1.1 Difference between deterministic and stochastic world

	deterministic world	stochastic world
Single variable: Temp of a sick man	$\mathbb{R}$ $T = 39°C$	random variable $\mathbb{E}$, Var, ...
Variables changing over time: T in first 3 days	$\mathbb{R}_+ \to \mathbb{R}$ $T(1) = 39$ $T(2) = 38.5$ $T(3) = 38$ $\vdots$	stochastic process

1.2 Difference between various fields of stochastics

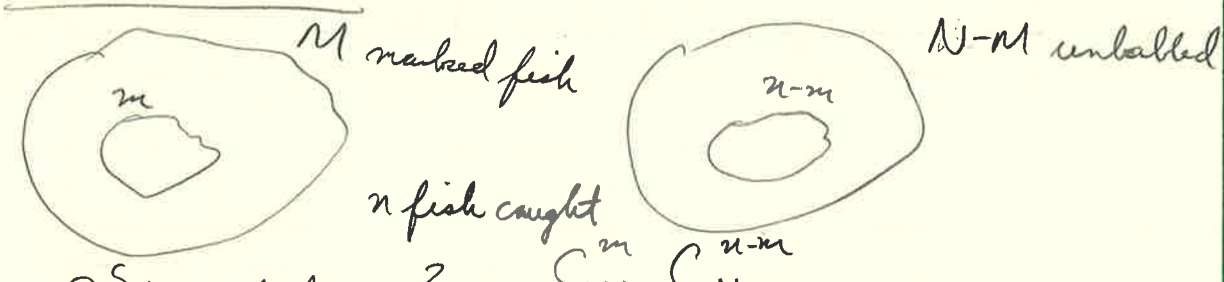
Stochastics
- probability theory
- mathematical statistics
- stochastic processes

Consider a pond that contains fish

Prob theory : # of fish at some given time (N)

$\mathbb{E}$, Var, or limit laws

Mathematical Stats :



$$P\{\#\text{ marked} = m\} = \frac{C_M^m \, C_{N-M}^{n-m}}{C_N^M}$$

Repeat $m_1, m_2, \dots, m_q$

$$\left(\begin{matrix}\text{log}\\ \text{likelihood}\end{matrix}\right) \sum_{k=1}^{q} P\{\#\text{ marked} = m_k\} \longrightarrow \max_N \quad (MLE)$$

1.3 Probability space $(\Omega, \mathcal{F}, P)$

General theory	Bernoulli Scheme $\left[\begin{matrix}1, \text{ success} \\ 0, \text{ failure}\end{matrix}\right]$ $(a_1, \dots, a_n)$, $a_i \in \{0,1\}$	$[0,1]$ Select point from
Ω - sample space	$\#\Omega = 2^n$, set of all vectors with components $\in \{0,1\}$	$\Omega = [0,1]$
$\mathcal{F}$ - σ-algebra 1) $\Omega \in \mathcal{F}$ 2) $A \in \mathcal{F} \Rightarrow \Omega \setminus A \in \mathcal{F}$ 3) $A_1, \dots, A_n, \dots \in \mathcal{F}$ $\Downarrow$ $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$	$\mathcal{F}$ = power set $\#\mathcal{F} = 2^{\#\Omega} = 2^{2^n}$	$P\{X \in [\alpha, \beta]\}$ $\Rightarrow [\alpha, \beta), (\alpha, \beta]$, $(\alpha, \beta), [\alpha, \beta), \{\beta\} \in \mathcal{F}$ Borel σ-algebra
P - probability measure 1) $P(\Omega) = 1$ 2) $A_1, A_2, \dots \in \mathcal{F}$ (disjoint) $\Rightarrow P\{\bigcup_i A_i\} = \sum_i P(A_i)$ $P : \mathcal{F} \to [0,1]$	$P\{1\} = p$ $P\{0\} = 1 - p$	$P\{[\alpha, \beta]\} = \beta - \alpha$

1.4 Definition of a stochastic function. Types of stochastic functions.

$(\Omega, \mathcal{F}, P)$

Random variable - $\underbrace{\text{measurable function}}_{}$ $\xi : \Omega \to \mathbb{R}$.

$$\forall B \in \mathcal{B}(\mathbb{R}) : \xi^{-1}(B) \subset \mathcal{F}$$

T - time

$X : T \times \Omega \to \mathbb{R}$ - random function, if $\forall t \in T$: $X(t, \cdot)$ is a random variable on $(\Omega, \mathcal{F}, P)$, denoted X_t

If $T = \mathbb{R}_+$, this is called a random process or stochastic process

$T = \mathbb{R}_+^n$, random field or stochastic field

$T = \mathbb{N}$ or $\mathbb{Z}$, discrete time stochastic process

$T = \mathbb{R}_+$ or $\mathbb{R}$, continuous time stochastic process

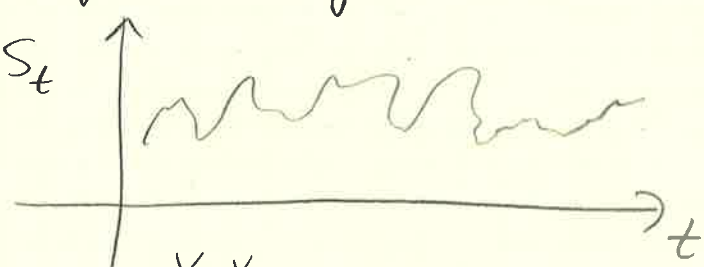
1.5 Trajectories and finite-dimensional distributions

$X : T \times \Omega \to \mathbb{R}$, $T = \mathbb{R}_+$

$\forall t \in T$: $X_t = X(t, \cdot)$ is a r.v. on $(\Omega, \mathcal{F}, P)$

Trajectory (= path)

X_t fix ω and get mapping $T \to \mathbb{R}$



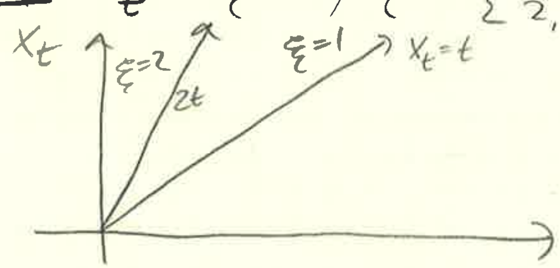
$X_1, X_2, \ldots,$

Finite-dimensional distribution $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$, $t_1, \ldots, t_n \in \mathbb{R}$

In mathematic stats, $X_{t_1}, X_{t_2} \ldots X_{t_n}$ are independent

In stochastic process, $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ are dependent

__Ex__: $X_t = \xi t$, $\xi = \begin{cases} 1, & \text{w.p. } 1/2 \\ 2, & \text{w.p. } 1/2 \end{cases}$



$$P\{X_{t_1} \le x_1, X_{t_2} \le x_2\} = \begin{cases} 0, & \min\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}\right) < 1 \\ 1/2, & \quad '' \quad \in [1,2] \\ 1, & \quad '' \quad \ge 2 \end{cases}$$

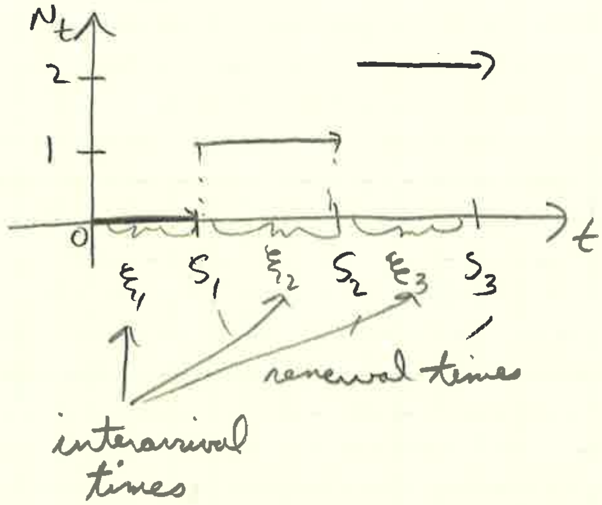
1.6 Renewal process. Counting process.

Renewal processes (discrete time)

$S_0 = 0,\ S_n = S_{n-1} + \xi_n$, where $\xi_1, \xi_2, \ldots$ – iid > 0 a.s.

