1) $X_t = \eta + t,\ \eta \sim F_\eta \qquad ,t_1, \dots, t_n \in \mathbb{R}_+$

$P\{X_{t_1} \le x_1, X_{t_2} \le x_2, \dots, X_{t_n} \le x_n\}$

$= P\{\eta \le x_1 - t_1,\ \eta \le x_2 - t_2, \dots, \eta \le x_n - t_n\}$

$\{\eta \le x_1 - t_1,\ \eta \le x_2 - t_2, \dots, \eta \le x_n - t_n\} = \{\eta \le \min(x_1 - t_1, x_2 - t_2, \dots, x_n - t_n)\}$

$\Rightarrow F(X_{t_1}, X_{t_2}, \dots, X_{t_n}) = F_\eta(\min(x_1 - t_1, x_2 - t_2, \dots, x_n - t_n))$

2) $\xi_n = S_n - S_{n-1} = \begin{cases} 1 & \text{w.p. } 1/2 \\ 2 & \text{w.p. } 1/2 \end{cases}$

Find $\mathbb{E}N_t$ at $t=3$

$S_n = \xi_1 + \dots + \xi_n$

$\mathbb{E}N_t = \mathbb{E}[\#\{n : S_n \le t\}]$

$\Rightarrow \mathbb{E}N_3 = \mathbb{E}[\#\{n : S_n \le 3\}] = \mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{1}\{S_n \le 3\}\right]$

$= \sum_{n=1}^{\infty} P\{S_n \le 3\} = P\{S_1 \le 3\} + P\{S_2 \le 3\} + P\{S_3 \le 3\} + \dots$

$P\{S_1 \le 3\} = 1$

$P\{S_2 \le 3\} = 1 - P\{\xi_1 = 2, \xi_2 = 2\} = 1 - \frac{1}{4} = \frac{3}{4}$

$P\{S_3 \le 3\} = P\{\xi_1 = 1, \xi_2 = 1, \xi_3 = 1\} = \frac{1}{8}$

$\mathbb{E}N_3 = 1 + \frac{3}{4} + \frac{1}{8} = \frac{15}{8}$

3) $S_n = S_{n-1} + \xi_n,\quad p_\xi(x) = \lambda e^{-\lambda x}$

$\mathcal{L}p(s) = \int_0^\infty \lambda e^{-\lambda x} e^{-sx}\,dx = \int_0^\infty \lambda e^{-(\lambda+s)x}\,dx = \frac{-\lambda}{\lambda+s} e^{-(\lambda+s)x}\Big|_0^\infty$

$= \frac{\lambda}{\lambda+s}$

$\mathcal{L}_u(s) = \frac{\mathcal{L}p(s)}{s(1-\mathcal{L}p(s))} = \frac{\frac{\lambda}{\lambda+s}}{s\left(1-\frac{\lambda}{\lambda+s}\right)} = \frac{\frac{\lambda}{\lambda+s}}{s\left(\frac{\lambda+s-\lambda}{\lambda+s}\right)} = \frac{\lambda}{s^2}$

$\Rightarrow u(t) = \lambda t$

4) $X_t = e^{\eta t^2}$, $t \in R_+$

Find $F(X_{t_1}, X_{t_2}, \dots, X_{t_n})$

$$F(X_{t_1}, X_{t_2}, \dots, X_{t_n}) = P\{X_{t_1} \le x_1, X_{t_2} \le x_2, \dots, X_{t_n} \le x_n\}$$

$$= P\{e^{\eta t_1^2} \le x_1, \dots, e^{\eta t_n^2} \le x_n\}$$

$$= P\left\{\eta \le \ln \frac{x_1}{t_1^2}, \dots, \eta \le \ln \frac{x_n}{t_n^2}\right\}$$

$$= P\left\{\eta \le \min\left(\ln \frac{x_1}{t_1^2}, \dots, \ln \frac{x_n}{t_n^2}\right)\right\}$$

$$= F_\eta\left(\min\left(\ln \frac{x_1}{t_1^2}, \dots, \ln \frac{x_n}{t_n^2}\right)\right)$$

