1.1 Difference between deterministic and stochastic world

	deterministic world	stochastic world
Single variable: Temp of a sick man	$\mathbb{R}$ $T = 39°C$	random variable $\mathbb{E}$, Var, ...
Variables changing over time: T in first 3 days	$\mathbb{R}_+ \to \mathbb{R}$ $T(1) = 39$ $T(2) = 38.5$ $T(3) = 38$ $\vdots$	stochastic process

1.2 Difference between various fields of stochastics

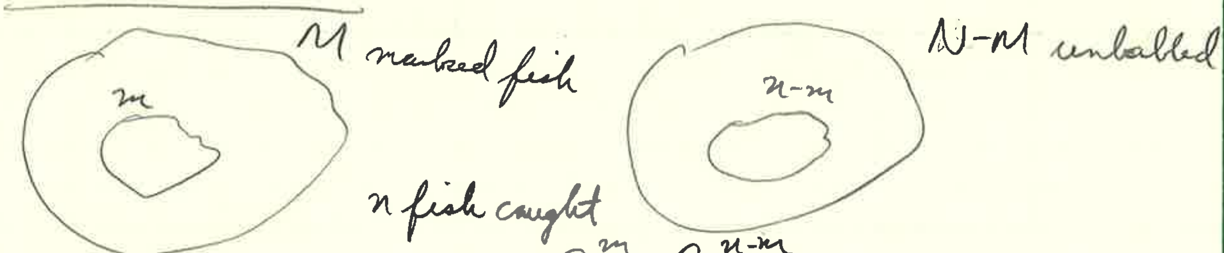
Stochastics
- probability theory
- mathematical statistics
- stochastic processes

Consider a pond that contains fish

Prob theory : # of fish at some given time (N)

$\mathbb{E}$, Var, or limit laws

Mathematical Stats :



$$P\{\#\text{ marked} = m\} = \frac{C_M^m \, C_{N-M}^{n-m}}{C_N^M}$$

Repeat $m_1, m_2, \ldots, m_q$

$$\left(\begin{array}{c}\text{log}\\ \text{likelihood}\end{array}\right) \sum_{k=1}^{q} P\{\#\text{ marked} = m_k\} \longrightarrow \max_N \quad (MLE)$$

1.3 Probability space $(\Omega, \mathcal{F}, P)$

General theory	Bernoulli Scheme $\left[\begin{matrix}1, \text{ success} \\ 0, \text{ failure}\end{matrix}\right]$ $(a_1, \dots, a_n)$, $a_i \in \{0,1\}$	$[0,1]$ Select point from
Ω - sample space	$\#\Omega = 2^n$, set of all vectors with components $\in \{0,1\}$	$\Omega = [0,1]$
$\mathcal{F}$ - σ-algebra 1) $\Omega \in \mathcal{F}$ 2) $A \in \mathcal{F} \Rightarrow \Omega \setminus A \in \mathcal{F}$ 3) $A_1, \dots, A_n, \dots \in \mathcal{F}$ $\Downarrow$ $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$	$\mathcal{F}$ = power set $\#\mathcal{F} = 2^{\#\Omega} = 2^{2^n}$	$P\{x \in [\alpha, \beta]\}$ $\Rightarrow [\alpha, \beta), (\alpha, \beta],$ $(\alpha, \beta), [\alpha, \beta), \{\beta\} \in \mathcal{F}$ Borel σ-algebra
P - probability measure 1) $P(\Omega) = 1$ 2) $A_1, A_2, \dots \in \mathcal{F}$ (disjoint) $\Rightarrow P\{\bigcup_i A_i\} = \sum_i P(A_i)$ $P: \mathcal{F} \to [0,1]$	$P\{1\} = p$ $P\{0\} = 1 - p$	$P\{[\alpha, \beta]\} = \beta - \alpha$

1.4 Definition of a stochastic function. Types of stochastic functions.

$(\Omega, \mathcal{F}, P)$

Random variable - $\underbrace{\text{measurable}}_{}$ function $\xi : \Omega \to \mathbb{R}$.

$$\forall B \in \mathcal{B}(\mathbb{R}) : \xi^{-1}(B) \subset \mathcal{F}$$

T - time

$X : T \times \Omega \to \mathbb{R}$ - random function, if $\forall t \in T : X(t, \cdot)$ is a random variable on $(\Omega, \mathcal{F}, P)$, denoted X_t

If $T = \mathbb{R}_+$, this is called a random process or stochastic process

$T = \mathbb{R}_+^n$, random field or stochastic field

$T = \mathbb{N}$ or $\mathbb{Z}$, discrete time stochastic process

$T = \mathbb{R}_+$ or $\mathbb{R}$, continuous time stochastic process

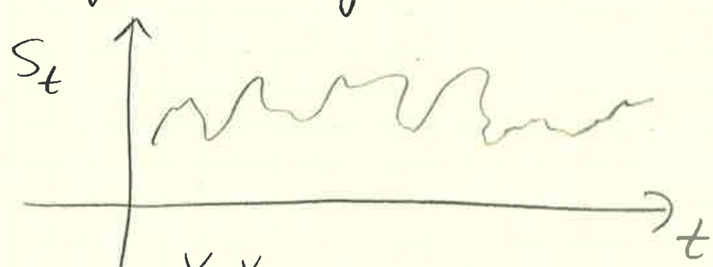
1.5 Trajectories and finite-dimensional distributions

$X : T \times \Omega \to \mathbb{R}$, $T = \mathbb{R}_+$

$\forall t \in T$: $X_t = X(t, \cdot)$ is a r.v. on $(\Omega, \mathcal{F}, P)$

Trajectory (= path)

X_t fix ω and get mapping $T \to \mathbb{R}$



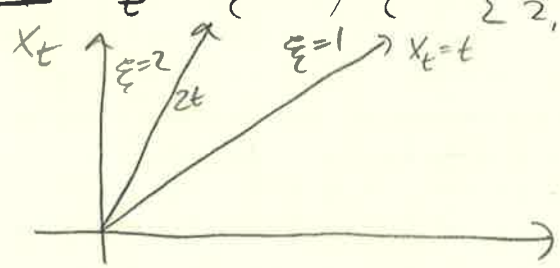
$X_1, X_2, \ldots,$

Finite-dimensional distribution $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$, $t_1, \ldots, t_n \in \mathbb{R}$

In mathematic stats, $X_{t_1}, X_{t_2}, \ldots X_{t_n}$ are independent

In stochastic process, $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ are dependent

<u>Ex</u>: $X_t = \xi t$, $\xi = \begin{cases} 1, & \text{w.p. } 1/2 \\ 2, & \text{w.p. } 1/2 \end{cases}$



$$P\{X_{t_1} \le x_1, X_{t_2} \le x_2\} = \begin{cases} 0, & \min\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}\right) < 1 \\ 1/2, & \quad '' \quad \in [1,2] \\ 1, & \quad '' \quad \ge 2 \end{cases}$$

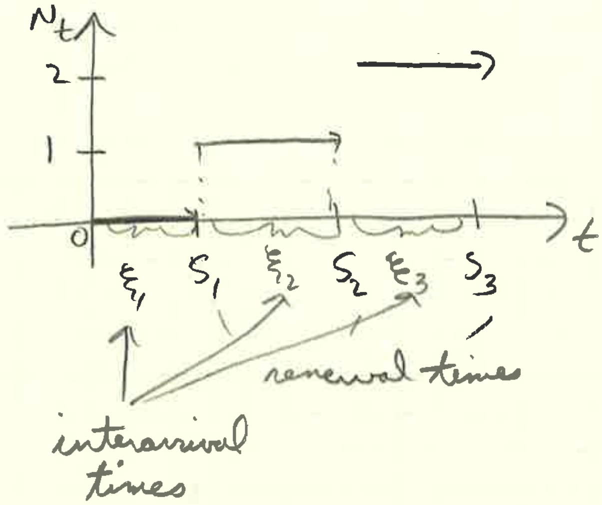
1.6 Renewal process. Counting process.

Renewal processes (discrete time)

$S_0 = 0,\ S_n = S_{n-1} + \xi_n$, where $\xi_1, \xi_2, \ldots$ – iid > 0 a.s.

$P\{\xi_i > 0\} = 1 \iff F(0) = 0$

$N_t = \underset{k}{\operatorname{argmax}} \{S_k \le t\}$ (Counting process)



$\{S_n > t\} = \{N_t < n\}$

$F \Rightarrow \mathbb{E} N_t \qquad S_n = \xi_1 + \cdots + \xi_n$

1.7. Convolution

<u>Convolution</u> $X \perp\!\!\!\perp Y$

$X \sim F_X,\ Y \sim F_Y$

$F_{X+Y}(x) = \int_{\mathbb{R}} F_X(x-y)\, dF(y) =: \underbrace{F_X * F_Y}_{}$ conv in terms of distribution functions

$X \sim p_X,\ Y \sim p_Y$ (If Y, X have densities)

$p_{X+Y}(x) = \int_{\mathbb{R}} p_X(x-y)\, p_Y(y)\, dy =: p_X * p_Y$ } conv in terms of densities

$S_n = \xi_1 + \cdots + \xi_n$

let $F^{n*} := \underbrace{F * \cdots * F}_{n}$

1) $F^{n*}(x) \le F^{n}(x)$ if $F(0)=0$

$\xi_1, \dots, \xi_n \overset{iid}{\sim} F$

$\{\xi_1 + \dots + \xi_n \le x\} \subset \{\xi_1 \le x, \dots, \xi_n \le x\}$ Since $\xi_i > 0$ a.s.

$$\underset{F^{n*}(x)}{\underset{\shortparallel}{P\{\xi_1 + \dots + \xi_n \le x\}}} \le \prod_{k=1}^{n} \underbrace{P\{\xi_k \le x\}}_{F(x)}$$

2) $F^{n*}(x) \ge F^{(n+1)*}(x)$

$\{\xi_1 + \dots + \xi_n \le x\} \supset \{\xi_1 + \dots + \xi_{n+1} \le x\}$

__Theorem:__ $S_n = S_{n-1} + \xi_n$ where $\xi_1, \xi_2, \dots \overset{iid}{\sim} F$, $F(0)=0$

(i) $\boxed{U(t) = \sum_{n=1}^{\infty} F^{n*}(t) < \infty}$

(2) $\boxed{EN_t = U(t)}$

__proof for (2)__

$$EN_t = E[\#\{n : S_n \le t\}]$$
$$= E\left[\sum_{n=1}^{\infty} \mathbb{1}\{S_n \le t\}\right] = \sum_{n=1}^{\infty} P\{S_n \le t\}$$
$$= \sum_{n=1}^{\infty} F^{n*}(t)$$

1.8 Laplace transform. Calculation of an expectation of a counting process (1)

__Laplace transform__

$f : \mathbb{R}_+ \to \mathbb{R}$: $\mathcal{L}_f(s) = \int_0^{\infty} e^{-sx} f(x)\,dx$

1) f - density of ξ, then $\mathcal{L}_f(s) = E[e^{-s\xi}]$

2) f_1, f_2 : $\mathcal{L}_{\underbrace{f_1 * f_2}_{\text{densities}}}(s) = \mathcal{L}_{f_1}(s) \cdot \mathcal{L}_{f_2}(s)$

3) F - distribution function, $F(0)=0$, $p = F'$

$$\mathcal{L}_F(s) = \frac{\mathcal{L}p(s)}{s}$$

$$\text{l.h.s.} = \int_{\mathbb{R}_+} F(x)\,\frac{d(e^{-sx})}{s} = \cancel{-\frac{F(x)e^{-sx}}{s}\Big|_0^{\infty}} + \frac{1}{s}\int_{\mathbb{R}_+} p(x)\,e^{-sx}\,dx$$

$$= \text{r.h.s.}$$

Ex

1) $$\mathcal{L}_{x^k}(s) = \int_{\mathbb{R}_+} x^n\,\frac{d(e^{-sx})}{s} = \frac{n}{s}\int_{\mathbb{R}_+} x^{n-1}e^{-sx}\,dx$$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdots\frac{2}{s}\int_{\mathbb{R}_+} e^{-sx}\,dx = \frac{n!}{s^n}$$

2) $$\mathcal{L}_{e^{ax}}(s) = \frac{1}{s-a},\ \text{if } a<s$$

1.9 Laplace transform. Calculation of an expectation of a counting process (2)

$F \to \mathbb{E}N_t$

$$\mathbb{E}N_t = \mathcal{U}(t) = \sum_{n=1}^{\infty} F^{n*}(t) = F(t) + \left(\sum_{n=1}^{\infty} F^{n*}(t)\right) * F(t)$$

$$\Leftrightarrow\quad \mathcal{U} = F + \mathcal{U} \underset{\text{distd fcn}}{*} F = F + \mathcal{U} \underset{\text{densities}}{*} p \quad \text{If } F' = p \text{ exists}$$

$$\int_{\mathbb{R}} \mathcal{U}(x-y)\,dF(y) = \int_{\mathbb{R}} \mathcal{U}(x-y)\,p(y)\,dy$$

$$\mathcal{L}_{\mathcal{U}}(s) = \underbrace{\mathcal{L}_F(s)}_{=\frac{\mathcal{L}_p(s)}{s}} + \mathcal{L}_{\mathcal{U}}(s)\,\mathcal{L}_p(s)$$

$$\boxed{\mathcal{L}_{\mathcal{U}}(s) = \frac{\mathcal{L}_p(s)}{s(1-\mathcal{L}_p(s))}}$$

1 $F \to \mathcal{L}_p$

2 $\mathcal{L}_p \to \mathcal{L}_{\mathcal{U}}$

3 $\mathcal{L}_{\mathcal{U}} \to \mathcal{U}$

1.10 *Laplace transform. Calculation of an expectation of a counting process (3)*

Example: $S_n = S_{n-1} + \xi_n$, $\xi_1, \xi_1, \ldots$ have density $p(x)$

$$p(x) = \frac{e^{-x}}{2} + e^{-2x}, \quad x > 0$$

$\mathbb{E}N_t = ?$

① $p \to \mathcal{L}p$: $\mathcal{L}p(s) = \frac{1}{2}\mathcal{L}_{e^{-x}}(s) + \mathcal{L}_{e^{-2x}}(s)$

$$= \frac{1}{2(s+1)} + \frac{1}{s+2} = \frac{3s+4}{2(s+1)(s+2)}$$

② $\mathcal{L}p \to \mathcal{L}_u$: $\mathcal{L}_u(s) = \frac{\mathcal{L}p(s)}{s(1-\mathcal{L}p(s))} = \frac{3s+4}{s^2(2s+3)}$

③ $\mathcal{L}_u \to u$: $\mathcal{L}_u(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{2s+3}$

$$= \frac{A(2s+3) + B(2s^2+3s) + Cs^2}{s^2(2s+3)}$$

$$3s+4 = (2B+C)s^2 + (2A+3B)s + 3A$$

$A = \frac{4}{3}$, $2A+3B = 3 \Leftrightarrow B = \frac{1}{9}$, $2B+C=0 \Leftrightarrow C = -\frac{2}{9}$

$$\boxed{u(t) = \frac{4}{3}t + \frac{1}{9}(1) - \frac{1}{9}e^{-3/2\, t}}$$

1.11 *Limit theorems for renewal processes*

$S_n = S_{n-1} + \xi_n$; $\xi_1, \xi_2, \ldots$ iid > 0 a.s.

Thm 1 $\mu = \mathbb{E}\xi_1 < \infty \Rightarrow \frac{N_t}{t} \xrightarrow{t\to\infty} \frac{1}{\mu}$ a.s.

