# §7 排序不等式

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**排序不等式 (Rearrangement Inequality)**: 设 $a_1 \le a_2 \le \cdots \le a_n, b_1 \le b_2 \le \cdots \le b_n, j_1, j_2, \cdots, j_n$ 是 $1, 2, \cdots, n$ 得任意排列, 则 $a_1b_n + a_2b_{n-1} + \cdots + a_{n-1}b_2 + a_nb_1 \le a_1b_{j_1} + a_2b_{j_2} + \cdots + a_nb_{j_n} \le a_1b_1 + a_2b_2 + \cdots + a_nb_n$, 等号成立当且仅当 $a_1 = a_2 = \cdots = a_n$ 或 $b_1 = b_2 = \cdots = b_n$.

**切比雪夫 (总和) 不等式 (Chebyshev's Sum Inequality)**: 设 $a_1 \le a_2 \le \cdots \le a_n, b_1 \le b_2 \le \cdots \le b_n$, 则 $n(a_1b_n + a_2b_{n-1} + \cdots + a_{n-1}b_2 + a_nb_1) \le (a_1 + a_2 + \cdots + a_{n-1} + a_n)(b_1 + b_2 + \cdots + b_{n-1} + b_n) \le n(a_1b_1 + a_2b_2 + \cdots + a_nb_n)$, 等号成立当且仅当 $a_1 = a_2 = \cdots = a_n$ 或 $b_1 = b_2 = \cdots = b_n$.

1. 证明排序不等式.

记 $S(k_1, k_2, \cdots, k_n) = \sum_{i=1}^n a_ib_{k_i}$,

若 $k_n \ne n$, 设 $k_p = n$,

$$\begin{aligned}
&S(k_1, k_2, \cdots, k_n) - S(k_1, k_2, \cdots, k_n, k_{p+1}, \cdots, k_{n-1}, k_p)\\
=&a_pb_{k_p} + a_nb_{k_n} - a_pb_{k_n} - a_nb_{k_n}\\
=&(a_p - a_n)(b_{k_p} - b_{k_n})\\
\le&0.
\end{aligned}$$

当 $S(k_1, k_2, \cdots, k_n)$ 最大时, $k_n = n$. 同理 $k_{n-1} = n - 1, \cdots, k_2 = 2, k_1 = 1$.

2. 证明切比雪夫不等式.

$$\begin{aligned}
(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) =&(a_1b_1 + a_2b_2 + \cdots + a_nb_n)\\
&+ (a_1b_2 + a_2b_3 + \cdots + a_nb_1)\\
&+ \cdots\\
&+ (a_1b_n + a_2b_1 + \cdots + a_nb_{n-1}).
\end{aligned}$$

3. 设 $a,b,c,\lambda\in\mathbb{R}^+$ 且 $a^{n-1}+b^{n-1}+c^{n-1}=1(n\geq 2)$. 求证:

$$\frac{a^n}{b+\lambda c}+\frac{b^n}{c+\lambda a}+\frac{c^n}{a+\lambda b}\geq\frac{1}{1+\lambda}.$$

$$\left(\frac{a^n}{b+\lambda c}+\frac{b^n}{c+\lambda a}+\frac{c^n}{a+\lambda b}\right)\left[a^{n-2}(b+\lambda c)+b^{n-2}(c+\lambda a)+c^{n-2}(a+\lambda b)\right]\geq\left(a^{n-1}+b^{n-1}+c^{n-1}\right)^2=1.$$

只需证

$$a^{n-2}(b+\lambda c)+b^{n-2}(c+\lambda a)+c^{n-2}(a+\lambda b)\leq 1+\lambda.$$

$$\begin{aligned}
a^{n-2}b+b^{n-2}c+c^{n-2}a&\leq a^{n-2}a+b^{n-2}b+c^{n-2}c+\lambda a^{n-2}c+b^{n-2}a+c^{n-2}b\\
&\leq(1+\lambda)\left(a^{n-2}a+b^{n-2}b+c^{n-2}c\right)\\
&=(1+\lambda)\left(a^{n-1}b^{n-1}c^{n-1}\right)\\
&=1+\lambda.
\end{aligned}$$

4. 设 $x_1,x_2,\cdots,x_n$ 与 $a_1,a_2,\cdots,a_n(n>2)$ 满足条件: (1) $x_1+x_2+\cdots+x_n=0$, (2) $|x_1|+|x_2|+\cdots+|x_n|=1$, (3) $a_1\geq a_2\geq\cdots\geq a_n$ 的两组任意实数. 为了使不等式 $|a_1x_1+a_2x_2+\cdots+a_nx_n|\leq A(a_1-a_n)$ 成立, 求数 $A$ 的最小值.

