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ECE236B homework #1 solutions

1. *Least squares fit of a circle to points.*

(a) Expanding the squares in the ith term of the cost function gives

$$\begin{aligned}(u_i - u_c)^2 + (v_i - v_c)^2 - R^2 &= -2u_iu_c - 2v_iv_c + u_c^2 + v_c^2 - R^2 + u_i^2 + v_i^2 \\ &= -2u_iu_c - 2v_iv_c + w + u_i^2 + v_i^2.\end{aligned}$$

This is linear in u_c, v_c, w, so we obtain a linear least squares problem with variables $x = (u_c, v_c, w)$ and

$$A = \begin{bmatrix} -2u_1 & -2v_1 & 1 \\ -2u_2 & -2v_2 & 1 \\ \vdots & \vdots & \vdots \\ -2u_m & -2v_m & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} -u_1^2 - v_1^2 \\ -u_2^2 - v_2^2 \\ \vdots \\ -u_m^2 - v_m^2 \end{bmatrix}.$$

(b) The property follows from the normal equations $A^T(Ax - b) = 0$. $Ax - b$ is an m-vector with components $-2u_iu_c - 2v_iv_c + w + u_i^2 + v_i^2$. Since the last column of A is all ones, the last equation of $A^T(Ax - b) = 0$ gives

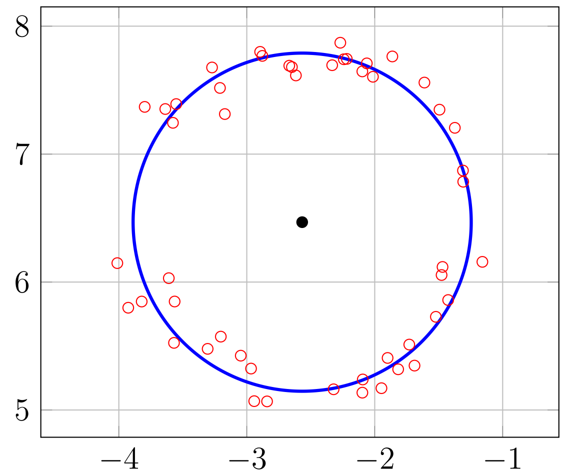
$$\begin{aligned} 0 &= \sum_{i=1}^m \left(-2u_iu_c - 2v_iv_c + w + u_i^2 + v_i^2\right) \\ &= \sum_{i=1}^m \left((u_i - u_c)^2 + (v_i - v_c)^2 + w - u_c^2 - v_c^2\right) \\ &= \sum_{i=1}^m \left((u_i - u_c)^2 + (v_i - v_c)^2\right) + m(w - u_c^2 - v_c^2).\end{aligned}$$

Therefore

$$u_c^2 + v_c^2 - w = \frac{1}{m}\sum_{i=1}^m \left((u_i - u_c)^2 + (v_i - v_c)^2\right) \geq 0.$$

(c) The solution for the problem is

$$R = 1.3214, \qquad u_c = -2.5671, \qquad v_c = 6.468.$$



2. *Schur complements.*

   (a) If $A = 0$, the matrix X is positive semidefinite if and only if for all u, v,

$$2u^T Bv + v^T Cv = \begin{bmatrix} u \\ v \end{bmatrix}^T \begin{bmatrix} 0 & B \\ B^T & C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \geq 0.$$

   Clearly, a sufficient condition is that $B = 0$ and C is positive semidefinite. Taking $u = 0$ shows that positive semidefiniteness of C is also necessary. To see that $B = 0$ is necessary, take any v with $Bv \neq 0$ and choose $u = -tBv$. The quadratic form then reduces to

$$-2t\|Bv\|_2^2 + v^T Cv$$

   which is negative for sufficiently large t.

   (b) We have

$$\begin{array}{rcl} AA^\dagger & = & Q_1\Lambda_1 Q_1^T Q_1 \Lambda_1^{-1} Q_1^T \\ & = & Q_1 Q_1^T \\ I - AA^\dagger & = & Q_1Q_1^T + Q_2Q_2^T - AA^\dagger \\ & = & Q_2Q_2^T. \end{array}$$

   The proofs of the identities $A^\dagger A = Q_1Q_1^T$ and $I - A^\dagger A = Q_2 Q_2^T$ are similar.

