1.1 Difference between deterministic and stochastic world

	deterministic world	stochastic world
Single variable: Temp of a sick man	$\mathbb{R}$ $T = 39^\circ C$	random variable $\mathbb{E}$, Var, ...
Variables changing over time: T in first 3 days	$\mathbb{R}_+ \to \mathbb{R}$ $T(1) = 39$ $T(2) = 38.5$ $T(3) = 38$ $\vdots$	stochastic process

1.2 Difference between various fields of stochastics

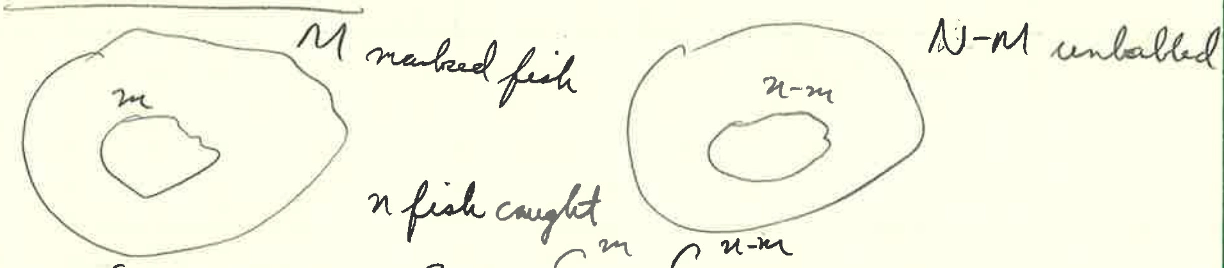
Stochastics
- probability theory
- mathematical statistics
- stochastic processes

Consider a pond that contains fish

Prob theory : # of fish at some given time (N)

$\mathbb{E}$, Var, or limit laws

Mathematical Stats :



$$P\{\#\text{ marked} = m\} = \frac{C_M^m \, C_{N-M}^{n-m}}{C_N^M}$$

Repeat $m_1, m_2, \dots, m_q$

$$\left(\text{log likelihood}\right) \sum_{k=1}^{q} P\{\#\text{ marked} = m_k\} \longrightarrow \max_N \quad (MLE)$$

1.3 Probability space $(\Omega, \mathcal{F}, P)$

General theory	Bernoulli Scheme $\begin{bmatrix} 1, & \text{success} \\ 0, & \text{failure} \end{bmatrix}$ $(a_1, \ldots, a_n)$, $a_i \in \{0,1\}$	$[0,1]$ Select point from
Ω - sample space	$\#\Omega = 2^n$, set of all vectors with components $\in \{0,1\}$	$\Omega = [0,1]$
$\mathcal{F}$ - σ-algebra 1) $\Omega \in \mathcal{F}$ 2) $A \in \mathcal{F} \Rightarrow \Omega \setminus A \in \mathcal{F}$ 3) $A_1, \ldots, A_n, \ldots \in \mathcal{F}$ $\Downarrow$ $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$	$\mathcal{F}$ = power set $\#\mathcal{F} = 2^{\#\Omega} = 2^{2^n}$	$P\{X \in [\alpha, \beta]\}$ $\Rightarrow [\alpha, \beta), (\alpha, \beta],$ $(\alpha, \beta), [\alpha, \beta), \{\beta\} \in \mathcal{F}$ Borel σ-algebra
P - probability measure 1) $P(\Omega) = 1$ 2) $A_1, A_2, \ldots \in \mathcal{F}$ (disjoint) $\Rightarrow P\{\bigcup_i A_i\} = \sum_i P(A_i)$ $P: \mathcal{F} \to [0,1]$	$P\{1\} = p$ $P\{0\} = 1 - p$	$P\{[\alpha, \beta]\} = \beta - \alpha$

1.4 Definition of a stochastic function. Types of stochastic functions.

$(\Omega, \mathcal{F}, P)$

Random variable - $\underbrace{\text{measurable}}_{}$ function $\xi: \Omega \to \mathbb{R}$.

$$\forall B \in \mathcal{B}(\mathbb{R}) : \xi^{-1}(B) \subset \mathcal{F}$$

T - time

$X: T \times \Omega \to \mathbb{R}$ - random function, if $\forall t \in T$: $X(t, \cdot)$ is a random variable on $(\Omega, \mathcal{F}, P)$, denoted X_t

If $T = \mathbb{R}_+$, this is called a random process or stochastic process

$T = \mathbb{R}_+^n$, random field or stochastic field

$T = \mathbb{N}$ or $\mathbb{Z}$, discrete time stochastic process

$T = \mathbb{R}_+$ or $\mathbb{R}$, continuous time stochastic process

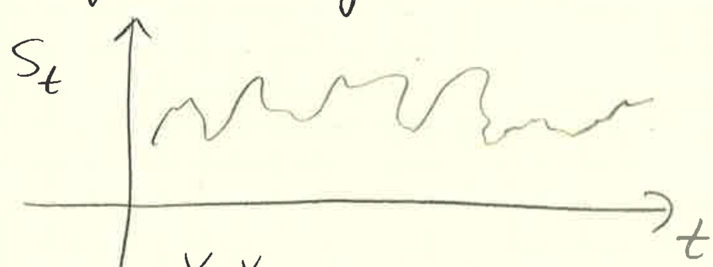
1.5 Trajectories and finite-dimensional distributions

$X : T \times \Omega \to \mathbb{R}$, $T = \mathbb{R}_+$

$\forall t \in T$: $X_t = X(t, \cdot)$ is a r.v. on $(\Omega, \mathcal{F}, P)$

Trajectory (= path)

X_t fix ω and get mapping $T \to \mathbb{R}$



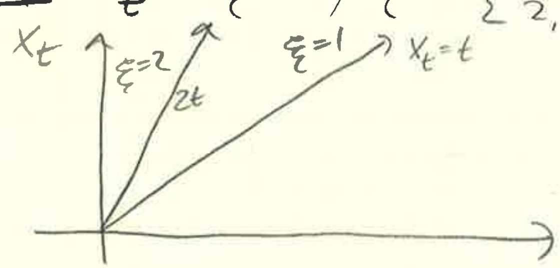
$X_1, X_2, \dots,$

Finite-dimensional distribution $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$, $t_1, \dots, t_n \in \mathbb{R}$

In mathematic stats, $X_{t_1}, X_{t_2} \dots X_{t_n}$ are independent

In stochastic process, $X_{t_1}, X_{t_2}, \dots, X_{t_n}$ are dependent

__Ex__: $X_t = \xi t$, $\xi = \begin{cases} 1, & \text{w.p. } 1/2 \\ 2, & \text{w.p. } 1/2 \end{cases}$



$P\{X_{t_1} \le x_1, X_{t_2} \le x_2\} =$
$$\begin{cases} 0, & \min\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}\right) < 1 \\ 1/2, & \quad " \quad \in [1,2] \\ 1, & \quad " \quad \ge 2 \end{cases}$$

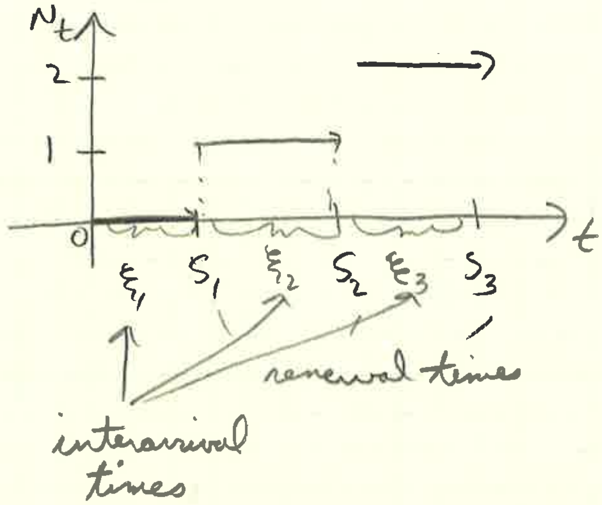
1.6 Renewal process. Counting process.

