2) $Y_{n+1} = \alpha Y_n + X_n,\ n=0,1,\dots \quad, Y_0 = 0,\ |\alpha|<1$

$X_n \overset{iid}{\sim} N(0,1),\ n=0,1,\dots$

$\mathbb{E} Y_1 = \alpha \mathbb{E} Y_0 + \mathbb{E} X_0 = 0$

Suppose this holds for $n=k$

$\mathbb{E} Y_{k+1} = \alpha\, \mathbb{E} Y_k + \mathbb{E} X_k = 0+0=0 \quad , \mathbb{E} Y_n = 0$

$Y_1 = \alpha Y_0 + X_0 \quad ; \operatorname{Var} Y_1 = 1$

$Y_2 = \alpha Y_1 + X_1 = \alpha X_0 + X_1$

$Y_3 = \alpha Y_2 + X_2 = \alpha(\alpha X_0 + X_1) + X_2 = \alpha^2 X_0 + \alpha X_1 + X_2$

$\Downarrow$

$$Y_n = \sum_{k=0}^{n-1} \alpha^k X_{n-k-1}$$

$$K(n,m) = \operatorname{cov}(Y_n, Y_m) = \operatorname{cov}\Big(\sum_{k=0}^{n-1}\alpha^k X_{n-k-1};\ \sum_{j=0}^{m-1}\alpha^j X_{m-j-1}\Big)$$

$$= \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} \alpha^{k+j} \operatorname{cov}(X_{n-k-1}, X_{m-j-1})$$

$$= \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} \alpha^{k+j}\, \mathbb{1}\{n-k-1 = m-j-1\}$$

$$= \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} \alpha^{k+j}\, \mathbb{1}\{n-m = k-j\}$$

$\Rightarrow Y_n$ is W.S.

$$\operatorname{Var} Y_n = \operatorname{Var}\Big(\sum_{k=0}^{n-1}\alpha^k X_{n-k-1}\Big) = \sum_{k=0}^{n-1}\alpha^{2k} = \frac{1}{1-\alpha^2}$$

3) W_t - Brownian motion. $X_t = (1-t)\, W_{t/(1-t)}$

$\mathbb{E} X_t = (1-t)\, \mathbb{E} W_{t/(1-t)} = 0$

$\operatorname{Var} X_t = (1-t)^2 \dfrac{t}{1-t} = (1-t)t \le 0$ for $t>1$

$$\operatorname{cov}(X_t, X_s) = \operatorname{cov}\big((1-t)\, W_{t/(1-t)};\ (1-s)\, W_{s/(1-s)}\big)$$

$$= (1-t)(1-s) \operatorname{cov}\big(W_{t/(1-t)}, W_{s/(1-s)}\big)$$

$$= (1-t)(1-s) \min\Big(\frac{t}{1-t}, \frac{s}{1-s}\Big) = K(t,s)$$

3cont'd) $K(t+h,s+h) = [1-(t+h)][1-(s+h)] \min\left(\frac{t+h}{1-(t+h)}, \frac{s+h}{1-(s+h)}\right)$

$\neq K(t,s) \Rightarrow$ not w.s

4) $X_t = W_{t+h} - W_t$, W_t - Brownian motion, $h>0$

$K(t,s) = \operatorname{cov}(X_t, X_s) = \operatorname{cov}(W_{t+h}-W_t, W_{s+h}-W_s)$

$= \operatorname{cov}(W_{t+h}, W_{s+h}) - \operatorname{cov}(W_t, W_{s+h}) - \operatorname{cov}(W_{t+h}, W_s) + \operatorname{cov}(W_t, W_s)$

$= \min(t+h, s+h) - \min(t, s+h) - \min(t+h, s) + \min(t,s)$

If $|t-s| > h \Rightarrow W_{t+h}-W_t \perp\!\!\!\perp W_{s+h}-W_s \Rightarrow K(t,s)=0$

If $|t-s| \le h \Rightarrow -h \le t-s \le h$

For $s \le t$: $t \le h+s \Rightarrow K(t,s) = s+h - t \cancel{-s} \cancel{+s} = h-(t-s)$

For $s > t$: $s \le t+h \Rightarrow K(t,s) = t+h - \cancel{t} - s \cancel{+t} = h-(s-t)$

$\Rightarrow K(t,s) = \begin{cases} h-|t-s| & , |t-s| \le h \\ 0 & , |t-s| > h \end{cases}$

6) X_t - w.s. , $\gamma(0)=2$, $\gamma(1)=\gamma(-1)=1$, $\gamma(n)=0$ all other n

$g_X(x) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ihx} \gamma(h) = \frac{1}{2\pi}\left[e^{ix} + 2 + e^{-ix}\right]$

$= \frac{1}{\pi}\left[1+\cos x\right]$

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