1.1 Difference between deterministic and stochastic world

	deterministic world	stochastic world
Single variable: Temp of a sick man	$\mathbb{R}$ $T = 39°C$	random variable $\mathbb{E}$, Var, ...
Variables changing over time: T in first 3 days	$\mathbb{R}_+ \to \mathbb{R}$ $T(1) = 39$ $T(2) = 38.5$ $T(3) = 38$ $\vdots$	stochastic process

1.2 Difference between various fields of stochastics

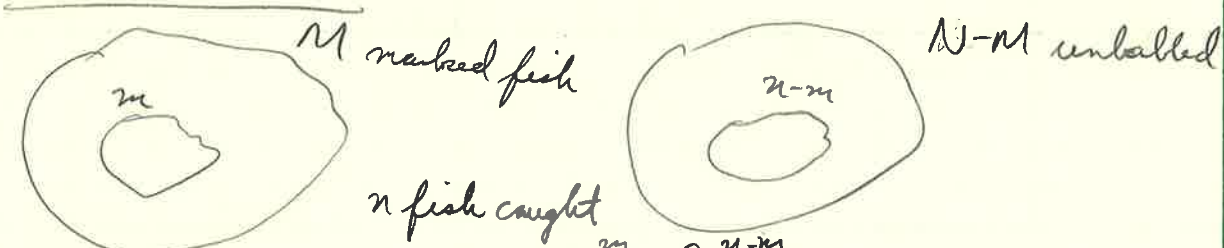
Stochastics
- probability theory
- mathematical statistics
- stochastic processes

Consider a pond that contains fish

Prob theory : # of fish at some given time (N)

$\mathbb{E}$, Var, or limit laws

Mathematical Stats :



$$P\{\#\text{ marked} = m\} = \frac{C_M^m \, C_{N-M}^{n-m}}{C_N^M}$$

Repeat $m_1, m_2, \dots, m_q$

$$\left(\begin{matrix}\text{log}\\ \text{likelihood}\end{matrix}\right) \sum_{k=1}^{q} P\{\#\text{ marked} = m_k\} \longrightarrow \max_N \quad (MLE)$$

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1.3 Probability space $(\Omega, \mathcal{F}, P)$

General theory	Bernoulli Scheme $\begin{bmatrix} 1, \text{ success} \\ 0, \text{ failure} \end{bmatrix}$ $(a_1, \dots, a_n)$, $a_i \in \{0,1\}$	$[0,1]$ Select point from
Ω - sample space	$\#\Omega = 2^n$, set of all vectors with components $\in \{0,1\}$	$\Omega = [0,1]$
$\mathcal{F}$ - σ-algebra 1) $\Omega \in \mathcal{F}$ 2) $A \in \mathcal{F} \Rightarrow \Omega \setminus A \in \mathcal{F}$ 3) $A_1, \dots, A_n, \dots \in \mathcal{F}$ $\Downarrow$ $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$	$\mathcal{F}$ = power set $\#\mathcal{F} = 2^{\#\Omega} = 2^{2^n}$	$P\{x \in [\alpha, \beta]\}$ $\Rightarrow$ $[\alpha, \beta)$, $(\alpha, \beta]$, (α, β), $[\alpha, \beta)$, $\{\beta\} \in \mathcal{F}$ Borel σ-algebra
P - probability measure 1) $P(\Omega) = 1$ 2) $A_1, A_2, \dots \in \mathcal{F}$ (disjoint) $\Rightarrow P\{\bigcup_i A_i\} = \sum_i P(A_i)$ $P : \mathcal{F} \to [0,1]$	$P\{1\} = p$ $P\{0\} = 1 - p$	$P\{[\alpha, \beta]\} = \beta - \alpha$

1.4 Definition of a stochastic function. Types of stochastic functions.

$(\Omega, \mathcal{F}, P)$

Random variable - $\underbrace{\text{measurable function}}$ $\xi : \Omega \to \mathbb{R}$.

$$\forall B \in \mathcal{B}(\mathbb{R}) : \xi^{-1}(B) \subset \mathcal{F}$$

T - time

$X : T \times \Omega \to \mathbb{R}$ - random function, if $\forall t \in T$: $X(t, \cdot)$ is a random variable on $(\Omega, \mathcal{F}, P)$, denoted X_t

If $T=\mathbb{R}_+$, this is called a random process or stochastic process

$T=\mathbb{R}_+^n$, random field or stochastic field

$T=\mathbb{N}$ or $\mathbb{Z}$, discrete time stochastic process

$T=\mathbb{R}_+$ or $\mathbb{R}$, continuous time stochastic process

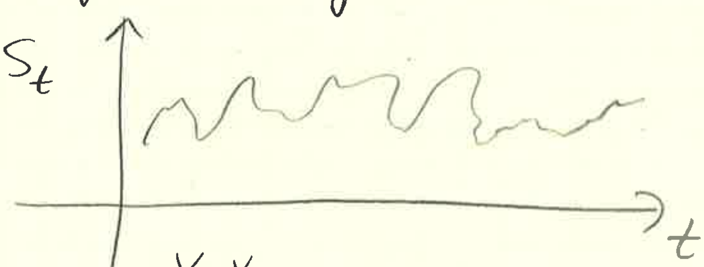
1.5 Trajectories and finite-dimensional distributions

$X: T\times\Omega \to \mathbb{R}$, $T=\mathbb{R}_+$

$\forall t\in T$: $X_t = X(t,\cdot)$ is a r.v. on $(\Omega, \mathcal{F}, P)$

Trajectory (= path)

X_t fix ω and get mapping $T\to\mathbb{R}$



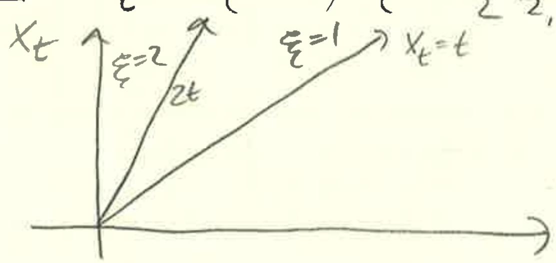
$X_1, X_2, \ldots,$

Finite-dimensional distribution $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$, $t_1,\ldots,t_n \in \mathbb{R}$

In mathematic stats, $X_{t_1}, X_{t_2} \ldots X_{t_n}$ are independent

In stochastic process, $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ are dependent

<u>Ex</u>: $X_t = \xi t$, $\xi = \begin{cases} 1, & \text{w.p. } 1/2 \\ 2, & \text{w.p. } 1/2 \end{cases}$



$$P\{X_{t_1}\le x_1, X_{t_2}\le x_2\} = \begin{cases} 0, & \min\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}\right) < 1 \\ 1/2, & \quad '' \quad \in [1,2] \\ 1, & \quad '' \quad \ge 2 \end{cases}$$

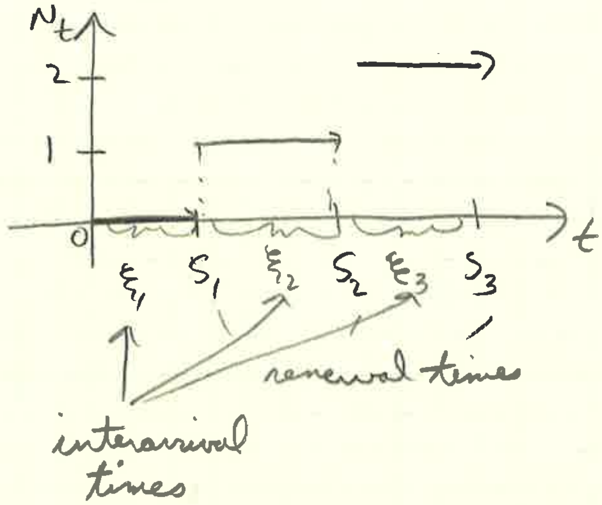
1.6 Renewal process. Counting process.

Renewal processes (discrete time)

$S_0 = 0, \; S_n = S_{n-1} + \xi_n$, where $\xi_1, \xi_2, \ldots$ – iid > 0 a.s.

$P\{\xi_i > 0\} = 1 \iff F(0) = 0$

$N_t = \underset{k}{\operatorname{argmax}} \{S_k \le t\}$ (Counting process)



$\{S_n > t\} = \{N_t < n\}$

$F \Rightarrow \mathbb{E} N_t \qquad S_n = \xi_1 + \cdots + \xi_n$

1.7. Convolution

<u>Convolution</u> $\quad X \perp\!\!\!\perp Y$

$X \sim F_X, \; Y \sim F_Y$

$F_{X+Y}(x) = \int_{\mathbb{R}} F_X(x-y)\, dF(y) =: \underbrace{F_X * F_Y}$ conv in terms of distribution functions

$X \sim p_X, \; Y \sim p_Y$ (If Y, X have densities)

$p_{X+Y}(x) = \int_{\mathbb{R}} p_X(x-y)\, p_Y(y)\, dy =: p_X * p_Y$ $\Big\}$ conv in terms of densities

$S_n = \xi_1 + \cdots + \xi_n$

let $F^{n*} := \underbrace{F * \cdots * F}_{n}$

1) $F^{n*}(x) \le F^{n}(x)$ if $F(0)=0$

$\xi_1, \dots, \xi_n \overset{iid}{\sim} F$

$\{\xi_1 + \dots + \xi_n \le x\} \subset \{\xi_1 \le x, \dots, \xi_n \le x\}$ Since $\xi_i > 0$ a.s.

$$\underbrace{P\{\xi_1 + \dots + \xi_n \le x\}}_{\shortparallel \atop F^{n*}(x)} \le \prod_{k=1}^{n} \underbrace{P\{\xi_k \le x\}}_{F(x)}$$

2) $F^{n*}(x) \ge F^{(n+1)*}(x)$

$\{\xi_1 + \dots + \xi_n \le x\} \supset \{\xi_1 + \dots + \xi_{n+1} \le x\}$

Theorem: $S_n = S_{n-1} + \xi_n$ where $\xi_1, \xi_2, \dots \overset{iid}{\sim} F$, $F(0)=0$

(i) $\boxed{U(t) = \sum_{n=1}^{\infty} F^{n*}(t) < \infty}$

(2) $\boxed{\mathbb{E}N_t = U(t)}$

proof for (2)

$$\mathbb{E}N_t = \mathbb{E}[\#\{n : S_n \le t\}]$$
$$= \mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{1}\{S_n \le t\}\right] = \sum_{n=1}^{\infty} P\{S_n \le t\}$$
$$= \sum_{n=1}^{\infty} F^{n*}(t)$$

1.8 Laplace transform. Calculation of an expectation of a counting process (1)

Laplace transform

$f : \mathbb{R}_+ \to \mathbb{R}$: $\mathcal{L}_f(s) = \int_0^{\infty} e^{-sx} f(x)\,dx$

1) f - density of ξ, then $\mathcal{L}_f(s) = \mathbb{E}[e^{-s\xi}]$

2) f_1, f_2 : $\mathcal{L}_{\underbrace{f_1 * f_2}_{\text{densities}}}(s) = \mathcal{L}_{f_1}(s) \cdot \mathcal{L}_{f_2}(s)$

3) F - distribution function, $F(0)=0$, $p = F'$

$$\mathcal{L}_F(s) = \frac{\mathcal{L}_p(s)}{s}$$

$$\text{l.h.s.} = \int_{\mathbb{R}_+} F(x)\,\frac{d(e^{-sx})}{s} = \cancel{-\frac{F(x)e^{-sx}}{s}\Big|_0^\infty} + \frac{1}{s}\int_{\mathbb{R}_+} p(x)\,e^{-sx}\,dx$$

$$= \text{r.h.s.}$$

Ex:

1) $$\mathcal{L}_{x^k}(s) = \int_{\mathbb{R}_+} x^n \,\frac{d(e^{-sx})}{s} = \frac{n}{s}\int_{\mathbb{R}_+} x^{n-1} e^{-sx}\,dx$$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdots\frac{2}{s}\int_{\mathbb{R}_+} e^{-sx}\,dx = \frac{n!}{s^n}$$

2) $$\mathcal{L}_{e^{ax}}(s) = \frac{1}{s-a}, \text{ if } a<s$$

1.9 Laplace transform. Calculation of an expectation of a counting process (2)

$F \to \mathbb{E}N_t$

$$\mathbb{E}N_t = \mathcal{U}(t) = \sum_{n=1}^{\infty} F^{n*}(t) = F(t) + \left(\sum_{n=1}^{\infty} F^{n*}(t)\right) * F(t)$$

$$\Leftrightarrow \quad \mathcal{U} = F + \mathcal{U} \underset{\text{dist fcn}}{*} F = F + \mathcal{U} \underset{\text{densities}}{*} p \quad \text{If } F' = p \text{ exists}$$

$$\int_{\mathbb{R}} \mathcal{U}(x-y)\,dF(y) = \int_{\mathbb{R}} \mathcal{U}(x-y)\,p(y)\,dy$$

$$\mathcal{L}_{\mathcal{U}}(s) = \underbrace{\mathcal{L}_F(s)}_{=\frac{\mathcal{L}_p(s)}{s}} + \mathcal{L}_{\mathcal{U}}(s)\,\mathcal{L}_p(s)$$

$$\boxed{\mathcal{L}_{\mathcal{U}}(s) = \frac{\mathcal{L}_p(s)}{s(1-\mathcal{L}_p(s))}}$$

1. $F \to \mathcal{L}_p$
2. $\mathcal{L}_p \to \mathcal{L}_{\mathcal{U}}$
3. $\mathcal{L}_{\mathcal{U}} \to \mathcal{U}$

1.10 Laplace transform. Calculation of an expectation of a counting process (3)

Example: $S_n = S_{n-1} + \xi_n$, $\xi_1, \xi_1, \ldots$ have density $p(x)$

$$p(x) = \frac{e^{-x}}{2} + e^{-2x}, \quad x > 0$$

$\mathbb{E}N_t = ?$

① $p \to \mathcal{L}p$: $\mathcal{L}p(s) = \frac{1}{2}\mathcal{L}_{e^{-x}}(s) + \mathcal{L}_{e^{-2x}}(s)$

$$= \frac{1}{2(s+1)} + \frac{1}{s+2} = \frac{3s+4}{2(s+1)(s+2)}$$

② $\mathcal{L}p \to \mathcal{L}_u$: $\mathcal{L}_u(s) = \dfrac{\mathcal{L}p(s)}{s(1-\mathcal{L}p(s))} = \dfrac{3s+4}{s^2(2s+3)}$

③ $\mathcal{L}_u \to u$: $\mathcal{L}_u(s) = \dfrac{A}{s^2} + \dfrac{B}{s} + \dfrac{C}{2s+3}$

$$= \frac{A(2s+3) + B(2s^2+3s) + Cs^2}{s^2(2s+3)}$$

$$3s+4 = (2B+C)s^2 + (2A+3B)s + 3A$$

$A = \frac{4}{3}$, $2A+3B = 3 \Leftrightarrow B = \frac{1}{9}$, $2B+C=0 \Leftrightarrow C = -\frac{2}{9}$

$$\boxed{u(t) = \frac{4}{3}t + \frac{1}{9}(1) - \frac{1}{9}e^{-3/2\, t}}$$

1.11 Limit theorems for renewal processes

$S_n = S_{n-1} + \xi_n$; $\xi_1, \xi_2, \ldots$ iid > 0 a.s.

