§4 均值不等式'

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1. 设正实数 a,b,c 满足 $abc=1$，求证：$\dfrac{a^{2021}}{a+b}+\dfrac{b^{2021}}{b+c}+\dfrac{c^{2021}}{c+a}\geq\dfrac{3}{2}$.

   法一：$\dfrac{a^{2021}}{a+b}+\dfrac{a+b}{4}+\underbrace{\dfrac{1}{2}+\cdots+\dfrac{1}{2}}_{2019个}\geq 2021\sqrt[2021]{\dfrac{a^{2021}}{2^{2021}}}=\dfrac{2021}{2}a$.

   $\therefore\dfrac{a^{2021}}{a+b}\geq\dfrac{2021}{2}a-\dfrac{a+b}{4}-\dfrac{2019}{2}$.

   同理 $\therefore\dfrac{b^{2021}}{b+c}\geq\dfrac{2021}{2}b-\dfrac{b+c}{4}-\dfrac{2019}{2},\dfrac{c^{2021}}{c+a}\geq\dfrac{2021}{2}c-\dfrac{c+a}{4}-\dfrac{2019}{2}$.

   于是有 左 $\geq\dfrac{2021}{2}(a+b+c)-\dfrac{a+b+c}{2}-\dfrac{2019}{2}\times 3=1010(a+b+c)-\dfrac{2019}{2}\times 3\geq\dfrac{3}{2}$.

   法二：由 Cauchy 不等式，有：$\left(\dfrac{a^{2021}}{a+b}+\dfrac{b^{2021}}{b+c}+\dfrac{c^{2021}}{c+a}\right)[(a+b)+(b+c)+(c+a)]\geq\left(a^{\frac{2021}{2}}+b^{\frac{2021}{2}}+c^{\frac{2021}{2}}\right)^2$.

   $\therefore$ 原式 $\geq\frac{\left(a^{\frac{2021}{2}}+b^{\frac{2021}{2}}+c^{\frac{2021}{2}}\right)^2}{2(a+b+c)}$.

   由幂平均不等式，有 $\sqrt[\frac{2021}{2}]{\dfrac{a^{\frac{2021}{2}}+b^{\frac{2021}{2}}+c^{\frac{2021}{2}}}{3}}\geq\dfrac{a+b+c}{3}$.

   于是原式 $\geq\dfrac{\left[3\left(\frac{a+b+c}{3}\right)^{\frac{2021}{2}}\right]^2}{2(a+b+c)}\geq\dfrac{3}{2}$.

2. 对于满足 $\sum\limits_{i=1}^{2015}x_i=2014$ 的非负实数 $x_1,x_2,\cdots,x_2015$，求 $\left(\sum\limits_{i=1}^{2015}x_i^i\right)_{\min}$.

   思路：$x_i+?+\cdots+?\geq i\sqrt[i]{x^i\cdot ?^{i-1}}=x_i\Rightarrow ?^{i-1}=\dfrac{1}{i^i},?=i^{-\frac{i}{i-1}}$.

   由均值不等式，有：$x_i^i+\underbrace{i^{-\frac{i}{i-1}}+\cdots+i^{-\frac{i}{i-1}}}_{i-1个}\geq x_i(i\geq 2)$,

   即 $x_i^i\geq x_i-(i-1)i^{-\frac{i}{i-1}}(i=2,3,\cdots,2015)$,

   等号成立当且仅当 $x_i=i^{-\frac{i}{i-1}}\in(0,1)$,

   $\therefore x_2,x_2\cdots x_2015\in(0,1)\Rightarrow x_1$ 非负.

   $\therefore\sum\limits_{i=1}^{2015}x_i^i\geq 2014-\sum\limits_{k=2}^{2015}(k-1)k^{-\frac{k}{k-1}}$.

3. 设 $a_i \in \mathbf{R}^+(i=1,2,\cdots,n)$ 且 $a_1+a_2+\cdots+a_n=1$，求证：$\dfrac{a_1^4}{a_1^3+a_1^2a_2+a_1a_2^2+a_2^3}+\dfrac{a_2^4}{a_2^3+a_2^2a_3+a_2a_3^2+a_3^3}+\cdots+\dfrac{a_n^4}{a_n^3+a_n^2a_1+a_na_1^2+a_1^3}\geq\dfrac{1}{4}$.

