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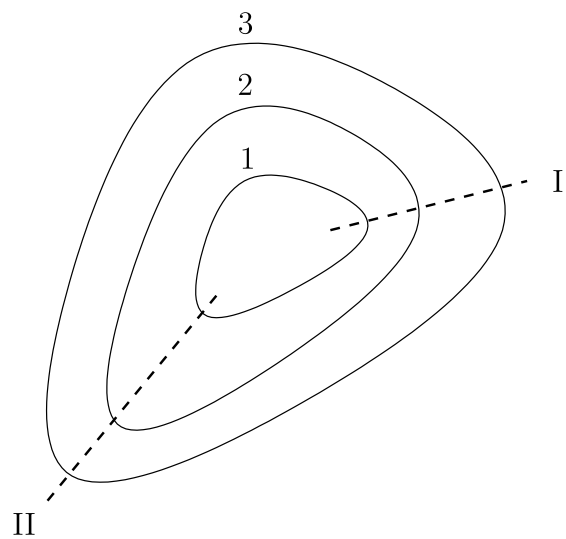
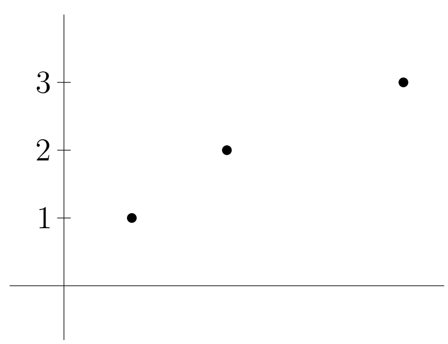
ECE236B homework #3 solutions

1. *Exercise T3.2.* The first function could be quasiconvex because the sublevel sets that are shown are convex. It is definitely not concave or quasiconcave because the superlevel sets are not convex. It is also not convex, for the following reason. We plot the function values along the dashed line labeled I.



Along this line the function passes through the points marked as black dots in the figure below. Clearly along this line segment, the function is not convex.



If we repeat the same analysis for the second function, we see that it could be concave (and therefore it could be quasiconcave). It cannot be convex or quasiconvex, because the sublevel sets are not convex.

2. *Exercise T3.19 (a).* We can express f as

$$\begin{array}{rcl} f(x) & = & \alpha_r(x_{[1]} + x_{[2]} + \cdots + x_{[r]}) + (\alpha_{r-1} - \alpha_r)(x_{[1]} + x_{[2]} + \cdots + x_{[r-1]}) \\ & & +(\alpha_{r-2} - \alpha_{r-1})(x_{[1]} + x_{[2]} + \cdots + x_{[r-2]}) + \cdots + (\alpha_1 - \alpha_2)x_{[1]}. \end{array}$$

This is a nonnegative sum of the convex functions

$$x_{[1]}, \qquad x_{[1]} + x_{[2]}, \qquad x_{[1]} + x_{[2]} + x_{[3]}, \qquad \ldots, \qquad x_{[1]} + x_{[2]} + \cdots + x_{[r]}.$$

3. *Exercise A2.20 (a).* f is the difference of a convex and a concave function. The first term is convex because it is the supremum of a family of linear functions of x. The second term is concave because it is the infimum of a family of linear functions of x.

4. *Exercise T3.22 (b).* We can express f as $f(x, u, v) = -\sqrt{u(v - x^T x/u)}$. The function $h(x_1, x_2) = -\sqrt{x_1 x_2}$ is convex on $\mathbf{R}^2_{++}$, and decreasing in each argument. The functions $g_1(u, v, x) = u$ and $g_2(u, v, x) = v - x^T x/u$ are concave. Therefore $f(u, v, x) = h(g(u, v, x))$ is convex.

5. *Exercise A2.32.*

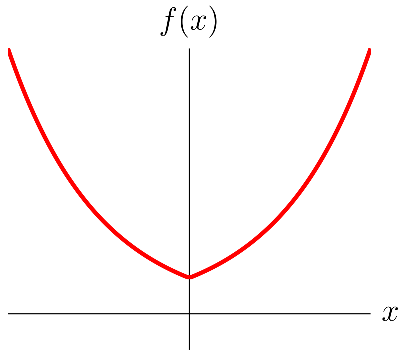
   (a) This follows from the composition rules: $f(x) = h(g_1(x), \ldots, g_n(x))$, with h nondecreasing in each argument and $g_k(x) = |x_k|$, a convex function.

