§5 柯西不等式

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Cauchy–Schwarz 不等式在 $\mathbf{R}^n$ 上的特殊化: $\forall n \in \mathbf{Z} \cap [2, +\infty), i \in \mathbf{Z} \cap [1, n], a_i \in \mathbf{R}, b_i \in \mathbf{R}$ 有:

$$\sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n b_i^2 \geq \left(\sum_{i=1}^n a_i b_i \right)^2,$$

等号成立当且仅当 $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$ (可以认为 $\dfrac{\neq 0}{0} =$ 无穷, $\dfrac{0}{0} =$ 任意实数).

Carlson 不等式 (Cauchy-Schwarz 不等式在二维矩阵上的推广): $\forall n, m \in \mathbf{Z} \cap [2, +\infty), i \in \mathbf{Z} \cap [1, n], j \in \mathbf{Z} \cap [1, m], a_{ij} \in \mathbf{R}^+$ 有:

$$\prod_{j=1}^m \left(\sum_{i=1}^n a_{ij} \right)^{\frac{1}{m}} \geq \sum_{j=1}^m \prod_{i=1}^n a_{ij}^{\frac{1}{m}},$$

等号成立当且仅当 $\forall i \in \mathbf{Z} \cap [1, n-1] : \dfrac{a_{i1}}{a_{(i+1)1}} = \dfrac{a_{i2}}{a_{(i+1)2}} = \cdots = \dfrac{a_{im}}{a_{(i+1)m}}$ (可以认为 $\dfrac{\neq 0}{0} =$ 无穷, $\dfrac{0}{0} =$ 任意实数).

1. 证明 $m = 3$ 时的 Carlson 不等式.

不妨设 $a_1^3 + a_2^3 + \cdots + a_n^3 = b_1^3 + b_2^3 + \cdots + b_n^3 = c_1^3 + c_2^3 + \cdots + c_n^3 = 1$, 则原不等式 $\Leftrightarrow 1 \geq (a_1 b_1 c_1 + a_2 b_2 c_2 + \cdots + a_n b_n c_n)^3$.

$$\sum_{i=1}^n a_i b_i c_i \leq \sum_{i=1}^n \frac{a_i^3 + b_i^3 + c_i^3}{3} = 1.$$

2. $x, y, z \in \mathbf{R}^+, x^2 + y^2 + z^2 = 1$. 求 $\left[\dfrac{x}{1-x^2} + \dfrac{y}{1-y^2} + \dfrac{z}{1-z^2} \right]_{\min}$.

$$\left(\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} \right) \left[x^3(1-x^2) + y^3(1-y^2) + z^3(1-z^2) \right] \geq (x^2 + y^2 + z^2)^2.$$

$\therefore$ 有:

$$\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} \geq \frac{1}{x^3(1-x^2) + y^3(1-y^2) + z^3(1-z^2)} \geq ? \frac{3}{2}\sqrt{3}.$$

$\therefore$ 只需证

$$x^3(1-x^2)+y^3(1-y^2)+z^3(1-z^2) \le \frac{2}{3\sqrt{3}}.$$

$$\begin{aligned} x^3(1-x^2) \le \frac{2}{3\sqrt{3}}x^2 \quad &\Leftrightarrow \quad x(1-x^2) \le \frac{2}{3\sqrt{3}} \\ &\Leftrightarrow \quad x^2(1-x^2)(1-x^2) \le \frac{4}{27} \\ &\Leftrightarrow \quad 2x^2(1-x^2)(1-x^2) \le \frac{8}{27} \end{aligned}$$

由均值不等式显然成立.

证毕.

另解:

令 k:

$$\frac{x}{1-x^2} \ge kx^2+m,$$

即

$$\frac{\sqrt{x}}{1-x} \ge kx+m(x\in[0,1)).$$

记

$$f(x)=\frac{\sqrt{x}}{1-x},$$

有

$$k=f'\left(\frac{1}{3}\right).$$

而

$$f'(x)=\frac{\frac{1}{2}\cdot\frac{1}{\sqrt{x}}\cdot(1-x)-\sqrt{x}(-1)}{(1-x)^2},$$

$\therefore k=\frac{3\sqrt{3}}{2}$,

代入有 $m=0$.

