GCSE Maths Knowledge Sheet

Eason's Mathematics Toolbox

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Version 1. April 1, 2023

Contents

What is this and why this?

Mathematics is one of my favourite subjects and it is very important whatever you are doing in the future. It is mostly about techiniques solving problems, but the knowledge behind all those techiniques are vital for understanding. To aid practice, I produced this document based on the syllabus.

This is more of an extension of the syllabus and the structure is exactly the same. However, it provides some sample answers for those questions in the syllabus and is a good way to refer to your self assessment based on the syllabus.

I am also an IGCSE student so errors are inevitable in this document. Feel free to email eason.syc@icloud.com to point out any mistakes or submit an issue on the GitHub page!

This document assumes prior knowledge in CIE IGCSE Mathematics.

Section 0 Prior Knowledge and Notations

§0.1 Set Notations

Definition 0.1 (set and elements). If x is an element of set S, we denote $x \in S$. Otherwise, $x \notin S$.

Definition 0.2 (set constructers). We have two set constructer notations:

$$\{x \mid P(x)\} = \{x : P(x)\}$$

defines a set containing all x satisfying condition $P(x)$.

$$\{x_1, x_2, \dots\}$$

defines a set with elements $x_1, x_2, \dots$.

Definition 0.3 (empty set, universal set). We use $\emptyset$ or $\varnothing$ to define the empty set (set with no elements) and use $\mathcal{E}$ to denote the universal set.

Definition 0.4 (cardinality). We use $n(S)$ to denote the number of elements in set S.

Definition 0.5 (complement). We use S' to define the complement of set S,

$$S' = \{x \in \mathcal{E} \mid x \notin S\}$$

Definition 0.6 (subset, proper subset). We denote

$$A \subseteq B$$

if $x \in A \implies x \in B$.

Furthermore, if $A \neq B$, we denote it as

$$A \subset B.$$

Definition 0.7 (union, intersection). We denote

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\},$$

and

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Definition 0.8 (number sets). The set $\mathbb{N}$ is the natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$.

The set $\mathbb{Z}$ is the integers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

The set $\mathbb{Q}$ is the rational numbers,

$$\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\right\}.$$

The set $\mathbb{R}$ is the set of real numbers.

Definition 0.9 (intervals). We define the intervals as follows:

$$\begin{aligned}
(a, b) &= \{x \mid a < x < b\}, \\
(a, b] &= \{x \mid a < x \leq b\}, \\
[a, b) &= \{x \mid a \leq x < b\}, \\
[a, b] &= \{x \mid a \leq x \leq b\}, \\
(a, +\infty) &= \{x \mid x > a\}, \\
[a, +\infty) &= \{x \mid x \geq a\}, \\
(-\infty, b) &= \{x \mid x < b\}, \\
(-\infty, b] &= \{x \mid x \leq b\}, \\
(-\infty, +\infty) &= \mathbb{R}.
\end{aligned}$$

§0.2 Relationship Symbols and Operations

Definition 0.10 (implies, implied by, equivilent). A implies B is denoted by $A \implies B$, B implies A is denoted by $A \impliedby B$, A and B are equivilent is denoted by $A \iff B$.

Definition 0.11 (sum and product). We define

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n,$$

and

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdots a_n.$$

Definition 0.12 (binominal coefficient and factorial). We define

$$n! = n \cdot (n-1) \cdots 1,$$

with $0! = 1$, hence defining

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

§0.3 Functions

Definition 0.13 (composite of two functions). We define

$$gf(x) = g(f(x)).$$

Definition 0.14 (derivative). We denote

$$\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} = f^{(n)}(x)$$

as the nth derivative of $f(x)$.

Section 1 Functions

Definition 1.1 (function, domain, image). A **fucntion** $f : A \to B$ is defined as a mapping which maps each element in A to exactly one element in B. Basically, a function is an operation on a thing which definitely produces another thing.

