STA219 Ass 7 12310401 王子恒

1. (1) $\bar{x} = (17.2+22.1+18.5+17.2+18.6+14.8+21.7+15.8+16.3+22.8)/10$
$= 18.5$

$S^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2$

$= 7.88$

$S = \sqrt{S^2} = 2.80$

(2) From lower to upper:
14.8, 15.8, 16.3, 17.2, 17.2, 18.5, 18.6, 21.7, 22.1, 22.8
$Q_1 = X_3 = 16.3$
$Q_3 = X_8 = 21.7$
$Q_3 - Q_1 = 5.4$

2. (1) $E(X_i) = \frac{\theta}{2}$

$$E(\hat{\theta}_1) = E(\tfrac{4}{3}X_{(3)}) = \frac{4}{3}E(X_{(3)}) = \frac{4}{3}\int_{-\infty}^{\infty} x f_{max}(x)dx = \frac{4}{3}\int_{-\infty}^{\infty} 3xf(x)[F(x)]^2 dx = 4\int_{-\infty}^{\infty}\frac{x}{\theta}\cdot\frac{x^2}{\theta^2}dx = 4\int_0^\theta \frac{x^3}{\theta^3}dx = \theta.$$

$$E(\hat{\theta}_2) = E(4X_{(1)}) = 4E(X_{(1)}) = 4\int_0^\theta x\cdot 3\frac{1}{\theta}\cdot(1-\frac{x}{\theta})^2 dx = 12\int_0^\theta \frac{x}{\theta} - \frac{2x^2}{\theta^2} + \frac{x^3}{\theta^3}dx = 12\cdot\left[\frac{\theta}{2} - \frac{2\theta}{3} + \frac{\theta}{4}\right] = \theta.$$

Thus $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased estimators of θ.

(2) $$Var(\hat{\theta}_1) = Var(\tfrac{4}{3}X_{(3)}) = \frac{16}{9}Var(X_{(3)}) = \frac{16}{9}\{E(X_{(3)}^2) - [E(X_{(3)})]^2\}$$

$E(X_{(3)}^2) = \int_0^\theta x^2\cdot\frac{3}{\theta}\cdot\frac{x^2}{\theta^2}dx = \frac{3}{5}\theta^2$, $E(X_{(3)}) = \frac{3}{4}\theta$.

$Var(\hat{\theta}_1) = \frac{16}{9}\left(\frac{3}{5}\theta^2 - \frac{9}{16}\theta^2\right) = \frac{1}{15}\theta^2$

$$Var(\hat{\theta}_2) = 16\, Var(X_{(1)})$$
$$= 16\left\{E(X_{(1)}^2) - [E(X_{(1)})]^2\right\}$$
$$E(X_{(1)}^2) = \int_0^\theta x^2 \cdot \frac{3}{\theta} \cdot \left(1 - \frac{x}{\theta}\right)^2 dx$$
$$= \int_0^\theta \frac{3x^2}{\theta} - \frac{6x^3}{\theta^2} + \frac{3x^4}{\theta^3}\, dx$$
$$= \left(1 - \frac{3}{2} + \frac{3}{5}\right)\theta^2 = \frac{1}{10}\theta^2$$
$$E(X_{(1)}) = \frac{1}{4}\theta$$
$$Var(\hat{\theta}_2) = 16\left(\frac{1}{10}\theta^2 - \frac{1}{16}\theta^2\right) = \frac{3}{5}\theta^2$$
$\because Var(\hat{\theta}_1) < Var(\hat{\theta}_2)$

$\therefore \hat{\theta}_1$ is more efficient.

3. $\mu = 7.3 \times 10^9 \qquad \sigma = 0.7 \times 10^9$

$5.2 \times 10^9 = \mu - 3\sigma$

$9.4 \times 10^9 = \mu + 3\sigma$

$\therefore P(|X - \mu| \geq 3\sigma) \leq \frac{1}{9}$

$\therefore P(|X - \mu| < 3\sigma) \geq 1 - \frac{1}{9} = \frac{8}{9}$

The lower bound is $\frac{8}{9}$

4. $$E(\hat{\mu}) = E\left(\frac{2}{n(n+1)} \sum k X_k\right)$$
$$= \frac{2}{n(n+1)} \sum k E(X_k)$$
$$= \frac{2}{n(n+1)} \cdot \frac{n(n+1)}{2} E(X_k)$$
$$= \mu$$

$$Var(\hat{\mu}) = Var\left(\frac{2}{n(n+1)} \sum k X_k\right)$$
$$= \frac{2}{n^2(n+1)^2} \cdot \sum k^2 Var(X_k)$$
$$= \frac{2}{n^2(n+1)^2} \cdot \frac{n(n+1)(2n+1)}{6} Var(X_k)$$
$$= \frac{2n+1}{3n(n+1)} Var(X_k) \to 0$$

Thus, $\hat{\mu}$ is a consistent estimator of μ

5. (1) Consider the 1st population moment of X:

$\mu_1 = E(X) = 3\theta + 7(1-\theta) = 7 - 4\theta$.