(Analog to SLLN)

$$\boxed{\text{SLLN: } \frac{\xi_1 + \cdots + \xi_n}{n} \xrightarrow[n\to\infty]{} \mu \text{ a.s.}}$$

Thm 2: (Analog of CLT) $\sigma^2 = \operatorname{Var}\xi_1 < \infty$

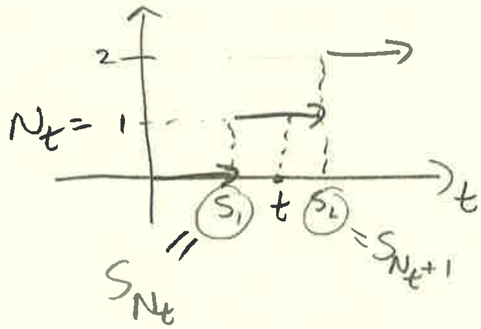
Then, $Z_t := \frac{N_t - t/\mu}{\sigma\sqrt{t}/\mu^{3/2}} \xrightarrow[t\to\infty]{d} N(0,1)$

$$P\{Z_t \le x\} \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

$$\boxed{\text{CLT: } \frac{\xi_1 + \cdots + \xi_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1)}$$

proof (thm 1)

$$S_{N_t} \le t \le S_{N_t+1}$$



$$\frac{N_t}{S_{N_t+1}} \le \frac{N_t}{t} \le \frac{N_t}{S_{N_t}}$$

$$\lim_{t\to\infty} \frac{N_t}{S_{N_t}} = \lim_{n\to\infty} \frac{n}{S_n} = \frac{1}{\mu} \quad \text{by SLLN}$$

$$\lim_{t\to\infty} \frac{N_t}{S_{N_t+1}} = \underbrace{\lim_{t\to\infty} \frac{N_t}{N_t+1}}_{=1} \cdot \underbrace{\lim_{t\to\infty} \frac{N_t+1}{S_{N_t+1}}}_{=1/\mu} = \frac{1}{\mu}$$

proof (thm 2)

$$P\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right\} \to \phi(x), \quad x \in R$$

$$P\{S_n \le \underbrace{n\mu + \sigma\sqrt{n}\, x}_{=t}\} \to \phi(x)$$

$\Leftrightarrow P\{N_t \ge n\}$ (set complements) $\qquad n\mu \approx t$, $n \approx t/\mu$ (for n large enough)

$$n = \frac{t}{\mu} - \frac{\sigma\sqrt{n}}{\mu}x \approx \frac{t}{n} - \frac{\sigma\sqrt{t}}{\mu^{3/2}}x$$

$$\Rightarrow P\{Z_t \ge -x\} \to \phi(x) \quad \Leftrightarrow P\{Z_t \le x\} = 1 - P\{Z_t \ge -x\} \to 1 - \phi(-x) = \phi(x)$$

Poisson Processes

2.1 Definition of a Poisson process as a special example of a renewal process. Exact forms of the distributions of the renewal process and the counting process (1)

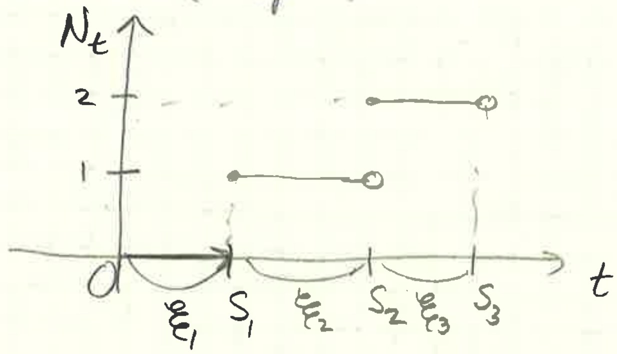
Renewal process

$S_0 = 0, \quad S_n = S_{n-1} + \xi_n, \quad \xi_1, \xi_2, \dots$ - i.i.d > 0 a.s., $\xi_i \sim F$

$N_t = \underset{k}{\operatorname{argmax}} \{ S_k \le t \}$ (Counting process)

$$U(t) = \mathbb{E} N_t = \sum_{n=1}^{\infty} F^{n*}(t)$$

$$\mathcal{L}_U(s) = \frac{\mathcal{L}_p(s)}{s(1-\mathcal{L}_p(s))} \quad : \quad p \to \mathcal{L}_p \to \mathcal{L}_U \to U \qquad (p = F')$$



$$\mathcal{L}_U(s) = \int_{\mathbb{R}_+} e^{-sx} U(x)\, dx$$

2.2 .... (2)

Poisson process

Def 1: A Poisson process is a renewal process s.t.

$$\xi_i \sim p(x) = \lambda e^{-\lambda x} \, \mathbb{1}\{x > 0\}, \quad \lambda \text{ - intensity or rate}$$

Thm (i): A distribution function of S_n

$$F_{S_n}(x) = \begin{cases} 1 - e^{-\lambda x} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!}, & x > 0 \\ 0, & x < 0 \end{cases}$$

$$p_{S_n}(x) = \lambda \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} \, \mathbb{1}\{x > 0\}$$

(ii) $\mathbb{P}\{N_t = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$, $N_t \sim \text{Poisson}(\lambda t)$

2.3 ... (3)

Proof (i) $n=1$: $S_1 = \xi_1$

$$p_{S_1}(x) = \lambda e^{-\lambda x}, \; x>0$$

$n \to n+1$

$$p_{S_{n+1}}(x) = \int_0^x p_{S_n}(x-y)\, p_{\xi_{n+1}}(y)\, dy$$

$$= \int_0^x \frac{\lambda^n (x-y)^{n-1}}{(n-1)!} e^{-\lambda(x-\cancel{y})} \lambda \cancel{e^{-\lambda y}} dy$$

$$= \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda x} \int_0^x (x-y)^{n-1} dy = \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda x} \frac{x^n}{n}$$

$$= \lambda \frac{(\lambda x)^n}{n!} e^{-\lambda x}$$

$\square$

2.4 ... (4)

proof (ii)

$$P\{N_t = n\} = P\{S_n \le t\} - P\{S_{n+1} \le t\} \quad =$$

$$\{N_t = n\} = \underbrace{\{S_n \le t\}}_{= A} \cap \underbrace{\{S_{n+1} > t\}}_{= B}$$

$$\left.\begin{array}{l} A \cap B = A \setminus B^c \\ \text{Here: } B^c \subset A \end{array}\right\} \Rightarrow P\{A \cap B\} = P\{A\} - P\{B^c\}$$

$$= \left(1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}\right) - \left(1 - e^{-\lambda t} \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!}\right)$$

$$= e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$\square$

2.5 Memoryless property

A r.v. X possesses the memoryless property iff

$P\{X > u+v\} = P\{X>u\}\, P\{X>v\}$. If $P\{X>v\}>0$, then

$$\boxed{P\{X > u+v \mid X > v\} = P\{X>u\}}$$

Thm 2: Let X be a r.v. with density $p(x)$, then

X- memoryless $\Leftrightarrow$ $p(x) = \lambda e^{-\lambda x}$

Ex busses arrive every 20 ± 2 minutes

$v = 19$ min, $u = 10$ min

(l.h.s.) $P\{X > 29 \mid X > 19\} = 0$ given the data

(r.h.s) $P\{X > 10\} = 1$

Thus, Poisson process is not appropriate

2.6 Other definitions of Poisson processes (1)

Def 2 N_t – an integer value process s.t.

0) $N_0 = 0$ a.s.

1) N_t has independent increments: $\forall t_0 < t_1 < \dots < t_n$,

$N_{t_1} - N_{t_0}, \dots, N_{t_n} - N_{t_{n-1}}$ are independent

2) N_t has stationary increments

$$N_t - N_s \overset{d}{=} N_{t-s}$$

3) $N_t - N_s \sim \text{Poisson}(\lambda(t-s))$, $t > s$

3) $\Rightarrow$ 2)

2.7 Other definitions of Poisson processes (2)

$$P\{N_{t+h} - N_t = 0\} = 1 - \lambda h + o(h), \quad h \to 0$$

$$P\{N_{t+h} - N_t = 1\} = \lambda h + o(h), \quad h \to 0$$

$$P\{N_{t+h} - N_t \geq 2\} = o(h), \quad h \to 0$$

$$\lim_{h \to 0} \frac{1 - P\{N_{t+h} - N_t = 0\}}{h} = \lim_{h \to 0} \frac{1 - e^{-\lambda h}}{h} = \lambda$$

Def 3 N_t is a Poisson process, if

0) $N_0 = 0$

1) N_t has independent increments

2) N_t has stationary increments

3') $$\lim_{h \to 0} \frac{P\{N_{t+h} - N_t \geq 2\}}{P\{N_{t+h} - N_t = 1\}} = 0$$

2.8 Non-homogeneous Poisson processes (1)

$N_t \sim Pois(\lambda t) \Rightarrow \mathbb{E}N_t = \lambda t$

Def: Let $\Lambda(t)$ be a differentiable, increasing function s.t. $\Lambda(0)=0$. Then, $X_t = N_t$ is a non-homogeneous Poisson process if,

0) $N_0 = 0$

1) N_t has independent increments

2) $\boxed{N_t - N_s \sim Pois(\Lambda(t) - \Lambda(s))}$

2.9 Non-homogeneous Poisson processes (2) (NHPP)

$\lambda(t) = \Lambda'(t)$ - intensity function

Properties NHPP:

1) $\mathbb{E}N_t = \Lambda(t)$

$\Lambda(t) = \alpha t^{\beta}, \alpha>0, \beta>0$ (for example)

2) if $\lambda(t) = \text{const} \Rightarrow \Lambda(t) = \text{const} \cdot t$

3) $\left.\begin{array}{l}\Lambda(t)\text{-differentiable} \Rightarrow \Lambda(t)\text{-continuous}\\ \Lambda(t)\text{-increasing}\end{array}\right\} \Rightarrow$

$\Rightarrow \exists \Lambda^{-1}(t)$. If $\text{Image}\,\Lambda(t) = \mathbb{R}_+$, $N_{\Lambda^{-1}(t)}$ - homogeneous P.P.

2.10 Relation between renewal theory and NHPP (1)

$S_n = \underset{t}{\operatorname{argmin}}\{N_t = n\}$, $\xi_n = S_n - S_{n-1}$

$\xi_1, \xi_2, \dots$ - iid ?

1) $p_{\xi_1}(x) = \lambda(x) e^{-\Lambda(x)}$

$P\{\xi_1 \le x\} = P\{S_1 \le x\} = P\{N_x \ge 1\} = 1 - P\{N_x = 0\} =$

$\{S_n > t\} = \{N_t < n\}$

$= 1 - e^{-\Lambda(x)}$

Take derivatives of both sides to finish proof □.

2.11 ... (2)

$S_k = \operatorname{argmin}_t \{N_t = k\}$

$\xi_k = S_k - S_{k-1}$

1) $\boxed{p_{\xi_1}(t) = \lambda(t) e^{-\Lambda(t)}}$

2) $\boxed{p_{\xi_2|\xi_1}(t|s) = \lambda(t+s) e^{-\Lambda(t+s)+\Lambda(s)}}$

$$F_{(\xi_1,\xi_2)}(s,t) = P\{\xi_1 \le s, \xi_2 \le t\} = \int_0^s P\{\cancel{\xi_1 \le s}, \xi_2 \le t \,|\, \xi_1 = y\} p_{\xi_1}(y)\,dy$$

Since $y<s$

$$= \int_0^s P\{N_{t+y} - N_y \ge 1 \,|\, \xi_1 = y\} p_{\xi_1}(y)\,dy$$

independent

$$= \int_0^s \left(1 - e^{-\Lambda(t+y)+\Lambda(y)}\right) \lambda(y) e^{-\Lambda(y)}\,dy$$

$$p_{(\xi_1,\xi_2)}(s,t) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s} F_{(\xi_1,\xi_2)}(s,t)\right)$$

$$= \frac{\partial}{\partial t}\left(\left(1 - e^{-\Lambda(t+s)+\Lambda(s)}\right) \lambda(s) e^{-\Lambda(s)}\right)$$

$$= \lambda(t+s) e^{-\Lambda(t+s)+\Lambda(s)} \lambda(s) e^{-\Lambda(s)}$$

Then $p_{\xi_2|\xi_1}(t|s) = \dfrac{p_{(\xi_1,\xi_2)}(s,t)}{p_{\xi_1}(s)}$ finishes the proof $\square$

2.12 ... (3)

$\xi_1, \xi_2, \ldots$ – i.i.d. ? (NHPP can be obtained from renewal process iff NHPP is homogeneous PP)

$$p_{\xi_1}(t) = p_{\xi_2|\xi_2}(t|s), \quad \forall t, s > 0$$

$$\lambda(t) e^{-\Lambda(t)} = \lambda(t+s) e^{-\Lambda(t+s)+\Lambda(s)}$$

$$\left(\int_0^T \cdots dt\right): \cancel{e^{-\Lambda(0)}} - e^{-\Lambda(T)} = \cancel{e^0} - e^{-\Lambda(T+S)+\Lambda(s)}$$

$$\boxed{\begin{array}{l}\Lambda(T) = \Lambda(T+S) - \Lambda(s), \quad \forall s, T > 0 \\ \Lambda(t) \text{ – increasing}\end{array}} \Rightarrow \Lambda(t) = \overset{\text{const}}{\overset{\parallel}{\lambda}} t$$

2.13 Elements of queuing theory. M/G/k systems (1)

$$\boxed{\begin{aligned} &P\{N_{t+h}-N_t=0\} = 1-\overset{\lambda(t)}{\lambda} h + o(h) \\ &P\{N_{t+h}-N_t=1\} = \overset{\lambda(t)}{\lambda} h + o(h) \\ &P\{N_{t+h}-N_t\geq 2\} = o(h) \end{aligned}}$$

M/G/k

I) Arrival Process: M - memoryless (Poisson)
D - deterministic
G - general

II) Service time (M, D, G)

III) A number of services $(1, 2, \dots, \infty)$

$M/G/\infty$ $\quad \tau > 0$ (time moment)

$N(t)$ — Customer arrivals →
- $N_1(t)$ - still being served at τ : $\lambda_1(t) = \lambda(1-G(\tau-t))$
- $N_2(t)$ - already completed by τ : $\lambda_2(t) = \lambda G(\tau-t)$

$$P\{N_1(t+\delta)-N_1(t)=1\} = P\{N(t+\delta)-N(t)=1\}\cdot\left(P\{Y>\tau-t\}+o(\delta)\right)$$
$$= (\delta\lambda+o(\delta))(1-G(\tau-t)+o(\delta))$$
$$= \boxed{\lambda\delta(1-G(\tau-t)+o(\delta))}$$

2.14 ... (2)

bernoulli scheme

$$P\{N_1(t)=n_1,\ N_2(t)=n_2\} = P\{N_1(t)=n_1,\ N_2(t)=n_2 \mid N(t)=n_1+n_2\}\cdot P\{N(t)=n_1+n_2\}$$
$$= C_{n_1+n_2}^{n_1}(1-G(\tau-t))^{n_1}G(\tau-t)^{n_2}\cdot e^{-\lambda t}\frac{(\lambda t)^{n_1+n_2}}{(n_1+n_2)!}$$
$$= \frac{\lambda t(1-G(\tau-t))^{n_1}}{n!}e^{-\lambda(1-G(\tau-t))}\cdot\frac{\lambda t(G(\tau-t))^{n_2}}{n_2!}e^{-\lambda G(\tau-t)}$$
$$= P\{N_1(t)=n_1\}\cdot P\{N_2(t)=n_2\}$$

Therefore $N_1 \perp\!\!\!\perp N_2$

2.15 Compound Poisson Processes (1)

$$X_t = \sum_{k=1}^{N_t} \xi_k \quad , \xi_1, \xi_2, \dots \text{ - iid.} \quad , N_t \text{ - P.P. with intensity } \lambda$$

and $\xi_1, \xi_2, \dots$ and N_t are independent

$\xi_1, \xi_2, \dots$ claim sizes

N_t - amount of claims until time t (Insurance interpretation)

X_t - aggregated claim amount

1) Probability generating function (BGF)

ξ - integer, ≥ 0 values

$$\boxed{\phi_\xi(u) = \mathbb{E}[u^\xi]}, \ |u| < 1$$

$$\xi_1 \perp\!\!\!\perp \xi_2 \Rightarrow \phi_{\xi_1+\xi_2}(u) = \phi_{\xi_1}(u)\,\phi_{\xi_2}(u)$$

2) Moment-generating function (MGF)

$$\boxed{\mathcal{L}_\xi(u) = \mathbb{E}[e^{-u\xi}]}, \ \xi \geq 0, \ u > 0$$

2.16 ... (2)

3) Characteristic function

$$\phi_\xi(u) = \mathbb{E}[e^{iu\xi}], u \in \mathbb{R}, \forall \xi \quad , \phi_\xi : \mathbb{R} \to \mathbb{C} \quad , \xi_1 \perp\!\!\!\perp \xi_2 \Rightarrow \phi_{\xi_1+\xi_2}(u) = \phi_{\xi_1}(u)\phi_{\xi_2}(u)$$

Thm $$\boxed{\phi_{X_t - X_s}(u) = e^{\lambda(t-s)(\phi_{\xi_1}(u) - 1)}}$$

Proof: $$lhs = \mathbb{E} e^{iu(X_t - X_s)} = \sum_{k=0}^{\infty} \mathbb{E}[\underbrace{e^{iu(X_t-X_s)}}_{\overset{d}{=}\ \xi_1+\dots+\xi_k} \,|\, \underbrace{\cancel{N_t - N_s = k}}_{\text{from } \perp\!\!\!\perp}]\, P\{N_t - N_s = k\}$$

$$= \sum_{k=0}^{\infty} (\phi_{\xi_1}(u))^k e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^k}{k!}$$

$\square$

2.17 ⋯ (3)

