§12 不等式中的归纳法

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1. 设 $a_1, a_2, \cdots, a_n$ 为非负数, 求证:

$$\sqrt{a_1+a_2+\cdots+a_n}+\sqrt{a_1+a_2+\cdots+a_{n-1}}+\cdots+\sqrt{a_1+a_2}+\sqrt{a_1} \geq \sqrt{n^2a_1+(n-1)^2a_2+\cdots+2^2a_{n-1}+a_n}.$$

当 $n=1$ 时, 原不等式显然成立.

若 $n=k, k\in\mathbb{N}^*$ 时原不等式成立, 即

$$\sqrt{a_1+a_2+\cdots+a_k}+\sqrt{a_1+a_2+\cdots+a_{k-1}}+\cdots+\sqrt{a_1+a_2}+\sqrt{a_1} \geq \sqrt{k^2a_1+(k-1)^2a_2+\cdots+2^2a_{k-1}+a_k}, \tag{1}$$

只需证当 $n=k+1$ 时

$$\sqrt{a_1+a_2+\cdots+a_{k+1}}+\sqrt{a_1+a_2+\cdots+a_k}+\cdots+\sqrt{a_1+a_2}+\sqrt{a_1} \geq \sqrt{(k+1)^2a_1+(k)^2a_2+\cdots+2^2a_k+a_{k+1}}. \tag{2}$$

(2) - (1), 即证

$$\sqrt{a_1+\cdots+a_{k+1}}+\sqrt{k^2a_1+(k-1)a_2+\cdots+a_k} \geq \sqrt{(k+1)^2a_1+k^2a_2+\cdots+2^2a_k+a_{k+1}}.$$

平方, 只需证

$$2\sqrt{a_1+\cdots+a_{k+1}}\sqrt{k^2a_1+\cdots+a_k} \geq 2ka_1+2(k-1)a_2+\cdots+2a_k.$$

由 Cauchy 不等式, 得证.

2. 设整数 $n\geq 2$, 且实数 $x_1, x_2, \cdots, x_n \in [0,1]$, 求证:

$$\sum_{1\leq k<l\leq n} kx_kx_l \leq \frac{n-1}{3}\sum_{k=1}^{n} kx_k.$$

当 $n=2$ 时, 左 $=x_1x_2$, 右 $=\dfrac{1}{3}(x_1+2x_2)$, 成立.

若 n 时成立, 只需证其对 $n+1$ 也成立, 只需证

$$x_{n+1}(x_1+2x_2+\cdots+nx_n)\le\frac{n}{3}(n+1)x_{n+1}+\frac{1}{3}(x_1+2x_2+\cdots+nx_n),$$

即证

$$3x_{n+1}(x_1+\cdots+nx_n)\le n(n+1)x_{n+1}+(x_1+\cdots+nx_n).$$

有

$$3kx_kx_{n+1}\le kx_k+2kx_{n+1},(k=1,2,\cdots,n)$$

相加即得证.

3. $x_1\ge 2x_2\ge\cdots\ge 2^{n-1}x_n\ge 0$,

$$\sum_{i=1}^n\frac{x_i}{\sqrt{i}}=1,$$

证明:

$$\sum_{i=1}^n x_i^2\le 1.$$

即证

$$\sum_{i=1}^n x_i^2\le\left(\sum_{i=1}^n\frac{x_i}{\sqrt{i}}\right)^2.$$

考虑 $n=1$, 显然成立.

假设 $n=k$ 时成立, 只需证 $n=k+1$ 时成立, 即证

$$x_{k+1}^2\le\frac{x_{k+1}}{\sqrt{x_{k+1}}}\left[\frac{x_{k+1}}{\sqrt{k+1}}+2\left(\frac{x_1}{\sqrt{1}}+\frac{x_2}{\sqrt{2}}+\cdots+\frac{x_k}{\sqrt{k}}\right)\right],$$

即证

$$\sqrt{k+1}x_{k+1}\le\frac{x_{k+1}}{\sqrt{k+1}}+2\sum_{i=1}^k\frac{x_i}{\sqrt{i}},$$

