



CS215 DISCRETE MATH

Dr. QI WANG

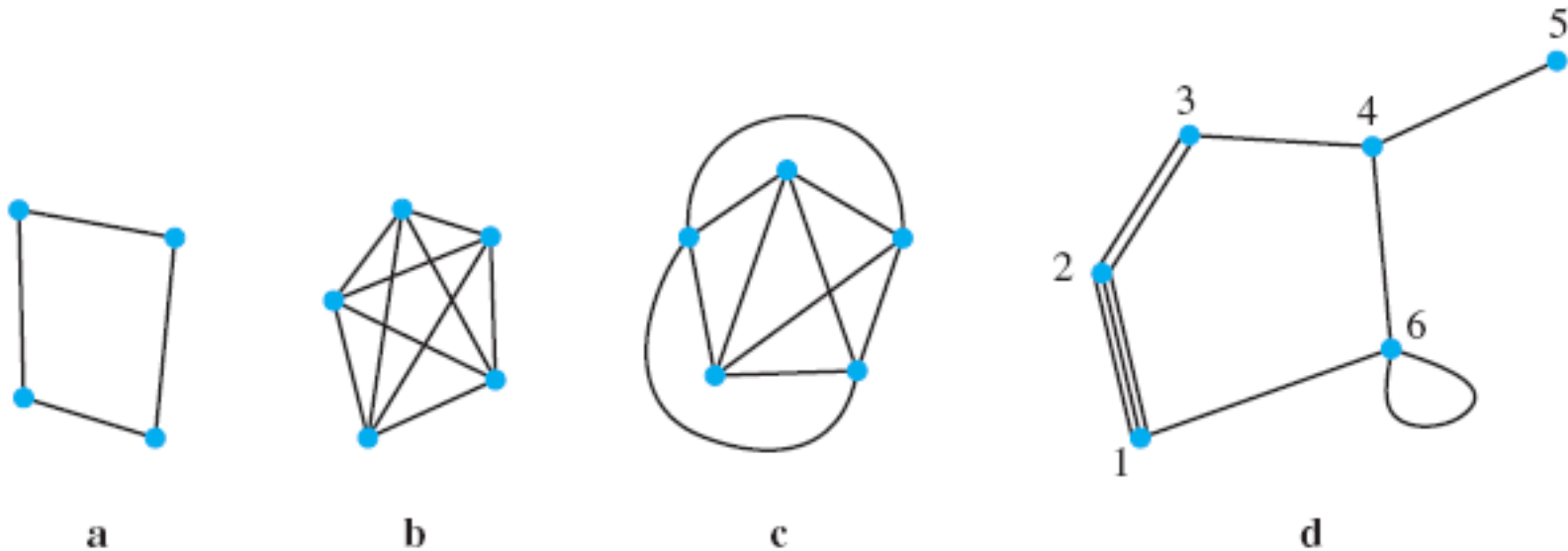
Department of Computer Science and Engineering

Office: Room413, CoE South Tower

Email: wangqi@sustech.edu.cn

Definition of a Graph

- **Definition.** A *graph* $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to be *incident to* (or *connect* its endpoints).



Complete Graphs

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K_1

K_2

K_3

K_4

K_5

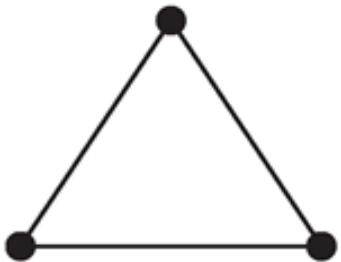
K_6

Cycles

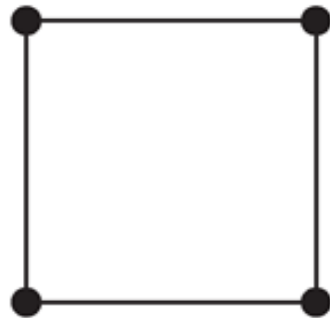
- A *cycle* C_n for $n \geq 3$ consists of n vertices v_1, v_2, \dots, v_n , and edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

Cycles

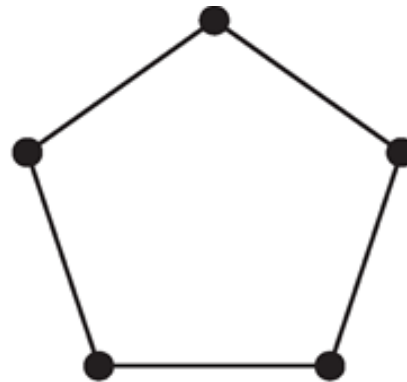
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C_3



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C_5



C_6

Wheels

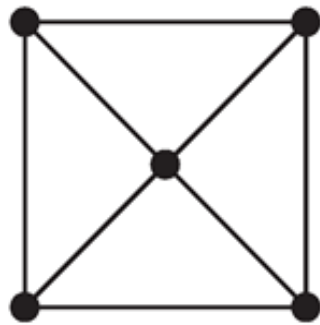
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Wheels

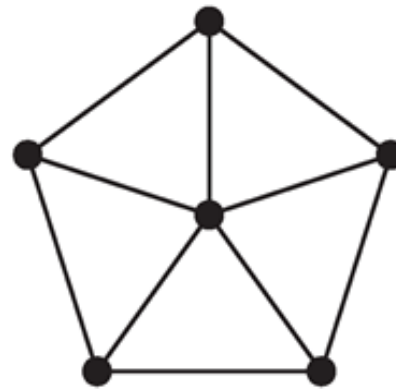
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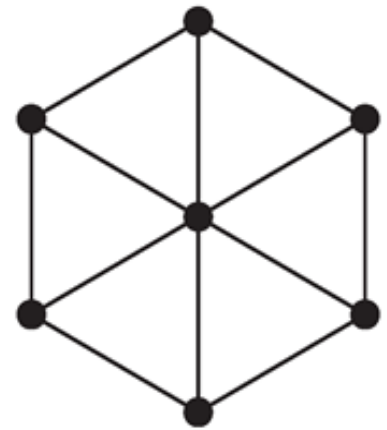
W_3



W_4



W_5



W_6

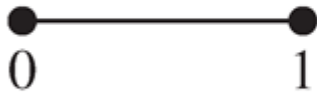
N -dimensional Hypercube

- An *n -dimensional hypercube*, or *n -cube*, Q_n is a graph with 2^n vertices representing all bit strings of length n , where there is an edge between two vertices that differ in exactly one bit position.

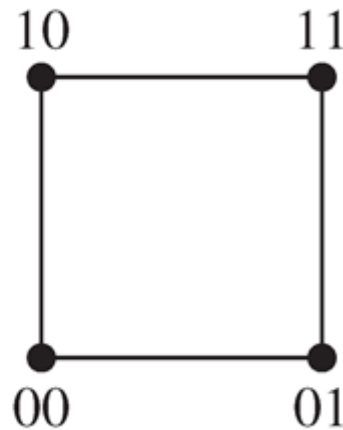


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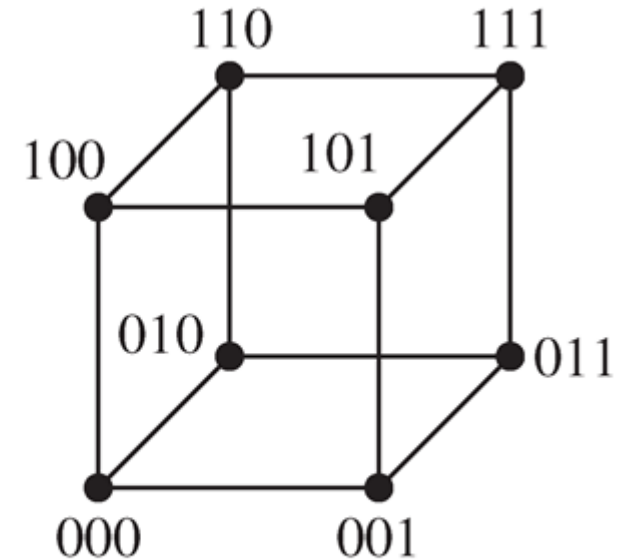
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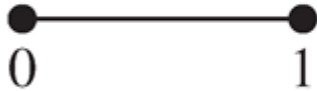
Q_2



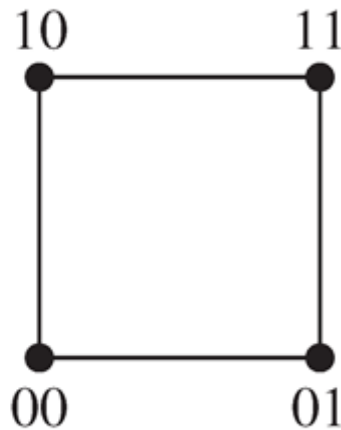
Q_3

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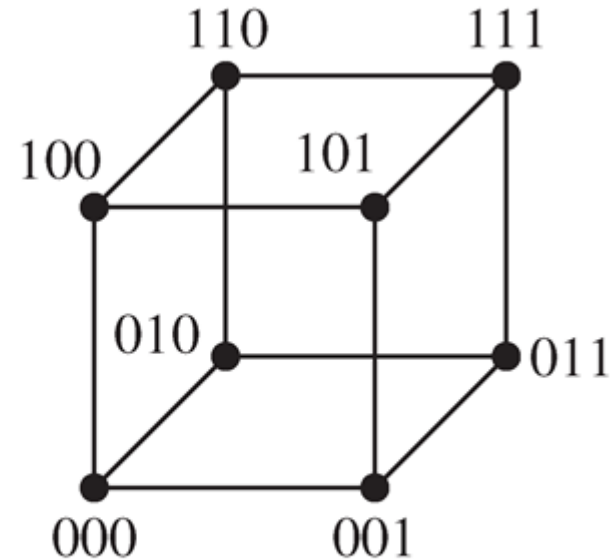
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Q_1



Q_2



Q_3

How many vertices? How many edges?

6 - 3

Bipartite Graphs

- **Definition** A simple graph G is *bipartite* if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2 .

Bipartite Graphs

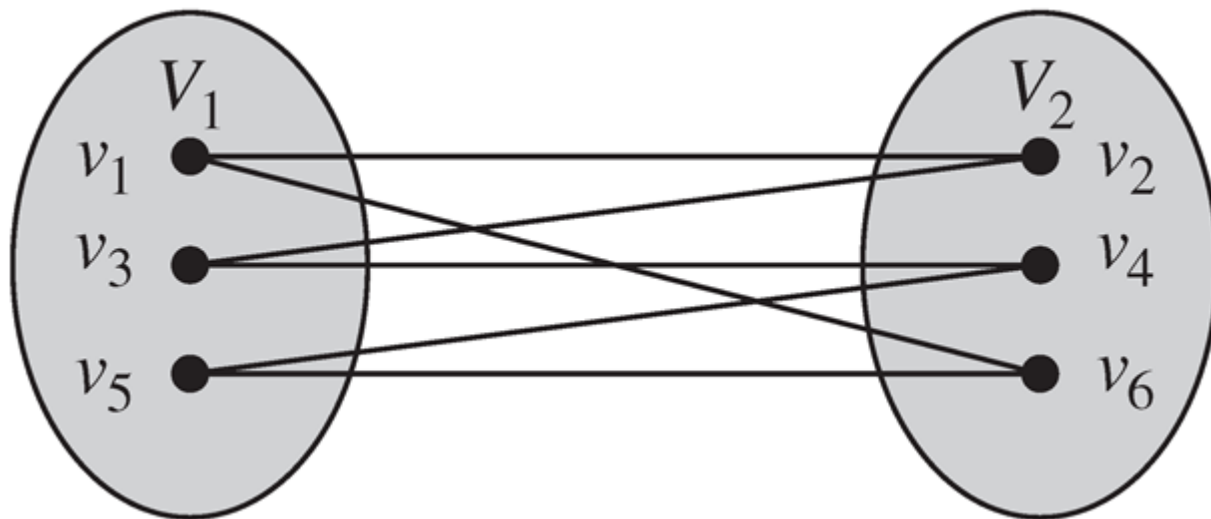
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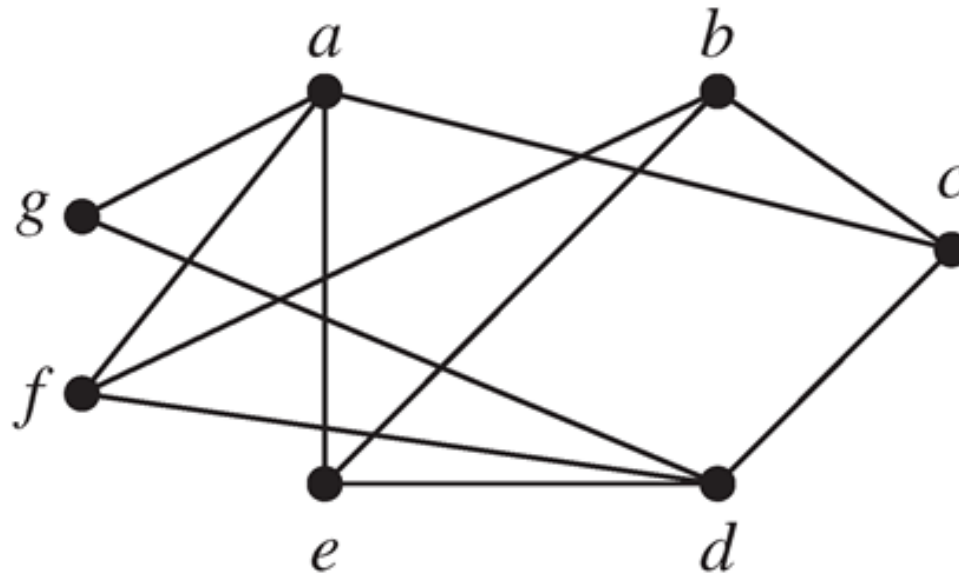
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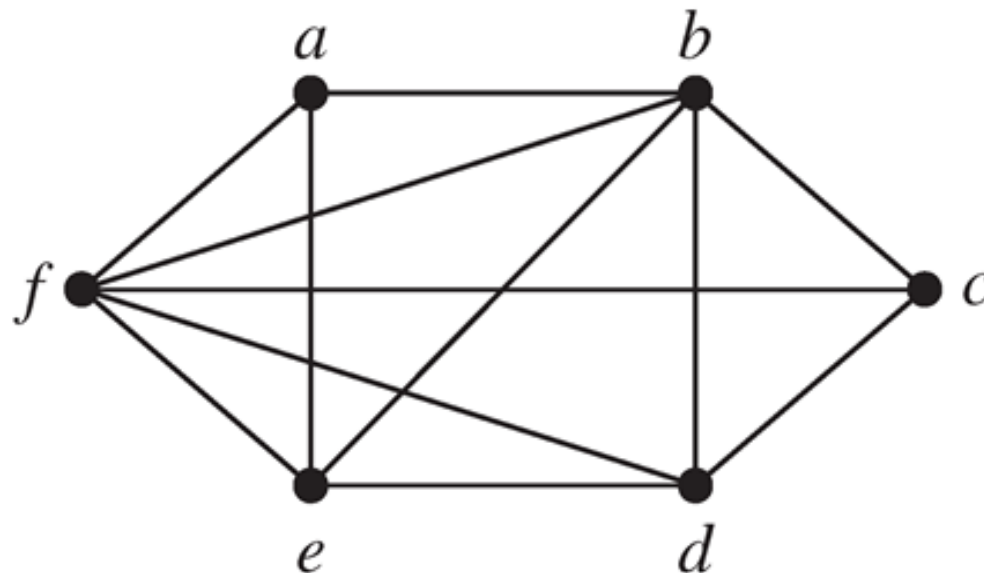
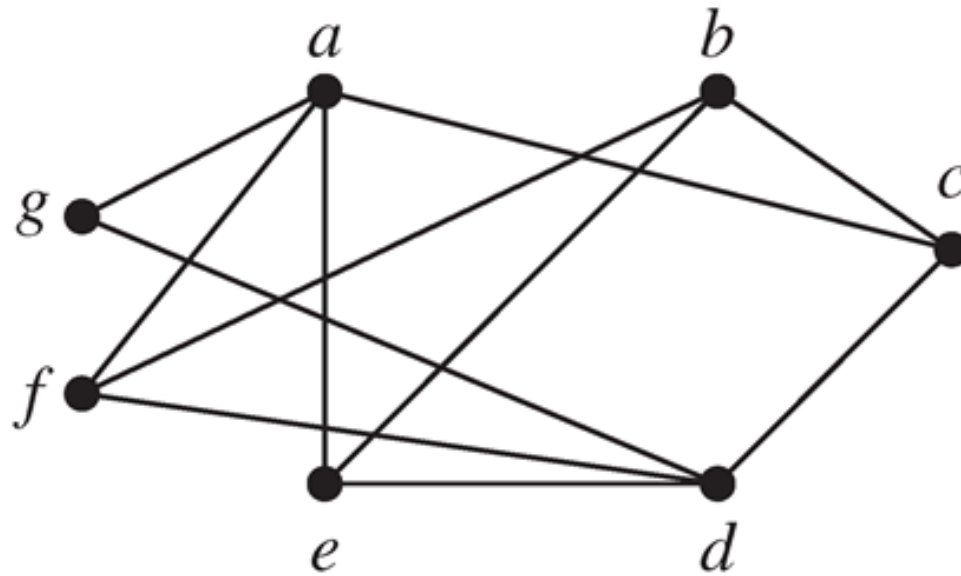
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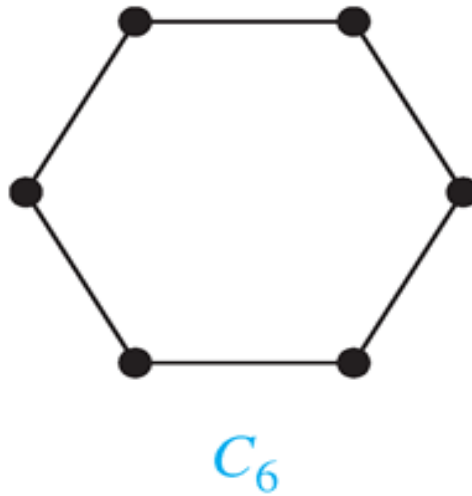