记 $M=\displaystyle\sum_{k=1}^{n}\frac{a_k^4}{a_k^3+a_k^2a_{k+1}+a_ka_{k+1}^2+a_{k+1}^3}, N=\sum_{k=1}^{n}\frac{a_{k+1}^4}{a_k^3+a_k^2a_{k+1}+a_ka_{k+1}^2+a_{k+1}^3}$,

则 $M-N=\displaystyle\sum_{k=1}^{n}\frac{a_k^4-a_{k+1}^4}{a_k^3+a_k^2a_{k+1}+a_ka_{k+1}^2+a_{k+1}^3}=\sum_{k=1}^{n}(a_k-a_{k+1})=0$,

即 $M=N$.

于是有

$$
\begin{aligned}
M &= \frac{1}{2}(M+N)\\
&= \frac{1}{2}\sum_{k=1}^{n}\frac{a_k^4+a_{k+1}^4}{a_k^3+a_k^2a_{k+1}+a_ka_{k+1}^2+a_{k+1}^3}\\
&= \frac{1}{2}\sum_{k=1}^{n}\frac{a_k^4+a_{k+1}^4}{(a_k+a_{k+1})(a_k^2+a_{k+1}^2)}\\
&\geq \frac{1}{2}\sum_{k=1}^{n}\frac{\frac{(a_k^2+a_{k+1}^2)^2}{2}}{(a_k+a_{k+1})(a_k^2+a_{k+1}^2)} && \text{(Cauchy–Schwarz inequality)}\\
&= \frac{1}{4}\sum\frac{a_k^2+a_{k+1}^2}{a_k+a_{k+1}}\\
&\geq \frac{1}{4}\sum\frac{a_k+a_{k+1}+1}{2} && \text{(Cauchy–Schwarz inequality)}\\
&= \tfrac{1}{4}\sum a_k\\
&= 4.
\end{aligned}
$$

4. 已知正实数 x,y,z 满足 $xyz+xy+yz+zx+x+y+z=7$. 求证：$xyz(x+y+z)\leq 3$.

$(xy+yz+zx)^2\geq 3xyz(x+y+z)$, 等号成立?

记 $M=xyz(x+y+z)$, 有：

$$
\begin{aligned}
7 &= xyz+(xy+yz+zx)+(x+y+z)\\
&\geq \sqrt{3M}+xyz+\frac{1}{3}(x+y+z)+\frac{2}{3}(x+y+z)\\
&\geq \sqrt{3M}+2\sqrt{\tfrac{1}{3}M}+\tfrac{2}{3}(x+y+z),
\end{aligned}
$$

而

$$
\begin{aligned}
(x+y+z)^4 &= (x+y+z)^3(x+y+z)\\
&\geq 27xyz(x+y+z) && rcl\\
&= 27M.
\end{aligned}
$$

$7\geq\sqrt{3M}+2\sqrt{\frac{1}{3}M}+\frac{2}{3}\sqrt[4]{27M}=f(M)$,

当 $M=3$ 时取等.

$f(M)$ 单调递增，$\therefore M\leq 3$.

5. 设 a,b,c 为正实数，求证：$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)\geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$.

法一：左 $=2+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$.

由均值不等式，有 $\frac{c}{b}+\frac{a}{c}+\frac{a}{c}\geq 3\frac{a}{\sqrt[3]{abc}}, \frac{a}{c}+\frac{b}{a}+\frac{b}{a}\geq 3\frac{b}{\sqrt[3]{abc}}, \frac{b}{a}+\frac{c}{b}+\frac{c}{b}\geq 3\frac{c}{\sqrt[3]{abc}}$.

相加，有 $3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq 3\frac{a+b+c}{\sqrt[3]{abc}}$.

$\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\geq \frac{a+b+c}{\sqrt[3]{abc}}$.

同理，$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq \frac{a+b+c}{\sqrt[3]{abc}}$.

于是得证.