$X_t = \sum_{k=1}^{N_t} \xi_k$, ξ can be any random variable

ξ: $\phi_\xi(u) = E[e^{iu\xi}]$

$\phi : \mathbb{R} \to \mathbb{C}$

$\xi_1 \perp\!\!\!\perp \xi_2 \Rightarrow \phi_{\xi_1+\xi_2}(u) = \phi_{\xi_1}(u)\,\phi_{\xi_2}(u)$

Thm: $\boxed{\phi_{X_t - X_s}(u) = e^{\lambda(t-s)(\phi_{\xi_1}(u)-1)} \quad , t > s \geq 0}$

proof

$lhs = \mathbb{E}[e^{iu(X_t - X_s)}]$

$= \sum_{k=0}^{\infty} \mathbb{E}[e^{iu\underbrace{(X_t-X_s)}_{\overset{d}{=} \xi_1+\cdots+\xi_k}} | \cancel{N_t - N_s = k}] \cdot P\{\underbrace{N_t - N_s = k}_{\sim Pois(\lambda(t-s))}\}$

since $\xi_1, \xi_2, \ldots \perp\!\!\!\perp N_t$

$= \sum_{k=0}^{\infty} [\phi_{\xi_1}(u)]^k \cdot e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^k}{k!}$ □

2.18 ⋯ (4)

Corollary: $\boxed{\begin{aligned} \mathbb{E}X_t &= \lambda t \mathbb{E}\xi_1 \\ Var X_t &= \lambda t \mathbb{E}\xi_1^2 \end{aligned}}$

proof $E[\xi^r] < \infty \Rightarrow \phi(u)$ is r-times differentiable at 0 and $\phi^{(r)}(0) = i^r \mathbb{E}\xi^r$

$\mathbb{E}X_t = \frac{\phi_{X_t}'(0)}{i} = \frac{\lambda t\, \phi_\xi'(0) \cdot \overbrace{\phi_{X_t}(0)}^{=1}}{i} = \lambda t \mathbb{E}\xi_1$ □

$\frac{\phi_\xi'(0)}{i} = \mathbb{E}\xi$

3.1 Definition of a Markov chain. Some examples

Def: A Markov chain $- S_n,\ n=0,1,2,\dots$

S - state space (countable)

$$\boxed{P\{S_n=j \mid S_{n-1}=i_{n-1},\dots,S_0=i_0\} = P\{S_n=j \mid S_{n-1}=i_{n-1}\}}$$

$i_0,\dots,i_{n-1},j \in S$ and $P\{S_{n-1}=i_{n-1},\dots,S_0=i_0\} \neq 0$

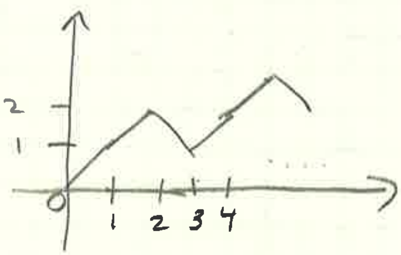
$$P\{S_n=i_n, S_{n-1}=i_{n-1},\dots,S_0=i_0\} = P\{S_n=i_n \mid S_{n-1}=i_{n-1},\dots,S_0=i_0\} \cdot P\{S_{n-1}=i_{n-1},\dots,S_0=i_0\}$$

$$= P\{S_n=i_n \mid S_{n-1}=i_{n-1}\} \cdot P\{S_{n-1}=i_{n-1},\dots,S_0=i_0\}$$

$$= P\{S_n=i_n \mid S_{n-1}=i_{n-1}\}\, P\{S_{n-1}=i_{n-1} \mid S_{n-2}=i_{n-2}\} \cdot \dots \cdot P\{S_1=i_1 \mid S_0=i_0\}\, P\{S_0=i_0\}$$

Ex ① Random walk (not a renewal process)

$S_0=0,\ S_n=S_{n-1}+\xi_n,\quad \xi_1,\xi_2,\dots$ - i.i.d. $\sim \begin{cases} 1, & \text{w.p. } p \\ -1, & \text{w.p. } 1-p \end{cases}$



$$P\{S_n=j \mid S_{n-1}=i_{n-1}\} = \begin{cases} p, & j=i_{n-1}+1 \\ 1-p, & j=i_{n-1}-1 \\ 0, & \text{otherwise} \end{cases}$$

② Taxis in the airport

1 taxi at any 1 moment, $n=1,2,3,\dots$

X_k = # people waiting for a taxi at time k

Y_k = # people arriving at k

$$X_k = Y_k + (X_{k-1}-1)_+ = \begin{cases} Y_k, & \text{if } X_{k-1}=0 \\ Y_k + X_{k-1}-1, & \text{if } X_{k-1}-1>0 \end{cases}$$

③ X_n: $P\{X_n=j \mid X_{n-1}=i_{n-1},\dots,X_0=i_0\} = P\{X_n=j \mid X_{n-1}=i_{n-1},\dots,X_{n-m}=i_{n-m}\}$

$m\in\mathbb{N}$, fixed (X_n is not a Markov chain)

$S_n=(X_n,\dots,X_{n-m-1})$, $n=(m-1), m, \dots$ $\qquad$ S_n is a Markov chain

3.2 Matrix representation of a Markov chain. Transition matrix. Chapman-Kolmogorov equation.

Matrix representation

$S = (1, 2, \dots, M)$

$P\{X_n = j \mid X_{n-1} = i\} = p_{ij}$ - homogeneous (no dependence on n)

$\underline{P} = (p_{ij})_{i,j=1}^{M}$ - transition matrix

$\left.\sum_{j=1}^{M} p_{ij} = 1, \ \forall i \ ; \ p_{ij} \geq 0 \right\}$ - stochastic matrix

$p_{ij}^{(m)} = P\{X_{n+m} = j \mid X_n = i\}$

$\underline{P}^{(m)} = (p_{ij}^{(m)})$ - m-step transition matrix

<u>Thm</u>: $\boxed{\underline{P}^{(m)} = \underline{P}^{M}}$

<u>proof</u>:
$$p_{ij}^{(m)} = \sum_{k=1}^{M} P\{X_{n+m} = j \mid X_{n+m-1} = k, X_n = i\} \cdot P\{X_{n+m-1} = k \mid X_n = i\}$$

(Markov property)
$$= \sum_{k=1}^{M} P\{X_{n+m} = j \mid X_{n+m-1} = k\} \, P\{X_{n+m-1} = k \mid X_n = i\}$$

$$= \sum_{k=1}^{M} p_{kj} \, p_{ik}^{(m-1)} \Rightarrow \underline{P}^{(m)} = \underline{P} \cdot \underline{P}^{(m-1)} = \dots = \underline{P}^{m} \qquad \square$$

$P\{X_k = j\} := \pi_j^{(k)}$, $\left(\pi_1^{(k)}, \dots, \pi_M^{(k)}\right) := \vec{\pi}^{(k)}$

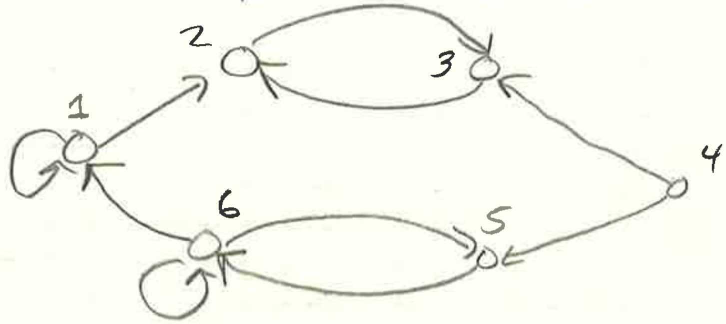
$$\pi_j^{(k)} = \sum_{i=1}^{M} P\{X_k = j \mid X_{k-1} = i\} \, P\{X_{k-1} = i\}$$

$$= \sum_{i=1}^{M} p_{ij} \, \pi_i^{(k-1)} \Rightarrow \vec{\pi}^{(k)} = \vec{\pi}^{(k-1)} \cdot \underline{P} = \vec{\pi}^{(0)} \underline{P}^{n}$$

$\vec{\pi}^*$ - stationary distribution for Markov chain if $\boxed{\vec{\pi}^* \underline{P} = \vec{\pi}^*}$

3.3 Graphical representation; Classification of states (1)

Graphical representation



1 node = 1 state

i,j-arc $\Leftrightarrow p_{ij} \neq 0$

$$\underline{P} = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ & & \cdots & & & \\ & & & & & \end{pmatrix}_{6\times 6}$$

Def (1) j is accessible from i $\exists$ walk (path) from i to j $(i \to j)$

$1 \to 3$; $1 \not\to 4$

(2) i and j communicate if $i \to j$ and $j \to i$ $(i \leftrightarrow j)$

$2 \leftrightarrow 3$

(3) $\underline{Y}$ - set, and relation $\sim$ called an equivalence relation

$a \sim a$, $a \in \underline{Y}$ - reflexivity

$a \sim b \Rightarrow b \sim a$, $a, b \in \underline{Y}$ - symmetry

$a \sim b$, $b \sim a \Rightarrow a \sim c$, $a, b, c \in \underline{Y}$ - transitivity

$\underline{Y} = \bigsqcup B_i$ ($\bigsqcup$ - disjoint union), B_i - equivalence classes

$\leftrightarrow$ $\quad \nwarrow \circ \nearrow \quad \Leftrightarrow \underline{P}\left(\cdots \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \cdots \right)$

$B_1, B_2, \ldots$ - equivalence classes

$$\boxed{\forall j \in B_i, \ \forall k \in S \begin{cases} k \in B_i, & k \leftrightarrow j \\ k \notin B_i, & k \not\leftrightarrow j \end{cases}}$$

$2 \leftrightarrow 3$, $5 \leftrightarrow 6$, 1, 4 are the four equivalence classes

3.4 ... (2)

Def: i is recurrent, $\forall j : i \to j \Rightarrow j \to i$

i is transient if it's not recurrent $\Leftrightarrow \exists j : i \to j, \ j \not\to i$

ex: ①, ④, ⑤, ⑥ - transient

②③ - recurrent

<u>Thm</u>: In 1 class of equivalence, all states are either recurrent or transient.

<u>proof</u> k-transient : $\exists j : k \to j,\ j \not\to k$

$i, k \in 1$ class $\Rightarrow$ $i \to k \to j$, but $j \not\to i$: $j \to i \to k$ is a contradiction $\square$

<u>3.5 ··· (3)</u>

<u>Def</u>: <u>Period</u> of a state i is $GCD\{n : p_{ii}(n) \neq 0\} =: d(i)$

$d(i) = 1 \Rightarrow i$-aperiodic

ex $d(1) = 1 = d(4) = d(5) = d(6)$

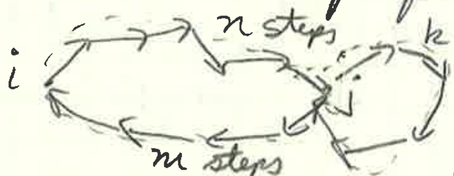
$d(2) = 2 = d(3)$

④ has no return, so $d(4) = 1$ by convention

<u>Thm</u>: All elements in 1 class of equivalence have the same period

<u>proof</u>:

$p_{ii}(n+m+k) \neq 0$



$p_{ii}(n+m) \neq 0 \Rightarrow n+m \mid d(i)$

$k : p_{jj}(k) \neq 0 \Rightarrow n+m+k \mid d(i)$

$$\Rightarrow k \mid d(i) \Rightarrow \left.\begin{array}{l} d(i) \mid d(j) \\ d(j) \mid d(i) \end{array}\right\} \Rightarrow d(i) = d(j)$$

$\square$

<u>3.6 Ergodic chains. Ergodic Theorem (1)</u>

<u>Matrix representation</u>

$\underline{P}$, $\vec{\Pi}^{(k)}$

$\underline{P}^{(m)} = \underline{P}^m$

<u>Ergodic Markov chains</u>:

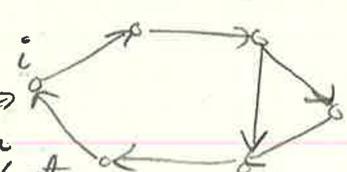
- 1 class of equivalence
- recurrent
- $d(i) = 1$ (aperiodic)

<u>Graphical representation</u>

classes of equivalence

recurrent / transient

$d(i)$ - period



return in 5 or 6 steps

$\therefore d(i) = 1$

<u>Prop</u>: Markov chain is ergodic $\Longleftrightarrow \exists m \in \mathbb{N} : p_{ij}(m) \neq 0, \forall i,j \in S$ (*)

If chain is ergodic, then (*) hold $\forall m \geq (M-1)^2+1$

<u>3.7 ... (2)</u>

<u>Ergodic theorem</u>: Let X_t - ergodic Markov chain, i.e. X_t has 1 class of equivalence, recurrent and aperiodic. Then,

$$\boxed{\exists \lim_{n\to\infty} p_{ij}(n) = \pi_j^* > 0 \text{ (doesn't depend on } i)}$$

$$\sum_{j=1}^{M} \pi_j^* = 1 \qquad \vec{\pi}^* = (\pi_1^*, \dots, \pi_M^*)$$

<u>Corr</u>(i) $\vec{\pi}^*$ - stationary distribution : $\vec{\pi}^* \underline{P} = \vec{\pi}^*$

(ii) $\lim_{n\to\infty} P\{X_n = j\} = \pi_j^*$ $\quad [\pi_j^{(0)}$ is arbitrary$]$

<u>proof</u> (i) $\quad i = 1, \dots, M$

$$(\pi^* \underline{P})_i = \sum_{j=1}^{M} \pi_j^* p_{ji} = \sum_{j=1}^{M} \lim_{n\to\infty} p_{kj}(n)\, p_{ji} \qquad (k \in 1, \dots, M)$$

$$= \lim_{n\to\infty} \sum_{j=1}^{M} \underbrace{p_{kj}(n)\, p_{ji}}$$

$$\underline{P}^{(n)} \underline{P} = \underline{P}^{n+1} = \underline{P}^{(n+1)}$$

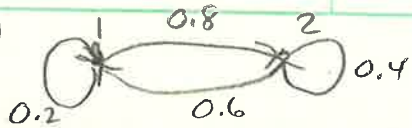
$$= \lim_{n\to\infty} p_{ki}(n+1) = \pi_i^* \qquad \square$$

<u>proof</u> (ii) $\lim_{n\to\infty} \pi_j^{(n)} = \lim_{n\to\infty} \sum_{k=1}^{M} \pi_k^{(0)} p_{kj}(n) \qquad \pi_j^{(0)}$ is arbitrary

$$\vec{\pi}^{(n)} = \vec{\pi}^{(0)} \underline{P}^{(n)}$$

$$= \sum_{k=1}^{M} \pi_k^{(0)} \underbrace{\lim_{n\to\infty} p_{kj}(n)}_{=\pi_j^*} = \pi_j^* \underbrace{\sum_{k=1}^{M} \pi_k^{(0)}}_{=1} = \pi_j^* \qquad \square$$

ex



$\underline{P} = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}$

$\vec{\pi}^* = (a, b)$; $\vec{\pi}^* \underline{P} = \vec{\pi}^*$

$$(a \ b)\begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix} = (a \ b)$$

$$\left.\begin{aligned} 0.2a + 0.6b &= a \\ 0.8a + 0.4b &= b \end{aligned}\right\} \Rightarrow a = \tfrac{3}{7},\ b = \tfrac{4}{7}$$

$P\{X_n = 1\} \to \frac{3}{7}$

$P\{X_n = 2\} \to \frac{4}{7}$

4.1 Random vector. Definition and main properties

$\xi \sim N(\mu, \sigma^2)$, $\rho(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $\sigma > 0, \mu \in R$

$$\phi(u) = e^{iu\mu - \frac{1}{2}u^2\sigma^2}$$

$$\left.\begin{array}{l} X_1, X_2 \sim N(0,1) \\ \operatorname{cor}(X_1, X_2) = 0 \end{array}\right\} \not\Rightarrow X_1 \perp\!\!\!\perp X_2$$

$$P\{X = \mu\} = 1 \Rightarrow \sigma = 0$$

<u>Def</u>: A random vector $\vec{X} = (X_1, \dots, X_n)$ is Gaussian iff $\forall (\lambda_1, \dots, \lambda_n) \in R^n$, $\sum_{k=1}^{n} \lambda_k X_k \sim N$

4.2 Gaussian vector. Definition and main properties

<u>Thm</u>: $\vec{X}$ - Gaussian iff any of the following holds:

(i) $\phi_{\vec{X}}(\vec{u}) = \mathbb{E}\left[e^{i\langle \vec{u}, \vec{X} \rangle}\right] = \exp\left\{ i\langle \vec{u}, \vec{\mu} \rangle - \frac{1}{2}\vec{u}^T C \vec{u} \right\}$

$\vec{\mu} \in R^n$; C - symmetric, positive semidefinite (size $n \times n$)