Thm 1 $\mu = \mathbb{E}\xi_1 < \infty \Rightarrow \dfrac{N_t}{t} \xrightarrow{t\to\infty} \dfrac{1}{\mu}$ a.s.

(Analog to SLLN)

$$\boxed{\text{SLLN: } \frac{\xi_1 + \cdots + \xi_n}{n} \xrightarrow[n\to\infty]{} \mu \text{ a.s.}}$$

Thm 2: (Analog of CLT) $\sigma^2 = \mathrm{Var}\,\xi_1 < \infty$

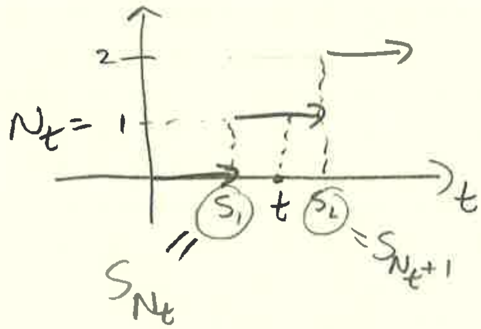
Then, $Z_t := \dfrac{N_t - t/\mu}{\sigma\sqrt{t}/\mu^{3/2}} \xrightarrow[t\to\infty]{d} N(0,1)$

$$P\{Z_t \le x\} \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du$$

$$\boxed{\text{CLT: } \frac{\xi_1 + \cdots + \xi_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1)}$$

proof (thm 1)

$$S_{N_t} \le t \le S_{N_t+1}$$



$$\frac{N_t}{S_{N_t+1}} \le \frac{N_t}{t} \le \frac{N_t}{S_{N_t}}$$

$$\lim_{t\to\infty} \frac{N_t}{S_{N_t}} = \lim_{n\to\infty} \frac{n}{S_n} = \frac{1}{\mu} \text{ by SLLN}$$

$$\lim_{t\to\infty} \frac{N_t}{S_{N_t+1}} = \underbrace{\lim_{t\to\infty} \frac{N_t}{N_t+1}}_{=1} \cdot \underbrace{\lim_{t\to\infty} \frac{N_t+1}{S_{N_t}+1}}_{=1/\mu} = \frac{1}{\mu}$$

proof (thm 2)

$$P\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right\} \to \phi(x), \; x \in R$$

$$P\{S_n \le \underbrace{n\mu + \sigma\sqrt{n}\, x}_{= t}\} \to \phi(x)$$

$$\Leftrightarrow P\{N_t \ge n\}$$
(set complements)

$$n\mu \approx t$$
$$n \approx t/\mu \text{ (for } n \text{ large enough)}$$

$$n = \frac{t}{\mu} - \frac{\sigma\sqrt{n}}{\mu} x \approx \frac{t}{n} - \frac{\sigma\sqrt{t}}{\mu^{3/2}} x$$

$$\Rightarrow P\{Z_t \ge -x\} \to \phi(x) \quad \Leftrightarrow P\{Z_t \le x\} = 1 - P\{Z_t \ge -x\} \to 1 - \phi(-x)$$
$$= \phi(x)$$

Poisson Processes

2.1 Definition of a Poisson process as a special example of a renewal process. Exact forms of the distributions of the renewal process and the counting process (1)

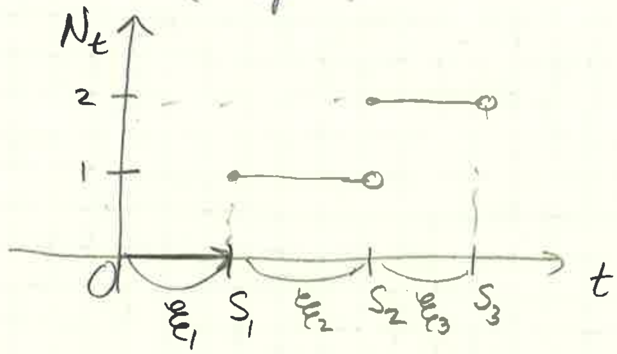
Renewal process

$S_0 = 0, \quad S_n = S_{n-1} + \xi_n, \quad \xi_1, \xi_2, \dots$ - i.i.d > 0 a.s., $\xi_i \sim F$

$N_t = \underset{k}{\operatorname{argmax}} \{S_k \le t\}$ (Counting process)

$U(t) = \mathbb{E} N_t = \sum_{n=1}^{\infty} F^{n*}(t)$

$\mathcal{L}_U(s) = \dfrac{\mathcal{L}_p(s)}{s(1-\mathcal{L}_p(s))}$: $p \to \mathcal{L}_p \to \mathcal{L}_U \to U$ $(p = F')$



$\mathcal{L}_U(s) = \int_{\mathbb{R}_+} e^{-sx} U(x)\,dx$

2.2 .... (2)

Poisson process

Def 1: A Poisson process is a renewal process s.t.

$\xi_i \sim p(x) = \lambda e^{-\lambda x} \mathbb{I}\{x > 0\}$, λ - intensity or rate

Thm (i): A distribution function of S_n

$$F_{S_n}(x) = \begin{cases} 1 - e^{-\lambda x} \sum_{k=0}^{n-1} \dfrac{(\lambda x)^k}{k!}, & x > 0 \\ 0, & x < 0 \end{cases}$$

$$p_{S_n}(x) = \lambda \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} \mathbb{I}\{x > 0\}$$

(ii) $\mathbb{P}\{N_t = n\} = e^{-\lambda t} \dfrac{(\lambda t)^n}{n!}$, $N_t \sim \text{Poisson}(\lambda t)$

2.3 ⋯ (3)

Proof (i) $n=1$: $S_1 = \xi_1$

$$p_{S_1}(x) = \lambda e^{-\lambda x}, \; x>0$$

$n \to n+1$

$$p_{S_{n+1}}(x) = \int_0^x p_{S_n}(x-y)\, p_{\xi_{n+1}}(y)\, dy$$

$$= \int_0^x \frac{\lambda^n (x-y)^{n-1}}{(n-1)!} e^{-\lambda(x-\cancel{y})} \lambda \cancel{e^{-\lambda y}}\, dy$$

$$= \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda x} \int_0^x (x-y)^{n-1} dy = \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda x} \frac{x^n}{n}$$

$$= \lambda \frac{(\lambda x)^n}{n!} e^{-\lambda x}$$

□

2.4 ⋯ (4)

proof (ii)

$$P\{N_t = n\} = P\{S_n \le t\} - P\{S_{n+1} \le t\} \quad =$$

$$\{N_t = n\} = \underbrace{\{S_n \le t\}}_{=A} \cap \underbrace{\{S_{n+1} > t\}}_{=B}$$

$$\left.\begin{array}{l} A\cap B = A \setminus B^c \\ \text{Here: } B^c \subset A \end{array}\right\} \Rightarrow P\{A\cap B\} = P\{A\} - P\{B^c\}$$

$$= \left(1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}\right) - \left(1 - e^{-\lambda t} \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!}\right)$$

$$= e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

□

2.5 Memoryless property

A r.v. X possesses the memoryless property iff

$P\{X > u+v\} = P\{X>u\}\, P\{X>v\}$. If $P\{X>v\}>0$, then

$$\boxed{P\{X>u+v \mid X>v\} = P\{X>u\}}$$

Thm 2: Let X be a r.v. with density $p(x)$, then

X- memoryless $\Leftrightarrow$ $p(x) = \lambda e^{-\lambda x}$

Ex busses arrive every 20 ± 2 minutes

$v = 19$ min, $u = 10$ min

(l.h.s.) $P\{X > 29 \mid X > 19\} = 0$ given the data

(r.h.s) $P\{X > 10\} = 1$

Thus, Poisson process is not appropriate

2.6. Other definitions of Poisson processes (1)

Def 2 N_t – an integer value process s.t.

0) $N_0 = 0$ a.s.

1) N_t has independent increments: $\forall t_0 < t_1 < \ldots < t_n$,

$N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}}$ are independent

2) N_t has stationary increments

$N_t - N_s \overset{d}{=} N_{t-s}$

3) $N_t - N_s \sim \text{Poisson}(\lambda(t-s))$, $t > s$

3) $\Rightarrow$ 2)

2.7 Other definitions of Poisson processes (2)

$$P\{N_{t+h} - N_t = 0\} = 1 - \lambda h + o(h), \quad h \to 0$$

$$P\{N_{t+h} - N_t = 1\} = \lambda h + o(h), \quad h \to 0$$

$$P\{N_{t+h} - N_t \geq 2\} = o(h), \quad h \to 0$$

$$\lim_{h \to 0} \frac{1 - P\{N_{t+h} - N_t = 0\}}{h} = \lim_{h \to 0} \frac{1 - e^{-\lambda h}}{h} = \lambda$$

Def 3 N_t is a Poisson process, if

0) $N_0 = 0$

1) N_t has independent increments

2) N_t has stationary increments

3') $\displaystyle \lim_{h \to 0} \frac{P\{N_{t+h} - N_t \geq 2\}}{P\{N_{t+h} - N_t = 1\}} = 0$

2.8 Non-homogeneous Poisson processes (1)

$N_t \sim Pois(\lambda t) \Rightarrow \mathbb{E}N_t = \lambda t$

<u>Def</u>: Let $\Lambda(t)$ be a differentiable, increasing function s.t. $\Lambda(0)=0$. Then, $X_t = N_t$ is a non-homogeneous Poisson process if,

0) $N_0 = 0$

1) N_t has independent increments

2) $\boxed{N_t - N_s \sim Pois(\Lambda(t) - \Lambda(s))}$

2.9 Non-homogeneous Poisson processes (2) (NHPP)

$\lambda(t) = \Lambda'(t)$ - intensity function

<u>Properties NHPP</u>:

1) $\mathbb{E}N_t = \Lambda(t)$

$\Lambda(t) = \alpha t^\beta, \alpha > 0, \beta > 0$ (for example)

2) if $\lambda(t) = \text{const} \Rightarrow \Lambda(t) = \text{const} \cdot t$

3) $\left.\begin{array}{l}\Lambda(t)\text{ - differentiable} \Rightarrow \Lambda(t)\text{ - continuous}\\ \Lambda(t)\text{ - increasing}\end{array}\right\} \Rightarrow$

$\Rightarrow \exists \Lambda^{-1}(t)$. If Image $\Lambda(t) = \mathbb{R}_+$, $N_{\Lambda^{-1}(t)}$ - homogeneous P.P.

2.10 Relation between renewal theory and NHPP (1)

$S_n = \underset{t}{\operatorname{argmin}}\{N_t = n\}$, $\xi_n = S_n - S_{n-1}$

$\xi_1, \xi_2, \ldots$ - iid ?

1) $p_{\xi_1}(x) = \lambda(x) e^{-\Lambda(x)}$

$$P\{\xi_1 \le x\} = P\{S_1 \le x\} \underset{\{S_n > t\} = \{N_t < n\}}{=} P\{N_x \ge 1\} = 1 - P\{N_x = 0\} =$$

$= 1 - e^{-\Lambda(x)}$

Take derivatives of both sides to finish proof $\square$

2.11 ••• (2)

$$S_k = \operatorname*{argmin}_t \{N_t = k\}$$

$$\xi_k = S_k - S_{k-1}$$

1) $\boxed{p_{\xi_1}(t) = \lambda(t)e^{-\Lambda(t)}}$

2) $\boxed{p_{\xi_2|\xi_1}(t|s) = \lambda(t+s)e^{-\Lambda(t+s)+\Lambda(s)}}$

$$F_{(\xi_1,\xi_2)}(s,t) = P\{\xi_1 \le s, \xi_2 \le t\} = \int_0^s P\{\underbrace{\cancel{\xi_1 \le s}}_{\text{since } y<s}, \xi_2 \le t \mid \xi_1 = y\}\, p_{\xi_1}(y)\,dy$$

$$= \int_0^s P\{N_{t+y} - N_y \ge 1 \mid \xi_1 = y\}\, p_{\xi_1}(y)\,dy$$

independent

$$= \int_0^s \left(1 - e^{-\Lambda(t+y)+\Lambda(y)}\right) \lambda(y)e^{-\Lambda(y)}\,dy$$

$$p_{(\xi_1,\xi_2)}(s,t) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s} F_{(\xi_1,\xi_2)}(s,t)\right)$$

$$= \frac{\partial}{\partial t}\left(\left(1 - e^{-\Lambda(t+s)+\Lambda(s)}\right)\lambda(s)e^{-\Lambda(s)}\right)$$

$$= \lambda(t+s)e^{-\Lambda(t+s)+\Lambda(s)}\,\lambda(s)e^{-\Lambda(s)}$$

Then $p_{\xi_2|\xi_1}(t|s) = \dfrac{p_{(\xi_1,\xi_2)}(s,t)}{p_{\xi_1}(s)}$ finishes the proof □

2.12 ••• (3)