即证

$$\frac{k}{\sqrt{k+1}}x_{k+1}\le 2\sum_{i=1}^k\frac{x_i}{\sqrt{i}}.$$

有

$$2^{i-1}x_i\ge 2^k\cdot x_{k+1},$$

故

$$x_i \geq 2^{k-i+1} x_{k+1},$$

$$2\sum_{i=1}^{k} \frac{x_i}{\sqrt{i}} \geq 2\sum_{i=1}^{k} \frac{2^k - i + 1}{\sqrt{i}} x_{k+1} \geq 2\frac{2^{k-1+1}}{\sqrt{1}} x_{k+1} = 2^{k+1} x_{k+1} \geq \frac{k}{\sqrt{k+1}} x_{k+1},$$

得证.

4. 已知 $a_1, a_2, \cdots, a_n \geq 0 (n \geq 4)$, 求证:

$$4\sum_{i=1}^{n} a_{i+1} a_i \leq \left(\sum_{i=1}^{n} a_i \right)^2,$$

其中 $a_{n+1} = a_1$.

考虑 $n = 4$,

$$左 = 4(a_1 + a_3)(a_2 + a_4) \leq [(a_1 + a_3) + (a_2 + a_4)]^2 = 右.$$

若 $n = k$ 时成立, 只需证 $n = k + 1$ 时成立.

不妨设 $a_{k+1} = \min\{a_1, a_2, \cdots, a_{k+1}\}$, 只需证

$$4(a_k a_{k+1} + a_{k+1} a_1 - a_k a_1) \leq a_{k+1}[a_{k+1} + 2(a_1 + \cdots + a_k)].$$

只需证

$$右 \geq a_{k+1}(a_{k+1} + 2a_1 + 2a_k) \geq 4(a_k a_{k+1} + a_{k+1} a_1 - a_k a_1),$$

即证

$$a_{k+1}^2 - 2(a_1 + a_k) a_{k+1} + 4a_1 a_k \geq 0,$$

即证

$$(a_{k+1} - 2a_1)(a_{k+1} - 2a_k) \geq 0,$$

得证.

5. 设正整数 $n \geq 4$, 集合 $\{x_1, x_2, \cdots, x_n\} = \{y_1, y_2, \cdots, y_n\} = \{1, 2, \cdots, n\}$, 则

$$\sum_{i=1}^{n} x_i y_i$$

的可能值有多少种?

不妨设 $x_1 = 1, x_2 = 2, \cdots, x_n = n$,

由排序不等式, 有

$$
\begin{aligned}
\sum_{i=1}^{n} x_i y_i &\geq \sum_{i=1}^{n} i(n+1-i) \\
&= (n+1)\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2 \\
&= (n+1)\cdot\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6},
\end{aligned}
$$

又

$$
\sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.
$$

有

$$
\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{k} i y_i,
$$

考虑是否能取遍区间内所有的值.

$n=4$ 时显然可以.

假设 $n=k$ 时可以, 下证 $n=k+1$ 时同样可以取遍.

取 $y_{k+1}=k+1$, 则

$$
\sum_{i=1}^{k+1} i y_i = \sum_{i=1}^{k} i y_i + (k+1)^2
$$

能取遍

$$
\left[\frac{k(k+1)(k+2)}{6} + (k+1)^2, \frac{(k+1)(k+2)(2k+3)}{6}\right]
$$

中所有整数.

取 $y_{k+1}=1$, 则有

$$
\sum_{i=1}^{k+1} i y_i = \sum_{i=1}^{k} i y_i + (k+1) = \sum_{i=1}^{k} i(y_i-1) + \frac{k(k+1)}{2} + (k+1),
$$

又 $y_1-1, y_2-1, \cdots, y_{k-1}$ 是 $1,2,\cdots,k$ 的一个排列, 由归纳假设, 有

$$
\sum_{i=1}^{k} i(y_i-1)
$$

能取遍

$$
\left[\frac{k(k+1)(k+2)}{6}, \frac{k(k+1)(2k+1)}{6} + \frac{(k+1)(k+2)}{2}\right]
$$

中所有整数.