(ii) $\vec{X} = A\vec{X}^\circ + \vec{\mu}$; $A \in Mat(n \times n)$, $\vec{X}^\circ$ - standard normal vector

<u>Remark</u>: $\vec{\mu} = (\mathbb{E}X_1, \dots, \mathbb{E}X_n)$

$$C = (c_{jk})_{j,k=1}^{n}; \quad c_{jk} = \operatorname{cov}(X_j, X_k)$$

$$\sum_{k,j=1}^{n} u_k c_{kj} u_j \geq 0, \ \forall u \in R^n \iff u^T C u \geq 0, \ \forall u \in R$$

$$\sum_{k,j=1}^{n} u_k \operatorname{cov}(X_j, X_k) u_j = \operatorname{cov}\left(\sum_{j=1}^{n} u_j X_j, \sum_{k=1}^{n} u_k X_k\right)$$

$$= \operatorname{var}\left(\sum_{j=1}^{n} u_j X_j\right) \geq 0$$

$$A = C^{1/2} : AA = C \Rightarrow \exists U : U^{-1} = U^T : C = U^T \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} U$$

$$A = U^T \begin{pmatrix} \sqrt{d_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{d_n} \end{pmatrix} U \Rightarrow C = AA^T$$

Proof: Def $\Leftrightarrow$ (i) $\Leftrightarrow$ (ii)

Def $\Rightarrow$ (i): $\langle \vec{u}, \vec{X} \rangle \sim N$

$$\phi_{\vec{X}}(\vec{u}) = \mathbb{E} e^{i\underbrace{\langle \vec{u}, \vec{X} \rangle}_{\xi} \cdot 1} = \phi_{\xi}(1) = e^{i\mu_\xi - \frac{1}{2}\sigma_\xi^2}$$

$$\mu_\xi = \mathbb{E}\left[\sum_{k=1}^n u_k X_k\right] = \sum_{k=1}^n u_k \mathbb{E}X_k = \sum_{k=1}^n u_k \mu_k = \langle \vec{\mu}, \vec{u} \rangle$$

$$\sigma_\xi^2 = \operatorname{cov}\left(\sum_{k=1}^n u_k X_k, \sum_{j=1}^n u_j X_j\right)$$

$$= \sum_{k=1}^n \sum_{j=1}^n u_k \operatorname{cov}(X_k, X_j) u_j = \vec{u}^T C \vec{u}$$

(i) $\Rightarrow$ Def By definition of ϕ for Gaussian

(ii) $\Rightarrow$ (i)

$$\vec{X}^\circ\text{-Gaussian} \Rightarrow \phi_{\vec{X}_0}(\vec{u}) = \exp\left\{-\tfrac{1}{2}\vec{u}^T\vec{u}\right\}$$

$$\phi_{\vec{X}}(\vec{u}) = \mathbb{E}\left[e^{i\langle \vec{u}, A\vec{X}^\circ + \vec{\mu} \rangle}\right] = \mathbb{E}\left[e^{i[\langle \vec{u}, \vec{\mu} \rangle + \langle \vec{u}, A\vec{X}^\circ \rangle]}\right]$$

$$= e^{i\langle \vec{u}, \vec{\mu} \rangle} \phi_{\vec{X}^\circ}(A^T\vec{u}) = e^{i\langle \vec{u}, \vec{\mu} \rangle} e^{-\frac{1}{2}\vec{u}^T \underbrace{AA^T}_{= C} \vec{u}}$$

$$= e^{i\langle \vec{u}, \vec{\mu} \rangle} e^{-\frac{1}{2}\vec{u}^T C \vec{u}}$$

(i) $\Rightarrow$ (ii) $A = C^{1/2}$ $\square$

4.3 Connection between independence of normal random variates and absence of correlation

Thm: Let $X_1, X_2 \sim N(0,1)$ and $\operatorname{cov}(X_1, X_2) = 0$, then

$$X_1 \perp\!\!\!\perp X_2 \Leftrightarrow (X_1, X_2) \text{ - Gaussian vector}$$

proof ($\Rightarrow$) $\lambda_1 X_1 + \lambda_2 X_2 \sim N \Rightarrow (X_1, X_2)$ - Gaussian vector

($\Leftarrow$) $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A = C^{1/2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = C$

$$(ii) \Rightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = A\vec{X}^\circ + \vec{\mu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X_1^\circ \\ X_2^\circ \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} X_1^\circ \\ X_2^\circ \end{pmatrix}$$ $\square$

<u>ex</u> $X_1 \sim N(0,1)$, $X_2 := |X_1| \cdot \xi$, $\xi = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$, $\xi \perp\!\!\!\perp X_1$

1) $X_2 \sim N(0,1)$, $x > 0$

$$P\{X_2 \le x\} = P\{|X_1| \le x \mid \xi = 1\} P\{\xi = 1\} + P\{|X_2| \ge -x \mid \xi = -1\} P\{\xi = -1\}$$

$$\overset{\xi \perp\!\!\!\perp X_1}{=} P\{|X_1| \le x\} \cdot 1/2 + \underbrace{P\{|X_2| \ge -x\}}_{1} \cdot \tfrac{1}{2}$$

$$= \frac{1}{2}\left[1 + P\{|X_1| \le x\}\right] = P\{X_1 \le x\}$$

2) $\underline{\operatorname{cov}(X_1 X_2) = 0}$

$$\underset{\parallel}{\operatorname{cov}}\ldots \; \mathbb{E}[X_1 X_2] - \mathbb{E}X_1 \, \mathbb{E}X_2 = \mathbb{E}[X_1 |X_1| \xi]$$

$$= \mathbb{E}[X_1 |X_1|] \underbrace{\mathbb{E}\xi}_{0} - 0 \cdot \mathbb{E}X_2$$

$$= 0 - 0 = 0 \qquad \square$$

3) X_1, X_2 are dependent

Assume $X_1 \perp\!\!\!\perp X_2 \Rightarrow (X_1, X_2)$ - Gaussian

$$\gamma = X_1 - X_2 = X_1 - |X_1|\xi \sim N$$

$\{\gamma > 0\}$ when $X_1 > 0$ and $\xi = -1$

$$P\{\gamma > 0\} \ge P\{X_1 > 0 \cap \xi = -1\} = P\{X_1 > 0\} P\{\xi = -1\}$$

$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$P\{\gamma = 0\} \ge 1/4$ since $\operatorname{var}(\gamma) \ne 0$

Thus γ is not Gaussian

4.4 Definition of a Gaussian process. Covariance function (1)

<u>Def</u>: A Gaussian process X_t is a stochastic process s.t.
$\forall t_1, t_2, \dots, t_n$: $(X_{t_1}, \dots, X_{t_n})$ – Gaussian vector

$m(t) = \mathbb{E}X_t$ - mathematical expectation $\quad m: \mathbb{R}_+ \to \mathbb{R}$

$K(t,s) = cov(X_t, X_s)$, $K: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$

$K(t,t) = Var\, X_t$ and $\boxed{K(t,s) = K(s,t)}$

<u>K - a positive semidefinite function</u> (p.s.d.)

$\forall (t_1, \dots, t_n) \in \mathbb{R}_+^n,$

$\forall (u_1, \dots, u_n) \in \mathbb{R}^n$

$$\boxed{\sum_{k=1}^{n} \sum_{j=1}^{n} u_k u_j K(t_k, t_j) \geq 0}$$

$$\Leftrightarrow cov\left(\sum_{k=1}^{n} u_k X_{t_k}, \sum_{j=1}^{n} u_j X_{t_j}\right) = var\left(\sum_{k=1}^{n} u_k X_{t_k}\right) \geq 0$$

4.5 Definition of a Gaussian process. Covariance function (2)

<u>Thm</u>: $m: \mathbb{R}_+ \to \mathbb{R}$, $K: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, K-symmetric and p.s.d.
Then, $\exists$ Gaussian process X_t: $\boxed{\mathbb{E}X_t = m(t),\ cov(X_t, X_s) = K(t,s)}$

Gaussian r.v. : $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$
Gaussian vector : $\vec{\mu} \in \mathbb{R}^n$, $C \in Mat(n,n)$ and symmetric and p.s.d.

Gaussian process : $m: \mathbb{R}_+ \to \mathbb{R}$, $K: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ (sym. and p.s.d.)

<u>Ex</u>: $K(t,s) = |t-s|$ is not p.s.d.

assume p.s.d. $\Rightarrow \exists X_t : cov(X_t, X_s) = |t-s|$

let $t=s$, $var\, X_t = 0 \Rightarrow X_t = f(t)$ - deterministic

$cov(X_t, X_s) = \mathbb{E}[X_t X_s] - \mathbb{E}X_t \mathbb{E}X_s = f(t)f(s) - f(t)f(s) = 0 \neq |t-s|$

<u>Contradiction</u> $\square$

<u>Ex</u> $K(t,s) = \min(t,s)$ is p.s.d.

$$\sum_{k,j=1}^{n} u_j u_k \min(t_j, t_k) \geq 0$$

$$f_t(x) = \mathbb{1}\{x \in [0,t]\}$$

$$\int_0^\infty \underbrace{f_t(x) f_s(x)}_{\substack{\shortparallel \\ 1 \text{ iff } x\in[0,t],\ x\in[0,s] \iff x \in [0, \min(t,s)]}} dx = \min(t,s)$$

$$\sum_{j,k=1}^{n} u_j u_k \int_0^\infty f_{t_j}(x) f_{t_k}(x) dx = \int_0^\infty \sum_{j,k=1}^{n} u_j u_k f_{t_j}(x) f_{t_k}(x) dx$$

$$= \int_{R_+} \sum_{j=1}^{n} u_j f_{t_j}(x) \sum_{k=1}^{n} u_k f_{t_k}(x) dx = \int_{R_+} \Big(\sum_{j=1}^{n} u_j f_{t_j}(x)\Big)^2 dx \geq 0$$

Then, $\boxed{K(t,s) = \min(t,s) \text{ is p.s.d.}}$ $\square$

4.6 Two definitions of a Brownian motion

Brownian motion = Wiener process
$(B_t = W_t)$

<u>Def 1</u>: B_t - a Gaussian process with $m(t) = 0$, $K(t,s) = \min(t,s)$

<u>Def 2</u>: (0) $B_0 = 0$ a.s.

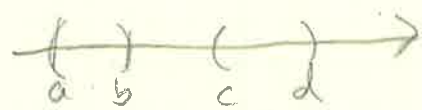
(1) B_t independent increments $[B_t \sim N(0,t)]$

(2) $B_t - B_s \sim N(0, t-s)$, $t > s \geq 0$

<u>Def 1 ⇒ Def 2</u>

(0) $\left.\begin{array}{l} \mathbb{E} B_0 = m(0) = 0 \\ \mathrm{Var}\, B_0 = K(0,0) = 0 \end{array}\right\} \Rightarrow B_0 = 0$ a.s.

(1)



$B_b - B_a \perp\!\!\!\perp B_d - B_c$

$$\mathrm{cov}(B_b - B_a, B_d - B_c) = \mathrm{cov}(B_b, B_d) - \mathrm{cov}(B_a, B_d) - \mathrm{cov}(B_b, B_c) + \mathrm{cov}(B_a, B_c)$$

$$= \min(b,d) - \min(a,d) - \min(b,c) + \min(a,c)$$

$$= b - a - b + a = 0$$

$$\lambda_1(B_b-B_a)+\lambda_2(B_d-B_c) = \lambda_1 B_b - \lambda_1 B_a + \lambda_2 B_d - \lambda_2 B_c \sim N(\cdot,\cdot)$$

$\Rightarrow \begin{bmatrix} B_b-B_a \\ B_d-B_c \end{bmatrix}$ - Gaussian vector

$\Rightarrow B_b - B_a \perp\!\!\!\perp B_d - B_c$

(2) (B_t, B_s) - Gaussian vector $\Rightarrow B_t - B_s \sim N$

$$E[B_t-B_s] = \mathbb{E}B_t - \mathbb{E}B_s = m(t)-m(s) = 0-0=0$$

$$\begin{aligned} Var[B_t-B_s] &= cov(B_t-B_s, B_t-B_s) = cov(B_t,B_t) - 2cov(B_t,B_s) + cov(B_s,B_s) \\ &= min(t,t) - 2min(t,s) + min(s,s) \\ &= t - 2s + s = t-s \quad \square \end{aligned}$$

<u>Def 2 $\Rightarrow$ Def 1</u> $\quad t_1 < t_2 < \cdots < t_n$

$$\sum_{k=1}^{n} \lambda_k B_{t_k} = \lambda_n(B_{t_n} - B_{t_{n-1}}) + (\lambda_n + \lambda_{n-1})B_{t_{n-1}} + \sum_{k=1}^{n-2} \lambda_k B_{t_k}$$

$$= \sum_{k=1}^{n} d_k (B_{t_k} - B_{t_{k-1}}) \sim N \quad (t_0 = 0)$$

$\Rightarrow (B_{t_1}, \ldots, B_{t_n})$ - Gaussian vector $\Rightarrow B_t$ - Gaussian

$B_t \sim N(0,t) \Rightarrow m(t) = \mathbb{E}B_t = 0 \qquad t > s$

$$\begin{aligned} K(t,s) = cov(B_t, B_s) &= cov(B_t - B_s + B_s, B_s) \\ &= cov(B_t - B_s, B_s) + cov(B_s, B_s) \\ &= \underbrace{cov(B_t - B_s, B_s - B_0)}_{=0} + cov(B_s, B_s) \\ &= var(B_s) = s \end{aligned}$$

If $s > t$, $K(t,s) = t$

$\therefore K(t,s) = min(t,s) \quad \square$

4.7 Modification of a process. Kolmogorov continuity theorem

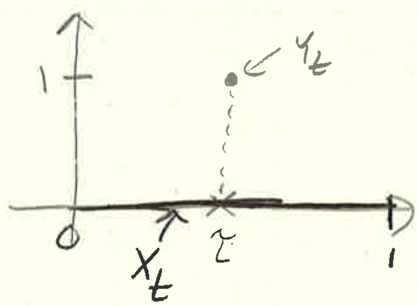
Kolmogorov continuity theorem

Def X_t, Y_t are stochastically equivalent if
$$P\{X_t = Y_t\} = 1 \quad , \forall t \geq 0$$

Ex: $X_t = 0$, $\forall t \in [0,1]$

$$Y_t = \mathbb{1}\{\tau = t\} \ , \ \tau \sim Unif(0,1)$$

$$P\{X_t = Y_t\} = P\{Y_t = 0\} = P\{t \neq \tau\} = 1$$



Thm If $\exists C, \alpha, \beta > 0$ s.t. $\mathbb{E}[|X_t - X_s|^\alpha] \leq C|t-s|^{1+\beta}$ $\forall t,s \in [a,b]$, then $\exists Y_t$ that is stochastically equivalent to X_t s.t. Y_t has continuous trajectories, i.e., X_t has a continuous modification.

Ex: $\mathbb{E}[|\underbrace{B_t - B_s}_{N(0,t-s)}|^4] = (t-s)^2 \underbrace{\mathbb{E}\xi^4}_{3} = 3(t-s)^2$

$$B_t - B_s \overset{d}{=} \sqrt{t-s}\ \xi, \ \xi \sim N(0,1)$$

$$\Rightarrow C = 3, \beta = 1, \alpha = 4$$

4.8 Main properties of Brownian Motion

(1) Quadratic variation

$t_0 = 0 \quad t_1 \quad t_2 \quad \cdots \quad t = t_n$

$$\lim_{n\to\infty} \sum_{k=1}^{n} (B_{t_k} - B_{t_{k-1}})^2 = t \qquad S_n$$

$\max|t_i - t_{i-1}| \to 0$ as $n \to \infty$

$$\mathbb{E}(S_n - t)^2 \xrightarrow[n\to\infty]{} 0 \iff \boxed{\lim_{n\to\infty} \sum_{k=1}^{\infty} (B_{t_k} - B_{t_{k-1}})^2 = t}$$ quadratic variation

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} |B_{t_k} - B_{t_{k-1}}| = \infty}$$ variation

② B_t - everywhere continuous, but nowhere differentiable

$$B_{t+h} \xrightarrow[h\to 0]{P} B_t, \ \forall t \geq 0$$

③ $\lim\limits_{t\to\infty} \frac{B_t}{t} = 0$ a.s.

$\lim\limits_{t\to\infty} \frac{B_t}{\sqrt{t}} = \infty$ a.s.

Law of iterated logarithm: $\boxed{\lim\limits_{t\to\infty} \frac{B_t}{\sqrt{2t\log(\log t)}} = 1}$

5.1 Two types of stationarity (1)

Stationarity

Def 1: X_t is (strictly) stationary if (S.S.)

$$(X_{t_1+h}, \dots, X_{t_n+h}) \overset{d}{=} (X_{t_1}, \dots, X_{t_n}),\ \forall (t_1, \dots, t_n) \in \mathbb{R}_+^n,\ \forall h>0$$

Def 2: X_t is (weakly) stationary if (wide-sense stationarity) (W.S.)