We call A the **domain** (the set which this function can operate on). (And B the co-domain.)

We define the set

$$\{f(x) \mid x \in A\}$$

as the **range** of the function, which is all the outputs of the function.

At this stage, B will be $\mathbb{R}$ and A will be a subset of $\mathbb{R}$.

Definition 1.2 (one-to-one, many-to-one). We call a function f **one-to-one**, or injective, when

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

This means that each output a function will produce can only appear by opeating on exactly one element.

If a function is not one-to-one, we call it **many-to-one**.

Definition 1.3 (function notations). The result that f maps an element of the domain x to is denoted as $f(x)$. As an example, if function f maps x to $\sin x$, then the following are equivilent:

1. $f(x) = \sin x$,
2. $f : x \mapsto \sin x$.

Definition 1.4 (inverse). A function's inverse, denoted as $f^{-1}(x)$, is defined from the range of $f(x)$ to the domain of $f(x)$, and satisfies that:

$$f^{-1}(f(x)) = f(f^{-1}(x)) = x.$$

Theorem 1.5 (unique inverse). If the inverse of a function exists, then it is unique.

Theorem 1.6 (condition for existance of inverse). The inverse of a function exists if and only if such function is one-to-one.

Remark. The previous theorem is true if and only if such inverse is defined from the range. If such inverse is defined from the co-domain then we also require the function to surjective (i.e. range equals to co-domain) hence bijective. This is a very useful concept (isomorphism)!

Theorem 1.7 (inverse graphs). The graph of a inverse of a function and the function itself is symmetric by the line $y = x$.

Definition 1.8 (composite). The composite of f with f denoted as f^2 is defined as follows:

$$f^2(x) = f(f(x)).$$

Definition 1.9 (modulus). The graph of $|f(x)|$ and $f(x)$ has a relationship as follows:

The graph of $|f(x)|$ reflects the part of the graph of $f(x)$ below the x axis with regards to the x axis (basically flip it up).

Section 2 Quadratic

Definition 2.1 (quadratic). A quadratic function f is defined as an element of $\mathbb{P}[x]$ where $\deg f(x) = 2$.

Just kidding. A quadratic function f is defined as

$$f(x) = ax^2 + bx + c$$

where $a \neq 0$.

Theorem 2.2 (extremum property). A quadratic function $f(x)$ has a maximum if and only if $a < 0$, and it has a minimum if and only if $a > 0$. The turning point (extremum point in this case) of a quadratic is

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4ac}\right).$$

Proof. We can show this by **completing the square**.

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + 2 \cdot \frac{b}{2a} \cdot x\right) + c \\ &= a\left[x^2 + 2 \cdot \frac{b}{2a} \cdot x + \left(\frac{b}{2a}\right)^2\right] - a \cdot \left(\frac{b}{2a}\right)^2 + c \\ &= a \cdot \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c. \end{aligned}$$

If $a > 0$, then we have

$$ax^2 + bx + c \geq -\frac{b^2}{4a} + c,$$

where the equal sign holds if and onlly if $x = -\frac{b}{2a}$.

Similar argument holds for $a < 0$. □

Proof. We can also show this by **differentiation**. □

Theorem 2.3 (roots). The roots (solutions) to the quadratic will be

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Theorem 2.4 (discriminant). The **discriminant** for the quadratic $ax^2 + bx + c$ is defined as

$$\Delta = b^2 - 4ac.$$

When $\Delta > 0$, the quadratic has two distinct real roots; when $\Delta = 0$, the quadratic has two equal real roots; when $\Delta < 0$, the quadratic has two complex roots which are complex conjugate of each other (i.e. sum to a real number).

Theorem 2.5 (quadratic and a line). The intersections for a quadratic and a line (which is not perpendicular to the x axis) can be found by equating their equations and solve the corresponding equation (which is a quadratic).

Theorem 2.6 (quadratic inequalities). A quadratic inequality can be solved by finding the two solutions (known as **critical values**).

