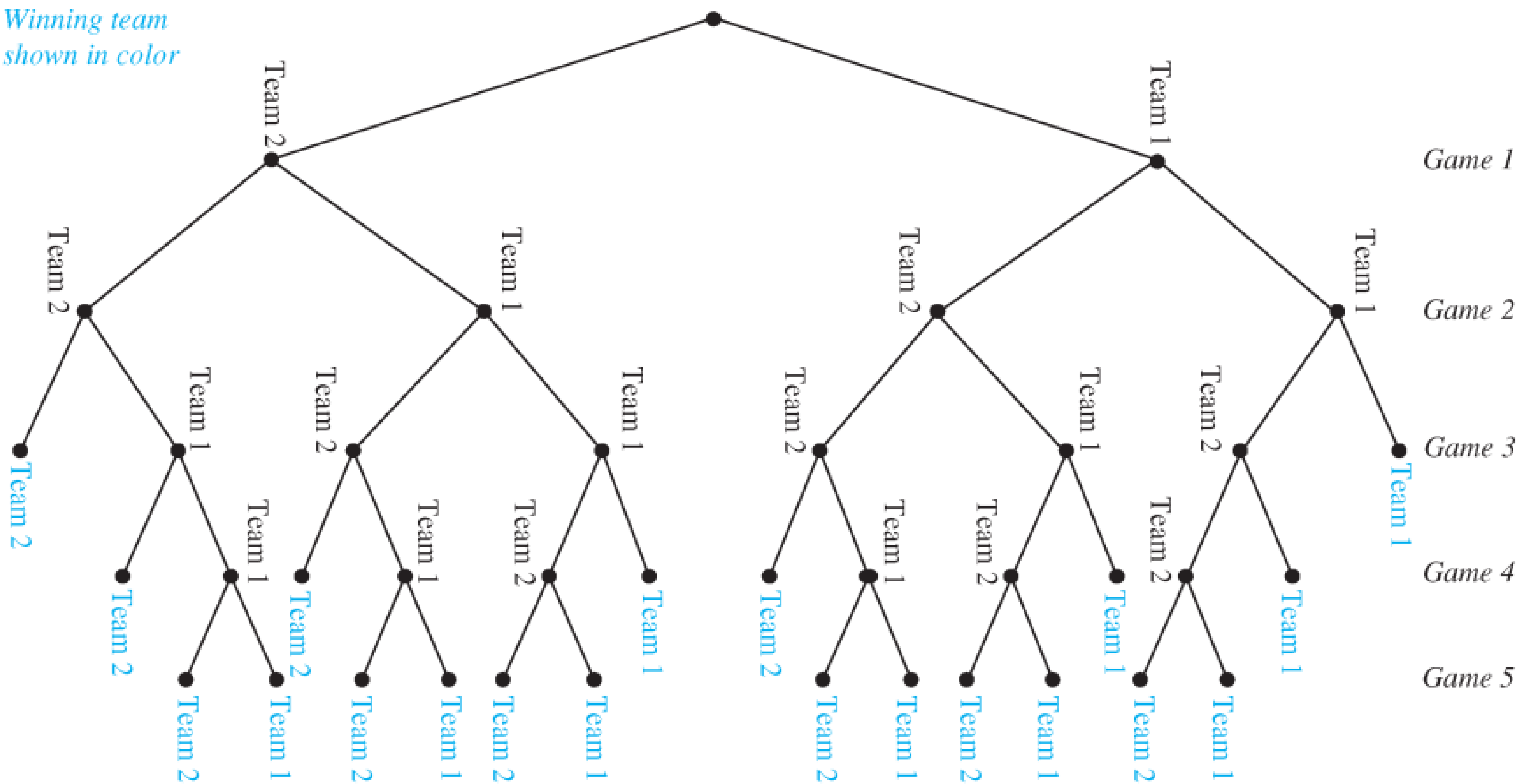

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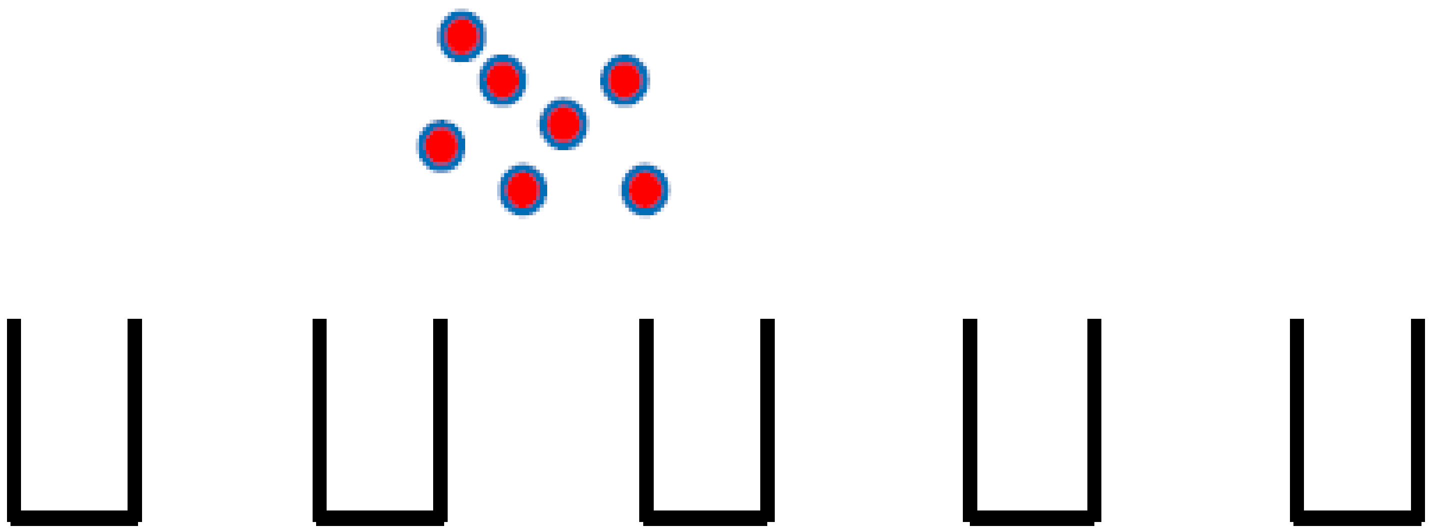

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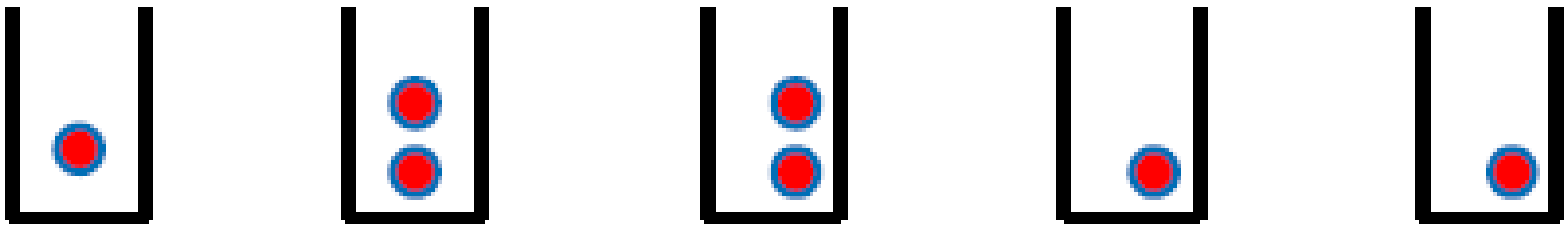

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  Example

  Assume that there are 367 students. Are there any two people who have the same birthday?

