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EE236B homework #1 solutions

1. (a) Expanding the squares in the ith term of the cost function gives

$$\begin{array}{rcl}(u_i - u_{\rm c})^2 + (v_i - v_{\rm c})^2 - R^2 & = & -2u_iu_{\rm c} - 2v_iv_{\rm c} + u_{\rm c}^2 + v_{\rm c}^2 - R^2 + u_i^2 + v_i^2 \\ & = & -2u_iu_{\rm c} - 2v_iv_{\rm c} + w + u_i^2 + v_i^2.\end{array}$$

This is linear in $u_{\rm c}$, $v_{\rm c}$, w, so we obtain a linear least-squares problem with variables $x = (u_{\rm c}, v_{\rm c}, w)$ and

$$A = \left[\begin{array}{ccc} -2u_1 & -2v_1 & 1 \\ -2u_2 & -2v_2 & 1 \\ \vdots & \vdots & \vdots \\ -2u_m & -2v_m & 1 \end{array}\right], \qquad b = \left[\begin{array}{c} -u_1^2 - v_1^2 \\ -u_2^2 - v_2^2 \\ \vdots \\ -u_m^2 - v_m^2 \end{array}\right].$$

(b) The property follows from the normal equations $A^T(Ax - b) = 0$. $Ax - b$ is an m-vector with components $-2u_iu_{\rm c} - 2v_iv_{\rm c} + w + u_i^2 + v_i^2$. Since the last column of A is all ones, the last equation of $A^T(Ax - b) = 0$ gives

$$\begin{array}{rcl} 0 & = & \sum_{i=1}^m \left(-2u_iu_{\rm c} - 2v_iv_{\rm c} + w + u_i^2 + v_i^2\right) \\ & = & \sum_{i=1}^m \left((u_i - u_{\rm c})^2 + (v_i - v_{\rm c})^2 + w - u_{\rm c}^2 - v_{\rm c}^2\right) \\ & = & \sum_{i=1}^m \left((u_i - u_{\rm c})^2 + (v_i - v_{\rm c})^2\right) + m(w - u_{\rm c}^2 - v_{\rm c}^2).\end{array}$$

Therefore

$$u_{\rm c}^2 + v_{\rm c}^2 - w = \frac{1}{m}\sum_{i=1}^m \left((u_i - u_{\rm c})^2 + (v_i - v_{\rm c})^2\right) \geq 0.$$

The solution for the test problem is

$$R = 1.3214, \qquad u_{\rm c} = -2.5671, \qquad v_{\rm c} = 6.468.$$

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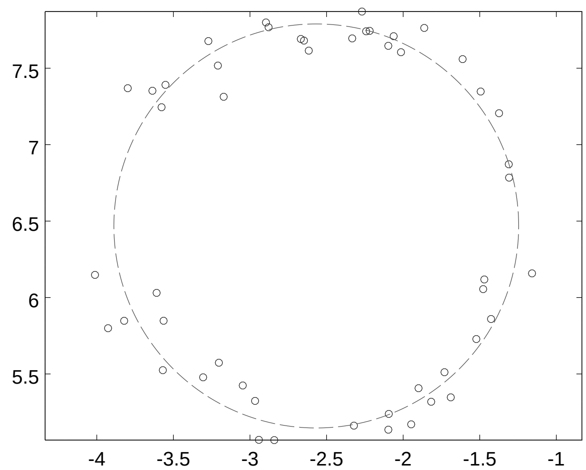
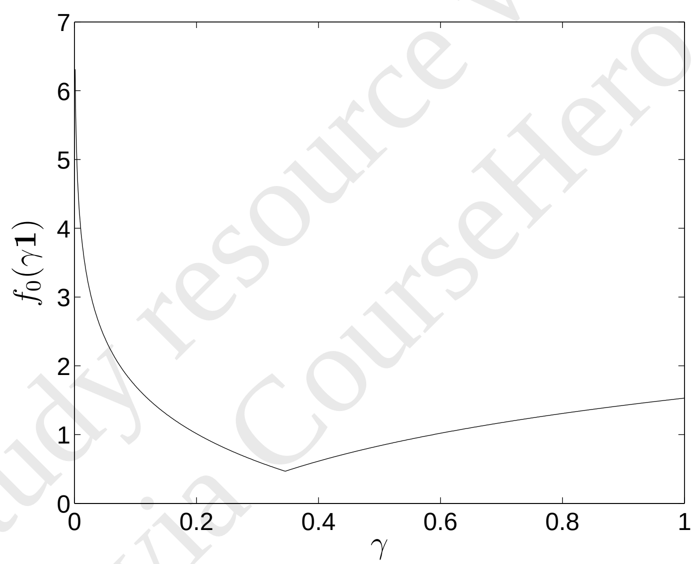


2. (a) *Equal lamp powers.* The figure shows $f_0(\gamma \mathbf{1}) = \max_k |\log(\gamma a_k^T \mathbf{1})|$ versus γ.



