§9 局部不等式与放缩法'

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1. 已知 $a_1, a_2, \cdots, a_n$ 为正实数, 且满足 $a_1 + a_2 + \cdots + a_n = 1$, 证明:
$$\frac{a_1^3}{a_1^2 + a_2 a_3} + \frac{a_2^3}{a_2^2 + a_3 a_4} + \cdots + \frac{a_{n-1}^3}{a_{n-1}^2 + a_n a_1} + \frac{a_n^3}{a_n^2 + a_1 a_2} \geq \frac{1}{2}.$$

$$\sum \frac{a_k^3}{a_k^2 + a_{k+1} a_{k+2}} \geq \frac{1}{2} \sum a_k,$$

令 $p, q(p + 2q = \frac{1}{2})$ 满足

$$\frac{a_k^3}{a_k^2 + a_{k+1} a_{k+2}} \geq p a_k + q(a_{k+1} + a_{k+2}),$$

有

$$a_k^3 \geq [p a_k + q(a_{k+1} + a_{k+2})] (a_k^2 + a_{k+1} a_{k+2}),$$

有

$$3 = 3p + p + 4q,$$

即

$$p = 1, q = -\frac{1}{4}.$$

故原不等式即证

$$\frac{a_k^3}{a_k^2 + a_{k+1} a_{k+2}} \geq a_k - \frac{1}{4}(a_{k+1} + a_{k+2}),$$

即证

$$0 \geq -\frac{1}{4} a_k^2 (a_{k+1} a_{k+2}) + a_k a_{k+1} a_{k+2} - \frac{1}{4}(a_{k+1} + a_{k+2}) a_{k+1} a_{k+2},$$

有

$$
\begin{aligned}
&\frac{1}{4}a_k^2(a_{k+1}+a_{k+2})+\frac{1}{4}(a_{k+1}+a_{k+2})a_{k+1}a_{k+2}\\
=&\frac{1}{4}(a_k^2a_{k+1}+a_{k+1}a_{k+2})^2+\frac{1}{4}(a_k^2a_{k+2}+a_{k+1}^2a_{k+1})\ge\frac{1}{2}a_ka_{k+1}a_{k+2}+\frac{1}{2}a_ka_{k+1}a_{k+2}\\
=&a_ka_{k+1}a_{k+2},
\end{aligned}
$$

得证.

2. 设 $n\ge 3$ 是整数, 求证: 对正整数 $x_1\le x_2\le\cdots\le x_n$, 有不等式

$$
\frac{x_nx_1}{x_2}+\frac{x_1x_2}{x_3}+\cdots+\frac{x_{n-1}x_n}{x_1}\ge x_1+x_2+\cdots+x_n.
$$

有

$$
\begin{gathered}
\frac{x_1x_2}{x_3}+x_3\ge x_1+x_2\\
\frac{x_2x_3}{x_4}+x_4\ge x_2+x_3\\
\vdots\\
\frac{x_{n-2}x_{n-1}}{x_n}+x_n\ge x_{n-2}+x_{n-1}\\
----------\\
\frac{x_{n-1}x_n}{x_1}+x_1\ge x_{n-1}+x_n\\
\frac{x_nx_1}{x_2}+x_2\ge x_n+x_1?
\end{gathered}
$$

只需证

$$
\left(\frac{x_{n-1}x_n}{x_1}+x_1\right)+\left(\frac{x_nx_1}{x_2}+x_2\right)\ge(x_{n-1}+x_n)+(x_n+x_1),
$$

即证

$$
\frac{x_{n-1}x_n}{x_1}+\frac{x_nx_1}{x_2}+x_2\ge x_{n-1}+2x_n.
$$

记

$$
f(x_1,x_2,x_{n-1},x_n)=\frac{x_{n-1}x_n}{x_1}+\frac{x_nx_1}{x_2}+x_2,
$$

$$
x_1\le x_2\Rightarrow x_1=x_2,f_{\min}=\frac{x_{n-1}+x_n}{x_2}+x_2+x_n\ge(x_{n-1}+x_n)+x_n,
$$

得证.

3. 设正整数 $n \geq 2$, 实数 $x_1, x_2, \cdots, x_n$ 满足 $\sum\limits_{i=1}^{n} x_i = 0, \sum\limits_{i=1}^{n} x_i^2 = 1$. 求证: 存在 $a, b \in \{x_1, x_2, \cdots, x_n\}$, 使得对任意的 $1 \leq j \leq n$, 有

$$a + 2x_j + nax_j^2 \leq \sum_{x=1}^{n} x_i^3 \leq b + 2x_k + nbx_k^2.$$

$$a = \min\{x_1, x_2, \cdots, x_n\}, b = \max\{x_1, x_2, \cdots, x_n\}.$$

$$(x_i - x_j)^2(x_i - a) \geq 0$$

$$\Updownarrow$$

$$x_i^3 - (a + 2x_j)x_i^2 + (2ax_j + x_j^2)x_i + ax_j^2 \geq 0$$

$$\Updownarrow$$

$$x_i^3 \geq ax_j^2 + 2x_j x_i^2 + ax_i^2 - (2ax_j + a_j^2)x_i.$$

将 i 从 1 到 n 相加,

$$\begin{aligned}\sum x_o^3 &\geq \sum_{i=1}^{n} ax_j^2 + 2x_j \sum_{i=1}^{n} x_i^2 + a\sum_{i=1}^{n} x_i^2 - (2ax_j + x_j^2)\sum_{i=1}^{n} x_i \\ &= nax_j^2 + 2x_j + a - 0 \\ &= \text{L.H.S.}.\end{aligned}$$

故得证. (b 同理)

4. 设正数 a, b, c 满足 $a + b + c = 1$, 求证:

$$\sqrt{1 + \frac{bc}{a}} + \sqrt{1 + \frac{ca}{b}} + \sqrt{1 + \frac{ab}{c}} \geq 2\sqrt{3}.$$

$$\sqrt{1+x} + \sqrt{1+y} + \sqrt{1+z} \geq 2\sqrt{3},$$

$$xyz = abc, xy = c^2, yz = a^2, zx = b^2,$$

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} = 1.$$

$$
\begin{aligned}
(\sqrt{1+x}+\sqrt{1+y}+\sqrt{1+z})^2 &= 3+(x+y+z)+2\sqrt{(1+x)(1+y)}+2\sqrt{(1+y)(1+z)}+2\sqrt{(1+z)(1+x)} \\
&\geq 3+(x+y+z)+2(1+\sqrt{xy})+2(1+\sqrt{yz})+2(1+\sqrt{zx}),
\end{aligned}
$$

$$
x+y+z \geq 1,
$$

得证.