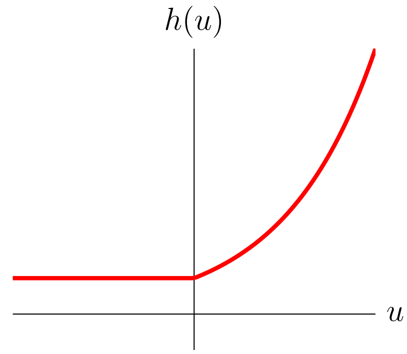
   (b) The conjugate is

$$\begin{array}{rcl} f^*(y) & = & \sup_x (y^T x - h(|x_1|, \ldots, |x_n|)) \\ & = & \sup_{u \succeq 0} \max_{s \in \{-1,1\}^n} (\sum_{k=1}^n y_k s_k u_k - h(u)) \\ & = & \sup_{u \succeq 0} (\sum_{k=1}^n |y_k| u_k - h(u)) \\ & = & \sup_u (\sum_k |y_k| u_k - h(u)) \\ & = & f^*(|y_1|, \ldots, |y_n|). \end{array}$$

   On line 2 we write x_k as $x_k = s_k u_k$ with u_k nonnegative and $s_k \in \{-1, 1\}$. Line 3 follows because the optimal choice of s_k is the one that maximizes $y_k s_k u_k$, *i.e.*, $s_k = \mathrm{sign}(y_k)$. On line 4 we use the property $h(u) = h(u_1^+, \ldots, u_n^+)$. Because of this property, allowing negative components in u does not increase the maximum value of $\sum_k |y_k| u_k - h(u)$.

   (c) The figure shows the functions $f(x) = \exp(|x|)$ and $h(u) = \exp(u^+)$.





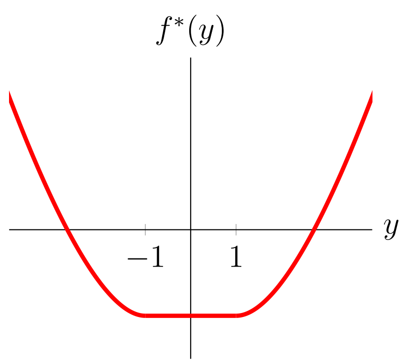
The conjugate of $f(x) = \exp(|x|)$ is

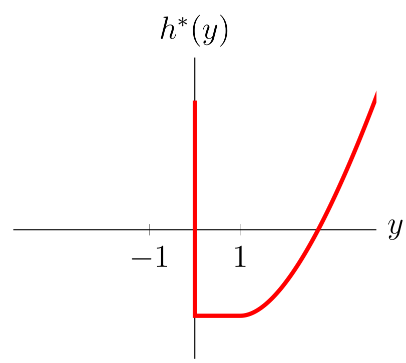
$$f^*(y) = \sup_x \left(yx - \exp(|x|)\right) = \begin{cases} |y| \log |y| - |y| & |y| \geq 1 \\ -1 & |y| \leq 1. \end{cases}$$

If $|y| \leq 1$, the maximum over x is at $x = 0$. If $y > 1$ we differentiate $yx - e^x$ to find that $x = \log y$ is optimal and the optimal value is $y \log y - y$. If $y < -1$ we differentiate $yx - e^{-x}$ to find that $x = -\log(-y)$ is optimal and the optimal value is $-y \log(-y) + y$.
The conjugate of $h(x) = \exp(x^+)$ is

$$h^*(y) = \sup_x \left(yx - \exp(x^+)\right) = \begin{cases} y \log y - y & y \geq 1 \\ -1 & 0 \leq y \leq 1 \\ +\infty & y < 0. \end{cases}$$

If $y > 1$, we differentiate $xy - e^x$ to find that $x = \log y$ is optimal. If $0 \leq y \leq 1$, the maximum over x is at $x = 0$. For $y < 0$, the function $xy - \exp(x^+)$ goes to $+\infty$ as $x \to -\infty$.