1. $m(t) = \mathbb{E}X_t = \text{const}$
2. $K(t,s) = \operatorname{cov}(X_t, X_s) = k(t+h, s+h)$, $\forall (t,s) \in \mathbb{R}_+^2$, $\forall h>0$

$\iff \gamma : \mathbb{R} \to \mathbb{R} : K(t,s) = \gamma(t-s)$

Properties of $\gamma(\cdot)$:

1) $\boxed{\gamma(0) \geq 0}$

$\gamma(0) = \operatorname{cov}(X_t, X_t) = \operatorname{Var}(X_t)$

2) $\boxed{|\gamma(t)| \leq \gamma(0)}$

$|\gamma(t)| = |\operatorname{cov}(X_t, X_0)| \leq \sqrt{\operatorname{Var} X_t} \cdot \sqrt{\operatorname{Var} X_0}$ Cauchy-Schwartz

$= \sqrt{\gamma(0)} \cdot \sqrt{\gamma(0)} = \gamma(0)$

3) $\boxed{\gamma \text{ is even}}$

$\gamma(t) = \operatorname{cov}(X_t, X_0) = \operatorname{cov}(X_0, X_t) = \gamma(-t)$

Ex:

(i) $\mathbb{E}X_t^2 < \infty$

X_t is strictly stationary $\Rightarrow$ X_t is weakly stationary

(ii) X_t - Gaussian process $\Rightarrow$ X_t - S.S. $\iff$ X_t - W.S.

① White noise process: X_t, $t = 0, \pm 1, \pm 2, \pm 3, \dots$

$\boxed{\mathbb{E}X_t = 0}$, $\operatorname{Var} X_t = \sigma^2$ $[WN(0,\sigma^2)]$

$K(t,s) = 0$, if $t \neq s$

$k(t,s) = \sigma^2 \mathbb{1}\{t=s\} = \gamma(t-s)$, $\boxed{\gamma(x) = \sigma^2 \mathbb{1}\{x=0\}}$ $\Rightarrow$ W.S.

a) $X_1, X_2, \dots$ - iid. noise
b) X_t - Gaussian
$\Big\}$ - X_t - S.S.

② Random walk

$S_n = S_{n-1} + \xi_n$, $\xi_1, \xi_2, \dots$ - iid. $\begin{cases} 1, & \text{w.p. } p \\ -1, & \text{w.p. } 1-p \end{cases}$

$S_0 = 0$

$S_n = \xi_1 + \dots + \xi_n$

$\mathbb{E} S_n = n \mathbb{E}\xi_1 = n(2p-1) \Rightarrow$ not W.S. $\Rightarrow$ not S.S.

a) $p = 1/2 \Rightarrow \mathbb{E} S_n = 0$

$$K(n,m) = \text{cov}(S_m + \xi_{m+1} + \dots + \xi_n, S_m) \quad (n>m)$$

$$= \underbrace{\text{cov}(S_m, S_m)}_{\text{Var } S_m = m \text{ Var } \xi_1} + \underbrace{\text{cov}(\xi_{m+1} + \dots + \xi_n, S_m)}_{0}$$

$$\text{Var } S_m = m \text{ Var } \xi_1 = \min(n,m) \text{ Var } \xi_1$$

$\Rightarrow$ not W.S. $\Rightarrow$ not S.S.

③ Brownian motion

$\mathbb{E} B_t = 0$

$\text{Var } B_t = t \qquad [\underbrace{B_t - B_s}_{B_s = 0} \sim N(0, \underbrace{t-s}_{t})]$

$\text{Var } B_t = t \neq \gamma(0) \Rightarrow$ not W.S. $\Rightarrow$ not W.S.

$K(t,s) = \min(t,s)$, $t > s$

$K(t+h, s+h) = s+h \neq s$

5.2 Two types of stationarity (2)

④ Moving average process : $MA(q)$

$\boxed{Y_t = a_0 X_t + a_1 X_{t-1} + \dots + a_q X_{t-q}}$ $(a_1, \dots, a_q) \in \mathbb{R}^q$; $a_0 = 1$

$X_t \sim WN(0, \sigma^2)$

$\mathbb{E} Y_t = 0$

$$K(t,s) = \text{cov}\left(\sum_{j=0}^{q} a_j X_{t-j}, \sum_{k=0}^{q} a_k X_{t-k}\right)$$

$$= \sum_{j=0}^{q}\sum_{k=0}^{q} a_j a_k \operatorname{cov}(X_{t-j}, X_{s-k}) = \sum_{j=0}^{q}\sum_{k=0}^{q} a_j a_k \sigma^2 \mathbb{1}\{t-s=j-k\}$$

MA(1): $\gamma(x) = C\,\mathbb{1}\{|x|=1\} + D\,\mathbb{1}\{x=0\}$

$\boxed{\text{MA}(q) \text{ is W.S.}}$

⑤ Autoregressive AR(p)

$$\boxed{Y_t = b_1 Y_{t-1} + \dots + b_p Y_{t-p} + \underbrace{\varepsilon_t}_{WN(0,\sigma^2)}}$$

$\operatorname{cov}(\varepsilon_t, Y_s) = 0$, $\forall t > s$

AR(1): $Y_t - bY_{t-1} = \varepsilon_t$, $b \in \mathbb{R}$

$$\boxed{Y_t = \sum_{j=0}^{\infty} b^j \varepsilon_{t-j}}, \quad \mathbb{E}Y_t = 0$$

$$K(t,s) = \sum_{j,k=0}^{\infty} b^{j+k} \sigma^2 \mathbb{1}\{t-s=j-k\}$$

If $t-s=0 \Rightarrow K(t,s) = \sum_{k=0}^{\infty} b^{2k} < \infty \Leftrightarrow |b|<1$

If $|b|<1$, Y_t – W.S.

5.3 Spectral density of a wide-sense stationary process (1)

Spectral density

Bochner–Khintchine theorem

$\phi(u)$ – char fcn $\Leftrightarrow$ 1) ϕ – continuous

$\phi: \mathbb{R} \to \mathbb{C}$ 2) ϕ – positive semi-definite $\sum_{j,k=1}^{n} z_j \bar{z}_k \phi(u_j - u_k) \geq 0$

$\forall (z_1, \dots, z_n) \in \mathbb{C}^n$,

$\forall (u_1, \dots, u_n) \in \mathbb{R}^n$

$\phi(u) = \mathbb{E} e^{iu\xi}$

3) $\phi(0) = 1$

1), 2) properties only $\Rightarrow \exists \mu: \phi(u) = \int e^{iux} \mu(dx)$

1), 2), $\int |\phi(u)| du < \infty \Rightarrow \phi(u) = \int e^{iux} s(x)\, dx$

X_t - w.s. $\Rightarrow \gamma: K(t,s) = \gamma(t-s)$

If γ is continuous, $\int |\gamma(u)| du < \infty$ Fourier transform

$\Rightarrow \exists g(x): \gamma(u) = \int e^{iux} g(x) dx = \mathcal{F}[g](u)$

$$\boxed{g(x) = \frac{1}{2\pi} \int e^{-iux} \gamma(u) du}$$ – spectral density

$$\boxed{g(x) = \frac{1}{2\pi} \sum_{h=-\infty}^{+\infty} e^{-ihx} \gamma(h)}$$ – discrete time spectral density

5.4 : Spectral density of a wide-sense stationary process (2)

Ex 1) $WN(0,\sigma^2) \Rightarrow \gamma(u) = \sigma^2 \mathbb{1}\{u=0\}$

$$g(x) = \frac{\sigma^2}{2\pi}$$

2) MA (1)

$$\gamma(u) = \begin{cases} 0, & |u|>1 \\ a\sigma^2, & |u|=1 \\ a(1+u^2)\sigma^2, & u=0 \end{cases}$$

$$g(x) = \frac{\sigma^2}{2\pi}\left(1+a^2+2a\cos(x)\right)$$

Prop: A real-valued function $g(x)$ is a spectral density of a stochastic process X_t iff on $[-\pi,\pi]$

1) $g(x) \geq 0$

2) g – even

3) $\int_{-\pi}^{\pi} g(x) dx < \infty$

5.5 Moving-average filters (1)

Filter: $X_t \to Y_t$

$Y_t = a_0 X_t + a_1 X_{t-1} + \dots + a_n X_{t-n}$

$Y_t = \int_{\mathbb{R}} e^{-\beta(t-s)} X_s \, ds$

1) Linearity: $\left.\begin{array}{l} X_t^{(1)} \to Y_t^{(1)} \\ X_t^{(2)} \to Y_t^{(2)} \end{array}\right\} \Rightarrow c_1 X_t^{(1)} + c_2 X_t^{(2)} \to c_1 Y_t^{(1)} + c_2 Y_t^{(2)}$

2) time-invariance

$$[X_t \to Y_t] \Rightarrow [X_{t+h} \to Y_{t+h}], \ \forall h > 0$$

$$\int_{\mathbb{R}} e^{-\beta(t-s)} X_{s+h}\, ds = \int_{\mathbb{R}} e^{-\beta((t+h)-(s+h))} X_{s+h}\, ds = Y_{t+h}$$

$$Y_t = \int_{\mathbb{R}} g(s) X_{t-s}\, ds \quad \text{(continuous time)}$$

$$Y_t = \sum_{h=-\infty}^{\infty} g(h) X_{t-h} \quad \text{(discrete time)}$$

5.6 *Moving-average filters (2)*

Thm $\boxed{X_t - \text{w.s. process}}$ with $EX_t = 0$, $g_X(\cdot)$ and

$$Y_t = \int_{\mathbb{R}} g(s) X_{t-s}\, ds, \text{ then}$$

(i) $\boxed{Y_t - \text{w.s. process}}$

(ii) $\boxed{g_Y(x) = g_X(x) \cdot |\mathcal{F}[g](x)|^2}$

$$\mathcal{F}[g](x) = \int_{\mathbb{R}} e^{iux} g(u)\, du$$

Proof: (i) $E Y_t = \int_{\mathbb{R}} g(s) \underbrace{E X_{t-s}}_{=0} ds = 0$

$$K_Y(t_1, t_2) = E\left[\int_{\mathbb{R}} g(s_1) X_{t_1-s_1}\, ds_1 \cdot \int_{\mathbb{R}} g(s_2) X_{t_2-s_2}\, ds_2\right]$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} g(s_1) g(s_2) \underbrace{E[X_{t_1-s_1} X_{t_2-s_2}]}_{\gamma_X(t_2-t_1-(s_2-s_1))} ds_1\, ds$$

$$\Rightarrow \gamma_Y(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} g(s_1) g(s_2)\, \gamma_X(x-(s_2-s_1))\, ds_1\, ds_2$$

(ii) $\gamma_Y(x) = \int_{\mathbb{R}} g(s_1) \int_{\mathbb{R}} \underbrace{\gamma_X(x+s_1-s_2) g(s_2)\, ds_2}_{[\gamma_X * g](x+s_1)} ds_1$

$$= \int_{\mathbb{R}} [\gamma_X * g](x+s_1)\, g(s_1)\, ds_1$$

let $g^\circ(x) := g(-x)$

$$= \int_{\mathbb{R}} [\gamma_X * g](x-s_1)\, \underbrace{g(-s_1)}_{= g^\circ(s_1)}\, ds_1 = \gamma_X * g * g^\circ(x)$$

$$\left(g_Y(x) = \frac{1}{2\pi} \mathcal{F}[\gamma_Y](-x) \right)$$

$$\frac{1}{2\pi} \mathcal{F}[\gamma_Y](x) = \frac{1}{2\pi} \mathcal{F}[\gamma_X](x) \cdot \mathcal{F}[g](x) \cdot \mathcal{F}[g^\circ](x)$$

$$\Rightarrow \underbrace{\frac{1}{2\pi} \mathcal{F}[\gamma_Y](-x)}_{= g_Y(x)} = \underbrace{\frac{1}{2\pi} \mathcal{F}[\gamma_X](-x)}_{= g_X(x)} \cdot \underbrace{\mathcal{F}[g](-x) \cdot \mathcal{F}[g^\circ](-x)}_{\text{complex conjugates}}$$

5.7 Moving-average filters (3)

X_n - W.S. (discrete time), g_X

$Y_n = a_1 X_{n-1} + a_2 X_{n-2}$. What are a_1, a_2 such that $\mathbb{E}[(X_n - Y_n)^2]$ is minimum?

$Z_n = X_n - Y_n = X_n - a_1 X_{n-1} - a_2 X_{n-2} \Rightarrow Z_n$ - W.S. from thm

$g_Z(x) = g_X(x) \,|\mathcal{F}[g](x)|^2$ from thm

$g(x) = \mathbb{1}\{x=0\} - a_1 \mathbb{1}\{x=1\} - a_2 \mathbb{1}\{x=2\}$

$\mathcal{F}[g](x) = 1 - a_1 e^{ix} - a_2 e^{2ix}$

$\underbrace{\mathrm{Var}\, Z_n}_{=} \to \min_{a_1, a_2}$

$$K_Z(n,n) = \gamma_Z(0) = \int_{\mathbb{R}} \overbrace{e^{i\cdot 0 \cdot x}}^{=1} g_Z(x)\, dx$$

$$= \int_{\mathbb{R}} g_X(x) \underbrace{|1 - a_1 e^{ix} - a_2 e^{2ix}|^2}_{= (1-a_1 e^{ix} - a_2 e^{2ix})(1 - a_1 e^{-ix} - a_2 e^{-2ix})} dx$$

$$= \sum_{i,j=1}^{2} B_{ij}\, a_i a_j + \sum_{i=1}^{2} C_i a_i + D \quad \left(\text{minimized w.r.t } a_1, a_2\right) D$$

Ergodicity

6.1 Notion of ergodicity. Examples

Ergodicity

Law of Large Numbers (LLN): $\xi_1, \xi_2, \dots$ - i.i.d.

$$\Rightarrow \frac{1}{N}\sum_{n=1}^{N} \xi_n \xrightarrow[n\to\infty]{P} \mathbb{E}\xi_1 \text{ if } \mathbb{E}\xi_1^2 < \infty \quad \text{(Classical)}$$

$$\text{If } \mathbb{E}\xi_1 < \infty \Rightarrow \frac{1}{N}\sum_{n=1}^{N} \xi_n \xrightarrow[n\to\infty]{P} \mathbb{E}\xi_1 \quad \text{(Khinchine)}$$

$$\text{SSLN: If } \mathbb{E}\xi_1 < \infty \Rightarrow \frac{1}{N}\sum_{n=1}^{N} \xi_n \xrightarrow[n\to\infty]{a.s.} \mathbb{E}\xi_1$$

X_t - discrete time stochastic process, $t = 1, 2, 3, \dots$ (not iid)

$$\frac{1}{T}\sum_{t=1}^{T} X_t \xrightarrow[T\to\infty]{\mathbb{P}} \text{constant} \Rightarrow X_t \text{ is ergodic}$$

$$\xi_n \xrightarrow{a.s.} \xi \iff P\{\omega : \xi_n(\omega) \to \xi(\omega)\} = 1 \text{ as } n \to \infty$$

$$\xi_n \xrightarrow{\mathcal{L}^2 \text{ (or mean squared)}} \xi \iff \mathbb{E}(\xi_n - \xi)^2 \to 0 \text{ as } n \to \infty$$

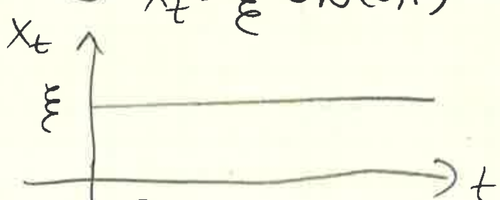
$$\xi_n \xrightarrow{P} \xi \iff \forall \varepsilon > 0,\ P\{|\xi_n - \xi| > \varepsilon\} \to 0 \text{ as } n \to \infty$$

$$\xi_n \xrightarrow{d} \xi \iff \mathbb{P}\{\xi_n \le x\} \to \mathbb{P}\{\xi \le x\} \text{ as } n \to \infty,\ \forall x \in \mathbb{R} \text{ - point}$$
of continuity of $\mathbb{P}\{\xi \le x\}$ (also called weak convergence)

$$\begin{array}{ccccc} a.s. & \Rightarrow & P & \Rightarrow & d \\ & \nearrow & & \nwarrow & \\ \mathcal{L}^2 & & & & d \to \text{constant a.s.} \end{array}$$

(weak convergence to a constant a.s.)