5) $S_n = S_{n-1} + \xi_n$, $p_\xi(x) = \begin{cases} \frac{1}{2} e^{-x}(x+1), & x \ge 0 \\ 0, & x < 0 \end{cases}$

Find EN_t

$$\mathcal{L}_p(s) = \frac{1}{2}\int_0^\infty (x+1)e^{-(s+1)x}dx = \frac{1}{2}\left\{\frac{1}{s+1} + \mathcal{L}_x(s+1)\right\} = \frac{1}{2}\left\{\frac{1}{s+1} + \frac{1}{(s+1)^2}\right\}$$

$$\mathcal{L}_u(s) = \frac{\frac{1}{2}\left(\frac{1}{s+1} + \frac{1}{(s+1)^2}\right)}{s\left(1 - \frac{1}{2}\left\{\frac{1}{s+1} + \frac{1}{(s+1)^2}\right\}\right)} = \frac{\frac{1}{2}\frac{s+2}{(s+1)^2}}{s\left(1 - \frac{1}{2}\frac{s+2}{(s+1)^2}\right)} = \frac{\frac{1}{2}\frac{s+2}{(s+1)^2}}{s\left(\frac{2(s+1)^2-(s+2)}{2(s+1)^2}\right)}$$

$$= \frac{s+2}{s(2(s^2+2s+1)-s-2)} = \frac{s+2}{s(2s^2+3s)} = \frac{s+2}{s^2(2s+3)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{2s+3}$$

$$A(2s+3) + Bs(2s+3) + Cs^2 = s+2$$

$$\Leftrightarrow (C+2B)s^2 + (2A+3B)s + 3A = s+2$$

$$A = \frac{2}{3}, \quad \frac{4}{3} + 3B = 1 \Leftrightarrow B = -\frac{1}{9}$$

$$C - \frac{2}{9} = 0 \Leftrightarrow C = \frac{2}{9}$$

$$\mathcal{L}_u(s) = \frac{2/3}{s^2} - \frac{\frac{1}{9}}{s} + \frac{2/9}{2s+3} \Leftrightarrow u(t) = \frac{2}{3}t - \frac{1}{9} + \frac{1}{9}e^{-3/2 t}$$

6) $P\{\eta > x\} = P\{\eta < -x\}$ and $P\{\eta = 0\} = 0$

$X_t = \xi^2 + t(\eta + t)$, $t \geq 0$

Find $P\{X_{t_2} > X_{t_1}\}$, $t_1 < t_2 \in \mathbb{R}_+$

$P\{X_{t_2} - X_{t_1} > 0\} = P\{t_2(\eta + t_2) - t_1(\eta + t_1) > 0\}$

$= P\{\eta(t_2 - t_1) > t_1^2 - t_2^2\} = P\left\{\eta < \frac{-(t_2^2 - t_1^2)}{t_2 - t_1}\right\}$

$= P\left\{\eta > \frac{t_2^2 - t_1^2}{t_2 - t_1}\right\}$

Since $\left\{\eta < \frac{-(t_2^2 - t_1^2)}{t_2 - t_1}\right\} \cup \left\{\eta > \frac{t_2^2 - t_1^2}{t_2 - t_1}\right\} \cup \{\eta = 0\} = \Omega$

and $P\left\{\eta < \frac{(t_2^2 - t_1^2)}{t_2 - t_1}\right\} + P\left\{\eta > \frac{t_2^2 - t_1^2}{t_2 - t_1}\right\} + \underbrace{P\{\eta = 0\}}_{0} = 1$

$P\left\{\eta < \frac{-(t_2^2 - t_1^2)}{t_2 - t_1}\right\} = \frac{1}{2}$ $\square$