CS215: Discrete Math (H)
2024 Fall Semester Written Assignment # 4
Due: Dec. 9th, 2024, please submit at the beginning of class

Q.1 Use induction to prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.
Solution: Base case: $n = 1$, $n^3 + 2n = 3$, which is divisible by 3.
Inductive hypothesis: Suppose that 3 divides $n^3 + 2n$.
Inductive step: We now prove that 3 divides $(n+1)^3 + 2(n+1)$. We have

$$\begin{aligned}(n+1)^3 + 2(n+1) &= (n^3 + 2n) + (3n^2 + 3n + 3)\\ &= (n^3 + 2n) + 3(n^2 + n + 1).\end{aligned}$$

Since $n^3 + 2n$ is divisible by 3 by i.h., and also $3(n^2 + n + 1)$ is divisible by 3. it follows that $(n+1)^3 + 2(n+1)$ is divisible by 3.
Conclusion: By mathematical induction, we prove the result.

□

Q.2 Suppose that a and b are real numbers with $0 < b < a$. Prove that if n is a positive integer, then $a^n - b^n \le na^{n-1}(a-b)$.
Solution: It turns out to be easier to think about the given statement as $na^{n-1}(a-b) \ge a^n - b^n$. The basic step ($n = 1$) is true since $a - b \ge a - b$. Assume that the inductive hypothesis, that $ka^{k-1}(a-b) \ge a^k - b^k$; we must show that $(k+1)a^k(a-b) \ge a^{k+1} - b^{k+1}$. We have

$$\begin{aligned}(k+1)a^k(a-b) &= k \cdot a \cdot a^{k-1}(a-b) + a^k(a-b)\\ &\ge a(a^k - b^k) + a^k(a-b)\\ &= a^{k+1} - ab^k + a^{k+1} - ba^k.\end{aligned}$$

To complete the proof we want to show that $a^{k+1} - ab^k + a^{k+1} - ba^k \ge a^{k+1} - b^{k+1}$. This inequality is equivalent to $a^{k+1} - ab^k - ba^k + b^{k+1} \ge 0$, which factors into $(a^k - b^k)(a - b) \ge 0$, and this is true, because we are given that $a > b$.

□

Q.3 A store gives out gift certificates in the amounts of \$10 and \$25. What amounts of money can you make using gift certificates from the store? Prove your answer using strong induction.

Solution:

By checking the first few values $10, 20, 25, 30, 35, 40, 45, 50, \ldots$, we guess that we can make \$$n$ in amount of money, where

$$n \in \{10\} \cup \{5m : \ m \geq 4 \text{ and } m \in \mathbb{Z}^+\}.$$

Let $P(n)$ be the statement "we can make \$$5m$ in gift certificate in amount of \$10 and \$25."

Base case: $m = 4, 5$, we can make \$20 and \$25 in gift certificate.

Inductive hypothesis: We can make \$$5k$ for $4 \leq k < m$.

Inductive step: We now prove $P(m)$ for $m \geq 6$. Note that $5m = 10 + 5(m-2)$. Since $4 \leq m-2 < m$, $P(m-2)$ is true. So we can make \$$5(m-2)$ in gift certificate. It then follows that we can \$$5m$ in gift certificate by adding an extra \$10 certificate.

□

Q.4 Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

Solution: The strong induction principle clearly implies ordinary induction, for if one has shown that $P(k) \to P(k+1)$, then it automatically follows that $[P(1) \wedge \cdots \wedge P(k)] \to P(k+1)$; in other words, strong induction can always be invoked whenever ordinary induction is used.

Conversely, suppose that $P(n)$ is a statement that one can prove using strong induction. Let $Q(n)$ be $P(1) \wedge \cdots \wedge P(n)$. Clearly $\forall n P(n)$ is logically equivalent to $\forall n Q(n)$. We show how $\forall n Q(n)$ can be proved using ordinary induction. First, $Q(1)$ is true because $Q(1) = P(1)$ and $P(1)$ is true by the basis step for the proof of $\forall n P(n)$ by strong induction. Now suppose that $Q(k)$ is true, i.e., $P(1) \wedge \cdots \wedge P(k)$ is true. By the proof of $\forall n P(n)$ by strong induction, it follows that $P(k+1)$ is true. But $Q(k) \wedge P(k+1)$ is just $Q(k+1)$. Thus, we have proved $\forall n Q(n)$ by ordinary induction.