6. *Exercise A3.17.* We solve the problem in CVX, using the formulation

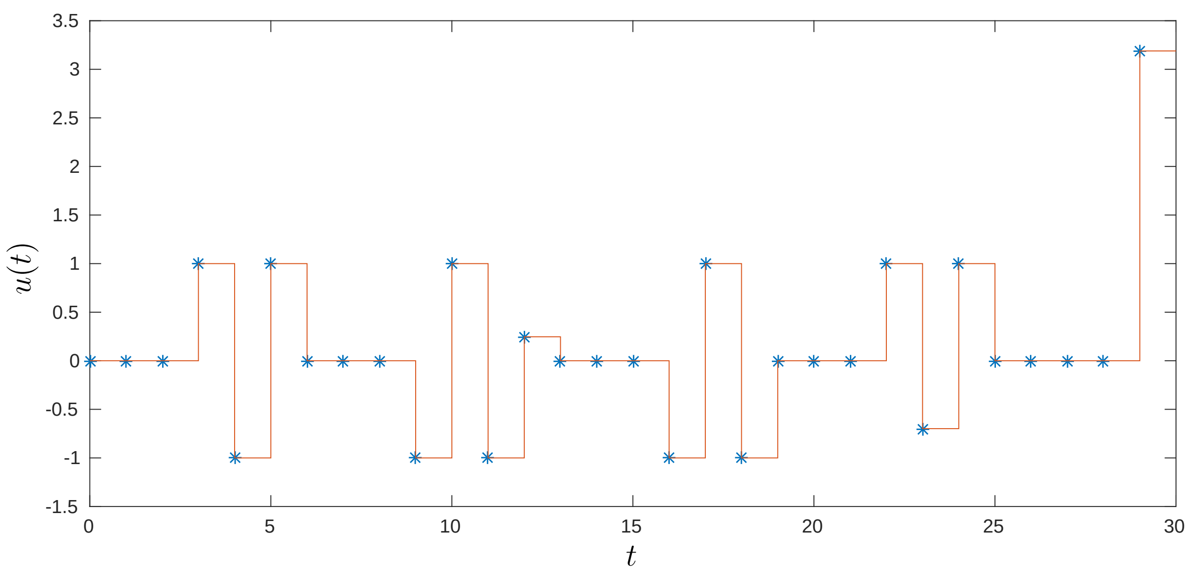
$$\begin{array}{ll} \text{minimize} & \sum_{t=0}^{N-1} \max\{|u(t)|, 2|u(t)| - 1\} \\ \text{subject to} & x(t+1) = Ax(t) + bu(t),\ t = 0, \ldots, n-1 \\ & x(0) = 0 \\ & x(N) = x_{\mathrm{des}}, \end{array}$$

with variables $u(0), \ldots, u(N-1)$, and $x(0), \ldots, x(N)$. The CVX code defines two matrix variables

$$X = \left[\begin{array}{cccc} x(0) & x(1) & \cdots & x(N) \end{array}\right], \qquad u = \left[\begin{array}{cccc} u(0) & u(1) & \cdots & u(N-1) \end{array}\right].$$

```
n = 3;
N = 30;
A = [ -1, 0.4, 0.8; 1, 0, 0; 0, 1, 0];
b = [ 1; 0; 0.3];
x0 = zeros(n,1);
xdes = [ 7; 2; -6];
cvx_begin
    variable X(n,N+1);
    variable u(1,N);
    minimize (sum(max(abs(u), 2*abs(u)-1)))
    subject to
        X(:,2:N+1) == A*X(:,1:N) + b*u;
        X(:,1) == x0;
        X(:,N+1) == xdes;
cvx_end
plot(0:N-1, u, '*');
hold on
stairs(0:N, [u, u(end)]);
axis([0, 30, -1.5, 3.5])
xlabel('t')
ylabel('u(t)')
hold off
```

The optimal actuator signal is shown in the figure.



