1.1 Difference between deterministic and stochastic world

	deterministic world	stochastic world
Single variable: Temp of a sick man	$\mathbb{R}$ $T = 39°C$	random variable $\mathbb{E}$, Var, ...
Variables changing over time: T in first 3 days	$\mathbb{R}_+ \to \mathbb{R}$ $T(1) = 39$ $T(2) = 38.5$ $T(3) = 38$ $\vdots$	stochastic process

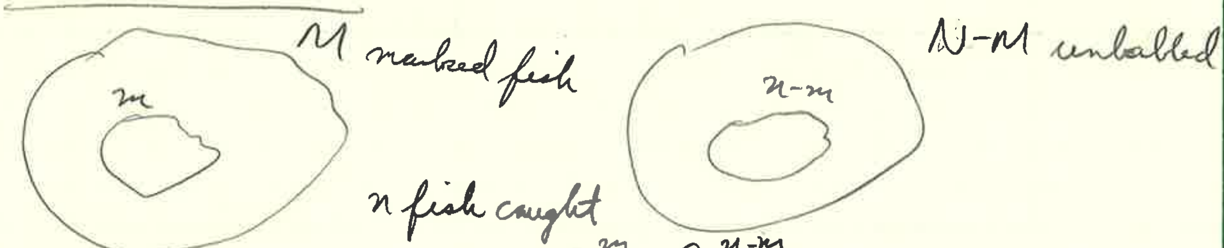
1.2 Difference between various fields of stochastics

Stochastics
- probability theory
- mathematical statistics
- stochastic processes

Consider a pond that contains fish

Prob theory : # of fish at some given time (N)
$\mathbb{E}$, Var, or limit laws

Mathematical Stats :



$$P\{\#\text{ marked} = m\} = \frac{C_M^m \, C_{N-M}^{n-m}}{C_N^M}$$

Repeat $m_1, m_2, \dots, m_q$

$$\left(\begin{matrix}\text{log} \\ \text{likelihood}\end{matrix}\right) \sum_{k=1}^{q} P\{\#\text{ marked} = m_k\} \longrightarrow \max_N \quad (MLE)$$

1.3 Probability space $(\Omega, \mathcal{F}, P)$

General theory	Bernoulli Scheme $\begin{bmatrix} 1, & \text{success} \\ 0, & \text{failure} \end{bmatrix}$ $(a_1, \dots, a_n)$, $a_i \in \{0,1\}$	$[0,1]$ Select point from
Ω - sample space	$\#\Omega = 2^n$, set of all vectors with components $\in \{0,1\}$	$\Omega = [0,1]$
$\mathcal{F}$ - σ-algebra 1) $\Omega \in \mathcal{F}$ 2) $A \in \mathcal{F} \Rightarrow \Omega \setminus A \in \mathcal{F}$ 3) $A_1, \dots, A_n, \dots \in \mathcal{F}$ $\Downarrow$ $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$	$\mathcal{F}$ = power set $\#\mathcal{F} = 2^{\#\Omega} = 2^{2^n}$	$P\{X \in [\alpha, \beta]\}$ $\Rightarrow [\alpha, \beta), (\alpha, \beta]$, $(\alpha, \beta), [\alpha, \beta), \{\beta\} \in \mathcal{F}$ Borel σ-algebra
P - probability measure 1) $P(\Omega) = 1$ 2) $A_1, A_2, \dots \in \mathcal{F}$ (disjoint) $\Rightarrow P\{\bigcup_i A_i\} = \sum_i P(A_i)$ $P: \mathcal{F} \to [0,1]$	$P\{1\} = p$ $P\{0\} = 1 - p$	$P\{[\alpha, \beta]\} = \beta - \alpha$

1.4 Definition of a stochastic function. Types of stochastic functions.

$(\Omega, \mathcal{F}, P)$

Random variable - $\underbrace{\text{measurable}}_{}$ function $\xi: \Omega \to \mathbb{R}$.

$$\forall B \in \mathcal{B}(\mathbb{R}) : \xi^{-1}(B) \subset \mathcal{F}$$

T - time

$X: T \times \Omega \to \mathbb{R}$ - random function, if $\forall t \in T$: $X(t, \cdot)$ is a random variable on $(\Omega, \mathcal{F}, P)$, denoted X_t