法二：

$$\begin{aligned}
\text{左} &= \frac{(a+b)(b+c)(c+a)}{abc} \\
&= \frac{(a+b+c)(ab+bc+ca)-abc}{abc} \\
&\geq \frac{(a+b+c)3\sqrt[3]{a^2b^2c^2}}{abc}-1 \\
&= 3\frac{a+b+c}{\sqrt[3]{abc}}-1 \\
&= 2\frac{a+b+c}{\sqrt[3]{abc}}+\frac{a+b+c}{\sqrt[3]{abc}}-1 \\
&\geq 2.
\end{aligned}$$

6. $x_1,x_2,\cdots,x_n$ 为非负实数，求证：$\left(\sum_{k=1}^{n}\frac{x_k}{k}\right)\left(\sum_{k=1}^{n}kx_k\right)\leq\frac{(n+1)^2}{4n}\left(\sum_{k=1}^{n}x_k\right)^2$.

$n=2, \left(x_1+\frac{x_2}{2}\right)(x_1+2x_2)\leq\frac{9}{8}(x_1+x_2)^2\Leftrightarrow\frac{1}{8}x_1^2+\frac{1}{8}x_2^2-\frac{1}{4}x_1x_2\geq 0$.

$n=3, \left(x_1+\frac{x_2}{2}+\frac{x_3}{3}\right)(x_1+2x_2+3x_3)\leq\frac{4}{3}(x_1+x_2+x_3)^2\Leftrightarrow\frac{1}{3}x_1^2+\frac{1}{3}x_2^2+\frac{1}{3}x_3^2+\frac{1}{6}x_1x_2+\frac{!}{2}x_2x_3-\frac{2}{3}x_3x_1\geq 0$.

猜想：等号成立条件 $x_1=x_n, x_2=\cdots=x_{n-1}=0$.

$$
\begin{aligned}
\left(\sum \frac{x_k}{k}\right)\left(\sum kx_k\right) &= \frac{1}{n}\left(n\sum \frac{x_k}{k}\right)\left(\sum kx_k\right) \\
&\le \frac{1}{n}\left(\frac{n\sum \frac{x_k}{k}+\sum kx_k}{2}\right)^2 \\
&= \frac{1}{n}\left[\frac{\sum\left(k+\frac{n}{k}\right)x_k}{2}\right]^2 \\
&\le \frac{1}{n}\left[\frac{\sum\left(1+\frac{n}{1}\right)x_k}{2}\right]^2 \qquad \forall k\in[1,n]: k+\frac{n}{k}\le n+1 \\
&= \frac{(n+1)^2}{4n}\left(\sum x_k\right)^2.
\end{aligned}
$$

7. $\left(\sum_{k=1}^{5} x_k\right)\left(\sum_{k=1}^{5}\frac{1}{x_k}\right)<26$. 求证：$x_1,x_2,x_3,x_4,x_5$ 中的任意三个数均能构成三角形的三边长.

反证法：假设结论不成立.

不妨设 $x_1\ge x_2+x_3$, 目标：$\left(\sum_{k=1}^{5} x_k\right)\left(\sum_{k=1}^{5}\frac{1}{x_k}\right)\ge 26$.

$$
\begin{aligned}
&\left(\sum_{k=1}^{5} x_k\right)\left(\sum_{k=1}^{5}\frac{1}{x_k}\right) \\
= \; & (x_1+x_2+x_3)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)+(x_4+x_5)\left(\frac{1}{x_4}+\frac{1}{x_5}\right) \\
& +\left(\frac{1}{x_4}+\frac{1}{x_5}\right)(x_1+x_2+x_3)+(x_4+x_5)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right) \\
\ge \; & (x_1+x_2+x_3)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)+(x_4+x_5)\left(\frac{1}{x_4}+\frac{1}{x_5}\right) \\
& +2\sqrt{(x_4+x_5)\left(\frac{1}{x_4}+\frac{1}{x_5}\right)(x_1+x_2+x_3)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)}
\end{aligned}
$$

令 $f(x_1,x_2,x_3)=(x_1+x_2+x_3)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)=1+(x_2+x_3)\left(\frac{1}{x_2}+\frac{1}{x_3}\right)+x_1\left(\frac{1}{x_2}+\frac{1}{x_3}\right)+\frac{1}{x_1}(x_2+x_3)$,
有 f 在 $x_1\in(0,\sqrt{x_2x_3}]$ 关于 x_1 单调递减，在 $x_1\in[\sqrt{x_2x_3},+\infty)$ 关于 x_1 单调递增.