Renewal processes (discrete time)

$S_0 = 0,\ S_n = S_{n-1} + \xi_n$, where $\xi_1, \xi_2, \ldots$ – iid > 0 a.s.

$P\{\xi_i > 0\} = 1 \iff F(0) = 0$

$N_t = \underset{k}{\operatorname{argmax}} \{S_k \le t\}$ (Counting process)



$\{S_n > t\} = \{N_t < n\}$

$F \Rightarrow \mathbb{E} N_t \qquad S_n = \xi_1 + \cdots + \xi_n$

1.7. Convolution

Convolution $\quad X \perp\!\!\!\perp Y$

$X \sim F_X,\ Y \sim F_Y$

$$F_{X+Y}(x) = \int_{\mathbb{R}} F_X(x-y)\, dF(y) =: \underbrace{F_X * F_Y}$$ conv in terms of distribution functions

$X \sim p_X,\ Y \sim p_Y$ (If Y, X have densities)

$$p_{X+Y}(x) = \int_{\mathbb{R}} p_X(x-y)\, p_Y(y)\, dy =: p_X * p_Y$$ $\Big\}$ conv in terms of densities

$S_n = \xi_1 + \cdots + \xi_n$

let $F^{n*} := \underbrace{F * \cdots * F}_{n}$

1) $F^{n*}(x) \leq F^{n}(x)$ if $F(0)=0$

$\xi_1, \dots, \xi_n \overset{iid}{\sim} F$

$\{\xi_1 + \dots + \xi_n \leq x\} \subset \{\xi_1 \leq x, \dots, \xi_n \leq x\}$ Since $\xi_i \geq 0$ a.s.

$$\underbrace{P\{\xi_1 + \dots + \xi_n \leq x\}}_{= F^{n*}(x)} \leq \prod_{k=1}^{n} \underbrace{P\{\xi_k \leq x\}}_{F(x)}$$

2) $F^{n*}(x) \geq F^{(n+1)*}(x)$

$\{\xi_1 + \dots + \xi_n \leq x\} \supset \{\xi_1 + \dots + \xi_{n+1} \leq x\}$

<u>Theorem</u>: $S_n = S_{n-1} + \xi_n$ where $\xi_1, \xi_2, \dots \overset{iid}{\sim} F$, $F(0)=0$

(i) $\boxed{U(t) = \sum_{n=1}^{\infty} F^{n*}(t) < \infty}$

(2) $\boxed{\mathbb{E}N_t = U(t)}$

<u>proof for (2)</u>

$$\mathbb{E}N_t = \mathbb{E}[\#\{n : S_n \leq t\}]$$
$$= \mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{1}\{S_n \leq t\}\right] = \sum_{n=1}^{\infty} P\{S_n \leq t\}$$
$$= \sum_{n=1}^{\infty} F^{n*}(t)$$

1.8 Laplace transform. Calculation of an expectation of a counting process (1)

<u>Laplace transform</u>

$f: \mathbb{R}_+ \to \mathbb{R}$: $\mathcal{L}_f(s) = \int_0^{\infty} e^{-sx} f(x)\,dx$

1) f - density of ξ, then $\mathcal{L}_f(s) = \mathbb{E}[e^{-s\xi}]$

2) f_1, f_2 : $\mathcal{L}_{\underbrace{f_1 * f_2}_{\text{densities}}}(s) = \mathcal{L}_{f_1}(s) \cdot \mathcal{L}_{f_2}(s)$

3) F - distribution function, $F(0)=0$, $p = F'$

$$\mathcal{L}_F(s) = \frac{\mathcal{L}_p(s)}{s}$$

$$\text{l.h.s.} = \int_{\mathbb{R}_+} F(x)\,\frac{d(e^{-sx})}{s} = \cancel{-\frac{F(x)e^{-sx}}{s}\Big|_0^{\infty}} + \frac{1}{s}\int_{\mathbb{R}_+} p(x)\,e^{-sx}\,dx$$

$$= \text{r.h.s.}$$

Ex !!

1) $\mathcal{L}_{x^k}(s) = \int_{\mathbb{R}_+} x^n \,\frac{d(e^{-sx})}{s} = \frac{n}{s}\int_{\mathbb{R}_+} x^{n-1} e^{-sx}\,dx$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdots\frac{2}{s}\int_{\mathbb{R}_+} e^{-sx}\,dx = \frac{n!}{s^n}$$

2) $\mathcal{L}_{e^{ax}}(s) = \frac{1}{s-a}$, if $a<s$

1.9 Laplace transform. Calculation of an expectation of a counting process (2)

$F \to \mathbb{E}N_t$

$$\mathbb{E}N_t = \mathcal{U}(t) = \sum_{n=1}^{\infty} F^{n*}(t) = F(t) + \left(\sum_{n=1}^{\infty} F^{n*}(t)\right) * F(t)$$

$$\Leftrightarrow \quad \mathcal{U} = F + \mathcal{U} \underset{\text{dist fcn}}{*} F = F + \mathcal{U} \underset{\text{densities}}{*} p \quad \text{If } F' = p \text{ exists}$$

$$\int_{\mathbb{R}} \mathcal{U}(x-y)\,dF(y) = \int_{\mathbb{R}} \mathcal{U}(x-y)\,p(y)\,dy$$

$$\mathcal{L}_{\mathcal{U}}(s) = \underbrace{\mathcal{L}_F(s)}_{= \frac{\mathcal{L}_p(s)}{s}} + \mathcal{L}_{\mathcal{U}}(s)\,\mathcal{L}_p(s)$$

$$\boxed{\mathcal{L}_{\mathcal{U}}(s) = \frac{\mathcal{L}_p(s)}{s(1-\mathcal{L}_p(s))}}$$

① $F \to \mathcal{L}_p$

② $\mathcal{L}_p \to \mathcal{L}_{\mathcal{U}}$

③ $\mathcal{L}_{\mathcal{U}} \to \mathcal{U}$

1.10 Laplace transform. Calculation of an expectation of a counting process (3)

Example: $S_n = S_{n-1} + \xi_n$, $\xi_1, \xi_1, \dots$ have density $p(x)$

$$p(x) = \frac{e^{-x}}{2} + e^{-2x}, \quad x > 0$$

$\mathbb{E}N_t = ?$

① $p \to \mathcal{L}p$: $\mathcal{L}p(s) = \frac{1}{2}\mathcal{L}_{e^{-x}}(s) + \mathcal{L}_{e^{-2x}}(s)$

$$= \frac{1}{2(s+1)} + \frac{1}{s+2} = \frac{3s+4}{2(s+1)(s+2)}$$

② $\mathcal{L}p \to \mathcal{L}_u$: $\mathcal{L}_u(s) = \frac{\mathcal{L}p(s)}{s(1-\mathcal{L}p(s))} = \frac{3s+4}{s^2(2s+3)}$

③ $\mathcal{L}_u \to u$: $\mathcal{L}_u(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{2s+3}$

$$= \frac{A(2s+3) + B(2s^2+3s) + Cs^2}{s^2(2s+3)}$$

$$3s+4 = (2B+C)s^2 + (2A+3B)s + 3A$$

$A = \frac{4}{3}$, $2A+3B = 3 \Leftrightarrow B = \frac{1}{9}$, $2B+C=0 \Leftrightarrow C = -\frac{2}{9}$

$$\boxed{u(t) = \frac{4}{3}t + \frac{1}{9}(1) - \frac{1}{9}e^{-3/2\, t}}$$

1.11 Limit theorems for renewal processes

$S_n = S_{n-1} + \xi_n$; $\xi_1, \xi_2, \dots$ iid > 0 a.s.