If $T = \mathbb{R}_+$, this is called a random process or stochastic process

$T = \mathbb{R}_+^n$, random field or stochastic field

$T = \mathbb{N}$ or $\mathbb{Z}$, discrete time stochastic process

$T = \mathbb{R}_+$ or $\mathbb{R}$, continuous time stochastic process

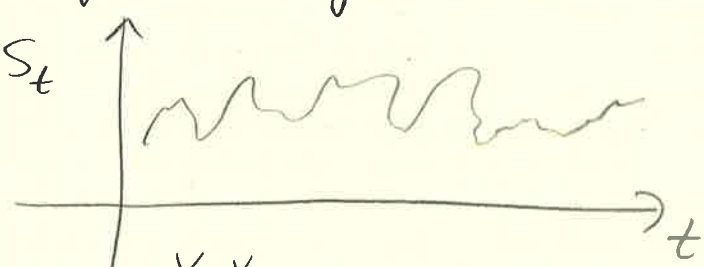
1.5 Trajectories and finite-dimensional distributions

$X : T \times \Omega \to \mathbb{R}$, $T = \mathbb{R}_+$

$\forall t \in T$: $X_t = X(t, \cdot)$ is a r.v. on $(\Omega, \mathcal{F}, P)$

Trajectory (= path)

X_t fix ω and get mapping $T \to \mathbb{R}$



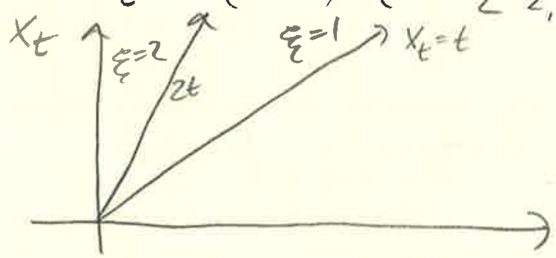
$X_1, X_2, \ldots,$

Finite-dimensional distribution $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$, $t_1, \ldots, t_n \in \mathbb{R}$

In mathematic stats, $X_{t_1}, X_{t_2} \ldots X_{t_n}$ are independent

In stochastic process, $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ are dependent

<u>Ex</u>: $X_t = \xi t$, $\xi = \begin{cases} 1, & \text{w.p. } 1/2 \\ 2, & \text{w.p. } 1/2 \end{cases}$



$$P\{X_{t_1} \le x_1, X_{t_2} \le x_2\} = \begin{cases} 0, & \min\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}\right) < 1 \\ 1/2, & \quad '' \quad \in [1,2] \\ 1, & \quad '' \quad \ge 2 \end{cases}$$

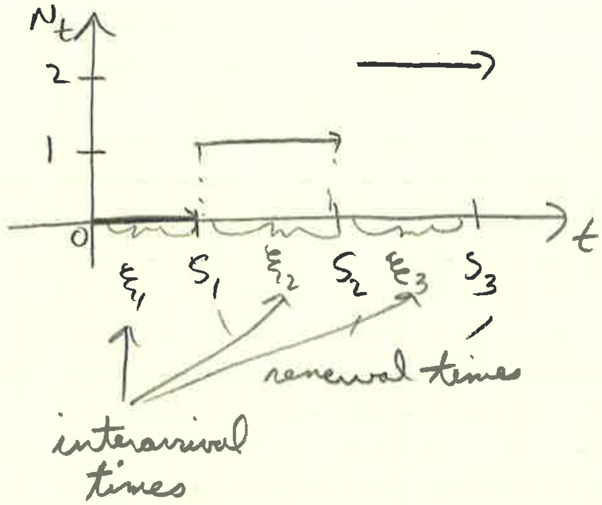
1.6 Renewal process. Counting process.

Renewal processes (discrete time)

$S_0 = 0,\ S_n = S_{n-1} + \xi_n$, where $\xi_1, \xi_2, \ldots$ – iid > 0 a.s.

$P\{\xi_i > 0\} = 1 \iff F(0) = 0$

$N_t = \underset{k}{\operatorname{argmax}} \{S_k \le t\}$ (Counting process)



$\{S_n > t\} = \{N_t < n\}$

$F \Rightarrow \mathbb{E} N_t \qquad S_n = \xi_1 + \cdots + \xi_n$

1.7. Convolution

<u>Convolution</u> $X \perp\!\!\!\perp Y$

$X \sim F_X,\ Y \sim F_Y$

$F_{X+Y}(x) = \int_{\mathbb{R}} F_X(x-y)\, dF(y) =: \underbrace{F_X * F_Y}$ conv in terms of distribution functions

$X \sim p_X,\ Y \sim p_Y$ (If Y, X have densities)

$p_{X+Y}(x) = \int_{\mathbb{R}} p_X(x-y)\, p_Y(y)\, dy =: p_X * p_Y$ } conv in terms of densities

