220302 同角三角比 题目选解

高一 (4) 班 邵亦成 48 号

Section 1 填空题

§1.1 Q8

8. $f(x)=\sin\frac{\pi x}{6}+\cos\frac{\pi x}{4}$, 求 $\sum\limits_{i=1}^{2022}f(i)$.

定义$g(x), h(x)$如下:

$$g(x)\overset{\text{def}}{=\!=}\sin\frac{\pi x}{6}, h(x)\overset{\text{def}}{=\!=}\cos\frac{\pi x}{4},$$

显然有:

$$\sum_{i=1}^{2022}f(i)=\sum_{i=1}^{2022}g(i)+\sum_{i=1}^{2022}h(i).$$

下求$\sum\limits_{i=1}^{2022}g(i)$.

考虑 $\sum\limits_{i=1}^{12}g(i)$:

$$\begin{aligned}\sum_{i=1}^{12}g(i)=&\sin\frac{\pi}{6}+\sin\frac{2\pi}{6}+\sin\frac{3\pi}{6}+\sin\frac{4\pi}{6}+\sin\frac{5\pi}{6}+\sin\frac{6\pi}{6}\\&+\sin\frac{7\pi}{6}+\sin\frac{8\pi}{6}+\sin\frac{9\pi}{6}+\sin\frac{10\pi}{6}+\sin\frac{11\pi}{6}+\sin\frac{12\pi}{6}\\=&\sin\frac{\pi}{6}+\sin\frac{2\pi}{6}+\sin\frac{3\pi}{6}+\sin\frac{4\pi}{6}+\sin\frac{5\pi}{6}+\sin\frac{6\pi}{6}\\&-\sin\frac{5\pi}{6}-\sin\frac{4\pi}{6}-\sin\frac{3\pi}{6}-\sin\frac{2\pi}{6}-\sin\frac{\pi}{6}-\sin\frac{0\pi}{6}\\=&\sin\frac{12\pi}{6}-\sin\frac{0\pi}{6}\\=&0.\end{aligned}$$

显然有:

$$g(x)=g(x+12),$$

又 $2022=168\times12+6$,
有:

$$\begin{aligned}\sum_{i=1}^{2022} g(i) &= 168\times 0+\sin\frac{\pi}{6}+\sin\frac{2\pi}{6}+\sin\frac{3\pi}{6}+\sin\frac{4\pi}{6}+\sin\frac{5\pi}{6}+\sin\frac{6\pi}{6}\\ &=0+\frac{1}{2}+\frac{\sqrt{3}}{2}+1+\frac{\sqrt{3}}{2}+\frac{1}{2}+0\\ &=2+\sqrt{3}.\end{aligned}$$

下求$\sum_{i=1}^{2022} h(i)$.

考虑 $\sum_{i=1}^{8} h(i)$:

$$\begin{aligned}\sum_{i=1}^{8} h(i) =& \cos\frac{\pi}{4}+\cos\frac{2\pi}{4}+\cos\frac{3\pi}{4}+\cos\frac{4\pi}{4}\\ &+\cos\frac{5\pi}{4}+\cos\frac{6\pi}{4}+\cos\frac{7\pi}{4}+\cos\frac{8\pi}{4}\\ =&\cos\frac{\pi}{4}+\cos\frac{2\pi}{4}-\cos\frac{\pi}{4}-\cos\frac{0\pi}{4}\\ &-\cos\frac{\pi}{4}-\cos\frac{2\pi}{4}+\cos\frac{\pi}{4}+\cos\frac{0\pi}{4}\\ =&0.\end{aligned}$$

显然有:

$$h(x)=h(x+8),$$

又 $2022=252\times 8+6$,

有:

$$\begin{aligned}\sum_{i=1}^{2022} h(i) &= 252\times 0+\cos\frac{\pi}{4}+\cos\frac{2\pi}{4}+\cos\frac{3\pi}{4}+\cos\frac{4\pi}{4}+\cos\frac{5\pi}{4}+\cos\frac{6\pi}{4}\\ &=0+0-\cos\frac{0\pi}{4}-\cos\frac{\pi}{4}-0\\ &=-1-\frac{\sqrt{2}}{2}.\end{aligned}$$

综上,

$$\begin{aligned}\sum_{i=1}^{2022} f(i) &= \sum_{i=1}^{2022} g(i)+\sum_{i=1}^{2022} h(i)\\ &=2+\sqrt{3}-1-\frac{\sqrt{2}}{2}\\ &=1-\frac{\sqrt{2}}{2}+\sqrt{3}.\end{aligned}$$

§1.2 Q10

10. 下列命题中, 正确的有:

(1) 若 $\sin\alpha\sqrt{1-\cos^2\alpha}-\cos\alpha\sqrt{1-\sin^2\alpha}=-1$, 则 $\alpha\in\mathrm{IV}$.

