Ass 3 12310401 王子恒

1. (1) $P(600 \le X \le 1200) = P\left(\frac{600-900}{200} \le \frac{X-900}{200} \le \frac{1200-900}{200}\right)$

$= P(-\frac{3}{2} \le Z \le \frac{3}{2}) = 1-2P(Z > \frac{3}{2}) = 1-2\times 0.0668 = 0.8664$

(2) According to the std normal table, the CDF of 0.03 is $Z=-1.88$

$\therefore \frac{X-900}{200} = -1.88$

$\Rightarrow X = 524$

The families with incomes below 524 coins will recieve food stamps.

2. $\Delta = 16-4X < 0 \Rightarrow X > 4$

$\Rightarrow P(X>4) = 0.5$

$\therefore \mu = 4$

3. $P(X>96) = P\left(\frac{X-72}{6} > \frac{96-72}{6}\right) = P(Z > \frac{24}{6}) = 0.023$

According to the std table, the CDF of 0.023 is $Z = -2.0$

$\therefore \frac{24}{6} = 2 \qquad 6 = 12.$

$P(60 \le X \le 84) = P\left(\frac{60-72}{12} \le \frac{X-72}{12} \le \frac{84-72}{12}\right) = P(-1 \le Z \le 1)$

$= 1-2P(X<-1) = 1-2\times 0.1587 = 0.6826$

4.

$d \sim \text{Uniform}(a,b)$

$S = \frac{\pi}{4}E(d^2) = \frac{\pi}{4}\frac{1}{b-a}\int_a^b d^2\,dd = \frac{\pi}{12}\frac{b^3-a^3}{b-a}$

$= \frac{\pi}{12}(b^2+a^2+ab).$

5. $\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$

$\because Z \sim N(0,1)$

$\therefore \Phi(z) \sim \text{Uniform}(0,1).$

$E(\Phi(Z)) = \int_{-\infty}^{\infty} \Phi(x)\, d\Phi(x) = \frac{1^2-0}{2} = \frac{1}{2}$

$Var(\Phi(Z)) = E^2(\Phi(Z)) - \int_{-\infty}^{\infty} \Phi^2(x)\, d\Phi(x) = \frac{1}{4} - \frac{1}{3} = \frac{1}{12}$

6. (1) $Y_1 = \begin{cases} X & X>0 \\ -X & X<0 \end{cases}$

$\therefore f_{Y_1}(y) = f_X(y) + f_X(-y) = 2f_X(y) = \sqrt{\frac{2}{\pi}}\, e^{-\frac{y^2}{2}} \quad y \geq 0$

$f_{Y_1}(y) = 0 \quad y<0 \qquad \therefore f_{Y_1}(y) = \begin{cases} \sqrt{\frac{2}{\pi}}\, e^{-\frac{y^2}{2}} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

(2) $F_{Y_2}(y) = P(Y_2 \leq y) = P(2X^2+1 \leq y)$

$\Rightarrow -\sqrt{\frac{y-1}{2}} \leq X \leq \sqrt{\frac{y-1}{2}}$

$\therefore F_{Y_2}(y) = F_X\left(\sqrt{\frac{y-1}{2}}\right) - F_X\left(-\sqrt{\frac{y-1}{2}}\right)$

$= 2F_X\left(\sqrt{\frac{y-1}{2}}\right) - 1$

$\therefore f_{Y_2} = 2 \cdot F_X\left(\sqrt{\frac{y-1}{2}}\right) \cdot \frac{d}{dy}\sqrt{\frac{y-1}{2}}$

$= 2 \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\cdot\frac{y-1}{2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2(y-1)}}$

$= \frac{1}{2\sqrt{\pi(y-1)}} \cdot e^{-\frac{y-1}{4}} \quad y \geq 1$

$f_{Y_2} = 0 \quad y<1 \qquad f_{Y_2} = \begin{cases} \frac{1}{2\sqrt{\pi(y-1)}}\, e^{-\frac{y-1}{4}} & y \geq 1 \\ 0 & \text{otherwise} \end{cases}$

7. $X \sim \exp(2)$

$f_X = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x<0 \end{cases} \qquad F_X = 1 - e^{-2x}$

$\therefore Y_1 = e^{-2x} = \frac{1}{2} f_X \qquad F_{Y_1}(y) = P(Y_1 \leq y) = P(e^{-2x} \leq y) = P(X \geq \frac{-\ln y}{2}) = 1 - P(X \leq \frac{-\ln y}{2})$

$= 1 - (1 - e^{-2\cdot\frac{-\ln y}{2}}) = y$ $\therefore$ The CDF of Y_1 is y

Thus, $f_{Y_1}(Y) = dF_{Y_1}(Y) = 1 \quad 0<y<1$

$\therefore Y_1 \sim \text{Uniform}(0,1)$

$Y_2 = 1 - Y_1 \quad \therefore Y_2 \sim \text{Uniform}(0,1)$

Or proof: $F_{Y_2}(Y) = P(Y_2 \leq y) = P(1 - e^{-2x} \leq y) = P(X \leq \frac{-\ln(1-y)}{2}) = 1 - e^{\frac{-\ln(1-y)}{2}} = 1-(1-y) = y$

Similarly as Y_1, $Y_2 \sim \text{Uniform}(0,1)$.

8. (1) $\int_{-\infty}^{\infty} f(x,y) = \int_0^{\infty}\int_0^{\infty} k e^{-(3x+4y)} dx dy = \frac{k}{3}\int_0^{\infty} e^{-4y} dy = \frac{k}{12} = 1$

$\Rightarrow k = 12$

(2) $F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v) du dv$

$= \int_0^y \int_0^x 12 e^{-(3u+4v)} du dv$

$= \int_0^y -4(e^{-3x-4v} - e^{-4v}) dv$

$= -4(e^{-3x}-1) \cdot \int_0^y e^{-4v} dv$

$= (e^{-3x}-1)(e^{-4v}-1)$.

(3) $P(X+Y \leq 1) = P(X \leq 1-Y) = \int_0^1 \left(\int_0^{1-y} 12 e^{-(3x+4y)} dx \right) dy$

$= \int_0^1 -4(e^{-3+3y-4y} - e^{-4y}) dy$

$= -4\int_0^1 (e^{-y-3} - e^{-4y}) dy$

$= -4 \cdot \left[-(e^{-4} - e^{-3}) + \frac{1}{4}(e^{-4} - 1) \right]$

$= 1 - 4e^{-3} + 3e^{-4} \approx 0.856$

9. $f_X(x) = \int_{-\infty}^{\infty} f(x,y) dy = \int_x^{\infty} e^{-y} dy = -(0 - e^{-x}) = e^{-x} \quad (x>0)$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y) dx = \int_0^y e^{-y} dx = y e^{-y} \quad (y>0)$.

$\therefore f_X(x) = \begin{cases} e^{-x} & x>0 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} y e^{-y} & y>0 \\ 0 & \text{otherwise.} \end{cases}$