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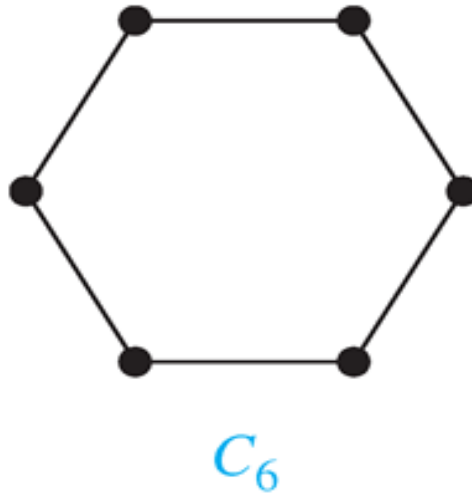
Bipartite Graphs

- **Example** Show that C_6 is bipartite.

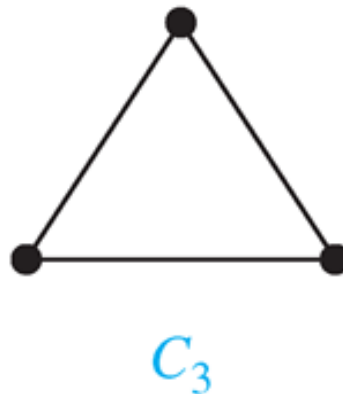


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Example Show that C_3 is not bipartite.

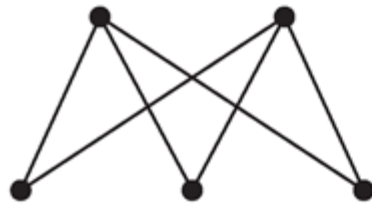


Complete Bipartite Graphs

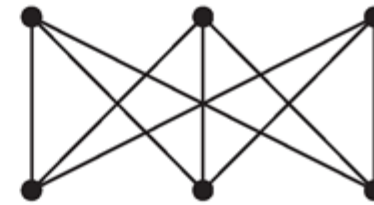
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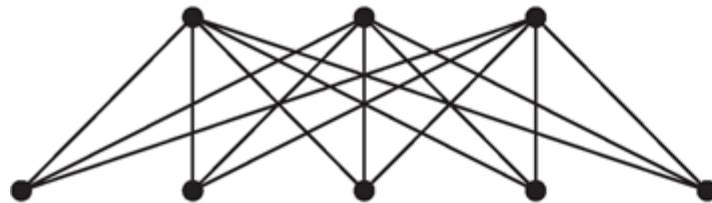
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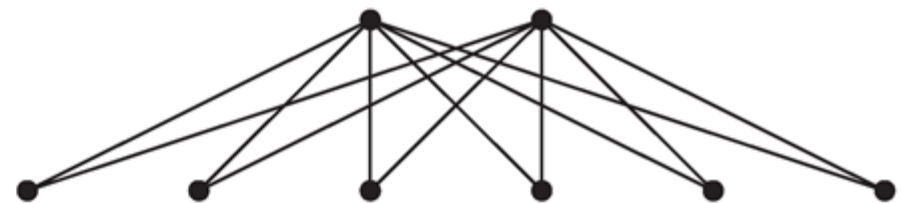
$K_{2,3}$



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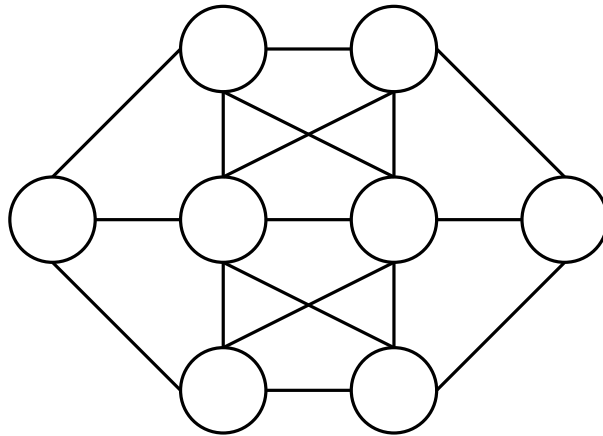
$K_{3,5}$



$K_{2,6}$

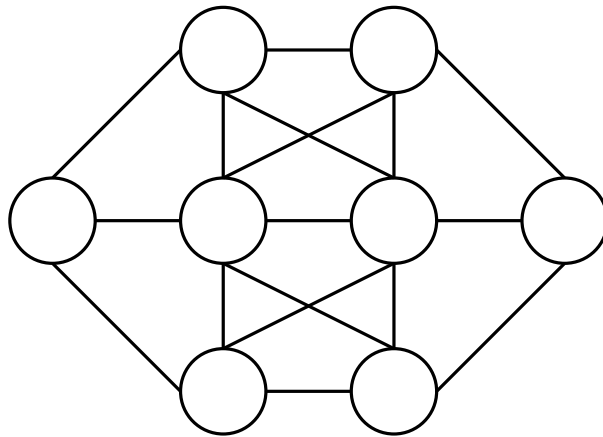
Puzzles using Graphs

- **The eight-circles problem** Place the letters A, B, C, D, E, F, G, H into the eight circles in the figure, in such a way that **no** letter is adjacent to a letter that is next to it in the alphabet.



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- **Six people at a party** Show that, in any gathering of six people, there are either three people who all know each other, or three people none of which knows either of the other two.

Bipartite Graphs and Matchings

- *Matching* the elements of one set to elements in another. A *matching* is a subset of E s.t. **no two edges are incident with the same vertex.**



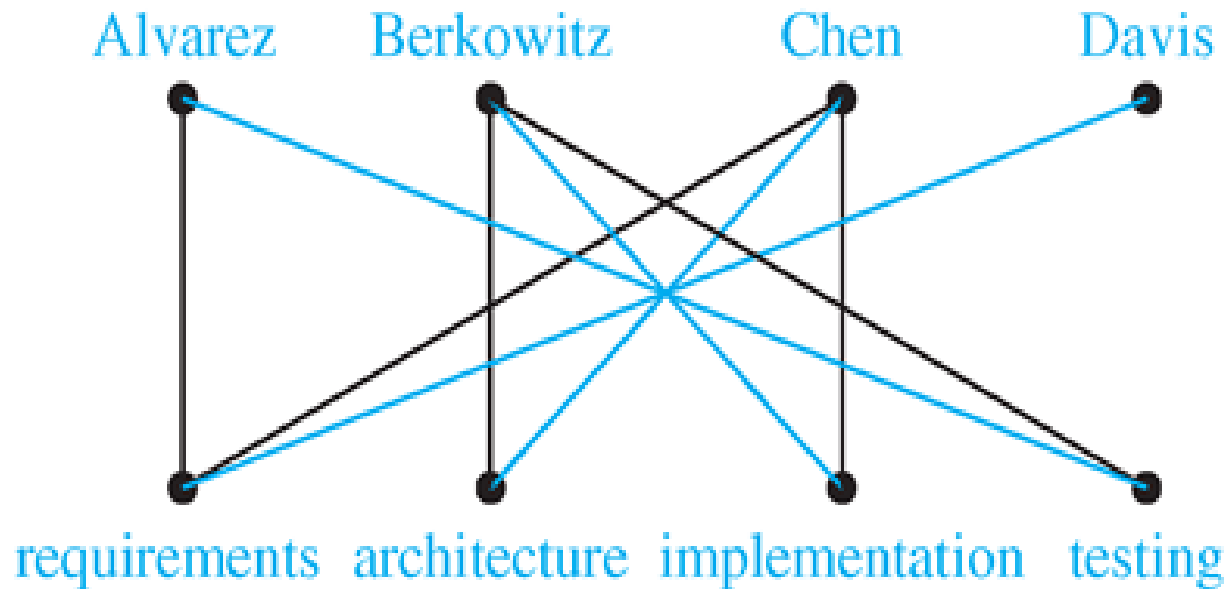
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Job assignments: vertices represent the jobs and the employees, edges link employees with those jobs they have been trained to do. A **common goal** is to **match jobs to employees so that the most jobs are done**.

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A matching M in a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) is a *complete matching from V_1 to V_2* if **every vertex in V_1 is the endpoint of an edge in the matching**, or equivalently, if $|M| = |V_1|$.

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Theorem (Hall's Marriage Theorem) The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a *complete matching from V_1 to V_2* if and only if $|N(A)| \geq |A|$ for **all subsets A of V_1** .

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Then, for every vertex $v \in A$, there is an edge in M connecting v to a vertex in V_2 . Thus, there are at least as many vertices in V_2 that are neighbors of vertices in V_1 as there are vertices in V_1 .

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Case (i): For all integers j with $1 \leq j \leq k$, the vertices in every set of j elements from W_1 are adjacent to at least $j + 1$ elements of W_2

Case (ii): For some integer j with $1 \leq j \leq k$, there is a subset W'_1 of j vertices such that there are exactly j neighbors of these vertices in W_2



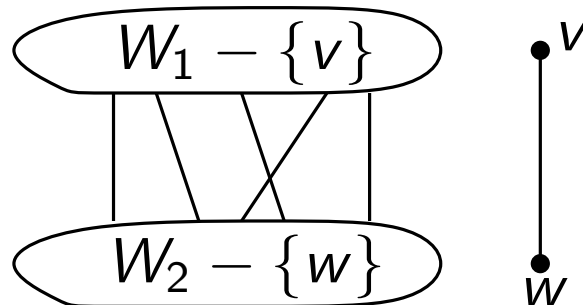
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If not, there is a subset B of t vertices with $1 \leq t \leq k + 1 - j$ s.t. $|N(B)| < t$.



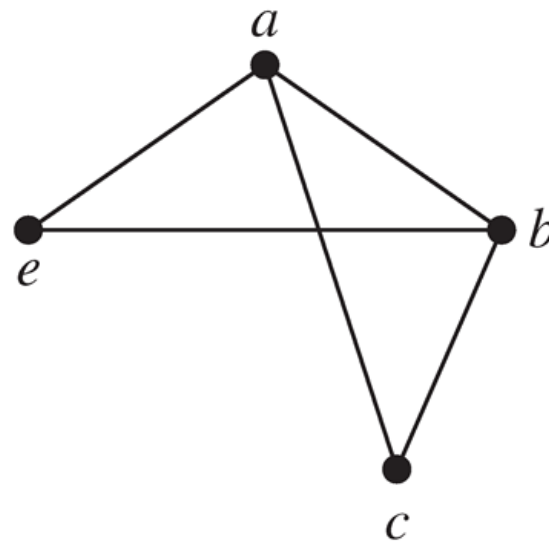
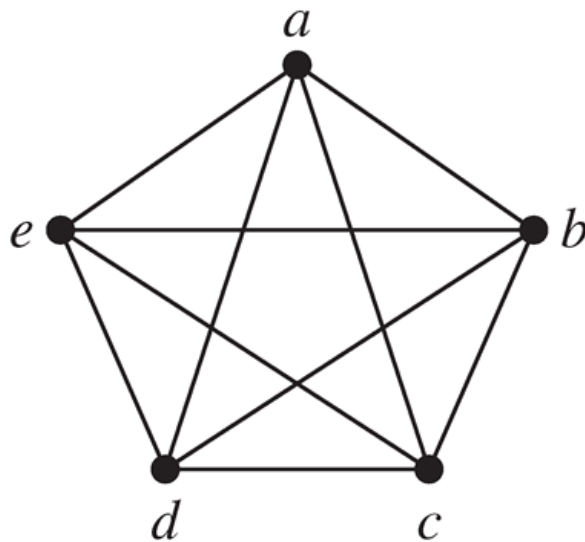
Subgraphs

- **Definition** A *subgraph of a graph* $G = (V, E)$ is a graph (W, F) , where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H \neq G$.