$\xi_1, \xi_2, \ldots$ – i.i.d. ? (NHPP can be obtained from renewal process iff NHPP is homogeneous PP)

$$p_{\xi_1}(t) = p_{\xi_2|\xi_2}(t|s), \quad \forall t, s > 0$$

$$\lambda(t)e^{-\Lambda(t)} = \lambda(t+s)e^{-\Lambda(t+s)+\Lambda(s)}$$

$$\left(\int_0^T \cdots dt\right): \quad \cancel{e^{-\Lambda(0)}} - e^{-\Lambda(T)} = \cancel{e^{0}} - e^{-\Lambda(T+S)+\Lambda(s)}$$

$$\boxed{\begin{array}{l}\Lambda(T) = \Lambda(T+S) - \Lambda(s), \quad \forall s, T > 0 \\ \Lambda(t) \text{ – increasing}\end{array}} \Rightarrow \Lambda(t) = \overset{\text{const}}{\overset{\shortparallel}{\lambda}}\, t$$

2.13 Elements of queuing theory. M/G/k systems (1)

$$\boxed{\begin{aligned} P\{N_{t+h}-N_t=0\} &= 1-\overset{\lambda(t)}{\lambda} h+o(h)\\ P\{N_{t+h}-N_t=1\} &= \overset{\lambda(t)}{\lambda} h+o(h)\\ P\{N_{t+h}-N_t\geq 2\} &= o(h)\end{aligned}}$$

M/G/k

I) Arrival Process: M - memoryless (Poisson)
D - deterministic
G - general

II) Service time (M, D, G)

III) A number of services $(1, 2, \dots, \infty)$

M/G/∞ $\quad \tau > 0$ (time moment)

$\xrightarrow[\text{Customer arrivals}]{N(t)}$
- $N_1(t)$ - still being served at τ : $\lambda_1(t) = \lambda(1-G(\tau-t))$
- $N_2(t)$ - already completed by τ : $\lambda_2(t) = \lambda G(\tau-t)$

$$P\{N_1(t+\delta)-N_1(t)=1\} = P\{N(t+\delta)-N(t)=1\}\cdot\left(P\{Y>\tau-t\}+o(\delta)\right)$$
$$= (\delta\lambda+o(\delta))(1-G(\tau-t)+o(\delta))$$
$$= \boxed{\lambda\delta(1-G(\tau-t)+o(\delta))}$$

2.14 ... (2)

bernoulli scheme

$$P\{N_1(t)=n_1,\ N_2(t)=n_2\} = P\{N_1(t)=n_1,\ N_2(t)=n_2 \mid N(t)=n_1+n_2\}$$
$$\cdot\ P\{N(t)=n_1+n_2\}$$
$$= C_{n_1+n_2}^{n_1}(1-G(\tau-t))^{n_1}G(\tau-t)^{n_2}\cdot e^{-\lambda t}\frac{(\lambda t)^{n_1+n_2}}{(n_1+n_2)!}$$
$$= \frac{\lambda t(1-G(\tau-t))^{n_1}}{n!}e^{-\lambda(1-G(\tau-t))}\cdot\frac{\lambda t(G(\tau-t))^{n_2}}{n_2!}e^{-\lambda G(\tau-t)}$$
$$= P\{N_1(t)=n_1\}\cdot P\{N_2(t)=n_2\}$$

Therefore $N_1 \perp\!\!\!\perp N_2$

2.15 Compound Poisson Processes (1)

$$X_t = \sum_{k=1}^{N_t} \xi_k \quad , \quad \xi_1, \xi_2, \dots \text{ - iid. }, \; N_t \text{ - P.P. with intensity } \lambda$$

and $\xi_1, \xi_2, \dots$ and N_t are independent

$\xi_1, \xi_2, \dots$ claim sizes

N_t - amount of claims until time t (Insurance interpretation)

X_t - aggregated claim amount

1) Probability generating function (PGF)

ξ - integer, ≥ 0 values

$$\boxed{\phi_\xi(u) = \mathbb{E}[u^\xi]}, \; |u| < 1$$

$$\xi_1 \perp\!\!\!\perp \xi_2 \Rightarrow \phi_{\xi_1+\xi_2}(u) = \phi_{\xi_1}(u)\,\phi_{\xi_2}(u)$$

2) Moment-generating function (MGF)

$$\boxed{\mathcal{L}_\xi(u) = \mathbb{E}\left[e^{-u\xi}\right]}, \; \xi \geq 0, \; u > 0$$

2.16 ... (2)

3) Characteristic function

$$\phi_\xi(u) = \mathbb{E}\left[e^{iu\xi}\right], u \in \mathbb{R}, \forall \xi \;, \; \phi_\xi : \mathbb{R} \to \mathbb{C} \;, \; \xi_1 \perp\!\!\!\perp \xi_2 \Rightarrow \phi_{\xi_1+\xi_2}(u) = \phi_{\xi_1}(u)\phi_{\xi_2}(u)$$

Thm $\boxed{\phi_{X_t - X_s}(u) = e^{\lambda(t-s)(\phi_{\xi_1}(u)-1)}}$

Proof: $$lhs = \mathbb{E}e^{iu(X_t-X_s)} = \sum_{k=0}^{\infty} \mathbb{E}\Big[\underbrace{e^{iu(X_t-X_s)}}_{\substack{\|d \\ \xi_1+\dots+\xi_k}} \Big| N_t - \cancel{N_s} = k\Big]\, P\{N_t - N_s = k\}$$ from $\perp\!\!\!\perp$

$$= \sum_{k=0}^{\infty} \left(\phi_{\xi_1}(u)\right)^k e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^k}{k!}$$

□

2.17 ⋯ (3)

$$X_t = \sum_{k=1}^{N_t} \xi_k \quad , \ \xi \text{ can be any random variable}$$

$$\xi : \ \phi_\xi(u) = E\left[e^{iu\xi}\right]$$

$$\phi : \mathbb{R} \to \mathbb{C}$$

$$\xi_1 \perp\!\!\!\perp \xi_2 \Rightarrow \phi_{\xi_1+\xi_2}(u) = \phi_{\xi_1}(u)\,\phi_{\xi_2}(u)$$

Thm: $\boxed{\phi_{X_t - X_s}(u) = e^{\lambda(t-s)\left(\phi_{\xi_1}(u) - 1\right)} \quad , \ t > s \geq 0}$

proof

$$\text{lhs} = \mathbb{E}\left[e^{iu(X_t - X_s)}\right]$$

$$= \sum_{k=0}^{\infty} \mathbb{E}\Big[e^{iu\underbrace{(X_t - X_s)}_{\overset{d}{=}\ \xi_1+\cdots+\xi_k}} \Big| N_t - \cancel{N_s = k}\Big] \cdot P\{\underbrace{N_t - N_s = k}_{\sim \text{Pois}(\lambda(t-s))}\}$$

since $\xi_1, \xi_2, \ldots \perp\!\!\!\perp N_t$

$$= \sum_{k=0}^{\infty} \left[\phi_{\xi_1}(u)\right]^k \cdot e^{-\lambda(t-s)} \frac{\left[\lambda(t-s)\right]^k}{k!} \qquad \square$$

2.18 ⋯ (4)

Corollary: $\boxed{\begin{array}{l} \mathbb{E}X_t = \lambda t\, \mathbb{E}\xi_1 \\ \text{Var}\, X_t = \lambda t\, \mathbb{E}\xi_1^2 \end{array}}$

proof $E[\xi^r] < \infty \Rightarrow \phi(u)$ is r-times differentiable at 0 and $\phi^{(r)}(0) = i^r\, \mathbb{E}\xi^r$

$$\mathbb{E}X_t = \frac{\phi_{X_t}'(0)}{i} = \frac{\lambda t\, \phi_\xi'(0) \cdot \overbrace{\phi_{X_t}(0)}^{=1}}{i}, \quad \frac{\phi_\xi'(0)}{i} = \mathbb{E}\xi, \quad = \lambda t\, \mathbb{E}\xi_1 \qquad \square$$

3.1 Definition of a Markov chain. Some examples

<u>Def</u>: A Markov chain $- S_n,\ n = 0, 1, 2, \dots$

S - state space (countable)

$$\boxed{P\{S_n = j \mid S_{n-1} = i_{n-1}, \dots, S_0 = i_0\} = P\{S_n = j \mid S_{n-1} = i_{n-1}\}}$$

$i_0, \dots, i_{n-1}, j \in S$ and $P\{S_{n-1} = i_{n-1}, \dots, S_0 = i_0\} \neq 0$

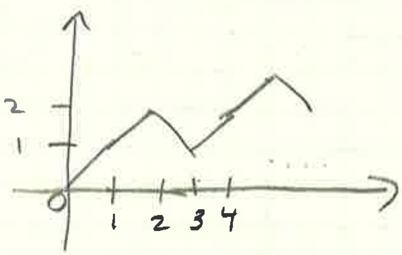
$$P\{S_n = i_n, S_{n-1} = i_{n-1}, \dots, S_0 = i_0\} = P\{S_n = i_n \mid S_{n-1} = i_{n-1}, \dots, S_0 = i_0\} \cdot P\{S_{n-1} = i_{n-1}, \dots, S_0 = i_0\}$$

$$= P\{S_n = i_n \mid S_{n-1} = i_{n-1}\} \cdot P\{S_{n-1} = i_{n-1}, \dots, S_0 = i_0\}$$

$$= P\{S_n = i_n \mid S_{n-1} = i_{n-1}\}\, P\{S_{n-1} = i_{n-1} \mid S_{n-2} = i_{n-2}\} \cdot \dots \cdot P\{S_1 = i_1 \mid S_0 = i_0\}\, P\{S_0 = i_0\}$$

<u>Ex ①</u> Random walk (not a renewal process)

$S_0 = 0,\ S_n = S_{n-1} + \xi_n,\ \ \xi_1, \xi_2, \dots$ - i.i.d. $\sim \begin{cases} 1, & \text{w.p. } p \\ -1, & \text{w.p. } 1-p \end{cases}$



$$P\{S_n = j \mid S_{n-1} = i_{n-1}\} = \begin{cases} p, & j = i_{n-1} + 1 \\ 1-p, & j = i_{n-1} - 1 \\ 0, & \text{otherwise} \end{cases}$$

② Taxis in the airport

1 taxi at any 1 moment, $n = 1, 2, 3, \dots$

X_k = # people waiting for a taxi at time k

Y_k = # people arriving at k

$$X_k = Y_k + (X_{k-1} - 1)_+ = \begin{cases} Y_k, & \text{if } X_{k-1} = 0 \\ Y_k + X_{k-1} - 1, & \text{if } X_{k-1} - 1 > 0 \end{cases}$$

③ X_n : $P\{X_n = j \mid X_{n-1} = i_{n-1}, \dots, X_0 = i_0\} = P\{X_n = j \mid X_{n-1} = i_{n-1}, \dots, X_{n-m} = i_{n-m}\}$

$m \in \mathbb{N}$, fixed (X_n is not a Markov chain)

$S_n = (X_n, \dots, X_{n-m-1})$, $n = (m-1), m, \dots$ $\quad S_n$ is a Markov chain

3.2 Matrix representation of a Markov chain. Transition matrix. Chapman-Kolmogorov equation.

Matrix representation

$S = (1, 2, \dots, M)$

$P\{X_n = j \mid X_{n-1} = i\} = p_{ij}$ - homogeneous (no dependence on n)

$\underline{P} = (p_{ij})_{i,j=1}^{M}$ - transition matrix

$\left.\sum_{j=1}^{M} p_{ij} = 1, \ \forall i \ ; \ p_{ij} \geq 0 \right\}$ - stochastic matrix

$p_{ij}^{(m)} = P\{X_{n+m} = j \mid X_n = i\}$

$\underline{P}^{(m)} = (p_{ij}^{(m)})$ - m-step transition matrix

<u>Thm</u>: $\boxed{\underline{P}^{(m)} = \underline{P}^{m}}$

<u>proof</u>:
$$p_{ij}^{(m)} = \sum_{k=1}^{M} P\{X_{n+m} = j \mid X_{n+m-1} = k, X_n = i\} \cdot P\{X_{n+m-1} = k \mid X_n = i\}$$

(Markov property)
$$= \sum_{k=1}^{M} P\{X_{n+m} = j \mid X_{n+m-1} = k\} \, P\{X_{n+m-1} = k \mid X_n = i\}$$

$$= \sum_{k=1}^{M} p_{kj} \, p_{ik}^{(m-1)} \Rightarrow \underline{P}^{(m)} = \underline{P} \cdot \underline{P}^{(m-1)} = \dots = \underline{P}^{m} \quad \square$$

$P\{X_k = j\} := \pi_j^{(k)}$, $\left(\pi_1^{(k)}, \dots, \pi_M^{(k)}\right) := \vec{\pi}^{(k)}$

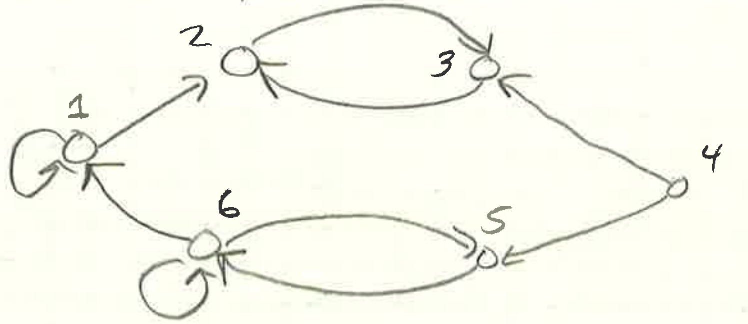
$$\pi_j^{(k)} = \sum_{i=1}^{M} P\{X_k = j \mid X_{k-1} = i\} \, P\{X_{k-1} = i\}$$

$$= \sum_{i=1}^{M} p_{ij} \, \pi_i^{(k-1)} \Rightarrow \vec{\pi}^{(k)} = \vec{\pi}^{(k-1)} \cdot \underline{P} = \vec{\pi}^{(0)} \underline{P}^{n}$$

$\vec{\pi}^{*}$ - stationary distribution for Markov chain if $\boxed{\vec{\pi}^{*} \underline{P} = \vec{\pi}^{*}}$

3.3 Graphical representation : Classification of states (1)

Graphical representation



1 node = 1 state

i,j-arc $\Leftrightarrow p_{ij} \neq 0$

$$\underline{P} = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ & & \cdots & & & \\ & & & & & \end{pmatrix}_{6\times 6}$$