Thm 1 $\mu = \mathbb{E}\xi_1 < \infty \Rightarrow \frac{N_t}{t} \xrightarrow{t\to\infty} \frac{1}{\mu}$ a.s.

(Analog to SLLN)

$$\boxed{\text{SLLN: } \frac{\xi_1 + \dots + \xi_n}{n} \xrightarrow[n\to\infty]{} \mu \text{ a.s.}}$$

Thm 2: (Analog of CLT) $\sigma^2 = \mathrm{Var}\,\xi_1 < \infty$

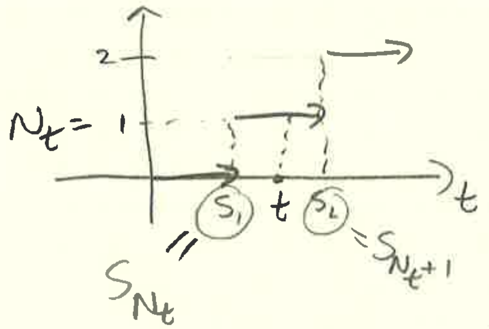
Then, $Z_t := \frac{N_t - t/\mu}{\sigma\sqrt{t}/\mu^{3/2}} \xrightarrow[t\to\infty]{d} N(0,1)$

$$P\{Z_t \le x\} \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du$$

$$\boxed{\text{CLT: } \frac{\xi_1 + \dots + \xi_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1)}$$

proof (thm 1)

$$S_{N_t} \le t \le S_{N_t+1}$$



$$\frac{N_t}{S_{N_t+1}} \le \frac{N_t}{t} \le \frac{N_t}{S_{N_t}}$$

$$\lim_{t\to\infty} \frac{N_t}{S_{N_t}} = \lim_{n\to\infty} \frac{n}{S_n} = \frac{1}{\mu} \text{ by SLLN}$$

$$\lim_{t\to\infty} \frac{N_t}{S_{N_t+1}} = \underbrace{\lim_{t\to\infty} \frac{N_t}{N_t+1}}_{=1} \cdot \underbrace{\lim_{t\to\infty} \frac{N_t+1}{S_{N_t+1}}}_{=1/\mu} = \frac{1}{\mu}$$

proof (thm 2)

$$P\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right\} \to \phi(x), \quad x \in R$$

$$P\{S_n \le \underbrace{n\mu + \sigma\sqrt{n}\, x}_{=t}\} \to \phi(x)$$

$\Leftrightarrow P\{N_t \ge n\}$ (set complements)

$N\mu \approx t$

$n \approx t/\mu$ (for n large enough)

$$n = \frac{t}{\mu} - \frac{\sigma\sqrt{n}}{\mu}x \approx \frac{t}{n} - \frac{\sigma\sqrt{t}}{\mu^{3/2}}x$$

$$\Rightarrow P\{Z_t \ge -x\} \to \phi(x) \quad \Leftrightarrow P\{Z_t \le x\} = 1 - P\{Z_t \ge -x\} \to 1 - \phi(-x) = \phi(x)$$

Poisson Processes

2.1 Definition of a Poisson process as a special example of a renewal process. Exact forms of the distributions of the renewal process and the counting process (1)

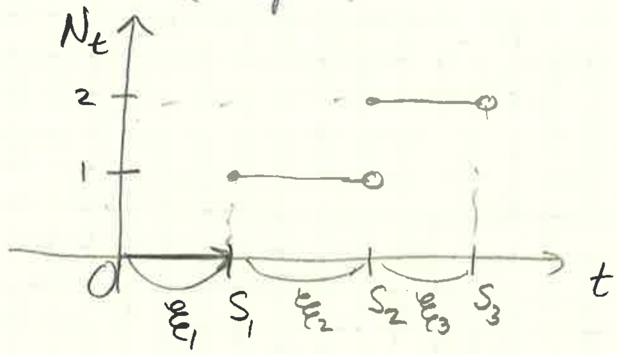
Renewal process

$S_0 = 0, \quad S_n = S_{n-1} + \xi_n, \quad \xi_1, \xi_2, \dots$ - i.i.d > 0 a.s., $\xi_i \sim F$

$N_t = \underset{k}{\operatorname{argmax}} \{ S_k \le t \}$ (Counting process)

$$U(t) = \mathbb{E} N_t = \sum_{n=1}^{\infty} F^{n*}(t)$$

$$\mathcal{L}_U(s) = \frac{\mathcal{L}_p(s)}{s(1-\mathcal{L}_p(s))} \quad : \quad p \to \mathcal{L}_p \to \mathcal{L}_U \to U \qquad (p = F')$$



$$\mathcal{L}_U(s) = \int_{\mathbb{R}_+} e^{-sx} U(x)\, dx$$

2.2 .... (2)

Poisson process

<u>Def 1</u>: A Poisson process is a renewal process s.t.

$$\xi_i \sim p(x) = \lambda e^{-\lambda x} \mathbb{I}\{x > 0\}, \quad \lambda \text{ - intensity or rate}$$

<u>Thm</u> (i): A distribution function of S_n

$$F_{S_n}(x) = \begin{cases} 1 - e^{-\lambda x} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!}, & x > 0 \\ 0, & x < 0 \end{cases}$$

$$p_{S_n}(x) = \lambda \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} \mathbb{I}\{x > 0\}$$

(ii) $\mathbb{P}\{N_t = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$, $N_t \sim \text{Poisson}(\lambda t)$

2.3 ··· (3)

Proof (i) $n=1$: $S_1 = \xi_1$

$$p_{S_1}(x) = \lambda e^{-\lambda x}, \; x>0$$

$n \to n+1$

$$p_{S_{n+1}}(x) = \int_0^x p_{S_n}(x-y)\, p_{\xi_{n+1}}(y)\, dy$$

$$= \int_0^x \frac{\lambda^n (x-y)^{n-1}}{(n-1)!} e^{-\lambda(x-\cancel{y})} \lambda \cancel{e^{-\lambda y}}\, dy$$

$$= \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda x} \int_0^x (x-y)^{n-1} dy = \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda x} \frac{x^n}{n}$$

$$= \lambda \frac{(\lambda x)^n}{n!} e^{-\lambda x}$$

□

2.4 ··· (4)

proof (ii)

$$P\{N_t = n\} = P\{S_n \le t\} - P\{S_{n+1} \le t\} \;=$$

$$\{N_t = n\} = \underbrace{\{S_n \le t\}}_{= A} \cap \underbrace{\{S_{n+1} > t\}}_{= B}$$

$$\left.\begin{array}{l} A \cap B = A \setminus B^c \\ \text{Here: } B^c \subset A \end{array}\right\} \Rightarrow P\{A \cap B\} = P\{A\} - P\{B^c\}$$

$$= \left(1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}\right) - \left(1 - e^{-\lambda t} \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!}\right)$$

$$= e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

□

2.5 Memoryless property

A r.v. X possesses the memoryless property iff

$P\{X > u+v\} = P\{X>u\}\, P\{X>v\}$. If $P\{X>v\}>0$, then

$$\boxed{P\{X > u+v \mid X > v\} = P\{X > u\}}$$

Thm 2: Let X be a r.v. with density $p(x)$, then

X- memoryless $\iff p(x) = \lambda e^{-\lambda x}$

Ex 4 busses arrive every 20 ± 2 minutes

$v = 19$ min, $u = 10$ min

(l.h.s.) $P\{X > 29 \mid X > 19\} = 0$ given the data

(r.h.s) $P\{X > 10\} = 1$

Thus, Poisson process is not appropriate

2.6 Other definitions of Poisson processes (1)

Def 2 N_t – an integer value process s.t.