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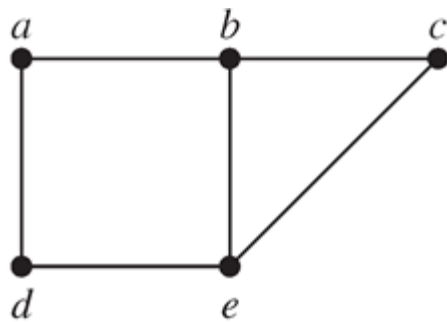
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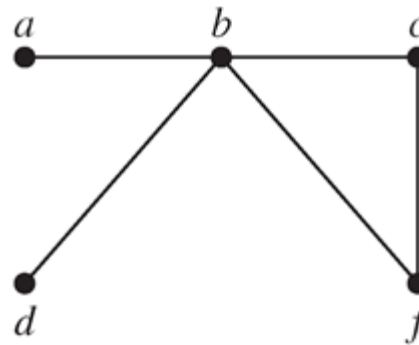


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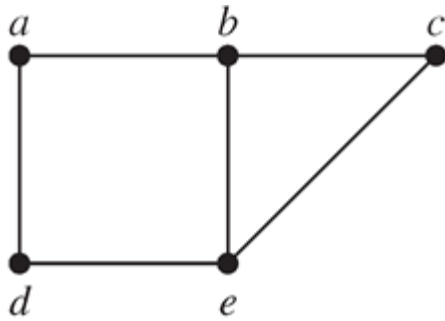
G_1



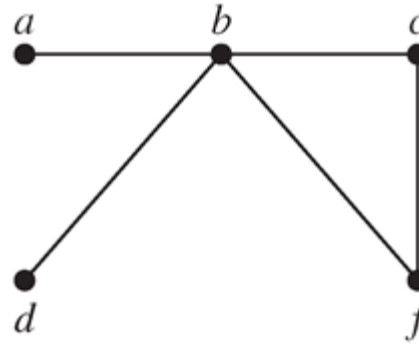
G_2

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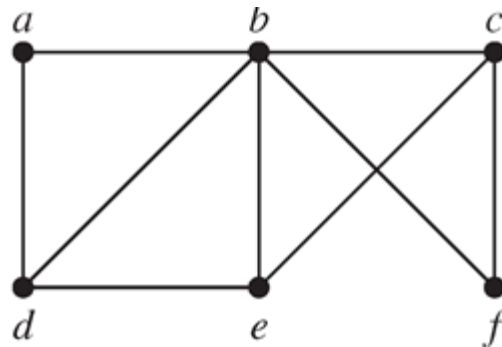
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G_1



G_2



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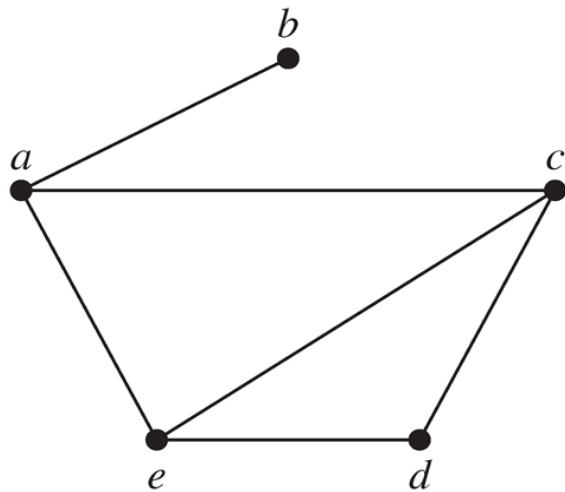
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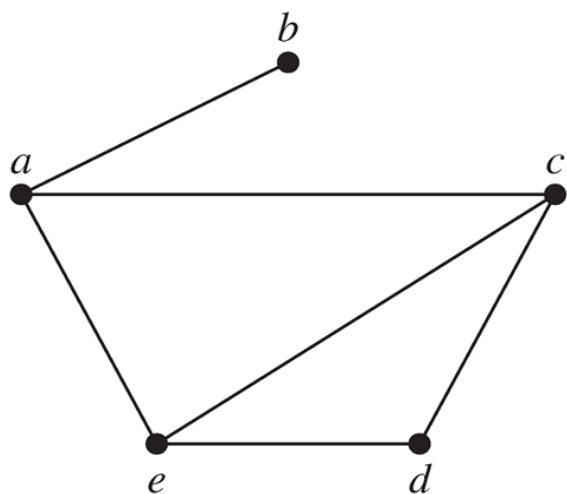
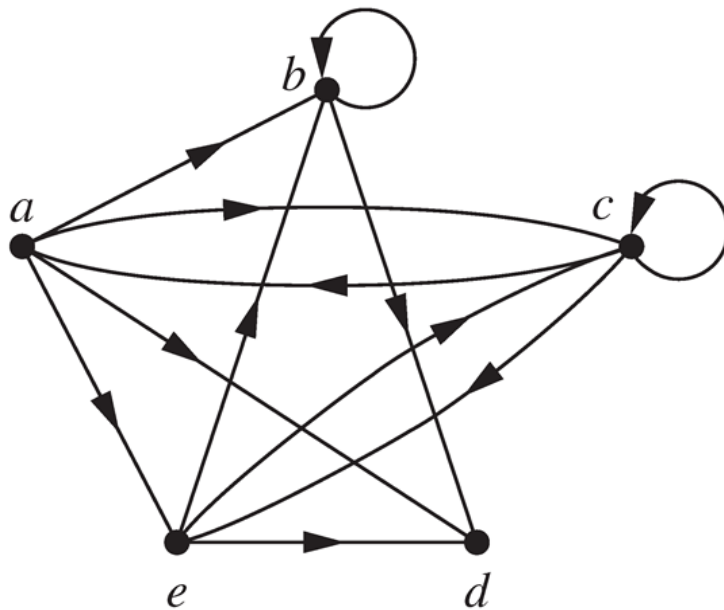


TABLE 1 An Adjacency List for a Simple Graph.

| Vertex | Adjacent Vertices |
|----------|-------------------|
| <i>a</i> | <i>b, c, e</i> |
| <i>b</i> | <i>a</i> |
| <i>c</i> | <i>a, d, e</i> |
| <i>d</i> | <i>c, e</i> |
| <i>e</i> | <i>a, c, d</i> |

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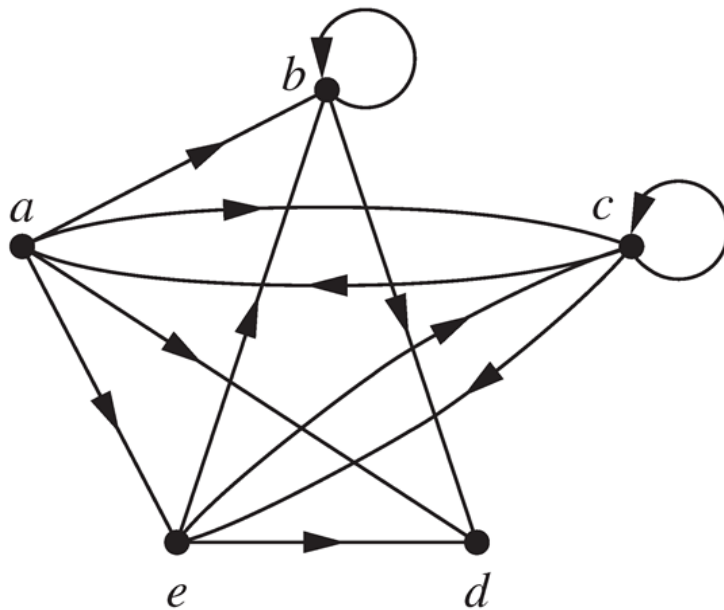


TABLE 2 An Adjacency List for a Directed Graph.

| <i>Initial Vertex</i> | <i>Terminal Vertices</i> |
|-----------------------|--------------------------|
| <i>a</i> | <i>b, c, d, e</i> |
| <i>b</i> | <i>b, d</i> |
| <i>c</i> | <i>a, c, e</i> |
| <i>d</i> | |
| <i>e</i> | <i>b, c, d</i> |

Adjacency Matrices

- **Definition** Suppose that $G = (V, E)$ is a simple graph with $|V| = n$. Arbitrarily list the vertices of G as v_1, v_2, \dots, v_n . The *adjacency matrix* \mathbf{A}_G of G , is the $n \times n$ zero-one matrix with 1 as its (i, j) -th entry when v_i and v_j are adjacent, and 0 as its (i, j) -th entry when they are not adjacent.



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$$\mathbf{A}_G = [a_{ij}]_{n \times n}, \text{ where}$$

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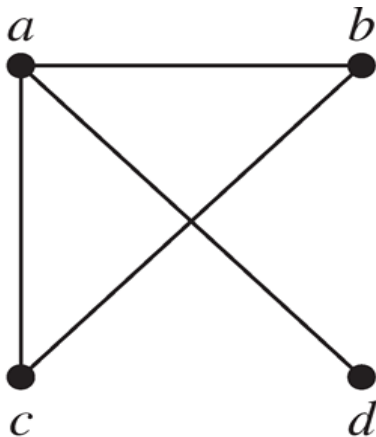


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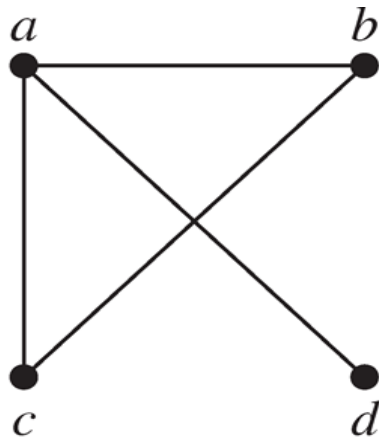


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$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

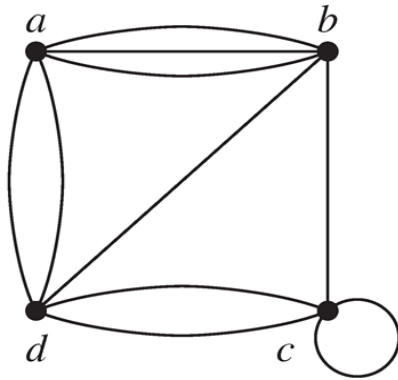
Adjacency Matrices

- Adjacency matrices can also be used to represent graphs with loops and multiple edges. The matrix is no longer a zero-one matrix.



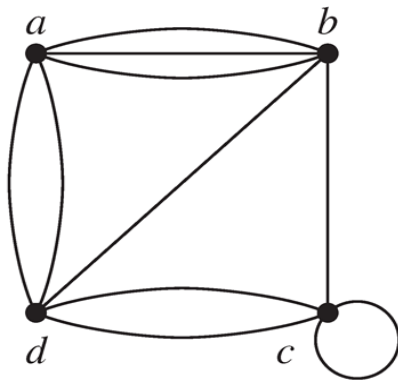
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$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Incidence Matrices

- **Definition** Let $G = (V, E)$ be an undirected graph with vertices v_1, v_2, \dots, v_n and edges e_1, e_2, \dots, e_m . The *incidence matrix* with respect to the ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [m_{ij}]_{n \times m}$, where

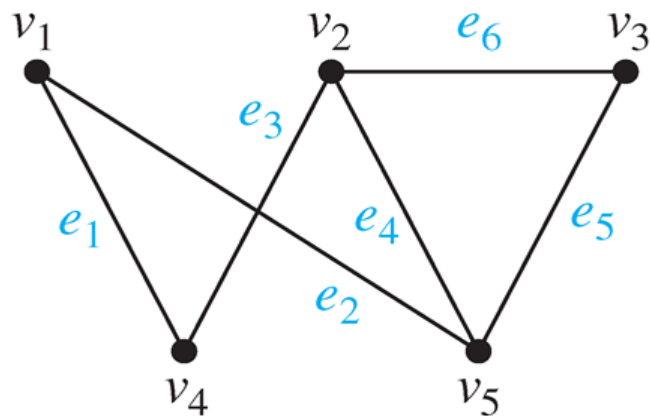
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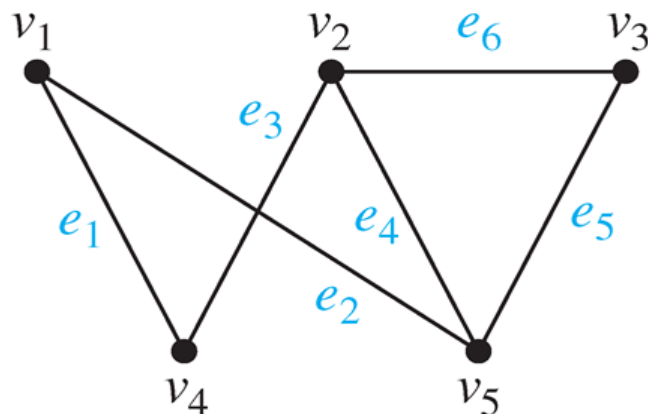
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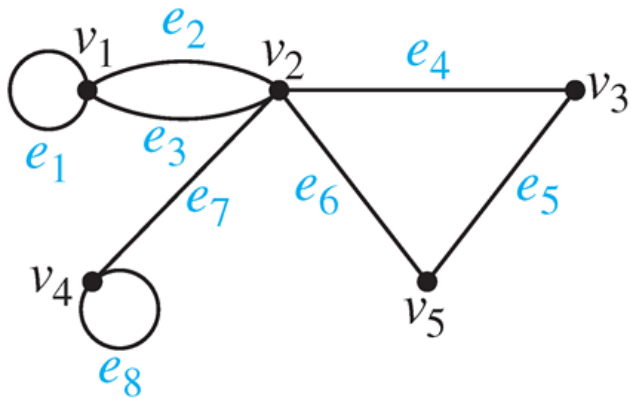


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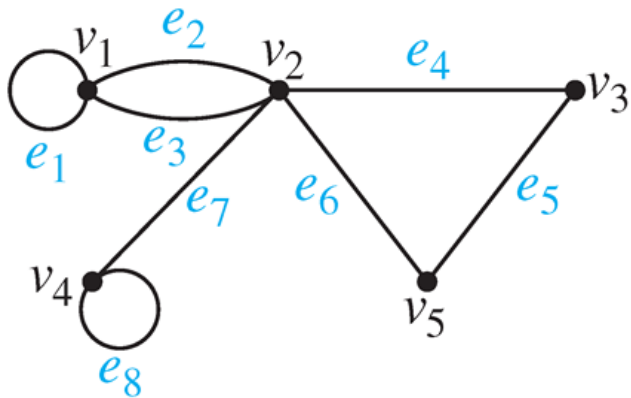
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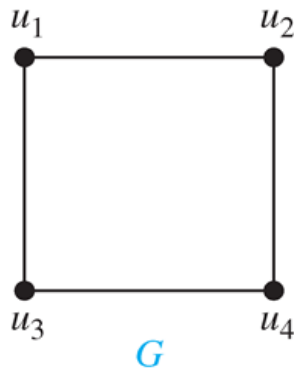
Isomorphism of Graphs

- **Definition** The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a **one-to-one** and **onto** function from V_1 to V_2 with the property that **a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2** , for all a and b in V_1 . Such a function is called an *isomorphism*.

