A Simple Problem

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Problem a, b and c are positive real numbers where $abc = 1$. Show that

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ac}{a^5 + c^5 + ac} \leq 1.$$

Proof As a, b are real numbers,

$$a^5 + b^5 \geq a^3b^2 + a^2b^3,$$

with equality if and only if $a = b$.
Hence,

$$\begin{aligned}\frac{ab}{a^5 + b^5 + ab} &= \frac{a^2b^2c}{a^5 + b^5 + a^2b^2c} \\ &\leq \frac{a^2b^2c}{a^3b^2 + a^2b^3 + a^2b^2c} \\ &= \frac{c}{a + b + c}.\end{aligned}$$

Similar results can be drawn, by replacing a, b with b, c and c, a:

$$\frac{bc}{b^5 + c^5 + bc} \leq \frac{a}{a + b + c}, \frac{ca}{c^5 + a^5 + ca} \leq \frac{b}{a + b + c}.$$

Then

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ac}{a^5 + c^5 + ac} \leq 1,$$

with equality if and only if $a = b = c$.