3. $\forall a,b,c\in\mathbf{R}^+$. 证明: $\displaystyle\sum_{\text{cyc}}\frac{a}{b+c}+\sqrt{\frac{\sum_{\text{cyc}}ab}{\sum_{\text{cyc}}a^2}} \ge \frac{5}{2}$.

$\displaystyle\left(\sum_{\text{cyc}}\frac{a}{b+c}\right)\left(\sum_{\text{cyc}}a(b+c)\right) \ge \left(\sum_{\text{cyc}}a\right)^2$.

$\therefore$ 有:

$$
\begin{aligned}
\text{左边} &\geq \frac{\sum_{\text{cyc}} a^2}{2\sum_{\text{cyc}} ab}+\sqrt{\frac{\sum_{\text{cyc}} ab}{\sum_{\text{cyc}} a^2}} \\
&= 1+\frac{a^2+b^2+c^2}{2(ab+bc+ca)}+\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}} \\
&= 1+\frac{a^2+b^2+c^2}{2(ab+bc+ca)}+\frac{1}{2}\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}}+\frac{1}{2}\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}} \\
&\geq 1+3\sqrt[3]{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}} \\
&= \frac{5}{2}.
\end{aligned}
$$

4. $x_1,x_2,x_3\in\mathbf{R}^*, x_1+x_2+x_3=1$. 求 $(x_1+3x_2+5x_3)\left(x_1+\frac{x_2}{3}+\frac{x_3}{5}\right)$ 的最值.

下求**最小值**.

由 Cauchy-Schwarz 不等式有:

$$(x_1+3x_2+5x_3)\left(x_1+\frac{x_2}{3}+\frac{x_3}{5}\right)\geq(x_1+x_2+x_3)^2=1,$$

等号成立当且仅当 $\frac{x_1}{x_1}=\frac{3x_2}{\frac{x_2}{3}}=\frac{5x_3}{\frac{x_3}{5}}$ 即 $x_1=1,x_2=0,x_3=0$.

下为**最大值**的草稿.

$$(x_1+3x_2+5x_3)\left(x_1+\frac{x_2}{3}+\frac{x_3}{5}\right)\leq M(x_1+x_2+x_3)^2,$$

即

$$(M-1)x_1^2+(M-1)x_2^2+(M-1)x_3^2\geq\left(\frac{10}{3}-2M\right)x_1x_2+\left(\frac{26}{5}-2M\right)x_1x_3+\left(\frac{34}{15}-2M\right)x_2x_3.$$

显然有 $M>1$. $M\geq\frac{13}{5}$ 显然成立.

考虑 $\frac{5}{3}\leq M\leq\frac{13}{5}$, 该不等式等价于

$$(M-1)x_1^2+(M-1)x_2^2+(M-1)x_3^2+\left(2M-\frac{10}{3}\right)x_1x_2+\left(2M-\frac{34}{15}\right)x_2x_3\geq\left(\frac{26}{5}-2M\right)x_1x_3.$$

令 $x_2=0$, 得

$$(M-1)x_1^2+(M-1)x_3^2\geq\left(\frac{26}{5}-2M\right)x_1x_3.$$

$\Delta\leq 0$, 即

$$\left(\frac{26}{5}-2M\right)^2\leq 4(M-1)^2,$$

即

$$\frac{16}{5}\cdot\left(\frac{36}{5}-4M\right)\leq 0,$$

即

$$M \geq \frac{9}{5}.$$

下求**最大值**.

$$(x_1+3x_2+5x_3)\left(x_1+\frac{x_2}{3}+\frac{x_3}{5}\right) \leq \frac{9}{5}(x_1+x_2+x_3)^2,$$

即

$$\frac{4}{5}x_1^2+\frac{4}{5}x_2^2+\frac{4}{5}x_3^2+\frac{4}{15}x_1x_2+\frac{4}{3}x_2x_3 \geq \frac{8}{5}x_2x_3$$

即

$$\frac{4}{5}(x_1-x_3)^2+\frac{4}{5}x_2^2+\frac{4}{15}x_1x_2+\frac{4}{3}x_2x_3 \geq 0$$

显然成立, 等号成立当且仅当 $x_1=x_3=\dfrac{1}{2}, x_2=0$.