  There are 5 bins and 12 objects. Then there must be a bin with at least 3 objects. Why?

Generalized Pigeonhole Principle

- If N objects are placed into k bins, then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

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Example

Assume there are 100 students. How many of them were born in the same month?

Bijections and Permutations

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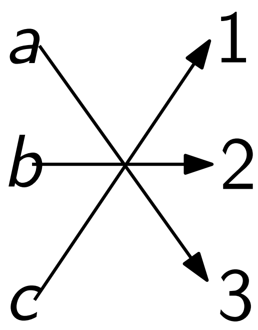

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  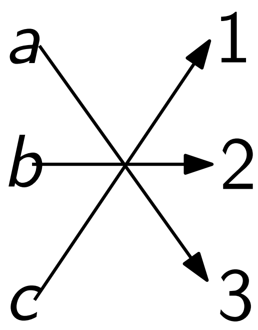

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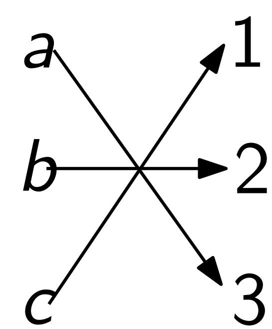

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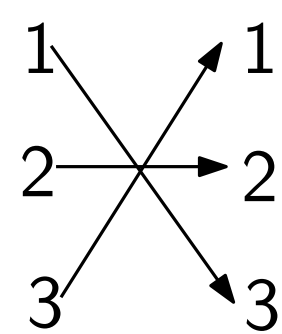

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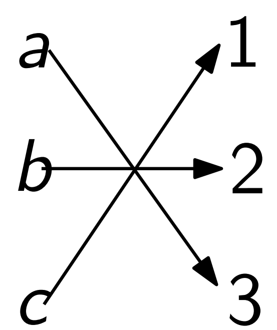

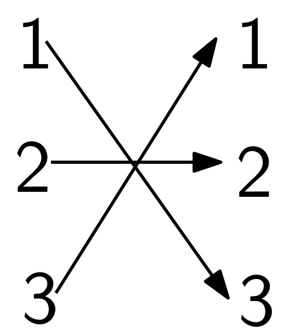

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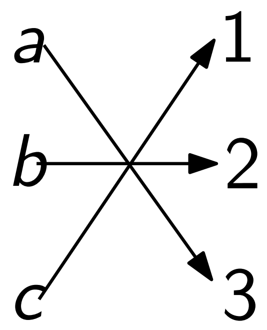
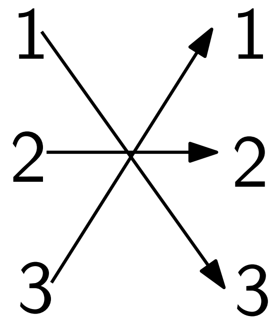
  Thus,
  the left and right sides must have the same size.





The Bijection Principle

- The following loop is a part of program to determine the number of triangles formed by n points in the plane.

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trianglecount = 0
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Among all iterations of line 5, what is the total number of times this line checks three points to see if they are collinear?

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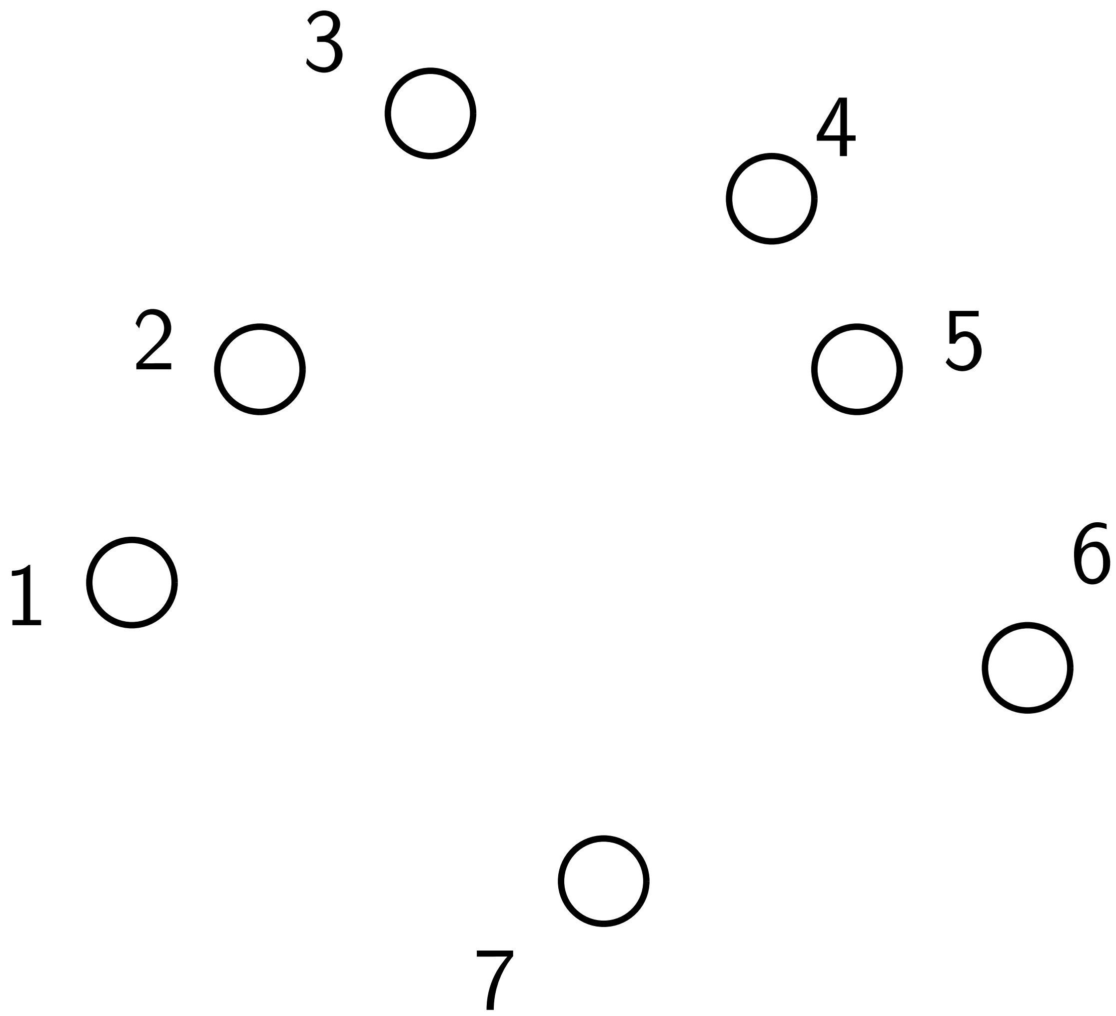