   (c) Suppose we sort the eigenvalues of A so that its eigenvalue decomposition can be written as

$$A = Q\Lambda Q^T = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}^T$$

   with Λ_1 positive diagonal. Following the hint, the question can be reduced to showing that the block matrix

$$\begin{bmatrix} \Lambda_1 & 0 & Q_1^T B \\ 0 & 0 & Q_2^T B \\ B^T Q_1 & B^T Q_2 & C \end{bmatrix}$$

is positive semidefinite. By the result in part (a), the matrix is positive semidefinite if and only if

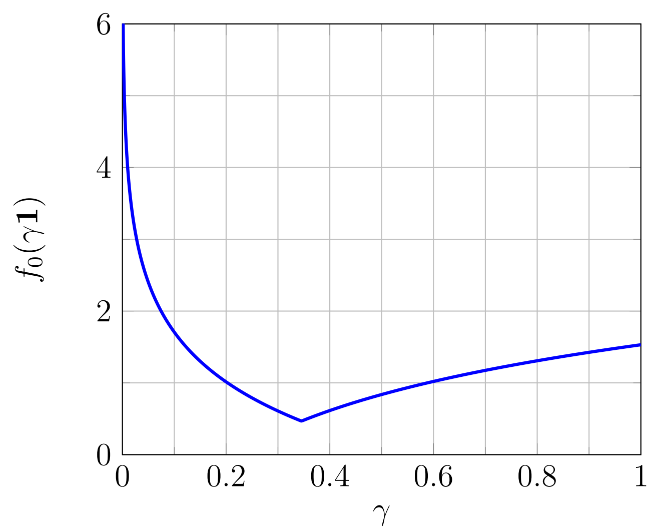
$$Q_2^T B = 0, \qquad \begin{bmatrix} \Lambda_1 & Q_1^T B \\ B^T Q_1 & C \end{bmatrix} \succeq 0.$$

The first condition is equivalent to $(I - AA^\dagger)B = 0$. Since Λ_1 is nonsingular, we can apply the Schur complement result for nonsingular A to the 2×2 block matrix. This gives the condition

$$C - B^T Q_1 \Lambda^{-1} Q_1^T B = C - B^T A^\dagger B \succeq 0.$$

3. *Illumination problem.*

(a) *Equal lamp powers.* The figure shows $f_0(\gamma \mathbf{1}) = \max_k |\log(\gamma a_k^T \mathbf{1})|$ versus γ.



The minimum is reached at $\gamma = 0.3453$.

(b) *Least squares with saturation.* We compute p as

```
p = A \ ones(n,1).
```

All coefficients of p are outside the feasible interval $[0, 1]$ and need to be rounded.

(c) *Regularized least squares.* We compute p by solving a least squares problem

```
p = [A; sqrt(rho)*eye(m)] \ [ones(n,1); sqrt(rho)*.5*ones(m,1)].
```

The smallest ρ that gives a feasible p is $\rho = 0.2190$.

(d) *Chebyshev approximation.* We solve this problem using `cvx`.

```
cvx_begin
   variable p(m)
   minimize (norm(A*p-b, inf))
```

```
        subject to
            p >= 0
            p <= 1
    cvx_end
```

(e) *Exact solution.*

```
    cvx_begin
        variable p(m)
        minimize (max([A*p; inv_pos(A*p)]))
        subject to
            p >= 0
            p <= 1
    cvx_end
```

The results are summarized in the following table.

	Equal power	Sat. LS	Weighted LS	Cheb.	Exact
p_1	0.3448	1	0.5004	1	1
p_2	0.3448	0	0.4778	0.1165	0.2023
p_3	0.3448	1	0.0833	0	0
p_4	0.3448	0	0.0000	0	0
p_5	0.3448	0	0.4561	1	1
p_6	0.3448	1	0.4354	0	0
p_7	0.3448	0	0.4598	1	1
p_8	0.3448	1	0.4307	0.0249	0.1882
p_9	0.3448	0	0.4034	0	0
p_{10}	0.3448	1	0.4526	1	1
$f_0(p)$	0.4693	0.8628	0.4439	0.4198	0.3575