§3 均值不等式

高一（6）班 邵亦成 26 号

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均值不等式：$\forall x_1,x_2,\cdots,x_n\in\mathbf{R}^+,n\in\mathbf{N}^*\cap[2,+\infty):\dfrac{1}{n}\sum\limits_{i=1}^{n}x_i\geq\sqrt[n]{\prod\limits_{i=1}^{n}x_i}$, 等号成立当且仅当 $x_1=x_2=\cdots=x_n$.

幂平均不等式：$\forall x_1,x_2,\cdots,x_n\in\mathbf{R}^+,n\in\mathbf{N}^*\cap[2,+\infty):f(p)=\left(\dfrac{x_1^p+x_2^p+\cdots+x_n^p}{n}\right)^{\frac{1}{p}}$ 单调递增.

考虑 $p=2,1,0,-1$，则有：$\sqrt{\dfrac{x_1^2+x_2^2+\cdots+x_n^2}{n}}\geq\dfrac{x_1+x_2+\cdots+x_n}{n}\geq\sqrt[n]{x_1x_2\cdots x_n}\geq\dfrac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}$.

注：$p=0$ 实质上为 $p\to0$，求解如下：

$$f(p)=\left(\frac{x_1^p+x_2^p+\cdots+x_n^p}{n}\right)^{\frac{1}{p}}=e^{\frac{1}{p}\ln\frac{x_1^p+x_2^p+\cdots+x_n^p}{n}},$$

$$\begin{aligned}\lim_{p\to0}\frac{1}{p}\ln\frac{x_1^p+x_2^p+\cdots+x_n^p}{n}&\xlongequal{\text{L. Hospital}}\lim_{p\to0}\frac{n}{x_1^p+x_2^p+\cdots+x_n^p}\cdot\frac{x_1^p\ln x_1+x_2^p\ln x_2+\cdots+x_n^p\ln x_n}{n}\\&=\frac{\ln x_1+\ln x_2+\cdots+\ln x_n}{n},\end{aligned}$$

$\therefore\lim\limits_{p\to0}f(p)=e^{\frac{\ln x_1+\ln x_2+\cdots+\ln x_n}{n}}=\sqrt[n]{x_1x_2\cdots x_n}$.

1. 证明均值不等式：$\forall x_1,x_2,\cdots,x_n\in\mathbf{R}^+,n\in\mathbf{N}^*\cap[2,+\infty):\dfrac{1}{n}\sum\limits_{i=1}^{n}x_i\geq\sqrt[n]{\prod\limits_{i=1}^{n}x_i}$.

   本题可选用数学归纳法解决.

   1° $n=2$ 时，显然成立.

   2° 假设 $n=k$ 时成立，证明对 $n=k+1$ 时成立.
   记 $\dfrac{1}{k}\sum\limits_{i=1}^{k}x_i=A_k$, $\sqrt[k]{\prod\limits_{i=1}^{k}x_i}=G_k$,
   则有 $A_k\geq G_k$.
   当 $n=k+1$ 时，

$$
\begin{aligned}
& (k+1)A_{k+1} \\
\geq\ & kG_k + x_{k+1} \\
=\ & kG_k + [x_{k+1} + (k-1)G_{k+1}] - (k-1)G_{k+1} \\
\geq\ & kG_k + k\sqrt[k]{x_{k+1}G_{k+1}^{k-1}} - (k-1)G_{k+1} \\
\geq\ & k \cdot 2\sqrt{G_k \cdot \sqrt[k]{x_{k-1}G_{k+1}^{k-1}}} - (k-1)G_{k+1} \\
=\ & 2k \cdot \sqrt[2k]{G_k^k \cdot x_{k+1} \cdot G_{k+1}^{k-1}} - (k-1)G_{k+1} \\
=\ & 2k \cdot \sqrt[2k]{G_{k+1}^{k+1} \cdot G_{k-1}^{k-1}} - (k-1)G_{k+1} \\
=\ & 2k \cdot G_{k+1} - (k-1)G_{k+1} \\
=\ & (k+1)G_{k+1}.
\end{aligned}
$$

即有 $A_{k+1} \geq G_{k+1}$.

于是有 $\forall x_1, x_2, \cdots, x_n \in \mathbf{R}^+, n \in \mathbf{N}^* \cap [2, +\infty) : \dfrac{1}{n}\sum_{i=1}^{n} x_i \geq \sqrt[n]{\prod_{i=1}^{n} x_i}$.