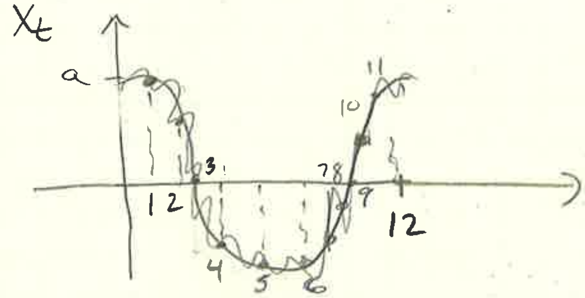
Ex 1 $X_t = \xi \sim N(0,1)$



$$\left.\begin{array}{l} m(t) = 0 \\ K(t,s) = \mathrm{Var}\,\xi = 1 \end{array}\right\} \Rightarrow \text{stationary}$$

$$\frac{1}{T}\sum_{t=1}^{T} X_t = \xi \ne C \Rightarrow \text{non-ergodic}$$

② $X_t = \varepsilon_t + a\cos\frac{\pi t}{6}$, $a \neq 0$, $\varepsilon_1, \varepsilon_2, \dots$ - i.i.d. $N(0,1)$



$m(t) = a\cos\frac{\pi t}{6} \neq \text{const}$
$\Rightarrow$ not stationary

$$\frac{1}{T}\sum_{t=1}^{T} X_t \sim N\Big(\frac{a}{T}\sum_{t=1}^{T}\cos\frac{\pi t}{6},\ \underbrace{\frac{1}{T}}_{\xrightarrow{T\to\infty} 0}\Big)$$

pt 1 = - pt 5
pt 2 = - pt 4
pt 3 = 0
⋮

first 3 elements

$$\Rightarrow \left|\frac{a}{T}\sum_{t=1}^{T}\cos\frac{\pi t}{6}\right| \leq \frac{a}{T}\cdot 3 \to 0 \text{ as } T\to\infty$$

$\Rightarrow$ ergodic

6.2 Ergodicity of wide-sense stationary processes

Proposition: X_t - discrete time s.p.; $\exists \alpha$ s.t. $|K(s,t)| < \alpha$

$C(T) := \operatorname{cov}(X_T, M_T)$, $M_T := \frac{1}{T}\sum_{t=1}^{T} X_t$

$$\Rightarrow \boxed{\operatorname{Var} M_T \xrightarrow[T\to\infty]{} 0 \iff C(T) \xrightarrow[T\to\infty]{} 0}$$

Corollary: X_t - w.s.; $\gamma(\cdot)$ - autocovariance function

(i) $\frac{1}{T}\sum_{r=0}^{T-1}\gamma(r) \xrightarrow[T\to\infty]{} 0 \Rightarrow X_t$ - ergodic

(ii) $\gamma(r) \xrightarrow[r\to\infty]{} 0 \Rightarrow X_t$ - ergodic

Proof (i) $\mathbb{E}X_t = c \Rightarrow \mathbb{E}M_T = c$

$$\operatorname{Var} M_T = \mathbb{E}\left[(M_T - c)^2\right] \overset{?}{\to} 0 \Rightarrow M_T \xrightarrow{\mathcal{L}^2} c \Rightarrow \underbrace{M_T \xrightarrow{P} c}_{\Downarrow\ X_t \text{ ergodic}}$$

$$C(T) = \operatorname{cov}\Big(X_T, \frac{1}{T}\sum_{t=1}^{T} X_t\Big)$$

$$= \frac{1}{T}\sum_{t=1}^{T}\operatorname{cov}(X_T, X_t) = \frac{1}{T}\sum_{t=1}^{T}\gamma(T-t) = \frac{1}{T}\sum_{r=0}^{T-1}\gamma(r) \xrightarrow[T\to\infty]{} 0$$

$\square$

proof (ii) Stolz-Cesaro thm: a_n, b_n - sequences in $\mathbb{R}$

b_n - strictly increasing to unbounded

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = g \Rightarrow \frac{a_n}{b_n} \xrightarrow[n\to\infty]{} g$$

$$a_n = \sum_{r=0}^{n-1} \gamma(r) ; \quad b_n = n$$

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \frac{\gamma(n-1)}{1} \to 0 = g$$

$$\Rightarrow \frac{1}{n} \sum_{r=0}^{n-1} \gamma(r) \xrightarrow[n\to\infty]{} 0 \Rightarrow X_t \text{ - ergodic} \quad \square$$

ex N_t - poisson process, λ.

$p > 0$: $X_t = N_{t+p} - N_t$

$\mathbb{E}X_t = \lambda(t+p) - \lambda t = \lambda p$

$$K(t,s) = \gamma(t-s), \text{ where } \gamma(r) = \begin{cases} \lambda(p - |r|), & |r| \le p \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \gamma(r) \xrightarrow[r\to\infty]{} 0 \overset{(ii)}{\Longrightarrow} X_t \text{ - ergodic}$$

ex $X_t = A\cos(\omega t) + B\sin(\omega t)$, A, B - r.v., $\omega = \frac{\pi}{20}$

$cov(A,B) = 0$

$\mathbb{E}A = \mathbb{E}B = 0$, $Var A = Var B = 1$

$$\left.\begin{aligned} &\mathbb{E}X_t = 0 \\ &K(t,s) = \cos(\omega(t-s)) \end{aligned}\right\} X_t \text{ - w.s.}$$

$\gamma(r) = \cos \omega r$

$$\left| \frac{1}{T} \sum_{r=0}^{T-1} \cos \omega r \right| \le \frac{10}{T} \to 0 \text{ as } T \to \infty$$

$\Rightarrow X_t$ - ergodic

A s.p. is stationary $\Leftrightarrow$ A - ergodic

This is incorrect. Ergodic implies stationary only.

6.3. Definition of a stochastic derivative

Stochastic derivative: X_t differentiable at $t=t_0$ if

$$\frac{X_{t_0+h}-X_{t_0}}{h} \xrightarrow[h\to 0]{\mathcal{L}^2} \eta =: X'_{t_0}$$

$$\mathbb{E}\left(\frac{X_{t_0+h}-X_{t_0}}{h}-\eta\right)^2 \xrightarrow[h\to 0]{} 0$$

Prop: $\mathbb{E}X_t^2<\infty$. Then X_t - differentiable at $t=t_0$ $\Longleftrightarrow$

$$\begin{cases} m(t)=\mathbb{E}X_t \text{ - differentiable at } t=t_0 \\ \dfrac{\partial^2}{\partial t\,\partial s}K(t,s)\ \exists \text{ at } (t_0,t_0) \end{cases}$$

Ex 1 X_t - w.s. $\Rightarrow m(t)=\text{const}$, $K(t,s)=\gamma(t-s)$

$$\left.\frac{\partial^2 K}{\partial t\,\partial s}\right|_{(t_0,t_0)} = -\gamma''(0)$$

Thus. w.s - differentiable $\Longleftrightarrow$ $\gamma''(0)$ exists

If $\gamma(r)=e^{-\alpha|r|}$ $\Rightarrow X_t$ - not differentiable

If $\gamma(r)=\cos(\alpha r)$ $\Rightarrow X_t$ - differentiable $\forall t$

2 Brownian Motion is not differentiable at any $t=t_0$

$K(t,s)=\min(t,s)$

$$\frac{K(t_0+h,t_0)-K(t_0,t_0)}{h} = \frac{\min(t_0,t_0+h)-t_0}{h} = \begin{cases} 0, & h>0 \\ 1, & h<0 \end{cases}$$

$$\Rightarrow \lim_{h\to 0}\frac{K(t_0+h,t_0)-K(t_0,t_0)}{h} \text{ doesn't exist}$$

3 X_t - independent increments, $X_0=0$ a.s.

$K(t,s)=\mathrm{Var}\,X_{\min(t,s)}$

Most of the time, not differentiable

$$K(t,s) = \mathrm{cov}(X_t, X_s) \overset{t>s}{=} \underbrace{\mathrm{cov}(X_t - X_s, \underset{X_s - X_0}{X_s})}_{=0} + \underbrace{\mathrm{cov}(X_s, X_s)}_{= \mathrm{Var} X_s}$$

$\Rightarrow$ in general, $K = \mathrm{Var} X_{\min(t,s)}$

6.4 Relation between differentiability and properties of the covariance function

Continuity in the mean-squared sense if $X_t \xrightarrow[t \to t_0]{\mathcal{L}^2} X_{t_0}$

$\Leftrightarrow \mathbb{E}(X_t - X_{t_0})^2 \xrightarrow[t \to t_0]{} 0$.

Let $\mathbb{E}X_t = 0 \Rightarrow$ Prop: (i) $K(t,s)$ is continuous at (t_0, t_0)

$\Downarrow$

X_t is cont in the m.s. at $t = t_0$

(ii) X_t is cont in m.s. sense at $t = t_0$ and $t = s_0$

$\Downarrow$

$K(t,s)$ is cont. at (t_0, s_0)

proof (i) $\mathbb{E}(X_t - X_{t_0})^2 = \mathbb{E}X_t^2 - 2\mathbb{E}X_t \mathbb{E}X_{t_0} + \mathbb{E}X_{t_0}^2$

$= K(t,t) - 2K(t,t_0) + K(t_0,t_0) \xrightarrow[t \to t_0]{} 0$

(ii) $K(t,s) - K(t_0,s_0) = K(t,s) + K(t_0,s) - K(t_0,s) - K(t_0,s_0)$

$= (K(t,s) - K(t_0,s)) + (K(t_0,s) - K(t_0,s_0))$

$|K(t,s) - K(t_0,s)| = |\mathbb{E}[(X_t - X_{t_0})X_s]| \leq \sqrt{\mathbb{E}(X_t - X_{t_0})^2}\sqrt{\mathbb{E}X_s^2} \xrightarrow[t \to t_0]{} 0$

Same for $K(t_0,s) - K(t_0,s_0)$

$\square$

Corollary: $K(t,s)$ is continuous at $(t_0,s_0) \Leftrightarrow K(t,s)$ is cont at (t_0,t_0)
(holds for any covariance function)

proof: $K(t,s)$ is cont at $(t_0,t_0) \overset{(i)}{\Rightarrow} X_t$ is cont m.s. at $t = t_0$

$\overset{(ii)}{\Rightarrow} K(t,s)$ is cont at (t_0,s_0)

$\square$

7.1 Different types of stochastic integrals. Integrals of type $\int X_t\,dt$ (1)

Stochastic integration

$$\int_a^b X_t\,dt\ ,\quad \int_a^b f(t)\,dW_t\ ,\quad \int_a^b X_t\,dW_t\ ,\quad \int_a^b X_t\,dH_t$$

$X_t : \Omega \times \mathbb{R}_+ \to \mathbb{R}$

Fix $\omega \Rightarrow \int_a^b X_t(\omega)\,dt = \lim_{\max_k |t_k - t_{k-1}| \to 0} \sum_{k=1}^n X_{t_{k-1}}(\omega)(t_k - t_{k-1})$

$t = a < t_1, \dots < t_n = b$ limit in m. sq. sense

$$\Rightarrow \quad \mathbb{E}\left(\sum_{k=1}^n X_{t_{k-1}}(\omega)(t_k - t_{k-1}) - \int_a^b X_t(\omega)\,dt\right)^2 \xrightarrow[\max_k |t_k - t_{k-1}| \to 0]{} 0$$

<u>Thm</u>: $m(t)$ - continuous, $K(t,s)$ - cont $\Rightarrow \exists \int_a^b X_t\,dt$

$$\mathbb{E}\int_a^b X_t\,dt \underset{\text{Fubini}}{=} \int_a^b \mathbb{E}X_t\,dt$$

$$\operatorname{Var}\int_a^b X_t\,dt \overset{\text{Thm}}{=} \int_a^b\int_a^b K(t,s)\,dt\,ds$$

7.2 Integrals of type $\int X_t\,dt$ (2)

Stochastic integrals $\int_a^b X_t\,dt$

$$\int_a^b X_t\,dt = \lim_{\max_i |t_i - t_{i-1}| \to 0} \sum_{k=1}^n X_{t_{k-1}}(t_k - t_{k-1})$$

limit in mean square sense

$t_0 = a \quad t_1 \quad t_2 \quad \dots \quad b = t_n$

<u>Thm</u>: X_t: Stochastic process with $\mathbb{E}X_t^2 < \infty$

$$\left.\begin{array}{l} m(t) - \text{continuous} \\ K(t,s) - \text{continuous} \end{array}\right\} \Rightarrow \int_a^b X_t\,dt \text{ exists}$$

$K(t,s)$ is continuous $\forall (t_0, s_0) \Leftrightarrow K(t,s)$ is continuous $\forall (t_0, t_0)$

i.e. $K(t,s)$ is continuous on the diagonal

(Special thing true for covariance functions)

1) $K(t,s)$ - cont at (t_0,t_0) $\Rightarrow$ X_t - cont at t_0 in mean square sense

i.e. $\mathbb{E}(X_t - X_{t_0})^2 \xrightarrow[t \to t_0]{} 0$

2) X_t is cont in t_0, s_0 $\Rightarrow$ $K(t_0,s_0)$ is continuous

Properties

1) $\mathbb{E}\left[\int_a^b X_t\,dt\right] = \int_a^b \mathbb{E}X_t\,dt$ (Fubini theorem)

2) $\mathbb{E}\left[\underbrace{\left(\int_a^b X_t dt\right)^2}\right] = \int_a^b\int_a^b \mathbb{E}[X_t X_s]\,dt\,ds$

$\overset{\shortparallel}{\int_a^b\int_a^b X_t X_s\,dt\,ds}$

3) $Var\left[\int X_t\,dt\right] = \int_a^b\int_a^b K(t,s)\,dt\,ds \overset{\text{symmetry}}{=} 2\int_a^b\int_a^s K(t,s)\,dt\,ds$

7.3 : Integrals of type $\int f(t)\,dW_t$ (1)

$\int_a^b f(t)\,dW_t$, W_t - Brownian motion (Wiener integral)

$f \in \mathcal{L}^2([a,b])$ (Hilbert space) $\Longleftrightarrow$ $\int_a^b f^2(x) < \infty$

Def: inner product of f,g:

$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$

a) $\langle f,g\rangle = \langle g,f\rangle$

b) $\langle a_1 f_1 + a_2 f_2, g\rangle = a_1\langle f_1,g\rangle + a_2\langle f_2,g\rangle$

c) $\langle f,f\rangle \geq 0$, $\langle f,f\rangle = 0 \Leftrightarrow f=0$

$f_n \xrightarrow{\mathcal{L}^2} f \Longleftrightarrow \langle f_n - f, f_n - f\rangle \xrightarrow[n\to\infty]{} 0$

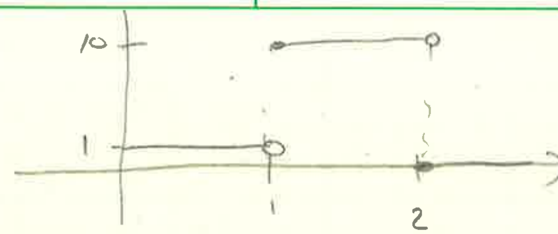
$\Longleftrightarrow \int_a^b (f_n(x) - f(x))^2\,dx \to 0$

Stage 1: Step function $\Longleftrightarrow$ $f(x) = \sum_{i=1}^n \alpha_i \mathbb{1}\{t_{i-1} \leq x < t_i\}$, $\alpha_i \in \mathbb{R}$

$a = t_0 < t_1 < \ldots < t_n = b$

$\int_a^b f(t)\,dW_t = \sum_{i=1}^n \alpha_i (W_{t_i} - W_{t_{i-1}})$

ex $f(t) = \begin{cases} 1 & , 0 \le t < 1 \\ 10 & , 1 \le t < 2 \\ 0 & , t \ge 2 \end{cases}$



$$\int_0^T f(t)\,dW_t = \begin{cases} W_T & , T<1 \\ W_1 + 10(W_T - W_1) & , 1 \le T < 2 \\ W_1 + 10(W_2 - W_1) & , T \ge 2 \end{cases}$$

Thm: $I(f) := \int_a^b f(t)\,dW_t$. If f - step function $\Rightarrow I(f) \sim N\left(0, \int_a^b f^2(x)\,dx\right)$

Proof:

$$\mathbb{E}[I(f)] = \sum_{i=1}^n \alpha_i \underbrace{\mathbb{E}[W_{t_i} - W_{t_{i-1}}]}_{0} = 0$$

$$Var[I(f)] \overset{\text{indep increments}}{=} \sum_{i=1}^n \alpha_i^2 \underbrace{Var(W_{t_i} - W_{t_{i-1}})}_{t_i - t_{i-1}} = \int_a^b f^2(x)\,dx$$

□

7.4 Integrals of type $\int f(t)\,dW_t$ (2)

$I(f) = \int_a^b f(t)\,dW_t$, $f \in \mathcal{L}^2(a,b)$

Stage 2: $f \in \mathcal{L}^2(a,b)$

f_n – step functions s.t. $f_n \xrightarrow{\mathcal{L}^2} f$: $\int_a^b (f_n(t) - f(t))^2\,dt \underset{n\to\infty}{\longrightarrow} 0$

$I(f) = \lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} \int_a^b f_n(t)\,dW_t$ (lim in mean squared sense)

$$\mathbb{E}\left[(I(f_n) - I(f))^2\right] \underset{n\to\infty}{\longrightarrow} 0$$

1. Why $I(f)$ does not depend on f_n?
2. Properties of $I(f)$?
3. Construction of f_n?

Thm: $f_n, \tilde{f}_n$ - sequences of step functions, $f_n \xrightarrow{\mathcal{L}^2} f$, $\tilde{f}_n \xrightarrow{\mathcal{L}^2} f$.

