§11 不等式中的恒等变形'

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1. 对于正实数 $a_1, a_2, \cdots, a_n$, 证明:

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \le \frac{n}{2(a_1 + a_2 + \cdots + a_n)} \sum_{i<j} a_i a_j.$$

有

$$2\sum_{i<j} a_i a_j = \left(\sum a_i\right)^2 - \left(\sum a_i^2\right),$$

故即证

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \le \frac{1}{4} n \sum a_i - \frac{n \sum a_i^2}{4 \sum a_i}.$$

考虑通过调和平均数证明: 由

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \le \sum \frac{a_i + a_j}{4} = \frac{n-1}{4} \sum a_i,$$

只需证

$$\frac{n-1}{4} \sum a_i \le \frac{1}{4} n \sum a_i - \frac{n \sum a_i^2}{4 \sum a_i},$$

即证

$$\sum a_i \ge n \frac{\sum a_i^2}{\sum a_i},$$

并非恒成立.

考虑化简不等式: 有

$$\begin{aligned}\sum \frac{a_i a_j}{a_i + a_j} &= \sum \frac{a_i + a_j}{4} - \frac{(a_i - a_j)^2}{4(a_i + a_j)} \\ &= \frac{n-1}{4} \sum a_i - \sum \frac{(a_i - a_j)^2}{4(a_i + a_j)},\end{aligned}$$

故即证

$$\frac{n-1}{4}\sum a_i-\sum\frac{(a_i-a_j)^2}{4(a_i+a_j)}\leq\frac{1}{4}n\sum a_i-\frac{n\sum a_i^2}{4\sum a_i},$$

即证

$$a_i\geq\frac{n\sum a_i^2}{4\sum a_i}-\sum\frac{(a_i-a_j)^2}{4(a_i+a_j)},$$

即证

$$-\sum\frac{(a_i-a_j)^2}{a_i+a_j}\leq\sum a_i-\frac{n\sum a_i^2}{\sum a_i}=\frac{\left(\sum a_i\right)^2-n\sum a_i^2}{\sum a_i}=-\frac{\sum(a_i-a_j)^2}{\sum a_i},$$

即证

$$\frac{\sum(a_i-a_j)^2}{\sum a_i}\leq\sum\frac{(a_i-a_j)^2}{a_i+a_j},$$

显然成立.

2. 设 n 为给定的正整数, $x_1,x_2,\cdots,x_n$ 为正实数, 证明:

$$\sum_{i=1}^{n}x_i\left[1-\left(\sum_{j=1}^{i}x_j\right)^2\right]\leq\frac{2}{3}.$$

记

$$S_i=\sum_{j=1}^{i}x_j,$$

令 $S_0=0$, 则有 $0<S_1<S_2<\cdots<S_n$,

即证

$$\sum_{i=1}^{n}\left(S_i-S_{i-1}\right)\left(1-S_i\right)^2\leq\frac{2}{3},$$

即证

$$S_n-\sum_{i=1}^{n}\left(S_i-S_{i-1}\right)S_i^2\leq\frac{2}{3},$$

即证

$$\sum_{i=1}^{n}\left(S_i-S_{i-1}\right)S_i^2\geq S_n-\frac{2}{3}.$$

又

$$
S_i^2 \geq \frac{S_i^2+S_{i-1}S_i+S_{i-1}^2}{3},
$$

故

$$
\sum_{i=1}^{n}\left(S_i-S_{i-1}\right)S_i^2 \geq \sum_{i=1}^{n}\frac{S_i^3-S_{i-1}^3}{3}=\frac{1}{3}S_n^3,
$$

只需证

$$
\frac{1}{3}S_n^3 \geq S_n-\frac{2}{3},
$$

成立.