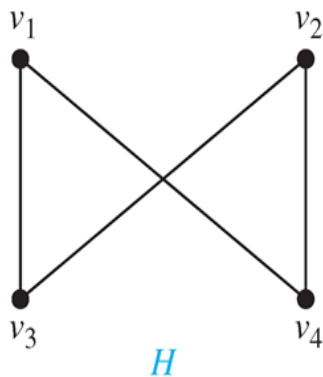


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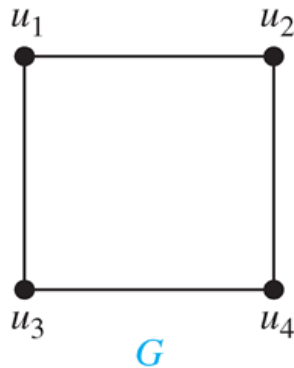


Are the two graphs **isomorphic**?



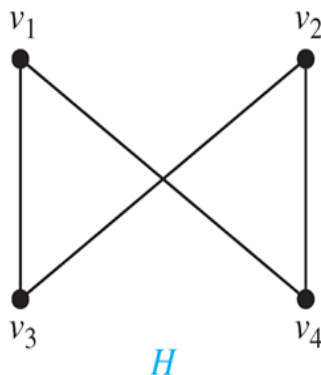
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Are the two graphs **isomorphic**?

Define a **one-to-one correspondence**:
 $f(u_1) = v_1, f(u_2) = v_4, f(u_3) = v_3,$ and
 $f(u_4) = v_2$



Isomorphism of Graphs

- It is usually **difficult** to determine whether two simple graphs are isomorphic **using brute force** since there are $n!$ possible **one-to-one correspondences**.



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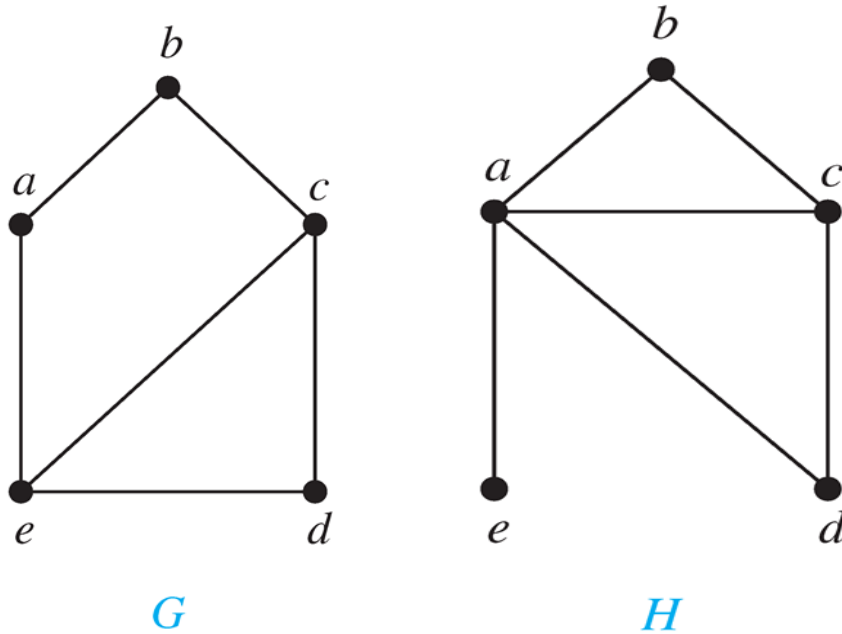
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- Useful **graph invariants** include **the number of vertices**, **number of edges**, **degree sequence**, etc.



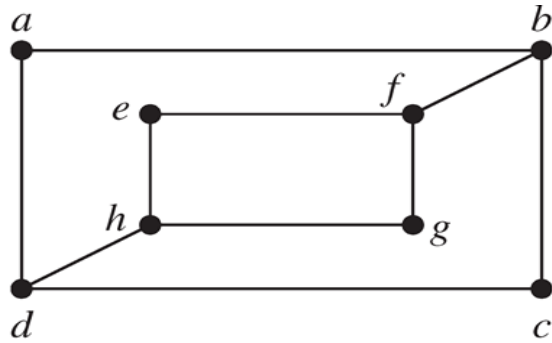
Isomorphism of Graphs

- **Example** Determine whether these two graphs are **isomorphic**.

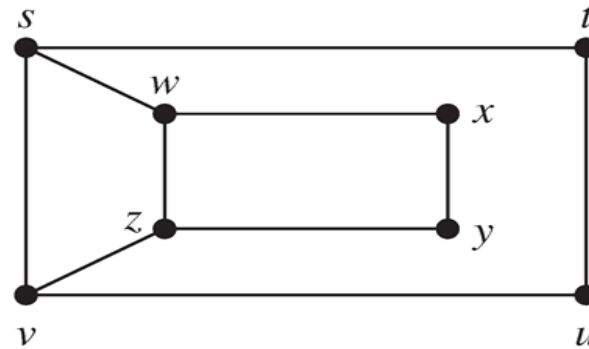


Isomorphism of Graphs

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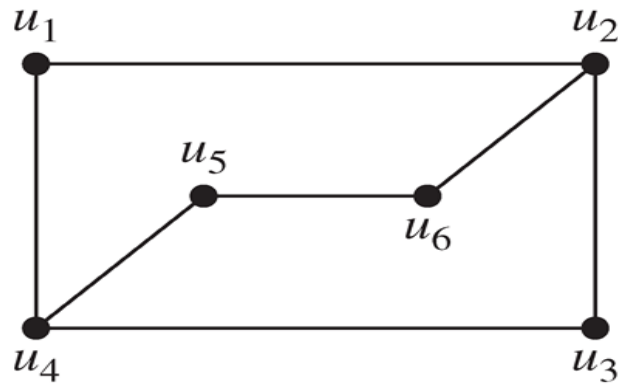
G



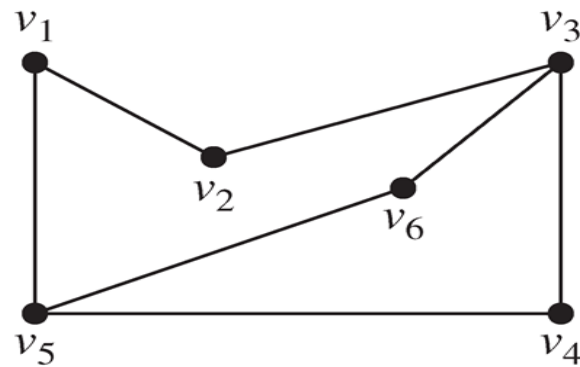
H

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G



H

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- **Definition** Let n be a nonnegative integer and G an undirected graph. A *path of length n* from u to v in G is a sequence of *n edges* e_1, e_2, \dots, e_n of G for which there exists a sequence $x_0 = u, x_1, \dots, x_{n-1}, x_n = v$ of vertices such that e_i has the endpoints x_{i-1} and x_i for $i = 1, \dots, n$. The path is a *circuit* if it *begins and ends at the same vertex*, i.e., if $u = v$ and has length greater than zero. A path or circuit is *simple* if it *does not* contain *repeating vertices*.

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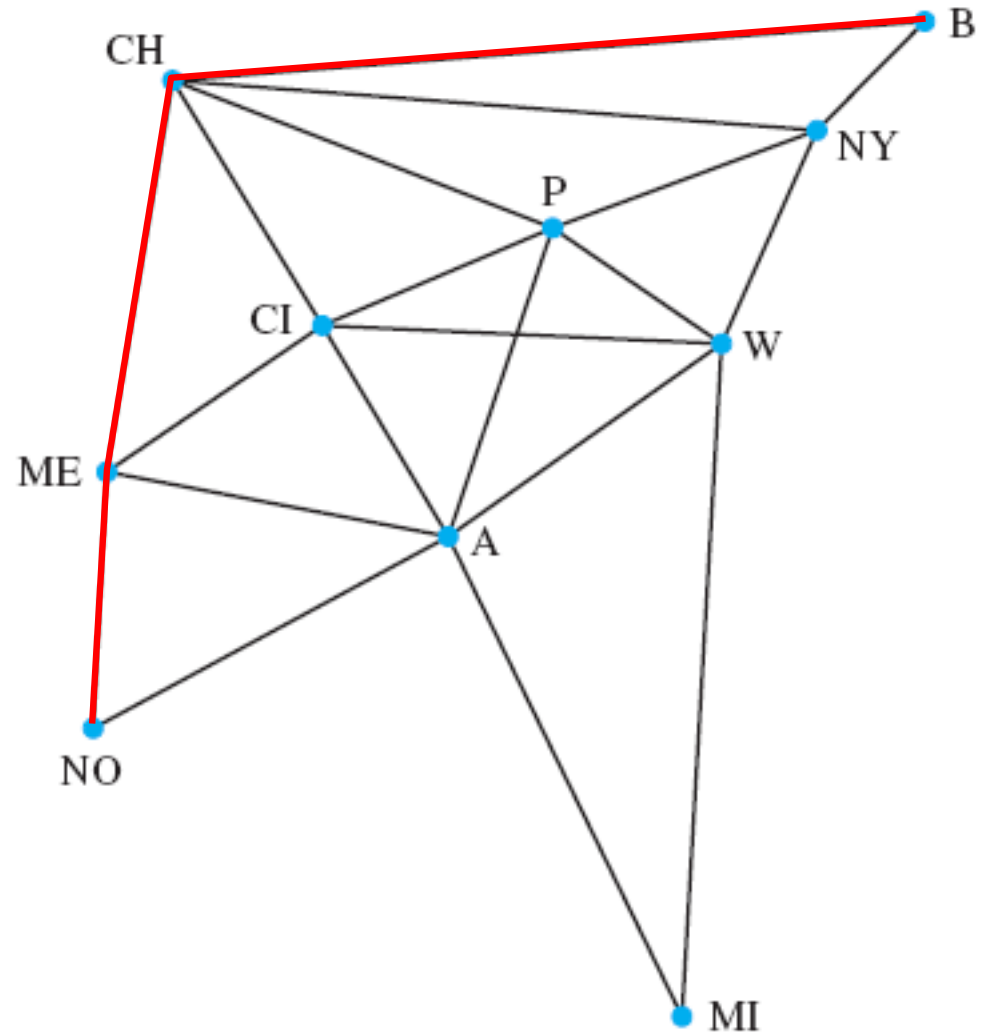


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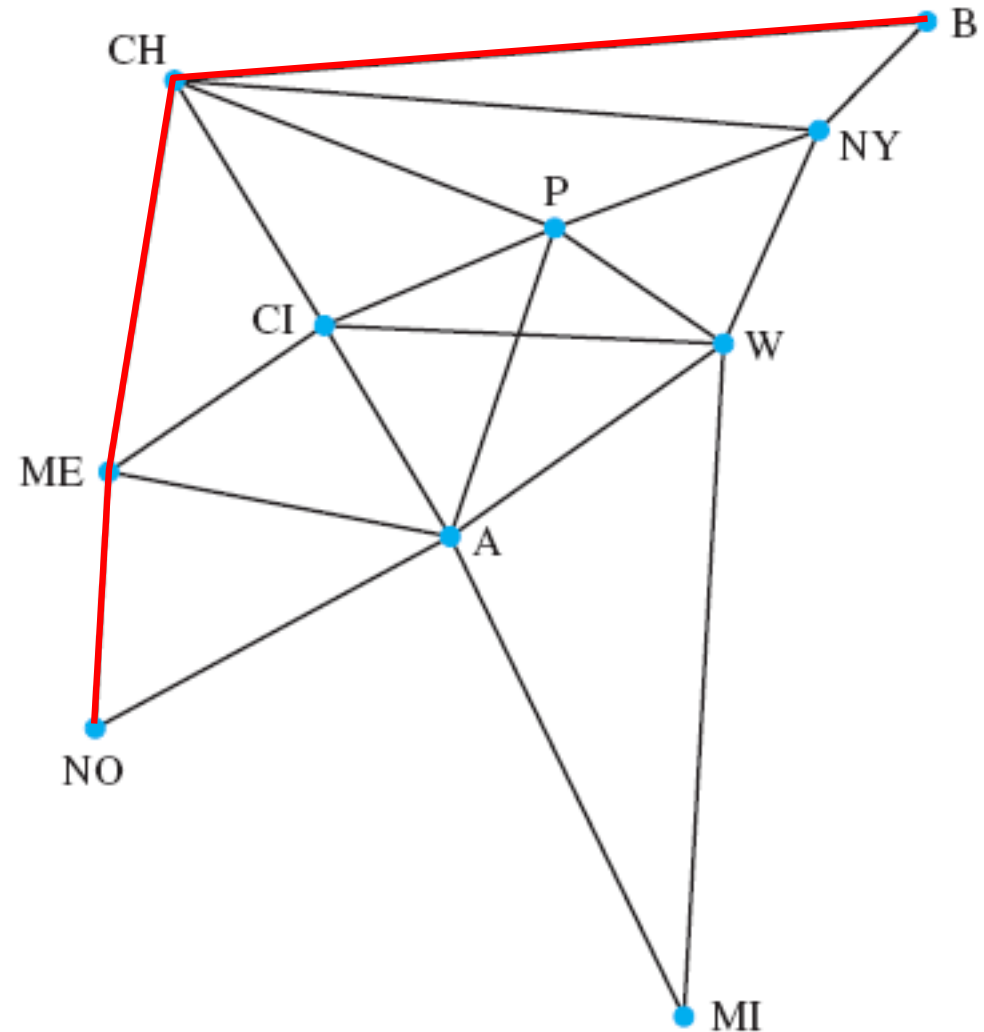
Length of a *path* = # of edges on path