□

Algorithm 1 twopower (n: positive integer, a: real number)

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if n = 1 then
    return a^2
else
    return twopower(n − 1, a)^2
end if
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Q.5 Devise a recursive algorithm to find a^{2^n}, where a is a real number and n is a positive integer. (use the equality $2^{2^{n+1}} = (a^{2^n})^2$)

Solution:

□

Q.6 Suppose that the function f satisfies the recurrence relation $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square greater than 1 and $f(2) = 1$.

(a) Find $f(16)$

(b) Find a big-O estimate for $f(n)$. [Hint: make the substitution $m = \log n$.]

Solution:

(a) $f(16) = 2f(4) + 4 = 2(2f(2) + 2) + 4 = 2(2 \cdot 1 + 2) + 4 = 12$.

(b) Let $m = \log n$, so that $n = 2^m$. Also, let $g(m) = f(2^m)$. Then our recurrence becomes $f(2^m) = 2f(2^{m/2}) + m$, since $\sqrt{2^m} = (2^m)^{1/2} = 2^{m/2}$. Rewriting this in terms of g we have $g(m) = 2g(m/2) + m$. Theorem 2 (with $a = 2, b = 2, c = 1$, and $d = 1$ now tells us that $g(m)$ is $O(m \log m)$. Since $m = \log n$, this means that our function is $O(\log n \cdot \log \log n)$.

□

Q.7 The running time of an algorithm A is described by the following recurrence relation:

$$S(n) = \begin{cases} b & n = 1 \\ 9S(n/2) + n^2 & n > 1 \end{cases}$$

where b is a positive constant and n is a power of 2. The running time of a competing algorithm B is described by the following recurrence relation:

$$T(n) = \begin{cases} c & n = 1 \\ aT(n/4) + n^2 & n > 1 \end{cases}$$

where a and c are positive constants and n is a power of 4. For the rest of this problem, you may assume that n is always a power of 4. You should also assume that $a > 16$. (Hint: you may use the equation $a^{\log_2 n} = n^{\log_2 a}$)

(a) Find a solution for $S(n)$. Your solution should be in *closed form* (in terms of b if necessary) and should *not* use summation.

(b) Find a solution for $T(n)$. Your solution should be in *closed form* (in terms of a and c if necessary) and should *not* use summation.

(c) For what range of values of $a > 16$ is Algorithm B at least as efficient as Algorithm A asymptotically ($T(n) = O(S(n))$)?

Solution:

(a) By repeated substitution, we get

$$\begin{aligned}
S(n) &= 9S(n/2) + n^2 \\
&= 9\left[9S\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2\right] + n^2 \\
&= 9^2 S\left(\frac{n}{2^2}\right) + \left(\frac{9}{4}\right) n^2 + n^2 \\
&= 9^2\left[9S\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2\right] + \left(\frac{9}{4}\right) n^2 + n^2 \\
&= 9^3 S\left(\frac{n}{2^3}\right) + \left(\frac{9}{4}\right)^2 n^2 + \left(\frac{9}{4}\right) n^2 + n^2 \\
&= \cdots \\
&= 9^{\log_2 n} S(1) + n^2 \sum_{i=0}^{\log_2 n - 1} \left(\frac{9}{4}\right)^i \\
&= bn^{\log_2 9} + \frac{4}{5} n^{\log_2 9} - \frac{4}{5} n^2 \\
&= \left(b + \frac{4}{5}\right) n^{\log_2 9} - \frac{4}{5} n^2,
\end{aligned}$$

where we are using the fact that

$$\left(\frac{9}{4}\right)^{\log_2 n} = \frac{9^{\log_2 n}}{n^2} = \frac{n^{\log_2 9}}{n^2}.$$

(b) Similar to (a), we get

$$\begin{aligned}
T(n) &= aT\left(\frac{n}{4}\right) + n^2 \\
&= a\left[aT\left(\frac{n}{4^2}\right) + \left(\frac{n}{4}\right)^2\right] + n^2 \\
&= a^2T\left(\frac{n}{4^2}\right) + \left(\frac{a}{16}\right)n^2 + n^2 \\
&= a^2\left[aT\left(\frac{n}{4^3}\right) + \left(\frac{n}{4^2}\right)^2\right] + \left(\frac{a}{16}\right)n^2 + n^2 \\
&= a^3T\left(\frac{n}{4^3}\right) + \left(\frac{a}{16}\right)^2 n^2 + \left(\frac{a}{16}\right)n^2 + n^2 \\
&= \cdots \\
&= a^{\log_4 n}T(1) + n^2 \sum_{i=0}^{\log_4 n-1} \left(\frac{a}{16}\right)^i \\
&= cn^{\log_4 a} + \frac{16}{a-16}n^{\log_4 a} - \frac{16}{a-16}n^2 \\
&= \left(c + \frac{16}{a-16}\right)n^{\log_4 a} - \frac{16}{a-16}n^2,
\end{aligned}$$