0) $N_0 = 0$ a.s.

1) N_t has independent increments: $\forall t_0 < t_1 < \dots < t_n$,

$N_{t_1} - N_{t_0}, \dots, N_{t_n} - N_{t_{n-1}}$ are independent

2) N_t has stationary increments

$N_t - N_s \overset{d}{=} N_{t-s}$

3) $N_t - N_s \sim \text{Poisson}(\lambda(t-s))$, $t > s$

3) $\Rightarrow$ 2)

2.7 Other definitions of Poisson processes (2)

$$P\{N_{t+h} - N_t = 0\} = 1 - \lambda h + o(h), \quad h \to 0$$

$$P\{N_{t+h} - N_t = 1\} = \lambda h + o(h), \quad h \to 0$$

$$P\{N_{t+h} - N_t \geq 2\} = o(h), \quad h \to 0$$

$$\lim_{h \to 0} \frac{1 - P\{N_{t+h} - N_t = 0\}}{h} = \lim_{h \to 0} \frac{1 - e^{-\lambda h}}{h} = \lambda$$

Def 3 N_t is a Poisson process, if

0) $N_0 = 0$

1) N_t has independent increments

2) N_t has stationary increments

3') $\displaystyle\lim_{h \to 0} \frac{P\{N_{t+h} - N_t \geq 2\}}{P\{N_{t+h} - N_t = 1\}} = 0$

Models related to Poisson processes

2.8 Non-homogeneous Poisson processes (1)

$N_t \sim Pois(\lambda t) \Rightarrow \mathbb{E}N_t = \lambda t$

Def: Let $\Lambda(t)$ be a differentiable, increasing function s.t. $\Lambda(0)=0$. Then, $X_t = N_t$ is a non-homogeneous Poisson process if,

0) $N_0 = 0$

1) N_t has independent increments

2) $\boxed{N_t - N_s \sim Pois(\Lambda(t)-\Lambda(s))}$

2.9 Non-homogeneous Poisson processes (2) (NHPP)

$\lambda(t) = \Lambda'(t)$ - intensity function

Properties NHPP:

1) $\mathbb{E}N_t = \Lambda(t)$

$\Lambda(t) = \alpha t^\beta, \alpha>0, \beta>0$ (for example)

2) if $\lambda(t) = \text{const} \Rightarrow \Lambda(t) = \text{const}\cdot t$

3) $\left.\begin{array}{l}\Lambda(t)\text{-differentiable} \Rightarrow \Lambda(t)\text{-continuous}\\ \Lambda(t)\text{-increasing}\end{array}\right\} \Rightarrow$

$\Rightarrow \exists \Lambda^{-1}(t)$. If Image $\Lambda(t) = \mathbb{R}_+$, $N_{\Lambda^{-1}(t)}$ - homogeneous P.P.

2.10 Relation between renewal theory and NHPP (1)

$S_n = \underset{t}{\operatorname{argmin}}\{N_t = n\}$, $\xi_n = S_n - S_{n-1}$

$\xi_1, \xi_2, \dots$ - iid ?

1) $p_{\xi_1}(x) = \lambda(x)e^{-\Lambda(x)}$

$P\{\xi_1 \le x\} = P\{S_1 \le x\} \underset{\{S_n > t\} = \{N_t < n\}}{=} P\{N_x \ge 1\} = 1 - P\{N_x = 0\} =$

$= 1 - e^{-\Lambda(x)}$

Take derivatives of both sides to finish proof $\square$

2.11 ··· (2)

$S_k = \underset{t}{\operatorname{argmin}}\{N_t = k\}$

$\xi_k = S_k - S_{k-1}$

1) $\boxed{p_{\xi_1}(t) = \lambda(t) e^{-\Lambda(t)}}$

2) $\boxed{p_{\xi_2|\xi_1}(t|s) = \lambda(t+s) e^{-\Lambda(t+s)+\Lambda(s)}}$

$$F_{(\xi_1,\xi_2)}(s,t) = P\{\xi_1 \le s, \xi_2 \le t\} = \int_0^s P\{\underbrace{\cancel{\xi_1 \le s}}_{\text{since } y<s}, \xi_2 \le t \,|\, \xi_1 = y\} p_{\xi_1}(y)\,dy$$

$$= \int_0^s P\{N_{t+y} - N_y \ge 1 \,|\, \xi_1 = y\} p_{\xi_1}(y)\,dy$$

independent

$$= \int_0^s \left(1 - e^{-\Lambda(t+y)+\Lambda(y)}\right) \lambda(y) e^{-\Lambda(y)}\,dy$$

$$p_{(\xi_1,\xi_2)}(s,t) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s} F_{(\xi_1,\xi_2)}(s,t)\right)$$

$$= \frac{\partial}{\partial t}\left(\left(1 - e^{-\Lambda(t+s)+\Lambda(s)}\right) \lambda(s) e^{-\Lambda(s)}\right)$$

$$= \lambda(t+s) e^{-\Lambda(t+s)+\Lambda(s)} \lambda(s) e^{-\Lambda(s)}$$

Then $p_{\xi_2|\xi_1}(t|s) = \dfrac{p_{(\xi_1,\xi_2)}(s,t)}{p_{\xi_1}(s)}$ finishes the proof $\square$

2.12 ··· (3)

$\xi_1, \xi_2, \ldots$ – i.i.d. ? (NHPP can be obtained from renewal process iff NHPP is homogeneous PP)

$$p_{\xi_1}(t) = p_{\xi_2|\xi_2}(t|s), \quad \forall t, s > 0$$

$$\lambda(t) e^{-\Lambda(t)} = \lambda(t+s) e^{-\Lambda(t+s)+\Lambda(s)}$$

$$\left(\int_0^T \ldots dt\right): \cancel{e^{-\Lambda(0)}} - e^{-\Lambda(T)} = \cancel{e^0} - e^{-\Lambda(T+S)+\Lambda(S)}$$

$$\boxed{\begin{array}{l}\Lambda(T) = \Lambda(T+S) - \Lambda(S), \quad \forall S, T > 0 \\ \Lambda(t) \text{ – increasing}\end{array}} \Rightarrow \Lambda(t) = \overset{\text{const}}{\overset{\shortparallel}{\lambda}} t$$

2.13 Elements of queuing theory. M/G/k systems (1)

$$\boxed{\begin{aligned} P\{N_{t+h}-N_t=0\} &= 1-\overset{\lambda(t)}{\lambda} h+o(h)\\ P\{N_{t+h}-N_t=1\} &= \overset{\lambda(t)}{\lambda} h+o(h)\\ P\{N_{t+h}-N_t\geq 2\} &= o(h)\end{aligned}}$$

M/G/k

I) Arrival Process: M - memoryless (Poisson)
D - deterministic
G - general

II) Service time (M, D, G)