Def (1) j is accessible from i $\exists$ walk (path) from i to j $(i \to j)$

$1 \to 3$; $1 \not\to 4$

(2) i and j communicate if $i \to j$ and $j \to i$ $(i \leftrightarrow j)$

$2 \leftrightarrow 3$

(3) $\underline{Y}$ - set, and relation $\sim$ called an equivalence relation

$a \sim a$, $a \in \underline{Y}$ - reflexivity

$a \sim b \Rightarrow b \sim a$, $a, b \in \underline{Y}$ - symmetry

$a \sim b$, $b \sim a \Rightarrow a \sim c$, $a, b, c \in \underline{Y}$ - transitivity

$\underline{Y} = \bigsqcup B_i$ ($\bigsqcup$ - disjoint union), B_i - equivalence classes

$\leftrightarrow$  $\Leftrightarrow \underline{P}\left(\cdots \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \cdots \right)$

$B_1, B_2, \ldots$ - equivalence classes

$$\boxed{\forall j \in B_i,\ \forall k \in S \begin{cases} k \in B_i, & k \leftrightarrow j \\ k \notin B_i, & k \not\leftrightarrow j \end{cases}}$$

$2 \leftrightarrow 3$, $5 \leftrightarrow 6$, 1, 4 are the four equivalence classes

3.4 ... (2)

Def: i is recurrent, $\forall j : i \to j \Rightarrow j \to i$

i is transient if it's not recurrent $\Leftrightarrow \exists j : i \to j,\ j \not\to i$

ex: 1, 4, 5, 6 - transient

2, 3 - recurrent

Thm: In 1 class of equivalence, all states are either recurrent or transient.

proof k-transient : $\exists j : k \to j,\ j \not\to k$

$i, k \in 1$ class $\Rightarrow$ $i \to k \to j$, but $j \not\to i$: $j \to i \to k$ is a contradiction $\square$

3.5 ... (3)

Def: Period of a state i is $GCD\{n : p_{ii}(n) \neq 0\} =: d(i)$

$d(i) = 1 \Rightarrow i$ - aperiodic

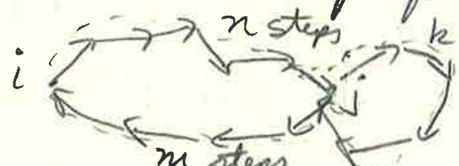
ex $d(1) = 1 = d(4) = d(5) = d(6)$

$d(2) = 2 = d(3)$

(4) has no return, so $d(4) = 1$ by convention

Thm: All elements in 1 class of equivalence have the same period

proof:



$p_{ii}(n+m+k) \neq 0$

$p_{ii}(n+m) \neq 0 \Rightarrow n+m \mid d(i)$

$k : p_{jj}(k) \neq 0 \Rightarrow n+m+k \mid d(i)$

$$\Rightarrow k \mid d(i) \Rightarrow \left.\begin{array}{l} d(i) \mid d(j) \\ d(j) \mid d(i) \end{array}\right\} \Rightarrow d(i) = d(j)$$

$\square$

3.6 Ergodic chains. Ergodic Theorem (1)

Matrix representation

$\underline{P}$, $\overrightarrow{\pi}^{(k)}$

$\underline{P}^{(m)} = \underline{P}^m$

Ergodic Markov chains:

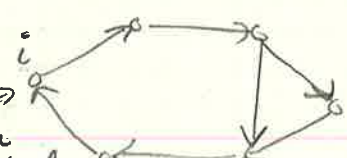
- 1 class of equivalence
- recurrent
- $d(i) = 1$ (aperiodic)

Graphical representation

classes of equivalence

recurrent / transient

$d(i)$ - period



return in 5 or 6 steps

$\therefore d(i) = 1$

Prop: Markov chain is ergodic $\Longleftrightarrow \exists m \in \mathbb{N} : p_{ij}(m) \neq 0, \forall i,j \in S$ (*)

If chain is ergodic, then (*) hold $\forall m \geq (M-1)^2+1$

3.7 ... (2)

Ergodic theorem: Let X_t - ergodic Markov chain, i.e. X_t has 1 class of equivalence, recurrent and aperiodic. Then,

$$\boxed{\exists \lim_{n\to\infty} p_{ij}(n) = \pi_j^* > 0 \text{ (doesn't depend on } i)}$$

$$\sum_{j=1}^{M} \pi_j^* = 1 \qquad \vec{\pi}^* = (\pi_1^*, \dots, \pi_M^*)$$

Corr (i) $\vec{\pi}^*$ - stationary distribution : $\vec{\pi}^* \underline{P} = \vec{\pi}^*$

(ii) $\lim_{n\to\infty} P\{X_n = j\} = \pi_j^*$ [$\pi_j^{(0)}$ is arbitrary]

proof (i) $i = 1, \dots, M$

$$(\pi^* \underline{P})_i = \sum_{j=1}^{M} \pi_j^* p_{ji} = \sum_{j=1}^{M} \lim_{n\to\infty} p_{kj}(n)\, p_{ji} \qquad (k \in 1, \dots, M)$$

$$= \lim_{n\to\infty} \sum_{j=1}^{M} \underbrace{p_{kj}(n)\, p_{ji}}$$

$$\underline{P}^{(n)} \underline{P} = \underline{P}^{n+1} = \underline{P}^{(n+1)}$$

$$= \lim_{n\to\infty} p_{ki}(n+1) = \pi_i^*$$

$\square$

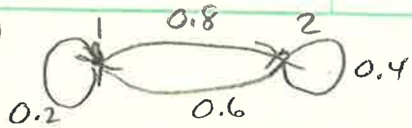
proof (ii) $\lim_{n\to\infty} \pi_j^{(n)} = \lim_{n\to\infty} \sum_{k=1}^{M} \pi_k^{(0)} p_{kj}(n)$ $\qquad \pi_j^{(0)}$ is arbitrary

$$\vec{\pi}^{(n)} = \vec{\pi}^{(0)} \underline{P}^{(n)}$$

$$= \sum_{k=1}^{M} \pi_k^{(0)} \underbrace{\lim_{n\to\infty} p_{kj}(n)}_{= \pi_j^*} = \pi_j^* \underbrace{\sum_{k=1}^{M} \pi_k^{(0)}}_{= 1} = \pi_j^*$$

$\square$

ex



$$\underline{P} = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}$$

$$\vec{\pi}^* = (a, b) \; ; \; \vec{\pi}^* \underline{P} = \vec{\pi}^*$$

$$(a \;\; b) \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix} = (a \;\; b)$$

$$\left. \begin{array}{l} 0.2a + 0.6b = a \\ 0.8a + 0.4b = b \end{array} \right\} \Rightarrow a = \frac{3}{7}, \; b = \frac{4}{7}$$

$$P\{X_n = 1\} \to \frac{3}{7}$$

$$P\{X_n = 2\} \to \frac{4}{7}$$

4.1 Random vector. Definition and main properties

$\xi \sim N(\mu,\sigma^2)$, $p(x)=\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $\sigma>0, \mu\in R$

$\phi(u)=e^{iu\mu-\frac{1}{2}u^2\sigma^2}$

$\left.\begin{array}{l} X_1, X_2 \sim N(0,1) \\ \operatorname{cov}(X_1,X_2)=0 \end{array}\right\} \not\Rightarrow X_1 \perp\!\!\!\perp X_2$

$P\{X=\mu\}=1 \Rightarrow \sigma=0$

Def: A random vector $\vec{X}=(X_1,\dots,X_n)$ is Gaussian iff $\forall (\lambda_1,\dots,\lambda_n)\in R^n$, $\sum_{k=1}^{n} \lambda_k X_k \sim N$

4.2 Gaussian vector. Definition and main properties

Thm: $\vec{X}$ - Gaussian iff any of the following holds:

(i) $\phi_{\vec{X}}(\vec{u}) = \mathbb{E}\left[e^{i\langle\vec{u},\vec{X}\rangle}\right] = \exp\left\{i\langle\vec{u},\vec{\mu}\rangle - \frac{1}{2}\vec{u}^T C\vec{u}\right\}$

$\vec{\mu}\in R^n$; C - symmetric, positive semidefinite (size $n\times n$)

(ii) $\vec{X}=A\vec{X}^\circ+\vec{\mu}$; $A\in Mat(n\times n)$, $\vec{X}^\circ$ - standard normal vector

Remark: $\vec{\mu}=(\mathbb{E}X_1,\dots,\mathbb{E}X_n)$

$C=(c_{jk})_{j,k=1}^{n}$; $c_{jk}=\operatorname{cov}(X_j,X_k)$

$\sum_{k,j=1}^{n} u_k c_{kj} u_j \geq 0$, $\forall u\in R^n \Leftrightarrow u^T C u\geq 0$, $\forall u\in R$

$\|$

$\sum_{k,j=1}^{n} u_k \operatorname{cov}(X_j,X_k) u_j = \operatorname{cov}\left(\sum_{j=1}^{n} u_j X_j, \sum_{k=1}^{n} u_k X_k\right)$

$= \operatorname{var}\left(\sum_{j=1}^{n} u_j X_j\right)\geq 0$

$A=C^{1/2}$: $AA=C \Rightarrow \exists U: U^{-1}=U^T: C=U^T\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n\end{pmatrix}U$

$A=U^T\begin{pmatrix} \sqrt{d_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{d_n}\end{pmatrix}U \Rightarrow C=AA^T$

Proof: Def $\Leftrightarrow$ (i) $\Leftrightarrow$ (ii)

Def $\Rightarrow$ (i): $\langle \vec{u}, \vec{X} \rangle \sim N$

$$\phi_{\vec{X}}(\vec{u}) = \mathbb{E} e^{i\underbrace{\langle \vec{u}, \vec{X}\rangle}_{\xi} \cdot 1} = \phi_{\xi}(1) = e^{i\mu_\xi - \frac{1}{2}\sigma_\xi^2}$$

$$\mu_\xi = \mathbb{E}\left[\sum_{k=1}^{n} u_k X_k\right] = \sum_{k=1}^{n} u_k \mathbb{E}X_k = \sum_{k=1}^{n} u_k \mu_k = \langle \vec{\mu}, \vec{u} \rangle$$

$$\sigma_\xi^2 = \operatorname{cov}\left(\sum_{k=1}^{n} u_k X_k, \sum_{j=1}^{n} u_j X_j\right)$$

$$= \sum_{k=1}^{n}\sum_{j=1}^{n} u_k \operatorname{cov}(X_k, X_j) u_j = \vec{u}^T C \vec{u}$$

(i) $\Rightarrow$ Def By definition of ϕ for Gaussian

(ii) $\Rightarrow$ (i)

$$\vec{X}^\circ\text{-Gaussian} \Rightarrow \phi_{\vec{X}_0}(\vec{u}) = \exp\left\{-\tfrac{1}{2}\vec{u}^T\vec{u}\right\}$$

$$\phi_{\vec{X}}(\vec{u}) = \mathbb{E}\left[e^{i\langle \vec{u}, A\vec{X}^\circ + \vec{\mu}\rangle}\right] = \mathbb{E}\left[e^{i[\langle \vec{u}, \vec{\mu}\rangle + \langle \vec{u}, A\vec{X}^\circ\rangle]}\right]$$

$$= e^{i\langle \vec{u}, \vec{\mu}\rangle}\phi_{\vec{X}^\circ}(A^T\vec{u}) = e^{i\langle \vec{u}, \vec{\mu}\rangle} e^{-\frac{1}{2}\vec{u}^T \underbrace{AA^T}_{\overset{\shortparallel}{C}} \vec{u}}$$

$$= e^{i\langle \vec{u}, \vec{\mu}\rangle} e^{-\frac{1}{2}\vec{u}^T C \vec{u}}$$

(i) $\Rightarrow$ (ii) $A = C^{1/2}$ $\square$

4.3 Connection between independence of normal random variates and absence of correlation

Thm: Let $X_1, X_2 \sim N(0,1)$ and $\operatorname{cov}(X_1, X_2) = 0$, then

$$X_1 \perp\!\!\!\perp X_2 \Leftrightarrow (X_1, X_2) \text{ - Gaussian vector}$$

Proof ($\Rightarrow$) $\lambda_1 X_1 + \lambda_2 X_2 \sim N \Rightarrow (X_1, X_2)$ - Gaussian vector

($\Leftarrow$) $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A = C^{1/2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = C$

$$\text{(ii)} \Rightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = A\vec{X}^\circ + \vec{\mu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X_1^\circ \\ X_2^\circ \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \overset{d}{=} \begin{pmatrix} X_1^\circ \\ X_2^\circ \end{pmatrix}$$

$\square$

ex $X_1 \sim N(0,1)$, $X_2 := |X_1| \cdot \xi$, $\xi = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$, $\xi \perp\!\!\!\perp X_1$

1) $X_2 \sim N(0,1)$, $x > 0$

$$P\{X_2 \le x\} = \mathbb{P}\{|X_1| \le x \mid \xi = 1\} P\{\xi = 1\} + P\{|X_2| \ge -x \mid \xi = -1\} P\{\xi = -1\}$$

$$\overset{\xi \perp\!\!\!\perp X_1}{=} P\{|X_1| \le x\} \cdot 1/2 + P\{\underbrace{|X_2| \ge -x}_{1}\} \cdot \frac{1}{2}$$

$$= \frac{1}{2}\left[1 + P\{|X_1| \le x\}\right] = P\{X_1 \le x\}$$

2) $\underbrace{\mathrm{cov}(X_1 X_2)}_{\parallel} = 0$

$$\mathbb{E}[X_1 X_2] - \mathbb{E}X_1\, \mathbb{E}X_2 = \mathbb{E}[X_1 |X_1| \xi]$$

$$= \mathbb{E}[X_1 |X_1|]\, \underbrace{\mathbb{E}\xi}_{= 0} - 0 \cdot \mathbb{E}X_2$$

$$= 0 - 0 = 0 \quad \square$$

3) X_1, X_2 are dependent

Assume $X_1 \perp\!\!\!\perp X_2 \Rightarrow (X_1 X_2)$ - Gaussian

$$\gamma = X_1 - X_2 = X_1 - |X_1|\xi \sim N$$

$\{\gamma > 0\}$ when $X_1 > 0$ and $\xi = -1$

$$P\{\gamma > 0\} \ge P\{X_1 > 0 \cap \xi = -1\} = P\{X_1 > 0\} P\{\xi = -1\}$$

$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$P\{\gamma = 0\} \ge 1/4$ since $\mathrm{var}(\gamma) \ne 0$

Thus γ is not Gaussian

4.4 Definition of a Gaussian process. Covariance function (1)

Def: A Gaussian process X_t is a stochastic process s.t.
$\forall t_1, t_2, \dots, t_n$: $(X_{t_1}, \dots, X_{t_n})$ – Gaussian vector

$m(t) = \mathbb{E}X_t$ - mathematical expectation $m: \mathbb{R}_+ \to \mathbb{R}$

$K(t,s) = \mathrm{cov}(X_t, X_s)$, $K: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$

$K(t,t) = \mathrm{Var}\, X_t$ and $\boxed{K(t,s) = K(s,t)}$

K - a positive semidefinite function (p.s.d.)