$\rightarrow \lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} I(\tilde{f}_n)$ [mean squared limits]

<u>proof</u> $I(f_n) - I(\tilde{f}_n) = I(\underbrace{f_n - \tilde{f}_n}_{\text{step fcn}}) \sim N\left(0, \int_a^b (f_n(x) - \tilde{f}_n(x))^2 dx\right)$

$$\mathbb{E}\left[(I(f_n) - I(\tilde{f}_n))^2\right] = \int_a^b (f_n(x) - \tilde{f}_n(x))^2 dx \xrightarrow[n\to\infty]{} 0 \quad \left(\text{since } f_n \xrightarrow{\mathcal{L}^2} f, \tilde{f}_n \xrightarrow{\mathcal{L}^2} f\right)$$

$\square$

1 answered

2. <u>Thm</u>: $\forall f \in \mathcal{L}^2(a,b)$, $I(f) \sim N\left(0, \int_a^b f^2(x)\, dx\right)$

<u>proof</u>

$I(f) = \lim_{n\to\infty} I(f_n)$, $I(f_n) \sim N\left(0, \int_a^b f_n^2(x)\, dx\right)$

For normal r.v.'s, $\mathbb{E}[\lim f_n] = \lim \mathbb{E}[f_n]$, $\mathrm{Var}[\lim f_n] = \lim[\mathrm{Var} f_n]$

$\Rightarrow I(f) \sim N\left(0, \lim_{n\to\infty} \int_a^b f_n^2(x)\, dx\right)$, $\lim_{n\to\infty} \int_a^b f_n^2(x) dx = \int_a^b f^2(x) dx$

$\square$

<u>7.5 Integrals of type $\int X_t\, dW_t$ (1)</u>

$I(X_t) := \int_a^b X_t\, dW_t$ (W_t - $\mathcal{F}_t$-Brownian motion)

Filtration - a sequence of σ-algebras $\mathcal{F}_t$ on $(\Omega, \mathcal{F}, \mathbb{P})$:

$\mathcal{F}_t \subset \mathcal{F}_s$, $\forall t \le s$

<u>$\mathcal{L}^2_{ad}([a,b], \Omega)$</u> (ad means adapted)

1) X_t - $\mathcal{F}_t$-adapted, i.e. X_t - $\mathcal{F}_t$-measurable, $\forall t$

$\{X_t \in B\} \subset \mathcal{F}_t$, $\forall t$, $\forall B \in \mathcal{B}(\mathbb{R})$

2) $\int_a^b \mathbb{E} X_t^2\, dt < \infty$

W_t - $\mathcal{F}_t$-Brownian motion if

1) W_t - $\mathcal{F}_t$-adapted

2) $W_t - W_s \perp\!\!\!\perp \mathcal{F}_s$, $\forall t > s$

Define $I(X_t)$:

1) Step processes: $\sum \xi_{i-1} \mathbb{1}\{t_{i-1} \le t < t_i\}$

2) $X_t \in \mathcal{L}^2_{ad}$

7.6: *Integrals of the type $\int X_t dW_t$ (2)*

$\int_a^b X_t dW_t$, $X_t \in \mathcal{L}^2_{ad}$, W_t - $\mathcal{F}_t$-Brownian motion

Stage 1: $X_t = \sum_{i=1}^n \xi_{i-1} \mathbb{1}\{t_{i-1} \le t < t_i\}$

$$I(X_t) = \sum_{i=1}^n \xi_{i-1}(W_{t_i} - W_{t_{i-1}})$$

Stage 2: $X_t \in \mathcal{L}^2_{ad}$; X_t^n - step processes

$$\int_a^b \mathbb{E}(X_t^n - X_t)^2 dt \xrightarrow[n\to\infty]{} 0$$

$$I(X_t) = \lim_{n\to\infty} I(X_t^n): \quad \mathbb{E}\left(I(X_t^n) - I(X_t)\right)^2 \xrightarrow[n\to\infty]{} 0$$

Thm: $m(t)$ is continuous, $K(t,s)$ is continuous

$$X_t^n = \sum_{i=1}^n X_{t_{i-1}} \mathbb{1}\{t_{i-1} \le t < t_i\}$$

$$X_t^n \longrightarrow X_t: \quad \int_a^b \mathbb{E}(X_t^n - X_t)^2 dt \to 0$$

proof:
$$\begin{aligned}\mathbb{E}(X_t - X_s)^2 &= \mathbb{E}X_t^2 - 2\mathbb{E}X_tX_s + \mathbb{E}X_s^2 \\ &= [K(t,t) + m^2(t)] - 2(K(t,s) + m(t)m(s)) \\ &\quad + [K(s,s) + m^2(s)] \xrightarrow[s\to t]{} 0\end{aligned}$$

$$\Rightarrow X_t^n \xrightarrow[n\to\infty]{} X_t \quad (\text{m. sq.})$$

$$\Rightarrow \int_a^b \lim \mathbb{E}(X_t^n - X_t)^2 dt \to 0 \underset{\text{Dominated convergence thm}}{\Rightarrow} \lim \int_a^b \mathbb{E}(X_t^n - X_t)^2 dt \to 0$$

Dominated convergence thm

$$\int \lim_{n\to\infty} f(n,t)dt = \lim_{n\to\infty} \int f(n,t)dt \quad \text{if } \exists M(t): |f(n,t)| \le M(t), \int M(t)dt < \infty$$

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$$\mathbb{E}(X_t^n - X_t)^2 \le 2\mathbb{E}(X_t^n)^2 + 2\mathbb{E}X_t^2 \qquad \left[(a-b)^2 \le 2a^2+2b^2\right]$$

$$\le 4 \max_{t\in[a,b]} \mathbb{E}X_t^2 = 4 \max_{t\in[a,b]} \left[k(t,t)+m^2(t)\right]$$

$\square$

(Ex) $$\int_0^t W_s\,dW_s = \lim_{n\to\infty} \sum_{i=1}^n W_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})$$

$$= \lim_{n\to\infty}\left(-\frac{1}{2}\underbrace{\sum_{i=1}^n (W_{t_i}-W_{t_{i-1}})^2}_{\text{quadratic variation of } W_t \to t} + \frac{1}{2}\underbrace{\sum_{i=1}^n W_{t_i}^2 - W_{t_{i-1}}^2}_{W_t^2}\right)$$

$$= -\frac{t}{2} + \frac{W_t^2}{2}$$

7.7 Integrals of the type $\int X_t\,dY_t$ where Y_t is an Itô process

$\int_a^b X_t\,dH_t$, H_t - Itô process

$$H_t = H_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s \iff dH_t = b_t\,dt + \sigma_t\,dW_t$$

$\mathcal{F}_t$ - filtration ; b_s, σ_s - processes adapted to $\mathcal{F}_s$

W_t - $\mathcal{F}_t$ - Brownian motion

H_0 - measurable w.r.t. $\mathcal{F}_0$

X_t : $\int_a^b |X_s b_s| + X_s^2\sigma_s^2\,ds < \infty$, then

$$\int_a^b X_t\,dH_t = \int_a^b b_s X_s\,ds + \int_a^b \sigma_s X_s\,dW_s$$

Thm: H_t - Itô process , $f(t,x)$ - twice continuously differentiable

Then,
$$f(t,H_t) = f(0,H_0) + \int_0^t \frac{\partial f}{\partial t}(s,H_s)\,ds + \int_0^t \frac{\partial f}{\partial x}(s,H_s)\,dH_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,H_s)\,\sigma_s^2\,ds$$

(Itô formula)

7.8 Itô's formula

$\int_0^t g(s,W_s)\,dW_s$, f - antiderivative of g w.r.t. 2^{nd} argument

i.e., $\frac{\partial f}{\partial x} = g$ $(\sigma^2 = 1 \text{ for } W_t)$

$H_t = W_t$

$$f(t,W_t) = f(0,\overset{0}{W_0}) + \int_0^t \frac{\partial f}{\partial t}(s,W_s)\,ds + \int_0^t g(s,W_s)\,dW_s + \frac{1}{2}\int_0^t \frac{\partial g}{\partial x}(s,W_s)\,ds$$

$$\int_0^t g(s,W_s)\,dW_s = f(t,W_t) - f(0,0) - \int_0^t \left[\frac{\partial f}{\partial t}(s,W_s) + \frac{1}{2}\frac{\partial g}{\partial x}(s,W_s)\right] ds$$

ex $\int_0^t W_s\,dW_s$: $g(t,x) = x$, $f(t,x) = \frac{1}{2}x^2 + h(t)$ doesn't count

$$\int_0^t W_s\,dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}\int_0^t ds = \frac{1}{2}W_t^2 - \frac{1}{2}t$$

7.9 Calculation of stochastic integrals using the Itô formula. Black-Scholes model

Black-Scholes model

$dX_t = X_t\mu\,dt + X_t\sigma\,dW_t$, $\sigma > 0$

$\Leftrightarrow X_t = X_0 + \int_0^t X_s\mu\,ds + \int_0^t X_s\sigma\,dW_s$

(Itô formula in differential form:
$$df(t,H_t) = \frac{\partial f}{\partial t}(t,H_t)\,dt + \frac{\partial f}{\partial x}(t,H_t)\,dH_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,H_t)\,\sigma_t^2\,dt$$)

$f(t,x) = \ln x$, $H_t = X_t$

$$d(\ln X_t) = 0 + \frac{1}{X_t}dX_t - \frac{1}{2X_t^2}(X_t\sigma)^2\,dt$$

$$\Rightarrow d(\ln X_t) = \frac{1}{X_t}[X_t\mu\,dt + X_t\sigma\,dW_t] - \frac{\sigma^2}{2}dt$$

$$= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\,dW_t$$

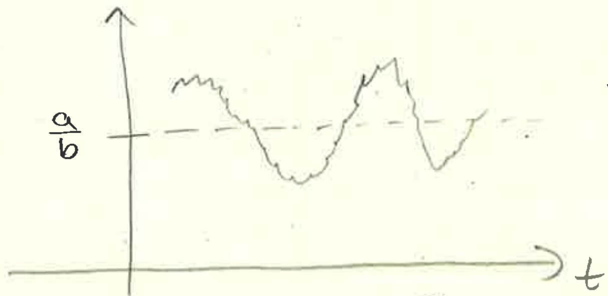
$$\Rightarrow \boxed{X_t = X_0\exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}}$$

X_t - a.s. continuous trajectories

7.10 Vasicek model: Application of the Itô formula to stochastic modeling

Vasicek model

$$dX_t = \underbrace{(a - bX_t)}_{b\left(\frac{a}{b} - X_t\right)} dt + c\, dW_t \quad , \; a \in \mathbb{R},\; b, c > 0.$$



b - speed of reversion

$$f(t,x) = x e^{bt} \quad \left(\frac{\partial^2 f}{\partial x^2} = 0\right)$$

$$d(X_t e^{bt}) = \cancel{bX_t e^{bt} dt} + e^{bt}\left[(a - \cancel{bX_t})dt + c\, dW_t\right] + \frac{1}{2}(0)$$

$$= a e^{bt} dt + c e^{bt} dW_t$$

$$\Rightarrow \boxed{X_t = e^{-bt} X_0 + \frac{a}{b}\left(1 - e^{-bt}\right) + \int_0^t e^{-bs} dW_s}$$

7.11 Ornstein-Uhlenbeck process. Application of the Itô formula to stochastic modelling

Ornstein - Uhlenbeck process

$$m\, dV_t = dW_t - \lambda V_t\, dt \quad , \; \lambda \text{ - friction coefficient}, \; m \text{ - mass}$$

V_t - velocity

$$f(t,x) = x e^{\frac{\lambda}{m} t}$$

$$\Rightarrow V_t = e^{-\frac{\lambda}{m} t}\left(V_0 + \frac{1}{m}\int_0^t e^{\frac{\lambda}{m} s} dW_s\right)$$

If $V_0 \sim N\left(0, \frac{1}{2\lambda m}\right) \perp\!\!\!\perp W_t \Rightarrow V_t$ - Gaussian process with

$$K(t,s) = \frac{m}{2\lambda} e^{-\frac{\lambda}{m}|t-s|} \quad \text{(stationary in both strict and weak senses)}$$

TOPS. 35500

Week 8: Lévy processes

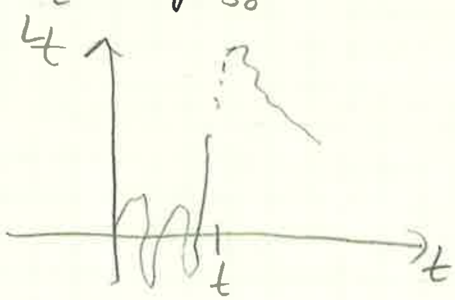
8.1 Definition of a Lévy process. Stochastic continuity and càdlàg paths

N_t	W_t	Lévy process
• $N_0 = 0$ a.s.	• $W_0 = 0$ a.s.	• $L_0 = 0$ a.s.
• independent increments: [$\forall t_0 < t_1 < \dots < t_n$, $X_{t_n} - X_{t_{n-1}}, \dots, X_1 - X_0$ are jointly indep]	• Same	• Same
• Stationary increments [$\forall t, s \geq 0$, $\forall h > 0$ $X_{t+h} - X_{s+h} \overset{d}{=} X_t - X_s$]	• Same	• Same
• $N_t - N_s \sim \text{Poisson}(\lambda(t-s))$	• $W_t - W_s \sim N(0, t-s), \forall t > s$	• $L_t - L_s \sim \mathcal{P}(t-s)$ ← infinitely divisible distributions

Assume L_t is stochastically continuous: $L_{t+h} \xrightarrow[h \to 0]{\mathbb{P}} L_t$

i.e. $\mathbb{P}\{|L_{t+h} - L_t| > \varepsilon\} \xrightarrow[h\to 0]{} 0$, $\forall \varepsilon > 0$

Ex. $L_t = \log \frac{S_t}{S_0}$, S_t is a stock price



L_t jumps at $t =$ May 1st at 2

$\mathbb{P}\{|L_{t+h} - L_t| > 1\} = 1 \not\to 0$

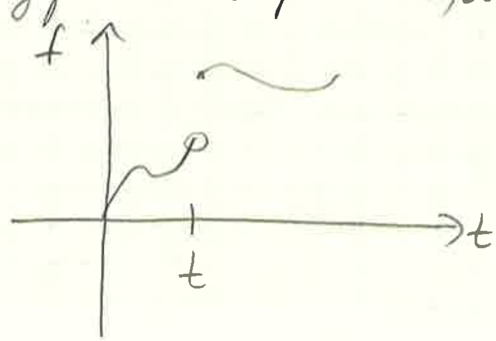
This is not a Lévy Process

Trajectory of Lévy processes: Càdlàg functions (right cont., left limits)

f - a càdlàg function

$\exists \lim_{\substack{s \to t \\ s < t}} f(s) =: f(t-)$

$\exists \lim_{\substack{s \to t \\ s > t}} f(s) =: f(t+) = f(t)$



8.2 Examples of Lévy processes. Calculation of the characteristic function in particular cases

Infinitely divisible distributions

ξ - inf. div. dist., if $\forall n \geq 2$, $\xi \stackrel{d}{=} Y_1 + \dots + Y_n$, $Y_1, \dots, Y_n$ - iid.

$\Leftrightarrow \phi_\xi(u) = (\phi_{Y_1}(u))^n \Rightarrow (\phi_\xi(u))^{1/n}$ - characteristic fcn. $\forall n$

<u>Proposition</u> (i) $\forall$ Lévy pr. L_t at any t has an inf. div. dist.