III) A number of services $(1, 2, \dots, \infty)$

$M/G/\infty$ $\qquad \tau > 0$ (time moment)

$\xrightarrow[\text{Customer arrivals}]{N(t)}$
- $N_1(t)$ - still being served at τ: $\lambda_1(t) = \lambda(1-G(\tau-t))$
- $N_2(t)$ - already completed by τ: $\lambda_2(t) = \lambda G(\tau-t)$

$$P\{N_1(t+\delta)-N_1(t)=1\} = P\{N(t+\delta)-N(t)=1\}\cdot\left(P\{Y>\tau-t\}+o(\delta)\right)$$
$$= (\delta\lambda+o(\delta))(1-G(\tau-t)+o(\delta))$$
$$= \boxed{\lambda\delta(1-G(\tau-t)+o(\delta))}$$

2.14 ... (2)

bernoulli scheme

$$P\{N_1(t)=n_1,\ N_2(t)=n_2\} = P\{N_1(t)=n_1,\ N_2(t)=n_2 \mid N(t)=n_1+n_2\}\cdot P\{N(t)=n_1+n_2\}$$

$$= C_{n_1+n_2}^{n_1}(1-G(\tau-t))^{n_1}G(\tau-t)^{n_2}\cdot e^{-\lambda t}\frac{(\lambda t)^{n_1+n_2}}{(n_1+n_2)!}$$

$$= \frac{\lambda t(1-G(\tau-t))^{n_1}}{n!}e^{-\lambda(1-G(\tau-t))}\cdot\frac{\lambda t(G(\tau-t))^{n_2}}{n_2!}e^{-\lambda G(\tau-t)}$$

$$= P\{N_1(t)=n_1\}\cdot P\{N_2(t)=n_2\}$$

Therefore $N_1 \perp\!\!\!\perp N_2$

2.15 Compound Poisson Processes (1)

$$X_t = \sum_{k=1}^{N_t} \xi_k \,,\quad \xi_1, \xi_2, \dots \text{ - iid.}\,,\ N_t \text{ - P.P. with intensity } \lambda$$

and $\xi_1, \xi_2, \dots$ and N_t are independent

$\xi_1, \xi_2, \dots$ claim sizes

N_t - amount of claims until time t (Insurance interpretation)

X_t - aggregated claim amount

1) Probability generating function (PGF)

ξ - integer, ≥ 0 values

$$\boxed{\phi_\xi(u) = \mathbb{E}[u^\xi]}, \ |u| < 1$$

$$\xi_1 \perp\!\!\!\perp \xi_2 \Rightarrow \phi_{\xi_1+\xi_2}(u) = \phi_{\xi_1}(u)\,\phi_{\xi_2}(u)$$

2) Moment-generating function (MGF)

$$\boxed{\mathcal{L}_\xi(u) = \mathbb{E}[e^{-u\xi}]}, \ \xi \geq 0, \ u > 0$$

2.16 ... (2)

3) Characteristic function

$$\phi_\xi(u) = \mathbb{E}[e^{iu\xi}], u \in \mathbb{R}, \forall \xi \,,\quad \phi_\xi : \mathbb{R} \to \mathbb{C} \,,\quad \xi_1 \perp\!\!\!\perp \xi_2 \Rightarrow \phi_{\xi_1+\xi_2}(u) = \phi_{\xi_1}(u)\phi_{\xi_2}(u)$$

Thm $$\boxed{\phi_{X_t - X_s}(u) = e^{\lambda(t-s)(\phi_{\xi_1}(u) - 1)}}$$

Proof: $$\text{lhs} = \mathbb{E} e^{iu(X_t - X_s)} = \sum_{k=0}^{\infty} \mathbb{E}[\underbrace{e^{iu(X_t-X_s)}}_{\overset{d}{=}\ \xi_1 + \dots + \xi_k} | \cancel{N_t - N_s = k}]\, P\{N_t - N_s = k\}$$

from $\perp\!\!\!\perp$

$$= \sum_{k=0}^{\infty} (\phi_{\xi_1}(u))^k e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^k}{k!}$$

$\square$

2.17 ⋯ (3)

$X_t = \sum_{k=1}^{N_t} \xi_k$, ξ can be any random variable

ξ: $\phi_\xi(u) = E\left[e^{iu\xi}\right]$

$\phi : \mathbb{R} \to \mathbb{C}$

$\xi_1 \perp\!\!\!\perp \xi_2 \Rightarrow \phi_{\xi_1+\xi_2}(u) = \phi_{\xi_1}(u)\,\phi_{\xi_2}(u)$

Thm: $\boxed{\phi_{X_t - X_s}(u) = e^{\lambda(t-s)\left(\phi_{\xi_1}(u) - 1\right)}, \quad t > s \geq 0}$

proof

$$lhs = \mathbb{E}\left[e^{iu(X_t - X_s)}\right]$$

$$= \sum_{k=0}^{\infty} \mathbb{E}\Big[e^{iu\underbrace{(X_t - X_s)}_{\overset{d}{=}\ \xi_1 + \dots + \xi_k}} \Big| N_t - \cancel{N_s} = k\Big] \cdot P\{\underbrace{N_t - N_s = k}_{\sim Pois(\lambda(t-s))}\}$$

since $\xi_1, \xi_2, \dots \perp\!\!\!\perp N_t$

$$= \sum_{k=0}^{\infty} \left[\phi_{\xi_1}(u)\right]^k \cdot e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^k}{k!}$$

$\square$

2.18 ⋯ (4)

Corollary: $\boxed{\begin{aligned} \mathbb{E} X_t &= \lambda t\, \mathbb{E}\xi_1 \\ Var\, X_t &= \lambda t\, \mathbb{E}\xi_1^2 \end{aligned}}$

proof $E[\xi^r] < \infty \Rightarrow \phi(u)$ is r-times differentiable at 0 and $\phi^{(r)}(0) = i^r\, \mathbb{E}\xi^r$

$$\mathbb{E}X_t = \frac{\phi'_{X_t}(0)}{i} = \frac{\lambda t\, \phi'_\xi(0) \cdot \overbrace{\phi_{X_t}(0)}^{=1}}{i} = \lambda t\, \mathbb{E}\xi_1$$

$\frac{\phi'_\xi(0)}{i} = \mathbb{E}\xi$

$\square$

3.1 Definition of a Markov chain. Some examples

<u>Def</u>: A Markov chain $-S_n$, $n=0,1,2,\dots$

S - state space (countable)

$$\boxed{P\{S_n=j \mid S_{n-1}=i_{n-1},\dots,S_0=i_0\} = P\{S_n=j \mid S_{n-1}=i_{n-1}\}}$$

$i_0,\dots,i_{n-1},j \in S$ and $P\{S_{n-1}=i_{n-1},\dots,S_0=i_0\}\neq 0$

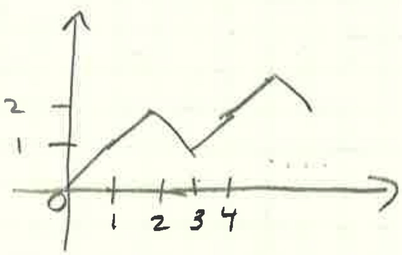
$$P\{S_n=i_n, S_{n-1}=i_{n-1},\dots,S_0=i_0\} = P\{S_n=i_n \mid S_{n-1}=i_{n-1},\dots,S_0=i_0\} \cdot P\{S_{n-1}=i_{n-1},\dots,S_0=i_0\}$$

$$= P\{S_n=i_n \mid S_{n-1}=i_{n-1}\}\cdot P\{S_{n-1}=i_{n-1},\dots,S_0=i_0\}$$

$$= P\{S_n=i_n \mid S_{n-1}=i_{n-1}\}\, P\{S_{n-1}=i_{n-1} \mid S_{n-2}=i_{n-2}\} \cdot \dots \cdot P\{S_1=i_1 \mid S_0=i_0\}\, P\{S_0=i_0\}$$