$\forall (t_1, \dots, t_n) \in \mathbb{R}_+^n$,

$\forall (u_1, \dots, u_n) \in \mathbb{R}^n$

$$\boxed{\sum_{k=1}^{n} \sum_{j=1}^{n} u_k u_j K(t_k, t_j) \geq 0}$$

$$\Leftrightarrow \mathrm{cov}\left(\sum_{k=1}^{n} u_k X_{t_k}, \sum_{j=1}^{n} u_j X_{t_j}\right) = \mathrm{var}\left(\sum_{k=1}^{n} u_k X_{t_k}\right) \geq 0$$

4.5 Definition of a Gaussian process. Covariance function (2)

Thm: $m: \mathbb{R}_+ \to \mathbb{R}$, $K: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, K - symmetric and p.s.d.
Then, $\exists$ Gaussian process X_t: $\boxed{\mathbb{E}X_t = m(t), \ \mathrm{cov}(X_t, X_s) = K(t,s)}$

Gaussian r.v.: $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$
Gaussian vector: $\vec{\mu} \in \mathbb{R}^n$, $C \in \mathrm{Mat}(n,n)$ and symmetric and p.s.d.

Gaussian process: $m: \mathbb{R}_+ \to \mathbb{R}$, $K: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ (sym. and p.s.d.)

Ex: $K(t,s) = |t-s|$ is not p.s.d.

assume p.s.d. $\Rightarrow \exists X_t: \mathrm{cov}(X_t, X_s) = |t-s|$

let $t = s$, $\mathrm{var}\, X_t = 0 \Rightarrow X_t = f(t)$ - deterministic

$\mathrm{cov}(X_t, X_s) = \mathbb{E}[X_t X_s] - \mathbb{E}X_t \mathbb{E}X_s = f(t)f(s) - f(t)f(s) = 0 \neq |t-s|$

Contradiction $\square$

Ex $K(t,s) = \min(t,s)$ is p.s.d.

$$\sum_{k,j=1}^{n} u_j u_k \min(t_j, t_k) \geq 0$$

$$f_t(x) = \mathbb{1}\{x \in [0,t]\}$$

$$\int_0^\infty \underbrace{f_t(x) f_s(x)}_{\substack{\shortparallel \\ 1 \text{ iff } x\in[0,t],\ x\in[0,s] \Leftrightarrow x\in[0,\min(t,s)]}} dx = \min(t,s)$$

$$\sum_{j,k=1}^{n} u_j u_k \int_0^\infty f_{t_j}(x) f_{t_k}(x) dx = \int_0^\infty \sum_{j,k=1}^{n} u_j u_k f_{t_j}(x) f_{t_k}(x) dx$$

$$= \int_{\mathbb{R}_+} \sum_{j=1}^{n} u_j f_{t_j}(x) \sum_{k=1}^{n} u_k f_{t_k}(x) dx = \int_{\mathbb{R}_+} \left(\sum_{j=1}^{n} u_j f_{t_j}(x) \right)^2 dx \geq 0$$

Then, $\boxed{K(t,s) = \min(t,s) \text{ is p.s.d.}}$ Δ

4.6 Two definitions of a Brownian motion

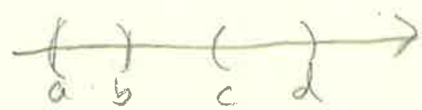
Brownian motion = Wiener process
$(B_t = W_t)$

Def 1: B_t - a Gaussian process with $m(t) = 0$, $K(t,s) = \min(t,s)$

Def 2: (0) $B_0 = 0$ a.s.
(1) B_t independent increments $[B_t \sim N(0,t)]$
(2) $B_t - B_s \sim N(0, t-s)$, $t > s \geq 0$

Def 1 $\Rightarrow$ Def 2

(0) $$\left.\begin{array}{l} \mathbb{E} B_0 = m(0) = 0 \\ \mathrm{Var}\, B_0 = K(0,0) = 0 \end{array}\right\} \Rightarrow B_0 = 0 \text{ a.s.}$$

(1)


$B_b - B_a \perp\!\!\!\perp B_d - B_c$

$$\mathrm{cov}(B_b - B_a, B_d - B_c) = \mathrm{cov}(B_b, B_d) - \mathrm{cov}(B_a, B_d) - \mathrm{cov}(B_b, B_c) + \mathrm{cov}(B_a, B_c)$$
$$= \min(b,d) - \min(a,d) - \min(b,c) + \min(a,c)$$
$$= b - a - b + a = 0$$

$$\lambda_1(B_b - B_a) + \lambda_2(B_d - B_c) = \lambda_1 B_b - \lambda_1 B_a + \lambda_2 B_d - \lambda_2 B_c \sim N(\cdot,\cdot)$$

$\Rightarrow \begin{bmatrix} B_b - B_a \\ B_d - B_c \end{bmatrix}$ - Gaussian vector

$\Rightarrow B_b - B_a \perp\!\!\!\perp B_d - B_c$

(2) (B_t, B_s) - Gaussian vector $\Rightarrow B_t - B_s \sim N$

$$E[B_t - B_s] = \mathbb{E}B_t - \mathbb{E}B_s = m(t) - m(s) = 0 - 0 = 0$$

$$\begin{aligned} Var[B_t - B_s] &= cov(B_t - B_s, B_t - B_s) = cov(B_t, B_t) - 2cov(B_t, B_s) + cov(B_s, B_s) \\ &= \min(t,t) - 2\min(t,s) + \min(s,s) \\ &= t - 2s + s = t - s \; \square \end{aligned}$$

<u>Def 2 $\Rightarrow$ Def 1</u> $\quad t_1 < t_2 < \cdots < t_n$

$$\sum_{k=1}^{n} \lambda_k B_{t_k} = \lambda_n(B_{t_n} - B_{t_{n-1}}) + (\lambda_n + \lambda_{n-1})B_{t_{n-1}} + \sum_{k=1}^{n-2} \lambda_k B_{t_k}$$

$$= \sum_{k=1}^{n} d_k (B_{t_k} - B_{t_{k-1}}) \sim N \qquad (t_0 = 0)$$

$\Rightarrow (B_{t_1}, \ldots, B_{t_n})$ - Gaussian vector $\Rightarrow B_t$ - Gaussian

$B_t \sim N(0,t) \Rightarrow m(t) = \mathbb{E}B_t = 0 \qquad t > s$

$$\begin{aligned} K(t,s) = cov(B_t, B_s) &= cov(B_t - B_s + B_s, B_s) \\ &= cov(B_t - B_s, B_s) + cov(B_s, B_s) \\ &= \underbrace{cov(B_t - B_s, B_s - B_0)}_{=0} + cov(B_s, B_s) \\ &= var(B_s) = s \end{aligned}$$

If $s > t$, $K(t,s) = t$

$\therefore K(t,s) = \min(t,s) \; \square$

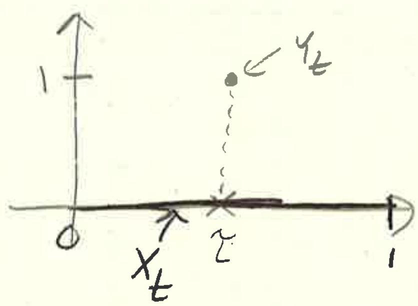
4.7 Modification of a process. Kolmogorov continuity theorem

Kolmogorov continuity theorem

Def X_t, Y_t are stochastically equivalent if
$$P\{X_t = Y_t\} = 1 \quad , \forall t \geq 0$$

Ex: $X_t = 0 \ , \forall t \in [0,1]$
$$Y_t = \mathbb{1}\{\tau = t\} \ , \tau \sim Unif(0,1)$$

$$P\{X_t = Y_t\} = P\{Y_t = 0\} = P\{t \neq \tau\} = 1$$



Thm If $\exists\, C, \alpha, \beta > 0$ s.t. $\mathbb{E}\left[|X_t - X_s|^\alpha\right] \leq C\,|t-s|^{1+\beta}$ $\forall\, t,s \in [a,b]$, then $\exists\, Y_t$ that is stochastically equivalent to X_t s.t. Y_t has continuous trajectories, i.e., X_t has a continuous modification.

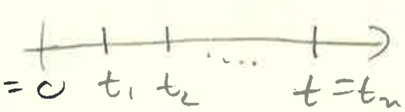
Ex: $$\mathbb{E}\big[|\underbrace{B_t - B_s}_{N(0,\,t-s)}|^4\big] = (t-s)^2 \underbrace{\mathbb{E}\,\xi^4}_{3} = 3(t-s)^2$$

$$B_t - B_s \overset{d}{=} \sqrt{t-s}\ \xi, \ \xi \sim N(0,1)$$

$$\Rightarrow C = 3,\ \beta = 1,\ \alpha = 4$$

4.8 Main properties of Brownian Motion

1) Quadratic variation



$$\lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} (B_{t_k} - B_{t_{k-1}})^2}_{S_n} = t$$

$\max |t_i - t_{i-1}| \to 0$ as $n \to \infty$

$$\mathbb{E}(S_n - t)^2 \xrightarrow[n\to\infty]{} 0 \iff \boxed{\lim_{n\to\infty} \sum_{k=1}^{\infty} (B_{t_k} - B_{t_{k-1}})^2 = t}$$ quadratic variation

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} |B_{t_k} - B_{t_{k-1}}| = \infty}$$ variation

(2) B_t - everywhere continuous, but nowhere differentiable

$$B_{t+h} \xrightarrow[h\to 0]{P} B_t, \quad \forall t \geq 0$$

(3) $\lim_{t\to\infty} \frac{B_t}{t} = 0$ a.s.

$\lim_{t\to\infty} \frac{B_t}{\sqrt{t}} = \infty$ a.s.

Law of iterated logarithm : $\boxed{\lim_{t\to\infty} \frac{B_t}{\sqrt{2t \log(\log t)}} = 1}$

5.1 Two types of stationarity (1)

Stationarity

Def 1: X_t is (strictly) stationary if (S.S.)

$$(X_{t_1+h}, \dots, X_{t_n+h}) \overset{d}{=} (X_{t_1}, \dots, X_{t_n}), \ \forall (t_1, \dots, t_n) \in \mathbb{R}_+^n, \ \forall h > 0$$

Def 2: X_t is (weakly) stationary if (wide-sense stationarity) (W.S.)

1. $m(t) = \mathbb{E}X_t = \text{const}$
2. $K(t,s) = cov(X_t, X_s) = k(t+h, s+h)$, $\forall (t,s) \in \mathbb{R}_+^2$, $\forall h > 0$

$\iff \gamma : \mathbb{R} \to \mathbb{R} : K(t,s) = \gamma(t-s)$

Properties of $\gamma(\cdot)$:

1) $\boxed{\gamma(0) \geq 0}$

$\gamma(0) = cov(X_t, X_t) = Var(X_t)$

2) $\boxed{|\gamma(t)| \leq \gamma(0)}$

$$|\gamma(t)| = |cov(X_t, X_0)| \leq \sqrt{Var X_t} \cdot \sqrt{Var X_0} \quad \text{Cauchy-Schwartz}$$
$$= \sqrt{\gamma(0)} \cdot \sqrt{\gamma(0)} = \gamma(0)$$

3) $\boxed{\gamma \text{ is even}}$

$$\gamma(t) = cov(X_t, X_0) = cov(X_0, X_t) = \gamma(-t)$$

Ex:

(i) $\mathbb{E}X_t^2 < \infty$

X_t is strictly stationary $\Rightarrow$ X_t is weakly stationary

(ii) X_t - Gaussian process $\Rightarrow$ X_t - S.S. $\iff$ X_t - W.S.

① White noise process: X_t, $t = 0, \pm 1, \pm 2, \pm 3, \dots$

$\boxed{\mathbb{E}X_t = 0}$, $Var X_t = \sigma^2$ $\quad [WN(0, \sigma^2)]$

$K(t,s) = 0$, if $t \neq s$

$K(t,s) = \sigma^2 \mathbb{1}\{t=s\} = \gamma(t-s)$, $\boxed{\gamma(x) = \sigma^2 \mathbb{1}\{x=0\}}$ $\Rightarrow$ W.S.

a) $X_1, X_2, \dots$ - i.i.d. noise
b) X_t - Gaussian
$\Big\}$ - X_t - S.S.

(2) Random walk

$S_n = S_{n-1} + \xi_n$, $\xi_1, \xi_2, \dots$ - i.i.d. $\begin{cases} 1, & \text{w.p. } p \\ -1, & \text{w.p. } 1-p \end{cases}$

$S_0 = 0$

$S_n = \xi_1 + \dots + \xi_n$

$\mathbb{E} S_n = n \mathbb{E} \xi_1 = n(2p-1) \Rightarrow$ not W.S. $\Rightarrow$ not S.S.

a) $p = 1/2 \Rightarrow \mathbb{E} S_n = 0$

$K(n,m) = \operatorname{cov}(S_m + \xi_{m+1} + \dots + \xi_n, S_m)$ $(n > m)$

$= \operatorname{cov}(S_m, S_m) + \underbrace{\operatorname{cov}(\xi_{m+1} + \dots + \xi_n, S_m)}_{=0}$

$\operatorname{Var} S_m = m \operatorname{Var} \xi_1 = \min(n,m) \operatorname{Var} \xi_1$

$\Rightarrow$ not w.s. $\Rightarrow$ not S.S.