$P\{\xi_i > 0\} = 1 \iff F(0) = 0$

$N_t = \operatorname{argmax}_k \{S_k \le t\}$ (Counting process)



$\{S_n > t\} = \{N_t < n\}$

$F \Rightarrow \mathbb{E} N_t \qquad S_n = \xi_1 + \cdots + \xi_n$

1.7. Convolution

<u>Convolution</u> $X \perp\!\!\!\perp Y$

$X \sim F_X,\ Y \sim F_Y$

$F_{X+Y}(x) = \int_{\mathbb{R}} F_X(x-y)\, dF(y) =: \underbrace{F_X * F_Y}$ conv in terms of distribution functions

$X \sim p_X,\ Y \sim p_Y$ (If Y, X have densities)

$p_{X+Y}(x) = \int_{\mathbb{R}} p_X(x-y)\, p_Y(y)\, dy =: p_X * p_Y$ } conv in terms of densities

$S_n = \xi_1 + \cdots + \xi_n$

let $F^{n*} := \underbrace{F * \cdots * F}_{n}$

1) $F^{n*}(x) \le F^{n}(x)$ if $F(0)=0$

$\xi_1, \dots, \xi_n \overset{iid}{\sim} F$

$\{\xi_1 + \dots + \xi_n \le x\} \subset \{\xi_1 \le x, \dots, \xi_n \le x\}$ Since $\xi_i > 0$ a.s.

$$\underbrace{P\{\xi_1 + \dots + \xi_n \le x\}}_{\parallel \; F^{n*}(x)} \le \prod_{k=1}^{n} \underbrace{P\{\xi_k \le x\}}_{F(x)}$$

2) $F^{n*}(x) \ge F^{(n+1)*}(x)$

$\{\xi_1 + \dots + \xi_n \le x\} \supset \{\xi_1 + \dots + \xi_{n+1} \le x\}$

Theorem: $S_n = S_{n-1} + \xi_n$ where $\xi_1, \xi_2, \dots \overset{iid}{\sim} F$, $F(0)=0$

(i) $\boxed{U(t) = \sum_{n=1}^{\infty} F^{n*}(t) < \infty}$

(2) $\boxed{\mathbb{E} N_t = U(t)}$

proof for (2)

$$\mathbb{E} N_t = \mathbb{E}\left[\#\{n : S_n \le t\}\right]$$

$$= \mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{1}\{S_n \le t\}\right] = \sum_{n=1}^{\infty} P\{S_n \le t\}$$

$$= \sum_{n=1}^{\infty} F^{n*}(t)$$

1.8 Laplace transform. Calculation of an expectation of a counting process (1)

Laplace transform

$f : \mathbb{R}_+ \to \mathbb{R}$: $\mathcal{L}_f(s) = \int_0^{\infty} e^{-sx} f(x)\,dx$

1) f - density of ξ, then $\mathcal{L}_f(s) = \mathbb{E}[e^{-s\xi}]$

2) f_1, f_2 : $\mathcal{L}_{\underbrace{f_1 * f_2}_{\text{densities}}}(s) = \mathcal{L}_{f_1}(s) \cdot \mathcal{L}_{f_2}(s)$

3) F - distribution function, $F(0)=0$, $p = F'$

$$\mathcal{L}_F(s) = \frac{\mathcal{L}_p(s)}{s}$$

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$$\text{l.h.s.} = \int_{\mathbb{R}_+} F(x)\,\frac{d(e^{-sx})}{s} = \cancel{-\frac{F(x)e^{-sx}}{s}\Big|_0^{\infty}} + \frac{1}{s}\int_{\mathbb{R}_+} p(x)\,e^{-sx}\,dx$$

$$= \text{r.h.s.}$$

Ex

1) $$\mathcal{L}_{x^k}(s) = \int_{\mathbb{R}_+} x^n \,\frac{d(e^{-sx})}{s} = \frac{n}{s}\int_{\mathbb{R}_+} x^{n-1} e^{-sx}\,dx$$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdots\frac{2}{s}\int_{\mathbb{R}_+} e^{-sx}\,dx = \frac{n!}{s^n}$$

2) $$\mathcal{L}_{e^{ax}}(s) = \frac{1}{s-a}, \text{ if } a<s$$

1.9 Laplace transform. Calculation of an expectation of a counting process (2)

$F \to \mathbb{E}N_t$

$$\mathbb{E}N_t = \mathcal{U}(t) = \sum_{n=1}^{\infty} F^{n*}(t) = F(t) + \Big(\sum_{n=1}^{\infty} F^{n*}(t)\Big) * F(t)$$

$$\Longleftrightarrow \quad \mathcal{U} = F + \mathcal{U} \underset{\text{dist fcn}}{*} F = F + \mathcal{U} \underset{\text{densities}}{*} p \quad \text{If } F' = p \text{ exists}$$

$$\int_{\mathbb{R}} \mathcal{U}(x-y)\,dF(y) = \int_{\mathbb{R}} \mathcal{U}(x-y)\,p(y)\,dy$$

$$\mathcal{L}_{\mathcal{U}}(s) = \underbrace{\mathcal{L}_F(s)}_{=\frac{\mathcal{L}_p(s)}{s}} + \mathcal{L}_{\mathcal{U}}(s)\,\mathcal{L}_p(s)$$

$$\boxed{\mathcal{L}_{\mathcal{U}}(s) = \frac{\mathcal{L}_p(s)}{s(1-\mathcal{L}_p(s))}}$$

① $F \to \mathcal{L}_p$

② $\mathcal{L}_p \to \mathcal{L}_{\mathcal{U}}$

③ $\mathcal{L}_{\mathcal{U}} \to \mathcal{U}$

1.10 *Laplace transform. Calculation of an expectation of a counting process (3)*

<u>Example</u>: $S_n = S_{n-1} + \xi_n$, $\xi_1, \xi_1, \dots$ have density $p(x)$

$$p(x) = \frac{e^{-x}}{2} + e^{-2x}, \quad x > 0$$

$\mathbb{E}N_t = ?$

① $p \to \mathcal{L}p$: $\mathcal{L}p(s) = \frac{1}{2}\mathcal{L}_{e^{-x}}(s) + \mathcal{L}_{e^{-2x}}(s)$

$$= \frac{1}{2(s+1)} + \frac{1}{s+2} = \frac{3s+4}{2(s+1)(s+2)}$$

② $\mathcal{L}p \to \mathcal{L}_u$: $\mathcal{L}_u(s) = \frac{\mathcal{L}p(s)}{s(1-\mathcal{L}p(s))} = \frac{3s+4}{s^2(2s+3)}$

③ $\mathcal{L}_u \to u$: $\mathcal{L}_u(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{2s+3}$

$$= \frac{A(2s+3) + B(2s^2+3s) + Cs^2}{s^2(2s+3)}$$

$$3s+4 = (2B+C)s^2 + (2A+3B)s + 3A$$

$A = \frac{4}{3}$, $2A+3B = 3 \Leftrightarrow B = \frac{1}{9}$, $2B+C=0 \Leftrightarrow C = -\frac{2}{9}$

$$\boxed{u(t) = \frac{4}{3}t + \frac{1}{9}(1) - \frac{1}{9}e^{-3/2\, t}}$$

1.11 *Limit theorems for renewal processes*

$S_n = S_{n-1} + \xi_n$; $\xi_1, \xi_2, \dots$ iid >0 a.s.

<u>Thm 1</u> $\mu = \mathbb{E}\xi_1 < \infty \Rightarrow \frac{N_t}{t} \xrightarrow{t\to\infty} \frac{1}{\mu}$ a.s.

