Another Simple Problem

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Problem a, b, c, p, q and r are positive real numbers where $abc = a + b + c$. Show that

$$\frac{2p}{\sqrt{1+a^2}} + \frac{2q}{\sqrt{1+b^2}} + \frac{2r}{\sqrt{1+c^2}} \leq \frac{pq}{r} + \frac{pr}{q} + \frac{qr}{p}. \tag{1}$$

Proof Let $a = \tan A, b = \tan B, c = \tan C$, where $A + B + C = \pi, 0 < A, B, C < \dfrac{\pi}{2}$,

$$\tan C = -\tan(A+B) = -\frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Hence,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Threfore,

$$\frac{2p}{\sqrt{1+a^2}} + \frac{2q}{\sqrt{1+b^2}} + \frac{2r}{\sqrt{1+c^2}} = 2p\cos A + 2q\cos B + 2r\cos C.$$

(1) is equlivant to the following inequality:

$$2p\cos A + 2q\cos B + 2r\cos C \leq \frac{pq}{r} + \frac{pr}{q} + \frac{qr}{p}. \tag{2}$$

According to the Wolstenholme's cyclic inequality

$$\forall \Delta ABC, x, y, z \in \mathbb{R}, x^2 + y^2 + z^2 \geq 2yz\cos A + 2zx\cos B + 2xy\cos C,$$

let $p = 2yz, q = 2zx, r = 2xy$, the following lemma can be drawn:

$$\forall \Delta ABC, p, q, r \in \mathbb{R}_+, p\cos A + q\cos B + r\cos C \leq \frac{1}{2}\left(\frac{pq}{r} + \frac{pr}{q} + \frac{qr}{p}\right). \tag{3}$$

(3) is equlivant to (2), which is equlivant to (1). Therefore, the original inequality is proven.