Abel 分步求和公式: 令 $S_k=\sum_{i=1}^{k}a_i$, 则

$$
\sum_{k=1}^{n}a_kb_k=\sum_{k=1}^{n-1}S_k\left(b_k-b_{k+1}\right)+b_nS_n.
$$

不妨令 $S_0=0$, 有

$$
\begin{aligned}
\sum_{k=1}^{n}a_kb_k &= \sum_{k=1}^{n}\left(S_k-S_{k-1}\right)b_k \\
&= \sum_{k=1}^{n-1}S_l\left(b_k-b_{k+1}\right)+S_0b_0+S_nb_n.
\end{aligned}
$$

3. 设 $b_1 \geq b_2 \geq \cdots \geq b_n > 0$, $m \leq \sum_{k=1}^{t}a_k \leq M(t=1,2,\cdots,n)$, 求证:

$$
b_1m \leq \sum_{k=1}^{n}a_kb_k \leq b_1M.
$$

$$
\begin{aligned}
\sum_{k=1}^{n}a_kb_k &= \sum_{k=1}^{n-1}S_k\left(b_k-b_{k+1}\right)+S_nb_n \\
&\leq \sum_{k=1}^{n-1}M\left(b_k-b_{k+1}\right)+M\cdot b_n \\
&= M\cdot b_1,
\end{aligned}
$$

左侧同理.

4. 已知 $a_1, a_2, \cdots, a_n$ 和 $b_1, b_2, \cdots, b_n$ 都是实数. 证明: 使得对任何满足 $x_1 \leq x_2 \leq \cdots \leq x_n$ 的实数, 不等式

$$\sum_{i=1}^{n} a_i x_i \leq \sum_{i=1}^{n} b_i x_i$$

恒成立的充要条件是

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i (k = 1, 2, \cdots, n-1)$$

且

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.$$

记

$$\sum_{i=1}^{k} (a_k - b_k) = S_k,$$

有

$$\begin{aligned}\sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} b_i x + i &= \sum_{i=1}^{n} (a_i - b_i) x_i \\ &= \sum_{i=1}^{n-1} S_i (x_i - x_{i+1}) + S_n x_n.\end{aligned}$$

故即证

$$\forall x_1 \leq x_2 \leq \cdots \leq x_n, \sum_{i=1}^{n} S_i (x_i - x_{i+1}) + S_n x_m \leq 0 \Leftrightarrow S_1, s_2, \cdots, S_{n-1} \geq 0, S_n = 0.$$

$\Leftarrow$ 显然成立.

$\Rightarrow$ 令 $x_1 = \cdots = x_n = 1$, $S_n \leq 0$; 令 $x_1 = x_2 = \cdots = x_n = -1$, $S_n \geq 0$. 故 $S_n = 0$.

令 $x_1 = \cdots = x_k = 0, x_{k=1} = \cdots = x_n = 1$, 得 $S_k(x_k - x_{k+1}) \leq 0, S_k \geq 0$.

5. 给定正整数 $n \geq 2$, 正数数列 $a_1, a_2, \cdots, a_n$ 满足 $a_k \geq a_1 + a_2 + \cdots + a_{k-1} (k = 2, 3, \cdots, n)$, 求 $\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n}$ 的最大值, 并求取得最大值的条件.

记

$$S_k = a_1 + \cdots + a_{k-1}, S_0 = 0,$$

有 $a_k \geq S_{k-1}$.

$$
\begin{aligned}
T=\text{ 原式} &= \sum_{k=1}^{n-1} \frac{S_k - S_{k-1}}{a_{k+1}} \\
&= \sum_{k=1}^{n-2} S_k \left(\frac{1}{a_{k+1}} - \frac{1}{a_{k+2}} \right) + \frac{S_{n-1}}{a_n} \\
&\le \sum_{k=1}^{n-2} a_{k+1} \left(\frac{1}{a_{k+1}} - \frac{1}{a_{k+2}} \right) + \frac{S_{n-1}}{a_n} \\
&= (n-2) - \sum_{k=1}^{n-2} \frac{a_{k+1}}{a_{k+2}} + \frac{S_{n-1}}{a_n} \\
&\le m - 2 - \left(T - \frac{a_1}{a_2} \right) + 1,
\end{aligned}
$$