<u>Ex ①</u> Random walk (not a renewal process)

$S_0=0$, $S_n=S_{n-1}+\xi_n$, $\xi_1,\xi_2,\dots$ - i.i.d. $\sim \begin{cases} 1, & \text{w.p. } p \\ -1, & \text{w.p. } 1-p \end{cases}$



$$P\{S_n=j \mid S_{n-1}=i_{n-1}\} = \begin{cases} p, & j=i_{n-1}+1 \\ 1-p, & j=i_{n-1}-1 \\ 0, & \text{otherwise} \end{cases}$$

② Taxis in the airport

1 taxi at any 1 moment, $n=1,2,3,\dots$

X_k = # people waiting for a taxi at time k

Y_k = # people arriving at k

$$X_k = Y_k + (X_{k-1}-1)_+ = \begin{cases} Y_k, & \text{if } X_{k-1}=0 \\ Y_k + X_{k-1}-1, & \text{if } X_{k-1}-1>0 \end{cases}$$

③ X_n: $P\{X_n=j \mid X_{n-1}=i_{n-1},\dots,X_0=i_0\} = P\{X_n=j \mid X_{n-1}=i_{n-1},\dots,X_{n-m}=i_{n-m}\}$

$m\in\mathbb{N}$, fixed (X_n is not a Markov chain)

$S_n=(X_n,\dots,X_{n-m-1})$, $n=(m-1),m,\dots$ $\quad S_n$ is a Markov chain

3.2 Matrix representation of a Markov chain. Transition matrix. Chapman–Kolmogorov equation.

Matrix representation

$S = (1, 2, \dots, M)$

$P\{X_n = j \mid X_{n-1} = i\} = p_{ij}$ – homogeneous (no dependence on n)

$\underline{P} = (p_{ij})_{i,j=1}^{M}$ – transition matrix

$\left.\sum_{j=1}^{M} p_{ij} = 1, \ \forall i \ ; \ p_{ij} \ge 0 \right\}$ – stochastic matrix

$p_{ij}^{(m)} = P\{X_{n+m} = j \mid X_n = i\}$

$\underline{P}^{(m)} = (p_{ij}^{(m)})$ – m-step transition matrix

Thm: $\boxed{\underline{P}^{(m)} = \underline{P}^{m}}$

proof: $p_{ij}^{(m)} = \sum_{k=1}^{M} P\{X_{n+m} = j \mid X_{n+m-1} = k, X_n = i\} \cdot P\{X_{n+m-1} = k \mid X_n = i\}$

(Markov property) $= \sum_{k=1}^{M} P\{X_{n+m} = j \mid X_{n+m-1} = k\} \, P\{X_{n+m-1} = k \mid X_n = i\}$

$= \sum_{k=1}^{M} p_{kj} \, p_{ik}^{(m-1)} \Rightarrow \underline{P}^{(m)} = \underline{P} \cdot \underline{P}^{(m-1)} = \dots = \underline{P}^{m}$ $\square$

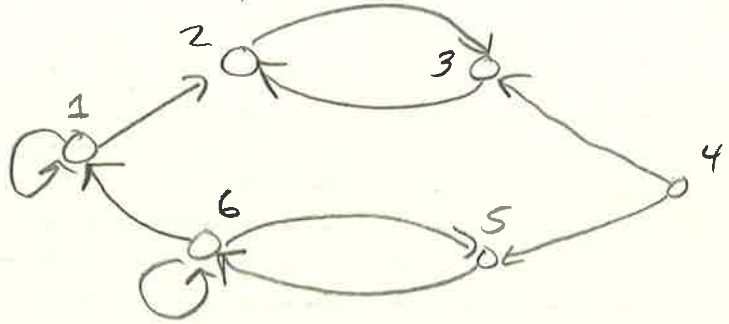
$P\{X_k = j\} := \pi_j^{(k)}$, $(\pi_1^{(k)}, \dots, \pi_M^{(k)}) := \vec{\pi}^{(k)}$

$\pi_j^{(k)} = \sum_{i=1}^{M} P\{X_k = j \mid X_{k-1} = i\} \, P\{X_{k-1} = i\}$

$= \sum_{i=1}^{M} p_{ij} \, \pi_i^{(k-1)} \Rightarrow \vec{\pi}^{(k)} = \vec{\pi}^{(k-1)} \cdot \underline{P} = \vec{\pi}^{(0)} \underline{P}^{n}$

$\vec{\pi}^*$ – stationary distribution for Markov chain if $\boxed{\vec{\pi}^* \underline{P} = \vec{\pi}^*}$

Graphical representation



1 node = 1 state

i,j-arc $\Leftrightarrow p_{ij} \neq 0$

$$\underline{P} = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ & & \cdots & & & \\ & & & & & \end{pmatrix}_{6\times 6}$$

<u>Def</u> (1) j is accessible from i $\exists$ walk (path) from i to j $(i \to j)$

$1 \to 3$; $1 \not\to 4$

(2) i and j communicate if $i \to j$ and $j \to i$ $(i \leftrightarrow j)$

$2 \leftrightarrow 3$

(3) $\underline{Y}$ - set, and relation $\sim$ called an equivalence relation

$a \sim a$, $a \in \underline{Y}$ - reflexivity

$a \sim b \Rightarrow b \sim a$, $a, b \in \underline{Y}$ - symmetry

$a \sim b$, $b \sim a \Rightarrow a \sim c$, $a, b, c \in \underline{Y}$ - transitivity

$\underline{Y} = \bigsqcup B_i$ ($\bigsqcup$ - disjoint union), B_i - equivalence classes

$\leftrightarrow$ $\Leftrightarrow$ $\underline{P}\left(\cdots \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \cdots \right)$

$B_1, B_2, \ldots$ - equivalence classes

$$\boxed{\forall j \in B_i,\ \forall k \in S \begin{cases} k \in B_i, & k \leftrightarrow j \\ k \notin B_i, & k \not\leftrightarrow j \end{cases}}$$

$2 \leftrightarrow 3$, $5 \leftrightarrow 6$, 1, 4 are the four equivalence classes

3.4 ... (2)

<u>Def</u>: i is recurrent, $\forall j : i \to j \Rightarrow j \to i$

i is transient if it's not recurrent $\Leftrightarrow \exists j : i \to j,\ j \not\to i$

<u>ex</u>: 1, 4, 5, 6 - transient

2, 3 - recurrent

<u>Thm</u>: In 1 class of equivalence, all states are either recurrent or transient.

