STA215 Ass5 12310401 王子恒

1. $f_X(x) = \lambda e^{-\lambda x}$ $f_Y(y) = \mu e^{-\mu y}$

$$P(Z=1) = \iint_{x\le y} f(x,y)\,dx\,dy$$
$$= \int_0^\infty \int_0^y f_X(x) f_Y(y)\,dx\,dy$$
$$= \int_0^\infty \int_0^y \lambda\mu\, e^{-\lambda x-\mu y}\,dx\,dy$$
$$= \int_0^\infty \lambda\mu \left(-\frac{1}{\lambda} e^{-\lambda x-\mu y}\right)\Big|_0^y\,dy$$
$$= \int_0^\infty \mu\left(e^{-\mu y} - e^{-\lambda y-\mu y}\right)dy$$
$$= 1 - \frac{\mu}{\lambda+\mu}$$
$$= \frac{\lambda}{\lambda+\mu}$$

$$P(Z=0) = 1 - P(Z=1) = \frac{\mu}{\lambda+\mu}$$

$$P(Z=z) = \begin{cases} \frac{\lambda}{\lambda+\mu} & z=1 \\ \frac{\mu}{\lambda+\mu} & z=0 \end{cases}$$

2. $$P(X\le z) = \sum_{k=0}^{z-1} (1-p)^k p = 1-(1-p)^z.$$

$$P(X\le z, Y\le z) = (1-(1-p)^z)^2.$$

$$P(Z\le z) = P(X\le z, Y\le z)$$
$$P(Z=z) = P(X\le z, Y\le z) - P(X\le z-1, Y\le z-1)$$
$$= (1-(1-p)^z)^2 - (1-(1-p)^{z-1})^2$$
$$= \left((1-p)^z + (1-p)^{z-1} - 2\right)\left((1-p)^z - (1-p)^{z-1}\right)$$
$$= (1-p)^{z-1}\left((1-p)^z + (1-p)^{z-1} - 2\right)(-p)$$

$$P(Z=z) = \begin{cases} (1-p)^{z-1}\left((1-p)^z + (1-p)^{z-1} - 2\right)(-p) & z=0 \\ 0 & \text{otherwise} \end{cases}$$

3. $T = MIN(X_1, X_2, X_3)$

$$F_T(t) = P(T \le t) = P(\min(X_1, X_2, X_3) \le t)$$
$$= 1 - P(\min(X_1, X_2, X_3) > t)$$
$$= 1 - P(X_1 > t)\,P(X_2 > t)\,P(X_3 > t)$$
$$= 1 - \left(1 - (1 - e^{-\lambda t})\right)^3$$
$$= 1 - e^{-3\lambda t}$$

$$f_T(t) = F_T(t)' = 3\lambda e^{-3\lambda t} \qquad t \ge 0$$

$$f_T(t) = \begin{cases} 3\lambda e^{-3\lambda t} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

4. (1) $E(X-Y) = E(X) - E(Y) = 0$

$$Var(X-Y) = Var(X) + Var(Y) - 2Conv(X, Y)$$
$$= 1 + 1 - 2\rho$$
$$= 2(1-\rho)$$

$$\therefore X - Y \sim N(0, 2(1-\rho))$$

$$f_Z(z) = f_{X-Y}(x-y) = \frac{1}{\sqrt{4\pi(1-\rho)}} \cdot e^{-\frac{z^2}{4(1-\rho)}}$$

(2) $W = X - Y \qquad V = XY$

$$Conv(W, V) = E[(X-Y)(XY)] - E(X-Y)E(XY)$$
$$= E(X^2Y - XY^2)$$
$$= E(X^2Y) - E(XY^2)$$
$$= E(X^2)E(Y) - E(X)E(Y^2)$$
$$= 0$$

$$Cor(W, V) = \frac{Conv(W, V)}{\sqrt{Var(W)Var(V)}} = 0.$$

5. $P_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi \sigma_x \sigma_Y \sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_x^2} - 2\rho\frac{xy}{\sigma_x\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right]\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_x\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{x}{\sigma_x} - \frac{y}{\sigma_Y}\right]^2\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_x\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{y^2(1-\rho^2)}{2\sigma_Y^2}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x}{\sigma_x} - \rho\frac{y}{\sigma_Y}\right) dx\right.$$

$$= \frac{1}{\sigma_Y\sqrt{2\pi}} \exp\left(-\frac{y^2}{2\sigma_Y^2}\right)$$

(2) $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)} = \dfrac{\frac{1}{2\pi\sigma_x\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_x^2} - 2\rho\frac{xy}{\sigma_x\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right]\right)}{\frac{1}{\sigma_Y\sqrt{2\pi}}\exp\left(-\frac{y^2}{2\sigma_Y^2}\right)}$

$$= \frac{1}{\sigma_x\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\rho\frac{\sigma_x}{\sigma_Y}y)^2}{\sigma_x^2}\right]\right).$$

