§6 柯西不等式'

高一（6）班 邵亦成 26 号

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1. $x_1, x_2, \cdots, x_n \geq 0$. 求证:

$$\frac{1}{a+x_1} + \frac{1}{1+x_1+x_2} + \cdots + \frac{1}{1+x_1+x_2+\cdots+x_n} \leq \sqrt{\sum_{i=1}^{n} \frac{1}{x_i}}.$$

$$左^2 = \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)\left[\frac{x_1}{(1+x_1)^2} + \frac{x_2}{(1+x_1+x_2)^2} + \cdots + \frac{x_n}{(1+x_1+x_2+\cdots+x_n)^2}\right]. \tag{1}$$

而

$$\begin{aligned}\frac{x_k}{(1+x_1+\cdots+x_k)^2} &\leq \frac{x_k}{(1+x_1+\cdots+x_{k-1})(1+x_1+\cdots+x_k)} \\ &= \frac{1}{1+x_1+\cdots+x_{k-1}} - \frac{1}{1+x_1+\cdots+x_k},\end{aligned}$$

故

$$\sum_{k=1}^{n} \frac{x_k}{(1+\cdots+x_k)^2} \leq \frac{1}{1} - \frac{1}{1+x_1+\cdots+x_n}.$$

代入 (1) 式, 有:

$$左^2 \leq \sum_{i=1}^{n} \frac{1}{x_i}.$$

Q.E.D.

2. 设 $a, b, c > 0$, 且 $a+b+c=3$. 求证:

$$\sqrt{\frac{b}{a^2+3}} + \sqrt{\frac{c}{b^2+3}} + \sqrt{\frac{a}{c^2+3}} \leq \frac{3}{2}\sqrt[4]{\frac{1}{abc}}.$$

$$\begin{aligned}左^2 &\leq (b+c+a)\left(\frac{1}{a^2+3} + \frac{1}{b^2+3} + \frac{1}{c^2+3}\right) \\ &= 3\left(\frac{1}{a^2+3} + \frac{1}{b^2+3} + \frac{1}{c^2+3}\right). \end{aligned} \tag{1}$$

又

$$\begin{aligned}\frac{1}{a^2+3}&=\frac{1}{a^2+1+1+1}\\&\le\frac{1}{4\sqrt[4]{a^2}}\\&=\frac{1}{4\sqrt{a}},\end{aligned}$$

故

$$\begin{aligned}\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}&\le\frac{1}{4}\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)\qquad x^2+y^2+z^2\ge xy+yz+zx\\&=\frac{1}{4}\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{\sqrt{abc}}\\&\le\frac{1}{4}\frac{a+b+c}{\sqrt{abc}}\\&=\frac{3}{4}\frac{1}{\sqrt{abc}}.\end{aligned}$$

代入 (1) 有

$$左^2\le\frac{9}{4}\sqrt{\frac{1}{abc}}.$$

Q.E.D.

3. 已知正实数 a,b,c,d 满足 $a+b+c+d=4$. 求证:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a}\ge 4+(a-d)^2.$$

奚同学失败的尝试:

由

$$(x_1^2+x_2^2+\cdots+x_n^2)(y_1^2+y_2^2+\cdots+y_n^2)-(x_1y_1+x_2y_2+\cdots+x_ny_n)^2=\sum_{1\le i<j\le n}(x_iy_j-x_jy_i)^2$$

有

$$\textbf{原不等式}\Leftrightarrow\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a}\right)(b+c+d+a)-(a+b+c+d)^2\ge 4(a-d)^2,$$

难以证明.

正确解答:

$$
\begin{aligned}
\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a}\right)-4 &= \left(\frac{a^2}{b}+b-2a\right)+\left(\frac{b^2}{c}+c-2b\right)+\left(\frac{c^2}{d}+d-2c\right)+\left(\frac{d^2}{A}+a-2d\right) \\
&= \left(\frac{a}{\sqrt{b}}-\sqrt{b}\right)^2+\left(\frac{b}{\sqrt{c}}-\sqrt{c}\right)^2+\left(\frac{c}{\sqrt{d}}-\sqrt{d}\right)^2+\left(\frac{d}{\sqrt{a}}-\sqrt{a}\right)^2 \\
&\geq \frac{[(a-b)+(b-c)+(c-d)+(d-a)]^2}{a+b+c+d} \\
&= (a-d)^2.
\end{aligned}
$$

Q.E.D.

4. 正实数 x, y, z 满足 $xyz \geq 1$. 求证:

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2} \geq 0.$$

$$\text{原不等式} \Leftrightarrow \sum \frac{x^2+y^2+z^2}{x^5+y^2+z^2} \leq 3. \qquad 1-\frac{x^5-x^2}{x^5+y^2+z^2}=\frac{x^2+y^2+z^2}{x^5+y^2+z^2}$$

只需证

$$\frac{x^2+y^2+z^2}{\frac{x^5}{xyz}+y^2+z^2} \leq 3. \tag{1}$$

由柯西不等式,

$$\left[\frac{x^5}{xyz}+y^2+z^2\right][yz+y^2+z^2] \geq (x^2+y^2+z^2)^2.$$

于是有

$$\frac{x^2+y^2+z^2}{\frac{x^5}{xyz}+y^2+z^2} \leq \frac{yz+y^2+z^2}{x^2+y^2+z^2},$$

故

$$
\begin{aligned}
\text{左} &\leq \frac{yz+y^2+z^2+zx+x^2+z^2+xy+x^2+y^2}{x^2+y^2+z^2} \\
&\leq 3,
\end{aligned}
$$