$\Rightarrow \hat{\theta} = \frac{7-M_1}{4} = \frac{7-\bar{x}}{4}$

$\bar{x} = (3\times 5 + 7\times 3)/8 = \frac{9}{2}$

$\Rightarrow \hat{\theta}_1 = \frac{5}{8} = 0.625$

(2) $E(\hat{\theta}_1) = E\left(\frac{7-\mu_1}{4}\right) = E(\theta) = \theta$. $\Rightarrow$ It's unbiased estimator

$Var(\hat{\theta}_1) = \frac{1}{16} Var(\bar{x}) = \frac{1}{16}\frac{Var(x)}{n} = \frac{1}{16n}\left(E(x^2) - [E(x)]^2\right)$

$E(x^2) = 9\theta + 49(1-\theta) = 49 - 40\theta$

$Var(\bar{\theta}_1) = \frac{1}{16\times 8}(49 - 40\theta - 49 + 56\theta - 16\theta^2) = \frac{\theta - \theta^2}{8}$

(3) $\mathcal{L}(\theta; x) = \theta^5(1-\theta)^3 = \theta^5 - 3\theta^6 + 3\theta^7 - \theta^8$

$0 = \frac{d\mathcal{L}(\theta; x)}{d\theta} = 5\theta^4 - 15\theta^5 + 21\theta^6 - 8\theta^7$

$= \theta^4(5 - 15\theta + 21\theta^2 - 8\theta^3)$

$\because 0 < \hat{\theta}_1 < 1$

$\therefore \hat{\theta}_1 = \frac{5}{8} = 0.625$

6. (1) $\bar{x} = \frac{150}{10} = 15$

$\mu_1 = E(X) = \int_0^\infty x f(x)\,dx$

$= \int_0^\infty \frac{x}{\theta} e^{-\frac{x}{\theta}}\,dx$

$= \theta \int_0^\infty y e^{-y}\,dy$

$= -\theta(1+y)e^{-y}\Big]_0^\infty$

$= \theta$

$\int y^2 e^{-y}$

$\int y^2 \cdot d(-e^{-y})$ dy

$-y^2 e^{-y} + \int (+e^{-y})\, d(2y)$

$-y^2 e^{-y} + \int 2y e^{-y}$

$\theta_1 = \bar{x} = 15$

(2) $Var(\hat{\theta}_1) = Var(\bar{x}) = \frac{Var(x)}{n} = \frac{E(X^2) - [E(X)]^2}{10}$

$E(x^2) = \int_0^\infty \frac{x^2}{\theta} e^{-\frac{x}{\theta}}\,dx = \theta^2 \int_0^\infty y^2 e^{-y}\,dy = \theta^2 \cdot \left[-y^2 e^{-y} - 2(1+y)e^{-y}\right]_0^\infty$

$= 2\theta^2$

$Var(\hat{\theta}_1) = \frac{\theta^2}{10} = \frac{\bar{x}^2}{10} = 22.5$ $\qquad \sigma = \sqrt{22.5} = 4.74$

(3) $\mathcal{L}(x;\theta) = \prod_{i=1}^{10} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} = \theta^{-n} \exp(-\frac{1}{\theta}\sum x_i)$.

$= \theta^{-n} e^{-\frac{150}{\theta}}$

$l(x;\theta) = \ln \mathcal{L}(x;\theta) = -10\ln(\theta) - \frac{150}{\theta}$

$\frac{\partial}{\partial\theta} l(\theta) = -\frac{10}{\theta} + \frac{150}{\theta^2} = 0$

$\hat{\theta}_2 = 15$

7. (1) $\mu = 42$. $\sigma = 5$

$\alpha = 1 - 95\% = 0.05$

$z_{0.025} \cdot \frac{\sigma}{\sqrt{n}} = 1.96 \times \frac{5}{8} = 1.225$

The interval is $(42 - 1.225, 42 + 1.225)$

which is $(40.775, 43.225)$

(2) $Y \sim N(40, 5^2)$

$P(40.775 \leq Y \leq 43.225) = P\left(\frac{40.775-40}{5} \leq Z \leq \frac{43.225-40}{5}\right)$

$= P(0.155 \leq Z \leq 0.645)$

$= \Phi(0.645) - \Phi(0.155)$

$= \Phi(0.65) - \Phi(0.16)$

$= 0.7422 - 0.5636$

$= 0.1736$

8. $E(A) = \mu_A = 1.64$ $\qquad$ $E(B) = \mu_B = 1.62$

$Var(A) = \sigma_A^2 = 0.2^2$ $\qquad$ $Var(B) = \sigma_B^2 = 0.4^2$

$\bar{A} \sim N(\mu_A, \frac{\sigma_A^2}{n_A})$ $\qquad$ $\bar{B} \sim N(\mu_B, \frac{\sigma_B^2}{n_B})$

$\bar{A} \sim N(1.64, 0.001)$ $\qquad$ $\bar{B} \sim N(1.62, 0.0032)$

$Z_{\frac{\alpha}{2}} = Z_{0.05} = 1.645$

$\bar{A} - \bar{B} \sim N(\mu_A - \mu_B, \frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}) = (0.02, 0.0042)$

The interval of $\bar{A} - \bar{B}$ is $(-0.087, 0.127)$