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ECE236B homework #5 solutions

1. *Exercise A3.11.*

   (a) We introduce a scalar variable t and write the problem as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & f(x) \leq t. \end{array}$$

We then use the Schur complement theorem to express this as an SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left[\begin{array}{cc} F(x) & c \\ c^T & t \end{array}\right] \succeq 0 \end{array}$$

with variables x, t. Note that there is a small difference between the two problems at the boundary of the domain, *i.e.*, for points x with $F(x)$ positive semidefinite but not positive definite. The linear matrix inequality in the SDP given above is equivalent to

$$F(x) \succeq 0, \qquad c \in \operatorname{range}(F(x)), \qquad c^T F(x)^\dagger c \leq t$$

where $F(x)^\dagger$ is the pseudo-inverse. The SDP is therefore equivalent to

$$\begin{array}{ll} \text{minimize} & c^T F(x)^\dagger c \\ \text{subject to} & F(x) \succeq 0 \\ & c \in \operatorname{range}(F(x)). \end{array}$$

If $F(x)$ is positive semidefinite but singular, and $c \in \operatorname{range}(F(x))$, the objective function $c^T F(x)^\dagger c$ is finite, whereas $f(x)$ is defined as $+\infty$ in the original problem. However this does not change the optimal value of the problem (unless the set $\{x \mid F(x) \succ 0\}$ is empty).

As an example, consider

$$c = \left[\begin{array}{c} 1 \\ 0 \end{array}\right], \qquad F(x) = \left[\begin{array}{cc} x & 0 \\ 0 & 1-x \end{array}\right].$$

Then the problem in the assignment is to minimize $1/x$, with domain $\{x \mid 0 < x < 1\}$. The optimal value is 1 and is not attained. The SDP reformulation is equivalent to minimizing $1/x$ subject to $0 \leq x \leq 1$. The optimal value is 1 and attained at $x = 1$.

(b) We first write the problem as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & f_i(x) \le t \quad i = 1, \ldots, K \end{array}$$

where $f_i(x) = c_i^T F(x)^{-1} c_i$ with domain $\{x \mid F(x) \succ 0\}$. We then use the Schur complement theorem to write this as an SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} F(x) & c_i \\ c_i^T & t \end{bmatrix} \succeq 0, \quad i = 1, \ldots, K. \end{array}$$

(c) The cost function can be expressed as

$$f(x) = \lambda_{\max}(F(x)^{-1}),$$

so $f(x) \le t$ if and only if $F(x)^{-1} \preceq tI$. Using a Schur complement we get

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} F(x) & I \\ I & tI \end{bmatrix} \succeq 0, \end{array}$$

with variables x and t. Alternatively, we can minimize the maximum eigenvalue of $F(x)^{-1}$ by maximizing the minimum eigenvalue of $F(x)$. This is an SDP

$$\begin{array}{ll} \text{minimize} & -t \\ \text{subject to} & F(x) \succeq tI \end{array}$$

with variables x and t.

(d) The cost function can be expressed as

$$f(x) = \bar{c}^T F(x)^{-1} \bar{c} + \mathbf{tr}(F(x)^{-1} S).$$

If we factor S as $S = \sum_{k=1}^m c_k c_k^T$ the problem is equivalent to

$$\text{minimize} \quad \bar{c}^T F(x)^{-1} \bar{c} + \sum_{k=1}^m c_k^T F(x)^{-1} c_k,$$

which we can write as an SDP

$$\begin{array}{ll} \text{minimize} & t_0 + \sum_k t_k \\ \text{subject to} & \begin{bmatrix} F(x) & \bar{c} \\ \bar{c}^T & t_0 \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} F(x) & c_k \\ c_k^T & t_k \end{bmatrix} \succeq 0, \quad k = 1, \ldots, m. \end{array}$$

The variables are t_0, t_1, $\ldots$, t_m, and x.

2. *Exercise A3.12.* The constraint is equivalent to $X \succeq B^T A^{-1} B$. Therefore $\mathbf{tr}\, X \geq \mathbf{tr}(B^T A^{-1} B)$ for all feasible X, with equality if $X = B^T A^{-1} B$. This shows that $X = B^T A^{-1} B$ is optimal.

