Mathematics 18.08.22 (2) Notes

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Contents

Section 1 Sequence

Example 1.1. **(Sequence)** Find the next two items: $2.1, 2.01, 2.001, 2.0001, \cdots$ $2.00001, 2.000001$.
Example 1.2. **(Sequence)** Find the next two items: $1, 1/2, 1/3, 1/4, 1/5, \cdots$. $1/6, 1/7$.

Problem 1.3. Find the next two items: $2/64, 5/32, 8/16, 11/8, \cdots$.
Solution. The numerator $+3$, the denominator $\div 2$. Therefore the answer is: $14/4, 17/2$.

Problem 1.4. Find the next two items: $1, 1, 2, 3, 5, 8, 13, 21, \cdots$.
Solution. Every item is the sum of the previous two itmes. Therefore the answer is: $13+21 = 34, 21+34 = 55$.

§1.1 Mathematical Notation

Problem 1.5. $u_n = 4n^2 - 1$ defines a sequence.

(1) Find u_1, u_2, u_3.
Solution. $u_1 = 4 \times 1^2 - 1 = 3, u_2 = 4 \times 2^2 - 1 = 15, u_3 = 4 \times 3^2 - 1 = 35$.

(2) What is n when $u_n = 1295$?
Solution. We have $u_n = 4n^2 - 1 = 1295$, thus $n^2 = 324$, and $n = 18$.

(3) Show 900 is not in the sequence.
Solution. $u_n = 900 = 4n^2 - 1 \Leftrightarrow 4n^2 = 901 \Leftrightarrow n^2 = 225.25 \notin \mathbb{N}$, while $n \in \mathbb{N}$.

§1.2 Other Way Around

Problem 1.6. Find the nth term for $2, 16, 54, 128, \cdots$.
Solution.

$$\begin{aligned} 2, 16, 54, 128, \cdots &\xrightarrow{\div 2} 1, 8, 27, 64, \cdots \\ &= n^3. \end{aligned}$$

$u_n = 2 \times n^3$.

Method 1.7. (Important Sequences) Remember these sequences:

n	1	2	3	4	5
Square Numbers n^2	1	4	9	16	25
Cube Numbers n^3	1	8	27	64	125
Triangular Numbers $n(n+1)/2$	1	3	6	10	15
All Below: Exponential 2^n	2	4	8	16	32
3^n	3	9	27	81	243
0.1^n	0.1	0.01	0.001	0.0001	0.00001
$(-1)^n$	-1	1	-1	1	-1

Problem 1.8. Find the nth term for $0, 2, 5, 9, 14, \cdots$.
Solution.

$$\begin{aligned} 0, 2, 5, 9, 14, \cdots &\xrightarrow{+1} 1, 3, 6, 10, 15, \cdots \\ &= \frac{n(n+1)}{2}. \end{aligned}$$

$u_n = n(n+1)/2 - 1 = (n^2 + n - 2)/2$.

Problem 1.9. Find the nth term for $4, 9, 16, 25, 36, \cdots$.
Solution. $u_n = (n-1)^2 = n^2 - 2n + 1$.

Problem 1.10. Find the nth term for $8, 16, 24, 32, 40, \cdots$.
Solution.

$$\begin{aligned} 8, 16, 24, 32, 40, \cdots &\xrightarrow{\div 8} 1, 2, 3, 4, 5, \cdots \\ &= n. \end{aligned}$$

$u_n = 8n$.

Problem 1.11. Find the nth term for $3, 12, 27, 48, 75, \cdots$.
Solution.

$$\begin{aligned} 3, 12, 27, 48, 75, \cdots &\xrightarrow{\div 3} 1, 4, 9, 16, 25, \cdots \\ &= n^2. \end{aligned}$$

$u_n = 3n^2$.

Problem 1.12. Find the nth term for $1, 0.1, 0.01, 0.001, 0.0001, \cdots$.
Solution. $u_n = 10 \times 0.1^n = 0.1^{n-1} = 10^{-n+1}$.

Problem 1.13. Find the nth term for $5, 0.5, 0.05, 0.005, \cdots$.

Solution. $u_n = 50 \times 0.1^n = 5 \times 0.1^{n-1} = 5 \times 10^{-n+1}$.