(3) Brownian motion

$\mathbb{E} B_t = 0$

$\operatorname{Var} B_t = t$ $\quad [B_t - B_s \sim N(0, t-s)]$, $B_s = 0$, $s = 0$, $t - s = t$

$\operatorname{Var} B_t = t \neq \gamma(0) \Rightarrow$ not W.S. $\Rightarrow$ not w.s.

$K(t,s) = \min(t,s)$, $t > s$

$K(t+h, s+h) = s+h \neq s$

5.2 Two types of stationarity (2)

(4) Moving average process : $MA(q)$

$\boxed{Y_t = a_0 X_t + a_1 X_{t-1} + \dots + a_q X_{t-q}}$ $(a_1, \dots, a_q) \in \mathbb{R}^q$; $a_0 = 1$

$X_t \sim WN(0, \sigma^2)$

$\mathbb{E} Y_t = 0$

$K(t,s) = \operatorname{cov}\left(\sum_{j=0}^{q} a_j X_{t-j}, \sum_{k=0}^{q} a_k X_{t-k}\right)$

$$= \sum_{j=0}^{q}\sum_{k=0}^{q} a_j a_k \operatorname{cov}(X_{t-j}, X_{s-k}) = \sum_{j=0}^{q}\sum_{k=0}^{q} a_j a_k \sigma^2 \mathbb{1}\{t-s=j-k\}$$

MA(1): $\gamma(x) = C\,\mathbb{1}\{|x|=1\} + D\,\mathbb{1}\{x=0\}$

$\boxed{MA(q) \text{ is W.S.}}$

⑤ Autoregressive AR(p)

$$\boxed{Y_t = b_1 Y_{t-1} + \dots + b_p Y_{t-p} + \varepsilon_t}$$

ε_t ~ $WN(0,\sigma^2)$

$\operatorname{cov}(\varepsilon_t, Y_s) = 0$, $\forall t > s$

AR(1): $Y_t - bY_{t-1} = \varepsilon_t$, $b \in \mathbb{R}$

$$\boxed{Y_t = \sum_{j=0}^{\infty} b^j \varepsilon_{t-j}}, \quad \mathbb{E}Y_t = 0$$

$$K(t,s) = \sum_{j,k=0}^{\infty} b^{j+k}\sigma^2 \mathbb{1}\{t-s=j-k\}$$

If $t-s=0 \Rightarrow K(t,s) = \sum_{k=0}^{\infty} b^{2k} < \infty \Leftrightarrow |b|<1$

If $|b|<1$, Y_t — W.S.

5.3 Spectral density of a wide-sense stationary process (1)

Spectral density

Bochner-Khintchine theorem

$\phi(u)$ - char fcn $\Leftrightarrow$ 1) ϕ - continuous

$\phi: \mathbb{R} \to \mathbb{C}$ 2) ϕ - positive semi-definite $\sum_{j,k=1}^{n} z_j \bar{z}_k \phi(u_j - u_k) \geq 0$

$\forall (z_1, \dots, z_n) \in \mathbb{C}^n$,

$\forall (u_1, \dots, u_n) \in \mathbb{R}^n$

$\phi(u) = \mathbb{E}e^{iu\xi}$

3) $\phi(0) = 1$

1), 2) properties only $\Rightarrow \exists \mu: \phi(u) = \int e^{iux}\mu(dx)$

1), 2), $\int |\phi(u)|du < \infty \Rightarrow \phi(u) = \int e^{iux} s(x)\,dx$

X_t - w.s. $\Rightarrow \gamma: K(t,s) = \gamma(t-s)$

If γ is continuous, $\int |\gamma(u)|\, du < \infty$ — Fourier transform

$\Rightarrow \exists g(x): \gamma(u) = \int e^{iux} g(x)\, dx = \mathcal{F}[g](u)$

$$\boxed{g(x) = \frac{1}{2\pi} \int e^{-iux} \gamma(u)\, du}$$ – spectral density

$$\boxed{g(x) = \frac{1}{2\pi} \sum_{h=-\infty}^{+\infty} e^{-ihx} \gamma(h)}$$ – discrete time spectral density

5.4: Spectral density of a wide-sense stationary process (2)

Ex 1) $WN(0,\sigma^2) \Rightarrow \gamma(u) = \sigma^2 \mathbb{1}\{u=0\}$

$g(x) = \frac{\sigma^2}{2\pi}$

2) MA(1)

$$\gamma(u) = \begin{cases} 0, & |u|>1 \\ a\sigma^2, & |u|=1 \\ a(1+a^2)\sigma^2, & u=0 \end{cases}$$

$$g(x) = \frac{\sigma^2}{2\pi}\left(1 + a^2 + 2a\cos(x)\right)$$

Prop: A real-valued function $g(x)$ on $[-\pi,\pi]$ is a spectral density of a stochastic process X_t iff

1) $g(x) \geq 0$
2) g – even
3) $\int_{-\pi}^{\pi} g(x)\, dx < \infty$

5.5 Moving-average filters (1)

Filter: $X_t \to Y_t$

$Y_t = a_0 X_t + a_1 X_{t-1} + \dots + a_n X_{t-n}$

$Y_t = \int_{\mathbb{R}} e^{-\beta(t-s)} X_s\, ds$

1) Linearity: $\left.\begin{matrix} X_t^{(1)} \to Y_t^{(1)} \\ X_t^{(2)} \to Y_t^{(2)} \end{matrix}\right\} \Rightarrow c_1 X_t^{(1)} + c_2 X_t^{(2)} \to c_1 Y_t^{(1)} + c_2 Y_t^{(2)}$

2) time-invariance

$$[X_t \to Y_t] \Rightarrow [X_{t+h} \to Y_{t+h}], \ \forall h>0$$

$$\int_{\mathbb{R}} e^{-\beta(t-s)} X_{s+h}\, ds = \int_{\mathbb{R}} e^{-\beta((t+h)-(s+h))} X_{s+h}\, ds = Y_{t+h}$$

$$Y_t = \int_{\mathbb{R}} g(s) X_{t-s}\, ds \quad \text{(continuous time)}$$

$$Y_t = \sum_{h=-\infty}^{\infty} g(h) X_{t-h} \quad \text{(discrete time)}$$

5.6 Moving-average filters (2)

Thm $\boxed{X_t - \text{w.s. process}}$ with $EX_t = 0$, $\gamma_X(\cdot)$ and

$$Y_t = \int_{\mathbb{R}} g(s) X_{t-s}\, ds, \text{ then}$$

(i) $\boxed{Y_t - \text{w.s. process}}$

(ii) $\boxed{\gamma_Y(x) = \gamma_X(x) \cdot |\mathcal{F}[g](x)|^2}$

$$\mathcal{F}[g](x) = \int_{\mathbb{R}} e^{iux} g(u)\, du$$

Proof: (i) $\mathbb{E}Y_t = \int_{\mathbb{R}} g(s) \underbrace{\mathbb{E}X_{t-s}}_{=0}\, ds = 0$

$$K_Y(t_1,t_2) = \mathbb{E}\left[\int_{\mathbb{R}} g(s_1) X_{t_1-s_1}\, ds_1 \cdot \int_{\mathbb{R}} g(s_2) X_{t_2-s_2}\, ds_2\right]$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} g(s_1) g(s_2) \underbrace{\mathbb{E}[X_{t_1-s_1} X_{t_2-s_2}]}_{\gamma_X(t_2-t_1-(s_2-s_1))} ds_1\, ds$$

$$\Rightarrow \gamma_Y(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} g(s_1) g(s_2) \gamma_X(x-(s_2-s_1))\, ds_1\, ds_2$$

(ii) $\gamma_Y(x) = \int_{\mathbb{R}} g(s_1) \underbrace{\int_{\mathbb{R}} \gamma_X(x+s_1-s_2) g(s_2)\, ds_2}_{[\gamma_X * g](x+s_1)} ds_1$

$$= \int_{\mathbb{R}} [\gamma_X * g](x+s_1)\, g(s_1)\, ds_1$$

let $g^{\circ}(x) := g(-x)$

$$= \int_{\mathbb{R}} [\gamma_X * g](x-s_1)\, \underbrace{g(-s_1)}_{= g^{\circ}(s_1)}\, ds_1 = \gamma_X * g * g^{\circ}(x)$$

$$\left(g_Y(x) = \frac{1}{2\pi} \mathcal{F}[\gamma_Y](-x) \right)$$

$$\frac{1}{2\pi} \mathcal{F}[\gamma_Y](x) = \frac{1}{2\pi} \mathcal{F}[\gamma_X](x) \cdot \mathcal{F}[g](x) \cdot \mathcal{F}[g^{\circ}](x)$$

$$\Rightarrow \underbrace{\frac{1}{2\pi} \mathcal{F}[\gamma_Y](-x)}_{= g_Y(x)} = \underbrace{\frac{1}{2\pi} \mathcal{F}[\gamma_X](-x)}_{= g_X(x)} \cdot \underbrace{\mathcal{F}[g](-x) \cdot \mathcal{F}[g^{\circ}](-x)}_{\text{complex conjugates}}$$

5.7 *Moving-average filters (3)*

X_n - W.S. (discrete time), g_X

$Y_n = a_1 X_{n-1} + a_2 X_{n-2}$. What are a_1, a_2 such that $\mathbb{E}[(X_n - Y_n)^2]$ is minimum?

$Z_n = X_n - Y_n = X_n - a_1 X_{n-1} - a_2 X_{n-2} \Rightarrow Z_n$ - W.S. from thm

$g_Z(x) = g_X(x) |\mathcal{F}[g](x)|^2$ from thm

$$g(x) = \mathbb{1}\{x=0\} - a_1 \mathbb{1}\{x=1\} - a_2 \mathbb{1}\{x=2\}$$

$$\mathcal{F}[g](x) = 1 - a_1 e^{ix} - a_2 e^{2ix}$$

$$\underbrace{\operatorname{Var} Z_n}_{=} \rightarrow \min_{a_1, a_2}$$

$$K_Z(n,n) = \gamma_Z(0) = \int_{\mathbb{R}} \underbrace{e^{i \cdot 0 \cdot x}}_{=1} g_Z(x)\, dx$$

$$= \int_{\mathbb{R}} g_X(x) \underbrace{|1 - a_1 e^{ix} - a_2 e^{2ix}|^2}_{= (1 - a_1 e^{ix} - a_2 e^{2ix})(1 - a_1 e^{-ix} - a_2 e^{-2ix})} dx$$

$$= \sum_{i,j=1}^{2} B_{ij}\, a_i a_j + \sum_{i=1}^{2} C_i a_i + D \quad \left(\text{minimized w.r.t. } a_1, a_2 \right) / D$$

6.1 Notion of ergodicity. Examples

Ergodicity

Law of Large Numbers (LLN): $\xi_1, \xi_2, \ldots$ - i.i.d.

$$\Rightarrow \frac{1}{N}\sum_{n=1}^{N} \xi_n \xrightarrow[n\to\infty]{P} \mathbb{E}\xi_1 \quad \text{if } \mathbb{E}\xi_1^2 < \infty \qquad \text{(Classical)}$$

$$\text{If } \mathbb{E}\xi_1 < \infty \Rightarrow \frac{1}{N}\sum_{n=1}^{N} \xi_n \xrightarrow[n\to\infty]{P} \mathbb{E}\xi_1 \qquad \text{(Khinchine)}$$

$$\text{SSLN: If } \mathbb{E}\xi_1 < \infty \Rightarrow \frac{1}{N}\sum_{n=1}^{N} \xi_n \xrightarrow[n\to\infty]{a.s.} \mathbb{E}\xi_1$$

X_t - discrete time stochastic process, $t = 1, 2, 3, \ldots$ (not iid)

$$\frac{1}{T}\sum_{t=1}^{T} X_t \xrightarrow[T\to\infty]{\mathbb{P}} \text{constant} \Rightarrow X_t \text{ is ergodic}$$

$$\xi_n \xrightarrow{a.s.} \xi \iff P\{\omega : \xi_n(\omega) \to \xi(\omega)\} = 1 \text{ as } n \to \infty$$

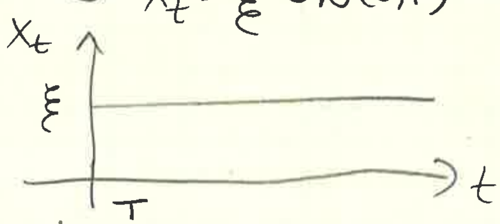
$$\xi_n \xrightarrow{\mathcal{L}^2 \text{ (or mean squared)}} \xi \iff \mathbb{E}(\xi_n - \xi)^2 \to 0 \text{ as } n \to \infty$$

$$\xi_n \xrightarrow{P} \xi \iff \forall \varepsilon > 0,\ P\{|\xi_n - \xi| > \varepsilon\} \to 0 \text{ as } n \to \infty$$

$$\xi_n \xrightarrow{d} \xi \iff \mathbb{P}\{\xi_n \le x\} \to \mathbb{P}\{\xi \le x\} \text{ as } n \to \infty,\ \forall x \in \mathbb{R} \text{ - point}$$
of continuity of $\mathbb{P}\{\xi \le x\}$ (also called weak convergence)

$$\text{a.s.} \Rightarrow P \Rightarrow d$$
$$\mathcal{L}^2 \Rightarrow P, \qquad d \to \text{constant a.s.} \Rightarrow P$$
(weak convergence to a constant a.s.)