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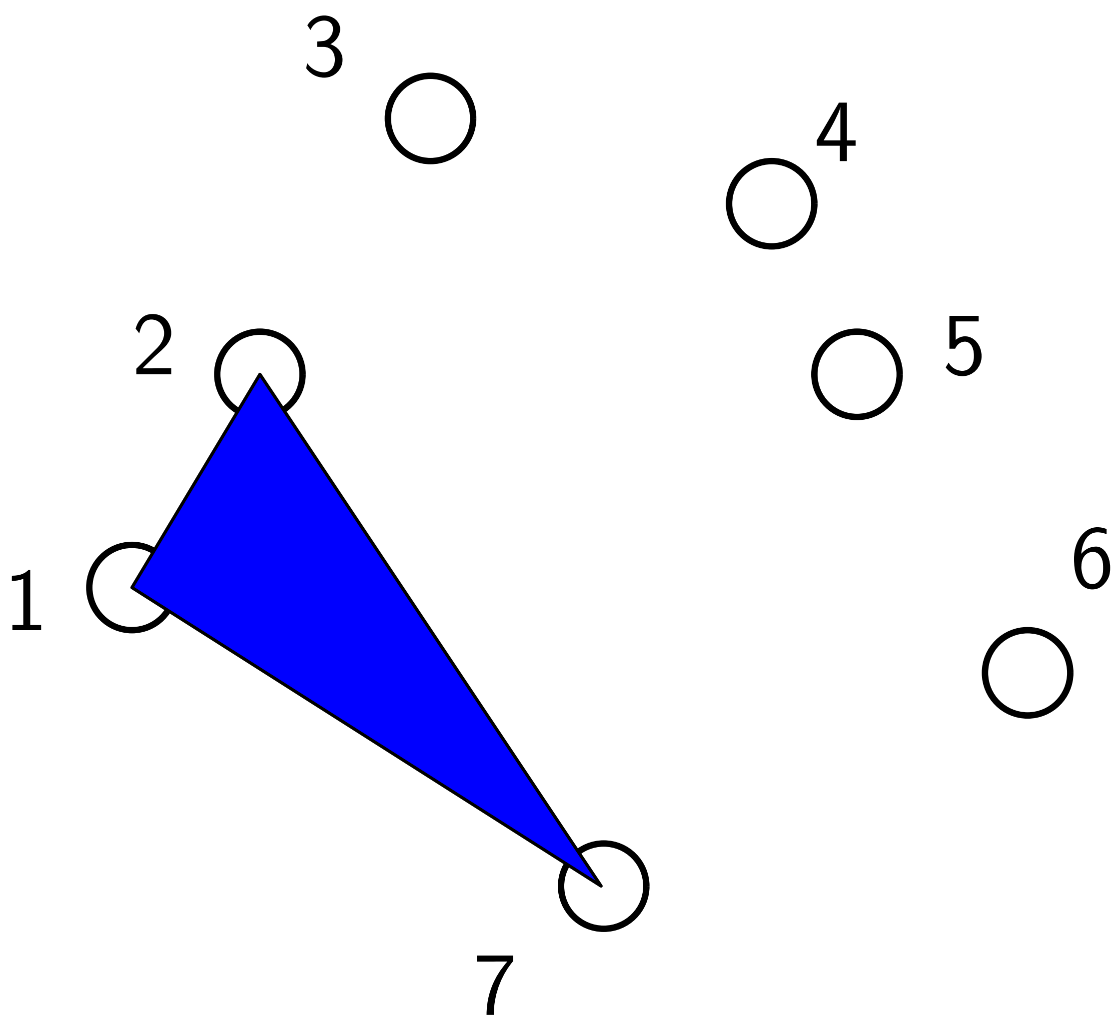

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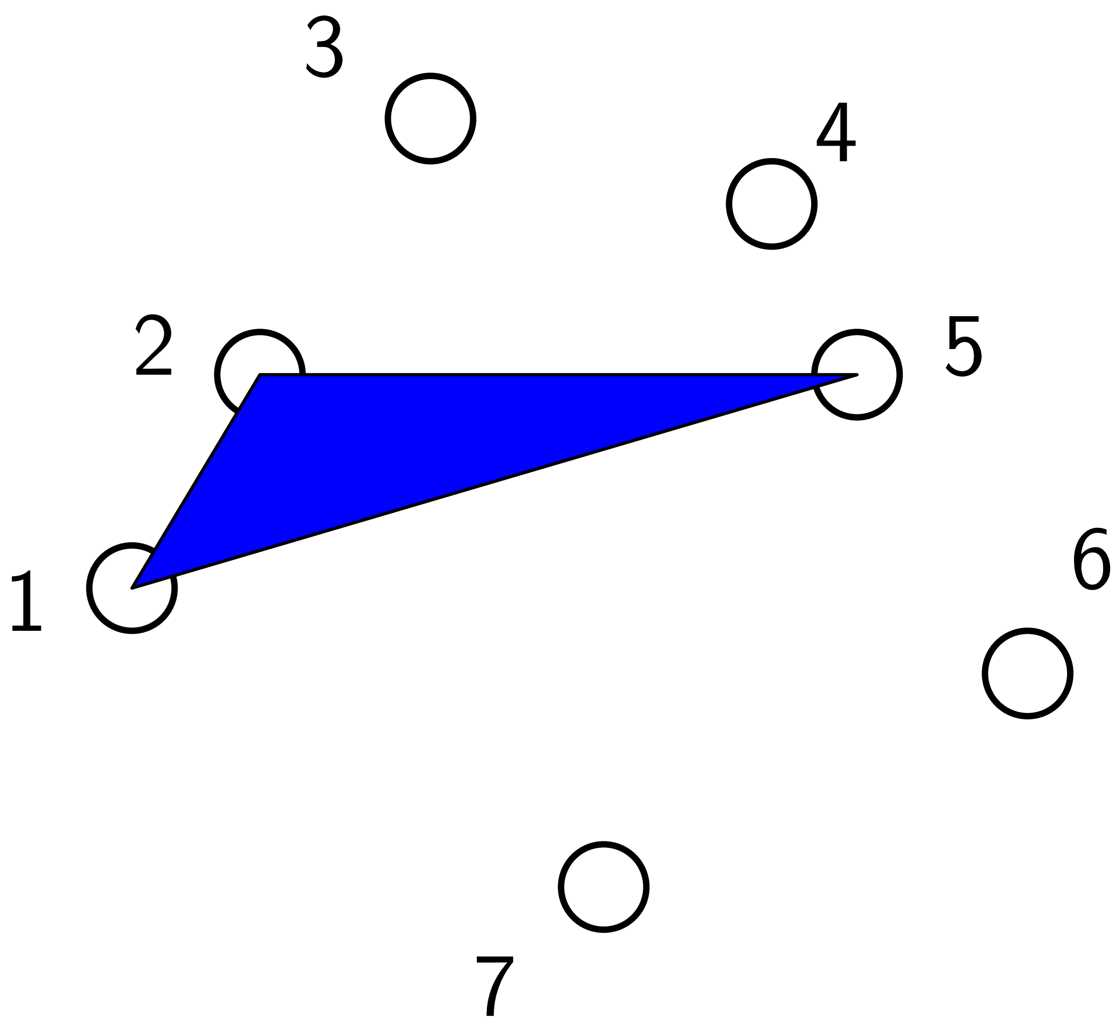