(2) 若 $\sqrt{\dfrac{1+\sin\alpha}{1-\sin\alpha}}=\tan\alpha+\sec\alpha$, 则 $\alpha\in\left(2k\pi-\dfrac{\pi}{2},2k\pi+\dfrac{\pi}{2}\right),k\in\mathbb{Z}$.

(3) 若 $\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}}-\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}=-2\cot\alpha$, 则 $\alpha\in(2k\pi-\pi,2k\pi),k\in\mathbb{Z}$.

(4) 若 $\alpha\in\mathrm{IV}$, 则 $\left[\exists\alpha:\left(\dfrac{\alpha}{2}\in\mathrm{II}\wedge\dfrac{\alpha}{4}\in\mathrm{II}\right)\right]\vee\left[\exists\alpha:\left(\dfrac{\alpha}{2}\in\mathrm{IV}\wedge\dfrac{\alpha}{4}\in\mathrm{IV}\right)\right]$.

(1) **若 $\sin\alpha\sqrt{1-\cos^2\alpha}-\cos\alpha\sqrt{1-\sin^2\alpha}=-1$, 则 $\alpha\in\mathrm{IV}$.**

$$\begin{aligned}\sin\alpha\sqrt{1-\cos^2\alpha}-\cos\alpha\sqrt{1-\sin^2\alpha}=-1&\Rightarrow\sin\alpha\left|\sin\alpha\right|-\cos\alpha\left|\cos\alpha\right|=-1\\&\Rightarrow\sin^2\alpha\cdot\operatorname{sgn}(\sin\alpha)-\cos^2\alpha\cdot\operatorname{sgn}(\cos\alpha)=-\sin^2\alpha-\cos^2\alpha.\end{aligned}$$

考虑$\sin\alpha=0\Leftrightarrow\alpha=k\pi,k\in\mathbb{Z}$, 有:

$$-\operatorname{sgn}(\cos\alpha)=-1,$$

即

$$\alpha=2k\pi,k\in\mathbb{Z}.$$

考虑$\cos\alpha=0\Leftrightarrow\alpha=k\pi+\dfrac{\pi}{2},k\in\mathbb{Z}$, 有:

$$\operatorname{sgn}(\sin\alpha)=-1,$$

即

$$\alpha=2k\pi+\frac{3}{2}\pi,k\in\mathbb{Z}.$$

考虑**其余的** α, 显然有 $\sin\alpha<0,\cos\alpha>0$, 即 $\alpha\in\mathrm{IV}$.

综上所述,

$$\alpha\in\bigcup_{k\in\mathbb{Z}}\left[2k\pi+\frac{3}{2}\pi,2k\pi+2\pi\right],$$

与原命题不符.

(2) **若 $\sqrt{\dfrac{1+\sin\alpha}{1-\sin\alpha}}=\tan\alpha+\sec\alpha$, 则 $\alpha\in\left(2k\pi-\dfrac{\pi}{2},2k\pi+\dfrac{\pi}{2}\right),k\in\mathbb{Z}$.**

$$
\begin{aligned}
\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} &= \sqrt{\frac{(1+\sin\alpha)^2}{(1-\sin\alpha)(1+\sin\alpha)}}\\
&= \sqrt{\frac{(1+\sin\alpha)^2}{\cos^2\alpha}}\\
&= \frac{1+\sin\alpha}{|\cos\alpha|}\\
&= \tan\alpha+\sec\alpha\\
&= \frac{\sin\alpha}{\cos\alpha}+\frac{1}{\cos\alpha}\\
&= \frac{1+\sin\alpha}{\cos\alpha}.
\end{aligned}
$$

于是 $\cos\alpha > 0(\sin\alpha=-1 \Rightarrow \tan\alpha, \sec\alpha \text{ DNE})$, 故有

$$
\alpha \in \bigcup_{k\in\mathbb{Z}} \left(2k\pi-\frac{\pi}{2}, 2k\pi+\frac{\pi}{2}\right),
$$

与原命题相符.

(3) **若** $\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}}-\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}=-2\cot\alpha$**, 则** $\alpha\in(2k\pi-\pi, 2k\pi), k\in\mathbb{Z}$**.**

$$
\begin{aligned}
\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}-\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} &= \sqrt{\frac{(1+\cos\alpha)^2}{(1-\cos\alpha)(1+\cos\alpha)}}-\sqrt{\frac{(1-\cos\alpha)^2}{(1+\cos\alpha)(1-\cos\alpha)}}\\
&= \frac{1+\cos\alpha}{|\sin\alpha|}-\frac{1-\cos\alpha}{|\sin\alpha|}\\
&= \operatorname{sgn}(\sin\alpha)\cdot 2\frac{\cos\alpha}{\sin\alpha}\\
&= -2\cot\alpha\\
&= -2\frac{\cos\alpha}{\sin\alpha}.
\end{aligned}
$$

于是 $\cos\alpha=0$ 或 $\sin\alpha<0$, 故有

$$
\alpha \in \left[\bigcup_{k\in\mathbb{Z}}(2k\pi-\pi, 2k\pi)\right]\cup\left[\bigcup_{k\in\mathbb{Z}}\left\{2k\pi-\frac{\pi}{2}, 2k\pi+\frac{\pi}{2}\right\}\right],
$$

与原命题不符.