显然

$$A_{\min}=\left(\frac{a_1x_1+a_2x_2+\cdots+a_nx_n}{a_1-a_n}\right)_{\max}.$$

不妨设 $a_1x_1+a_2x_2+\cdots+a_nx_n\geq 0$. (若 $<0$, 将 $x_i$ 与 $-x_i$ 替换), 设 $x_{k_1}\geq x_{k_2}\geq\cdots\geq x_{k_t}\geq 0>x_{k_{t+1}}\geq\cdots\geq x_{k_n}$, 则有

$$\sum_{i=1}^{n}x_{k_i}=\frac{1}{2},\sum_{i=k+1}^{n}x_{k_i}=-\frac{1}{2},$$

$$\begin{aligned}
a_1x_1+a_2x_2+\cdots+a_nx_n&\leq a_1x_{k_1}+a_2x_{k_2}+\cdots+a_nx_{k_n}\\
&=a_1(x_{k_1}+x_{k_2}+\cdots+x_{k_t})+a_n(x_{k_{t+1}}+\cdots+x_{k_n})\\
&=\frac{1}{2}(a_1-a_n)\\
\Rightarrow A_{\min}&=\frac{1}{2},
\end{aligned}$$

等号成立条件: $x_1=\frac{1}{2},x_n=-\frac{1}{2},x_2=\cdots=x_{n-1}=0$.

5. 设 $x, y, z \in \mathbb{R}^+$, 求证:

$$\frac{x}{\sqrt{y+z}} + \frac{y}{\sqrt{z+x}} + \frac{z}{\sqrt{x+y}} \geq \sqrt{\frac{3}{2}(x+y+z)}$$

.

不妨设 $x \geq y \geq z$, 则有

$$\frac{1}{\sqrt{y+z}} \geq \frac{1}{\sqrt{z+x}} \geq \frac{1}{\sqrt{x+y}},$$

有

$$左 \geq \left[\frac{1}{3}(x+y+z)\left(\frac{1}{\sqrt{y+z}} + \frac{1}{\sqrt{z+x}} + \frac{1}{\sqrt{x+y}}\right)\right]^2.$$

只需证

$$\left[\frac{1}{3}(x+y+z)\left(\frac{1}{\sqrt{y+z}} + \frac{1}{\sqrt{z+x}} + \frac{1}{\sqrt{x+y}}\right)\right]^2 \geq \frac{3}{2}.$$

设 $y+z=a, z+x=b, x+y=c$,

$$\begin{aligned} 左 &= \frac{1}{9} \cdot \frac{a+b+c}{2}\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right)^2 \\ &\geq \frac{1}{2}(1+1+1)^3 \\ &= \frac{27}{2}. \end{aligned}$$

卡尔松不等式

6. 设 $a \leq b \leq c \leq d \leq e$, 且 $a+b+c+d+e=1$. 求证:

$$ad+dc+cb+be+ea \leq \frac{1}{5}.$$

$$2(ad+dc+cb+be+ea) = a(d+e)+b(c+e)+c(b+d)+d(a+c)+e(a+b).$$

$$a \leq b \leq c \leq d \leq e \Rightarrow d+e \geq c+e \geq b+d \geq a+c \geq a+b,$$

由切比雪夫不等式,

$$a(d+e)+b(c+e)+c(b+d)+d(a+c)+e(a+b) \leq \frac{(a+b+c+d+e)[d+e+a+e+b+d+a+c+a+b]}{5} = \frac{2}{5}.$$

7. $x_1, x_2, \cdots x_n > 0$, $\sum\limits_{i=1}^{n} x_i = 1 (n \geq 2)$. 求证:

$$\sum_{i=1}^{n} \frac{1}{1-x_i} \cdot \sum_{1 \leq i < j \leq n} x_i x_j \leq \frac{n}{2}.$$

$$\begin{aligned} 左 &= \sum_{i=1}^{n} \frac{1-x_i}{\cdot} \frac{1}{2} \sum_{i=1}^{n} x_i(x_1 + x_2 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n) \\ &= \frac{1}{2} \sum_{i=1}^{n} \frac{1}{1-x_i} \cdot \sum_{i=1}^{n} x_i(1-x_i). \end{aligned}$$

不妨设 $x_1 \geq \cdots \geq x_n$, 则有 $\dfrac{1}{1-x_1} \geq \cdots \geq \dfrac{1}{1-x_n}$.

由切比雪夫不等式,

$$\sum_{i=1}^{n} \frac{1}{1-x_i} \cdot \sum_{i=1}^{n} x_i(1-x_i) \leq n \sum_{i=1}^{n} \frac{1}{1-x_i} \cdot x_i(1-x_i) = n.$$

8. 已知实数 $x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 < \cdots < y_n$ 满足 $\sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} y_i = 0$. 求证: (1) $\sum\limits_{i=1}^{n} x_i y_i \geq \dfrac{n}{n-1} \max\limits_{1 \leq i \leq n} x_i y_i$, (2) $(n-1)^2 \left(\sum\limits_{i=1}^{n} x_i y_i\right)^2 \geq \left(\sum\limits_{i=1}^{n} x_i^2\right)^2 \left(\sum\limits_{i=1}^{n} y_i^2\right)^2$.