   It follows that the optimal value of the SDP can be expressed as $\mathbf{tr}(B^T A^{-1} B) = \inf_X F(X, A, B)$ where the function $F : \mathbf{S}^n \times \mathbf{S}^n \times \mathbf{S}^n \to \mathbf{R}$ is defined as

   $$F(X, A, B) = \mathbf{tr}\, X,$$

   with domain

   $$\mathbf{dom}\, F = \left\{ (X, A, B) \in \mathbf{S}^m \times \mathbf{S}^m \times \mathbf{S}^m \;\middle|\; A \succ 0, \quad \left[\begin{array}{cc} A & B \\ B^T & X \end{array} \right] \succeq 0 \right\}.$$

   The function F is convex, jointly in A, B, X, because its domain is convex and on its domain it is linear. Therefore the infimum of $F(X, A, B)$ over X, which is $\mathbf{tr}(B^T A^{-1} B)$, is convex in A, B.

3. *Exercise A3.21 (a).* First consider the conditions

   $$y \leq \sqrt{z_1 z_2}, \qquad y, z_1, z_2 \geq 0.$$

   This is equivalent to

   $$y \geq 0, \qquad \left\| \left[\begin{array}{c} 2y \\ z_1 - z_2 \end{array} \right] \right\|_2 \leq z_1 + z_2.$$

   The equivalence can be seen by expanding the norm inequality, which gives

   $$4y^2 + (z_1 - z_2)^2 \leq (z_1 + z_2)^2, \qquad z_1 + z_2 \geq 0.$$

   This simplifies to $y^2 \leq z_1 z_2$ and $z_1, z_2 \geq 0$.

   Next consider the second constraint in the problem statement,

   $$y \leq (z_1 z_2 \cdots z_n)^{1/n}, \qquad y, z_1, \ldots, z_n \geq 0.$$

   First suppose $n = 2^l$. Introduce variables y_{ij} for $i = 1, \ldots, l-1$, and $j = 1, \ldots, 2^i$, and write the constraint as the following set of inequalities:

   $$\begin{array}{c} z \succeq 0, \\ y_{l-1,j} \leq (z_{2j-1} z_{2j})^{1/2}, \qquad y_{l-1,j} \geq 0, \qquad j = 1, \ldots, 2^{l-1} \\ y_{ij} \leq \left(y_{i+1,2j-1} y_{i+1,2j} \right)^{1/2}, \qquad y_{ij} \geq 0, \qquad i = 1, \ldots, l-2, \qquad j = 1, \ldots, 2^i \\ y \leq (y_{11} y_{12})^{1/2}, \qquad y \geq 0. \end{array}$$

   Then apply the second order cone formulation as in the case $n = 2$.

   For general n we write the constraint as

   $$y \leq (y^{m-n} z_1 \cdots z_n)^{1/m}, \qquad y \geq 0, \qquad z \succeq 0,$$

   where m is the smallest power of two that is greater than or equal to n, and then use the previous formulation.

4. *Exercise A3.26.* Make a change of variables $y_j = x_j^2$. Because $x_j \geq 0$, we can recover x_j as $x_j = y_j^{1/2}$. The quadratic functions f_i can be written in terms of y as

$$f_i(\sqrt{y_1}, \ldots, \sqrt{y_n}) = \frac{1}{2}\sum_{j=1}^{n}(P_i)_{jj}y_j + \sum_{j>k}(P_i)_{jk}(y_j y_k)^{1/2} + \sum_{j=1}^{n}(q_i)_j y_j^{1/2}.$$

The first terms $(P_i)_{jj}y_j$ are linear. In the other terms $y_j^{1/2}$ and $(y_j y_k)^{1/2}$ (the geometric mean of y_j and y_k) are concave. Since $(P_i)_{jk} \leq 0$ and $(q_i)_j \leq 0$, these terms are convex in y. Thus the QCQP becomes a convex problem in y:

$$\begin{array}{ll} \text{minimize} & (1/2)\sum_{j=1}^{n}(P_0)_{jj}y_j + \sum_{j>k}(P_0)_{jk}(y_j y_k)^{1/2} + \sum_{j=1}^{n}(q_0)_j y_j^{1/2} \\ \text{subject to} & \sum_{j=1}^{n}(P_i)_{jj}y_j + \sum_{j>k}(P_i)_{jk}(y_j y_k)^{1/2} + \sum_{j=1}^{n}(q_i)_j y_j^{1/2}, \quad i = 1, \ldots, m \\ & y \succeq 0. \end{array}$$

This problem can be expressed as an SOCP, using the technique of exercise 3.21.