For the quadratic $f(x) = ax^2 + bx + c$ where $a > 0$ ($a < 0$ can be considered similarly),

1. $\Delta > 0$. Let the two roots be x_1 and x_2.

   The solution set to $f(x) > 0$ is
   $$(-\infty, x_1) \cup (x_2, +\infty).$$
   The solution set to $f(x) < 0$ is
   $$(x_1, x_2).$$

2. $\Delta = 0$. Let the root be x_r.

   The solution set to $f(x) > 0$ is
   $$(-\infty, x_r) \cup (x_r, +\infty).$$
   The solution set to $f(x) < 0$ is $\emptyset$.

3. $\Delta < 0$. The solution set to $f(x) > 0$ is $\mathbb{R}$ and the solution set to $f(x) < 0$ is $\emptyset$.

Section 3 Equations, inequalities and graphs

Theorem 3.1 (type $|ax + b| = c, a \neq 0, c \geq 0$). The solutions to the equation

$$|ax + b| = c$$

is

$$x_1 = \frac{c - b}{a}, x_2 = \frac{-c - b}{a}.$$

Proof. The solution to this can be shown by dividing it into cases, where $ax + b = -c$ or $ax + b = c$. □

Proof. This solution can also be shown by squaring both sides to get rid of the absolute value and using quadratic solving methods. I would not prefer it in the first place. □

Theorem 3.2 (generalise: $|f(x)| = c \geq 0$). The solution to this equation is the same as the solutions to $f(x) = \pm c$.

Theorem 3.3 (type $|ax + b| = |cx + d|$). The solutions to the equation

$$|ax + b| = |cx + d|$$

is

$$x_{1,2} = \frac{(cd - ab) \pm 2(ad - bc)}{a^2 - c^2}.$$

Proof. We can show it by squaring both sides getting

$$(ax + b)^2 = (cx + d)^2 \implies (a^2 - c^2)x^2 + 2(ab - cd)x + (b^2 - d^2) = 0.$$

The discriminant will be

$$\begin{aligned}\Delta &= [2(ab-cd)]^2 - 4(a^2-c^2)(b^2-d^2)\\ &= 4[(a^2b^2 - 2abcd + c^2d^2) - a^2b^2 - c^2d^2 + a^2d^2 + b^2c^2]\\ &= 4[a^2d^2 - 2abcd + b^2c^2]\\ &= 4(ad-bc)^2.\end{aligned}$$

Hence, solutions will be

$$\begin{aligned}x_{1,2} &= \frac{-2(ab-cd) \pm \sqrt{\Delta}}{2(a^2-c^2)}\\ &= \frac{(cd-ab) \pm 2(ad-bc)}{a^2-c^2}.\end{aligned}$$

□

Proof. We can also show this by considering order relationship between $-\frac{b}{a}, -\frac{d}{c}, x$ and expanding absolute values. I would not prefer it in the first place. □

Theorem 3.4 (type $|f(x)| = |g(x)|$). This solution will be the same as $[f(x)]^2 = [g(x)]^2$.

Theorem 3.5 (type $|ax+b| > / \leq c, c \geq 0$). Solutions to $|ax+b| > c$ with previous mentioned restrictions will be the same as the solution to

$$(ax+b)^2 - c^2 > 0.$$

I would prefer to square everything if this is a inequality, but we should be careful whether this operation is equivilent or not (will it introduce more solutions? will it ignore certain solutions?)

Theorem 3.6 (type $|ax+b| \leq |cx+d|$). Solutions to this will be equivilent to

$$(a^2-c^2)x^2 + 2(ab-cd)x + (b^2-d^2) \leq 0.$$

We can also do this by considering the relationship between $x, -\frac{b}{a}, -\frac{d}{c}$, but I think this way is easier.