(Analog to SLLN)

$$\boxed{\text{SLLN: } \frac{\xi_1 + \dots + \xi_n}{n} \xrightarrow[n\to\infty]{} \mu \text{ a.s.}}$$

<u>Thm 2</u>: (Analog of CLT) $\sigma^2 = \operatorname{Var}\xi_1 < \infty$

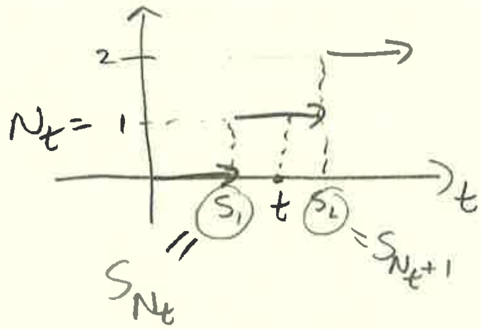
Then, $Z_t := \frac{N_t - t/\mu}{\sigma\sqrt{t}/\mu^{3/2}} \xrightarrow[t\to\infty]{d} N(0,1)$

$$P\{Z_t \le x\} \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

$$\boxed{\text{CLT: } \frac{\xi_1 + \dots + \xi_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1)}$$

proof (thm 1)

$$S_{N_t} \le t \le S_{N_t+1}$$



$$\frac{N_t}{S_{N_t+1}} \le \frac{N_t}{t} \le \frac{N_t}{S_{N_t}}$$

$$\lim_{t\to\infty} \frac{N_t}{S_{N_t}} = \lim_{n\to\infty} \frac{n}{S_n} = \frac{1}{\mu} \text{ by SLLN}$$

$$\lim_{t\to\infty} \frac{N_t}{S_{N_t+1}} = \underbrace{\lim_{t\to\infty} \frac{N_t}{N_t+1}}_{=1} \cdot \underbrace{\lim_{t\to\infty} \frac{N_t+1}{S_{N_t}+1}}_{=1/\mu} = \frac{1}{\mu}$$

proof (thm 2)

$$P\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right\} \to \phi(x), \ x \in R$$

$$P\{S_n \le \underbrace{n\mu + \sigma\sqrt{n}\, x}_{= t}\} \to \phi(x)$$

$$\Leftrightarrow P\{N_t \ge n\}$$

(set complements)

$N\mu \approx t$
$n \approx t/\mu$ (for n large enough)

$$n = \frac{t}{\mu} - \frac{\sigma\sqrt{n}}{\mu} x \approx \frac{t}{n} - \frac{\sigma\sqrt{t}}{\mu^{3/2}} x$$

$$\Rightarrow P\{Z_t \ge -x\} \to \phi(x) \quad \Leftrightarrow P\{Z_t \le x\} = 1 - P\{Z_t \ge -x\} \to 1 - \phi(-x) = \phi(x)$$

Poisson Processes

2.1 Definition of a Poisson process as a special example of a renewal process. Exact forms of the distributions of the renewal process and the counting process (1)

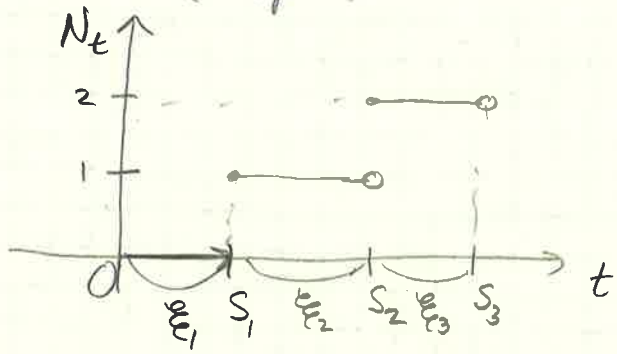
Renewal process

$S_0 = 0, \quad S_n = S_{n-1} + \xi_n, \quad \xi_1, \xi_2, \dots$ - i.i.d > 0 a.s., $\xi_i \sim F$

$N_t = \operatorname{argmax}_k \{S_k \le t\}$ (Counting process)

$U(t) = \mathbb{E} N_t = \sum_{n=1}^{\infty} F^{n*}(t)$

$\mathcal{L}_U(s) = \dfrac{\mathcal{L}_p(s)}{s(1-\mathcal{L}_p(s))}$: $p \to \mathcal{L}_p \to \mathcal{L}_U \to U$ $(p = F')$



$\mathcal{L}_U(s) = \int_{\mathbb{R}_+} e^{-sx} U(x)\, dx$

2.2 .... (2)

Poisson process

Def 1: A Poisson process is a renewal process s.t.

$$\xi_i \sim p(x) = \lambda e^{-\lambda x} \mathbb{I}\{x > 0\}$$, λ - intensity or rate

Thm (i): A distribution function of S_n

$$F_{S_n}(x) = \begin{cases} 1 - e^{-\lambda x} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!}, & x > 0 \\ 0, & x < 0 \end{cases}$$

$$p_{S_n}(x) = \lambda \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} \mathbb{I}\{x > 0\}$$

(ii) $\mathbb{P}\{N_t = n\} = e^{-\lambda t} \dfrac{(\lambda t)^n}{n!}$, $N_t \sim \text{Poisson}(\lambda t)$

2.3 ⋯ (3)

Proof (i) $n=1$: $S_1 = \xi_1$

$$p_{S_1}(x) = \lambda e^{-\lambda x}, \ x>0$$

$n \to n+1$

$$p_{S_{n+1}}(x) = \int_0^x p_{S_n}(x-y)\, p_{\xi_{n+1}}(y)\,dy$$

$$= \int_0^x \frac{\lambda^n (x-y)^{n-1}}{(n-1)!} e^{-\lambda(x-\cancel{y})} \lambda \cancel{e^{-\lambda y}}\,dy$$

$$= \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda x} \int_0^x (x-y)^{n-1}\,dy = \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda x} \frac{x^n}{n}$$

$$= \lambda \frac{(\lambda x)^n}{n!} e^{-\lambda x}$$

□

2.4 ⋯ (4)

proof (ii)

$$P\{N_t = n\} = P\{S_n \le t\} - P\{S_{n+1} \le t\} \quad =$$

$$\{N_t = n\} = \underbrace{\{S_n \le t\}}_{=A} \cap \underbrace{\{S_{n+1} > t\}}_{=B}$$

$$\left.\begin{array}{l} A \cap B = A \setminus B^c \\ \text{Here: } B^c \subset A \end{array}\right\} \Rightarrow P\{A \cap B\} = P\{A\} - P\{B^c\}$$

$$= \left(1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}\right) - \left(1 - e^{-\lambda t} \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!}\right)$$

$$= e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

□

2.5 Memoryless property

A r.v. X possesses the memoryless property iff

$P\{X > u+v\} = P\{X>u\}\, P\{X>v\}$. If $P\{X>v\}>0$, then

$$\boxed{P\{X > u+v \mid X > v\} = P\{X>u\}}$$

Thm 2: Let X be a r.v. with density $p(x)$, then

X- memoryless $\iff p(x) = \lambda e^{-\lambda x}$

Ex busses arrive every 20 ± 2 minutes

$v = 19$ min, $u = 10$ min

(l.h.s.) $P\{X > 29 \mid X > 19\} = 0$ given the data

(r.h.s) $P\{X > 10\} = 1$

Thus, Poisson process in not appropriate

2.6 Other definitions of Poisson processes (1)

Def 2 N_t – an integer value process s.t.

0) $N_0 = 0$ a.s.

1) N_t has independent increments: $\forall t_0 < t_1 < \dots < t_n$,

$N_{t_1} - N_{t_0}, \dots, N_{t_n} - N_{t_{n-1}}$ are independent

2) N_t has stationary increments

$N_t - N_s \overset{d}{=} N_{t-s}$

3) $N_t - N_s \sim \text{Poisson}(\lambda(t-s))$, $t > s$

3) $\Rightarrow$ 2)

2.7 Other definitions of Poisson processes (2)

$$P\{N_{t+h} - N_t = 0\} = 1 - \lambda h + o(h), \quad h \to 0$$

$$P\{N_{t+h} - N_t = 1\} = \lambda h + o(h), \quad h \to 0$$

$$P\{N_{t+h} - N_t \geq 2\} = o(h), \quad h \to 0$$

$$\lim_{h \to 0} \frac{1 - P\{N_{t+h} - N_t = 0\}}{h} = \lim_{h \to 0} \frac{1 - e^{-\lambda h}}{h} = \lambda$$

Def 3 N_t is a Poisson process, if

0) $N_0 = 0$

1) N_t has independent increments

2) N_t has stationary increments

3') $$\lim_{h \to 0} \frac{P\{N_{t+h} - N_t \geq 2\}}{P\{N_{t+h} - N_t = 1\}} = 0$$

Models related to Poisson processes

2.8 Non-homogeneous Poisson processes (1)

$N_t \sim Pois(\lambda t) \Rightarrow \mathbb{E}N_t = \lambda t$

Def: Let $\Lambda(t)$ be a differentiable, increasing function s.t. $\Lambda(0)=0$. Then, $X_t = N_t$ is a non-homogeneous Poisson process if,

0) $N_0 = 0$

1) N_t has independent increments

2) $\boxed{N_t - N_s \sim Pois(\Lambda(t) - \Lambda(s))}$

2.9 Non-homogeneous Poisson processes (2) (NHPP)

$\lambda(t) = \Lambda'(t)$ - intensity function

Properties NHPP:

1) $\mathbb{E}N_t = \Lambda(t)$

$\Lambda(t) = \alpha t^\beta, \alpha>0, \beta>0$ (for example)

2) if $\lambda(t) = \text{const} \Rightarrow \Lambda(t) = \text{const} \cdot t$

3) $\left.\begin{array}{l}\Lambda(t)\text{-differentiable} \Rightarrow \Lambda(t)\text{-continuous} \\ \Lambda(t)\text{-increasing}\end{array}\right\} \Rightarrow$

$\Rightarrow \exists \Lambda^{-1}(t)$. If $\text{Image}\,\Lambda(t) = \mathbb{R}_+$, $N_{\Lambda^{-1}(t)}$ - homogeneous P.P.

