Assignment 01 - Errors and function evaluation

ESO208 - Computational methods in engineering — 2024-25 I

Instructor: Balaji Devaraju [`dbalaji`] — **Issue: 07.08.2024**

Deadline: 20.08.2024 (20:00 hrs)

1. A sensor measures the soil temperature upto six significant figures. Suppose the true temperature is `25.123456` °C. Due to the precision limitations of the sensor, the recorded temperature is `25.123450` °C.

   i. Calculate the absolute and relative error in the recorded temperature.

   ii. If the temperature needs to be known within an absolute error of `0.000005` °C, does the recorded temperature meet this requirement?

   iii. Suppose the sensor measures temperature 5 times, at the same time and place, each time with a different error uniformly distributed within ±`0.000003`°C. Calculate the absolute errors for each measurement and the total cumulative error for these 5 measurements.

   iv. Analyze the effect of rounding off in the context of cumulative error and discuss potential mitigation strategies.

2. You are given the exponential function e^x and its Maclaurin series expansion:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

   i. Compute e^2 using the first four terms of the Maclaurin series expansion with a fixed-point system of four decimal places.

   ii. Compare the result with the approximate value of $e^2 \approx$ `7.389056099` and calculate the truncation error.

   iii. Use the given expression for the truncation error to estimate the truncation error for e^2:

$$R_4(x) = \frac{e^c \cdot x^5}{5!}$$

   This expression is known as the Lagrange form of the remainder and it represents the error introduced when the series is truncated after the fourth term. Here, c is some value between 0 and x.

   iv. Plot the truncation error for e^x for x ranging from `-2` to `-2` with an interval of `0.5` using the first four terms. Discuss the behavior of the error.

Programming Problems

3. Vincenty's distance formula calculates the geodetic distance between two points on the Earth's surface on a reference ellipsoid. The main steps of the formula are:

$$\begin{aligned}
&U_1 = \arctan\left((1-f)\tan\phi_1\right)\\
&U_2 = \arctan\left((1-f)\tan\phi_2\right)\\
&\lambda = L\\
&\sin U_1 = \sin\phi_1 \cos U_1, \quad \cos U_1 = \cos\phi_1 \cos U_1\\
&\sin U_2 = \sin\phi_2 \cos U_2, \quad \cos U_2 = \cos\phi_2 \cos U_2
\end{aligned}$$

   Where:

- ϕ_1 and λ_1 are the latitude and longitude of the first point,
- ϕ_2 and λ_2 are the latitude and longitude of the second point,
- f is the flattening of the ellipsoid ($f = \texttt{1/298.257223563}$),
- L is the difference in longitude ($L = \lambda_2 - \lambda_1$).

The iterative formula then continues as follows:

$$\sin\sigma = \sqrt{(\cos U_2 \sin\lambda)^2 + (\cos U_1 \sin U_2 - \sin U_1 \cos U_2 \cos\lambda)^2}$$
$$\cos\sigma = \sin U_1 \sin U_2 + \cos U_1 \cos U_2 \cos\lambda$$
$$\sigma = \arctan\left(\frac{\sin\sigma}{\cos\sigma}\right)$$
$$\sin\alpha = \frac{\cos U_1 \cos U_2 \sin\lambda}{\sin\sigma}$$
$$\cos^2\alpha = 1 - \sin^2\alpha$$
$$\cos 2\sigma_m = \cos\sigma - \frac{2\sin U_1 \sin U_2}{\cos^2\alpha}$$

The distance is finally given by:

$$\Delta\sigma = \frac{f}{16}\cos^2\alpha\left[4 + f(4 - 3\cos^2\alpha)\right]$$
$$s = b\sigma\,(1 + \Delta\sigma)$$

Where:

- $b = \texttt{6356752.3142}$ meters is the semi-minor axis of the ellipsoid,
- s is the distance between the two points.

i. Write a function to compute the geodesic distance between two points given their latitude and longitude using Vincenty's formula (See Appendix for procedure). Inputs should be latitude and longitude of two points and output should be the distance.

ii. Implement the function for your hometown and IIT Kanpur to calculate the distance. (**Hint**: You can use Google Maps or your phone GPS to get the coordinates of your hometown and IIT Kanpur and mention the source from where you have got those coordinates).

iii. Introduce a small change $\delta\phi = \texttt{0.0001}$ in the latitude of the hometown and recalculate the distance.

iv. Perform a backward error analysis to determine the relative change in the latitude that would result in the recalculated distance matching the original calculated distance:

- Calculate the original distance s_{original} using the coordinates of your hometown and IIT Kanpur.
- Introduce a small change ($\delta\phi = \texttt{0.0001}$) in the latitude of your hometown and recalculate the distance to get $s_{\text{perturbed}}$.
- Determine the relative change in latitude, $\frac{\delta\phi}{\phi_1}$, needed to make the recalculated distance match the original distance.

v. Create a plot where the x-axis represents small perturbations in the latitude of the first point ($\delta\phi$) and the y-axis represents the corresponding geodesic distance calculated using Vincenty's formula. Highlight the base distance (without perturbation) for reference.