Path



Path

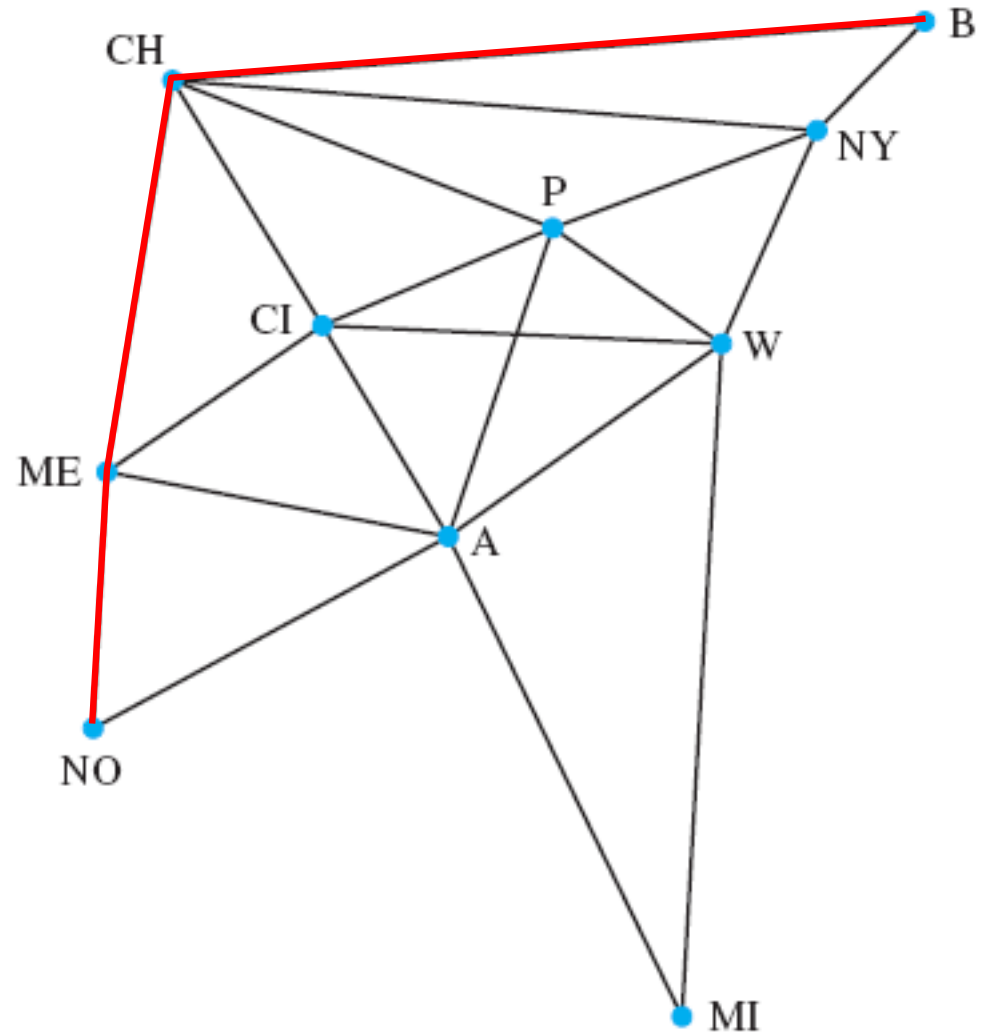
Path from Boston to New Orleans is B, CH, ME, NO



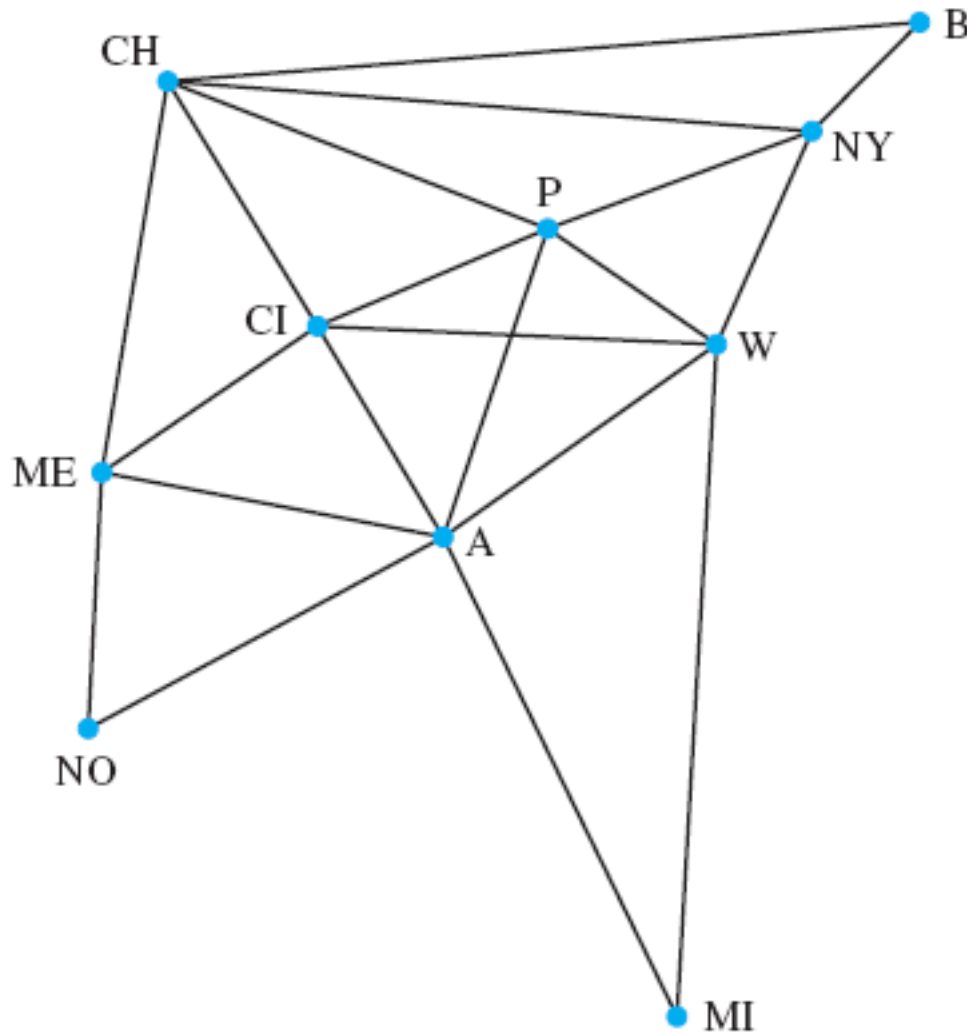
Path

Path from Boston to New Orleans is B, CH, ME, NO

This path has length 3.



Connectivity



Company decides to lease only **minimum number** of communication lines it needs to be able to send a message from any city to any other city by using any number of intermediate cities.

What is the **minimum** number of lines it needs to lease?

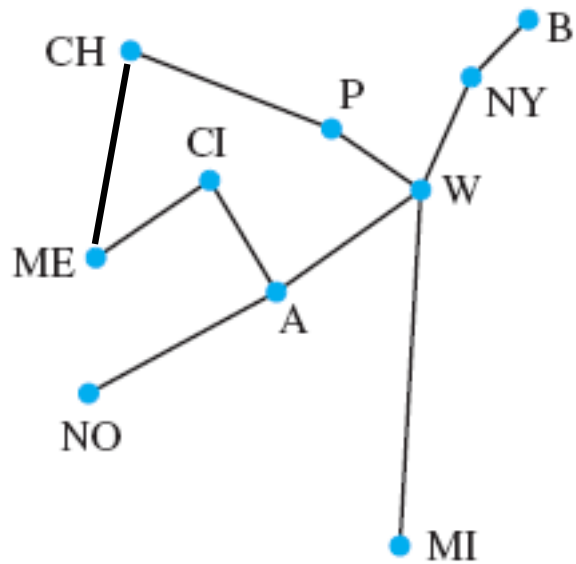
Connectivity

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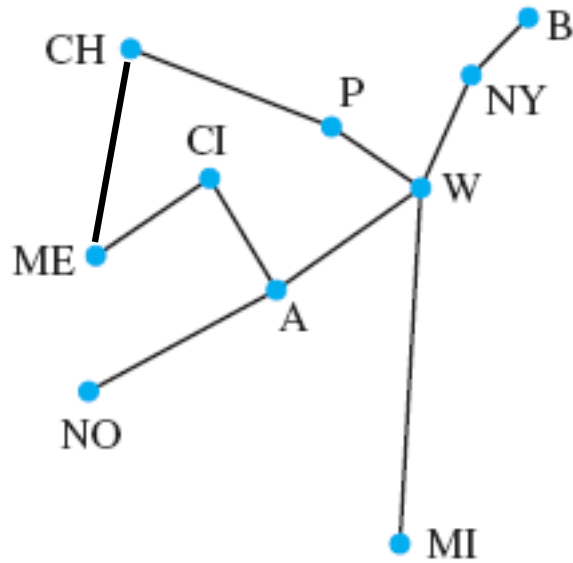
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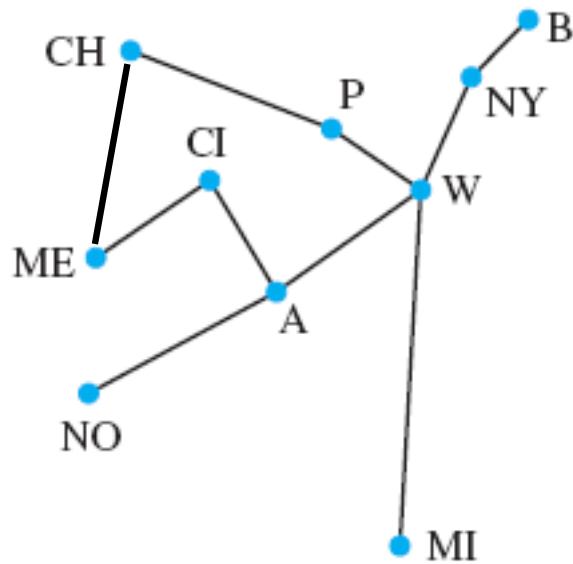


Too many.

Could throw away edge **CI**, **A**, and still have a solution.

Connectivity

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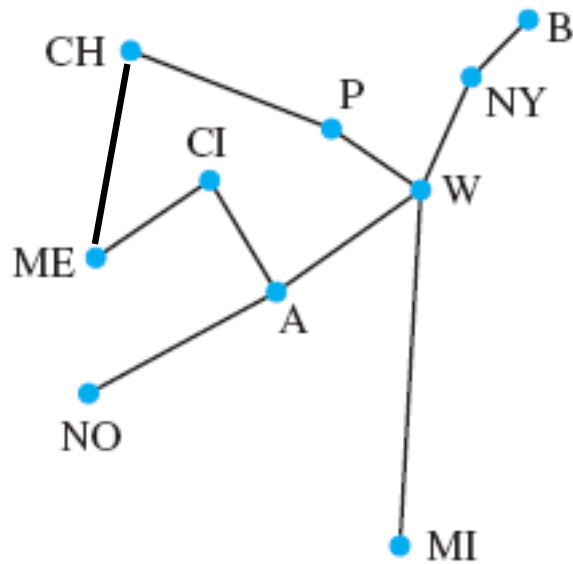
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Choosing 8 edges?

Connectivity

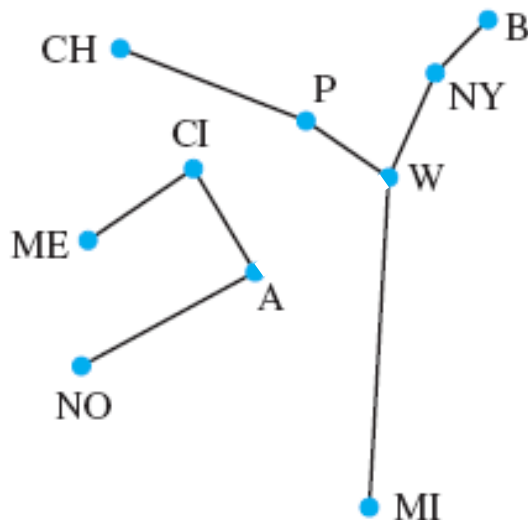
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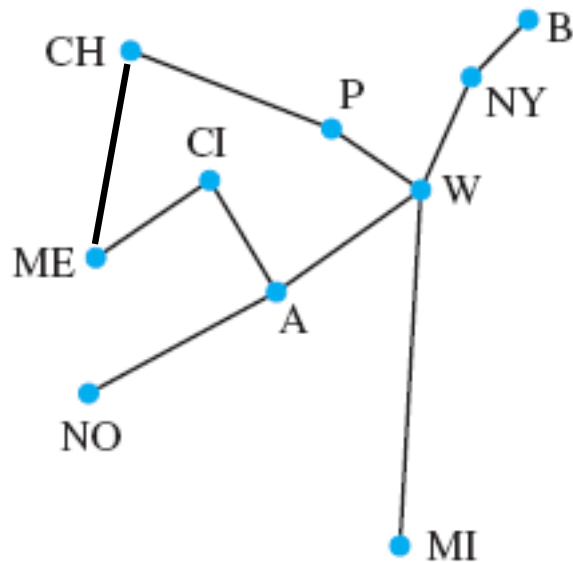
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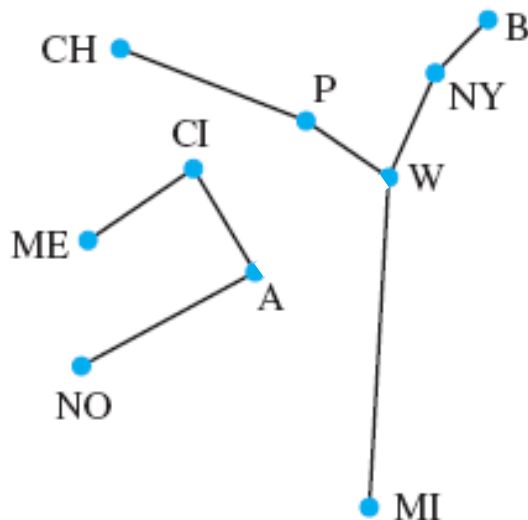
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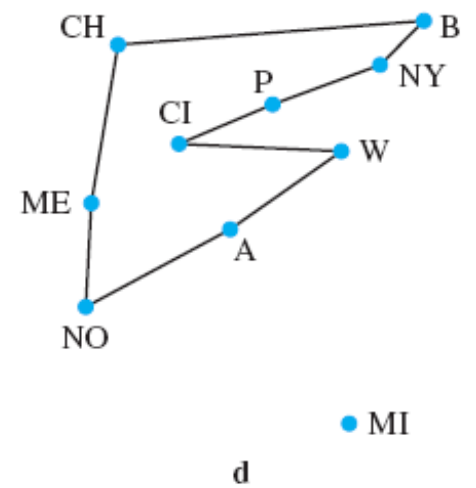
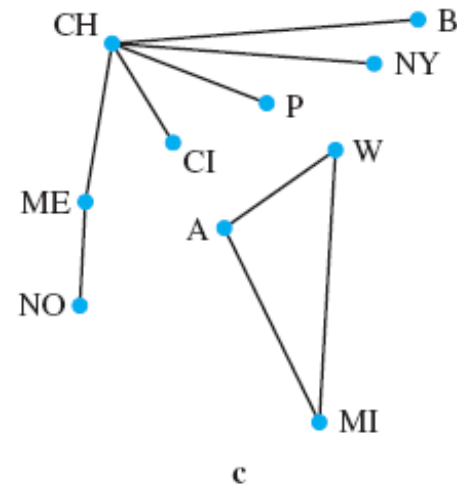
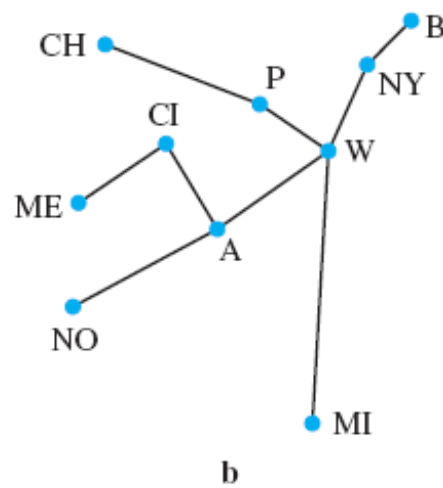
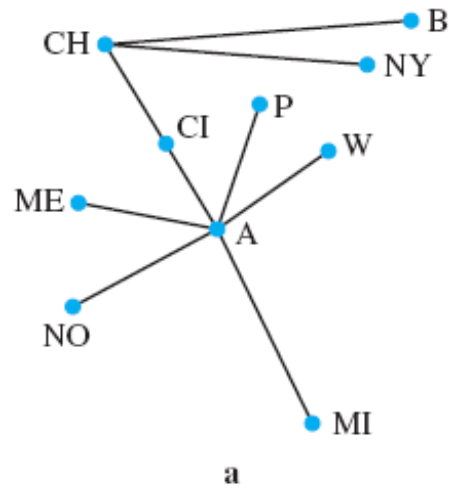
There is **no path** from, e.g., **NO** to **B**.

