Assignment 04 - Fourier

ESO208 - Computational methods in engineering | 2024-25 I
Instructor: Balaji Devaraju [dbalaji] | **Issue: 16.10.2024**
Deadline: 23.10.2024 (20:00 hrs)

1. **Orthogonal Functions:** One way of expressing Fourier expansion for any function is to use Sines and Cosines as the basis functions. Like vector algebra, orthogonality of these basis functions is an important aspect of Fourier series. Analogous to vector algebra, we define that any combination(s) of these basis functions are orthogonal if their inner product is zero. Inner product is defined as the integral of product of these basis functions within the limits of defined period (0 to L).

   Using this information, prove the following: **[10 * 3 = 30 marks]**

   i.
   $$\int_0^L \sin\left(\frac{2\pi nx}{L}\right)\cos\left(\frac{2\pi mx}{L}\right)dx = 0 \quad \forall\, n, m$$

   ii.
   $$\int_0^L \cos\left(\frac{2\pi nx}{L}\right)\cos\left(\frac{2\pi mx}{L}\right)dx = \begin{cases} 0 & \text{for } n \neq m \\ \frac{L}{2} & \text{for } n = m \end{cases}$$

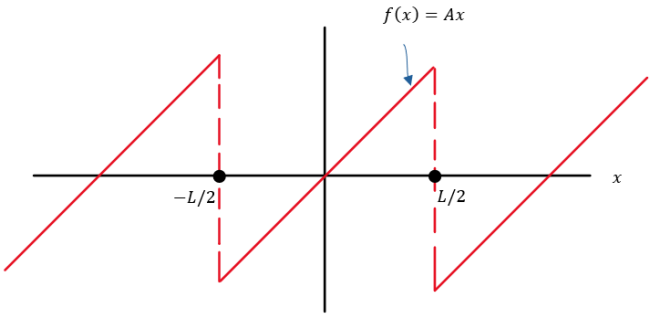
   iii.
   $$\int_0^L \sin\left(\frac{2\pi nx}{L}\right)\sin\left(\frac{2\pi mx}{L}\right)dx = \begin{cases} 0 & \text{for } n \neq m \\ \frac{L}{2} & \text{for } n = m \end{cases}$$

   where, $0 \leq x \leq L$, L is the period for basis functions, and n and m are some integers.

2. **Fourier Series of Sawtooth Functions:** Using Sines and Cosines, find the Fourier series for the periodic function shown below. It is defined by:

   $$f(x) = Ax \quad \text{for} \quad -\frac{L}{2} < x < \frac{L}{2}$$

   where L is the period of the function. **[10 + 5+ 10 = 25 marks]**



   i. For $x = \frac{L}{4}$, show that $\frac{\pi}{4}$ can be approximated by an infinite series:

   $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

ii. Using **Python**, draw the subplots for including the following number of terms of Fourier series: 1 term, 5 terms, 20 terms, 100 terms, and 250 terms. For simplicity, assume $A = 1$ and $L = 1$. Comment on how using a greater number of terms affects the representation of the function using the Fourier series.

3. **Fourier Transform of Gaussian function:** Fourier transform is used to express a non-periodic function as a series involving exponential terms. The key difference between the mathematical representation of Fourier series and Fourier transform is the use of discrete sum and continuous integral, respectively. Any non-periodic function $f(x)$ can be expressed as:

$$f(x) = \int_{-\infty}^{\infty} C(k) e^{ikx}\, dk$$

where $C(k)$ is the Fourier transform of $f(x)$:

$$C(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

i. Show that the Fourier transform of the Gaussian function:

$$f(x) = Ae^{-ax^2}$$

remains Gaussian with the form:

$$C(k) = \frac{1}{2\sqrt{\pi a}} Ae^{-\frac{k^2}{4a}}$$

[10 marks]

ii. Use **Python** to comment on the dependence of $f(x)$ and $C(k)$ on a for $a = 0.5, 1, 2, 10$. **[5 marks]**

4. **Programming Question:** A square wave is defined as:

$$f(x) = \begin{cases} A & \text{for } -a \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

i. Use **Python** to visualize $f(x)$ and its Fourier transform $C(k)$. For simplicity, assume $A = 1$ and $a = 1$. **[10 + 10 = 20 marks]**

ii. Plot $C(k)$ for $a = 10$, $A = 1$ and $a = 40$, $A = 1$. What do you observe regarding the oscillating pattern in $C(k)$? **[10 marks]**

Instructions:

(a) Flowchart is not required in any of the questions. While submitting the final file, make sure that all questions are sequentially arranged. For questions 2 and 3, make sure that sub-parts for Python are placed at appropriate order. This will ensure that all your questions are checked by us. Don't submit multiple files. **From now on, we will not consider submissions made by any other mode (or format). Kindly adhere to prescribed format. Final submission should be made as ONE pdf file.**

(b) Please read questions carefully. You may have to use python for some parts of questions 2 and 3. These programming questions (or sub-parts) will be explicitly stated.

(c) Question 4 is to be solved using Python.

(d) Total marks of the assignment = 100.

(e) You may try https://onlineconvertfree.com/convert-format/ipynb-to-pdf/ to convert your ipynb file to pdf. It is working well in my case.