where we are using the fact that

$$\left(\frac{a}{16}\right)^{\log_4 n} = \frac{a^{\log_4 n}}{n^2} = \frac{n^{\log_4 a}}{n^2}.$$

(c) For $T(n) = O(S(n))$, we should have

$$\begin{aligned}
n^{\log_4 a} &\leq n^{\log_2 9} \\
\log_4 a &\leq \log_2 9 \\
a &\leq 9^2 = 81.
\end{aligned}$$

So the range of values is $16 < a \leq 81$.

□

Q.8 How many bit strings of length 6 have <u>at least</u> one of the following properties:

- start with 010
- start with 11
- end with 00

State clearly how you count and get your answer.

Solution:

Let A denote the set of bit strings that start with 010, and B, C are the sets of bit strings with the latter two properties, respectively. We need to find the size of $A \cup B \cup C$. Note that $A \cap B = \emptyset$, as the same as $A \cap B \cap C$. By the Inclusion-Exclusion Principle, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap C| - |B \cap C| - |B \cap C| = 2^3 + 2^4 + 2^4 - 2^1 - 2^2 = 34.$$

$\square$

Q.9 Suppose that $n \geq 1$ is an integer.

(a) How many functions are there from the set $\{1, 2, \ldots, n\}$ to the set $\{1, 2, 3\}$?

(b) How many of the functions in part (a) are one-to-one functions?

(c) How many of the functions in part (a) are onto functions?

Solution:

(a) There are 3^n functions.

(b) If $n \leq 3$, there are $P(3, n)$ one-to-one functions. Hence, there are 3 when $n = 1$, 6 when $n = 2$, and 6 when $n = 3$. If $n > 3$, then there are 0 injective functions; there cannot be a one-to-one function from A to B if $|A| > |B|$.

(c) By the Inclusion-Exclusion Principle, we have

$$\begin{aligned}
\# &= \#\{f : f(A) \subseteq \{1,2,3\}\} - \#\{f : f(A) \subseteq \{1,2\}\} - \#\{f : f(A) \subseteq \{1,3\}\} \\
&\quad -\#\{f : f(A) \subseteq \{2,3\}\} + \{f : f(A) \subseteq \{1\}\} + \{f : f(A) \subseteq \{2\}\} \\
&\quad \#\{f : f(A) \subseteq \{3\}\} \\
&= 3^n - 2^n - 2^n - 2^n + 1 + 1 + 1 \\
&= 3^n - 3 \cdot 2^n + 3.
\end{aligned}$$

□

Q.10 How many 6-card poker hands consist of exactly 2 pairs? That is two of one rank of card, two of another rank of card, one of a third rank, and one of a fourth rank of card? Recall that a deck of cards consists of 4 suits each with one card of each of the 13 ranks. You should leave your answer as an equation.

Solution: First, we choose the ranks of the 2 pairs, noting that the order we pick these two ranks does not matter, so there are $\binom{13}{2}$ options here. Next we pick the 2 suits for the first pair, $\binom{4}{2}$ and the suits for the second pair $\binom{4}{2}$. Then we decide which 2 ranks of the remaining 11 to use for the other cards, $\binom{11}{2}$, and finally choose each of their suits $\binom{4}{1}\binom{4}{1}$. Altogether, by the product rule, this gives $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{2}\binom{4}{1}\binom{4}{1}$ hands.