The minimum is reached at $\gamma = 0.3453$.

(b) *Least-squares with saturation.* We compute p as

```
p = A \ ones(n,1).
```

All coefficients of p are outside the feasible interval $[0, 1]$ and need to be rounded.

(c) *Regularized least-squares.* We compute p by solving a least-squares problem

```
p = [A; sqrt(rho)*eye(m)] \ [ones(n,1); sqrt(rho)*.5*ones(m,1)].
```

The smallest ρ that gives a feasible p is $\rho = 0.2190$.

(d) *Chebyshev approximation.* We solve this problem using `cvx`.

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```
cvx_begin
   variable p(m)
   minimize (norm(A*p-b, inf))
   subject to
       p >= 0
       p <= 1
cvx_end
```

(e) *Piecewise-linear approximation.* We solve this problem using `cvx`.

```
cvx_begin
   variable p(m)
   minimize (max( [A*x;  2/0.5 - 1/0.5^2 * (A*x); ...
       2/0.8 - 1/0.8^2 * (A*x); 2 - A*x ]));
   subject to
       p >= 0
       p <= 1
cvx_end
```

(f) *Exact solution.*

```
cvx_begin
   variable p(m)
   minimize (max([A*p; inv_pos(A*p)]))
   subject to
       p >= 0
       p <= 1
cvx_end
```

The results are summarized in the following table.

	Equal power	Sat. LS	Weighted LS	Cheb.	PWL	Exact
p_1	0.3448	1	0.5004	1	1	1
p_2	0.3448	0	0.4778	0.1165	0.1896	0.2023
p_3	0.3448	1	0.0833	0	0	0
p_4	0.3448	0	0.0000	0	0	0
p_5	0.3448	0	0.4561	1	1	1
p_6	0.3448	1	0.4354	0	0	0
p_7	0.3448	0	0.4598	1	1	1
p_8	0.3448	1	0.4307	0.0249	0.1640	0.1882
p_9	0.3448	0	0.4034	0	0	0
p_{10}	0.3448	1	0.4526	1	1	1
$f_0(p)$	0.4693	0.8628	0.4439	0.4198	0.3664	0.3575

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3. (a) If $A = 0$, the matrix X is positive semidefinite if and only if for all u, v,

$$2u^TBv + v^TCv = \begin{bmatrix} u \\ v \end{bmatrix}^T \begin{bmatrix} 0 & B \\ B^T & C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \geq 0.$$

Clearly, a sufficient condition is that $B = 0$ and C is positive semidefinite. Taking $u = 0$ shows that positive semidefiniteness of C is also necessary. To see that $B = 0$ is necessary, take any v with $Bv \neq 0$ and choose $u = -tBv$. The quadratic form then reduces to

$$-2t\|Bv\|_2^2 + v^TCv$$

which is negative for sufficiently large t.

(b) We have

$$\begin{aligned} AA^\dagger &= Q_1\Lambda_1Q_1^TQ_1\Lambda_1^{-1}Q_1^T \\ &= Q_1Q_1^T \\ I - AA^\dagger &= Q_1Q_1^T + Q_2Q_2^T - AA^\dagger \\ &= Q_2Q_2^T. \end{aligned}$$

The proofs of the identities $A^\dagger A = Q_1Q_1^T$ and $I - A^\dagger A = Q_2Q_2^T$ are similar.

(c) Suppose we sort the eigenvalues of A so that its eigenvalue decomposition can be written as

$$A = Q\Lambda Q^T = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}^T$$

with Λ_1 positive diagonal. Following the hint, the question can be reduced to showing that the block matrix

$$\begin{bmatrix} \Lambda_1 & 0 & Q_1^TB \\ 0 & 0 & Q_2^TB \\ B^TQ_1 & B^TQ_2 & C \end{bmatrix}$$

is positive semidefinite. By the result in part (a), the matrix is positive semidefinite if and only if

$$Q_2^TB = 0, \qquad \begin{bmatrix} \Lambda_1 & Q_1^TB \\ B^TQ_1 & C \end{bmatrix} \succeq 0.$$

The first condition is equivalent to $(I - AA^\dagger)B = 0$. Since Λ_1 is nonsingular, we can apply the Schur complement result for nonsingular A to the 2×2 block matrix. This gives the condition

$$C - B^TQ_1\Lambda^{-1}Q_1^TB = C - B^TA^\dagger B \succeq 0.$$

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