于是原式 $\leq \dfrac{9}{5}$.

5. 求 $\sqrt{x+27}+\sqrt{13-x}+\sqrt{x}$ 的最值.

$D=[0,13]$.

下为**最大值**的草稿.

$$\left(\sqrt{x+27}+\sqrt{13-x}+\sqrt{x}\right)^2 \leq [1(x+27)+m(13-x)+nx]\left(1+\frac{1}{m}+\frac{1}{n}\right),$$

有 $m=n+1$,

等号成立当且仅当 $x+27=m^2(13-x)=n^2x, x=\dfrac{13m^2-27}{m^2+1}=\dfrac{27}{n^2-1}$, 有 $n=2, m=3$.

下求**最大值**.

$$\left(\sqrt{x+27}+\sqrt{13-x}+\sqrt{x}\right)^2 \leq [(x+27)+3(13-x)+2x]\left[1+\frac{1}{3}+\frac{1}{2}\right]=121,$$

等号成立当且仅当 $x=9$.

下求**最小值**.

$$\sqrt{x+27}+\sqrt{13-x}+\sqrt{x}=\sqrt{x+27}+\sqrt{13+2\sqrt{x(13-x)}} \geq \sqrt{27}+\sqrt{13}=3\sqrt{3}+\sqrt{13},$$

等号成立当且仅当 $x=0$.

6. $a,b,c,d\in\mathbf{R}, a+2b+3c+4d=\sqrt{10}$. 求 $\left[a^2+b^2+c^2+d^2+(a+b+c+d)^2\right]_{\min}$.

草稿.

$$\begin{aligned}
&\left[a^2+b^2+c^2+d^2+(a+B+c+d)^2\right]\left[(k-1)^2+(2k-1)^2+(3k-1)^2+(4k-1)^2+1^2\right]\\
\geq\ & \left[(k-1)a+(2k-1)b+(3k-1)c+(4k-1)d+(a+b+c+d)\right]^2\\
=\ & k^2(a+2b+3c+4d)^2\\
=\ & 10k^2,
\end{aligned}$$

等号成立当且仅当 $\dfrac{a}{k-1}=\dfrac{b}{2k-1}=\dfrac{c}{3k-1}=\dfrac{d}{4k-1}=\dfrac{a+b+c+d}{1}$.

有 $\dfrac{a+b+c+d}{10k-4}=\dfrac{a+b+c+d}{1}\Rightarrow k=\dfrac{1}{2}$.

过程:

$$\begin{aligned}
&\left[a^2+b^2+c^2+d^2+(a+b+c+d)^2\right]\left[\frac{1}{4}+0+\frac{1}{4}+1+1\right]\\
\geq\ & \left[-\frac{1}{2}a+0+\frac{1}{2}c+d+(a+b+c+d)\right]^2\\
=\ & \frac{5}{2}
\end{aligned}$$

$$\Rightarrow a^2+b^2+c^2+d^2+(a+b+c+d)^2\geq 1,$$

等号成立当且仅当 $a=-\dfrac{1}{2}m, b=0, c=\dfrac{1}{2}m, d=m$ 即 $a=\dfrac{\sqrt{10}}{10}, b=0, c=\dfrac{\sqrt{10}}{10}, d=\dfrac{\sqrt{10}}{5}$.