□

Q.11 How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

Solution: First we count the number of bit strings of length 10 that contain five consecutive 0s. We will count based on where the string of five or more consecutive 0s starts. If it starts in the first bit, then the first five bits are all 0s, but there is free choice for the last five bits, therefore there are $2^5 = 32$ such strings. If it starts in the second bit, then the first bit must be a 1, the next five bits are all 0s, but there is free choice for the last four bits; therefore there are $2^4 = 16$ such strings. If it starts in the third bit, then the second bit must be a 1 but the first bit and the last three bits are arbitrary; therefore there are $2^4 = 16$ such strings. Similarly, there are 16 such strings that have the consecutive 0s starting in each of positions four, five, and six. This gives us a total of $32 + 5 \cdot 16 = 112$ strings that contain five consecutive

0s. Symmetrically there are 112 strings that contain five consecutive 1s. Clearly there are exactly two strings that contain both (0000011111 and 1111100000). Therefore by the inclusion-exclusion principle, the answer is $112 + 112 - 2 = 222$.

□

Q.12 Consider the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 10.$$

with five variables.

(1) Count the number of integer solutions, with $x_1 \geq 3$, $x_2 \geq 0$, $x_3 \geq -2$, $x_4 \geq 0$, and $x_5 \geq 0$.

(2) Count the number of integer solutions, with $0 \leq x_1 \leq 5$ and $x_2, x_3, x_4, x_5 \geq 0$.

Solution:

(1) Let $y_1 = x_1 - 3$, $y_2 = x_2$, $y_3 = x_3 + 2$, $y_4 = x_4$, and $y_5 = x_5$. Then the number of integer solutions is equal to the number of integer solutions of the equation
$$y_1 + y_2 + y_3 + y_4 + y_5 = 9,$$
with all y_i's ≥ 0. Then the number is $\binom{13}{4} = 715$.

(2) We first count the number of integer solutions with $x_1 \geq 6$ and $x_2, x_3, x_4, x_5 \geq 0$. Let $y_1 = x_1 - 6$ and $y_i = x_i$ for $2 \leq i \leq 5$, we have the number of integer solutions is $\binom{8}{4}$. Thus, the number of integer solutions with $0 \leq x_1 \leq 5$ and $x_2, x_3, x_4, x_5 \geq 0$ is equal to the number of integer solutions with $x_1, x_2, x_3, x_4, x_5 \geq 0$ minus $\binom{8}{4}$. This leads to
$$\binom{14}{4} - \binom{8}{4} = 931.$$

Q.13 How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \bmod 5 = a_2 \bmod 5$ and $b_1 \bmod 5 = b_2 \bmod 5$.

Solution:

Working modulo 5 there are 25 pairs: $(0,0),(0,1),\ldots,(4,4)$. Thus, we could have 25 ordered pairs of integers (a,b) such that no two of them were equal when reduced modulo 5. The pigeonhole principle, however, guarantees that if we have 26 such pairs, then at least two of them will have the same coordinates, modulo 5.

$\square$

Q.14 Show that if p is a prime and k is an integer such that $1 \leq k \leq p-1$, then p divides $\binom{p}{k}$.

Solution:

We know that

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}.$$

Clearly p divides the numerator. On the other hand, p cannot divide the denominator, since the prime factorizations of these factorials contains only numbers less than p. Therefore the factor p does not cancel when this fraction is reduced to lowest terms (i.e., to a whole number), so p divides $\binom{p}{k}$.

$\square$

Q.15 Prove the hockeystick identity

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

(a) using a combinatorial argument

(b) using Pascal's identity.

Solution:

(a) $\binom{n+r+1}{r}$ counts the number of ways to choose a sequence of r 0s and $n+1$ 1s by choosing the positions of the 0s. Alternatively, suppose that the $(j+1)$st term is the last term equal to 1, so that $n \leq j \leq n+r$. Once we have determined where the last 1 is, we decide where the 0s are to be placed in the j spaces before the last 1. There are n 1s and $j-n$ 0s in this range. By the sum rule it follows that there are $\sum_{j=n}^{n+r} \binom{j}{j-n} = \sum_{k=0}^{r} \binom{n+k}{k}$ ways to this.

(b) Let $P(r)$ be the statement to be proved. The basis step is the equation $\binom{n}{0} = \binom{n+1}{0}$, which is just $1 = 1$. Assume that $P(r)$ is true. Then

$$\begin{aligned} &\sum_{k=0}^{r+1} \binom{n+k}{k} \\ &= \sum_{k=0}^{r} \binom{n+k}{k} + \binom{n+r+1}{r+1} \\ &= \binom{n+r+1}{r} + \binom{n+r+1}{r+1} \\ &= \binom{n+r+2}{r+1}, \end{aligned}$$

using the inductive hypothesis and Pascal's identity.