$S_n = \xi_1 + \cdots + \xi_n$

let $F^{n*} := \underbrace{F * \cdots * F}_{n}$

1) $F^{n*}(x) \le F^n(x)$ if $F(0)=0$

$\xi_1, \dots, \xi_n \overset{iid}{\sim} F$

$\{\xi_1 + \dots + \xi_n \le x\} \subset \{\xi_1 \le x, \dots, \xi_n \le x\}$ Since $\xi_i > 0$ a.s.

$$\underbrace{P\{\xi_1 + \dots + \xi_n \le x\}}_{F^{n*}(x)} \le \prod_{k=1}^{n} \underbrace{P\{\xi_k \le x\}}_{F(x)}$$

2) $F^{n*}(x) \ge F^{(n+1)*}(x)$

$\{\xi_1 + \dots + \xi_n \le x\} \supset \{\xi_1 + \dots + \xi_{n+1} \le x\}$

Theorem: $S_n = S_{n-1} + \xi_n$ where $\xi_1, \xi_2, \dots \overset{iid}{\sim} F$, $F(0)=0$

(i) $\boxed{U(t) = \sum_{n=1}^{\infty} F^{n*}(t) < \infty}$

(2) $\boxed{EN_t = U(t)}$

proof for (2)

$$EN_t = E[\#\{n : S_n \le t\}]$$
$$= E\left[\sum_{n=1}^{\infty} \mathbb{1}\{S_n \le t\}\right] = \sum_{n=1}^{\infty} P\{S_n \le t\}$$
$$= \sum_{n=1}^{\infty} F^{n*}(t)$$

1.8 Laplace transform. Calculation of an expectation of a counting process (1)

Laplace transform

$f : \mathbb{R}_+ \to \mathbb{R}$: $\mathcal{L}_f(s) = \int_0^{\infty} e^{-sx} f(x)\,dx$

1) f - density of ξ, then $\mathcal{L}_f(s) = E[e^{-s\xi}]$

2) f_1, f_2 : $\mathcal{L}_{\underbrace{f_1 * f_2}_{\text{densities}}}(s) = \mathcal{L}_{f_1}(s) \cdot \mathcal{L}_{f_2}(s)$

3) F - distribution function, $F(0)=0$, $p = F'$

$$\mathcal{L}_F(s) = \frac{\mathcal{L}_p(s)}{s}$$

$$\text{l.h.s.} = \int_{\mathbb{R}_+} F(x)\,\frac{d(e^{-sx})}{s} = \cancel{-\frac{F(x)e^{-sx}}{s}\Big|_0^{\infty}} + \frac{1}{s}\int_{\mathbb{R}_+} p(x)\,e^{-sx}\,dx$$

$$= \text{r.h.s.}$$

Ex

1) $$\mathcal{L}_{X^k}(s) = \int_{\mathbb{R}_+} x^n \frac{d(e^{-sx})}{s} = \frac{n}{s}\int_{\mathbb{R}_+} x^{n-1} e^{-sx}\,dx$$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdots\frac{2}{s}\int_{\mathbb{R}_+} e^{-sx}\,dx = \frac{n!}{s^n}$$

2) $$\mathcal{L}_{e^{ax}}(s) = \frac{1}{s-a}, \text{ if } a<s$$

1.9 Laplace transform. Calculation of an expectation of a counting process (2)

$F \to \mathbb{E}N_t$

$$\mathbb{E}N_t = \mathcal{U}(t) = \sum_{n=1}^{\infty} F^{n*}(t) = F(t) + \left(\sum_{n=1}^{\infty} F^{n*}(t)\right) * F(t)$$

$$\Leftrightarrow \quad \mathcal{U} = F + \mathcal{U} \underset{\text{dist fcn}}{*} F = F + \mathcal{U} \underset{\text{densities}}{*} p \quad \text{If } F' = p \text{ exists}$$

$$\int_{\mathbb{R}} \mathcal{U}(x-y)\,dF(y) = \int_{\mathbb{R}} \mathcal{U}(x-y)\,p(y)\,dy$$

$$\mathcal{L}_{\mathcal{U}}(s) = \underbrace{\mathcal{L}_F(s)}_{=\frac{\mathcal{L}_p(s)}{s}} + \mathcal{L}_{\mathcal{U}}(s)\,\mathcal{L}_p(s)$$

$$\boxed{\mathcal{L}_{\mathcal{U}}(s) = \frac{\mathcal{L}_p(s)}{s(1-\mathcal{L}_p(s))}}$$

1 $F \to \mathcal{L}_p$

2 $\mathcal{L}_p \to \mathcal{L}_{\mathcal{U}}$

3 $\mathcal{L}_{\mathcal{U}} \to \mathcal{U}$

1.10 *Laplace transform. Calculation of an expectation of a counting process (3)*