故

$$2T \le n - 1 + \frac{a_1}{a_2} \le n, T \le \frac{n}{2},$$

等号成立条件:

$$a_n = 2^{n-2} a_1.$$

6. 设 $a_i, b_i > 0 (1 \le i \le n+1), b_{i+1} - b_i \ge \delta > 0$($\delta$ 为常数), 若 $\sum\limits_{i=1}^{n} a_i = 1$, 证明:

$$\sum_{i=1}^{n} \frac{i \sqrt[i]{a_1 a_2 \cdots a_i b_1 b_2 \cdots b_i}}{b_i b_{i+1}} \le \frac{1}{\delta}.$$

$$i \sqrt[i]{a_1 a_2 \cdots a_i b_1 b_2 \cdots b_i} \le a_1 b_1 + \cdots + a_i b_i,$$

记

$$S_i = a_1 b_1 + a_2 b_2 + \cdots + a_i b_i,$$

又

$$\frac{1}{b_i} - \frac{1}{b_{i+1}} = \frac{b_{i+1} - b_i}{b_i b_{i+1}} \ge \frac{\delta}{b_i b_{i+1}} \Rightarrow \frac{1}{b_i b_{i+1}} \le \frac{1}{\delta} \left(\frac{1}{b_i} - \frac{1}{b_{i+1}} \right),$$

有

$$
\begin{aligned}
\text{原式} &\le \sum_{i=1}^{n} \frac{S_i}{b_i b_{i+1}} \\
&\le \sum_{i=1}^{n} \frac{1}{\delta} S_i \left(\frac{1}{b_i} - \frac{1}{b_{i+1}}\right) \\
&= \frac{1}{\delta}\left(\sum_{i=2}^{n} \frac{S_i - S_{i-1}}{b_i} - \frac{S_n}{b_{n+1}} + \frac{S_1}{b_1}\right) \\
&= \frac{1}{\delta}\left(\sum_{i=2}^{n} a_i - \frac{S_n}{b_{n+1}} + a_1\right) \\
&= \frac{1}{\delta}\left(1 - \frac{S_n}{b_{n+1}}\right) \\
&< \frac{1}{\delta}.
\end{aligned}
$$

7. 已知正数 $x_1, x_2, \cdots, x_n$ 和 $y_1, y_2, \cdots, y_n$ 满足 $x_1 > x_2 > \cdots > x_n$, $y_1 > y_2 > \cdots > y_n$, 且 $x_1 > y_1$, $x_1 + x_2 > y_1 + y_2$, $\cdots$, $x_1 + x_2 + \cdots + x_n > y_1 + y_2 + \cdots + y_n$. 求证: 对任意正整数 k, 有 $x_1^k + x_2^k + \cdots + x_n^k > y_1^k + y_2^k + \cdots + y_n^k$.

设 $S_i = x_1 + \cdots + x_i, T_i = y_1 + \cdots + y_i, S_i > T_i$,

$$
\begin{aligned}
\sum_{i=1}^{n}(x_i^k - y_i^k) &= \sum_{i=1}^{n}(x_i - y_i)\left(x_i^{k-1} + x_i^{k-2} y_i + \cdots + y_i^{k-1}\right) \\
&= \sum_{i=1}^{n-1}(S_i - T_i)(c_i - c_{i+1}) + (S_n - T_n) c_n > 0.
\end{aligned}
$$

8. 设 $x_i \ge 0 (i = 1, 2, \cdots, n)$, 且

$$
\sum_{i=1}^{n} x_i^2 + 2 \sum_{1 \le k < j \le n} \sqrt{\frac{k}{j}} x_k x_j = 1,
$$

求 $\sum\limits_{i=1}^{n} x_i$ 的最大值和最小值.

$$
\sqrt{\frac{k}{j}} < 1 \Rightarrow 1 \le \sum_{i=1}^{n} x_i^2 + 2 \sum_{1 \le k < j \le n} x_k x_j = \left(\sum_{i=1}^{n} x_i\right)^2 \Rightarrow \sum_{i=1}^{n} x_i \ge 1, x_1 = 1, x_2 = \cdots = x_n = 0.
$$