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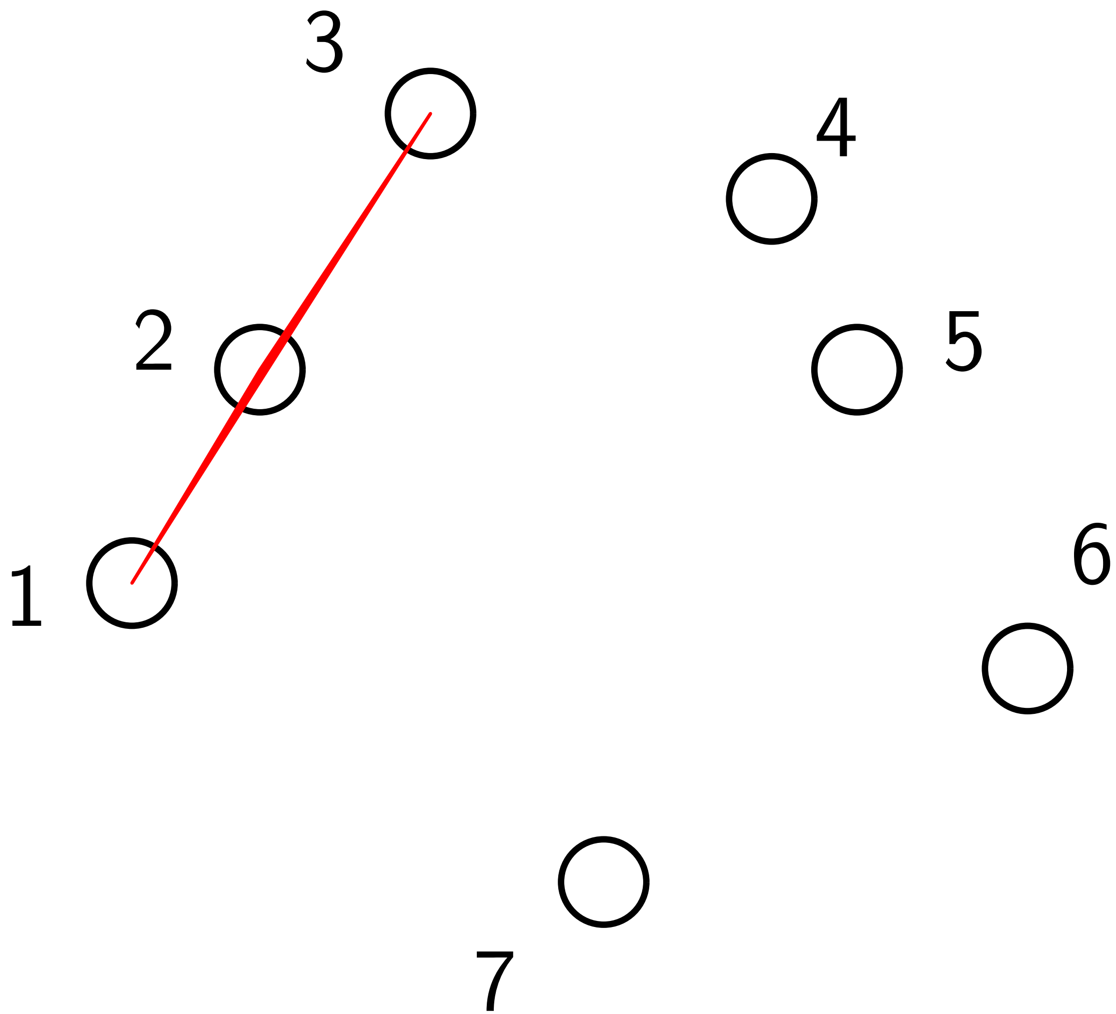

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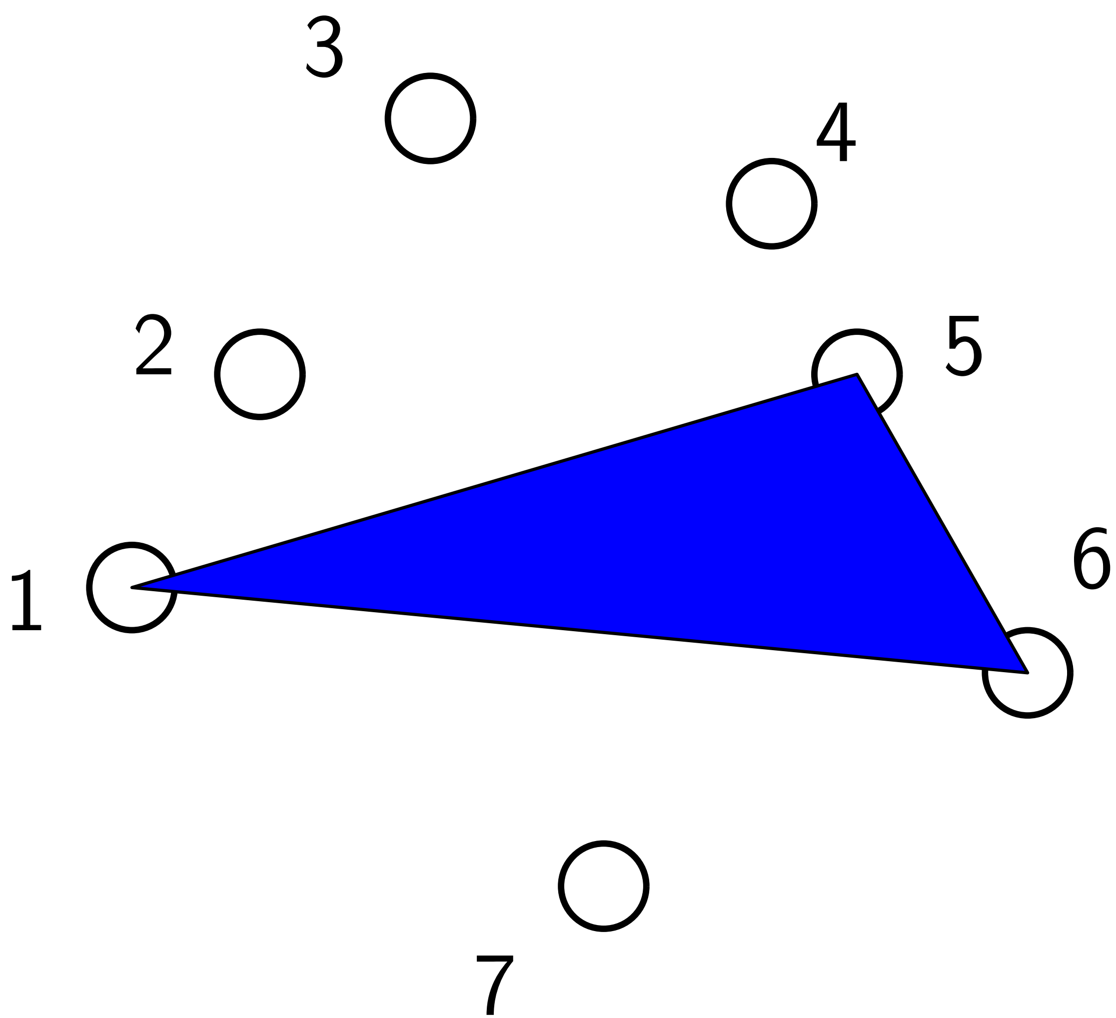