5. *Exercise A14.8.*

   (a) To find the minimum fuel thrust profile for a given K, we solve

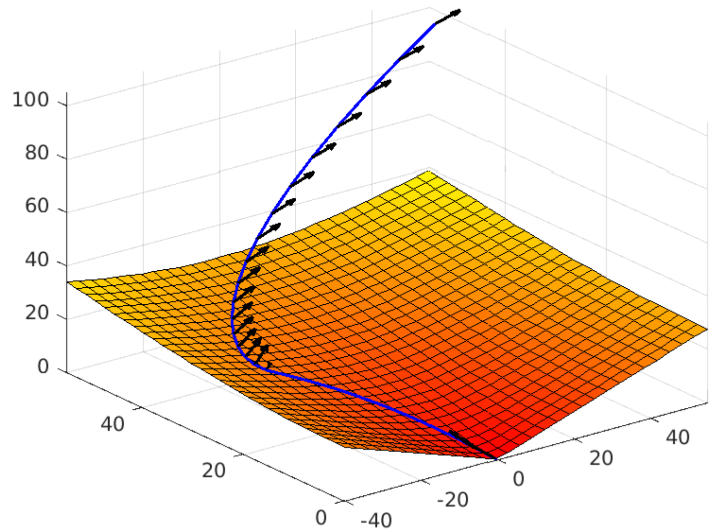
$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{K}\|f_k\|_2 \\ \text{subject to} & v_{k+1} = v_k + (h/m)f_k - hge_3, \quad k = 1, \ldots, K \\ & p_{k+1} = p_k + (h/2)(v_k + v_{k+1}), \quad k = 1, \ldots, K \\ & \|f_k\|_2 \leq F^{\max}, \quad k = 1, \ldots, K \\ & \alpha\sqrt{(p_k)_1^2 + (p_k)_2^2} \leq (p_k)_3, \quad k = 1, \ldots, K \\ & p_{K+1} = 0, \quad v_{K+1} = 0 \\ & p_1 = p(0), \quad v_1 = \dot{p}(0), \end{array}$$

   with variables $p_1, \ldots, p_{K+1}$, $v_1, \ldots, v_{K+1}$, and $f_1, \ldots, f_K$. This is a convex optimization problem.

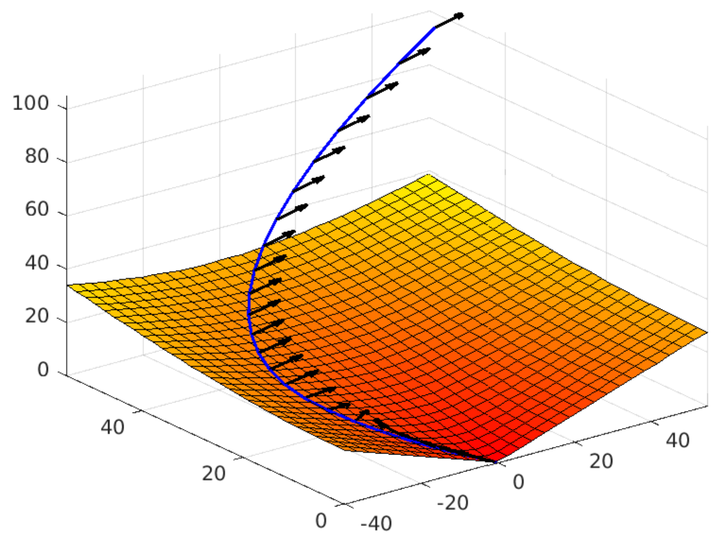
   (b) The solution is the smallest K for which the problem in part (a) is feasible. We can find the smallest K by solving the problem in part (a) for a sequence of decreasing values of K until the problem becomes infeasible. We can also use bisection on K.

   (c) For part (a) the optimal total fuel consumption is 193.0. For part (b) the minimum touchdown time is $K = 25$. The following plots show the optimal trajectories. The blue line shows the position of the spacecraft, the black arrows show the thrust profile, and the colored surface shows the glide slope constraint.