记 $y_i = \frac{x_i}{\sqrt{i}}$, 则 $x_i = \sqrt{i} y_i (y_i \ge 0)$,

$$
\sum_{i=1}^{n} i y_i^2 + 2 \sum_{1 \le k < j \le n} k y_k \cdot y_j = 1,
$$

$$
y_n^2 + (y_n + y_{n-1})^2 + \cdots + (y_n + y_{n-1} + \cdots + y_1)^2 = 1.
$$

记 $S_i = y_n + \cdots + y_i$, 则 $S_n^2 + \cdots + S_1^2 = 1$, 显然有 $S_1 \ge S_2 \ge \cdots \ge S_n, S_{n+1} = 0$,

$$
\begin{aligned}
\sum_{i=1}^{n} x_i &= \sum_{i=1}^{n} \sqrt{i}\cdot(S_i - S_{i+1}) \\
&= \sum_{i=2}^{n} S_i\left(\sqrt{i}-\sqrt{i-1}\right) + S_1 - \sqrt{n}S_{n+1} \\
&= \sum_{i=1}^{n} S_i\left(\sqrt{i}-\sqrt{i-1}\right),
\end{aligned}
$$

$$
\begin{aligned}
\left(\sum_{i=1}^{n} x_i\right)^2 &= \left[\sum_{i=1}^{n} S_i\left(\sqrt{i}-\sqrt{i-1}\right)\right]^2 \\
&\le \left(\sum_{i=1}^{n} S_i^2\right)\cdot\sum_{i=1}^{n}\left(\sqrt{i}-\sqrt{i-1}\right)^2 && \text{柯西不等式} \\
&= \sum_{i=1}^{n}\left(\sqrt{i}-\sqrt{i-1}\right)^2,
\end{aligned}
$$

等号成立条件

$$
\frac{S_1}{\sqrt{1}-\sqrt{0}} = \frac{S_2}{\sqrt{2}-\sqrt{1}} = \cdots = \frac{S_n}{\sqrt{n}-\sqrt{n-1}}.
$$

9. 设 $n \in \mathbb{N}^*, S \subseteq \{1, 2, \cdots, n\}, S \neq \emptyset$. 求证:

$$
\left(\sum_{i\in S} a_i\right)^2 \le \sum_{1\le i\le j\le n} (a_i + a_{i+1} + \cdots + a_j)^2. \tag{$*$}
$$

记 $S_k = a_1 + a_2 + \cdots + a_k (k \in 1, 2, \cdots, n), S_0 = 0,$

$$
\begin{aligned}
\text{右} &= \sum_{1\le i\le j\le n} \left(S_j - S_{i-1}\right)^2 \\
&= \sum_{0\le i<j\le n} \left(S_j - S_i\right)^2 \\
&= (n+1)\sum_{i=0}^{n} S_i^2 - \left(\sum_{i=0}^{n} S_i\right)^2,
\end{aligned}
$$

$$
\text{左} = \left[\sum_{i\in S} (S_i - S_{i-1})\right]^2,
$$

$$
\sum_{i\in S}(S_i - S_{i-1}) = \sum_{0\le j\le n} \lambda_i S_i (\lambda_i \in \{-1, 0, 1\}),
$$

$$
\left(\sum_{0\le j\le n} \lambda_i S_i\right)^2 \le \left(\sum_{0\le i\le n} \lambda_i\right)^2 \left(\sum_{0\le i\le n} S_i\right)^2 \le (n+1)\sum_{i=0}^{n} S_i^2.
$$

不妨设 $\sum\limits_{i=0}^{n} S_i = 0$, 若 $\sum\limits_{i=0}^{n} S_i \neq 0$, 令 $S_k' = S_k - \dfrac{\sum_{i=0}^{n} S_i}{n+1}$ 替换, 不等式等价.