Connectivity

- Choosing 9 edges:

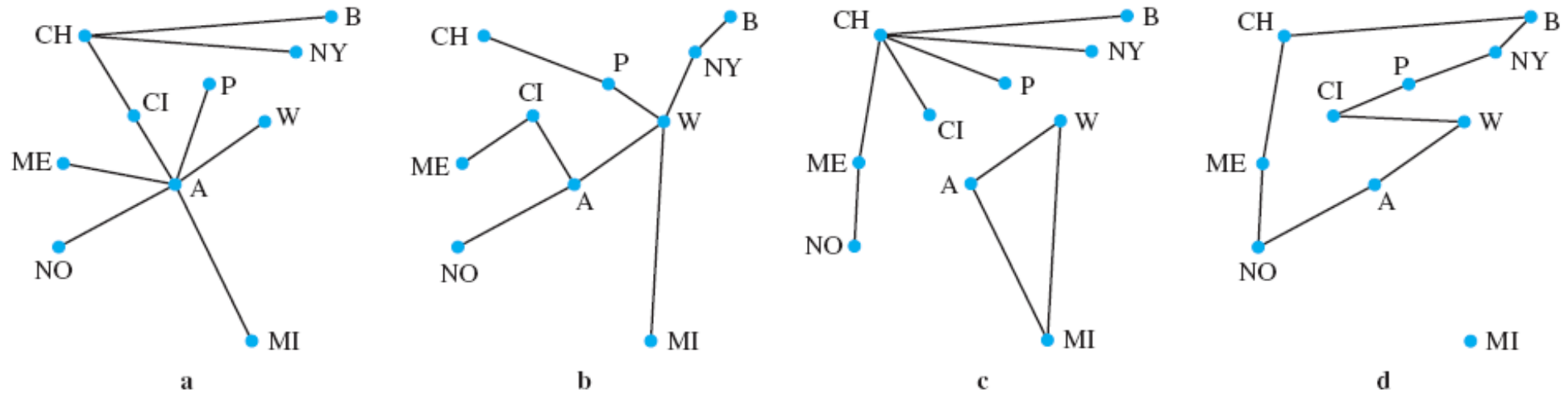
Connectivity

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Connectivity

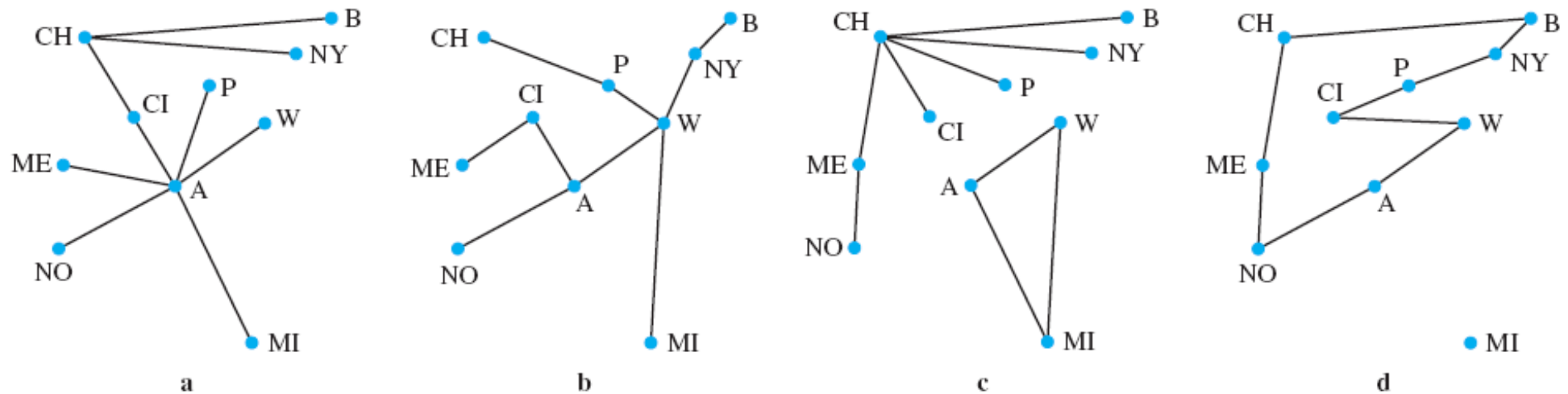
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Two vertices are *connected* if there is a path between them.

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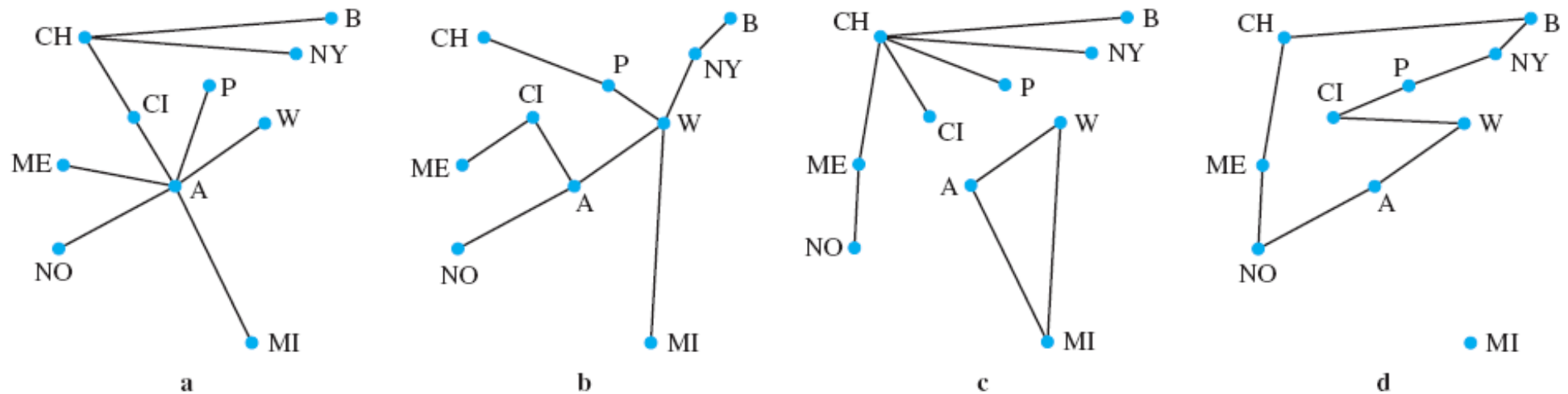


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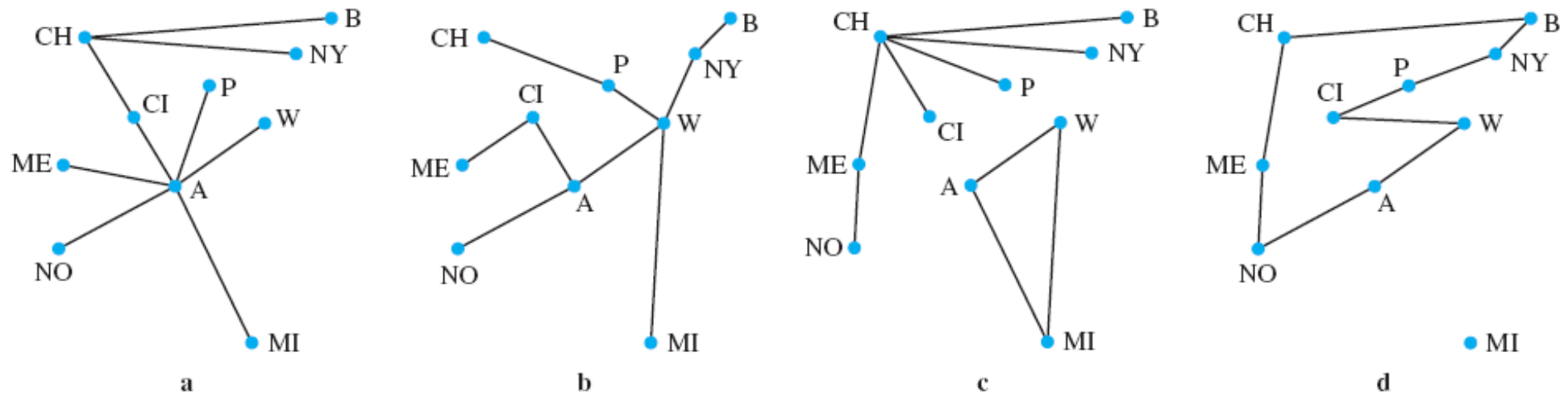
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Example: (a) and (b) are *connected*, (c) and (d) are *disconnected*.

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- **Lemma** If there is a path between two distinct vertices x and y of a graph G , then there is a simple path between x and y in G .



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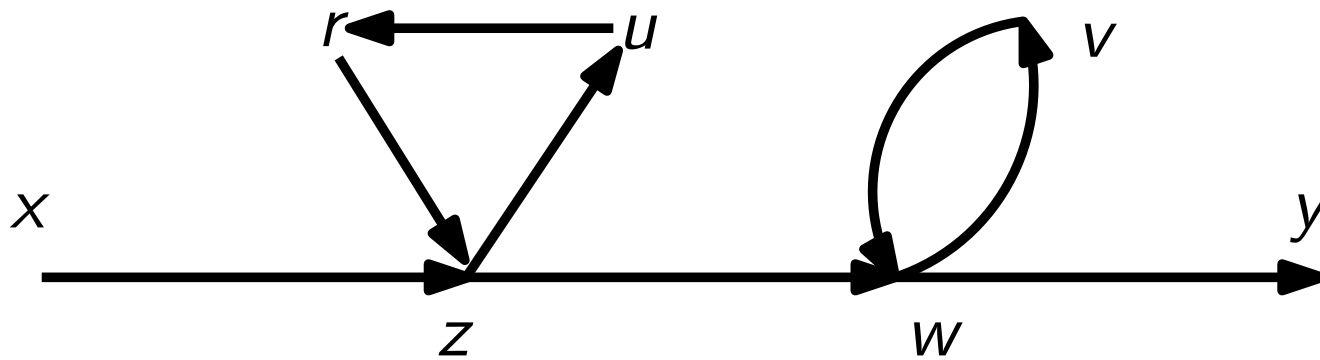
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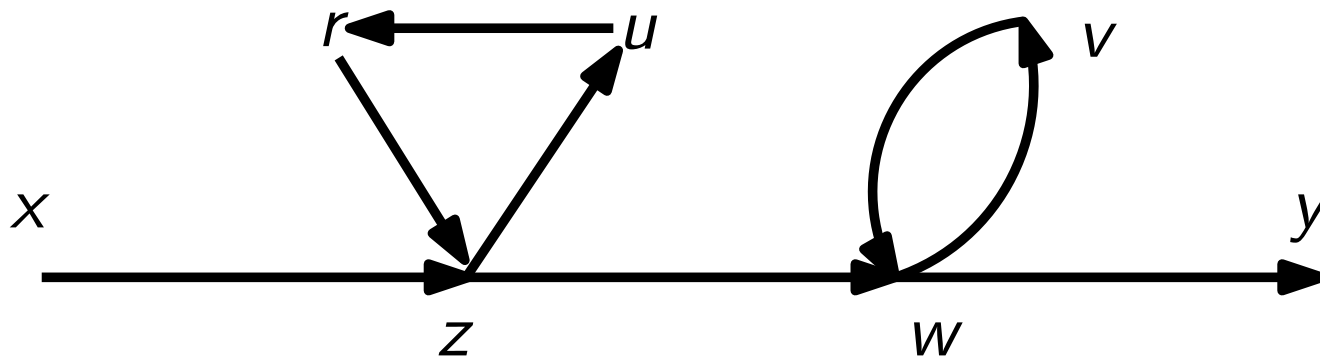
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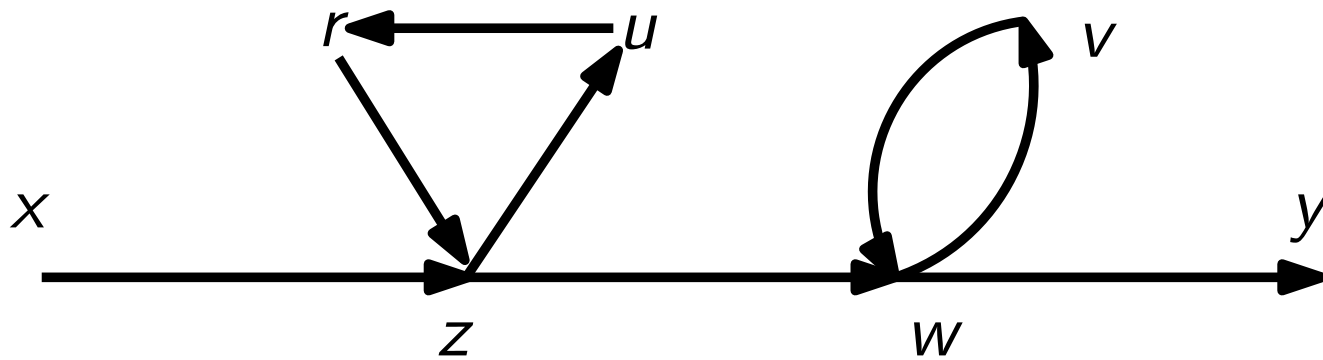
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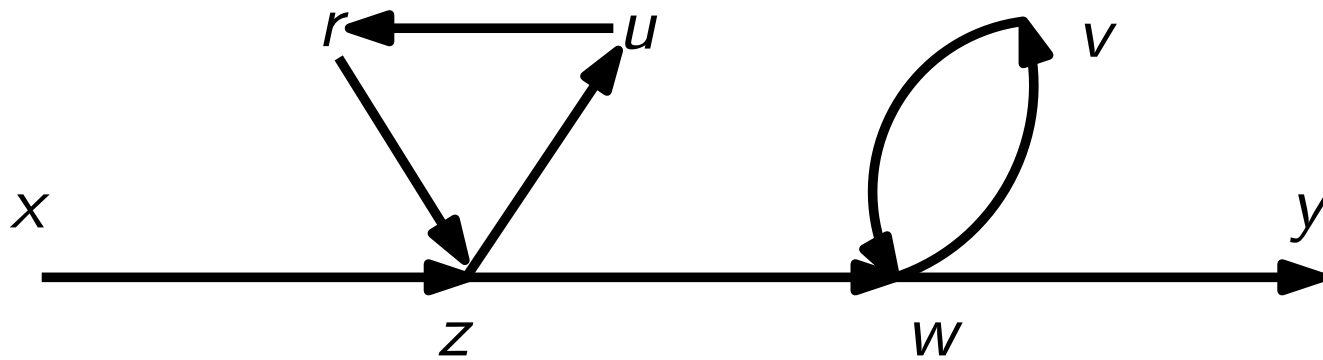


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Theorem There is a simple path between every pair of distinct vertices of a connected undirected graph.