又有 $x_1\ge x_2+x_3$，于是 $f(x_1,x_2,x_3)\ge f(x_2+x_3,x_2,x_3)\ge 1+4+4+1=10$.

于是有

$$
\begin{aligned}
&\left(\sum_{k=1}^{5} x_k\right)\left(\sum_{k=1}^{5}\frac{1}{x_k}\right) \\
\ge \; & (x_1+x_2+x_3)\left(\tfrac{1}{x_1}+\tfrac{1}{x_2}+\tfrac{1}{x_3}\right)+(x_4+x_5)\left(\tfrac{1}{x_4}+\tfrac{1}{x_5}\right) \\
& +2\sqrt{(x_4+x_5)\left(\frac{1}{x_4}+\frac{1}{x_5}\right)(x_1+x_2+x_3)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)} \\
= \; & f+4\sqrt{f}+4 \\
\ge \; & 26.
\end{aligned}
$$

即假设不成立，原命题成立.

8. 证明：$\dfrac{a+\sqrt{ab}+\sqrt[3]{abc}+\sqrt[4]{abcd}}{4} \le \sqrt[4]{a\cdot\dfrac{a+b}{2}\cdot\dfrac{a+b+c}{3}\cdot\dfrac{a+b+c+d}{4}}$，其中 a,b,c,d 均为正数.

记

$$M=\sqrt[4]{a\cdot\frac{a+b}{2}\cdot\frac{a+b+c}{3}\cdot\frac{a+b+c+d}{4}},$$

于是只需证明

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}+\sqrt[4]{abcd}}{M}\le 4.$$

有

$$\frac{a}{M}=\sqrt[4]{\frac{a}{a}\cdot\frac{a}{\frac{a+b}{2}}\cdot\frac{a}{\frac{a+b+c}{3}}\cdot\frac{a}{\frac{a+b+c+d}{4}}}\le\frac{1}{4}\left(\frac{a}{a}+\frac{2a}{a+b}+\frac{3a}{a+b+c}+\frac{4a}{a+b+c+d}\right),$$

$$\frac{\sqrt{ab}}{M}=\sqrt[4]{\frac{a}{a}\cdot\frac{2a}{\frac{a+b}{2}}\cdot\frac{3b}{\frac{a+b+c}{3}}\cdot\frac{4b}{\frac{a+b+c+d}{4}}}\le\frac{1}{4}\left(\frac{a}{a}+\frac{2a}{a+b}+\frac{3b}{a+b+c}+\frac{4b}{a+b+c+d}\right),$$

$$\frac{\sqrt[3]{abc}}{M}=\sqrt[4]{\frac{a}{a}\cdot\frac{2b}{\frac{a+b}{2}}\cdot\frac{3\sqrt[3]{abc}}{\frac{a+b+c}{3}}\cdot\frac{4c}{\frac{a+b+c+d}{4}}}\le\frac{1}{4}\left(\frac{a}{a}+\frac{2b}{a+b}+\frac{3\sqrt[3]{abc}}{a+b+c}+\frac{4c}{a+b+c+d}\right),$$

$$\frac{\sqrt[4]{abd}}{M}=\sqrt[4]{\frac{a}{a}\cdot\frac{2b}{\frac{a+b}{2}}\cdot\frac{3c}{\frac{a+b+c}{3}}\cdot\frac{4d}{\frac{a+b+c+d}{4}}}\le\frac{1}{4}\left(\frac{a}{a}+\frac{2b}{a+b}+\frac{3c}{a+b+c}+\frac{4d}{a+b+c+d}\right).$$

相加，有

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}+\sqrt[4]{abcd}}{M}\le\frac{1}{4}\left(4+4+3+\frac{3\sqrt[3]{abc}}{a+b+c}+4\right)\le 4.$$