§10 多元不等式'

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1. 已知实数 $x_1, x_2, \cdots, x_{100}$ 满足 $x_1 + x_2 + \cdots + x_{100} = 1$, 且 $|x_{k+1} - x_k| \leq \frac{1}{50}$, $k = 1, 2, \cdots, 99$. 求证: 存在整数 $1 \leq i_1 < i_2 < \cdots < i_{50} \leq 100$, 使得

$$\frac{49}{100} \leq x_{i_1} + x_{i_2} + \cdots + x_{i_{50}} \leq \frac{51}{100}.$$

即考虑数对 $(x_1, x_2), (x_3, x_4), \cdots, (x_{99}, x_{100})$.

记

$$b_k = \max\{x_{2k-1}, x_{2k}\}, c_k = \min\{x_{2k-1}, x_{2k}\},$$

则有

$$b_1 + b_{@} + \cdots + b_{50} \geq \frac{1}{2}, c_1 + c_2 + \cdots + c_{50} \leq \frac{1}{2}.$$

考察

$$b_1 + b_2 + \cdots + b_{50}, c_1 + b_2 + \cdots + b_{50}, c_1 + c_2 + \cdots + b_{50}, \cdots, c_1 + c_2 + \cdots + c_{50},$$

有相邻两个数之差 $\leq \frac{1}{50}$ (介值定理).

故得证.

2. 已知实数 p, q, r, s 满足 $p + q + r + s = 9, p^2 + q^2 + r^2 + s^2 = 21, p \geq q \geq r \geq s$, 证明:

$$pq - rs \geq 2.$$

设 $p = x + \alpha, q = x - \alpha, r = y + \beta, s = y - \beta$,

即证

$$p - \alpha^2 + \beta^2 \geq 2.$$

等号成立条件

$$p = 3, q = 2 = r = s, m = \frac{1}{4}, \alpha = \frac{1}{2}, \beta = 0.$$

由已知, 有

$$\begin{cases} x+y &= \dfrac{9}{2} \\ x^2+y^2+\alpha^2+\beta^2 &= \dfrac{21}{2} \\ \alpha \geq 0, \beta \geq 0, & x \geq y+\alpha+\beta \end{cases}$$

故有

$$2m \geq \alpha+\beta.$$

设 $x=\dfrac{9}{4}+m, y=\dfrac{9}{4}-m$, 则有

$$2\left(\frac{82}{46}+m^2\right)+\alpha^2+\beta^2=\frac{21}{2},$$

即

$$2m^2+\alpha^2+\beta^2=\frac{3}{8}.$$

即证

$$2m^2+9m+2\beta^2 \geq \frac{19}{8},$$

即证

$$2m^2+9m \geq \frac{19}{8}-2\beta^2,$$

只需证

$$2m^2+9m \geq \frac{19}{8},$$

即证

$$m \geq \frac{1}{4}.$$

有

$$\begin{aligned} \frac{3}{8} &= 2m^2+\alpha^2+\beta^2 \\ &\leq 2m^2+(\alpha+\beta)^2 \\ &\leq 6m^2, \end{aligned}$$

即

$$m \geq \frac{1}{4}.$$

则有原不等式成立.

3. 给定正整数 $n \geq 2$, 求最大的实数 λ, 使得对任意的正实数 $a_1, a_2, \cdots, a_n$ 有

$$\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{2} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2 + \lambda(a_1 - a_n)^2.$$

有

$$\begin{aligned}
\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} - \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2 &= \frac{1}{n^2} \sum_{1 \leq i < j \leq n} (a_i - a_j^2) \\
&\geq \frac{1}{n^2} \left\{(a_1 - a_n)^2 + \sum_{k=2}^{n-1} \left[(a_1 - a_k)^2 + (a_k - a_n)^2\right]\right\} \\
&\geq \frac{1}{n^2} \left[(a_1 - a_n)^2 + \sum_{k=2}^{n-1} \frac{(a_1 - a_n)^2}{2}\right] \\
&= \frac{1}{2n}(a_1 - a_n)^2.
\end{aligned}$$

故 $\lambda = \dfrac{1}{2n}$ 符合条件.

下证 $\lambda \leq \dfrac{1}{2n}$.

令 $a_1 = 1, a_2 = a_3 = \cdots = a_{n-1} = 0, a_n = -1$, 得:

$$\frac{2}{n} \geq 0 + 4\lambda,$$

即

$$\lambda \leq \frac{1}{2n}.$$

故有

$$\lambda_{\max} = \frac{1}{2n}.$$