GCSE Maths Knowledge Sheet

Eason's Mathematics Toolbox

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Contents

What is this and why this?

This document assumes prior knowledge in CIE IGCSE Mathematics.

Section 1 Functions

Definition 1.1 (function, domain, image). A **fucntion** $f : A \to B$ is defined as a mapping which maps each element in A to exactly one element in B. Basically, a function is an operation on a thing which definitely produces another thing.

We call A the **domain** (the set which this function can operate on). (And B the co-domain.)

We define the set

$$\{f(x) \mid x \in A\}$$

as the **range** of the function, which is all the outputs of the function.

At this stage, B will be $\mathbb{R}$ and A will be a subset of $\mathbb{R}$.

Definition 1.2 (one-to-one, many-to-one). We call a function f **one-to-one**, or injective, when

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

This means that each output a function will produce can only appear by opeating on exactly one element.

If a function is not one-to-one, we call it **many-to-one**.

Definition 1.3 (function notations). The result that f maps an element of the domain x to is denoted as $f(x)$. As an example, if function f maps x to $\sin x$, then the following are equivilant:

1. $f(x) = \sin x$,
2. $f : x \mapsto \sin x$.

Definition 1.4 (inverse). A function's inverse, denoted as $f^{-1}(x)$, is defined from the range of $f(x)$ to the domain of $f(x)$, and satisfies that:

$$f^{-1}(f(x)) = f(f^{-1}(x)) = x.$$

Theorem 1.5 (unique inverse). If the inverse of a function exists, then it is unique.

Theorem 1.6 (condition for existance of inverse). The inverse of a function exists if and only if such function is one-to-one.

Remark. The previous theorem is true if and only if such inverse is defined from the range. If such inverse is defined from the co-domain then we also require the function to surjective (i.e. range equals to co-domain) hence bijective. This is a very useful concept (isomorphism)!

Theorem 1.7 (inverse graphs). The graph of a inverse of a function and the function itself is symmetric by the line $y = x$.

Definition 1.8 (composite). The composite of f with f denoted as f^2 is defined as follows:

$$f^2(x) = f(f(x)).$$

Definition 1.9 (modulus). The graph of $|f(x)|$ and $f(x)$ has a relationship as follows:

The graph of $|f(x)|$ reflects the part of the graph of $f(x)$ below the x axis with regards to the x axis (basically flip it up).

Section 2 Quadratic

Definition 2.1 (quadratic). A quadratic function f is defined as an element of $\mathbb{P}[x]$ where $\deg f(x) = 2$.

Just kidding. A quadratic function f is defined as

$$f(x) = ax^2 + bx + c$$

where $a \neq 0$.

Theorem 2.2 (extremum property). A quadratic function $f(x)$ has a maximum if and only if $a < 0$, and it has a minimum if and only if $a > 0$. The turning point (extremum point in this case) of a quadratic is

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4ac}\right).$$

Proof. We can show this by **completing the square**.

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x\right)+c \\ &= a\left(x^2+2\cdot\frac{b}{2a}\cdot x\right)+c \\ &= a\left[x^2+2\cdot\frac{b}{2a}\cdot x+\left(\frac{b}{2a}\right)^2\right]-a\cdot\left(\frac{b}{2a}\right)^2+c \\ &= a\cdot\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c. \end{aligned}$$

If $a>0$, then we have

$$ax^2+bx+c \geq -\frac{b^2}{4a}+c,$$

where the equal sign holds if and onlly if $x=-\frac{b}{2a}$.

Similar argument holds for $a<0$. □

Proof. We can also show this by **differentiation**. □

Theorem 2.3 (roots). The roots (solutions) to the quadratic will be

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Theorem 2.4 (discriminant). The **discriminant** for the quadratic ax^2+bx+c is defined as

$$\Delta=b^2-4ac.$$

When $\Delta>0$, the quadratic has two distinct real roots; when $\Delta=0$, the quadratic has two equal real roots; when $\Delta<0$, the quadratic has two complex roots which are complex conjugate of each other (i.e. sum to a real number).

Theorem 2.5 (quadratic and a line). The intersections for a quadratic and a line (which is not perpendicular to the x axis) can be found by equating their equations and solve the corresponding equation (which is a quadratic).

Definition 2.6 (intervals). We define the intervals as follows:

$$\begin{aligned} (a,b) &= \{x \mid a<x<b\}, \\ (a,b] &= \{x \mid a<x\leq b\}, \\ [a,b) &= \{x \mid a\leq x<b\}, \\ [a,b] &= \{x \mid a\leq x\leq b\}, \\ (a,+\infty) &= \{x \mid x>a\}, \\ [a,+\infty) &= \{x \mid x\geq a\}, \\ (-\infty,b) &= \{x \mid x<b\}, \\ (-\infty,b] &= \{x \mid x\leq b\}, \\ (-\infty,+\infty) &= \mathbb{R}. \end{aligned}$$

Theorem 2.7 (quadratic inequalities). A quadratic inequality can be solved by finding the two solutions (known as **critical values**).

For the quadratic $f(x) = ax^2 + bx + c$ where $a > 0$ ($a < 0$ can be considered similarly),

1. $\Delta > 0$. Let the two roots be x_1 and x_2.

   The solution set to $f(x) > 0$ is
   $$(-\infty, x_1) \cup (x_2, +\infty).$$
   The solution set to $f(x) < 0$ is
   $$(x_1, x_2).$$

2. $\Delta = 0$. Let the root be x_r.

   The solution set to $f(x) > 0$ is
   $$(-\infty, x_r) \cup (x_r, +\infty).$$
   The solution set to $f(x) < 0$ is $\emptyset$.

3. $\Delta < 0$. The solution set to $f(x) > 0$ is $\mathbb{R}$ and the solution set to $f(x) < 0$ is $\emptyset$.

Afterwords