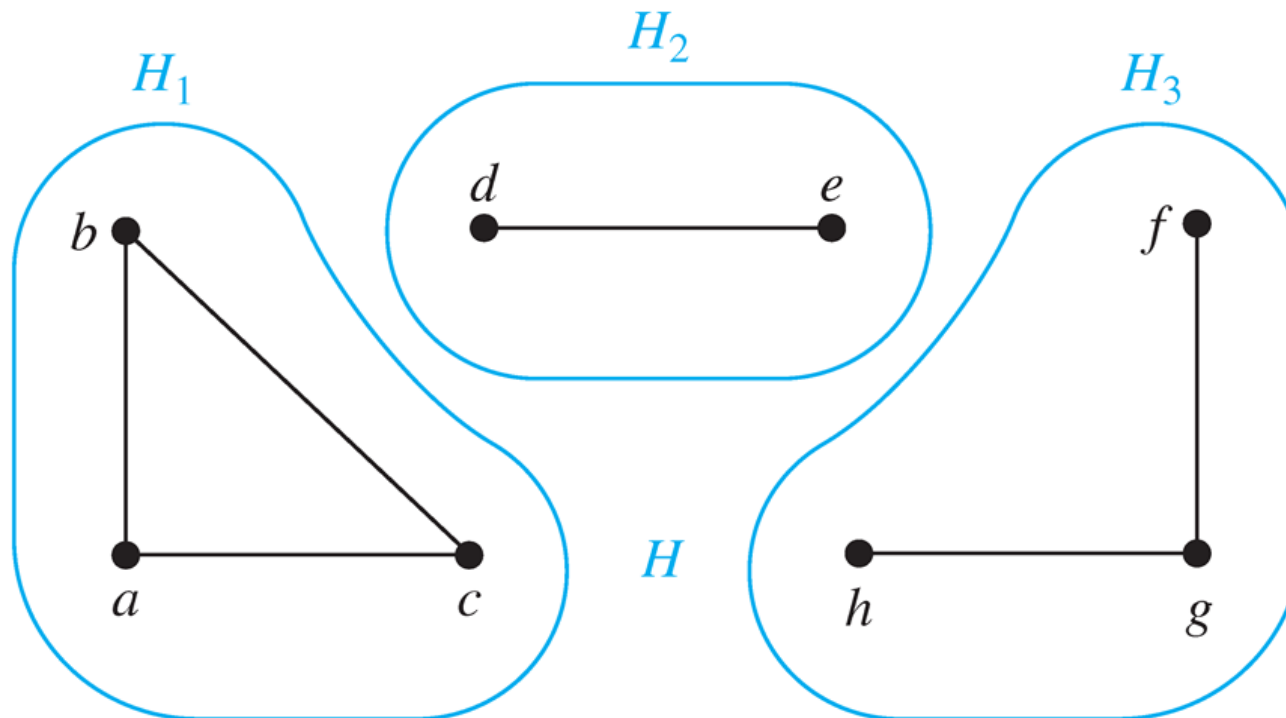
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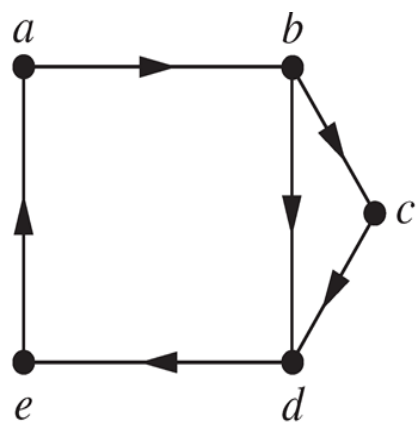
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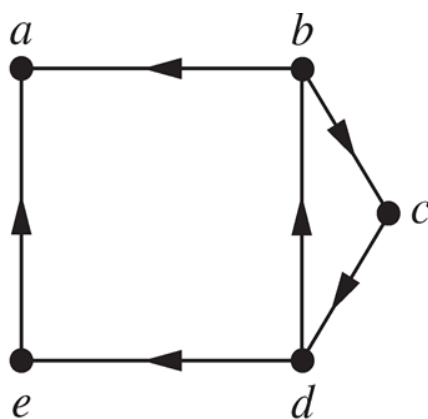
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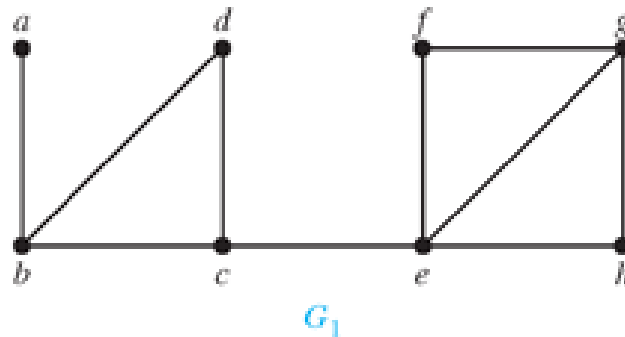
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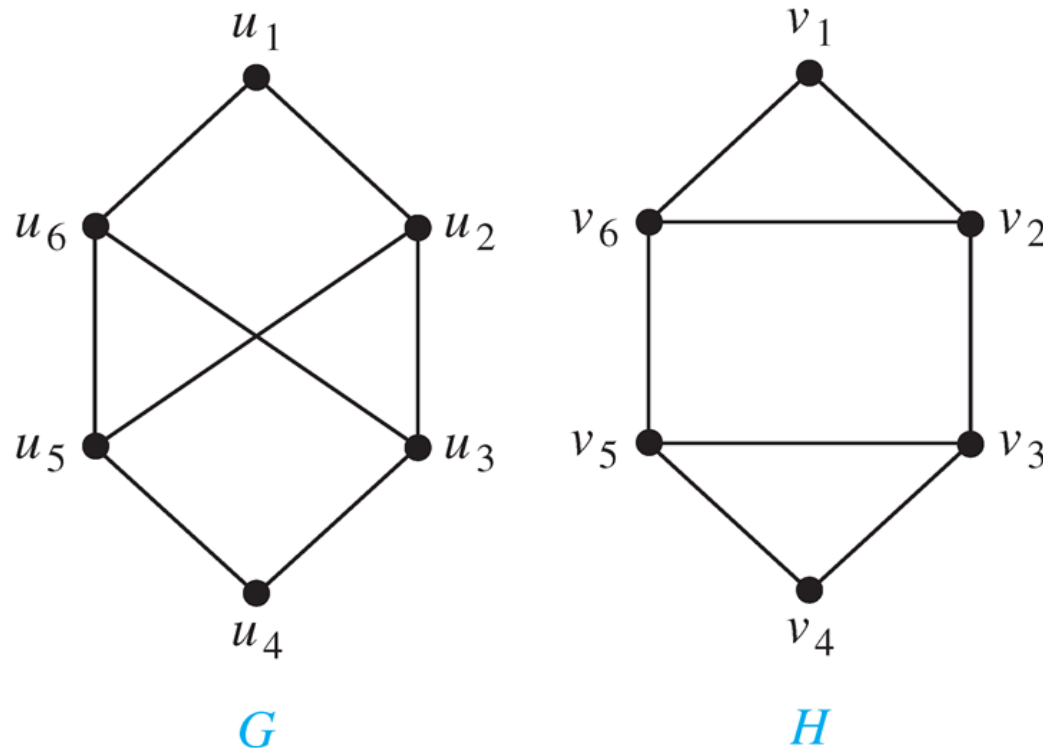
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- The existence of a simple circuit of length k is **isomorphic invariant**. In addition, **paths** can be used to construct mappings that may be **isomorphisms**.



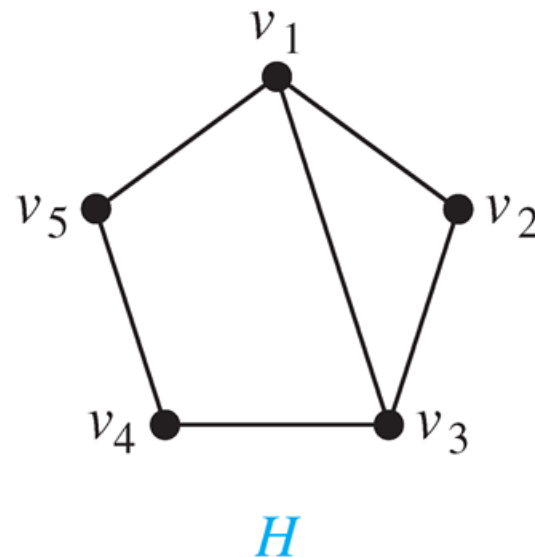
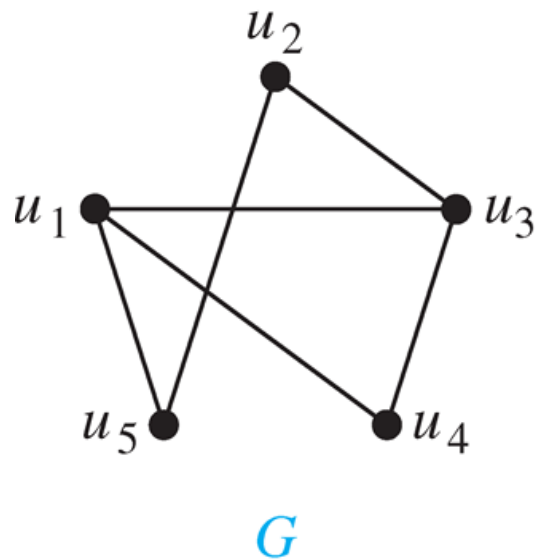
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Counting Paths between Vertices

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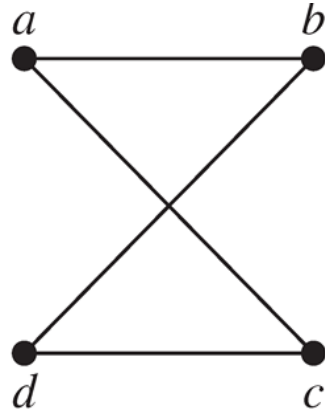
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$\mathbf{A}^{r+1} = \mathbf{A}^r \mathbf{A}$, the (i, j) -th entry of \mathbf{A}^{r+1} equals $b_{i1}a_{1j} + b_{i2}a_{2j} + \dots + b_{in}a_{nj}$, where b_{ik} is the (i, k) -th entry of \mathbf{A}^r .



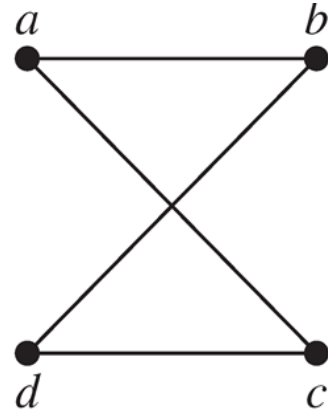
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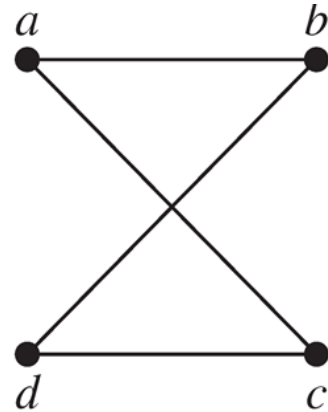
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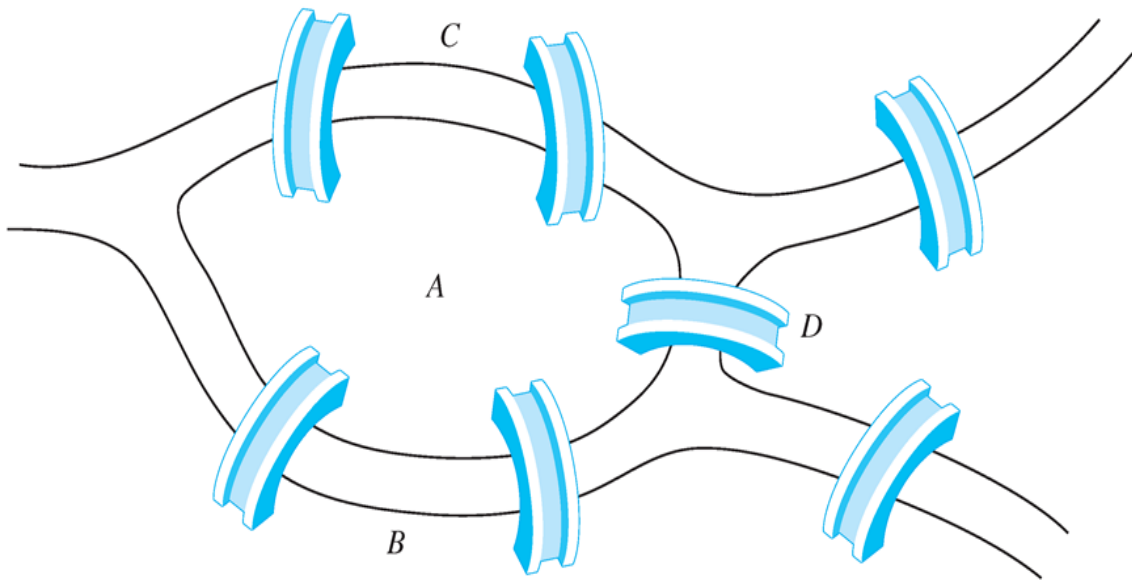