即 (1) 成立, **Q.E.D.**

5. 设 $x_i > 0, x_i y_i - z_i^2 > 0 (i = 1, 2, \cdots, n)$, 证明:

$$\frac{n^3}{\sum_{i=1}^n x_i \sum_{i=1}^n y_i - \left(\sum_{i=1}^n z_i^2\right)^2} \leq \sum_{i=1}^n \frac{1}{x_i y_i - z_i^2}.$$

由卡尔松不等式, 有

$$\left(\sum \frac{1}{x_iy_i - z_i^2}\right)\left[\sum\left(\sqrt{x_iy_i} + z_i\right)\right]\left[\sum\left(\sqrt{x_iy_i} - z_i\right)\right] \le \sum x_i - \sum y_i - \left(\sum z_i\right)^2.$$

又

$$\left(\sum \frac{1}{x_iy_i - z_i^2}\right)\left[\sum\left(\sqrt{x_iy_i} + z_i\right)\right]\left[\sum\left(\sqrt{x_iy_i} - z_i\right)\right] \ge (1+1+\cdots+1)^3 = n^3,$$

即

$$\frac{n^3}{\sum_{i=1}^n x_i \sum_{i=1}^n y_i - \left(\sum_{i=1}^n z_i^2\right)^2} \le \sum_{i=1}^n \frac{1}{x_iy_i - z_i^2}.$$

Q.E.D

6. 设 x, y, z 满足 $xy + yz + zx = x + y + z$, 求证:

$$\frac{1}{x^2+y+1} + \frac{1}{y^2+z+1} + \frac{1}{z^2+x+1} \le 1.$$

$$(x^2+y+1)(1+y+z^2) \ge (x+y+z)^2,$$

故

$$\frac{1}{x^2+y+1} \le \frac{1+y+z^2}{(x+y+z)^2},$$

$$\begin{aligned}
\text{左} &\le \frac{x^2+y^2+z^2+x+y+z+3}{(x+y+z)^2} \\
&= \frac{(x+y+z)^2-(x+y+z)+3}{(x+y+z)^2} \\
&\text{......}
\end{aligned}$$

Q.E.D

7. 设 $a, b, c > 0, (a+b-c)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right) = 4$. 求证:

$$\left(a^4+b^4+c^4\right)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right) \ge 2304.$$

$$
\begin{aligned}
(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) &= 3+c\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{c}(a+b) \\
&\geq 3+2\sqrt{(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)},
\end{aligned}
$$

即

$$
\begin{aligned}
&(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 9 \\
\Rightarrow &\frac{b}{a}+\frac{a}{b} \geq 7 \\
\Rightarrow &\frac{b^2}{a^2}+\frac{a^2}{b^2} \geq 47.
\end{aligned}
$$

$$
\begin{aligned}
\left(a^4+b^4+c^4\right)\left(\frac{1}{b^4}+\frac{1}{a^4}+\frac{1}{c^4}\right) &\geq \left(\frac{a^2}{b^2}+\frac{b^2}{a^2}+1\right)^2 \\
&\geq 48^2 \\
&= 2304.
\end{aligned}
$$

8. 设 $x,y,z\in\mathbb{R}$. 求证:

$$
(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) \geq (xy+yz+zx)^2.
$$

分两类讨论.

1° $x,y,z\geq 0$, 由卡尔松不等式, 易证.

2° $x,t\geq,z<0$, 有

$$
(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) \geq (x^2+xy+y^2)\cdot\frac{3}{4}y^2\cdot\frac{3}{4}x^2,
$$

即证

$$
\frac{3}{4}y^2\cdot\frac{3}{4}x^2 \geq (xy)^3,
$$

即证

$$
\frac{9}{16}(x^2+xy+y^2) \geq xy,
$$

即证

$$
9x^2+9y^2 \geq 7xy,
$$

显然成立.

于是有

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) \geq (xy+yz+zx)^2.$$

综上,

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) \geq (xy+yz+zx)^2. \qquad \textbf{Q.E.D}$$

9. 正数 $a_{ij}(i=1,2,\cdots n, j=1,2,\cdots,n)$ 满足 $a_{ij}\cdot a_{ji}=1$. 记 $c_i=\sum_{k=1}^n a_{ki}$. 求证:

$$\sum \frac{1}{c_i} \leq 1.$$

设

$$c=\sum_{j=1}^n \frac{1}{c_j}.$$

对任意的 $i\in[1,n]\cap\mathbb{Z}$ 及 $x_i\in\mathbb{R}^+$, 由柯西不等式得

$$\sum_{j=1}^n \frac{x_i^2}{a_{ji}} \geq \frac{\left(\sum_{j=1}^n x_j\right)^2}{\sum_{j=1}^n a_{ji}}.$$

而对丁任意的 $i,j\in[1,n]\cap\mathbb{Z}$, 均有 $a_{ij}a_{ji}=1$. 于是, 对于任意的 $i\in[1,n]\cap\mathbb{Z}$ 都有

$$\sum_{j=1}^n \frac{a_{ij}}{c_j^2} \geq \frac{c^2}{c_i}.$$

于是有

$$\sum_{i=1}^n\sum_{j=1}^n \frac{a_{ij}}{c_j^2} \geq c^2\sum_{i=1}^n \frac{1}{c_i}=c^3.$$

又

$$\sum_{i=1}^n\sum_{j=1}^n \frac{a_{ij}}{c_j^2}=\sum_{j=1}^n\sum_{i=1}^n \frac{a_{ij}}{c_j^2}=\sum_{j=1}^n\left(\frac{1}{c_j^2}\sum_{i=1}^n a_{ij}\right)=\sum_{j=1}^n \frac{c_j}{c_j^2}=\sum_{j=1}^n \frac{1}{c_j}=c.$$

于是有 $c\geq c^3$.

又 $c>0$, 有

$$c=\sum \frac{1}{c_i} \leq 1. \qquad \textbf{Q.E.D}$$