(4) **若** $\alpha\in\mathrm{IV}$**, 则** $\left[\exists\alpha:\left(\frac{\alpha}{2}\in\mathrm{II}\wedge\frac{\alpha}{4}\in\mathrm{II}\right)\right]\vee\left[\exists\alpha:\left(\frac{\alpha}{2}\in\mathrm{IV}\wedge\frac{\alpha}{4}\in\mathrm{IV}\right)\right]$.

$$\begin{aligned}\alpha \in \mathrm{IV} \Rightarrow \alpha &\in \bigcup_{k\in\mathbb{Z}}\left(2k\pi+\frac{3}{2}\pi, 2k\pi+2\pi\right)\\ \Rightarrow \frac{\alpha}{2} &\in \bigcup_{k\in\mathbb{Z}}\left(k\pi+\frac{3}{4}\pi, k\pi+\pi\right)\\ \Rightarrow \frac{\alpha}{4} &\in \bigcup_{k\in\mathbb{Z}}\left(\frac{k}{2}\pi+\frac{3}{8}\pi, \frac{k}{2}\pi+\frac{\pi}{2}\right).\end{aligned}$$

考虑$k=4n(n\in\mathbb{Z})$, 有 $\frac{\alpha}{2}\in \mathrm{II}, \frac{\alpha}{4}\in \mathrm{I}$.
考虑$k=4n+1(n\in\mathbb{Z})$, 有 $\frac{\alpha}{2}\in \mathrm{IV}, \frac{\alpha}{4}\in \mathrm{II}$.
考虑$k=4n+2(n\in\mathbb{Z})$, 有 $\frac{\alpha}{2}\in \mathrm{II}, \frac{\alpha}{4}\in \mathrm{III}$.
考虑$k=4n+3(n\in\mathbb{Z})$, 有 $\frac{\alpha}{2}\in \mathrm{IV}, \frac{\alpha}{4}\in \mathrm{IV}$.
与原命题不符.

综上, 真命题只有 (2).

Section 2 附加题

§2.1 Q15

15. 记 $f(\alpha)=\sin(\cos(\alpha)), g(\alpha)=\cos(\sin(\alpha))$,

(1) 解不等式: $f(\alpha)g(\alpha)>0$.

(2) 当 $\alpha\in\left(0,\frac{\pi}{2}\right)$, 证明: $f(\alpha)<g(\alpha)$.

(1) 解不等式: $f(\alpha)g(\alpha)>0$.

$$\begin{aligned} f(\alpha)g(\alpha) &> 0\\ \sin(\cos(\alpha))\cos(\sin(\alpha)) &> 0\\ \sin(\cos(\alpha)) &> 0 \qquad\qquad (\sin(\alpha)\in[-1,1], \forall x\in[-1,1]:\cos(\alpha)>0)\\ \cos(\alpha) &\in [-1,1]\cap\bigcup_{k\in\mathbb{Z}}(2k\pi, 2k\pi+\pi)\\ \cos(\alpha) &\in (0,1]\\ \alpha &\in \bigcup_{k\in\mathbb{Z}}\left(2k\pi-\frac{\pi}{2}, 2k\pi+\frac{\pi}{2}\right).\end{aligned}$$

(2) 当 $\alpha \in \left(0, \dfrac{\pi}{2}\right)$, 证明: $f(\alpha) < g(\alpha)$.

要证:

$$\forall \alpha \in \left(0, \frac{\pi}{2}\right): f(\alpha) < g(\alpha),$$

只需证:

$$\max_{\alpha \in \left(0, \frac{\pi}{2}\right)} f(\alpha) < \min_{\alpha \in \left(0, \frac{\pi}{2}\right)} g(\alpha),$$

由于 $f(\alpha), g(\alpha)$ 均连续, 只需证:

$$\max_{\alpha \in \left[0, \frac{\pi}{2}\right]} f(\alpha) < \min_{\alpha \in \left[0, \frac{\pi}{2}\right]} g(\alpha).$$

$\cos(\alpha)$ 在 $\alpha \in \left[0, \dfrac{\pi}{2}\right]$ 上单调递减且有 $\cos(\alpha) \in [0, 1]$, $\sin(\alpha)$ 在 $\alpha \in [0, 1]$ 上单调递增, 故 $f(\alpha)$ 在 $\alpha \in \left[0, \dfrac{\pi}{2}\right]$ 单调递减.