7. *Exercise A2.17.*

   (a) We can express g as

   $$g(x) = \inf_{x_1,\ldots,x_m} \left(f_1(x_1) + \cdots + f_m(x_m) + \phi(x_1,\ldots,x_m,x)\right),$$

   where $\phi(x_1,\ldots,x_m,x)$ is 0 when $x_1+\cdots+x_m = x$, and ∞ otherwise. The function on the right-hand side is convex in $x_1, \ldots, x_m, x$, so by the partial minimization rule, so is g.

   (b) We have

   $$\begin{aligned} g^*(y) &= \sup_x \,(y^T x - f(x)) \\ &= \sup_x \,(y^T x - \inf_{x_1+\cdots+x_m=x} \sum_{k=1}^m f_k(x_k)) \\ &= \sup_x \sup_{x=x_1+\cdots+x_m} (y^T x - \sum_{k=1}^m f_k(x_k)) \\ &= \sup_{x_1,\ldots,x_m} \sum_{k=1}^m (y^T x_k - f_k(x_k)) \\ &= \sum_{k=1}^m \sup_{x_k} (y^T x_k - f_k(x_k)) \\ &= f_1^*(y) + \cdots + f_m^*(y). \end{aligned}$$

8. *Exercise A2.20 (b).* $h(u) = \exp(1/u)$ is convex and decreasing on $\mathbf{R}_{++}$:

   $$h'(u) = -\frac{1}{u^2}e^{1/u}, \qquad h''(u) = \frac{2}{u^3}e^{1/u} + \frac{1}{u^4}e^{1/u} > 0.$$

   Therefore the composition $h(-f_i(x)) = \exp(-1/f_i(x))$ is convex if f_i is convex.

9. *Exercise A2.21.*

   (a) Suppose S is expressed as a convex combination of permutation matrices:

   $$S = \sum_k \theta_k P_k$$

   with $\theta_k \geq 0$, $\sum_k \theta_k = 1$, and permutation matrices P_k. From convexity and symmetry of f,

   $$f(Sx) = f(\sum_k \theta_k P_k x) \leq \sum_k \theta_k f(P_k x) = \sum_k \theta_k f(x) = f(x).$$

   The inequality is Jensen's inequality. The second equality follows from symmetry of f.

(b) The diagonal elements of $Y = Q\,\mathbf{diag}(\lambda)Q^T$ are given by

$$Y_{ii} = \sum_{j=1}^{n} Q_{ij}^2 \lambda_j.$$

From $QQ^T = I$, we have $\sum_j Q_{ij}^2 = 1$. From $Q^TQ = I$, we have $\sum_i Q_{ij}^2 = 1$.

(c) Combining the results in parts (a) and (b), we conclude that for any symmetric X, the inequality

$$f(\mathbf{diag}(X)) \leq f(\lambda(X))$$

holds. If V is orthogonal, then $\lambda(X) = \lambda(V^TXV)$. Therefore also

$$f(\mathbf{diag}(V^TXV)) \leq f(\lambda(X))$$

for all orthogonal V. Moreover, this inequality holds with equality if $V = Q$, where Q is the matrix of eigenvectors of X. Hence

$$f(\lambda(X)) = \sup_{V \in \mathcal{V}} f(\mathbf{diag}(V^TXV)).$$

This shows that $f(\lambda(X))$ is convex because it is the supremum of a family of convex functions of X.

10. *Exercise T3.23 (a).*

(a) The first and second derivatives of f are

$$f'(x) = e^{-h(x)}, \qquad f''(x) = -h'(x)e^{-h(x)}.$$

Log-concavity requires $f''(x)f(x) \leq (f'(x))^2$, *i.e.*,

$$\begin{aligned} -h'(x)e^{-h(x)} \int_{-\infty}^{x} e^{-h(t)}\,dt &\leq e^{-2h(x)} \\ -h'(x) \int_{-\infty}^{x} e^{-h(t)}\,dt &\leq e^{-h(x)}. \end{aligned}$$

This is obviously true if $h'(x) \geq 0$.