<u>proof</u> k-transient : $\exists j : k \to j,\ j \not\to k$

$i, k \in 1$ class $\Rightarrow$ $i \to k \to j$, but $j \not\to i$: $j \to i \to k$ is a contradiction $\square$

3.5 ... (3)

<u>Def</u>: <u>Period</u> of a state i is $GCD\{n : p_{ii}(n) \neq 0\} =: d(i)$

$d(i) = 1 \Rightarrow i$-aperiodic

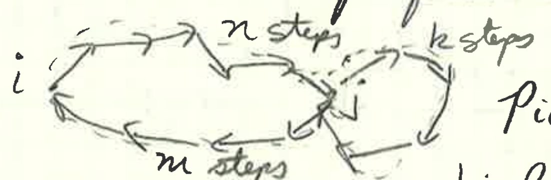
ex $d(1) = 1 = d(4) = d(5) = d(6)$

$d(2) = 2 = d(3)$

④ has no return, so $d(4) = 1$ by convention

<u>Thm</u>: All elements in 1 class of equivalence have the same period

<u>proof</u>:



$p_{ii}(n+m+k) \neq 0$

$p_{ii}(n+m) \neq 0 \Rightarrow n+m \mid d(i)$

$k : p_{jj}(k) \neq 0 \Rightarrow n+m+k \mid d(i)$

$$\Rightarrow k \mid d(i) \Rightarrow \left.\begin{matrix} d(i) \mid d(j) \\ d(j) \mid d(i) \end{matrix}\right\} \Rightarrow d(i) = d(j)$$

$\square$

3.6 Ergodic chains. Ergodic Theorem (1)

<u>Matrix representation</u>

$\underline{P}$, $\vec{\pi}^{(k)}$

$\underline{P}^{(m)} = \underline{P}^m$

<u>Ergodic Markov chains</u>:

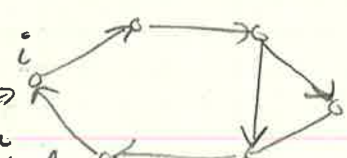
- 1 class of equivalence
- recurrent
- $d(i) = 1$ (aperiodic)

<u>Graphical representation</u>

classes of equivalence

recurrent / transient

$d(i)$ - period



return in 5 or 6 steps

$\therefore d(i) = 1$

Prop: Markov chain is ergodic $\Leftrightarrow \exists m \in \mathbb{N} : p_{ij}(m) \neq 0, \forall i,j \in S$ $(*)$

If chain is ergodic, then $(*)$ hold $\forall m \geq (M-1)^2+1$

3.7 ... (2)

Ergodic theorem: Let X_t - ergodic Markov chain, i.e. X_t has 1 class of equivalence, recurrent and aperiodic. Then,

$$\boxed{\exists \lim_{n\to\infty} p_{ij}(n) = \pi_j^* > 0 \text{ (doesn't depend on } i)}$$

$$\sum_{j=1}^{M} \pi_j^* = 1 \qquad \vec{\pi}^* = (\pi_1^*, \dots, \pi_M^*)$$

Corr (i) $\vec{\pi}^*$ - stationary distribution : $\vec{\pi}^* \underline{P} = \vec{\pi}^*$

(ii) $\lim_{n\to\infty} P\{X_n = j\} = \pi_j^*$ [$\pi_j^{(0)}$ is arbitrary]

proof (i) $i = 1, \dots, M$

$$(\pi^* \underline{P})_i = \sum_{j=1}^{M} \pi_j^* p_{ji} = \sum_{j=1}^{M} \lim_{n\to\infty} p_{kj}(n) p_{ji} \qquad (k \in 1, \dots, M)$$

$$= \lim_{n\to\infty} \sum_{j=1}^{M} \underbrace{p_{kj}(n) p_{ji}}_{\underline{P}^{(n)}\underline{P} = \underline{P}^{n+1} = \underline{P}^{(n+1)}}$$

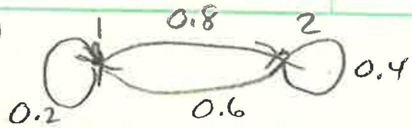
$$= \lim_{n\to\infty} p_{ki}(n+1) = \pi_i^* \qquad \square$$

proof (ii) $\lim_{n\to\infty} \pi_j^{(n)} = \lim_{n\to\infty} \sum_{k=1}^{M} \pi_k^{(0)} p_{kj}(n)$ $\qquad \pi_j^{(0)}$ is arbitrary

$\vec{\pi}^{(n)} = \vec{\pi}^{(0)} \underline{P}^{(n)}$

$$= \sum_{k=1}^{M} \pi_k^{(0)} \underbrace{\lim_{n\to\infty} p_{kj}(n)}_{= \pi_j^*} = \pi_j^* \underbrace{\sum_{k=1}^{M} \pi_k^{(0)}}_{= 1} = \pi_j^* \qquad \square$$

ex



$$\underline{P} = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}$$

$$\vec{\pi}^* = (a, b) \; ; \quad \vec{\pi}^* \underline{P} = \vec{\pi}^*$$

$$(a \;\; b)\begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix} = (a \;\; b)$$

$$\left.\begin{array}{l} 0.2a + 0.6b = a \\ 0.8a + 0.4b = b \end{array}\right\} \Rightarrow a = \frac{3}{7}, \; b = \frac{4}{7}$$

$$P\{X_n = 1\} \rightarrow \frac{3}{7}$$

$$P\{X_n = 2\} \rightarrow \frac{4}{7}$$

Gaussian Processes

4.1 Random vector. Definition and main properties

$\xi \sim N(\mu, \sigma^2)$, $\rho(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $\sigma > 0$, $\mu \in R$

$$\phi(u) = e^{iu\mu - \frac{1}{2}u^2\sigma^2}$$

$$\left.\begin{array}{l} X_1, X_2 \sim N(0,1) \\ \operatorname{cor}(X_1, X_2) = 0 \end{array}\right\} \not\Rightarrow X_1 \perp\!\!\!\perp X_2$$

$$P\{X = \mu\} = 1 \Rightarrow \sigma = 0$$

Def: A random vector $\vec{X} = (X_1, \dots, X_n)$ is Gaussian iff $\forall (\lambda_1, \dots, \lambda_n) \in R^n$, $\sum_{k=1}^{n} \lambda_k X_k \sim N$

4.2 Gaussian vector. Definition and main properties

Thm: $\vec{X}$ - Gaussian iff any of the following holds:

(i) $\phi_{\vec{X}}(\vec{u}) = \mathbb{E}\left[e^{i\langle \vec{u}, \vec{X}\rangle}\right] = \exp\left\{ i\langle \vec{u}, \vec{\mu}\rangle - \frac{1}{2}\vec{u}^T C \vec{u} \right\}$

$\vec{\mu} \in R^n$; C - symmetric, positive semidefinite (size $n \times n$)

(ii) $\vec{X} = A\vec{X}^\circ + \vec{\mu}$; $A \in Mat(n \times n)$, $\vec{X}^\circ$ - standard normal vector

Remark: $\vec{\mu} = (\mathbb{E}X_1, \dots, \mathbb{E}X_n)$

$$C = (c_{jk})_{j,k=1}^{n}; \quad c_{jk} = \operatorname{cov}(X_j, X_k)$$

$$\sum_{k,j=1}^{n} u_k C_{kj} u_j \geq 0, \ \forall u \in R^n \iff u^T C u \geq 0, \ \forall u \in R$$

$$\sum_{k,j=1}^{n} u_k C_{kj} u_j = \sum_{k,j=1}^{n} u_k \operatorname{cov}(X_j, X_k) u_j = \operatorname{cov}\left(\sum_{j=1}^{n} u_j X_j, \sum_{k=1}^{n} u_k X_k\right)$$

$$= \operatorname{var}\left(\sum_{j=1}^{n} u_j X_j\right) \geq 0$$

$$A = C^{1/2} : AA = C \Rightarrow \exists U : U^{-1} = U^T : C = U^T \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} U$$

$$A = U^T \begin{pmatrix} \sqrt{d_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{d_n} \end{pmatrix} U \Rightarrow C = AA^T$$