2.10 Relation between renewal theory and NHPP (1)

$S_n = \underset{t}{\operatorname{argmin}}\{N_t = n\}$, $\xi_n = S_n - S_{n-1}$

$\xi_1, \xi_2, \ldots$ - iid ?

1) $p_{\xi_1}(x) = \lambda(x) e^{-\Lambda(x)}$

$P\{\xi_1 \le x\} = P\{S_1 \le x\} = P\{N_x \ge 1\} = 1 - P\{N_x = 0\} =$

$\{S_n > t\} = \{N_t < n\}$

$= 1 - e^{-\Lambda(x)}$

Take derivatives of both sides to finish proof □

2.11 ⋯ (2)

$S_k = \underset{t}{\mathrm{argmin}}\{N_t = k\}$

$\xi_k = S_k - S_{k-1}$

1) $\boxed{p_{\xi_1}(t) = \lambda(t) e^{-\Lambda(t)}}$

2) $\boxed{p_{\xi_2|\xi_1}(t|s) = \lambda(t+s) e^{-\Lambda(t+s)+\Lambda(s)}}$

$$F_{(\xi_1,\xi_2)}(s,t) = P\{\xi_1 \le s, \xi_2 \le t\} = \int_0^s P\{\underbrace{\cancel{\xi_1 \le s},}_{\text{Since } y<s} \xi_2 \le t \,|\, \xi_1 = y\} p_{\xi_1}(y)\,dy$$

$$= \int_0^s P\{N_{t+y} - N_y \ge 1 \,|\, \xi_1 = y\} p_{\xi_1}(y)\,dy$$

independent

$$= \int_0^s \left(1 - e^{-\Lambda(t+y)+\Lambda(y)}\right) \lambda(y) e^{-\Lambda(y)}\,dy$$

$$p_{(\xi_1,\xi_2)}(s,t) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s} F_{(\xi_1,\xi_2)}(s,t)\right)$$

$$= \frac{\partial}{\partial t}\left(\left(1 - e^{-\Lambda(t+s)+\Lambda(s)}\right) \lambda(s) e^{-\Lambda(s)}\right)$$

$$= \lambda(t+s) e^{-\Lambda(t+s)+\Lambda(s)} \lambda(s) e^{-\Lambda(s)}$$

Then $p_{\xi_2|\xi_1}(t|s) = \dfrac{p_{(\xi_1,\xi_2)}(s,t)}{p_{\xi_1}(s)}$ finishes the proof □

2.12 ⋯ (3)

$\xi_1, \xi_2, \ldots$ – i.i.d. ? (NHPP can be obtained from renewal process iff NHPP is homogeneous PP)

$$p_{\xi_1}(t) = p_{\xi_2|\xi_2}(t|s), \quad \forall t, s > 0$$

$$\lambda(t) e^{-\Lambda(t)} = \lambda(t+s) e^{-\Lambda(t+s)+\Lambda(s)}$$

$$\left(\int_0^T \ldots dt\right): \cancel{e^{-\Lambda(0)}} - e^{-\Lambda(T)} = \cancel{e^0} - e^{-\Lambda(T+S)+\Lambda(S)}$$

$$\boxed{\begin{array}{l}\Lambda(T) = \Lambda(T+S) - \Lambda(S), \quad \forall S, T > 0 \\ \Lambda(t) - \text{increasing}\end{array}} \Rightarrow \Lambda(t) = \overset{\text{const}}{\overset{\|}{\lambda}}\, t$$

2.13 Elements of queuing theory. M/G/k systems (1)

$$\boxed{\begin{aligned} P\{N_{t+h}-N_t=0\} &= 1-\overset{\lambda(t)}{\lambda} h+o(h) \\ P\{N_{t+h}-N_t=1\} &= \overset{\lambda(t)}{\lambda} h+o(h) \\ P\{N_{t+h}-N_t\geq 2\} &= o(h) \end{aligned}}$$

M / G / k

I) Arrival Process : M - memoryless (Poisson)
D - deterministic
G - general

II) Service time (M, D, G)

III) A number of services $(1, 2, \ldots, \infty)$

$M/G/\infty$ $\qquad \tau > 0$ (time moment)

$\xrightarrow[\text{Customer arrivals}]{N(t)}$ $\to N_1(t)$ - still being served at τ : $\lambda_1(t)=\lambda(1-G(\tau-t))$

$\to N_2(t)$ - already completed by τ : $\lambda_2(t)=\lambda G(\tau-t)$

$$P\{N_1(t+\delta)-N_1(t)=1\} = P\{N(t+\delta)-N(t)=1\}\cdot\left(P\{Y>\tau-t\}+o(\delta)\right)$$
$$=(\delta\lambda+o(\delta))(1-G(\tau-t)+o(\delta))$$
$$=\boxed{\lambda\delta(1-G(\tau-t)+o(\delta))}$$

2.14 ... (2)

bernoulli scheme

$$P\{N_1(t)=n_1,\ N_2(t)=n_2\} = P\{N_1(t)=n_1,\ N_2(t)=n_2 \mid N(t)=n_1+n_2\}\cdot P\{N(t)=n_1+n_2\}$$

$$=C_{n_1+n_2}^{n_1}(1-G(\tau-t))^{n_1}G(\tau-t)^{n_2}\cdot e^{-\lambda t}\frac{(\lambda t)^{n_1+n_2}}{(n_1+n_2)!}$$

$$=\frac{\lambda t(1-G(\tau-t))^{n_1}}{n!}e^{-\lambda(1-G(\tau-t))}\cdot\frac{\lambda t(G(\tau-t))^{n_2}}{n_2!}e^{-\lambda G(\tau-t)}$$

$$=P\{N_1(t)=n_1\}\cdot P\{N_2(t)=n_2\}$$

Therefore $N_1 \perp\!\!\!\perp N_2$

2.15 Compound Poisson Processes (1)

$$X_t = \sum_{k=1}^{N_t} \xi_k \quad , \xi_1, \xi_2, \dots \text{ - iid.} \quad , N_t \text{ - P.P. with intensity } \lambda$$

and $\xi_1, \xi_2, \dots$ and N_t are independent

$\xi_1, \xi_2, \dots$ claim sizes

N_t - amount of claims until time t (Insurance interpretation)

X_t - aggregated claim amount

1) Probability generating function (BGF)

ξ - integer, ≥ 0 values

$$\boxed{\phi_\xi(u) = \mathbb{E}[u^\xi]}, \ |u| < 1$$

$$\xi_1 \perp\!\!\!\perp \xi_2 \Rightarrow \phi_{\xi_1+\xi_2}(u) = \phi_{\xi_1}(u)\, \phi_{\xi_2}(u)$$

2) Moment-generating function (MGF)

$$\boxed{\mathcal{L}_\xi(u) = \mathbb{E}[e^{-u\xi}]}, \ \xi \geq 0, \ u > 0$$

2.16 ... (2)

3) Characteristic function

$$\phi_\xi(u) = \mathbb{E}[e^{iu\xi}], u \in \mathbb{R}, \forall \xi \ , \ \phi_\xi : \mathbb{R} \to \mathbb{C} \ , \ \xi_1 \perp\!\!\!\perp \xi_2 \Rightarrow \phi_{\xi_1+\xi_2}(u) = \phi_{\xi_1}(u)\phi_{\xi_2}(u)$$

Thm $\boxed{\phi_{X_t - X_s}(u) = e^{\lambda(t-s)(\phi_{\xi_1}(u) - 1)}}$

Proof: $lhs = \mathbb{E} e^{iu(X_t - X_s)} = \sum_{k=0}^{\infty} \mathbb{E}[\underbrace{e^{iu(X_t-X_s)}}_{\overset{d}{=}\ \xi_1+\dots+\xi_k} | N_t - \cancel{N_s} = k]\, P\{N_t - N_s = k\}$ from $\perp\!\!\!\perp$

$$= \sum_{k=0}^{\infty} (\phi_{\xi_1}(u))^k e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^k}{k!}$$

$\square$

2.17 ⋯ (3)