   The first plot shows the minimum fuel trajectory for part (a). Notice that for a portion of the trajectory the thrust is exactly equal to zero.



The second plot is a minimum time trajectory for part (b).



The following MATLAB code was used.

```
spacecraft_landing_data;

% Part (a)
cvx_begin
    variables p(3, K+1) v(3, K+1) f(3, K)
    minimize (sum(norms(f)))
```

```
    subject to
        v(:,2:K+1) == v(:,1:K) + (h/m)*f - h*g*[zeros(2,K); ones(1,K)];
        p(:,2:K+1) == p(:,1:K) + (h/2)*(v(:,1:K) + v(:,2:K+1));
        p(:,1) == p0;
        v(:,1) == v0;
        p(:,K+1) == 0;
        v(:,K+1) == 0;
        p(3, :) >= alpha*norms(p(1:2,:));
        norms(f) <= Fmax;
cvx_end
min_fuel = cvx_optval*gamma*h;
p_minf = p;  v_minf = v;  f_minf = f;

% Part (b); we use a linear search, but bisection is faster
Ki = K;
while(1)
    cvx_begin
        variables p(3,Ki+1) v(3,Ki+1) f(3,Ki)
        minimize(sum(norms(f)))
        subject to
            v(:,2:Ki+1) == v(:,1:Ki) +(h/m)*f - ...
                h*g*[zeros(2,Ki); ones(1,Ki)];
            p(:,2:Ki+1) == p(:,1:Ki) + (h/2)*(v(:,1:Ki)+v(:,2:Ki+1));
            p(:,1) == p0;
            v(:,1) == v0;
            p(:,Ki+1) == 0;
            v(:,Ki+1) == 0;
            p(3,:) >= alpha*norms(p(1:2,:));
            norms(f) <= Fmax;
    cvx_end
    if(strcmp(cvx_status,'Infeasible') == 1)
        Kmin = Ki+1;
        break;
    end
    Ki = Ki-1;
    p_mink = p;  v_mink = v;  f_mink = f;
end

% plot the glide cone
x = linspace(-40,55,30); y = linspace(0,55,30);
[X,Y] = meshgrid(x,y);
Z = alpha*sqrt(X.^2+Y.^2);
```

```
figure; colormap autumn; surf(X,Y,Z);
axis([-40,55,0,55,0,105]);
grid on; hold on;

% plot minimum fuel trajectory for part (a)
plot3(p_minf(1,:), p_minf(2,:), p_minf(3,:), 'b', 'linewidth', 1.5);
quiver3(p_minf(1,1:K), p_minf(2,1:K), p_minf(3,1:K), ...
        f_minf(1,:), f_minf(2,:), f_minf(3,:), 0.3, 'k', 'linewidth',1.5);

% plot minimum time trajectory for part (b)
figure; colormap autumn; surf(X,Y,Z);
axis([-40,55,0,55,0,105]); grid on; hold on;
plot3(p_mink(1,:), p_mink(2,:), p_mink(3,:), 'b', 'linewidth', 1.5);
quiver3(p_mink(1,1:Kmin), p_mink(2,1:Kmin), p_mink(3,1:Kmin),...
        f_mink(1,:), f_mink(2,:), f_mink(3,:), 0.3, 'k', 'linewidth', 1.5);
```

6. *Exercise A3.13.* We first show that the matrix is nonsingular. Assume

$$\begin{bmatrix} A^{-1} & I \\ B^{-1} & -I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From the first equation, $y = -A^{-1}x$. Substituting this in the second equation gives $B^{-1}x + A^{-1}x = 0$, and therefore $x^T(B^{-1} + A^{-1})x = 0$. Since A and B are positive definite this implies $x = 0$. If $x = 0$, then also $y = -A^{-1}x = 0$. This shows that the columns are linearly independent.