2. 求证：$\left(1+\dfrac{1}{n}\right)^n < \left(1+\dfrac{1}{n+1}\right)^{n+1}$，其中 $n \in \mathbf{N}^*$.

$$
\begin{aligned}
\text{左} &= \left(1+\tfrac{1}{n}\right)^n \cdot 1 \\
&\leq \left[\tfrac{n\cdot\left(1+\frac{1}{n}\right)+1}{n+1}\right]^{n+1} \\
&= \left(\tfrac{n+2}{n+1}\right)^{n+1} \\
&= \left(1+\tfrac{1}{n+1}\right)^{n+1}.
\end{aligned}
$$

等号成立当且仅当 $1+\frac{1}{n} = 1$ 无法取到.

$\therefore$ 左 $<$ 右.

求证：$\left(1+\frac{1}{n}\right)^n < 3$.

$$
\begin{aligned}
\left(1+\tfrac{1}{n}\right)^n &= C_n^0 \cdot 1^n + C_n^1 \cdot 1^{n-1}\left(\tfrac{1}{n}\right)^1 + C_n^2 \cdot 1^{n-2}\left(\tfrac{1}{n}\right)^2 + \cdots + C_n^n\left(\tfrac{1}{n}\right)^n \\
&= 2 + \tfrac{n(n-1)}{2!}\cdot\tfrac{1}{n^2} + \tfrac{n(n-1)(n-2)}{3!}\cdot\tfrac{1}{n^3} + \cdots + \tfrac{n!}{n!}\cdot 1n^n \\
&< 2 + \tfrac{1}{2!} + \tfrac{1}{3!} + \cdots + \tfrac{1}{n!} \\
&< 2 + \tfrac{1}{1\times 2} + \tfrac{1}{2\times 3} + \cdots + \tfrac{1}{(n-1)n} \\
&= 2 + \left(1-\tfrac{1}{n}\right) \\
&< 3.
\end{aligned}
$$

数列单调递增 $+$ 数列收敛 $\Rightarrow$ 数列存在极限.

$$
\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \approx 2.718.
$$

3. 设 $a,b,c\in\mathbf{R}^+, abc=1$. 求证：$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq 1$.

法一：（未完成）

$$\begin{aligned} S &= (a-abc+ac)(b-abc+ab)(c-abc+bc) \\ &= (1-bc+c)(1-ac+a)(1-ab+b) \\ &= \left(1-\frac{1}{a}+c\right)\left(1-\frac{1}{b}+a\right)\left(1-\frac{1}{c}+b\right), \end{aligned}$$

$$\begin{aligned} S^2 &= \left[a^2-\left(1-\frac{1}{b}\right)^2\right]\left[b^2-\left(1-\frac{1}{c}\right)^2\right]\left[c^2-\left(1-\frac{1}{a}\right)^2\right] \\ &\leq a^2b^2c^2 \quad \text{（负数?）} \\ &= 1. \end{aligned}$$

若 $a^2-\left(1-\dfrac{1}{b}\right)^2, b^2-\left(1-\dfrac{1}{c}\right)^2, c^2-\left(1-\dfrac{1}{a}\right)^2$ 均为非负，则 $S^2\leq 1$ 有 $S\leq 1$.

若三者中有两负一正，? .

法二：设 $a=\dfrac{y}{x}, b=\dfrac{x}{z}, c=\dfrac{z}{y}(x,y,z\in\mathbf{R}^+)$，

$$\begin{aligned} \text{左} &= \left(\frac{y}{z}-1+\frac{z}{x}\right)\left(\frac{x}{z}-1+\frac{y}{z}\right)\left(\frac{z}{y}-1+\frac{x}{y}\right) \\ &= (y+z-x)(x+y-z)(z+x-y). \end{aligned}$$

若三者均为非负，则有

$$\left\{\begin{array}{lcl} (y+z-x)(x+y-z) & \leq & y^2 \\ (x+y-z)(z+x-y) & \leq & x^2 \\ (z+x-y)(y+z-x) & \leq & z^2 \end{array}\right. \Rightarrow (y+z-x)^2(x+y-z)^2(z+x-y)^2\leq x^2y^2z^2$$

若有一个负数，则左 $<0\leq 1$.

得证.

4. $a,b,c\in\mathbf{R}^+, abc=1$. 求证：$a+b+c\leq a^2+b^2+c^2$.

$$\begin{aligned} a^2+b^2+c^2 &\geq \tfrac{(a+b+c)^2}{3} \\ &= \tfrac{a+b+c}{3}\cdot 3\sqrt[3]{abc} \\ &= a+b+c. \end{aligned}$$

5. 设 $a_1,a_2,\cdots,a_{2016}\in\mathbf{R}, 9a_i>11a_{i+1}^2(i=1,2,\cdots,2015)$. 求:$\left[\left(a_1-a_2^2\right)\left(a_2-a_3^2\right)\cdots\left(a_{2015}-a_{2016}^2\right)\left(a_{2016}-a_1^2\right)\right]_{\max}$.