$X_t = \sum_{k=1}^{N_t} \xi_k$, ξ can be any random variable

ξ: $\phi_\xi(u) = E[e^{iu\xi}]$

$\phi : \mathbb{R} \to \mathbb{C}$

$\xi_1 \perp\!\!\!\perp \xi_2 \Rightarrow \phi_{\xi_1+\xi_2}(u) = \phi_{\xi_1}(u)\,\phi_{\xi_2}(u)$

Thm: $\boxed{\phi_{X_t - X_s}(u) = e^{\lambda(t-s)(\phi_{\xi_1}(u)-1)} \quad , t > s \geq 0}$

proof

$$lhs = \mathbb{E}[e^{iu(X_t - X_s)}]$$

$$= \sum_{k=0}^{\infty} \mathbb{E}[e^{iu\underbrace{(X_t-X_s)}_{\overset{d}{=}\ \xi_1+\cdots+\xi_k}} \mid \cancel{N_t - N_s = k}] \cdot P\{\underbrace{N_t - N_s = k}_{\sim \text{Pois}(\lambda(t-s))}\}$$

since $\xi_1, \xi_2, \dots \perp\!\!\!\perp N_t$

$$= \sum_{k=0}^{\infty} [\phi_{\xi_1}(u)]^k \cdot e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^k}{k!}$$

□

2.18 ⋯ (4)

Corollary: $\boxed{\begin{aligned} \mathbb{E}X_t &= \lambda t\, \mathbb{E}\xi_1 \\ \text{Var}\, X_t &= \lambda t\, \mathbb{E}\xi_1^2 \end{aligned}}$

proof $E[\xi^r] < \infty \Rightarrow \phi(u)$ is r-times differentiable at 0 and $\phi^{(r)}(0) = i^r\, \mathbb{E}\xi^r$

$$\mathbb{E}X_t = \frac{\phi'_{X_t}(0)}{i} = \frac{\lambda t\, \overbrace{\phi_\xi'(0)}^{} \cdot \overbrace{\phi_{X_t}(0)}^{=1}}{i} = \lambda t\, \mathbb{E}\xi_1$$

$\frac{\phi_\xi'(0)}{i} = \mathbb{E}\xi$

□

3.1 Definition of a Markov chain. Some examples

Def: A Markov chain $-S_n,\ n=0,1,2,\ldots$

S - state space (countable)

$$\boxed{P\{S_n=j \mid S_{n-1}=i_{n-1},\ldots,S_0=i_0\} = P\{S_n=j \mid S_{n-1}=i_{n-1}\}}$$

$i_0,\ldots,i_{n-1},j\in S$ and $P\{S_{n-1}=i_{n-1},\ldots,S_0=i_0\}\neq 0$

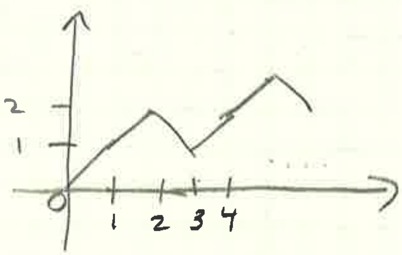
$$P\{S_n=i_n, S_{n-1}=i_{n-1},\ldots,S_0=i_0\} = P\{S_n=i_n \mid S_{n-1}=i_{n-1},\ldots,S_0=i_0\}\cdot P\{S_{n-1}=i_{n-1},\ldots,S_0=i_0\}$$

$$= P\{S_n=i_n \mid S_{n-1}=i_{n-1}\}\cdot P\{S_{n-1}=i_{n-1},\ldots,S_0=i_0\}$$

$$= P\{S_n=i_n \mid S_{n-1}=i_{n-1}\}\, P\{S_{n-1}=i_{n-1} \mid S_{n-2}=i_{n-2}\}\cdot\ldots\cdot P\{S_1=i_1 \mid S_0=i_0\}\, P\{S_0=i_0\}$$

Ex ① Random walk (not a renewal process)

$S_0=0,\ S_n=S_{n-1}+\xi_n,\quad \xi_1,\xi_2,\ldots$ - i.i.d. $\sim \begin{cases} 1, & \text{w.p. } p \\ -1, & \text{w.p. } 1-p \end{cases}$



$$P\{S_n=j \mid S_{n-1}=i_{n-1}\} = \begin{cases} p, & j=i_{n-1}+1 \\ 1-p, & j=i_{n-1}-1 \\ 0, & \text{otherwise} \end{cases}$$

② Taxis in the airport

1 taxi at any 1 moment, $n=1,2,3,\ldots$

X_k = # people waiting for a taxi at time k

Y_k = # people arriving at k

$$X_k = Y_k+(X_{k-1}-1)_+ = \begin{cases} Y_k, & \text{if } X_{k-1}=0 \\ Y_k+X_{k-1}-1, & \text{if } X_{k-1}-1>0 \end{cases}$$

③ X_n: $P\{X_n=j \mid X_{n-1}=i_{n-1},\ldots,X_0=i_0\} = P\{X_n=j \mid X_{n-1}=i_{n-1},\ldots,X_{n-m}=i_{n-m}\}$

$m\in\mathbb{N}$, fixed (X_n is not a Markov chain)

$S_n=(X_n,\ldots,X_{n-m-1})$, $n=(m-1),m,\ldots$ $\quad$ S_n is a Markov chain

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3.2 Matrix representation of a Markov chain. Transition matrix. Chapman – Kolmogorov equation.

Matrix representation

$S = (1, 2, \dots, M)$

$$P\{X_n = j \mid X_{n-1} = i\} = p_{ij} \text{ - homogeneous (no dependence on } n\text{)}$$

$$\underline{P} = (p_{ij})_{i,j=1}^{M} \text{ - transition matrix}$$

$$\left.\sum_{j=1}^{M} p_{ij} = 1,\ \forall i \ ;\quad p_{ij} \geq 0 \right\} \text{- stochastic matrix}$$

$$p_{ij}^{(m)} = P\{X_{n+m} = j \mid X_n = i\}$$

$$\underline{P}^{(m)} = (p_{ij}^{(m)}) \text{ - } m\text{-step transition matrix}$$

Thm: $\boxed{\underline{P}^{(m)} = \underline{P}^{M}}$

proof:
$$p_{ij}^{(m)} = \sum_{k=1}^{M} P\{X_{n+m} = j \mid X_{n+m-1} = k, X_n = i\} \cdot P\{X_{n+m-1} = k \mid X_n = i\}$$

(Markov property)
$$= \sum_{k=1}^{M} P\{X_{n+m} = j \mid X_{n+m-1} = k\}\, P\{X_{n+m-1} = k \mid X_n = i\}$$

$$= \sum_{k=1}^{M} p_{kj}\, p_{ik}^{(m-1)} \Rightarrow \underline{P}^{(m)} = \underline{P} \cdot \underline{P}^{(m-1)} = \dots = \underline{P}^{m} \quad \square$$

$$P\{X_k = j\} := \pi_j^{(k)}, \quad (\pi_1^{(k)}, \dots, \pi_M^{(k)}) := \vec{\pi}^{(k)}$$

$$\pi_j^{(k)} = \sum_{i=1}^{M} P\{X_k = j \mid X_{k-1} = i\}\, P\{X_{k-1} = i\}$$

$$= \sum_{i=1}^{M} p_{ij}\, \pi_i^{(k-1)} \Longrightarrow \vec{\pi}^{(k)} = \vec{\pi}^{(k-1)} \cdot \underline{P} = \vec{\pi}^{(0)} \underline{P}^{n}$$

$\vec{\pi}^*$ - stationary distribution for Markov chain if $\boxed{\vec{\pi}^* \underline{P} = \vec{\pi}^*}$

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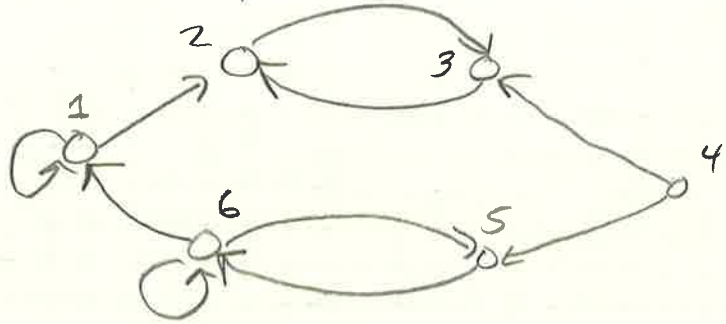
Graphical representation



1 node = 1 state

i,j-arc $\Leftrightarrow p_{ij} \neq 0$

$$\underline{P} = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ & & \cdots & & & \\ & & & & & \end{pmatrix}_{6\times 6}$$