Following the hint, we write the constraint as

$$\begin{bmatrix} A^{-1} & B^{-1} \\ I & -I \end{bmatrix} \begin{bmatrix} X & X \\ X & X \end{bmatrix} \begin{bmatrix} A^{-1} & I \\ B^{-1} & -I \end{bmatrix} \preceq \begin{bmatrix} A^{-1} & B^{-1} \\ I & -I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} A^{-1} & I \\ B^{-1} & -I \end{bmatrix}.$$

After working out the matrix products we get

$$\begin{bmatrix} (A^{-1}+B^{-1})X(A^{-1}+B^{-1}) & 0 \\ 0 & 0 \end{bmatrix} \preceq \begin{bmatrix} A^{-1}+B^{-1} & 0 \\ 0 & A+B \end{bmatrix}.$$

This shows that the SDP is equivalent to

$$\begin{array}{ll} \text{maximize} & \mathbf{tr}\, X \\ \text{subject to} & X \preceq (A^{-1}+B^{-1})^{-1}. \end{array}$$

We have $\mathbf{tr}\, X \leq \mathbf{tr}((A^{-1}+B^{-1})^{-1})$ for all feasible X, with equality if $X = (A^{-1} + B^{-1})^{-1}$. This proves that the optimal value is equal

$$f(A,B) = \mathbf{tr}((A^{-1}+B^{-1})^{-1}).$$

From this we conclude that $f(A,B) = \mathbf{tr}((A^{-1}+B^{-1}))$ is concave, for the following reason. Define a function $F : \mathbf{S}^n \times \mathbf{S}^n \times \mathbf{S}^n \to \mathbf{R}$ with

$$\mathbf{dom}\, F = \left\{ (X,A,B) \,\middle|\, \begin{bmatrix} X & X \\ X & X \end{bmatrix} \preceq \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\}.$$

and $F(X,A,B) = -\mathbf{tr}(X)$ on its domain. This function is convex jointly in (X,A,B), because its domain is a convex set and on its domain it is linear. Therefore the function $\inf_X F(X,A,B) = -f(A,B)$ is convex.

7. *Exercise A3.14.* Using Schur complements, we can write the constraint as

$$XA^{-1}X \preceq B,$$

and

$$(A^{-1/2}XA^{-1/2})^2 = A^{-1/2}XA^{-1}XA^{-1/2} \preceq A^{-1/2}BA^{-1/2}.$$

Using the hint, this implies that

$$\begin{array}{rcl} A^{-1/2}XA^{-1/2} & \preceq & (A^{-1/2}BA^{-1/2})^{1/2} \\ X & \preceq & A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}. \end{array}$$

We conclude that every feasible X satisfies $X \preceq G(A,B)$, and hence $\mathbf{tr}\, X \leq \mathbf{tr}\, G(A,B)$. Moreover, $X = G(A,B)$ is feasible because

$$\begin{array}{rcl} XA^{-1}X & = & A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}A^{-1}A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \\ & = & A^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2} \\ & = & B. \end{array}$$

From this we conclude that $f(A,B) = \mathbf{tr}(G(A,B))$ is concave, for the following reason. Define a function $F : \mathbf{S}^n \times \mathbf{S}^n \times \mathbf{S}^n \to \mathbf{R}$ with

$$\mathbf{dom}\, F = \left\{ (X,A,B) \,\middle|\, \begin{bmatrix} A & X \\ X & B \end{bmatrix} \succeq 0 \right\}.$$

and $F(X,A,B) = -\mathbf{tr}(X)$ on its domain. This function is convex jointly in (X,A,B), because its domain is a convex set and on its domain it is linear. Therefore the function $\inf_X F(X,A,B) = -f(A,B)$ is convex.

8. *Exercise A3.21 (b,c).*

(b) For the first function, write the constraint as

$$y \leq (t^s)^{1/r}, \qquad y \geq x, \qquad y \geq 0, \qquad t \geq 0$$

where $\alpha = r/s$ and r and s are integers, and then apply part (a) with $z_k = t$ for $k = 1,\ldots,s$ and $z_k = 1$ for $k = s+1,\ldots,r$.

For the second function, we express the constraint as

$$1 \leq (x^r t^s)^{1/(r+s)}, \qquad x \geq 0, \qquad t \geq 0,$$

where $\alpha = -r/s$ and r and s are integers, and then apply the formulation of (a).

(c) First express the problem as

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{m} t_k \\ \text{subject to} & -y \preceq Ax - b \preceq y \\ & y_k^p \leq t_k, \quad k = 1, \ldots, m \end{array}$$

and apply part (b).

9. *Exercise A14.4.* We note two difficulties with the problem as stated. First the expression $R^2 - r^2$ in the cost function and the first two constraint functions make these functions non-posynomial. Second, an expression of the form $\sqrt{w^2 + h^2}$ is not allowed in a posynomial function.