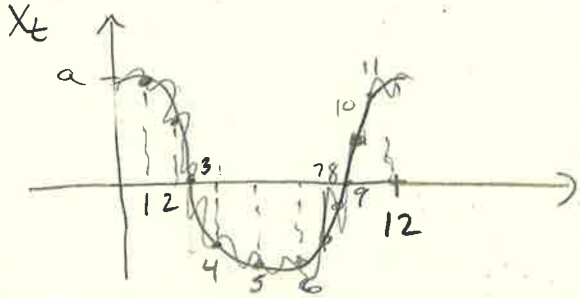
Ex ① $X_t = \xi \sim N(0,1)$



$m(t) = 0$
$K(t,s) = \operatorname{Var}\xi = 1$ $\Rightarrow$ stationary

$$\frac{1}{T}\sum_{t=1}^{T} X_t = \xi \neq C \Rightarrow \text{non-ergodic}$$

② $X_t = \varepsilon_t + a\cos\frac{\pi t}{6}$, $a \neq 0$, $\varepsilon_1, \varepsilon_2, \dots$ - i.i.d. $N(0,1)$



$m(t) = a\cos\frac{\pi t}{6} \neq$ const

$\Rightarrow$ not stationary

$$\frac{1}{T}\sum_{t=1}^{T} X_t \sim N\left(\frac{a}{T}\sum_{t=1}^{T}\cos\frac{\pi t}{6}, \ \frac{1}{T}\right)$$

$\frac{1}{T} \xrightarrow{T\to\infty} 0$

pt 1 = − pt 5
pt 2 = − pt 4
pt 3 = 0
⋮

first 3 elements

$$\Rightarrow \left|\frac{a}{T}\sum_{t=1}^{T}\cos\frac{\pi t}{6}\right| \le \frac{a}{T}\cdot 3 \to 0 \text{ as } T\to\infty$$

$\Rightarrow$ ergodic

6.2 Ergodicity of wide-sense stationary processes

Proposition: X_t - discrete time s.p. ; $\exists \alpha$ s.t. $|K(s,t)| < \alpha$

$C(T) := \mathrm{cov}(X_T, M_T)$, $M_T := \frac{1}{T}\sum_{t=1}^{T} X_t$

$$\Rightarrow \boxed{\mathrm{Var}\, M_T \xrightarrow[T\to\infty]{} 0 \iff C(T) \xrightarrow[T\to\infty]{} 0}$$

Corollary: X_t - w.s. ; $\gamma(\cdot)$ - autocovariance function

(i) $\frac{1}{T}\sum_{r=0}^{T-1}\gamma(r) \xrightarrow[T\to\infty]{} 0 \Rightarrow X_t$ - ergodic

(ii) $\gamma(r) \xrightarrow[r\to\infty]{} 0 \Rightarrow X_t$ - ergodic

Proof (i) $\mathbb{E}X_t = c \Rightarrow \mathbb{E}M_T = c$

$$\mathrm{Var}\, M_T = \mathbb{E}\left[(M_T - c)^2\right] \overset{?}{\to} 0 \Rightarrow M_T \xrightarrow{\mathcal{L}^2} c \Rightarrow M_T \xrightarrow{P} c$$

$\Downarrow$

X_t ergodic

$$C(T) = \mathrm{cov}\left(X_T, \frac{1}{T}\sum_{t=1}^{T} X_t\right)$$

$$= \frac{1}{T}\sum_{t=1}^{T}\mathrm{cov}(X_T, X_t) = \frac{1}{T}\sum_{t=1}^{T}\gamma(T-t) = \frac{1}{T}\sum_{r=0}^{T-1}\gamma(r) \xrightarrow[T\to\infty]{} 0$$

$\square$

proof (ii)

Stolz-Cesaro thm: a_n, b_n - sequences in $\mathbb{R}$

b_n - strictly increasing & unbounded

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = g \Rightarrow \frac{a_n}{b_n} \xrightarrow[n\to\infty]{} g$$

$$a_n = \sum_{r=0}^{n-1} \gamma(r) \ ; \ b_n = n$$

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \frac{\gamma(n-1)}{1} \to 0 = g$$

$$\Rightarrow \frac{1}{n} \sum_{r=0}^{n-1} \gamma(r) \xrightarrow[n\to\infty]{} 0 \Rightarrow X_t \text{ - ergodic} \quad \square$$

ex N_t - poisson process, λ.

$p > 0$: $X_t = N_{t+p} - N_t$

$\mathbb{E}X_t = \lambda(t+p) - \lambda t = \lambda p$

$K(t,s) = \gamma(t-s)$, where $\gamma(r) = \begin{cases} \lambda(p - |r|), & |r| \le p \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow \gamma(r) \xrightarrow[r\to\infty]{} 0 \overset{(ii)}{\Longrightarrow} X_t \text{ - ergodic}$$

ex $X_t = A\cos(\omega t) + B\sin(\omega t)$, A, B - r.v., $\omega = \frac{\pi}{20}$

$\operatorname{cov}(A,B) = 0$

$\mathbb{E}A = \mathbb{E}B = 0$, $\operatorname{Var} A = \operatorname{Var} B = 1$

$$\left.\begin{array}{l} \mathbb{E}X_t = 0 \\ K(t,s) = \cos(\omega(t-s)) \end{array}\right\} X_t \text{ - w.s.}$$

$\gamma(r) = \cos \omega r$

$$\left| \frac{1}{T} \sum_{r=0}^{T-1} \cos \omega r \right| \le \frac{10}{T} \to 0 \text{ as } T \to \infty$$

$\Rightarrow X_t$ - ergodic

A s.p. is stationary $\Leftrightarrow$ A - ergodic

This is incorrect. Ergodic implies stationary only.

6.3. Definition of a stochastic derivative

Stochastic derivative: X_t differentiable at $t=t_0$ if

$$\frac{X_{t_0+h}-X_{t_0}}{h} \xrightarrow[h\to 0]{\mathcal{L}^2} \eta =: X'_{t_0}$$

$$\mathbb{E}\left(\frac{X_{t_0+h}-X_{t_0}}{h}-\eta\right)^2 \xrightarrow[h\to 0]{} 0$$

Prop: $\mathbb{E}X_t^2<\infty$. Then X_t - differentiable at $t=t_0$ $\iff$

$$\begin{cases} m(t)=\mathbb{E}X_t \text{ - differentiable at } t=t_0 \\ \frac{\partial^2}{\partial t\,\partial s}K(t,s)\ \exists \text{ at } (t_0,t_0) \end{cases}$$

Ex ① X_t - w.s. $\Rightarrow m(t)=\text{const}$, $K(t,s)=\gamma(t-s)$

$$\left.\frac{\partial^2 K}{\partial t\,\partial s}\right|_{(t_0,t_0)} = -\gamma''(0)$$

Thus w.s - differentiable $\iff$ $\gamma''(0)$ exists

If $\gamma(r)=e^{-\alpha|r|}$ $\Rightarrow X_t$ - not differentiable

If $\gamma(r)=\cos(\alpha r)$ $\Rightarrow X_t$ - differentiable $\forall t$

② Brownian Motion is not differentiable at any $t=t_0$

$K(t,s)=\min(t,s)$

$$\frac{K(t_0+h,t_0)-K(t_0,t_0)}{h} = \frac{\min(t_0,t_0+h)-t_0}{h} = \begin{cases} 0, & h>0 \\ 1, & h<0 \end{cases}$$

$$\Rightarrow \lim_{h\to 0}\frac{K(t_0+h,t_0)-K(t_0,t_0)}{h} \text{ doesn't exist}$$

③ X_t - independent increments, $X_0=0$ a.s.

$K(t,s)=\operatorname{Var} X_{\min(t,s)}$

Most of the time, not differentiable

$$K(t,s) = cov(X_t, X_s) \overset{t>s}{=} \underbrace{cov(X_t - X_s, \underset{X_s - X_0}{X_s})}_{=0} + \underbrace{cov(X_s, X_s)}_{=Var X_s}$$

$\Rightarrow$ in general, $K = Var X_{min(t,s)}$

6.4 Relation between differentiability and properties of the covariance function

Continuity in the mean-squared sense if $X_t \xrightarrow[t \to t_0]{\mathcal{L}^2} X_{t_0}$

$\Leftrightarrow \mathbb{E}(X_t - X_{t_0})^2 \xrightarrow[t \to t_0]{} 0$.

Let $\mathbb{E}X_t = 0 \Rightarrow$ Prop: (i) $K(t,s)$ is continuous at (t_0, t_0)

$\Downarrow$

X_t is cont in the m.s. at $t = t_0$

(ii) X_t is cont in m.s. sense at $t = t_0$ and $t = s_0$

$\Downarrow$

$K(t,s)$ is cont. at (t_0, s_0)

proof (i) $\mathbb{E}(X_t - X_{t_0})^2 = \mathbb{E}X_t^2 - 2\mathbb{E}X_t X_{t_0} + \mathbb{E}X_{t_0}^2$

$= K(t,t) - 2K(t,t_0) + K(t_0,t_0) \xrightarrow[t \to t_0]{} 0$

(ii) $K(t,s) - K(t_0,s_0) = K(t,s) + K(t_0,s) - K(t_0,s) - K(t_0,s_0)$

$= (K(t,s) - K(t_0,s)) + (K(t_0,s) - K(t_0,s_0))$

$|K(t,s) - K(t_0,s)| = |\mathbb{E}[(X_t - X_{t_0})X_s]| \leq \sqrt{\mathbb{E}(X_t - X_{t_0})^2}\sqrt{\mathbb{E}X_s^2} \xrightarrow[t \to t_0]{} 0$

Same for $K(t_0,s) - K(t_0,s_0)$

$\square$

Corollary: $K(t,s)$ is continuous at $(t_0,s_0) \Leftrightarrow K(t,s)$ is cont at (t_0,t_0)
(holds for any covariance function)

proof: $K(t,s)$ is cont at $(t_0,t_0) \overset{(i)}{\Rightarrow} X_t$ is cont m.s. at $t = t_0$

$\overset{(ii)}{\Rightarrow} K(t,s)$ is cont at (t_0,s_0)

$\square$

7.1 Different types of stochastic integrals. Integrals of type $\int X_t\,dt$ (1)

Stochastic integration

$$\int_a^b X_t\,dt, \quad \int_a^b f(t)\,dW_t, \quad \int_a^b X_t\,dW_t, \quad \int_a^b X_t\,dH_t$$

$X_t : \Omega \times \mathbb{R}_+ \to \mathbb{R}$

Fix $\omega \Rightarrow \int_a^b X_t(\omega)\,dt = \lim_{\max_k |t_k - t_{k-1}| \to 0} \sum_{k=1}^n X_{t_{k-1}}(\omega)(t_k - t_{k-1})$

$t = a < t_1, \dots < t_n = b$ → limit in m.sq. sense

$$\Rightarrow \quad \mathbb{E}\left(\sum_{k=1}^n X_{t_{k-1}}(\omega)(t_k - t_{k-1}) - \int_a^b X_t(\omega)\,dt\right)^2 \xrightarrow[\max_k |t_k - t_{k-1}| \to 0]{} 0$$

<u>Thm</u>: $m(t)$ - continuous, $K(t,s)$ - cont $\Rightarrow \exists \int_a^b X_t\,dt$

$$\mathbb{E}\int_a^b X_t\,dt \underset{\uparrow}{=} \int_a^b \mathbb{E}X_t\,dt$$

Fubini Thm

$$\mathrm{Var}\int_a^b X_t\,dt = \int_a^b\int_a^b K(t,s)\,dt\,ds$$

7.2 Integrals of type $\int X_t\,dt$ (2)

Stochastic integrals $\int_a^b X_t\,dt$

$$\int_a^b X_t\,dt = \lim_{\max_i |t_i - t_{i-1}| \to 0} \sum_{k=1}^n X_{t_{k-1}}(t_k - t_{k-1})$$

→ limit in mean square sense

$t_0 = a \quad t_1 \quad t_2 \quad \dots \quad b = t_n$

<u>Thm</u>: X_t: Stochastic process with $\mathbb{E}X_t^2 < \infty$

$$\left.\begin{array}{l} m(t) - \text{continuous} \\ K(t,s) - \text{continuous} \end{array}\right\} \Rightarrow \int_a^b X_t\,dt \text{ exists}$$

$K(t,s)$ is continuous $\forall (t_0, s_0) \Leftrightarrow K(t,s)$ is continuous $\forall (t_0, t_0)$

i.e. $K(t,s)$ is continuous on the diagonal

(Special thing true for covariance functions)

1) $K(t,s)$ - cont at (t_0,t_0) $\Rightarrow$ X_t - cont at t_0 in mean square sense

i.e. $\mathbb{E}(X_t - X_{t_0})^2 \xrightarrow[t\to t_0]{} 0$

2) X_t is cont in t_0, s_0 $\Rightarrow$ $K(t_0,s_0)$ is continuous

Properties

1) $\mathbb{E}\left[\int_a^b X_t\,dt\right] = \int_a^b \mathbb{E}X_t\,dt$ (Fubini theorem)

2) $\mathbb{E}\left[\underbrace{\left(\int_a^b X_t\,dt\right)^2}_{\int_a^b\int_a^b X_t X_s\,dt\,ds}\right] = \int_a^b\int_a^b \mathbb{E}[X_t X_s]\,dt\,ds$

3) $Var\left[\int X_t\,dt\right] = \int_a^b\int_a^b K(t,s)\,dt\,ds \overset{\text{symmetry}}{=} 2\int_a^b\int_a^s K(t,s)\,dt\,ds$

7.3 : Integrals of type $\int f(t)\,dW_t$ (1)

$\int_a^b f(t)\,dW_t$, W_t - Brownian motion (Wiener integral)

$f \in \mathcal{L}^2([a,b])$ (Hilbert space) $\iff \int_a^b f^2(x) < \infty$

Def: inner product of f,g:

$$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$$

a) $\langle f,g\rangle = \langle g,f\rangle$

b) $\langle a_1f_1 + a_2f_2, g\rangle = a_1\langle f_1,g\rangle + a_2\langle f_2,g\rangle$

c) $\langle f,f\rangle \geq 0$, $\langle f,f\rangle = 0 \iff f=0$

$f_n \xrightarrow{\mathcal{L}^2} f \iff \langle f_n - f, f_n - f\rangle \xrightarrow[n\to\infty]{} 0$

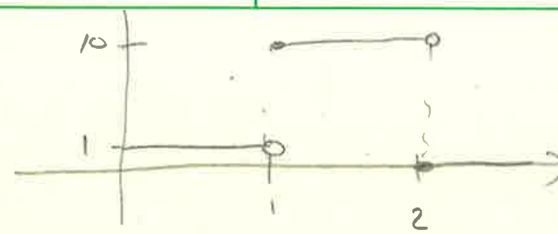
$\iff \int_a^b (f_n(x) - f(x))^2\,dx \to 0$

Stage 1: Step function $\iff f(x) = \sum_{i=1}^n \alpha_i \mathbb{1}\{t_{i-1} \leq x < t_i\}$, $\alpha_i \in \mathbb{R}$