4. Manning's equation is used to estimate the velocity or flow rate of water in open channels. It is given by:

$$Q = \frac{1}{n} A R^{2/3} S^{1/2}$$

where:

- Q is the flow rate (m^3/s) (dimensionless),
- n is the Manning's roughness coefficient,
- A is the cross-sectional area of flow (m^2),
- R is the hydraulic radius (m),
- S is the slope of the energy grade line (dimensionless).

i. In the table, area and wetted perimeter from three different sensors have been measured. Write a function to compute the flow rate Q using Manning's equation for given values of n, A, R, and S. Using the values for each month from all three sensors, compute the flow rate Q using Manning's equation with $n = \texttt{0.035}$ and $S = \texttt{0.001}$. Plot the flow rates for all months and sensors.

ii. For each month, using the hydraulic radius R values from all three sensors, truncate R to three decimal places. Recalculate the flow rate Q using the truncated R values. Determine and plot the truncation errors by comparing the recalculated flow rates with those obtained using the original R values for all months and sensors. Also, calculate and plot the errors in the flow rates due to the truncation of R. Compare the initial flow rate and the flow rate obtained using the truncated R values.

iii. For each month, using the hydraulic radius R values from all three methods, compute the flow rate Q using Manning's equation. Introduce a small perturbation ($\Delta R = \texttt{0.0001}\,\mathrm{m}$) to R and recalculate Q. Calculate and plot the forward error in the flow rate and the backward error in the hydraulic radius for each month and sensor.

5. Consider the function $f(x) = \sin x$ and its Maclaurin series expansion around $x = 0$:

$$f(x) = \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

i. Evaluate $f(x)$ at $x = \frac{\pi}{6}$ using the first four terms of the Maclaurin series expansion.

ii. Introduce a small perturbation $\delta x = 10^{-6}$ to x. Evaluate $f(x)$ at $x = \frac{\pi}{6} + \delta x$ using the first four terms of the Maclaurin series expansion.

iii. Estimate and analyze the condition number of the function $f(x) = \sin x$:

(a) Estimate the condition number of $f(x) = \sin x$ at $x = \frac{\pi}{6}$ using the series expansion. The condition number $K_{f(x)}$ of a function $f(x)$ at a point x is given by:

$$K_{f(x)} = \left| \frac{x f'(x)}{f(x)} \right|$$

where $f'(x)$ is the derivative of $f(x)$.

(b) Generalize the calculation of the condition number K_f for $f(x) = \sin x$ at different values of x:

$$K_{f(x)} = \left| \frac{x \cos x}{\sin x} \right|$$

(c) Plot the condition number K_f as a function of x for x ranging from 0 to 2π.

(d) Based on the plot, identify the range of x values for which the problem is well-conditioned (i.e., where the condition number is reasonably small).

(e) Discuss the sensitivity of the function $f(x) = \sin x$ to small changes in x based on the calculated condition number. Explain for what range of x the function $f(x)$ is well-conditioned and why this is important for numerical computations.

Appendix

Vincenty's Formula for Geodetic Distance

Vincenty's formula calculates the geodetic distance between two points on the Earth's surface on a reference ellipsoid. The main steps of the formula are outlined below:

Constants and Initial Values

$$
\begin{aligned}
f &= \frac{1}{298.257223563} \quad \text{(flattening)} \\
b &= 6356752.3142 \text{ meters} \quad \text{(semi-minor axis)}
\end{aligned}
$$

Reduced Latitudes

$$
\begin{aligned}
U_1 &= \arctan((1-f)\tan\phi_1) \\
U_2 &= \arctan((1-f)\tan\phi_2)
\end{aligned}
$$