(ii) $\forall$ inf div. dist. $\exists$ Lévy pr L_t where L_1 has this dist.

$$L_t = \sum_{k=1}^{n} \underbrace{\left(L_{t\cdot\frac{k}{n}} - L_{t\frac{k-1}{n}} \right)}_{\substack{\| d \\ L_{t/n}}}$$

(Ex) $\xi \sim N(\mu, \sigma^2)$ - inf. div. dist $\Longleftrightarrow \mu t + \sigma W_t \Leftarrow$ Lévy process

$\xi \stackrel{d}{=} Y_1 + \dots + Y_n$, $Y_k \stackrel{iid}{\sim} N\left(\frac{\mu}{n}, \frac{\sigma^2}{n}\right)$

$$\phi_\xi(u) = e^{i\mu u - \frac{1}{2}\sigma^2 u^2}$$

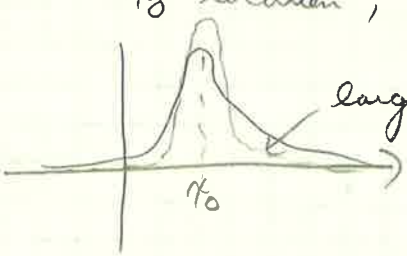
$$(\phi_\xi(u))^{1/n} = e^{i\frac{\mu}{n}u - \frac{1}{2}\frac{\sigma^2}{n}u^2}$$

(Ex) Cauchy distribution - inf. div. dist.

$$p(x) = \frac{1}{\pi\gamma\left(1 + \frac{(x-x_0)^2}{\gamma^2}\right)} \quad \text{[no expectation]}$$

x_0 - location, γ - scale

$$\phi(u) = e^{x_0 iu - \gamma|u|}$$



$$(\phi(u))^{1/n} = e^{\frac{x_0}{n}iu - \frac{\gamma}{n}|u|}$$

(Ex) Gamma distribution $\alpha > 0$, $\beta > 0$ - inf. div. dist.

$$p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0$$

skewness $= \frac{2}{\sqrt{\alpha}}$, kurtosis $= \frac{6}{\alpha}$, β-scale: $X \sim \Gamma(\alpha, \beta)$, $\lambda > 0$

$$\frac{X}{\lambda} \sim \Gamma(\alpha, \beta\lambda)$$

$\phi(u) = \left(1 - \frac{iu}{\beta}\right)^{-\alpha} \Rightarrow (\phi(u))^{1/n} = \left(1 - \frac{iu}{\beta}\right)^{-\alpha/n} \sim \Gamma\left(\frac{\alpha}{n}, \beta\right)$

Normal dist. } → also stable
Cauchy dist. }
Gamma dist.
Exponential dist.
Negative binomial
Geometric dist. → not stable
Poisson
Compound Poisson — $\phi(u) = e^{\lambda(\tilde{\phi}(u)-1)}$

} infinitely divisible distributions

Stable distribution: $\xi_1, \dots \xi_n$ - iid $\sim \xi$

$\xi_1 + \dots + \xi_n \stackrel{d}{=} a_n \xi + b_n$

Stable distribution $\Rightarrow$ inf. div. dist., but converse is not true

8.3 Relation to the infinitely divisible distributions

Bernoulli and Uniform are <u>not</u> inf. div.

Properties of inf. div. dist

$\boxed{1)\ \phi(u) = 0 \text{ does not have any } \mathbb{R} \text{ solutions}}$

$\xi \sim Unif(a,b) \Rightarrow \phi_\xi(u) = \int_a^b e^{iux} \frac{1}{b-a} dx \underset{u \neq 0}{=} \frac{1}{b-a} \frac{e^{iua}}{iu}\left(e^{iu(b-a)} - 1\right)$

$\phi_\xi(u) = 0 \Leftrightarrow e^{iu(b-a)} - 1 = 0$

$\Leftrightarrow u = \frac{2\pi k}{b-a}, \ k \in \mathbb{Z}/\{0\}$

$\boxed{2)\ supp(\xi) \text{ is unbounded}}$

$\quad \| $

$supp(\mathbb{P}_\xi)$ where $\mathbb{P}_\xi(B) = P\{\xi \in B\}$

for μ-measure, $supp(\mu) = \{x : \forall \text{ open set } G \text{ containing } x,\ \mu(G) > 0\}$

Let $\xi \sim Bernoulli(p)$, $\xi = \begin{cases} 1, & w.p.\ p \\ 0, & w.p.\ 1-p \end{cases}$ ~~$(+)$ $(+)$ $(+)$~~ not in $supp(\xi)$

$supp(\xi) = \{0, 1\}$

8.4 Characteristic exponent (same as cumulant)

<u>Prop</u>: $\forall$ Lévy process L_t, $\exists \psi : \mathbb{R} \to \mathbb{C}$: $\phi_{L_t}(u) = \mathbb{E}[e^{iuL_t}] = e^{t\cdot\psi(u)}$

Ex 1) $L_t = \mu t \Rightarrow \phi_{L_t}(u) = e^{iu\mu t} \Rightarrow \psi(u) = iu\mu$

2) $L_t = \sigma W_t \Rightarrow \psi(u) = -\frac{1}{2}\sigma^2 u^2$

3) $L_t = \sum_{k=1}^{N_t} \xi_k \Rightarrow \psi(u) = \lambda(\phi_{\xi_1}(u) - 1) = \int (e^{iux} - 1)\lambda \underbrace{F_{\xi_1}(dx)}_{f_\xi(x)dx}$

4) $L_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} \xi_k \Rightarrow \boxed{\psi(u) = iu\mu - \frac{1}{2}u^2\sigma^2 + \int (e^{iux} - 1)\lambda F_\xi(dx)}$

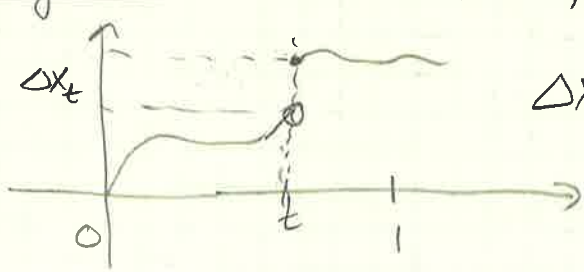
<u>Corollaries</u>

1. The dist of L_t is determined by L_t at one time moment, say L_1
   $\phi_{L_1}(u) = e^{\psi(u)} \Rightarrow \psi(u)$ - known $\Rightarrow \phi_{L_t}(u)$ - known

2. $\mathbb{E}L_t = t\,\mathbb{E}L_1$, $\text{Var}\,L_t = t\,\text{Var}\,L_1$, $K(t,s) = \min(t,s)\cdot \text{Var}\,L_1$
   $\Downarrow$
   L_t - typically not stationary

8.5 Properties of a Lévy process which directly follow from the existence of the characteristic exponent

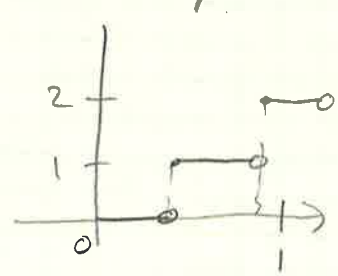
<u>Lévy measure</u> : $\forall B \subset \mathbb{R}\setminus\{0\}$, $\nu(B) = \mathbb{E}[\#\text{ jumps of size } \Delta X_t \in B \text{ for } t \in [0,1]]$



$\Delta X_t = X_t - X_{t-}$

Ex 1) Brownian motion : $\nu \equiv 0$

2) Poisson process : $\nu(B) = \begin{cases} 0, & 1 \notin B \\ \underset{\lambda}{\underset{\parallel}{\mathbb{E}N_1}}, & 1 \in B \end{cases} = \lambda \mathbb{1}\{1 \in B\}$



3) CPP $\left(\sum_{k=1}^{N_t} \xi_k\right) \Rightarrow \nu(B) = \lambda \mathbb{P}\{\xi_1 \in B\}$

if ξ_1 has a density $p_\xi \Rightarrow \nu(B) = \int_B \underbrace{\lambda p_\xi(x)}_{g(x)\text{ - Lévy density}} dx \quad , B \in \mathcal{B}(\mathbb{R})$

$$\boxed{\int_{|x|<1} x^2 \underbrace{\nu(dx)}_{\substack{\shortparallel \\ g(x)dx \text{ if } \nu \text{ has a density}}} < \infty} \quad \text{and} \quad \boxed{\int_{|x|>1} \nu(dx) < \infty}$$

8.6 Lévy-Khintchine representation and Lévy-Khintchine triplet (1)

Lévy-Khintchine theorem

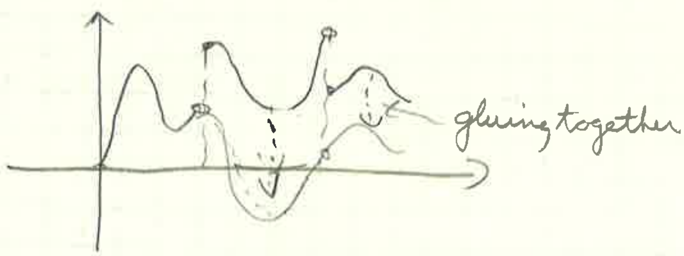
$$\psi(u) = iu\mu - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\,\mathbb{1}\{|x|<1\}\right)\nu(dx)$$

$\mu \in \mathbb{R}$, $\sigma \geq 0 \in \mathbb{R}$, ν - Lévy measure

(μ, σ, ν) - Lévy triplet

$X_t = \underbrace{\mu t + \sigma W_t}_{\text{cont part}} + \underbrace{J_t}_{\text{jump part}}$ For any Lévy process X_t

$$J_t \approx \underbrace{\sum_{\substack{0<s<t \\ |\Delta X_s|>1}} \Delta X_s}_{\text{CPP}} + \underbrace{\lim_{\varepsilon \to 0} \underbrace{\sum_{\substack{0<s<t \\ \varepsilon<|\Delta X_s|\leq 1}} \Delta X_s}_{\text{CPP}}}_{\text{not CPP}}$$



8.7 Lévy-Khintchine representation and Lévy-Khintchine triplet (2)

① X_t - of bounded variation

$$\sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}| \xrightarrow[\max|t_i - t_{i-1}| \to 0]{} \text{exists}$$

X_t - of bounded variation iff $\sigma = 0$, $\int_{|x|<1} x\nu(dx) < \infty$

$$\phi_{X_t}(u) = \exp\left\{t \cdot \left(iu\hat{\mu} + \int \left(e^{iux} - 1\right)\nu(dx)\right)\right\} \quad , \hat{\mu} = \mu - \int_{|x|<1} x\nu(dx)$$

② X_t - CPP $\iff \sigma = 0\,,\ \int_{\mathbb{R}} \nu(dx) = \nu(\mathbb{R}) < \infty$

Since $\int_{|x|<1} x\,\nu(dx) \le \int_{\mathbb{R}} \nu(dx) < \infty \Rightarrow \phi_{X_t}(u)$ is same as for bounded variation

③ X_t - subordinator, i.e. $X_t \ge 0$ a.s. $\iff X_t \ge X_s$ a.s. $\forall t > s$

$$X_t - X_s \overset{d}{=} X_{t-s} \ge 0$$

$\sigma = 0,\ \nu(\mathbb{R}_-) = 0,\ \int_0^1 x\,\nu(dx) < \infty \iff X_t$ - subordinator

and $\phi_{X_t}(u)$ is same as for bounded variation

8.8 Lévy-Khintchine representation and Lévy-Khintchine triplet (3)

$$\int_{|x|<1} x^2\,\nu(dx) < \infty\,,\quad \int_{|x|>1} \nu(dx) < \infty$$

$$\int_{\mathbb{R}} \left(e^{iux} - 1 - iux\,\mathbb{1}\{|x|<1\}\right)\nu(dx) =: J$$

$$J = \int_{|x|<1} \underbrace{(e^{iux} - 1 - iux)}_{= O(x^2)\cdot u^2}\nu(dx) + \int_{|x|\ge 1} \underbrace{(e^{iux} - 1)}_{\le 2 \text{ in absolute value}}\nu(dx) < \infty$$

Is it possible to find $r < 2$ s.t. $\int_{|x|<1} x^r\,\nu(dx) < \infty$?

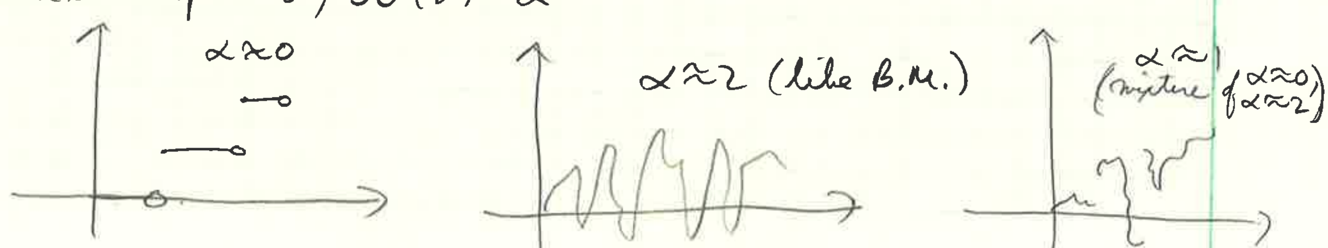
$\inf\left\{ r : \int_{|x|<1} x^r\,\nu(dx) < \infty \right\} =: BG(\nu)$, Blumenthal-Getoor index

S_t - Lévy process if $\forall a \ge 0, \exists\, b : \mathbb{R}_+ \to \mathbb{R} : \{S_{at}\}_{t\ge 0} \overset{d}{=} \{a^{1/\alpha} S_t + b(t)\}_{t \ge 0}$

$\alpha \in (0, 2]$, called α-stable

B.M. is $\alpha = 2$ stable

For a stable process, $BG(\nu) = \alpha$



8.9 Modeling of jump-type dynamics. Lévy-based models

X_t : $X_\Delta, X_{2\Delta}, \dots, X_{n\Delta}$ – data given on a Δ-step grid

how to estimate ν

X_t – bounded variation

$$\phi_{X_\Delta}(u) = \exp\left\{\Delta\left(iu\tilde{\mu} + \int_{\mathbb{R}}(e^{iux}-1)\,g(x)\,dx\right)\right\}$$

$$\phi'_{X_\Delta}(u) = \phi_{X_\Delta}(u)\,\Delta i\left(\tilde{\mu} + \int_{\mathbb{R}} e^{iux}x\,g(x)\,dx\right)$$

$$\phi''_{X_\Delta}(u) = \phi'_{X_\Delta}(u)\,\frac{\phi'_{X_\Delta}(u)}{\phi_{X_\Delta}(u)} + \phi_{X_\Delta}(u)\cdot(-\Delta)\int_{\mathbb{R}} \underbrace{e^{iux}x^2 g(x)\,dx}_{\mathcal{F}[x^2 g(x)](u)}$$

$$\mathcal{F}[x^2 g(x)](u) = \frac{1}{-\Delta}\left(\frac{\phi''_{X_\Delta}(u)}{\phi_{X_\Delta}(u)} - \left(\frac{\phi'_{X_\Delta}(u)}{\phi_{X_\Delta}(u)}\right)^2\right)$$

$$X_\Delta, X_{2\Delta}, \dots, X_{n\Delta} \longrightarrow \phi, \phi', \phi''$$

$$\hat{\phi}(u) = \frac{1}{n}\sum_{k=1}^{n} e^{iu(X_{k\Delta} - X_{(k-1)\Delta})} \quad \text{(consistent estimator)}$$

$$\Downarrow$$

$$\mathcal{F}[x^2 g(x)](u) \xrightarrow{\mathcal{F}^{-1}} g(x)$$

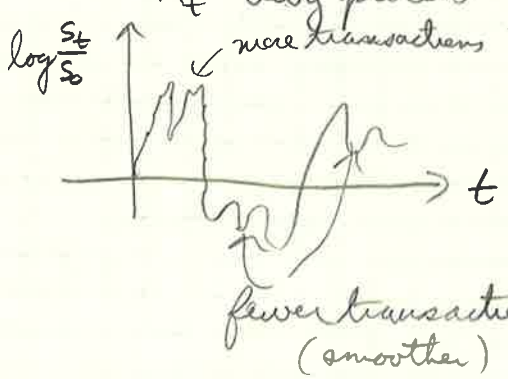
8.10 Time-changed stochastic processes. Monroe theorem

Lévy-based models

Stylized facts

① stochastic time change

X_t – Lévy process, $t \in T(s)$ – subordinator $\Big\} \Rightarrow X_{T(s)}$



$$\log\frac{S_t}{S_0} = W_{T(s)}$$

$T(s)$ = cumulative amount of transactions from $[0, S]$

Monroe's theorem: $\{W_{T(s)}\} = \{\text{all semimartingales}\}$

In this case $\boxed{\phi_{X_{T(s)}}(u) = \mathcal{L}_{T(s)}(-\psi(u))}$

(2) Stochastic volatility (σ is not constant)

$$d(\ln S_t) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\, dW_t$$

$$\sigma \to V_t \geq 0 \quad (V_t \text{ is stochastic volatility})$$

Cox-Ingersoll-Ross process

$$dV_t = (a - bV_t)dt + c\sqrt{V_t}\, dt, \quad a,b,c > 0$$