Problem 1.14. Find the nth term for $3, 4, 11, 30, 67, \cdots$.
Solution.

$$\begin{aligned} 3, 4, 11, 30, 67, \cdot &\xrightarrow{-3} 0, 1, 8, 27, 64 \\ &= (n-1)^3. \end{aligned}$$

$u_n = (n-1)^3 + 3 = n^3 - 3n^2 + 3n + 2$.

Method 1.15. (Trying Out) Try out after writing the formulae.

Problem 1.16. Find the nth term for $5, 20, 45, 80, 125, \cdots$.
Solution.

$$\begin{aligned} 5, 20, 45, 80, 125, \cdots &\xrightarrow{\div 5} 1, 4, 9, 16, 25, \cdots \\ &= n^2. \end{aligned}$$

$u_n = 5n^2$.

Problem 1.17. Find the nth term for $1, 15, 53, 127, 249, \cdots$.
Solution.

$$\begin{aligned} 1, 15, 53, 127, 249, \cdots &\xrightarrow{+1} 2, 16, 54, 128, 250, \cdots \\ &\xrightarrow{\div 2} 1, 8, 27, 64, 125, \cdots \\ &= n^3. \end{aligned}$$

$u_n = 2n^3 - 1$.

Problem 1.18. Find the nth term for $3, 5, 9, 17, 33, \cdots$.
Solution.

$$\begin{aligned} 3, 5, 9, 17, 33, \cdots &\xrightarrow{-1} 2, 4, 8, 16, 32, \cdots \\ &= 2^n. \end{aligned}$$

$u_n = 2^n + 1$.

Problem 1.19. Find the nth term for $2, 6, 12, 20, 30, \cdots$.
Solution.

$$\begin{aligned} 2, 6, 12, 20, 30, \cdots &\xrightarrow{\div 2} 1, 3, 6, 10, 15, \cdots \\ &= \frac{n(n+1)}{2}. \end{aligned}$$

$u_n = n(n+1) = n^2 + n$.

Problem 1.20. Find the nth term for $1/3, 2/7, 4/11, 8/15, 16/19, \cdots$.
Solution. Numerator: 2^n; Denominator: $4n - 1$. $u_n = 2^n/(4n-1)$.

Problem 1.21. Find the nth term for $0.1, -0.8, 2.7, -6.4, 12.5, \cdots$.
Solution.

$$0.1, -0.9, 2.7, -6.4, 12.5, \cdots \xrightarrow{\times 10} 1, -9, 27, -64, 125, \cdots$$
$$\xrightarrow{\div(-1)^{n+1}} 1, 9, 27, 64, 125, \cdots .$$

$u_n = 0.1(-1)^{n+1}n^3$.

Problem 1.22. Find the nth term for $-2, 4, -8, 16, -32, \cdots$.
Solution.

$$-2, 4, -8, 16, -32, \cdots \xrightarrow{\div(-1)^n} 2, 4, 8, 16, 32, \cdots$$
$$= 2^n.$$

$u_n = (-2)^n$.

Problem 1.23. Find the nth term for $0.01, 0.04, 0.09, 0.16, 0.25, \cdots$.
Solution.

$$0.01, 0.04, 0.09, 0.16, 0.25, \cdots \xrightarrow{\times 100} 1, 4, 9, 16, 25$$
$$= n^2.$$

$u_n = 0.01n^2$.

§1.3 Using Difference

Method 1.24. (Using Difference)

(1) Constant difference ⇒ Highest power is one;

(2) Difference is constant difference ⇒ Highest power is two;

(3) Third difference is same ⇒ Highest power is three;

(4) nth difference is same ⇒ Highest power is n.

Problem 1.25. Find the nth term for $0, 6, 24, 60, 120, \cdots$.
Solution. First Difference: $6, 18, 36, 60$. Second Difference: $12, 18, 24$. Third Difference: $6, 6$. Therefore it is cubic.

Study sequence n^3. $1, 8, 27, 64, 125$. First Difference: $7, 19, 37, 61$. Second Difference: $12, 18, 24$. Same as the sequence. Therefore the second highest is linear.

$$0, 6, 24, 60, 120, \cdots \xrightarrow{-n^3} -1, -2, -3, -4, -5, \cdots$$
$$\xrightarrow{\times -1} 1, 2, 3, 4, 5, \cdots$$
$$= n.$$

$u_n = n^3 - n$.