Initial Difference in Longitude

$$L = \lambda_2 - \lambda_1$$

Sine and Cosine of Reduced Latitudes

$$
\begin{aligned}
\sin U_1 &= \sin\phi_1 \cos U_1 \\
\cos U_1 &= \cos\phi_1 \cos U_1 \\
\sin U_2 &= \sin\phi_2 \cos U_2 \\
\cos U_2 &= \cos\phi_2 \cos U_2
\end{aligned}
$$

Iterative Calculations

$$\lambda_{\text{initial}} = L$$

For each iteration, update λ as follows:

$$
\begin{aligned}
\sin(\lambda) &= \sin(\lambda_{\text{old}}) \\
\cos(\lambda) &= \cos(\lambda_{\text{old}}) \\
\sin(\sigma) &= \sqrt{(\cos(U_2)\sin(\lambda))^2 + (\cos(U_1)\sin(U_2) - \sin(U_1)\cos(U_2)\cos(\lambda))^2} \\
\cos(\sigma) &= \sin(U_1)\sin(U_2) + \cos(U_1)\cos(U_2)\cos(\lambda) \\
\sigma &= \arctan 2(\sin(\sigma), \cos(\sigma)) \\
\sin(\alpha) &= \frac{\cos(U_1)\cos(U_2)\sin(\lambda)}{\sin(\sigma)} \\
\cos^2(\alpha) &= 1 - \sin^2(\alpha) \\
\cos(2\sigma_m) &= \cos(\sigma) - \frac{2\sin(U_1)\sin(U_2)}{\cos^2(\alpha)} \quad \text{if } \cos^2(\alpha) \neq 0 \\
C &= \frac{f}{16}\cos^2(\alpha)(4 + f(4 - 3\cos^2(\alpha))) \\
\lambda_{\text{new}} &= L + (1-C) f \sin(\alpha) \left(\sigma + C\sin(\sigma)\left(\cos(2\sigma_m) + C\cos(\sigma)(-1 + 2\cos^2(2\sigma_m))\right)\right)
\end{aligned}
$$

Convergence Check

Check for convergence:

$$\text{If } |\lambda_{\text{new}} - \lambda_{\text{old}}| < \text{tolerance}, \text{ the iteration has converged.}$$

Why Do We Iterate λ?

1. The geodetic problem involves nonlinear equations due to the trigonometric functions of the latitudes and longitudes. Directly solving these equations is not feasible because the relationship between the geographic coordinates and the geodetic distance is inherently nonlinear.

2. The angular separation σ between two points is a function of λ. Accurate computation of σ is crucial for determining the geodetic distance. The iterative method refines the value of λ, ensuring that σ is calculated accurately:

$$\sin\sigma = \sqrt{(\cos U_2 \sin\lambda)^2 + (\cos U_1 \sin U_2 - \sin U_1 \cos U_2 \cos\lambda)^2}$$

$$\cos\sigma = \sin U_1 \sin U_2 + \cos U_1 \cos U_2 \cos\lambda$$

3. The correction factor C accounts for the Earth's flattening and azimuth. It depends on λ and must be refined iteratively to ensure accurate calculations:

$$C = \frac{f}{16}\cos^2(\alpha)(4 + f(4 - 3\cos^2(\alpha)))$$

4. The iterative process refines λ until the change between successive iterations is smaller than a predefined tolerance. This ensures that the calculations of σ, α, and ultimately the distance s are accurate:

$$\lambda_{\text{new}} = L + (1 - C) f \sin(\alpha) \left(\sigma + C\sin(\sigma)\left(\cos(2\sigma_m) + C\cos(\sigma)(-1 + 2\cos^2(2\sigma_m))\right)\right)$$

Final Calculations

Once λ has converged, the final calculations for the geodetic distance s are performed:

$$u^2 = \cos^2(\alpha)\frac{a^2 - b^2}{b^2}$$

$$A = 1 + \frac{u^2}{16384}\left(4096 + u^2\left(-768 + u^2(320 - 175u^2)\right)\right)$$

$$B = \frac{u^2}{1024}\left(256 + u^2\left(-128 + u^2(74 - 47u^2)\right)\right)$$

$$\Delta\sigma = B\sin(\sigma)\left(\cos(2\sigma_m) + \frac{B}{4}\left(\cos(\sigma)(-1 + 2\cos^2(2\sigma_m)) - \frac{B}{6}\cos(2\sigma_m)(-3 + 4\sin^2(\sigma))(-3 + 4\cos^2(2\sigma_m))\right)\right)$$

$$s = bA(\sigma - \Delta\sigma)$$