6. n 为给定正整数, 已知正整数 $a_1, a_2, \cdots, a_n (a_1 \le a_2 \le \cdots \le a_n)$ 满足

$$\sum_{i=1}^{n} \frac{1}{a_i} < 1,$$

求证: 当

$$\sum_{i=1}^{n} \frac{1}{a_i}$$

取得最大值时, 有 $a_1 = 2ma_{k+1} = a_1 a_2 \cdots a_k + 1 (k = 1, 2, \cdots, n-1)$.

设 $\gamma_1 = 2, \gamma_{k+1} = \gamma_1 \gamma_2 \cdots \gamma_{k+1}$, 则有

$$\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \cdots + \frac{1}{\gamma_n} = 1 - \frac{1}{\gamma_1 \gamma_2 \cdots \gamma_n}.$$

要证

$$\sum_{i=1}^{n} \frac{1}{a_i} \le 1 - \frac{1}{a_1 a_2 \cdots a_n} \le 1 - \frac{1}{\gamma_1 \gamma_2 \cdots \gamma_n} = \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_n},$$

只需证

$$a_1 a_2 \cdots a_n \le \gamma_1 \gamma_2 \cdots \gamma_n.$$

$n = 2$ 时显然成立.

假设 $n \le k$ 时成立, 则 $n = k + 1$ 时, 若命题不成立, 只需证

$$m \sqrt[n]{\frac{a_1 a_2 \cdots a_n}{\gamma_1 \gamma_2 \cdots \gamma_n}} \le \frac{a_1}{\gamma_1} + \frac{a_2}{\gamma_2} + \cdots + \frac{a_n}{\gamma_n} \le n.$$

有

$$\sum_{i=1}^{k+1} \frac{1}{a_i} < 1,$$

$$\sum_{i=1}^{k+1} \frac{1}{a_i} > \sum_{i=1}^{n} \frac{1}{\gamma_i}. \tag{$*$}$$

由归纳假设, 易得

$$\begin{gathered}
\frac{1}{a_1} \le \frac{1}{\gamma_1}, \\
\frac{1}{a_1} + \frac{1}{a_2} \le \frac{1}{\gamma_1} + \frac{1}{\gamma_2}, \\
\vdots \\
\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \le \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \cdots + \frac{1}{\gamma_n}.
\end{gathered}$$

记 $A_i = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_i}, \Gamma_i = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \cdots + \frac{1}{\gamma_i}$,

则有

$$A_1 \le \Gamma_1, A_2 \le \Gamma_2, \cdots, A_k \le \Gamma_k, A_{k+1} > \Gamma_{k+1},$$

则

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{a_i}{\gamma_i} &= \sum_{i=1}^{k+1} \left(\Gamma_i - \Gamma_{i-1}\right) a_i \\
&= \sum_{i=1}^{k} \Gamma_i (a_i - a_{i+1}) + \Gamma_{k+1} a_{k+1} \\
&\le \sum_{i=1}^{k} A_i (a_i - a_{i-1}) + A_{k+1} a_{k+1} \\
&= \sum_{i=1}^{k+1} (A_i - A_{i-1}) a_i \\
&= k+1.
\end{aligned}$$

由均值不等式, 有

$$\prod_{i=1}^{k+1} \frac{a_i}{\gamma_i} \le \left(\frac{\sum_{i=1}^{k+1} \frac{a_i}{\gamma_i}}{k+1} \right)^{k+1} = 1,$$

故有

$$\prod_{i=1}^{k+1} a_i \le \prod_{i=1}^{k+1} \gamma_i,$$

则

$$\frac{1}{a_1} + \cdots + \frac{1}{a_{k+1}} \le 1 - \frac{1}{a_1 a_2 \cdots a_n} \le 1 - \frac{1}{\gamma_1 \gamma_2 \cdots \gamma_{k+1}} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \cdots + \frac{1}{\gamma_{k+1}},$$

与 (*) 矛盾.

故命题在 $n = k+1$ 时成立, 得证.