STA219: Probability and Statistics for Engineering

Assignment 5

Note: The assignment can be answered in Chinese or English, either is fine. Please provide derivation and computation details, not just the final answer. Please submit a PDF file on BB.

1. (10 points) X and Y are independent random variables, and $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$. Define random variable Z as follows:

$$Z = \begin{cases} 1, & if\ X \leq Y, \\ 0, & if\ X > Y. \end{cases}$$

Derive the PMF of Z.

2. (10 points) Suppose X and Y are independent, identically distributed random variables, and follow geometric distribution, which is $P(X = k) = (1-p)^{k-1}p,\ k = 1,2,\ldots$. Derive the PMF of $Z = \max(X, Y)$.

3. (10 points) Consider a device equipped with 3 identical electrical components that operate independently. The operating times of these components follow an exponential distribution with parameter λ. The device functions normally only when all 3 components are working properly. Please find the PDF of the operating time T of the device.

4. (20 points) Suppose X and Y are bivariate normally distributed with means $\mu_X = \mu_Y = 0$, variances $\sigma_X^2 = \sigma_Y^2 = 1$, and correlation coefficient ρ.
   (1) Please derive the distribution and PDF of $X - Y$. (10 points)
   (2) Please calculate the covariance and correlation coefficient of $X - Y$ and XY. (10 points)

5. (15 points) The PDF of zero mean bivariate normal random vector $(X, Y)^T$ is of the form

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_X^2} - 2\rho\frac{xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right]\right).$$

   (1) Please derive the marginal density function of Y. (5 points)
   (2) Prove that the conditional PDF of X given $Y = y$ is normal, and identify its conditional mean and variance. (10 points)

(For both questions, please provide a detailed proof. Do not directly use the conclusion from the slide.)

6. (20 points) The annual revenues of Company X and Company Y are positively correlated since the correlation coefficient between the two revenues is 0.65. The annual revenue of Company X is, on average, 4500 with standard deviation 1500. The annual revenue of Company Y is, on average, 5500 with standard deviation 2000. Assume that X and Y are bivariate normally distributed.
   (1) Calculate the probability that annual revenue of Company X is less than 6800 given that the annual revenue of Company Y is 6800. (10 points)
   (2) Calculate the probability that the annual revenue of Company X is greater than that of Company Y given that their total revenue is 12000. (10 points)

7. (15 points) Since both the normal distribution and the Poisson distribution can be used to approximate binomial distribution, can we use normal distribution to approximate the Poisson distribution?
   (1) First derive the normal approximation to the Poisson distribution $Poisson(\lambda)$. (5 points)
   (2) Use Python to plot the Poisson distribution and its corresponding normal approximation under different values of λ (similar to the plots on page 51 of the slide). Explain in which cases the normal approximation works well and in which cases it does not work well. (10 points)
   (Please provide both the plots and the code.)

   (既然正态分布和泊松分布都能近似二项分布，那正态分布是不是也可以近似泊松分布？请给出泊松分布$Poisson(\lambda)$的正态近似，并用 Python 代码画出 λ 取不同的值时与其近似正态分布的图形(与 PPT 上类似的图)，说明什么情况下正态近似比较好，什么情况下不太好。)