week4 Relation

> Chapter9 section 9.1,9.2,9.5,9.6

Definition

Relationships between elements of sets are represented using the structure called a relation, which is just a subset of the Cartesian product of the sets.

Definition

Let A and B be sets. A *binary relation* from A to B is a subset of $A \times B$.

We use $a\ R\ b$ or $R(a, b)$ to denote that $(a, b) \in R$.

Different Relations

Binary Relations

Let A and B be sets. A *binary relation* from A to B is a subset of $A \times B$.

Reflexive Relations

Definition

A relation R on a set A is called *reflexive*, if $(a, a) \in R$ for every element $a \in A$.

Symmetric Relations

Definition

A relation R on a set A is called *symmetric*, if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

Antisymmetric Relations

Definition

A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

Antisymmetric and Symmetric is not opposite

> if only (a,a) in R, it is both Symmetric and Antisymmetric

Transitive Relations

Definition

A relation R on a set A is called *transitive*, if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

Combining Relations: Composite

There is another way that relations are combined that is analogous to the composition of functions.

Definition

Let R be a relation from a set A to a set B and S a relation from B to a set C. The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Exercise

What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

> (1,0) (1,1) (2,1) (2,2) (3,0) (3,1)

Composing a Relation with Itself

Definition

Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Theorem

Theorem

The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

Equivalence Relations

Equivalence relation: a reflexive, symmetric, and transitive relation.

The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Equivalence Classes

$[a]_R$ (equivalence class of a with respect to R): the set of all elements of A that are equivalent to a

$$[a]_R = \{s | (a, s) \in R\}.$$

$$R = \{(x, y) | x = y(mod4)\}$$

$$[0]R = \{-12, -8, -4, 0, 4, 8, 12......\}$$

Partition

A partition of a set S is a collection of disjoint nonempty subsets of S t hat have S as their union.

- $A_i = \phi$ for $i \in I$
- $A_i \cap A_j = \phi$ when$i \neq q$
- $\cup_{i \in I} A_i = S$

example

$$[0]R, [1]R, [2]R, [3]R$$

Partial Orderings

Partial ordering: a relation that is reflexive, antisymmetric, and transitive

A set *S* together with a partial ordering *R* is called a partially ordered set, or **poset**, and is denoted by (*S**, R*). Members of *S* are called elements of the poset.

$$poset : (Z, >=)$$

$$the\ same\ as\ \{(x, y) | x >= y\}$$

- $\leq$
- $|$
- $\subseteq$

Comparable V.S. Incomparable

if (a,b) and (b,a) is both in poset, a,b are comparable.

Comparable: the elements a and b in the poset $(A, \preceq)$ are comparable if $a \preceq b$ or $b \preceq a$
Incomparable: When a and b are elements of S such that neither $a \preceq b$ nor $b \preceq a$, a and b are called incomparable.