**(1)**

$$\begin{aligned} x_1 y_1 &= (x_2 + \cdots + x_n)(y_2 + \cdots + y_n) \\ &\leq (n-1)(x_2 y_2 + x_3 y_3 + \cdots + x_n y_n) \\ &= (n-1) \left(\sum_{i=1}^{n} x_i y_i - x_1 y_1\right). \end{aligned}$$

同理, $n(x_n y_n) \leq (n-1) \sum_{i=1}^{n} x_i y_i$.

**(2)**

$$(n-1)^2 \left(\sum_{i=1}^{n} x_i y_i\right)^2 \geq (n x_1 y_1)(n x_n y_n).$$

只需证

$$-n x_1 x_n \geq \sum_{i=1}^{n} x_i^2.$$

($y$ 同理.)

$$(x_i - x_1)(x_i - x_n) \leq 0 \Rightarrow x_i^2 - (x_1 + x_n) x_i + x_1 x_n \leq 0.$$

求和, 有:

$$\sum_{i=1}^{n} x_i^2 - (x_1 + x_n)\sum_{i=1}^{n} x_i + n x_1 x_n \le 0,$$

即有:

$$\sum_{i=1}^{n} x_i^2 \le -n x_1 x_n.$$

9. 设 $x_1, x_2, \cdots, x_n > 0$, 求证:

$$\frac{1}{\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n}} - \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \ge \frac{1}{n}.$$

即证

$$\frac{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} - \frac{1}{1+x_1} - \frac{1}{1+x_2} - \cdots - \frac{1}{1+x_n}}{\left(\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n}\right)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)} \ge \frac{1}{n},$$

不妨设 $x_1 \ge x_2 \ge \cdots \ge x_n$, 则有 $\dfrac{1}{1+x_1} \le \dfrac{1}{1+x_2} \le \cdots \le \dfrac{1}{1+x_n}$.

由切比雪夫不等式,

$$\frac{\sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n} \frac{1}{1+x_i}}{\sum_{i=1}^{n} \frac{1}{x_i} \cdot \sum_{i=1}^{n} \frac{1}{1+x_i}} \le n \sum_{i=1}^{n} \frac{1}{x_i} \frac{1}{x_i + 1} = n \left( \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n} \frac{1}{1+x_i} \right),$$

即

$$\frac{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} - \frac{1}{1+x_1} - \frac{1}{1+x_2} - \cdots - \frac{1}{1+x_n}}{\left(\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n}\right)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)} \ge \frac{1}{n}.$$

10. $n \ge 2, 0 \le x_1 \le x_2 \le \cdots \le x_n, x_1 \ge \dfrac{x_2}{2} \ge \cdots \ge \dfrac{x_n}{n}$, 证明:

$$\frac{\sum\limits_{i=1}^{n} x_i}{n \sqrt[n]{\prod\limits_{i=1}^{n} x_i}} \le \frac{n+1}{2\sqrt[n]{n!}}.$$

$$\frac{1}{x_1} + \frac{2}{x_2} + \cdots + \frac{n}{x_n} \ge n \sqrt[n]{\frac{n!}{\prod\limits_{i=1}^{n} x_i}},$$

$$原不等式 \Leftrightarrow n\sqrt[n]{\frac{n!}{\prod\limits_{i=1}^{n}x_i}}\leq\frac{(n+1)n^2}{2\sum\limits_{i=1}^{n}x_i},$$

只需证

$$\sum_{i=1}^{n}\frac{i}{x_i}\leq\frac{(n+1)n^2}{2\sum\limits_{i=1}^{n}x_i}.$$

由切比雪夫不等式,

$$\sum_{i=1}^{n}\frac{i}{x_i}\cdot\sum_{i=1}^{n}x_i\leq n\sum_{i=1}^{n}i=\frac{n^2(n+1)}{2}.$$

11. 设 $x_1,x_2,\cdots,x_n>0,k\geq 1$, 求证:

$$\sum_{i=1}^{n}\frac{1}{1+x_i}\sum_{i=1}^{n}x_i\leq\sum_{i=1}^{n}\frac{x_i^{k+1}}{1+x_i}\sum_{i=1}^{n}\frac{1}{x_i^k}.$$

不妨设 $x_1\geq\cdots\geq x_n$, 有

$$\frac{1}{x_1^k}\leq\frac{1}{x_2^k}\leq\cdots\leq\frac{1}{x_n^k},$$

则有

$$\frac{x_1^{k+1}}{1+x_1}\geq\frac{x_2^{k+1}}{1+x_2}\geq\cdots\geq\frac{x_n^{k=1}}{1+x_n}.$$

由切比雪夫不等式, 有

$$\sum x_i\frac{x_i^k}{1+x_i}\geq\frac{\left(\sum x_i\right)\left(\sum\frac{x_i^k}{1+x_i}\right)}{n},$$

$$右\geq\left(\sum x_i\right)\frac{1}{n}\left(\sum\frac{x_i^k}{1+x_i}\right)\left(\sum\frac{1}{x_i^k}\right).$$

由切比雪夫不等式, 有

$$\frac{1}{n}\left(\sum\frac{x_i^k}{1+x_i}\right)\left(\sum\frac{1}{x_i^k}\right)\geq\sum\frac{1}{1+x_i}.$$