STA 215 Ass 4 12310401 王子恒

1. $f(x,y) = f_{X|Y}(x|y)\cdot f_Y(y) = \begin{cases} 15x^2y & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$

$$P(X>0.5) = \int_{0.5}^{1}\int_{0.5}^{y} f(x,y)\,dx\,dy$$

$$= \int_{0.5}^{1}\int_{0.5}^{y} 15x^2y\,dx\,dy$$

$$= \int_{0.5}^{1} 5\left(y^4 - \frac{y}{8}\right)dy$$

$$= \left(1-\frac{5}{16}\right) - \left(\frac{1}{32} - \frac{5}{64}\right)$$

$$= \frac{47}{64}$$

2. $f_X(x) = \begin{cases} a+\frac{1}{9}+c & x=x_1 \\ \frac{4}{9}+b & x=x_2 \end{cases}$ $\quad f_Y(y) = \begin{cases} a+\frac{1}{9} & y=y_1 \\ b+\frac{1}{9} & y=y_2 \\ c+\frac{1}{3} & y=y_3 \end{cases}$

$f(x=x_1, y=y_1) = f_X(x=x_1)\cdot f_Y(y=y_1) \Rightarrow a = \left(a+\frac{1}{9}+c\right)\left(a+\frac{1}{9}\right)$.

$f(x=x_1, y=y_2) = f_X(x=x_1)\cdot f_Y(y=y_2) \Rightarrow \frac{1}{9} = \left(a+\frac{1}{9}+c\right)\left(b+\frac{1}{9}\right)$

$f(x=x_1, y=y_3) = f_X(x=x_1)\cdot f_Y(y=y_3) \Rightarrow c = \left(a+\frac{1}{9}+c\right)\cdot\left(c+\frac{1}{3}\right)$.

$\Rightarrow \frac{a}{c} = \frac{a+\frac{1}{9}}{c+\frac{1}{3}} \Rightarrow ac+\frac{1}{3}a = ac+\frac{1}{9}c \Rightarrow a = \frac{1}{3}c$.

$\Rightarrow a = \left(a+\frac{1}{9}+3a\right)\left(a+\frac{1}{9}\right) \Rightarrow 4a^2 - \frac{4}{9}a + \frac{1}{81} = 0$.

$\left(2a-\frac{1}{9}\right)^2 = 0 \qquad a = \frac{1}{18} \qquad \Rightarrow c = \frac{1}{6}$

$\frac{1}{9} = \left(\frac{1}{18}+\frac{1}{9}+\frac{1}{6}\right)\left(b+\frac{1}{9}\right) \Rightarrow b = \frac{2}{9}$

$$\begin{cases} a = \frac{1}{18} \\ b = \frac{2}{9} \\ c = \frac{1}{6} \end{cases}$$

3. $Cov(U,V) = Cov(2X+Y, 2X-Y)$

$= Cov(2X, 2X-Y) + Cov(Y, 2X-Y)$

$= Cov(2X, 2X) - Cov(2X, Y) + Cov(Y, 2X) - Cov(Y,Y)$

$= 4\,Var(X) - Var(Y)$

$= 4\lambda - \lambda$

$= 3\lambda$

$Var(U) = Var(2X) + Var(Y) = 4\lambda + \lambda = 5\lambda$

$Var(V) = Var(2X) + Var(Y) = 4\lambda + \lambda = 5\lambda$.

$\therefore \rho_{XY} = \dfrac{Cov(U,V)}{\sqrt{Var(U)(Var(V))}} = \dfrac{3\lambda}{5\lambda} = \dfrac{3}{5}$

4. $X \sim binomial(n, \frac{1}{2}) \qquad Y \sim binomial(n, \frac{1}{2})$

$P_X(X=x) = C_n^x (\frac{1}{2})^n \qquad P_Y(Y=y) = C_n^y (\frac{1}{2})^n$

$X+Y = n$.