(b) Taking exponentials and integrating both sides of $-h(t) \leq -h(x) - h'(x)(t-x)$ gives

$$\begin{aligned} \int_{-\infty}^{x} e^{-h(t)}\,dt &\leq e^{xh'(x)-h(x)} \int_{-\infty}^{x} e^{-th'(x)}\,dt \\ &= e^{xh'(x)-h(x)}e^{-xh'(x)}/(-h'(x)) \\ &= \frac{e^{-h(x)}}{-h'(x)} \end{aligned}$$

and therefore

$$-h'(x) \int_{-\infty}^{x} e^{-h(t)}\,dt \leq e^{-h(x)}.$$

11. *Exercise A2.30.* We solve the optimization problem

$$\text{minimize} \quad \sum_{i=1}^{n} \left(|y_i| + \frac{1}{2}(y_i - x_i)^2 \right)$$

by optimizing over each y_i separately. The function

$$\tilde{f}(y_i) = |y_i| + \frac{1}{2}(y_i - x_i)^2$$

is convex, piecewise-quadratic, and nondifferentiable at $y_i = 0$. The left and right derivatives at the origin are $-1 - x_i$ and $1 - x_i$, respectively. There are three cases to consider.

- $x_i > 1$. The left and right derivative of $\tilde{f}$ at $y_i = 0$ are negative, and therefore the function reaches its minimum at some positive y_i. We can determine the optimum by setting the derivative to zero:

  $$\tilde{f}'(y_i) = 1 + y_i - x_i = 0$$

  so $y_i = x_i - 1$ and

  $$\inf_{y_i} \left(|y_i| + \frac{1}{2}(y_i - x_i)^2 \right) = x_i - \frac{1}{2}.$$

- $x_i < -1$. The left and right derivative of $\tilde{f}$ at $y_i = 0$ are positive, and the function reaches its minimum at some negative y_i. We determine the optimum by setting the derivative to zero:

  $$\tilde{f}'(y_i) = -1 + y_i - x_i = 0$$

  so $y_i = 1 + x_i$ and

  $$\inf_{y_i} \left(|y_i| + \frac{1}{2}(y_i - x_i)^2 \right) = -x_i - \frac{1}{2}.$$

- $-1 \leq x_i \leq 1$. The left derivative at zero is non-positive and the right derivative is non-negative. Therefore $y_i = 0$ is optimal and

  $$\inf_{y_i} \left(|y_i| + \frac{1}{2}(y_i - x_i)^2 \right) = \frac{1}{2}x_i^2.$$

To summarize,

$$\inf_{y_i} \left(|y_i| + \frac{1}{2}(y_i - x_i)^2 \right) = \begin{cases} x_i^2 & |x_i| \leq 1 \\ x_i - 1/2 & x_i > 1 \\ -x_i - 1/2 & x_i < -1. \end{cases}$$

This is the function $\phi(x_i)$ in the assignment.

12. *Exercise T3.55.*

(a) The first and second derivatives of f are

$$f'(x) = e^{-h(x)}, \qquad f''(x) = -h'(x)e^{-h(x)}.$$

Log-concavity requires $f''(x)f(x) \leq (f'(x))^2$, *i.e.*,

$$\begin{array}{rcl} -h'(x)e^{-h(x)} \displaystyle\int_{-\infty}^{x} e^{-h(t)}\, dt & \leq & e^{-2h(x)} \\ -h'(x) \displaystyle\int_{-\infty}^{x} e^{-h(t)}\, dt & \leq & e^{-h(x)}. \end{array}$$

This is obviously true if $h'(x) \geq 0$.

(b) Taking exponentials and integrating both sides of $-h(t) \leq -h(x) - h'(x)(t-x)$ gives

$$\begin{array}{rcl} \displaystyle\int_{-\infty}^{x} e^{-h(t)}\, dt & \leq & e^{xh'(x)-h(x)} \displaystyle\int_{-\infty}^{x} e^{-th'(x)}\, dt \\ & = & e^{xh'(x)-h(x)} e^{-xh'(x)}/(-h'(x)) \\ & = & \dfrac{e^{-h(x)}}{-h'(x)}. \end{array}$$

Therefore

$$-h'(x) \int_{-\infty}^{x} e^{-h(t)}\, dt \leq e^{-h(x)}.$$