Theorem 3.7 (graph of $p(x) = k(x-a)(x-b)(x-c), k \neq 0$). The graph of $p(x)$ will satisfies the follows:

$$\lim_{x\to+\infty} p(x) = -\lim_{x\to-\infty} p(x) = +\infty(\text{if } k > 0), = -\infty(\text{if } k < 0),$$

which shows the trends of the graph of $p(x)$ when it approaches infinity (go further to the left/right) has the same symbol as k.

Furthermore, the intersections of $p(x)$ and x-axis will be a, b, c, since intersection with x-axis implies $p(x) = 0$ (and they are the only ones due to the fundamental theorem of algebra).

Section 4 Indices and surds

In this section we will recall the full definition of the power of a positive number.

Definition 4.1 (a^b where $a > 0, b \in \mathbb{N}$). We inductively define it by base case

$$a^0 = 1,$$

and

$$a^b = a^{b-1} \cdot a \text{ for } b \geq 1,$$

in the positive direction, and

$$a^b = a^{b+1} \cdot \frac{1}{a} \text{ for } b \leq -1$$

in the negative direction.

Definition 4.2 (a^b where $a > 0, b = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N}$). We define it as

$$a^b = a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

Theorem 4.3 (calculation properties of powers). We have

$$\begin{aligned} a^b \cdot a^c &= a^{b+c}, \\ (a^b)^c = (a^c)^b &= a^{bc}, \\ (ab)^c &= a^c b^c. \end{aligned}$$

Readers should verify that the previous two definitions are actually well-defined under those properties (because I believe those properties are the reason why we define it as previously defined).

For those $b \in \mathbb{Q}'$ (i.e. b is a irrational number), we define it by a limit of rational numbers (Cauchy Sequence).

To rationalise a fraction, we use the following theorem (in fact, a method)

Theorem 4.4 (rationalising denominator). We have

$$\begin{aligned} \frac{k}{\sqrt{a}} &= \frac{k\sqrt{a}}{a}, \\ \frac{k}{\sqrt{a} - \sqrt{b}} &= \frac{k(\sqrt{a} + \sqrt{b})}{a - b}, \\ \frac{k}{\sqrt{a} + \sqrt{b}} &= \frac{k(\sqrt{a} - \sqrt{b})}{a - b}. \end{aligned}$$

Section 5 Factors of polynomials

Definition 5.1 (polynomial). A polynomial is an element of the linear space/communicative ring $\mathbb{P}[x]$.

Just kidding. A **polynomial** $p(x)$ can be expressed as a sum

$$p(x) = \sum_{i=0}^{n} k_i x^i$$

for some non-negative integer n which we call it the degree (DANGER ZONE! degree of the polynomial 0 is negative infinity), and real number k_is which are not all 0.

Definition 5.2 (root). A **root** x_0 of a polynomial $p(x)$ satisfies that $p(x_0) = 0$.

Theorem 5.3 (factor theorem). x_0 is a root of $p(x)$ if and only if $(x - x_0)$ is a factor of $p(x)$.

Proof. By basic properties of division, let $p(x) = q(x)(x - x_0) + r(x)$.

If x_0 is a root of $p(x)$, then by definition we have $p(x_0) = r(x) = 0$ which means $p(x)$ has a remainder of 0 upon division by $x - x_0$.

If $x - x_0$ is a factor of $p(x)$, we have $r(x) = 0$ and $p(x) = q(x)(x - x_0)$. Plugging in $x = x_0$ will give use $p(x) = 0$ hence x_0 is a root of $p(x)$. □

Theorem 5.4 (remainder theorem). The remainder of $p(x)$ divided by $(x - x_0)$ will be equal to $p(x_0)$.

Proof. Proof is similar to previous one. The reader should verify so. □

Section 6 Simultaneous equations

Section 7 Straight line graphs

Section 8 Logarithmic and exponential functions

Section 9 Circular measure

Section 10 Trigonometry

Section 11 Differentiation and integration

Section 12 Vectors in two dimensions

Section 13 Permutations and combinations

Section 14 Series

Afterwords