$\square$

Q.16 For $0 \leq k \leq n$, show that

$$\sum_{r=k}^{n} \binom{n}{r}\binom{r}{k} = \binom{n}{k} 2^{n-k}.$$

Your proof may be either combinatorial or algebraic.

Solution: Note that

$$\begin{aligned}\binom{n}{r}\binom{r}{k} &= \frac{n!}{r!\cdot(n-r)!}\frac{r!}{k!(r-k)!}\\ &= \frac{n!}{k!(r-k)!(n-r)!}\\ &= \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(r-k)!(n-r)!}\\ &= \binom{n}{k}\binom{n-k}{r-k}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\sum_{r=k}^{n}\binom{n}{r}\binom{r}{k} &= \sum_{r=k}^{n}\binom{n}{k}\binom{n-k}{r-k}\\ &= \binom{n}{k}\sum_{r=k}^{n}\binom{n-k}{r-k}\\ &= \binom{n}{k}\sum_{s=0}^{n-k}\binom{n-k}{s} \qquad \text{let } s=r-k\\ &= \binom{n}{k}2^{n-k}.\end{aligned}$$

For a combinatorial proof, observe that the right-hand side is equal to the number of ways of doing the following with a set X of size n:

- Pick a subset A of X consisting of k elements. This can be done in $\binom{n}{k}$ ways.
- Pick any subset B of $X \setminus A$. This can be done in 2^{n-k} ways.

An equivalent way of forming the sets A and B is to do the following:

- For any r such that $k \le r \le n$, pick a subset C of X consisting of r elements. For a given r, this can be done in $\binom{n}{r}$ ways.
- Pick a subset A of C consisting of k elements, and let $B = C \setminus A$. This can be done in $\binom{r}{k}$ ways.

Summing over all possible r, we obtain the desired result.

Q.17 Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.
Solution:

The characteristic equation is $r^3 - 2r^2 - r + 2 = 0$. This factors as $(r-1)(r+1)(r-2) = 0$, so the roots are $1, -1$, and 2. Therefore the general solution is $a_n = \alpha_1 + \alpha_2(-1)^n + \alpha_3 2^n$. Plugging in initial conditions gives $3 = \alpha_1 + \alpha_2 + \alpha_3$, $6 = \alpha_1 - \alpha_2 + 2\alpha_3$, and $0 = \alpha_1 + \alpha_2 + 4\alpha_3$. The solution to this system of equations is $\alpha_1 = 6$, $\alpha_2 = -1$ and $\alpha_3 = -1$. Therefore, the answer is $a_n = 6 - 2(-1)^n - 2^n$.

$\square$

Q.18 A computer system considers a string of decimal digits $(0, 1, \ldots, 9)$ to be a **valid** code word if and only if it contains an **odd number of zero digits**. For example, 12030 and 11111 are **not** valid, but 29046 is. Let $V(n)$ denote the number of valid n-digit code words. Find a recurrence relation for $V(n)$ with initial cases, and give a closed-form solution to this recurrence relation. Please explain how you find the recurrence relation. (Hint: notice that the number of non-valid code words is equal to $10^n - V(n)$.)
Solution: There are two ways to construct a valid code of length n from a string of $n-1$ digits:

(a) take a valid code of length $n-1$, append a number between 1 and 9: there are $9V(n-1)$ ways;

(b) take a non-valid code of length $n-1$, append a 0: there are $10^{n-1} - V(n-1)$ ways.

In total, we have

$$V(n) = 9V(n-1) + 10^{n-1} - V(n-1) = 10^{n-1} + 8V(n-1),$$

with initial cases $V(1) = 1$.

By iterating this recurrence, we have

$$\begin{aligned}
V(n) &= 8V(n-1) + 10^{n-1} \\
&= 8(8V(n-2) + 10^{n-2}) + 10^{n-1} \\
&= 8^2V(n-2) + 8 \cdot 10^{n-2} + 10^{n-1} \\
&= 8^2(8V(n-3) + 10^{n-3}) + 8 \cdot 10^{n-2} + 10^{n-1} \\
&= 8^3V(n-3) + 8^2 \cdot 10^{n-3} + 8 \cdot 10^{n-2} + 10^{n-1} \\
&= \vdots \\
&= 8^{n-1}V(1) + 8^{n-2}10^1 + 8^{n-3}10^2 + \cdots + 8 \cdot 10^{n-2} + 10^{n-1} \\
&= 8^{n-1}\left(1 + \frac{5}{4} + (\frac{5}{4})^2 + \cdots + (\frac{5}{4})^{n-1}\right) \\
&= 8^{n-1} \cdot \frac{1 - (\frac{5}{4})^n}{1 - \frac{5}{4}} \\
&= 5 \cdot 10^{n-1} - 4 \cdot 8^{n-1}.
\end{aligned}$$