$Cov(X,Y) = Cov(X, n-X) = Cov(X, n) - Cov(X, -X)$

$= -Var(X) = -\frac{n}{4}$

$\rho_{XY} = \dfrac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \dfrac{-\frac{n}{4}}{\frac{n}{4}} = -1$

5(1) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-x}^{x} 1\,dy = 2x,\ 0<x<1, \qquad f_X(x) = 0$ otherwise

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{|y|}^{1} 1\,dx = 1-|y|\ \ -1<y<1 \qquad f_Y(y) = 0$ otherwise

$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}$

$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_{-1}^{1} y - y|y|\,dy = 0$.

$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dy\,dx = \int_0^1 \int_{-x}^{x} xy\,dy\,dx = \int_0^1 0\,dx = 0$

$Cov(X,Y) = E(XY) - E(X)E(Y) = 0$.

(2) $f(X=x, Y=y) = \begin{cases} 1 & |y|<x,\ 0<x<1 \\ 0 & \text{otherwise.} \end{cases}$

$f(X=x)\cdot f(Y=y) = \begin{cases} 2x(1-|y|) & |y|<x,\ 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$f_{XY}(X=x, Y=y) \neq f_Y(X=x)\cdot f_Y(Y=y)$

$\therefore X$ & Y are not independent.

6. $f_X(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$ $f_Y(x) = \begin{cases} 1 & 0<y<1 \\ 0 & \text{otherwie} \end{cases}$

$F_X(x) = \begin{cases} 0 & x \le 0 \\ x & 0<x<1 \\ 1 & x \ge 1 \end{cases}$ $F_Y(x) = \begin{cases} 0 & y \le 0 \\ y & 0<y<1 \\ 1 & y \ge 1. \end{cases}$

$$f_T(t) = \int_{-\infty}^{\infty} f(t-y, y)dt$$
$$= \int_{-\infty}^{\infty} f_X(t-y) f_Y(y)dy$$
$$= \int_0^1 f_X(t-y)dy$$

when $0<t\le 1$ $f_T(t) = \int_0^t 1\,dx = t$

when $1<t\le 2$ $f_T(t) = \int_{t-1}^1 1\,dx = 2-t$

$$\therefore f_T(t) = \begin{cases} t & 0<t\le 1 \\ 2-t & 1<t\le 2 \\ 0 & \text{otherwise} \end{cases}$$

7. $X_1 \sim N(45,10)$ $X_2 \sim N(50,12)$ $X_3 \sim N(75,14)$

$E(T) = E(X_1) + E(X_2) + E(X_3) = 170.$

$Var(T) = Var(X_1) + Var(X_2) + Var(X_3) = 36.$

$T \sim N(170, 36).$

$P(T \le 170) = P(Z \le \frac{180-170}{\sqrt{36}}) = P(Z \le \frac{5}{3}) \approx 0.9515$

8. $\lambda = \frac{1}{25}$

$x_1, x_2 \cdots x_{40} \sim Exp(\lambda)$

$T = X_1 + X_2 + \cdots + X_{40} \sim Gamma(40, \lambda).$

$E(T) = \frac{40}{\lambda} = 1000$

$Var(T) = \frac{40}{\lambda^2} = 25000$

$\because 40 > 30$ According to CLT. we can consume $T \sim N(1000, \sqrt{25000})$.

$\therefore P(T > 900) = P(\frac{T-1000}{\sqrt{25000}} > \frac{900-1000}{\sqrt{25000}}) = P(Z > -0.632) = P(Z < 0.632) = 0.736.$

9. $X_1, X_2 \cdots X_{500} \sim Bernoulli(0.8)$

$T = X_1 + X_2 + \cdots + X_{500}.$ $E(T) = 0.8 \times 500 = 400$ $Var(T) = 0.8 \times 0.2 \times 500 = 80$

$\because 500 > 30$ $\therefore T \sim N(400, \sqrt{80})$

z-score $= 0.99$ when $Z = 2.33$

$\frac{T-400}{\sqrt{80}} = 2.33 \Rightarrow T = 421 \Rightarrow P = 2T = 842\,kW.$