STA219: Probability and Statistics for Engineering

Assignment 6

Note: The assignment can be answered in Chinese or English, either is fine. Please provide derivation and computation details, not just the final answer. Please submit a PDF file on BB.

Part I: Calculations and derivations by hand

1. (10 points) Bob and Carl have just learned the Law of Large Numbers. It turns out that they have a different understanding of what the law says.

   Bob: The Law of Large Numbers says that in the long run, a fair coin will land heads as often as it lands tails.

   Carl: I don't think that's what it says. The Law of Large Numbers says that the fraction of heads will get closer and closer to 1/2, which is the expected value of each toss.

   Bob: Isn't that the same as what I said?

   Carl: No, you said, "The coin will land heads as often as it lands tails," which implies that the difference between the number of heads and the number of tails will get smaller as we toss the coin more and more. I don't think the number of heads will be close to the number of tails.

   Bob: If the fraction of heads is close to 1/2, then the number of heads must be close to the number of tails. How could it be otherwise?

   Who is right: Bob or Carl?

   (1) Calculate the variance of

   $$\text{number of heads in } n \text{ tosses} - \text{number of tails in } n \text{ tosses}$$

   as a function of n. (5 points)

   (2) Considering this calculation, do you agree with Bob that the difference between the number of heads and the number of tails approaches 0 as the number of tosses increases? (5 points)

2. (15 points) Let $X_1, X_2, \dots, X_n$ be a sequence of independent and identically distributed random variables. Define $Y_n = (X_1 + X_2 + \cdots + X_n)/n$. Show that the sequence $Y_1, \dots, Y_n$ converges in probability to some limit, and identify the limit, for each of the following cases:

   (1) X_i follow the Poisson distribution with parameter 3. (5 points)

   (2) X_i are uniformly distributed over $[-1,3]$. (5 points)

   (3) X_i follow the exponential distribution with parameter 5. (5 points)

3. (15 points) Continued with Example 4.5 (3), prove that if $U_1 \sim \mathrm{U}[0,1]$ and $U_2 \sim \mathrm{U}[0,1]$ are independent, and let

$$\begin{cases} Z_1 = \sqrt{-2\ln(U_1)}\cos(2\pi U_2) \\ Z_2 = \sqrt{-2\ln(U_1)}\sin(2\pi U_2) \end{cases},$$

then Z_1 and Z_2 are a pair of independent standard normal random variables.
(Hint: Show that $P(Z_1 \le a, Z_2 \le b) = \Phi(a)\Phi(b)$ for all a and b, which requires variable substitution in a double integral.)

4. (10 points) Explain how to generate numbers from the following distributions based on a uniform distribution random number generator:
   (1) Geometric(p); (5 points)
   (2) Standard Cauchy distribution, of which the PDF is $f(x) = 1/[\pi(1+x^2)]$, $x \in (-\infty, \infty)$. (5 points)

Part II: Implementations by Python

Note: Please provide both the results and the code, and present them in one PDF file.

5. (15 points) Continued with Problem 4, **generate 10000 numbers** from the following distributions based on a uniform distribution random number generator in Python, **plot the histogram** of the generated numbers, and **compare it with the theoretical PMF/PDF** of the following distributions:
   (1) Geometric(0.5); (5 points)
   (2) Standard Cauchy distribution. (10 points)

6. (15 points) Apply the rejection sampling technique to sample from

$$f^*(x) = 0.6\exp\{-(x+5)^2/2\} + 0.4\exp\{-(x-1)^2/0.5\},\ x \in (-\infty, \infty).$$

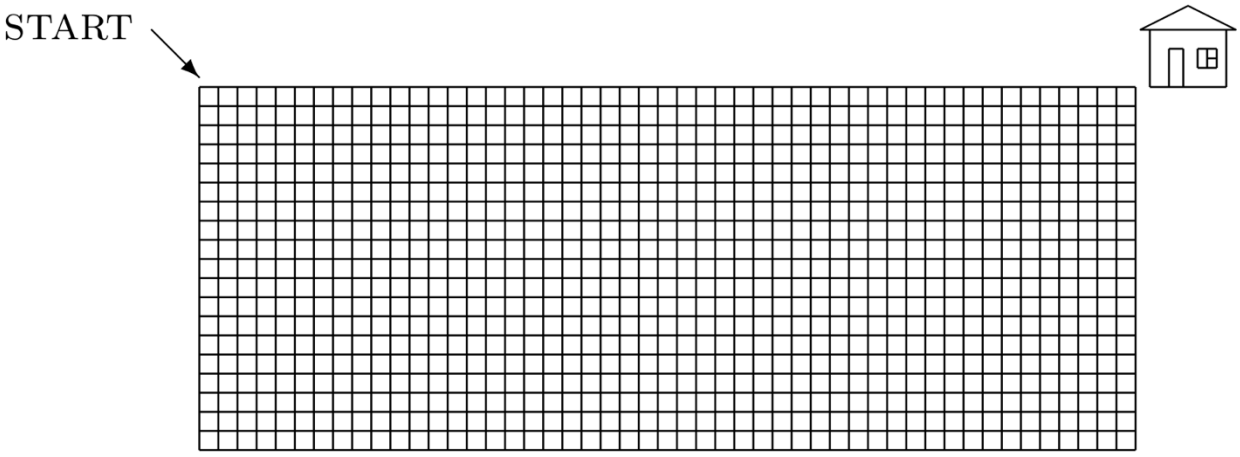
   (1) Choose an appropriate proposal distribution and plot it to show that it covers the target distribution. (5 points)
   (2) Apply the rejection sampling method to generate 500,000 samples, plot the histogram of the generated samples, and compare it with the theoretical PDF $f(x)$ (normalized $f^*(x)$). (5 points)
   (3) What is the acceptance proportion? (5 points)

7. (20 points) A forest consists of $1{,}000$ trees forming a perfect 20×50 rectangle as in the figure below. The northwestern (top-left) corner tree catches fire. Wind blows from the northwest, therefore trees can only catch fire from its buring left and above neighbors (火只会向东或向南蔓延). Assume the probability that any tree catches fire from its burning left neighbor is 0.8, and the probabilities to catch fire from trees immediately to the above is 0.3.

   (1) Conduct a Monte Carlo study to estimate the probability that more than 30% of the forest will eventually be burning. With probability 0.95, your answer should differ from the true value by no more than 0.005. (10 points)

   (You should first determine the value of n that satisfies the requirement, and then conduct a Monte Carlo study to estimate the probability.)

   (2) Based on the same study, predict the total number of affected trees X. (5 points)



   (3) What is the corresponding standard deviation of X. (5 points)