$a = t_0 < t_1 < \ldots < t_n = b$

$$\int_a^b f(t)\,dW_t = \sum_{i=1}^n \alpha_i (W_{t_i} - W_{t_{i-1}})$$

ex $f(t) = \begin{cases} 1 & , 0 \le t < 1 \\ 10 & , 1 \le t < 2 \\ 0 & , t \ge 2 \end{cases}$



$$\int_0^T f(t)\,dW_t = \begin{cases} W_T & , T < 1 \\ W_1 + 10(W_T - W_1) & , 1 \le T < 2 \\ W_1 + 10(W_2 - W_1) & , T \ge 2 \end{cases}$$

Thm: $I(f) := \int_a^b f(t)\,dW_t$. If f - step function $\Rightarrow I(f) \sim N\left(0, \int_a^b f^2(x)\,dx\right)$

Proof:

$$\mathbb{E}[I(f)] = \sum_{i=1}^{n} \alpha_i \underbrace{\mathbb{E}\left[W_{t_i} - W_{t_{i-1}}\right]}_{\overset{\shortparallel}{0}} = 0$$

$$\mathrm{Var}[I(f)] \overset{\text{indep increments}}{=} \sum_{i=1}^{n} \alpha_i^2 \underbrace{\mathrm{Var}\left(W_{t_i} - W_{t_{i-1}}\right)}_{t_i - t_{i-1}} = \int_a^b f^2(x)\,dx$$

$\square$

7.4 Integrals of type $\int f(t)\,dW_t$ (2)

$$I(f) = \int_a^b f(t)\,dW_t \quad , f \in \mathcal{L}^2(a,b)$$

Stage 2: $f \in \mathcal{L}^2(a,b)$

f_n – step functions s.t. $f_n \xrightarrow{\mathcal{L}^2} f$: $\int_a^b (f_n(t) - f(t))^2\,dt \xrightarrow[n\to\infty]{} 0$

$$I(f) = \lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} \int_a^b f_n(t)\,dW_t \qquad \text{(lims in mean squared sense)}$$

$$\mathbb{E}\left[(I(f_n) - I(f))^2\right] \xrightarrow[n\to\infty]{} 0$$

1. Why $I(f)$ does not depend on f_n?
2. Properties of $I(f)$?
3. Construction of f_n?

Thm: $f_n, \tilde{f}_n$ - sequences of step functions, $f_n \xrightarrow{\mathcal{L}^2} f$, $\tilde{f}_n \xrightarrow{\mathcal{L}^2} f$.

$\rightarrow \lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} I(\tilde{f}_n)$ [mean squared limits]

proof $I(f_n) - I(\tilde{f}_n) = I(\underbrace{f_n - \tilde{f}_n}_{\text{step fcn}}) \sim N\left(0, \int_a^b (f_n(x) - \tilde{f}_n(x))^2 dx\right)$

$$\mathbb{E}\left[(I(f_n) - I(\tilde{f}_n))^2\right] = \int_a^b (f_n(x) - \tilde{f}_n(x))^2 dx \xrightarrow[n\to\infty]{} 0 \quad \left(\text{since } f_n \xrightarrow{\mathcal{L}^2} f, \tilde{f}_n \xrightarrow{\mathcal{L}^2} f\right)$$

$\square$

(1) answered

(2) Thm: $\forall f \in \mathcal{L}^2(a,b)$, $I(f) \sim N\left(0, \int_a^b f^2(x)\,dx\right)$

proof

$I(f) = \lim_{n\to\infty} I(f_n)$, $I(f_n) \sim N\left(0, \int_a^b f_n^2(x)\,dx\right)$

For normal r.v.'s, $\mathbb{E}[\lim f_n] = \lim \mathbb{E}[f_n]$, $\mathrm{Var}[\lim f_n] = \lim [\mathrm{Var} f_n]$

$\Rightarrow I(f) \sim N\left(0, \lim_{n\to\infty} \int_a^b f_n^2(x)\,dx\right)$, $\lim_{n\to\infty} \int_a^b f_n^2(x)\,dx = \int_a^b f^2(x)\,dx$

$\square$

7.5 Integrals of type $\int X_t\, dW_t$ (1)

$I(X_t) := \int_a^b X_t\, dW_t$ (W_t - $\mathcal{F}_t$-Brownian motion)

Filtration – a sequence of σ-algebras $\mathcal{F}_t$ on $(\Omega, \mathcal{F}, \mathbb{P})$:

$\mathcal{F}_t \subset \mathcal{F}_s$, $\forall t \le s$

$\mathcal{L}^2_{ad}([a,b], \Omega)$ (ad means adapted)

1) X_t - $\mathcal{F}_t$-adapted, i.e. X_t - $\mathcal{F}_t$-measurable, $\forall t$

$\{X_t \in B\} \subset \mathcal{F}_t$, $\forall t$, $\forall B \in \mathcal{B}(\mathbb{R})$

2) $\int_a^b \mathbb{E} X_t^2\, dt < \infty$

W_t - $\mathcal{F}_t$-Brownian motion if

1) W_t - $\mathcal{F}_t$-adapted

2) $W_t - W_s \perp\!\!\!\perp \mathcal{F}_s$, $\forall t > s$

Define $I(X_t)$:

1) Step processes: $\sum \xi_{i-1} \mathbb{1}\{t_{i-1} \le t < t_i\}$

2) $X_t \in \mathcal{L}^2_{ad}$

7.6: Integrals of the type $\int X_t dW_t$ (2)

$\int_a^b X_t dW_t$, $X_t \in \mathcal{L}^2_{ad}$, W_t - $\mathcal{F}_t$-Brownian motion

Stage 1: $X_t = \sum_{i=1}^n \xi_{i-1} \mathbb{1}\{t_{i-1} \le t < t_i\}$

$$I(X_t) = \sum_{i=1}^n \xi_{i-1}(W_{t_i} - W_{t_{i-1}})$$

Stage 2: $X_t \in \mathcal{L}^2_{ad}$; X_t^n - step processes

$$\int_a^b \mathbb{E}(X_t^n - X_t)^2 dt \xrightarrow[n\to\infty]{} 0$$

$$I(X_t) = \lim_{n\to\infty} I(X_t^n): \quad \mathbb{E}\left(I(X_t^n) - I(X_t)\right)^2 \xrightarrow[n\to\infty]{} 0$$

Thm: $m(t)$ is continuous, $K(t,s)$ is continuous

$$X_t^n = \sum_{i=1}^n X_{t_{i-1}} \mathbb{1}\{t_{i-1} \le t < t_i\}$$

$$X_t^n \longrightarrow X_t: \quad \int_a^b E(X_t^n - X_t)^2 dt \to 0$$

proof: $\mathbb{E}(X_t - X_s)^2 = \mathbb{E}X_t^2 - 2\mathbb{E}X_tX_s + \mathbb{E}X_s^2$

$$= \left[K(t,t) + m^2(t)\right] - 2\left(K(t,s) + m(t)m(s)\right) + \left[K(s,s) + m^2(s)\right] \xrightarrow[s\to t]{} 0$$

$$\Rightarrow X_t^n \xrightarrow[n\to\infty]{} X_t \quad (m.sq.)$$

$$\Rightarrow \int_a^b \lim \mathbb{E}(X_t^n - X_t)^2 dt \to 0 \Rightarrow \lim \int_a^b \mathbb{E}(X_t^n - X_t)^2 dt \to 0$$

Dominated convergence thm

Dominated convergence thm

$$\int \lim_{n\to\infty} f(n,t)\,dt = \lim_{n\to\infty} \int f(n,t)\,dt \quad \text{if } \exists M(t): |f(n,t)| \le M(t), \int M(t)\,dt < \infty$$

$$\mathbb{E}(X_t^n - X_t)^2 \le 2\mathbb{E}(X_t^n)^2 + 2\mathbb{E}X_t^2 \qquad [(a-b)^2 \le 2a^2+2b^2]$$

$$\le 4 \max_{t\in[a,b]} \mathbb{E}X_t^2 = 4 \max_{t\in[a,b]} [k(t,t)+m^2(t)]$$

$\square$

(Ex) $$\int_0^t W_s\,dW_s = \lim_{n\to\infty} \sum_{i=1}^n W_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})$$

$$= \lim_{n\to\infty}\Big(-\frac{1}{2}\underbrace{\sum_{i=1}^n (W_{t_i}-W_{t_{i-1}})^2}_{\text{quadratic variation of } W_t \to t} + \frac{1}{2}\underbrace{\sum_{i=1}^n W_{t_i}^2 - W_{t_{i-1}}^2}_{W_t^2}\Big)$$

$$= -\frac{t}{2} + \frac{W_t^2}{2}$$

7.7 Integrals of the type $\int X_t\,dY_t$ where Y_t is an Itô process

$\int_a^b X_t\,dH_t$, H_t - Itô process

$$H_t = H_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s \iff dH_t = b_t\,dt + \sigma_t\,dW_t$$

$\mathcal{F}_t$ - filtration; b_s, σ_s - processes adapted to $\mathcal{F}_s$

W_t - $\mathcal{F}_t$-Brownian motion

H_0 - measurable w.r.t. $\mathcal{F}_0$

X_t: $\int_a^b |X_s b_s| + X_s^2\sigma_s^2\,ds < \infty$, then

$$\int_a^b X_t\,dH_t = \int_a^b b_s X_s\,ds + \int_a^b \sigma_s X_s\,dW_s$$

Thm: H_t - Itô process, $f(t,x)$ - twice continuously differentiable

Then, $$f(t,H_t) = f(0,H_0) + \int_0^t \frac{\partial f}{\partial t}(s,H_s)\,ds + \int_0^t \frac{\partial f}{\partial x}(s,H_s)\,dH_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,H_s)\,\sigma_s^2\,ds$$

(Itô formula)

7.8 Itô's formula

$\int_0^t g(s,W_s)\,dW_s$, f - antiderivative of g w.r.t. 2^{nd} argument

i.e., $\frac{\partial f}{\partial x} = g$ $\quad (\sigma^2 = 1$ for $W_t)$

$H_t = W_t$

$$f(t,W_t) = f(0,\overset{0}{W_0}) + \int_0^t \frac{\partial f}{\partial t}(s,W_s)\,ds + \int_0^t g(s,W_s)\,dW_s + \frac{1}{2}\int_0^t \frac{\partial g}{\partial x}(s,W_s)\,ds$$

$$\int_0^t g(s,W_s)\,dW_s = f(t,W_t) - f(0,0) - \int_0^t \left[\frac{\partial f}{\partial t}(s,W_s) + \frac{1}{2}\frac{\partial g}{\partial x}(s,W_s)\right] ds$$

ex $\int_0^t W_s\,dW_s$: $g(t,x) = x$, $f(t,x) = \frac{1}{2}x^2 + h(t)$ doesn't count

$$\int_0^t W_s\,dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}\int_0^t ds = \frac{1}{2}W_t^2 - \frac{1}{2}t$$

7.9 Calculation of stochastic integrals using the Itô formula. Black-Scholes model

Black-Scholes model

$dX_t = X_t\mu\,dt + X_t\sigma\,dW_t$, $\sigma > 0$

$\iff X_t = X_0 + \int_0^t X_s\mu\,ds + \int_0^t X_s\sigma\,dW_s$

(Itô formula in differential form:

$$df(t,H_t) = \frac{\partial f}{\partial t}(t,H_t)\,dt + \frac{\partial f}{\partial x}(t,H_t)\,dH_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,H_t)\,\sigma_t^2\,dt$$)

$f(t,x) = \ln x$, $H_t = X_t$

$$d(\ln X_t) = 0 + \frac{1}{X_t}dX_t - \frac{1}{2X_t^2}(X_t\sigma)^2\,dt$$

$$\Rightarrow d(\ln X_t) = \frac{1}{X_t}[X_t\mu\,dt + X_t\sigma\,dW_t] - \frac{\sigma^2}{2}dt$$

$$= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\,dW_t$$

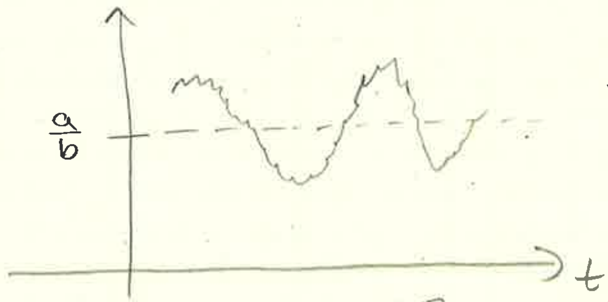
$$\Rightarrow \boxed{X_t = X_0\exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}}$$

X_t - a.s. continuous trajectories

7.10 Vasicek model: Application of the Itô formula to stochastic modeling

Vasicek model

$$dX_t = \underbrace{(a - bX_t)}_{b\left(\frac{a}{b} - X_t\right)}\,dt + c\,dW_t\ , \quad a \in \mathbb{R},\ b, c > 0.$$



b - speed of reversion

$$f(t,x) = xe^{bt} \quad \left(\frac{\partial^2 f}{\partial x^2} = 0\right)$$

$$d(X_t e^{bt}) = \cancel{bX_t e^{bt}\,dt} + e^{bt}\left[(a - \cancel{bX_t})\,dt + c\,dW_t\right] + \frac{1}{2}(0)$$

$$= ae^{bt}\,dt + ce^{bt}\,dW_t$$

$$\Rightarrow \boxed{X_t = e^{-bt}X_0 + \frac{a}{b}\left(1 - e^{-bt}\right) + \int_0^t e^{-bs}\,dW_s}$$

7.11 Ornstein-Uhlenbeck process. Application of the Itô formula to stochastic modelling

Ornstein - Uhlenbeck process

$$m\,dV_t = dW_t - \lambda V_t\,dt\ , \quad \lambda\text{ - friction coefficient},\ m\text{ - mass}$$

V_t - velocity

$$f(t,x) = xe^{\frac{\lambda}{m}t}$$

$$\Rightarrow V_t = e^{-\frac{\lambda}{m}t}\left(V_0 + \frac{1}{m}\int_0^t e^{\frac{\lambda}{m}s}\,dW_s\right)$$

If $V_0 \sim N\left(0, \frac{1}{2\lambda m}\right) \perp\!\!\!\perp W_t \Rightarrow V_t$ - Gaussian process with

$$K(t,s) = \frac{m}{2\lambda}e^{-\frac{\lambda}{m}|t-s|} \quad \text{(stationary in both strict and weak senses)}$$