thus the PDF of X gives $Y=y$ is normal,

$\mu(x|y) = \rho\frac{\sigma_x}{\sigma_Y}y$

$\sigma^2(x|y) = \sigma_x^2(1-\rho^2).$

Q6. (1) $\mu_X = 4500$, $\sigma_x = 1500$, $\mu_Y = 5500$, $\sigma_Y = 2000$

$\rho = 0.65$

$(X=x|Y=y) = y \sim N(\mu_{x|Y}, \sigma^2_{x|Y}).$

where $\mu_{x|Y} = \mu_x + \rho\frac{\sigma_x}{\sigma_Y}(y-\mu_Y) = 4500 + 0.65 \times \frac{1500}{2000}(6800-5500) = 5133.75$

$\sigma^2_{x|Y} = \sqrt{\sigma_x^2(1-\rho^2)} = \sqrt{1500^2(1-0.65^2)} = 1139.85$

$Z = \dfrac{6800-\mu_{x|Y}}{\sigma_{x|Y}} = \dfrac{6800-5133.75}{1140} \approx 1.46.$

$\Rightarrow P(Z<1.46) = 0.9279$

(2) $P(X>Y \mid X+Y=12000) = P(X-Y>0 \mid X+Y=12000)$

$W = X-Y \quad V = X+Y$

$\begin{pmatrix} W \\ V \end{pmatrix} = \begin{pmatrix} X-Y \\ X+Y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} -1000 \\ 1000 \end{pmatrix}, \begin{pmatrix} 1533^2 & -1750000 \\ -1750000 & 3186^2 \end{pmatrix}\right)$

$\rho = \dfrac{-1750000}{1533 \times 3186} = -0.358$

$W|V=12000 \sim N\left(-1000 - 0.358 \times \frac{1533}{3186} \times 2000,\ (1-0.358^2) \times 1533^2\right) = N(-1344.5, 1431.4^2)$

$P(W>0|V=12000) = 1 - \Phi\left(\frac{1344.5}{1431.4}\right) = 1 - 0.8264 = 0.1736$

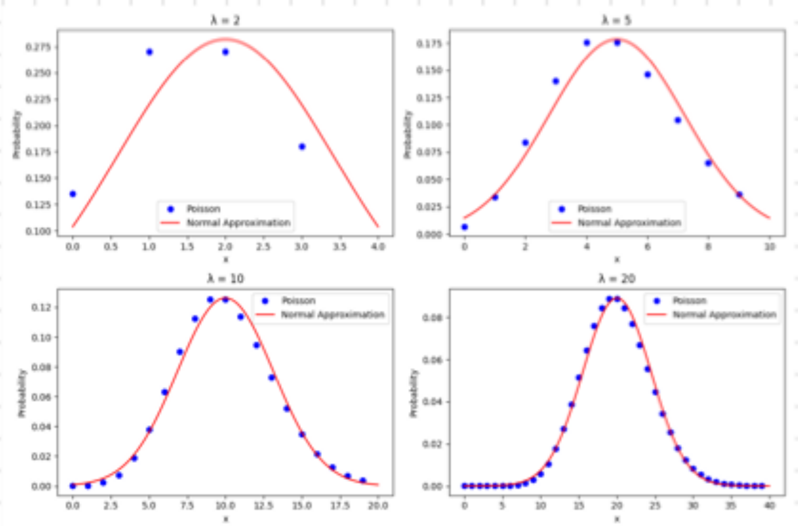
7.(1) Assume $X \sim \text{Poisson}(\lambda)$

$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$ according to CLT

$\Rightarrow \mu = \lambda, \quad \sigma^2 = \lambda \Rightarrow X \sim N(\lambda, \lambda)$

(2)

```python
import numpy as np
import scipy.stats as stats
import matplotlib.pyplot as plt

lambdas = [2, 5, 10, 20]

plt.figure(figsize=(12, 8))

for i, lambda_ in enumerate(lambdas):

    poisson = stats.poisson(lambda_)
    x_poisson = np.arange(0, 2 * lambda_)
    y_poisson = poisson.pmf(x_poisson)


    normal = stats.norm(lambda_, np.sqrt(lambda_))
    x_normal = np.linspace(0, 2 * lambda_, 1000)
    y_normal = normal.pdf(x_normal)


    plt.subplot(2, 2, i + 1)
    plt.plot(x_poisson, y_poisson, 'bo', label='Poisson')
    plt.plot(x_normal, y_normal, 'r-', label='Normal Approximation')
    plt.title(f'λ = {lambda_}')
    plt.xlabel('x')
    plt.ylabel('Probability')
    plt.legend()

plt.tight_layout()
plt.savefig('/Users/earendelh/Codes/Python/poisson_normal_approximation.png')
```



when λ is small, the normal approximation is not great.
when λ is larger the normal approximation is better.