To address the first issue, we substitute a new variable u for $R^2 - r^2$ in the objective and the first two constraints, and eliminate the variable R from the constraints $1.1r \leq R \leq R_{\mathrm{max}}$. The inequality $R \geq 1.1r$ can be written in terms of r and u as

$$R^2 - 1.1^2 r^2 = u - 0.21r^2 \geq 0,$$

or $0.21r^2/u \leq 1$ in the standard posynomial form. The upper bound $R \leq R_{\mathrm{max}}$ can be written as $R^2 = u + r^2 \leq R_{\mathrm{max}}^2$, or $u/R_{\mathrm{max}}^2 + r^2/R_{\mathrm{max}}^2 \leq 1$. This gives

$$\begin{array}{ll} \text{minimize} & 2\pi u \sqrt{w^2 + h^2} \\ \text{subject to} & F_1 (2\sigma\pi)^{-1} u^{-1} h^{-1} \sqrt{w^2 + h^2} \leq 1 \\ & F_2 (2\sigma\pi)^{-1} u^{-1} w^{-1} \sqrt{w^2 + h^2} \leq 1 \\ & w_{\mathrm{max}}^{-1} w \leq 1, \quad w_{\mathrm{min}} w^{-1} \leq 1 \\ & h_{\mathrm{max}}^{-1} h \leq 1, \quad h_{\mathrm{min}} h^{-1} \leq 1 \\ & 0.21 r^2 u^{-1} \leq 1 \\ & R_{\mathrm{max}}^2 u^{-1} + R_{\mathrm{max}}^2 r^2 \leq 1 \end{array}$$

(with domain $\{(u, r, w, h) \mid (u, r, w, h) \succ 0\}$). We can now square the cost function and the two sides of the first two inequalities to obtain a geometric program with variables

u, r, w, h:

$$\begin{array}{ll}
\text{minimize} & 4\pi^2(u^2w^2 + u^2h^2) \\
\text{subject to} & F_1^2(2\sigma\pi)^{-2}(u^{-2}h^{-2}w^2 + u^{-2}) \le 1 \\
& F_2^2(2\sigma\pi)^{-2}(u^{-2} + u^{-2}w^{-2}h^2) \le 1 \\
& w_{\max}^{-1}w \le 1, \quad w_{\min}w^{-1} \le 1 \\
& h_{\max}^{-1}h \le 1, \quad h_{\min}h^{-1} \le 1 \\
& 0.21r^2u^{-1} \le 1 \\
& R_{\max}^2u^{-1} + R_{\max}^2r^2 \le 1,
\end{array}$$

with domain $\{(u, r, w, h) \mid (u, r, w, h) \succ 0\}$. The optimal value of R can be calculated as $R = \sqrt{u + r^2}$.

Another solution is to replace $\sqrt{w^2 + h^2}$ by a new variable L. To enforce the equality $L = \sqrt{w^2 + h^2}$, we cannot add an equality constraint $L = \sqrt{w^2 + h^2}$, because it is not monomial. However we can add an inequality constraint $L \ge \sqrt{w^2 + h^2}$ (written as $w^2/L^2 + h^2/L^2 \le 1$) and note that at the optimum it will be satisfied with equality. (If $L > \sqrt{w^2 + h^2}$ at the optimum, we would able to decrease L while keeping the other variables fixed. This would give a feasible point with a lower value of the cost function.) This second solution results in the following geometric program with variables r, w, h, u, L:

$$\begin{array}{ll}
\text{minimize} & 2\pi uL \\
\text{subject to} & F_1(2\sigma\pi)^{-1}h^{-1}u^{-1}L \le 1 \\
& F_2(2\sigma\pi)^{-1}w^{-1}u^{-1}L \le 1 \\
& w_{\max}^{-1}w \le 1, \quad w_{\min}w^{-1} \le 1 \\
& h_{\max}^{-1}h \le 1, \quad h_{\min}h^{-1} \le 1 \\
& 0.21r^2u^{-1} \le 1 \\
& R_{\max}^2u^{-1} + R_{\max}^2r^2 \le 1 \\
& w^2L^{-2} + h^2L^{-2} \le 1.
\end{array}$$