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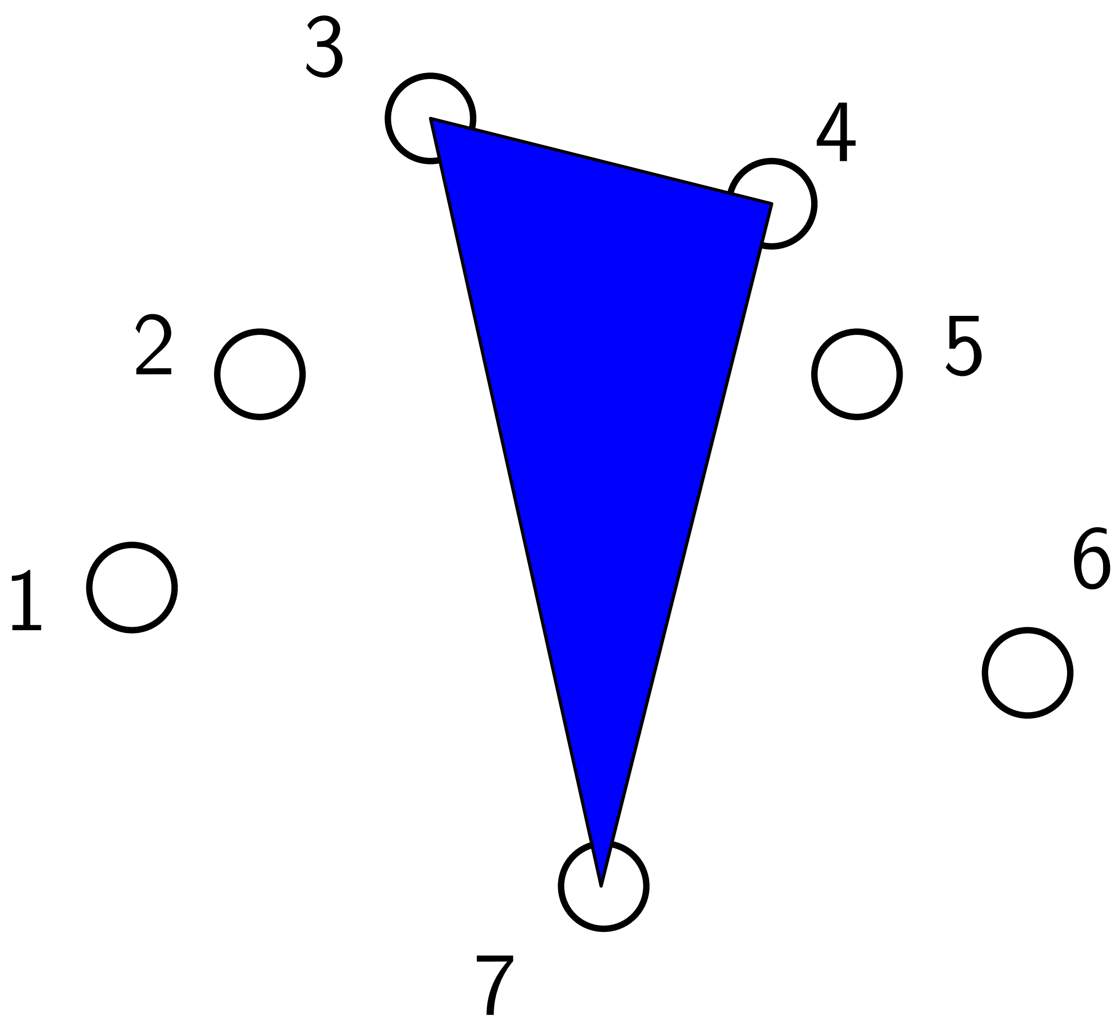

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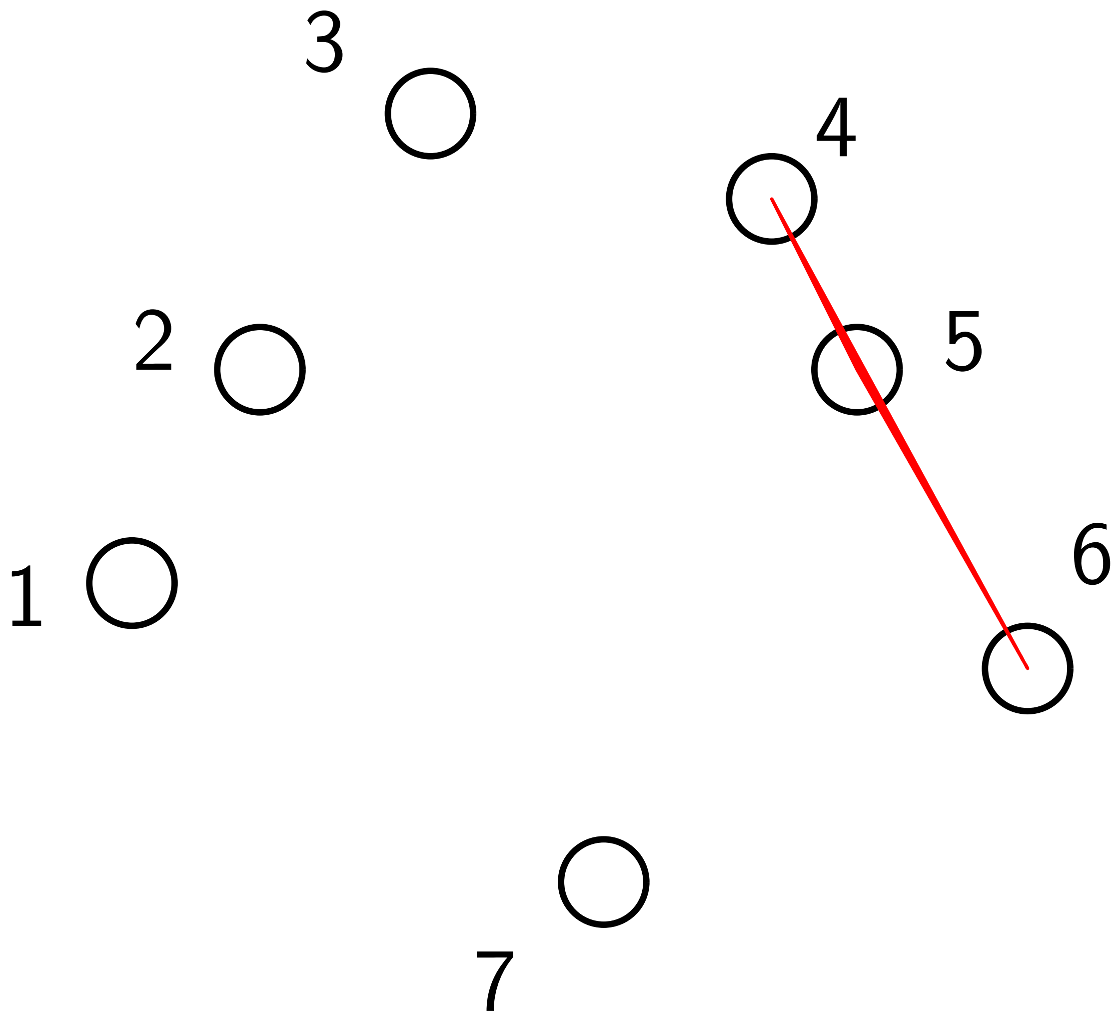