由已知，$a_i-a_{i+1}^2\geq\dfrac{11}{9}a_{i+1}^2-a_{i+1}^2\geq 0(i\in 1,2,\cdots,2015)$.

求最大值，不妨令 $a_{2016}-a_1^2>0$

原式 $\leq\left[\dfrac{\sum_{k=1}^{2016}\left(a_k-a_{k+1}^2\right)}{2016}\right]^{2016}=\left[\dfrac{\sum_{k=1}^{2016}a_k-\sum_{k=1}^{2016}a_{k+1}^2}{2016}\right]^{2016}$.

$a_k-a_k^2\leq\frac{1}{4}(k=1,2,\cdots,2016)$

$\therefore \dfrac{\sum_{k=1}^{2016}\left(a_k-a_k^2\right)}{2016} \leq \dfrac{1}{4}$,

$\therefore$ 原式 $\leq \dfrac{1}{4^{2016}}$,

当且仅当 $a_1=a_2=\cdots=a_{2016}=\frac{1}{2}$.

6. 设 $a_1,a_2,\cdots,a_n(n\geq 2)$ 为正实数，有 $a_1+a_2+\cdots+a_n<1$. 求证:

$$\frac{a_1a_2\cdots a_n[1-(a_1+a_2+\cdots+a_n)]}{(a_1+a_2+\cdots+a_n)(1-a_1)(1-a_2)\cdots(1-a_n)} \leq \frac{1}{n^{n+1}}$$

设 $l=a_1+a_2+\cdots+a_{n+1}$, 有 $a_{n+1}>0$.

$$\begin{aligned}
\text{左} &= \frac{a_1a_2\cdots a_na_{n+1}}{(a_1+a_2+\cdots+a_n)(a_2+a_3+\cdots+a_{n+1})(a_1+a_3+\cdots+a_{n+1})\cdots(a_1+a_2+\cdots+a_{n-1}+a_{n+1})}\\
&\leq \frac{\prod_{k=1}^{n+1}a_k}{n\sqrt[n]{a_1a_2\cdots a_n}\cdot n\sqrt[n]{a_2a_3\cdots a_{n+1}}\cdots n\sqrt[n]{a_1a_2\cdots a_{n-1}a_{n+1}}}\\
&= \frac{\prod_{k=1}^{n+1}a_k}{n^{n+1}\prod_{k=1}^{n+1}a_k}\\
&= \frac{1}{n^{n+1}}.
\end{aligned}$$

7. $a,b,c,d,e\geq -1, a+b+c+d+e=5, S=(a+b)(b+c)(c+d)(d+e)(e+a)$. 求 $S_{\min}$.

考虑 $S<0$，考虑正负号.

(1) 两正三负三个负数 ≥ -2，两个正数之和 ≤ 16.

$\Sigma=10$, 两个正数之积 ≤ 64.

考虑 $a=b=c=d=-1, e=9$ 有 $S_{\min 1}=-512$

(2) 四正一负负数 ≥ -2，四个正数之和 $\leq 12 \Rightarrow$ 之积 ≤ 34 有 $S\geq -162$.

(3) 五负（舍）

综上，$S_{\min}=-512, a=b=c=d=-1, e=9$.

8. $a,b,c\in(0,1],\lambda\in\mathbf{R}$, 有 $\dfrac{\sqrt{3}}{\sqrt{a+b+c}}\geq 1+\lambda(1-a)(1-b)(1-c)$ 恒成立，求 $\lambda_{\max}$.

即 $\dfrac{\frac{\sqrt{3}}{\sqrt{a+b+c}}-1}{(1-a)(1-b)(1-c)}\geq\lambda$,

即 左$_{\min}\geq\lambda$.

左 $\geq \dfrac{\frac{\sqrt{3}}{\sqrt{a+b+c}}-1}{\left(\frac{3-a-b-c}{3}\right)^3}$.

记 $a+b+c=3k, k\in(0,1]$,

左 $\geq \dfrac{\frac{\sqrt{3}}{\sqrt{3}k}-1}{(1-k)^3}=\dfrac{\frac{1}{k}-1}{(1-k^2)^3}=\dfrac{1}{k(1-k)^2(1+k)^3}$.

记 $1+k=t$,

原式 $=(t-1)(2-t)^2t^3=t^6-5t^5+8t^4-4t^3\triangleq g(t)$,

$g'(t)=6t^5-25t^4+32t^3-12t^2=t^2(6t^3-25t^2+32t-12)=t^2(t-2)(2t-3)(3t-2)=0$,

解得 $t=\dfrac{3}{2},g(t)=\dfrac{1}{2}$,

$\therefore$ 左 $\geq\dfrac{64}{27}$,

即 $\lambda_{\max}=\dfrac{64}{27}$.