□

Q.19 Let $\mathbf{A}_n$ be the $n \times n$ matrix with 2's on its main diagonal, 1's in all positions next to a diagonal element, and 0's everywhere else. Find a recurrence relation for d_n, the determinant of $\mathbf{A}_n$. Solve this recurrence relation to find a formula for d_n.

Solution:

We can compute the first few terms by hand. For $n = 1$, the matrix is just the number 2, so $d_1 = 2$. For $n = 2$, the matrix is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and its determinant is clearly $d_2 = 4 - 1 = 3$. For $n = 3$, the matrix is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

and we get $d_3 = 4$. For the general case, our matrix is

$$\mathbf{A}_n = \begin{bmatrix} 2 & 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 \end{bmatrix}.$$

To compute the determinant, we expand along the top row. This gives us a value of 2 times the determinant of the matrix obtained by deleting the first row and first column minus the determinant of the matrix obtained by deleting the first row and second column. The first of these smaller matrices is just $\mathbf{A}_{n-1}$, with determinant d_{n-1}. The second of these smaller matrices has just one nonzero entry in its first column, so we expand its determinant along the first column and see that it equals d_{n-2}. Therefore our recurrence relation is $d_n = 2d_{n-1} - d_{n-2}$, with initial conditions as computed at the start of this solution. If we compute a few more terms we are led to the conjecture that $d_n = n + 1$. If we show that this satisfies the recurrence, then we have proved that it is indeed the solution. And sure enough, $n+1 = 2n-(n-1)$.

$\square$

Q.20 Use generating functions to prove Pascal's identity: $C(n, r) = C(n-1, r) + C(n-1, r-1)$ when n and r are positive integers with $r < n$. [Hint: Use the identity $(1 + x)^n = (1 + x)^{n-1} + x(1 + x)^{n-1}$.]

Solution:

First we note, as the hint suggests, that $(1 + x)^n = (1 + x)(1 + x)^{n-1} = (1 + x)^{n-1} + x(1 + x)^{n-1}$. Expanding both sides of this equality using the binomial theorem, we have

$$\begin{aligned} \sum_{r=0}^{n} C(n, r)x^r &= \sum_{r=1}^{n-1} C(n-1, r)x^r + \sum_{r=0}^{n-1} C(n-1, r)x^{r+1} \\ &= \sum_{r=0}^{n-1} C(n-1, r)x^r + \sum_{r=1}^{n} C(n-1, r-1)x^r. \end{aligned}$$

Thus,

$$1+\left(\sum_{r=1}^{n-1} C(n,r)x^r\right)+x^n = 1+\left(\sum_{r=1}^{n-1}(C(n-1,r)+C(n-1,r-1))x^r\right)+x^n.$$

Comparing these two expressions, coefficient by coefficient, we see that $C(n,r)$ must equal $C(n-1,r)+C(n-1,r-1)$ for $1 \le r \le n-1$, as desired.

$\square$

Q.21 Use generating functions to prove Vandermonde's identity:

$$C(m+n,r) = \sum_{k=0}^{r} C(m,r-k)C(n,k),$$

whenever m, n, and r are nonnegative integers with r not exceeding either m or n. [Hint: Look at the coefficient of x^r in both sides of $(1+x)^{m+n} = (1+x)^m(1+x)^n$.]

Solution: Applying the binomial theorem to the equality $(1+x)^{m+n} = (1+x)^m(1+x)^n$, shows that $\sum_{r=0}^{m+n} C(m+n,r)x^r = \sum_{r=1}^{m} C(m,r)x^r \cdot \sum_{r=0}^{n} C(n,r)x^r = \sum_{r=0}^{m+n}[\sum_{k=0}^{r} C(m,r-k)C(n,k)]x^r$. Comparing coefficients gives the desired identity.

$\square$

Q.22 Generating functions are very useful, for example, provide an approach to solving linear recurrence relations. Read pp. 537-548 of the textbook. [You do not need to write anything for this problem on your submitted assignment paper.]