$\sin(\alpha)$ 在 $\alpha \in \left[0, \dfrac{\pi}{2}\right]$ 上单调递增且有 $\sin(\alpha) \in [0, 1]$, $\cos(\alpha)$ 在 $\alpha \in [0, 1]$ 上单调递减, 故 $g(\alpha)$ 在 $\alpha \in \left[0, \dfrac{\pi}{2}\right]$ 单调递减.

于是有

$$\max_{\alpha \in \left[0, \frac{\pi}{2}\right]} f(\alpha) = f(0) = \sin(1), \quad \min_{\alpha \in \left[0, \frac{\pi}{2}\right]} g(\alpha) = g\left(\frac{\pi}{2}\right) = \cos(1).$$

由 $\sin(\alpha)$ 在 $\alpha \in \left[0, \dfrac{\pi}{2}\right]$ 上单调增, $\cos(\alpha)$ 在 $\alpha \in \left[0, \dfrac{\pi}{2}\right]$ 上单调减, $\sin\left(\dfrac{\pi}{4}\right) = \cos\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$, 而有 $1 > \dfrac{\pi}{4}$, 则有

$$\cos(1) < \sin(1),$$

此证法不可行. (但是对于 $\forall x \in I: f(x) < A$ 的题证明 $\max\limits_{x \in I} f(x) < A$ 依然不失作为一个好的做法存在)

考虑到 $x \sim \sin x (x \to 0)$ (即 x 和 $\sin x$ 是 $x \to 0$ 的等价无穷小, 即 $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$), 构造 $\cos x$ 作为中间变量, 利用放缩法解决.

下证: $\forall \alpha \in \left(0, \dfrac{\pi}{2}\right): f(\alpha) < \cos(\alpha)$.

要证:

$$\forall \alpha \in \left(0, \frac{\pi}{2}\right): f(\alpha) < \cos(\alpha),$$

即证:

$$\forall \alpha \in \left(0, \frac{\pi}{2}\right): \sin(\cos(\alpha)) < \cos(\alpha),$$

即证:

$$\forall x \in (0,1) : \sin(x) < x.$$

下证: $\forall \alpha \in \left(0, \frac{\pi}{2}\right) : \cos(\alpha) < g(\alpha)$.
要证:

$$\forall \alpha \in \left(0, \frac{\pi}{2}\right) : \cos(\alpha) < g(\alpha),$$

即证:

$$\forall \alpha \in \left(0, \frac{\pi}{2}\right) : \cos(\alpha) < \cos(\sin(\alpha)),$$

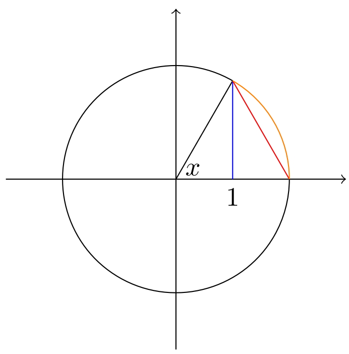
只需证:

$$\forall \alpha \in \left(0, \frac{\pi}{2}\right) : \alpha > \sin(\alpha). \qquad (\cos(x)\text{在}x \in \left(0, \frac{\pi}{2}\right)\text{上单调递减}, \alpha \in \left(0, \frac{\pi}{2}\right), \sin(\alpha)\ (0,1) \subset \left(0, \frac{\pi}{2}\right))$$

因此要证原命题, 只需证

$$\forall x \in \left(0, \frac{\pi}{2}\right) : \sin(x) < x.$$

使用单位圆进行此命题的证明:



显然有

$$\sin(x) < l < x.$$

故得证.
又考虑到统一外函数三角名:

$$g(\alpha) = \cos(\sin(\alpha)) = \sin\left(\frac{\pi}{2} - \sin(\alpha)\right).$$

即证:

$$\forall x \in \left(0, \frac{\pi}{2}\right) : \sin(\cos(\alpha)) < \sin\left(\frac{\pi}{2} - \sin(\alpha)\right),$$

即证:

$$\forall x \in \left(0, \frac{\pi}{2}\right) : \cos(\alpha) < \frac{\pi}{2} - \sin(\alpha),$$

即证:

$$\forall x \in \left(0, \frac{\pi}{2}\right) : \sin(\alpha) + \cos(\alpha) < \frac{\pi}{2}.$$

由**辅助角公式**:

$$a\sin(x) + b\cos(x) = \sqrt{a^2+b^2}\sin\left(x + \arctan\frac{a}{b}\right)(a > 0)$$

有

$$\sin(x) + \cos(x) = \sqrt{2}\sin\left(\frac{\pi}{4} + x\right) \leq \sqrt{2} < \frac{\pi}{2},$$

得证.

Section 3 特别致谢

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