Def (1) j is accessible from i $\exists$ walk (path) from i to j $(i \to j)$

$1 \to 3$; $1 \not\to 4$

(2) i and j communicate if $i \to j$ and $j \to i$ $(i \leftrightarrow j)$

$2 \leftrightarrow 3$

(3) $\underline{Y}$ - set, and relation $\sim$ called an equivalence relation

$a \sim a$, $a \in \underline{Y}$ - reflexivity

$a \sim b \Rightarrow b \sim a$, $a, b \in \underline{Y}$ - symmetry

$a \sim b$, $b \sim a \Rightarrow a \sim c$, $a, b, c \in \underline{Y}$ - transitivity

$\underline{Y} = \bigsqcup B_i$ ($\bigsqcup$ - disjoint union), B_i - equivalence classes

$\leftrightarrow$  $\Leftrightarrow \underline{P}\left(\cdots \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \cdots \right)$

$B_1, B_2, \ldots$ - equivalence classes

$$\boxed{\forall j \in B_i,\ \forall k \in S \quad \begin{cases} k \in B_i, & k \leftrightarrow j \\ k \notin B_i, & k \not\leftrightarrow j \end{cases}}$$

$2 \leftrightarrow 3$, $5 \leftrightarrow 6$, 1, 4 are the four equivalence classes

3.4 ... (2)

Def: i is recurrent, $\forall j : i \to j \Rightarrow j \to i$

i is transient if it's not recurrent $\Leftrightarrow \exists j : i \to j,\ j \not\to i$

ex: 1, 4, 5, 6 - transient

2, 3 - recurrent

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Thm: In 1 class of equivalence, all states are either recurrent or transient.

proof k-transient : $\exists j : k \to j,\ j \not\to k$

$i, k \in 1$ class $\Rightarrow$ $i \to k \to j$, but $j \not\to i$: $j \to i \to k$ is a contradiction $\square$

3.5 ... (3)

Def: *Period* of a state i is $GCD\{n : p_{ii}(n) \neq 0\} =: d(i)$

$d(i) = 1 \Rightarrow i$ - aperiodic

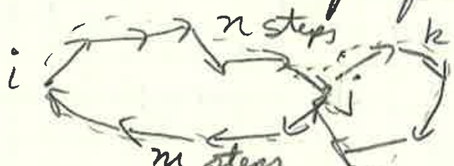
ex $d(1) = 1 = d(4) = d(5) = d(6)$

$d(2) = 2 = d(3)$

(4) has no return, so $d(4) = 1$ by convention

Thm: All elements in 1 class of equivalence have the same period

proof:



$p_{ii}(n+m+k) \neq 0$

$p_{ii}(n+m) \neq 0 \Rightarrow n+m \mid d(i)$

$k : p_{jj}(k) \neq 0 \Rightarrow n+m+k \mid d(i)$

$$\Rightarrow k \mid d(i) \Rightarrow \left.\begin{array}{l} d(i) \mid d(j) \\ d(j) \mid d(i) \end{array}\right\} \Rightarrow d(i) = d(j)$$

$\square$

3.6 Ergodic chains. Ergodic Theorem (1)

Matrix representation

$\underline{P}$, $\overrightarrow{\pi}^{(k)}$

$\underline{P}^{(m)} = \underline{P}^m$

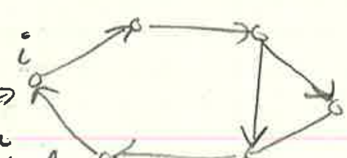
Ergodic Markov chains:
- 1 class of equivalence
- recurrent
- $d(i) = 1$ (aperiodic)

Graphical representation

classes of equivalence

recurrent / transient

$d(i)$ - period



return in 5 or 6 steps

$\therefore d(i) = 1$

Prop: Markov chain is ergodic $\Leftrightarrow \exists m \in \mathbb{N} : p_{ij}(m) \neq 0, \forall i,j \in S$ (*)

If chain is ergodic, then (*) hold $\forall m \geq (M-1)^2+1$

3.7 ... (2)

Ergodic theorem: Let X_t - ergodic Markov chain, i.e. X_t has 1 class of equivalence, recurrent and aperiodic. Then,

$$\boxed{\exists \lim_{n\to\infty} p_{ij}(n) = \pi_j^* > 0 \text{ (doesn't depend on } i)}$$

$$\sum_{j=1}^{M} \pi_j^* = 1 \qquad \vec{\pi}^* = (\pi_1^*, \ldots, \pi_M^*)$$

Corr (i) $\vec{\pi}^*$ - stationary distribution : $\vec{\pi}^* \underline{P} = \vec{\pi}^*$

(ii) $\lim_{n\to\infty} P\{X_n = j\} = \pi_j^*$ [$\pi_j^{(0)}$ is arbitrary]

proof (i) $i = 1, \ldots, M$

$$(\pi^* \underline{P})_i = \sum_{j=1}^{M} \pi_j^* p_{ji} = \sum_{j=1}^{M} \lim_{n\to\infty} p_{kj}(n)\, p_{ji} \qquad (k \in 1, \ldots, M)$$

$$= \lim_{n\to\infty} \sum_{j=1}^{M} \underbrace{p_{kj}(n)\, p_{ji}}_{\underline{P}^{(n)}\underline{P} = \underline{P}^{n+1} = \underline{P}^{(n+1)}}$$

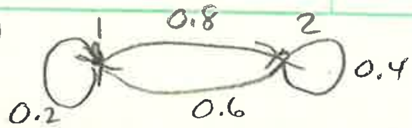
$$= \lim_{n\to\infty} p_{ki}(n+1) = \pi_i^* \qquad \square$$

proof (ii) $\lim_{n\to\infty} \pi_j^{(n)} = \lim_{n\to\infty} \sum_{k=1}^{M} \pi_k^{(0)} p_{kj}(n)$ $\qquad \pi_j^{(0)}$ is arbitrary

$\vec{\pi}^{(n)} = \vec{\pi}^{(0)} \underline{P}^{(n)}$

$$= \sum_{k=1}^{M} \pi_k^{(0)} \underbrace{\lim_{n\to\infty} p_{kj}(n)}_{= \pi_j^*} = \pi_j^* \underbrace{\sum_{k=1}^{M} \pi_k^{(0)}}_{=1} = \pi_j^* \qquad \square$$

ex



$$\underline{P} = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}$$

$$\vec{\pi}^* = (a, b) \; ; \; \vec{\pi}^* \underline{P} = \vec{\pi}^*$$

$$(a \;\; b) \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix} = (a \;\; b)$$

$$\left. \begin{aligned} 0.2a + 0.6b &= a \\ 0.8a + 0.4b &= b \end{aligned} \right\} \Rightarrow a = \tfrac{3}{7}, \; b = \tfrac{4}{7}$$

$$P\{X_n = 1\} \rightarrow \tfrac{3}{7}$$

$$P\{X_n = 2\} \rightarrow \tfrac{4}{7}$$

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4.1 Random vector. Definition and main properties

$\xi \sim N(\mu,\sigma^2)$, $p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $\sigma > 0$, $\mu \in R$

$$\phi(u) = e^{iu\mu - \frac{1}{2}u^2\sigma^2}$$

$$\left.\begin{array}{l} X_1, X_2 \sim N(0,1) \\ \mathrm{cor}(X_1, X_2) = 0 \end{array}\right\} \not\Rightarrow X_1 \perp\!\!\!\perp X_2$$

$$P\{X = \mu\} = 1 \Rightarrow \sigma = 0$$

<u>Def</u>: A random vector $\vec{X} = (X_1, \dots, X_n)$ is Gaussian iff $\forall (\lambda_1, \dots, \lambda_n) \in R^n$, $\sum_{k=1}^{n} \lambda_k X_k \sim N$

4.2 Gaussian vector. Definition and main properties

<u>Thm</u>: $\vec{X}$ - Gaussian iff any of the following holds:

(i) $\phi_{\vec{X}}(\vec{u}) = \mathbb{E}\left[e^{i\langle \vec{u}, \vec{X}\rangle}\right] = \exp\left\{ i\langle \vec{u}, \vec{\mu}\rangle - \frac{1}{2}\vec{u}^T C \vec{u} \right\}$

$\vec{\mu} \in R^n$; C - symmetric, positive semidefinite (size $n \times n$)

(ii) $\vec{X} = A\vec{X}^\circ + \vec{\mu}$; $A \in Mat(n \times n)$, $\vec{X}^\circ$ - standard normal vector

