Assignment 03 -Linear Algebra

ESO208 - Computational methods in engineering — 2024-25 I
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Deadline: 28.09.2024 (20:00 hrs)

Manual Problems

1. Consider a 2D Vector space $\mathbb{R}^2$:

   i. Compute L_2 norm of $\mathbf{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

   ii. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Determine whether $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

   iii. Write the rotation matrix $R(\theta)$ for $\theta = \pi/4$.

   iv. Apply the above rotation matrix $R(\pi/4)$ to $\mathbf{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

   v. Compute the Frobenius norm of the rotation matrix $R(\pi/4)$.

2. Consider the following set of equations:

$$\begin{bmatrix} 10^{-5} & 10^{-5} & 1 \\ 10^{-5} & -10^{-5} & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \times 10^{-5} \\ -2 \times 10^{-5} \\ 1 \end{bmatrix}$$

   i. Solve the system using Gaussian elimination without pivoting using 3-digit floating-point arithmetic with rounding.

   ii. Perform complete pivoting and carry out Gaussian elimination steps once again using 3-digit floating-point arithmetic with rounding.

   iii. Rewrite the set of equations after scaling and equilibration.

   iv. Solve the system using Gaussian elimination without pivoting using 3-digit floating-point arithmetic with rounding.

3. The Avengers are preparing for an impending battle against a formidable enemy. They must set up energy shields at three key locations (Stark Tower, Wakanda, and the Sanctum Sanctorum) to protect vital assets. The shields require energy from three sources: Tony Stark's Arc Reactor, Wakanda's Vibranium Core, and the Time Stone. However, due to the complex nature of energy distribution, the efficiency of energy transfer varies depending on the source-location pair. The team needs to determine the optimal energy distribution to each location such that all shields are activated with the least amount of energy wasted.

   Problem Statement: The energy transfer from the three sources to the three locations can be represented by the following system of linear equations:

$$\begin{aligned} 3x_1 + 4x_2 + 2x_3 &= 60 \quad \text{(Stark Tower)} \\ 2x_1 + 3x_2 + 3x_3 &= 48 \quad \text{(Wakanda)} \\ 5x_1 + 1x_2 + 4x_3 &= 72 \quad \text{(Sanctum Sanctorium)} \end{aligned}$$

   Where,

   – x_1: energy supplied by Tony Stark's arc reactor

– x_2: energy supplied by Wakanda's Vibranium core

– x_3: energy supplied by Time Stone

Objectives: Determine the energy distribution to Stark Tower, Wakanda, and the Sanctum Sanctorum using the following numerical methods:

i. Gauss Elimination

ii. Gauss-Jordan Elimination

iii. LU decomposition by using Gauss elimination

Comment on stability, accuracy, and the most suitable method for the calculations.

Programming Problems

1. Write a general Python function to solve the tridiagonal equations $Ax = b$ by Crout's method (Do not use inbuilt functions), taking tolerance to be 10^{-10}, where:

$$A = \begin{bmatrix} 6 & 2 & 0 & 0 & 0 \\ -1 & 7 & 2 & 0 & 0 \\ 0 & -2 & 8 & 2 & 0 \\ 0 & 0 & 3 & 7 & -2 \\ 0 & 0 & 0 & 3 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 27 \\ -3 \\ 4 \\ -3 \\ 1 \end{bmatrix}$$

2. A well-known example of an ill-conditioned matrix is the Hilbert matrix:

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(a) Write a program that specializes in solving the equations $Ax = b$ by Doolittle's decomposition method, where A is the Hilbert matrix of arbitrary size $n \times n$, and

$$b_i = \sum_{j=1}^{n} A_{ij}.$$

The program should have no input apart from n. Do not use inbuilt functions to generate the matrix or solving the matrix. In the output file use the last digit of your Roll Number as the value of n. If your last digit is 0 : n = the last non-zero digit of your Roll number

(b) By running the program, determine the largest n for which the solution is within 6 significant figures of the exact solution given as follows:

$$x = \begin{bmatrix} 1 & 1 & 1 & \cdots \end{bmatrix}^T.$$

(c) Find the inverse of the Hilbert matrix of the largest order n obtained in previous question by making use of the Doolittle Algorithm.

3. In a chemical plant, a distillation column is used to separate a mixture of multiple components (e.g., different hydrocarbons) based on their boiling points. The goal is to determine the concentrations of each component in different stages of the column to achieve a desired separation. For an arbitrary system, combining the mass balance equations for all components on all stages, equilibrium relationships, and overall material balances, you obtain a system of linear equations that can be expressed using a matrix given below:

$$
\begin{bmatrix}
2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\
-1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

where n is the total number of unknowns (number of components times stages in the distillation column).

(a) Without using the inbuilt functions, write a python function to solve the matrix using the Jacobi method (taking the tolerance value of 10^{-9}).

(b) Write a computer program to solve the n simultaneous equations represented by the above matrix using:

i. Gauss-Seidel method.

ii. Gauss-Seidel method with relaxation.

The program should work with any value of n. If there are 4 components formed in 5 stages, use the above program to solve the system of linear equations thus formed for a tolerance of 10^{-9}.

(c) Find the optimum relaxation coefficient for the Gauss–Seidel method with a relaxation scheme.

(d) Comment on the convergence of the above three schemes used and substantiate your answer with a suitable reason.

(e) If the chemical used was changed, it resulted in a new system of equations, and the new coefficient matrix is formed by replacing the diagonal elements from 2 to 4. Solve the newly formed system of linear equations by

(i) Jacobi Method

(ii) Gauss-Seidel Method

(iii) Gauss-Seidel with Relaxation.

How is the new system of equations different from the system given above? Compute the optimum relaxation factor for this new system. Comment on the convergence of the new system of linear equations with proper reasoning.

Bonus Problem

A new iterative scheme is proposed for solving the system of linear equations: $Ax = b$. The coefficient matrix $A = [a_{ij}]$ is split as follows:

$$A = D - E$$

where,

$$
D = \begin{bmatrix}
a_{11} & 0 & \cdots & & \\
 & a_{22} & & & \\
 & & \ddots & & \\
a_{k1} & a_{k2} & \cdots & a_{kk} & \cdots \\
\vdots & & & & a_{nn}
\end{bmatrix}, \quad E = -(A - D)
$$

where $1 \leq k \leq n$, all other entries are zero, and

$$E = -(A - D),$$

The iterative scheme is given by:

$$x^{(k+1)} = Tx^{k} + D^{-1}b$$

Where

$$T = D^{-1}E$$

(a) Show that the infinity norm of the iterative matrix of new scheme is less than or equal to the infinity norm of the iterative matrix of the Jacobi method. Solve this manually.

(b) Let the matrix A be given as:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Develop a Python function to solve the matrix A using the above-given iterative scheme.