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$3 - 4 - 7$: yes

$4 - 5 - 6$: no

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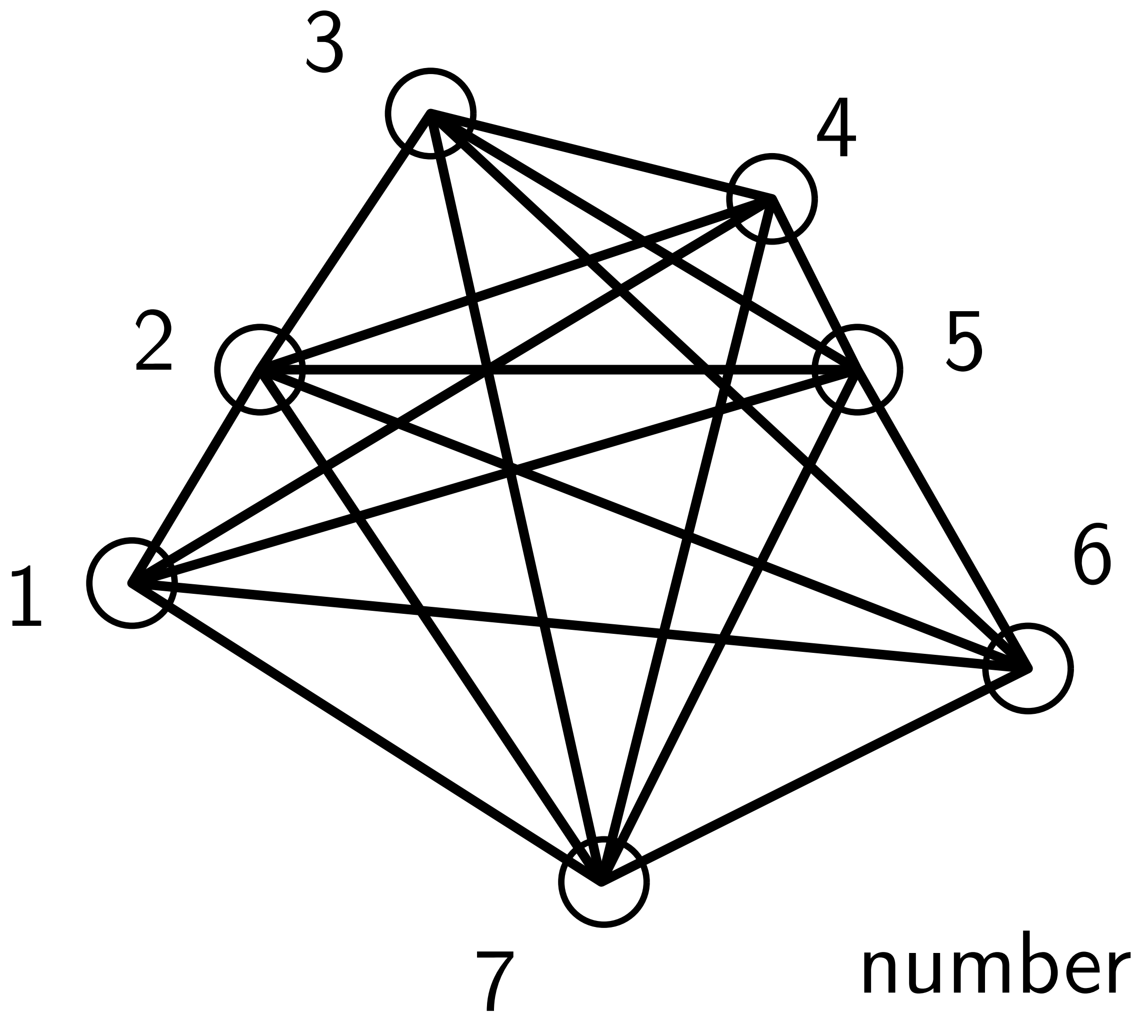

$1-2-7$: yes

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$1-5-6$: yes

$3-4-7$: yes

$4-5-6$: no

number of triangles: 33

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Thus each triple i, j, k with $i < j < k$ is examined exactly once.

For example, if $n = 4$, then triples (i, j, k) used by algorithm are (1,2,3), (1,2,4), (1,3,4), and (2,3,4).

Counting Triangles

- Want to compute the number of *increasing triples* (i, j, k) with $1 \le i < j < k \le n$.

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  Claim: Number of increasing triples is exactly the same as number of 3-element subsets from $\{1, 2, \ldots, n\}$

Counting Triangles

- Want to compute the number of
  increasing triples (i, j, k) with $1 \leq i < j < k \leq n$.

  Claim: Number of increasing triples is exactly the same as number of 3-element subsets from $\{1, 2, \ldots, n\}$

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f is a bijection because
f is one-to-one
if $(i, j, k) \neq (i', j', k') \Rightarrow f(\,(i, j, k)\,) \neq f(\,(i', j', k'))$
f is onto
if γ is a 3-element subset then it can be written as $\gamma = \{i, j, k\}$
where $i < j < k$ so $f(\,(i, j, k)\,) = \gamma$.