Example: $S_n = S_{n-1} + \xi_n$, $\xi_1, \xi_1, \dots$ have density $p(x)$

$$p(x) = \frac{e^{-x}}{2} + e^{-2x}, \quad x > 0$$

$\mathbb{E} N_t = ?$

① $p \to \mathcal{L}p$: $\mathcal{L}p(s) = \frac{1}{2}\mathcal{L}_{e^{-x}}(s) + \mathcal{L}_{e^{-2x}}(s)$

$$= \frac{1}{2(s+1)} + \frac{1}{s+2} = \frac{3s+4}{2(s+1)(s+2)}$$

② $\mathcal{L}p \to \mathcal{L}_u$: $\mathcal{L}_u(s) = \frac{\mathcal{L}p(s)}{s(1-\mathcal{L}p(s))} = \frac{3s+4}{s^2(2s+3)}$

③ $\mathcal{L}_u \to u$: $\mathcal{L}_u(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{2s+3}$

$$= \frac{A(2s+3) + B(2s^2+3s) + Cs^2}{s^2(2s+3)}$$

$$3s+4 = (2B+C)s^2 + (2A+3B)s + 3A$$

$A = \frac{4}{3}$, $2A+3B = 3 \Leftrightarrow B = \frac{1}{9}$, $2B+C=0 \Leftrightarrow C = -\frac{2}{9}$

$$\boxed{u(t) = \frac{4}{3}t + \frac{1}{9}(1) - \frac{1}{9}e^{-3/2\, t}}$$

1.11 *Limit theorems for renewal processes*

$S_n = S_{n-1} + \xi_n$; $\xi_1, \xi_2, \dots$ iid > 0 a.s.

Thm 1 $\mu = \mathbb{E}\xi_1 < \infty \Rightarrow \frac{N_t}{t} \xrightarrow{t\to\infty} \frac{1}{\mu}$ a.s.

(Analog to SLLN)

$$\boxed{\text{SLLN: } \frac{\xi_1 + \dots + \xi_n}{n} \xrightarrow[n\to\infty]{} \mu \text{ a.s.}}$$

Thm 2: (Analog of CLT) $\sigma^2 = \operatorname{Var} \xi_1 < \infty$

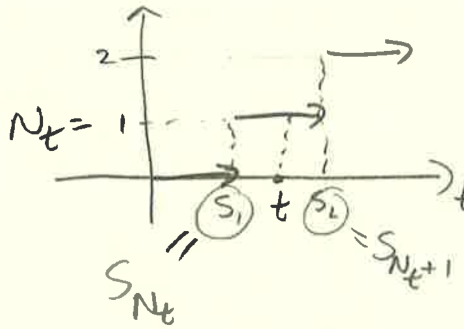
Then, $Z_t := \frac{N_t - t/\mu}{\sigma\sqrt{t}/\mu^{3/2}} \xrightarrow[t\to\infty]{d} N(0,1)$

$$P\{Z_t \le x\} \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

$$\boxed{\text{CLT: } \frac{\xi_1 + \dots + \xi_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1)}$$

proof (thm 1)

$$S_{N_t} \le t \le S_{N_t+1}$$



$$\frac{N_t}{S_{N_t+1}} \le \frac{N_t}{t} \le \frac{N_t}{S_{N_t}}$$

$$\lim_{t\to\infty} \frac{N_t}{S_{N_t}} = \lim_{n\to\infty} \frac{n}{S_n} = \frac{1}{\mu} \text{ by SLLN}$$

$$\lim_{t\to\infty} \frac{N_t}{S_{N_t+1}} = \underbrace{\lim_{t\to\infty} \frac{N_t}{N_t+1}}_{1} \cdot \underbrace{\lim_{t\to\infty} \frac{N_t+1}{S_{N_t+1}}}_{1/\mu} = \frac{1}{\mu}$$

proof (thm 2)

$$P\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right\} \to \phi(x), \; x \in R$$

$$P\{S_n \le \underbrace{n\mu + \sigma\sqrt{n}\, x}_{t}\} \to \phi(x)$$

$\Leftrightarrow P\{N_t \ge n\}$ (set complements)

$n\mu \approx t$
$n \approx t/\mu$ (for n large enough)

$$n = \frac{t}{\mu} - \frac{\sigma\sqrt{n}}{\mu}x \approx \frac{t}{n} - \frac{\sigma\sqrt{t}}{\mu^{3/2}}x$$

$$\Rightarrow P\{Z_t \ge -x\} \to \phi(x) \quad \Leftrightarrow P\{Z_t \le x\} = 1 - P\{Z_t \ge -x\} \to 1 - \phi(-x) = \phi(x)$$