<u>Remark</u>: $\vec{\mu} = (\mathbb{E}X_1, \dots, \mathbb{E}X_n)$

$$C = (c_{jk})_{j,k=1}^{n};\ c_{jk} = \mathrm{cov}(X_j, X_k)$$

$$\sum_{k,j=1}^{n} u_k C_{kj} u_j \geq 0,\ \forall u \in R^n \iff u^T C u \geq 0,\ \forall u \in R$$

$$\sum_{k,j=1}^{n} u_k C_{kj} u_j = \sum_{k,j=1}^{n} u_k \mathrm{cov}(X_j, X_k) u_j = \mathrm{cov}\left(\sum_{j=1}^{n} u_j X_j, \sum_{k=1}^{n} u_k X_k\right)$$

$$= \mathrm{var}\left(\sum_{j=1}^{n} u_j X_j\right) \geq 0$$

$$A = C^{1/2}:\ AA = C \Rightarrow \exists U : U^{-1} = U^T : C = U^T \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} U$$

$$A = U^T \begin{pmatrix} \sqrt{d_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{d_n} \end{pmatrix} U \Rightarrow C = AA^T$$

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Proof: Def $\Leftrightarrow$ (i) $\Leftrightarrow$ (ii)

Def $\Rightarrow$ (i): $\langle \vec{u}, \vec{X} \rangle \sim N$

$$\phi_{\vec{X}}(\vec{u}) = \mathbb{E} e^{i \underbrace{\langle \vec{u}, \vec{X} \rangle}_{\xi} \cdot 1} = \phi_{\xi}(1) = e^{i\mu_\xi - \frac{1}{2}\sigma_\xi^2}$$

$$\mu_\xi = \mathbb{E}\left[\sum_{k=1}^n u_k X_k\right] = \sum_{k=1}^n u_k \mathbb{E}X_k = \sum_{k=1}^n u_k \mu_k = \langle \vec{\mu}, \vec{u} \rangle$$

$$\sigma_\xi^2 = \operatorname{cov}\left(\sum_{k=1}^n u_k X_k, \sum_{j=1}^n u_j X_j\right)$$

$$= \sum_{k=1}^n \sum_{j=1}^n u_k \operatorname{cov}(X_k, X_j) u_j = \vec{u}^T C \vec{u}$$

(i) $\Rightarrow$ Def By definition of ϕ for Gaussian

(ii) $\Rightarrow$ (i)

$$\vec{X}^\circ\text{-Gaussian} \Rightarrow \phi_{\vec{X}_0}(\vec{u}) = \exp\left\{-\frac{1}{2}\vec{u}^T\vec{u}\right\}$$

$$\phi_{\vec{X}}(\vec{u}) = \mathbb{E}\left[e^{i\langle \vec{u}, A\vec{X}^\circ + \vec{\mu} \rangle}\right] = \mathbb{E}\left[e^{i[\langle \vec{u}, \vec{\mu} \rangle + \langle \vec{u}, A\vec{X}^\circ \rangle]}\right]$$

$$= e^{i\langle \vec{u}, \vec{\mu} \rangle} \phi_{\vec{X}^\circ}(A^T\vec{u}) = e^{i\langle \vec{u}, \vec{\mu} \rangle} e^{-\frac{1}{2}\vec{u}^T \underbrace{AA^T}_{= C} \vec{u}}$$

$$= e^{i\langle \vec{u}, \vec{\mu} \rangle} e^{-\frac{1}{2}\vec{u}^T C \vec{u}}$$

(i) $\Rightarrow$ (ii) $A = C^{1/2}$ $\square$

4.3 Connection between independence of normal random variates and absence of correlation

Thm: Let $X_1, X_2 \sim N(0,1)$ and $\operatorname{cov}(X_1, X_2) = 0$, then

$$X_1 \perp\!\!\!\perp X_2 \Leftrightarrow (X_1, X_2) \text{ - Gaussian vector}$$

proof ($\Rightarrow$) $\lambda_1 X_1 + \lambda_2 X_2 \sim N \Rightarrow (X_1, X_2)$ - Gaussian vector

($\Leftarrow$) $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A = C^{1/2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = C$

$$\text{(ii)} \Rightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = A\vec{X}^\circ + \vec{\mu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X_1^\circ \\ X_2^\circ \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \overset{d}{=} \begin{pmatrix} X_1^\circ \\ X_2^\circ \end{pmatrix} \square$$

ex $X_1 \sim N(0,1)$, $X_2 := |X_1| \cdot \xi$, $\xi = \begin{cases} 1 & w.p. \ 1/2 \\ -1 & w.p. \ 1/2 \end{cases}$, $\xi \perp\!\!\!\perp X_1$

1) $X_2 \sim N(0,1)$, $x > 0$

$$P\{X_2 \le x\} = \mathbb{P}\{|X_1| \le x \mid \xi = 1\} P\{\xi = 1\} + P\{|X_2| \ge -x \mid \xi = -1\} P\{\xi = -1\}$$

$$\overset{\xi \perp\!\!\!\perp X_1}{=} P\{|X_1| \le x\} \cdot 1/2 + P\underbrace{\{|X_2| \ge -x\}}_{=1} \cdot \frac{1}{2}$$

$$= \frac{1}{2}\left[1 + P\{|X_1| \le x\}\right] = P\{X_1 \le x\}$$

2) $\text{cov}(X_1 X_2) = 0$

$$\text{cov}(X_1 X_2) = \mathbb{E}[X_1 X_2] - \mathbb{E}X_1 \, \mathbb{E}X_2 = \mathbb{E}[X_1 |X_1| \xi]$$

$$= \mathbb{E}[X_1 |X_1|] \underbrace{\mathbb{E}\xi}_{=0} - 0 \cdot \mathbb{E}X_2$$

$$= 0 - 0 = 0 \quad \square$$

3) X_1, X_2 are dependent

Assume $X_1 \perp\!\!\!\perp X_2 \Rightarrow (X_1 X_2)$ - Gaussian

$$\gamma = X_1 - X_2 = X_1 - |X_1|\xi \sim N$$

$\{\gamma > 0\}$ when $X_1 > 0$ and $\xi = -1$

$$P\{\gamma > 0\} \ge P\{X_1 > 0 \cap \xi = -1\} = P\{X_1 > 0\} P\{\xi = -1\}$$

$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$P\{\gamma = 0\} \ge 1/4$ since $\text{var}(\gamma) \ne 0$

Thus γ is not Gaussian

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4.4 Definition of a Gaussian process. Covariance function (1)

Def: A Gaussian process X_t is a stochastic process s.t.
$\forall t_1, t_2, \dots, t_n$: $(X_{t_1}, \dots, X_{t_n})$ – Gaussian vector

$m(t) = \mathbb{E}X_t$ – mathematical expectation $\quad m: \mathbb{R}_+ \to \mathbb{R}$

$K(t,s) = \operatorname{cov}(X_t, X_s)$, $\quad K: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$

$K(t,t) = \operatorname{Var} X_t$ and $\boxed{K(t,s) = K(s,t)}$

K – a positive semidefinite function (p.s.d.)

$\forall (t_1, \dots, t_n) \in \mathbb{R}_+^n$,
$\forall (u_1, \dots, u_n) \in \mathbb{R}^n$

$$\boxed{\sum_{k=1}^{n} \sum_{j=1}^{n} u_k u_j K(t_k, t_j) \geq 0}$$

$$\Longleftrightarrow \operatorname{cov}\Big(\sum_{k=1}^{n} u_k X_{t_k}, \sum_{j=1}^{n} u_j X_{t_j}\Big) = \operatorname{var}\Big(\sum_{k=1}^{n} u_k X_{t_k}\Big) \geq 0$$

4.5 Definition of a Gaussian process. Covariance function (2)

Thm: $m: \mathbb{R}_+ \to \mathbb{R}$, $K: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, K – symmetric and p.s.d.
Then, $\exists$ Gaussian process X_t: $\boxed{\mathbb{E}X_t = m(t), \ \operatorname{cov}(X_t, X_s) = K(t,s)}$

Gaussian r.v.: $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$
Gaussian vector: $\vec{\mu} \in \mathbb{R}^n$, $C \in Mat(n,n)$ and symmetric and p.s.d.

Gaussian process: $m: \mathbb{R}_+ \to \mathbb{R}$, $K: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ (sym. and p.s.d.)