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  We actually already saw that $|X| = |Y| = \binom{n}{2}$

The Bijection Principle

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  Currently, we started with the problem of counting the # of increasing triples and changed it to the problem of counting the # of 3-element sets from $\{1, 2, \ldots, n\}$

Inclusion-Exclusion Principle

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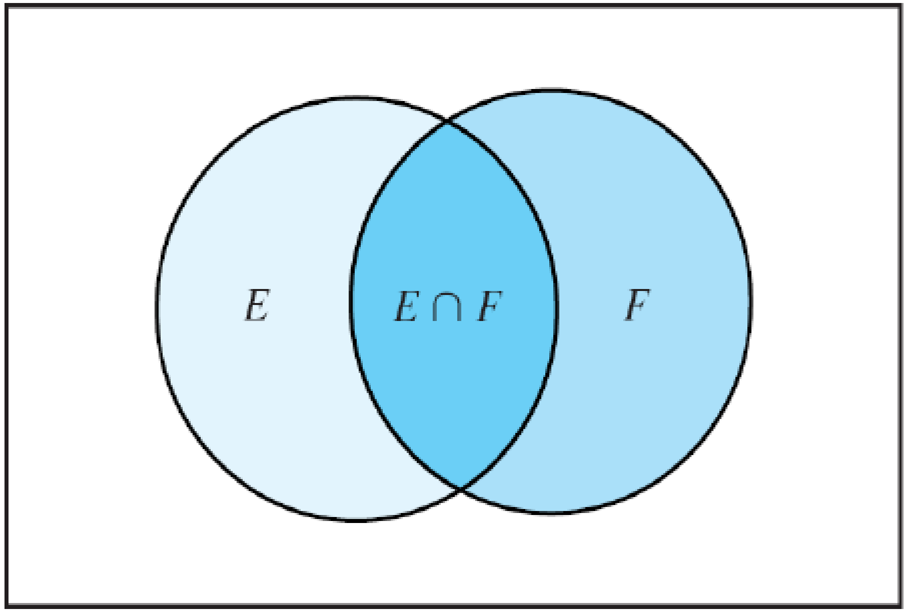

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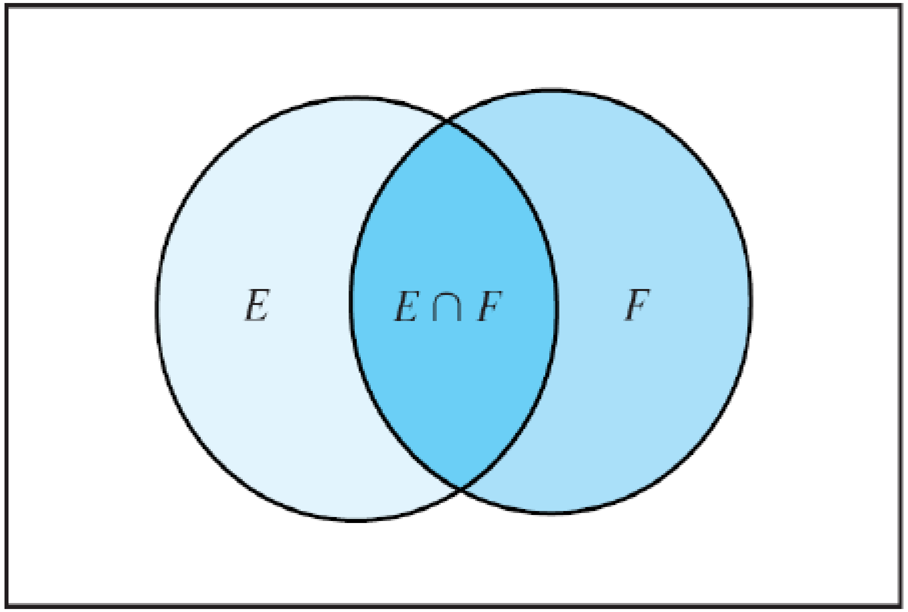

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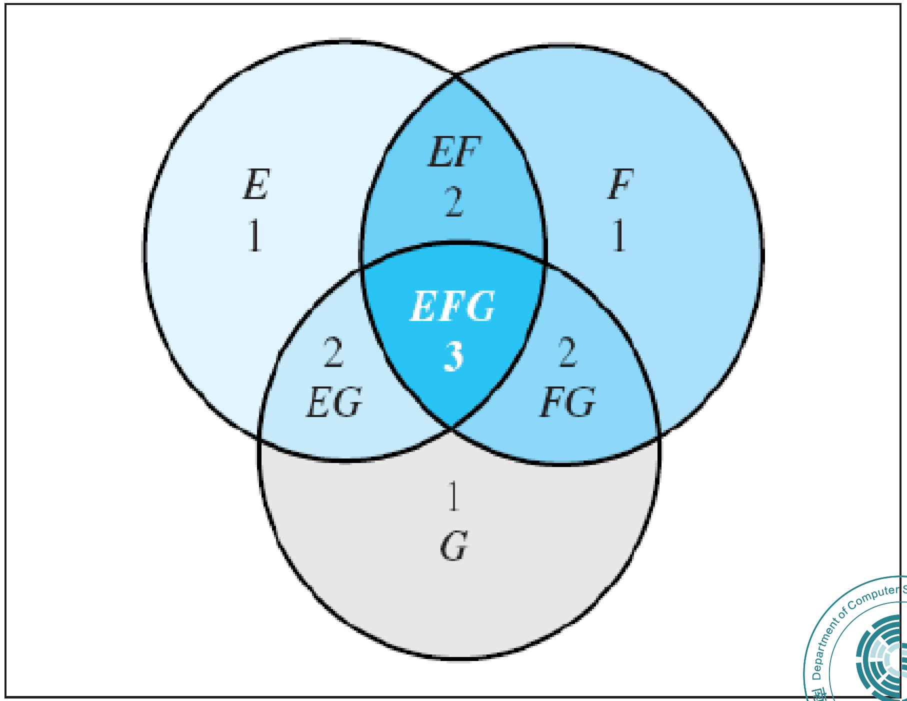