Proof: Def $\Leftrightarrow$ (i) $\Leftrightarrow$ (ii)

Def $\Rightarrow$ (i): $\langle \vec{u}, \vec{X} \rangle \sim N$

$$\phi_{\vec{X}}(\vec{u}) = \mathbb{E} e^{i\underbrace{\langle \vec{u}, \vec{X} \rangle}_{\xi}\cdot 1} = \phi_\xi(1) = e^{i\mu_\xi - \frac{1}{2}\sigma_\xi^2}$$

$$\mu_\xi = \mathbb{E}\left[\sum_{k=1}^n u_k X_k\right] = \sum_{k=1}^n u_k \mathbb{E}X_k = \sum_{k=1}^n u_k \mu_k = \langle \vec{\mu}, \vec{u} \rangle$$

$$\sigma_\xi^2 = \operatorname{cov}\left(\sum_{k=1}^n u_k X_k, \sum_{j=1}^n u_j X_j\right)$$
$$= \sum_{k=1}^n \sum_{j=1}^n u_k \operatorname{cov}(X_k, X_j) u_j = \vec{u}^T C \vec{u}$$

(i) $\Rightarrow$ Def By definition of ϕ for Gaussian

(ii) $\Rightarrow$ (i)

$$\vec{X}^\circ\text{-Gaussian} \Rightarrow \phi_{\vec{X}_0}(\vec{u}) = \exp\left\{-\tfrac{1}{2}\vec{u}^T\vec{u}\right\}$$

$$\phi_{\vec{X}}(\vec{u}) = \mathbb{E}\left[e^{i\langle \vec{u}, A\vec{X}^\circ + \vec{\mu}\rangle}\right] = \mathbb{E}\left[e^{i[\langle \vec{u}, \vec{\mu}\rangle + \langle \vec{u}, A\vec{X}^\circ\rangle]}\right]$$
$$= e^{i\langle \vec{u}, \vec{\mu}\rangle}\phi_{\vec{X}^\circ}(A^T\vec{u}) = e^{i\langle \vec{u}, \vec{\mu}\rangle} e^{-\frac{1}{2}\vec{u}^T \underbrace{AA^T}_{\overset{\shortparallel}{C}}\vec{u}}$$
$$= e^{i\langle \vec{u}, \vec{\mu}\rangle} e^{-\frac{1}{2}\vec{u}^T C \vec{u}}$$

(i) $\Rightarrow$ (ii) $A = C^{1/2}$ $\square$

4.3 Connection between independence of normal random variates and absence of correlation

Thm: Let $X_1, X_2 \sim N(0,1)$ and $\operatorname{cov}(X_1, X_2) = 0$, then
$$X_1 \perp\!\!\!\perp X_2 \Leftrightarrow (X_1, X_2) \text{ - Gaussian vector}$$

Proof ($\Rightarrow$) $\lambda_1 X_1 + \lambda_2 X_2 \sim N \Rightarrow (X_1, X_2)$ - Gaussian vector

($\Leftarrow$) $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A = C^{1/2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = C$

$$\text{(ii)} \Rightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = A\vec{X}^\circ + \vec{\mu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X_1^\circ \\ X_2^\circ \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \overset{d}{=} \begin{pmatrix} X_1^\circ \\ X_2^\circ \end{pmatrix}$$ $\square$

ex $X_1 \sim N(0,1)$, $X_2 := |X_1| \cdot \xi$, $\xi = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$, $\xi \perp\!\!\!\perp X_1$

1) $X_2 \sim N(0,1)$, $x > 0$

$$P\{X_2 \le x\} = P\{|X_1| \le x \mid \xi = 1\} P\{\xi = 1\} + P\{|X_2| \ge -x \mid \xi = -1\} P\{\xi = -1\}$$

$$\overset{\xi \perp\!\!\!\perp X_1}{=} P\{|X_1| \le x\} \cdot 1/2 + \underbrace{P\{|X_2| \ge -x\}}_{=1} \cdot \frac{1}{2}$$

$$= \frac{1}{2}\left[1 + P\{|X_1| \le x\}\right] = P\{X_1 \le x\}$$

2) $cov(X_1 X_2) = 0$

$$\overset{\|}{\mathbb{E}[X_1 X_2] - \mathbb{E}X_1 \, \mathbb{E}X_2} = \mathbb{E}[X_1 |X_1| \xi]$$

$$= \mathbb{E}[X_1 |X_1|] \underbrace{\mathbb{E}\xi}_{=0} - 0 \cdot \mathbb{E}X_2$$

$$= 0 - 0 = 0 \quad \square$$

3) X_1, X_2 are dependent

Assume $X_1 \perp\!\!\!\perp X_2 \Rightarrow (X_1 X_2)$ - Gaussian

$$\gamma = X_1 - X_2 = X_1 - |X_1| \xi \sim N$$

$\{\gamma > 0\}$ when $X_1 > 0$ and $\xi = -1$

$$P\{\gamma > 0\} \ge P\{X_1 > 0 \cap \xi = -1\} = P\{X_1 > 0\} P\{\xi = -1\}$$

$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$P\{\gamma = 0\} \ge 1/4$ since $var(\gamma) \ne 0$

Thus γ is not Gaussian

4.4 Definition of a Gaussian process. Covariance function (1)

Def: A Gaussian process X_t is a stochastic process s.t.

$\forall t_1, t_2, \dots, t_n$: $(X_{t_1}, \dots, X_{t_n})$ – Gaussian vector

$m(t) = \mathbb{E}X_t$ – mathematical expectation $m: \mathbb{R}_+ \to \mathbb{R}$

$K(t,s) = cov(X_t, X_s)$, $K: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$

$K(t,t) = Var\, X_t$ and $\boxed{K(t,s) = K(s,t)}$

K – a positive semidefinite function (p.s.d.)

$\forall (t_1, \dots, t_n) \in \mathbb{R}_+^n$,

$\forall (u_1, \dots, u_n) \in \mathbb{R}^n$

$$\boxed{\sum_{k=1}^{n} \sum_{j=1}^{n} u_k u_j K(t_k, t_j) \geq 0}$$

$$\Leftrightarrow cov\left(\sum_{k=1}^{n} u_k X_{t_k}, \sum_{j=1}^{n} u_j X_{t_j}\right) = var\left(\sum_{k=1}^{n} u_k X_{t_k}\right) \geq 0$$

4.5 Definition of a Gaussian process. Covariance function (2)

Thm: $m: \mathbb{R}_+ \to \mathbb{R}$, $K: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, K – symmetric and p.s.d.

Then, $\exists$ Gaussian process X_t: $\boxed{\mathbb{E}X_t = m(t),\ cov(X_t, X_s) = K(t,s)}$

Gaussian r.v.: $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$

Gaussian vector: $\vec{\mu} \in \mathbb{R}^n$, $C \in Mat(n,n)$ and symmetric and p.s.d.

Gaussian process: $m: \mathbb{R}_+ \to \mathbb{R}$, $K: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ (sym. and p.s.d.)

Ex: $K(t,s) = |t-s|$ is not p.s.d.

assume p.s.d. $\Rightarrow \exists X_t : cov(X_t, X_s) = |t-s|$

let $t=s$, $var\, X_t = 0 \Rightarrow X_t = f(t)$ – deterministic

$cov(X_t, X_s) = \mathbb{E}[X_t X_s] - \mathbb{E}X_t \mathbb{E}X_s = f(t)f(s) - f(t)f(s) = 0 \neq |t-s|$

Contradiction. $\square$