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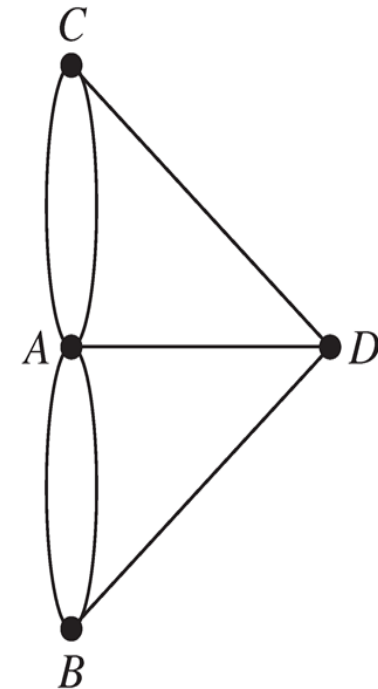
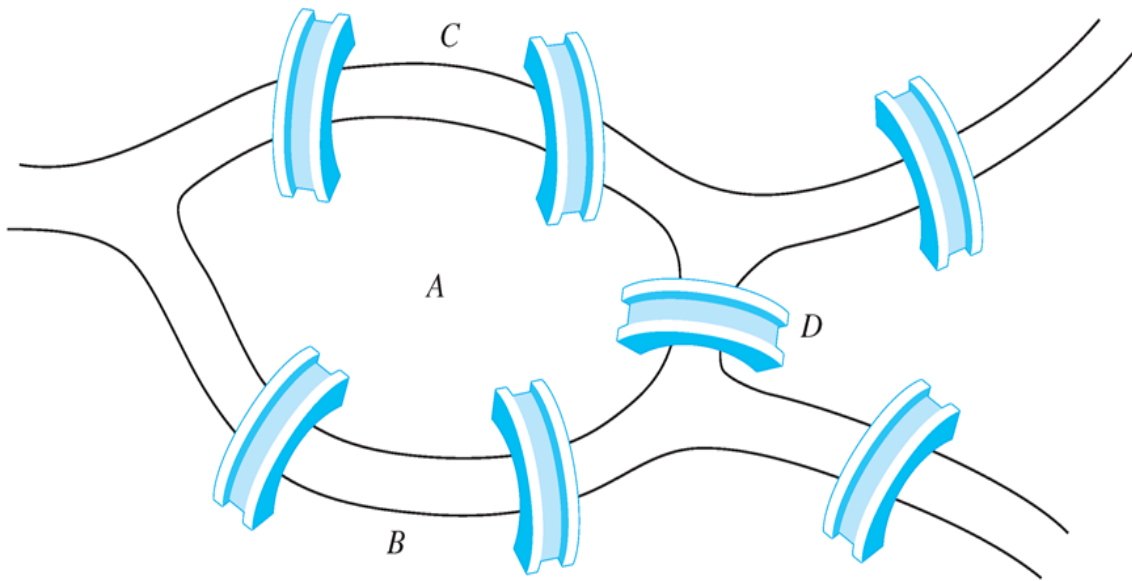
■ Königsberg seven-bridge problem

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Euler Paths and Circuits

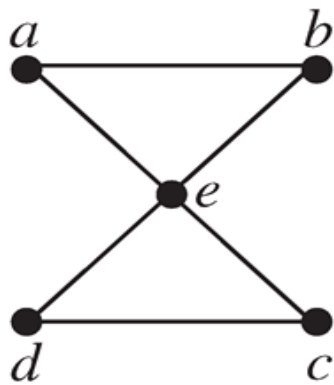
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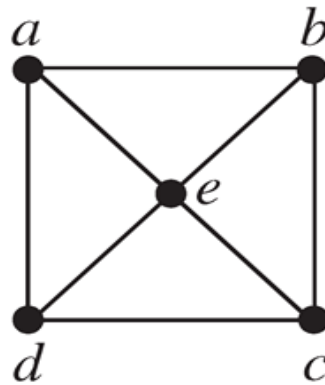
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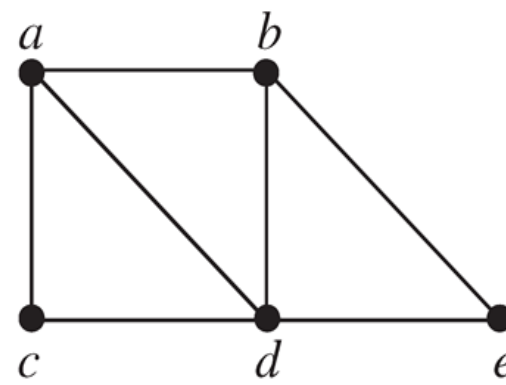
Example Which of the undirected graphs have an Euler circuit? Of those that do not, which have an Euler path?



G_1



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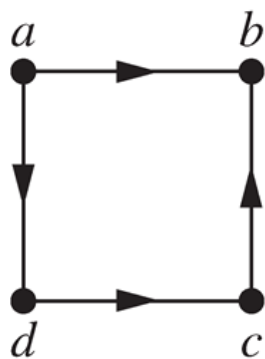


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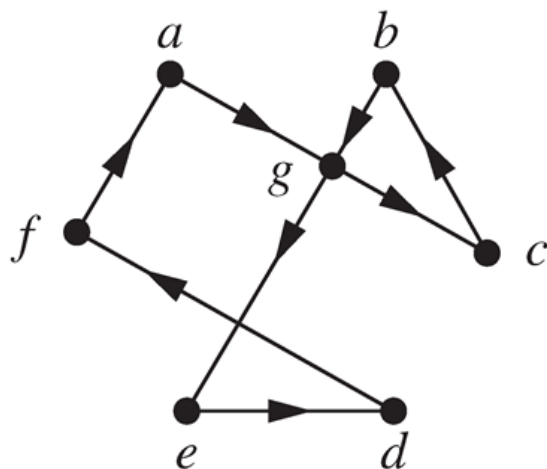
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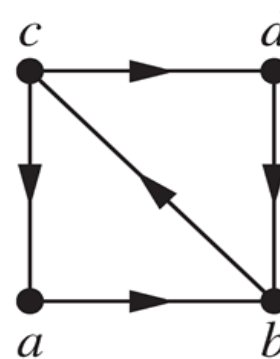
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- ◇ The initial vertex and the final vertex of an Euler path have odd degree.



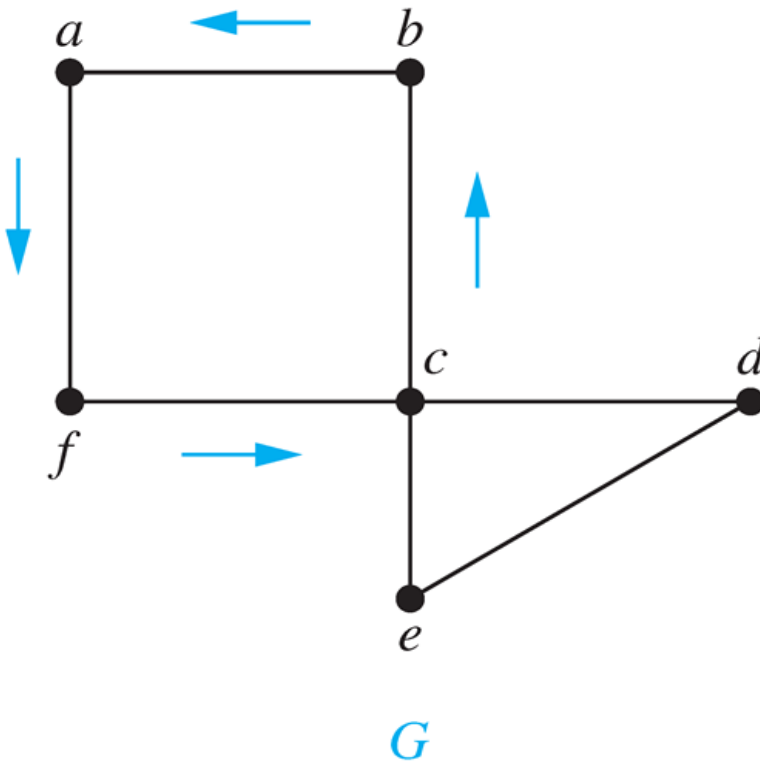
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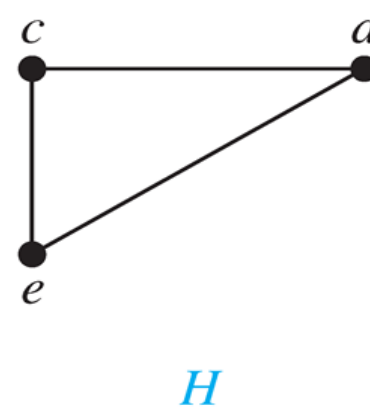
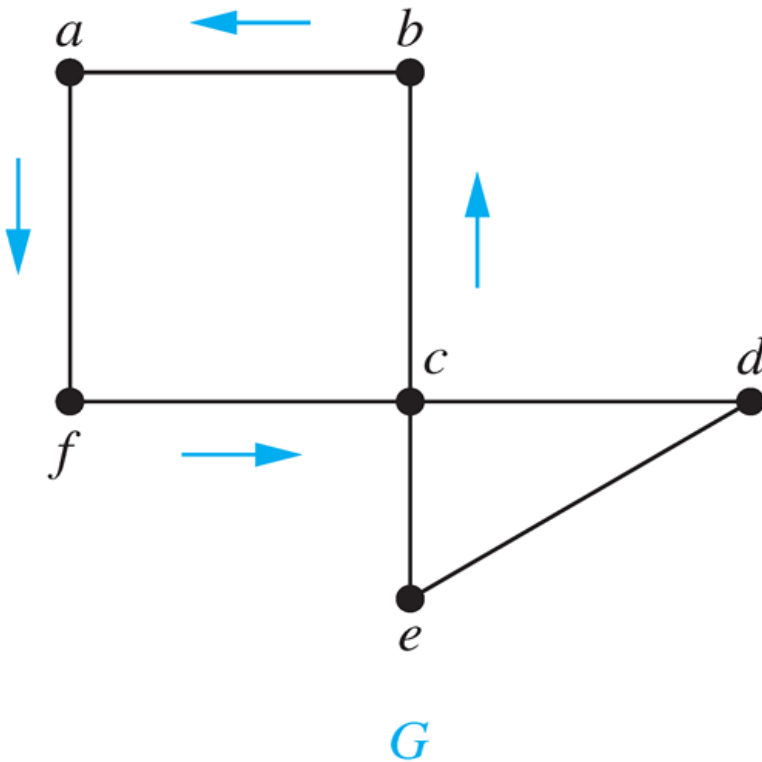
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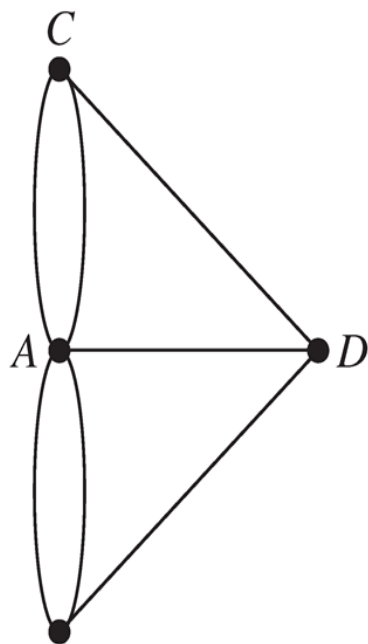
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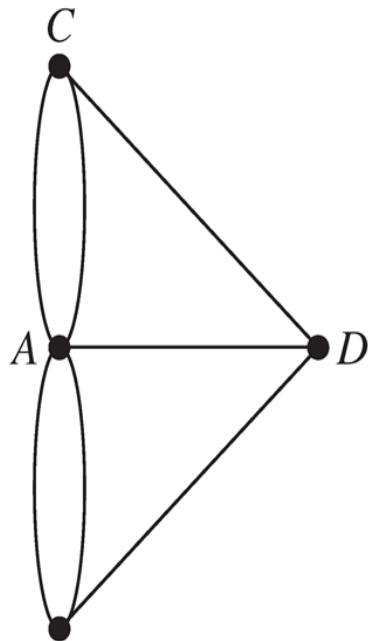
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Euler Circuits and Paths

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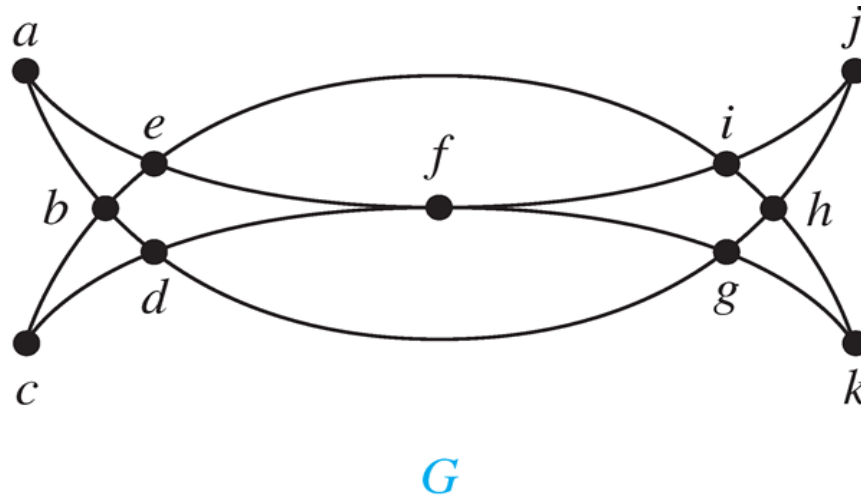
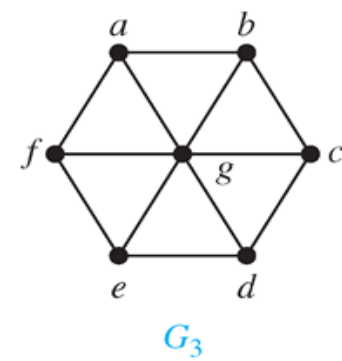
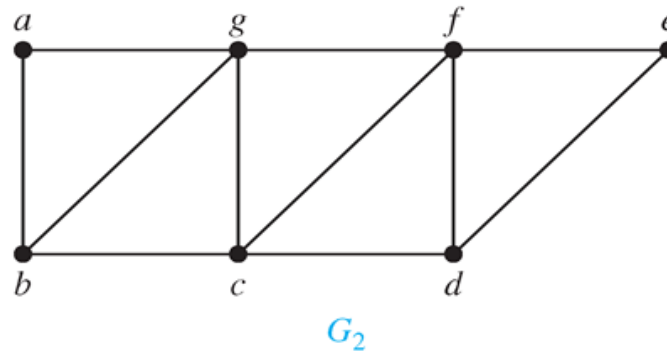
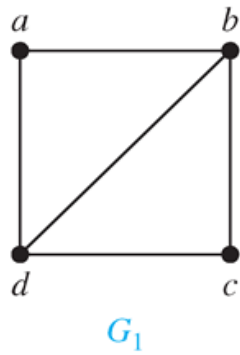


FIGURE 6 Mohammed's Scimitars.

Euler Circuits and Paths

■ Example



Applications of Euler Paths and Circuits

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48 - 5 \in NPC



Next Lecture

- Graph theory II ...