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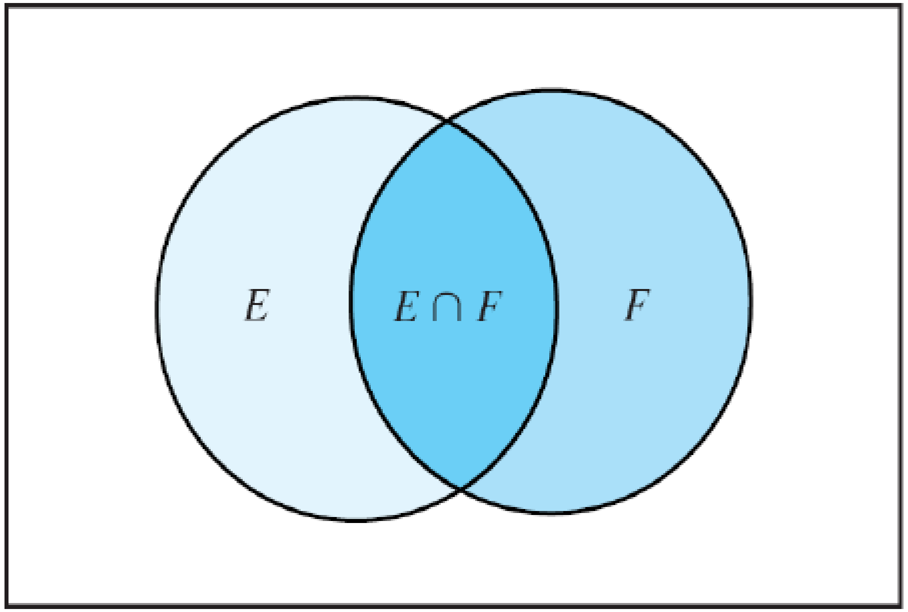
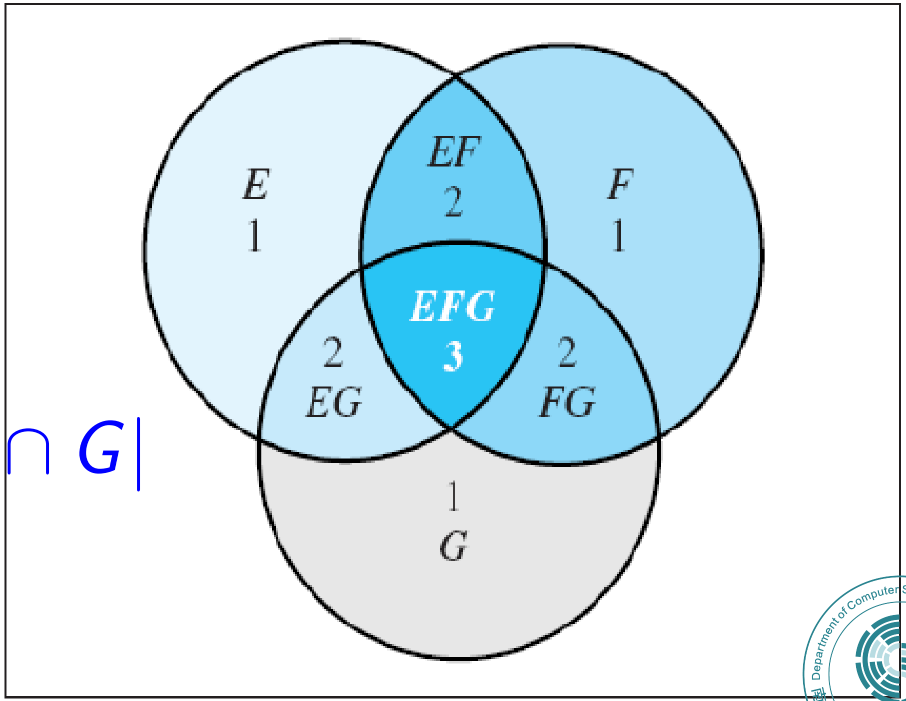


Three sets

$$\begin{aligned} &|E \cup F \cup G| \\ = &|E| + |F| + |G| \\ &-|E \cap F| - |E \cap G| - |F \cap G| \\ &+|E \cap F \cap G| \end{aligned}$$



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- $$|\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

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Inductive Hypothesis

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  For the third term, by distributive law,

$$\left|\left(\cup_{i=1}^{n-1} E_i\right) \cap E_n\right| = \left|\cup_{i=1}^{n-1}(E_i \cap E_n)\right| = \left|\cup_{i=1}^{n-1} G_i\right|$$

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Some discussion:
first summation sums $(-1)^{k+1}|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$ over all lists $i_1, i_2, \ldots, i_k$ that do not contain n
$|E_n|$ and second summation together sum $(-1)^{k+1}|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$ over all lists $i_1, i_2, \ldots, i_k$ that do contain n

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$= \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}(n-k)^m$

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  Note that the case of $k = n$ is special;
  An n-element permutation of a set N of size $|N| = n$ is what we earlier simply called a permutation.

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Ex: When $n=4$, there are $4 \times 3 \times 2 = 24$ 3 -element permutations of $\{1,2,3,4\}$

$$L = \{123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243\\ 312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432\}.$$

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Note: This type of "dictionary" ordering of tuples (assuming that we treat numbers the same as letters) is called a *lexicographic ordering* and is used quite often.

k-Element Permutations of a Set

- **Theorem** If N is a positive integer and k is an integer with $1 \leq k \leq n$, then there are
  $P(n, k) = n(n-1)(n-2)\cdots(n-k+1)$
  k-element permutations with n distinct elements.

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  $(\#$ 3-element perms$) = 6 \times (\#$ 3-element subsets$)$

  $$P(n, 3) = 3! \cdot C(n, 3)$$

Binomial Coefficient

- **Theorem** For integers n and k with $0 \leq k \leq n$, the number of k-element subsets of an n-element set is
$$\binom{n}{k} = C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}.$$

  This is the number of k-combinations of a set with n elements.

Some Properties of Binomial Coefficients

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of k-element subsets of an n-element set.

  $\binom{n}{0} = 1$ only one set of size 0.

  $\binom{n}{n} = 1$ only one set of size n.

  $\binom{n}{k} = \binom{n}{n-k}$ Obvious from equation. Can you think of a simple bijection that explains this?

Some Properties of Binomial Coefficients (cont.)

- $$\boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n}$$

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  Let P = set of all subsets of $\{1,2,\ldots,n\}$

  S_i = set of all i-subsets of $\{1,2,\ldots,n\}$

  $$\Rightarrow |P| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i}$$

Some Properties of Binomial Coefficients (cont.)

- Let $L = L_1 L_2 \ldots L_n$ be a list of size n from $\{0, 1\}$

  If $\mathcal{L} = \text{set of all such lists} \Rightarrow \ |\mathcal{L}| = 2^n$

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Ex: $n = 5$

$$f(10101) = \{1, 3, 5\}, \ f(11101) = \{1, 2, 3, 5\}, \ f(00000) = \emptyset$$

Binomial Coefficients

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

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Second half of each row is the reverse of the first half.

Sum of items on n-th row is 2^n

Pascal's Triangle

Take the table

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
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5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

and shift each row slightly so that middle element is in middle

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

What is the next row in the table?

Pascal's Triangle

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Pascal's Triangle

							1							
						1		1						
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		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Pascal identity

Each (non-1) entry in Pascal's Triangle is the sum of the two entries directly above it (to left and to right).

Pascal's Triangle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

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  We will use a combinatorial proof.

A Combinatorial Proof

- $\binom{n}{k}$ is the number of k-element subsets of an n-element set.

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$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

Therefore, each term (left and right) represents the number of subsets of a particular size chosen from an appropriately sized set.

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  Number of k-subsets of an $(n-1)$-element set.

  Try to use sum principle to explain relationship among these three terms.

  Example: $n = 5$, $k = 2$

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

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Set S_1 of 2-subsets of S

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\},$$
$$\{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}.$$

A Combinatorial Proof

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

Consider $S = \{A, B, C, D, E\}$.

Set S_1 of 2-subsets of S can be partitioned into 2 disjoint parts.

S_2 the 2-subsets that contain E and

S_3, the set of 2-subsets that do not contain E.

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \\ \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}.$$

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Let S_1 be set of all k-element subsets.

A Combinatorial Proof

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Proof: Apply sum rule.

Let S_1 be set of all k-element subsets.

To apply sum rule, partition S_1 into S_2 and S_3.

Let S_2 be set of k-element subsets that contain x_n.

Let S_3 be set of k-element subsets that don't contain x_n.

Blaise Pascal

Born 1623; Died 1662

French Mathematician

A Founder of Probability Theory

Inventor of one of the first mechanical calculating machines

Pascal Programming Language named for him

Next Lecture

- counting II ...