Ex: $K(t,s) = |t-s|$ is not p.s.d.
assume p.s.d. $\Rightarrow \exists X_t: \operatorname{cov}(X_t, X_s) = |t-s|$
let $t=s$, $\operatorname{var} X_t = 0 \Rightarrow X_t = f(t)$ – deterministic
$\operatorname{cov}(X_t, X_s) = \mathbb{E}[X_t X_s] - \mathbb{E}X_t \mathbb{E}X_s = f(t)f(s) - f(t)f(s) = 0 \neq |t-s|$
Contradiction $\square$

Ex $K(t,s) = \min(t,s)$ is p.s.d.

$$\sum_{k,j=1}^{n} u_j u_k \min(t_j, t_k) \geq 0$$

$$f_t(x) = \mathbb{1}\{x \in [0,t]\}$$

$$\int_0^\infty \underbrace{f_t(x) f_s(x)}_{\substack{\shortparallel \\ 1 \text{ iff } x\in[0,t],\ x\in[0,s] \iff x \in [0,\min(t,s)]}} dx = \min(t,s)$$

$$\sum_{j,k=1}^{n} u_j u_k \int_0^\infty f_{t_j}(x) f_{t_k}(x) dx = \int_0^\infty \sum_{j,k=1}^{n} u_j u_k f_{t_j}(x) f_{t_k}(x) dx$$

$$= \int_{R_+} \sum_{j=1}^{n} u_j f_{t_j}(x) \sum_{k=1}^{n} u_k f_{t_k}(x) dx = \int_{R_+} \left(\sum_{j=1}^{n} u_j f_{t_j}(x)\right)^2 dx \geq 0$$

Then, $\boxed{K(t,s) = \min(t,s) \text{ is p.s.d.}}$ Δ

4.6 Two definitions of a Brownian motion

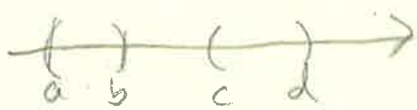
Brownian motion = Wiener process
$(B_t = W_t)$

Def 1: B_t - a Gaussian process with $m(t) = 0$, $K(t,s) = \min(t,s)$

Def 2: (0) $B_0 = 0$ a.s.
(1) B_t independent increments $[B_t \sim N(0,t)]$
(2) $B_t - B_s \sim N(0, t-s)$, $t > s \geq 0$

Def 1 $\Rightarrow$ Def 2

(0) $\left.\begin{array}{l} \mathbb{E} B_0 = m(0) = 0 \\ \mathrm{Var}\, B_0 = K(0,0) = 0 \end{array}\right\} \Rightarrow B_0 = 0$ a.s.

(1) 

$B_b - B_a \perp\!\!\!\perp B_d - B_c$

$$\mathrm{cov}(B_b - B_a, B_d - B_c) = \mathrm{cov}(B_b, B_d) - \mathrm{cov}(B_a, B_d) - \mathrm{cov}(B_b, B_c) + \mathrm{cov}(B_a, B_c)$$

$$= \min(b,d) - \min(a,d) - \min(b,c) + \min(a,c)$$

$$= b - a - b + a = 0$$

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$$\lambda_1(B_b - B_a) + \lambda_2(B_d - B_c) = \lambda_1 B_b - \lambda_1 B_a + \lambda_2 B_d - \lambda_2 B_c \sim N(\cdot,\cdot)$$

$\Rightarrow \begin{bmatrix} B_b - B_a \\ B_d - B_c \end{bmatrix}$ - Gaussian vector

$\Rightarrow B_b - B_a \perp\!\!\!\perp B_d - B_c$

(2) (B_t, B_s) - Gaussian vector $\Rightarrow B_t - B_s \sim N$

$$E[B_t - B_s] = \mathbb{E}B_t - \mathbb{E}B_s = m(t) - m(s) = 0 - 0 = 0$$

$$\begin{aligned} \mathrm{Var}[B_t - B_s] &= \mathrm{cov}(B_t - B_s, B_t - B_s) = \mathrm{cov}(B_t, B_t) - 2\mathrm{cov}(B_t, B_s) + \mathrm{cov}(B_s, B_s) \\ &= \min(t,t) - 2\min(t,s) + \min(s,s) \\ &= t - 2s + s = t - s \quad \square \end{aligned}$$

<u>Def 2 $\Rightarrow$ Def 1</u> $\quad t_1 < t_2 < \cdots < t_n$

$$\sum_{k=1}^{n} \lambda_k B_{t_k} = \lambda_n (B_{t_n} - B_{t_{n-1}}) + (\lambda_n + \lambda_{n-1}) B_{t_{n-1}} + \sum_{k=1}^{n-2} \lambda_k B_{t_k}$$

$$= \sum_{k=1}^{n} \alpha_k (B_{t_k} - B_{t_{k-1}}) \sim N \qquad (t_0 = 0)$$

$\Rightarrow (B_{t_1}, \ldots, B_{t_n})$ - Gaussian vector $\Rightarrow B_t$ - Gaussian

$B_t \sim N(0,t) \Rightarrow m(t) = \mathbb{E}B_t = 0 \qquad t > s$

$$\begin{aligned} K(t,s) = \mathrm{cov}(B_t, B_s) &= \mathrm{cov}(B_t - B_s + B_s, B_s) \\ &= \mathrm{cov}(B_t - B_s, B_s) + \mathrm{cov}(B_s, B_s) \\ &= \underbrace{\mathrm{cov}(B_t - B_s, B_s - B_0)}_{=0} + \mathrm{cov}(B_s, B_s) \\ &= \mathrm{var}(B_s) = s \end{aligned}$$

If $s > t$, $K(t,s) = t$

$\therefore K(t,s) = \min(t,s) \quad \square$

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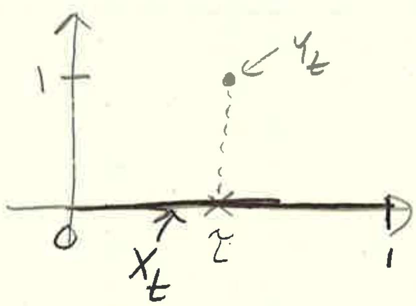
4.7 Modification of a process. Kolmogorov continuity theorem

Kolmogorov continuity theorem

Def X_t, Y_t are stochastically equivalent if
$$P\{X_t = Y_t\} = 1 \quad , \forall t \geq 0$$

Ex: $X_t = 0 \ , \forall t \in [0,1]$
$$Y_t = \mathbb{1}\{\tau = t\} \ , \ \tau \sim Unif(0,1)$$
$$P\{X_t = Y_t\} = P\{Y_t = 0\} = P\{t \neq \tau\} = 1$$



Thm If $\exists\, C, \alpha, \beta > 0$ s.t. $\mathbb{E}\left[|X_t - X_s|^\alpha\right] \leq C\,|t-s|^{1+\beta}$ $\forall\, t,s \in [a,b]$, then $\exists\, Y_t$ that is stochastically equivalent to X_t s.t. Y_t has continuous trajectories, i.e., X_t has a continuous modification.

Ex: $\mathbb{E}\left[|\underbrace{B_t - B_s}_{N(0,\,t-s)}|^4\right] = (t-s)^2 \underbrace{\mathbb{E}\,\xi^4}_{3} = 3(t-s)^2$
$$B_t - B_s \overset{d}{=} \sqrt{t-s}\ \xi, \ \xi \sim N(0,1)$$
$$\Rightarrow C = 3, \ \beta = 1, \ \alpha = 4$$

4.8 Main properties of Brownian Motion

① Quadratic variation

$$t_0 = 0 \quad t_1 \quad t_2 \quad \cdots \quad t = t_n$$

$$\lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} (B_{t_k} - B_{t_{k-1}})^2}_{S_n} = t$$

$\max |t_i - t_{i-1}| \to 0$ as $n \to \infty$

$$\mathbb{E}(S_n - t)^2 \xrightarrow[n\to\infty]{} 0 \iff \boxed{\lim_{n\to\infty} \sum_{k=1}^{\infty} (B_{t_k} - B_{t_{k-1}})^2 = t}$$ quadratic variation

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} |B_{t_k} - B_{t_{k-1}}| = \infty}$$ variation

② B_t - everywhere continuous, but nowhere differentiable

$$B_{t+h} \xrightarrow[h\to 0]{P} B_t, \quad \forall t \geq 0$$

③ $\lim\limits_{t\to\infty} \dfrac{B_t}{t} = 0$ a.s.

$\lim\limits_{t\to\infty} \dfrac{B_t}{\sqrt{t}} = \infty$ a.s.

Law of iterated logarithm : $\boxed{\lim\limits_{t\to\infty} \dfrac{B_t}{\sqrt{2t\log(\log t)}} = 1}$

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