7. $a_1,a_2\cdots a_n$ 有 $\displaystyle\sum_{i=1}^{n}a_i=0$. 求证:

$$\max_{1\leq k\leq n}\left(a_k^2\right)\leq\frac{n}{3}\sum_{i=1}^{n-1}\left(a_i-a_{i+1}\right)^2.$$

只需证

$$\forall 1\leq k\leq n: a_k^2\leq\frac{n}{3}\sum_{i=1}^{n-1}\left(a_i-a_{i+1}\right)^2.$$

记

$$d_i=a_{i+1}-a_i,$$

即证

$$a_k^2\leq\frac{n}{3}\sum_{i=1}^{n-1}d_i^2.$$

有

$$
\begin{aligned}
a_1 &= a_k-(d_{k-1}+\cdots+d_1),\\
&\vdots\\
a_{k-2} &= a_k-(d_{k-1}+d_{k-2}),\\
a_{k-1} &= a_k-d_{k-1},\\
a_k &= a_k,\\
a_{k+1} &= a_k+d_k,\\
a_{k+2} &= a_k+d_k+d_{k+1},\\
&\vdots\\
a_n &= a_k+d_k+d_{k+1}+\cdots+d_{n-1}.
\end{aligned}
$$

又有

$$
\begin{aligned}
\sum_{i=1}^n a_i &= na_k+(n-k)d_k+(n-k-1)d_{k+1}+d_{n-1}-(k-1)d_{k-1}-(k-2)d_{k-2}-\cdots-d_1\\
&= 0,
\end{aligned}
$$

故

$$
\begin{aligned}
(na_k)^2 &= [d_1+2d_2+\cdots+(k-1)d_{k-1}-(n-k)d_k-(n-k-1)d_{k+1}-\cdots-d_{n-1}]^2\\
&\le \left(d_1^2+d_2^2+\cdots+d_{n-1}^2\right)\left(1^2+2^2+\cdots+(k-1)^2+(n-k)^2+(n-k-1)^2+\cdots+1^2\right)\\
&\le \sum_{i=2}^{n-1} d_i^2\sum_{i=1}^{n-1} i^2\\
&= \sum_{i=1}^{n-1} d_i^2\frac{(n-1)n(2n-1)}{6}\\
&\le \sum_{i=1}^{n-1} d_i^2\left(\frac{n^3}{3}\right).
\end{aligned}
$$

于是有

$$
a_k^2\le\frac{n}{3}\sum_{i=1}^{n-1}\left(a_i-a_{i+1}\right)^2.
$$

得证.

8. $a_1,a_2\cdots a_n\in\mathbf{R}^+$ 有 $(a_1^2+a_2^2+\cdots+a_n^2)^2>(n-1)(a_1^4+a_2^4+\cdots+a_n^4)$. 求证: 任意三个 a_i 均能构成 Δ 的三边长.

使用反证法. 若命题不成立, 不妨设 $a_1\ge a_2+a_3$, 目标: $(a_1^2+a_2^2+\cdots+a_n^2)^2\le(n-1)(a_1^4+a_2^4+\cdots+a_n^4)$.

$$
\begin{aligned}
\text{左边} &= \left[\frac{a_1^2+a_2^2+a_3^2}{2}+\frac{a_1^2+a_2^2+a_3^2}{2}+a_4^2+\cdots+a_n^2\right]^2 \\
&\le (n-1)\left[\frac{(a_1^2+a_2^2+a_3^2)^2}{4}+\frac{(a_1^2+a_2^2+a_3^2)^2}{4}+a_4^4+\cdots+a_n^4\right].
\end{aligned}
$$

只需证

$$\frac{(a_1^2+a_2^2+a_3^2)^2}{2}\le a_1^4+a_2^4+a_3^4,$$

只需证

$$2a_1^2a_2^2+2a_2^2a_3^2+2a_3^2a_1^2-a_1^4-a_2^4-a_3^4\le 0,$$

只需证

$$(a_1+a_2+a_3)(a_1+a_2-a_3)(a_1+a_3-a_2)(a_2+a_3-a_1)\le 0$$

显然成立.

于是得证.

9. $a,b,c>0,a+b+c=3$. 求证:

$$\sum_{\text{cyc}}\frac{a^2+3b^2}{ab^2(4-ab)}\ge 4.$$

令

$$M=\sum_{\text{cyc}}\frac{a^2}{ab^2(4-ab)}=\sum_{\text{cyc}}\frac{a}{b^2(4-ab)},N=\sum_{\text{cyc}}\frac{3b^2}{ab^2(4-ab)}=3\sum_{\text{cyc}}\frac{1}{a(4-ab)},$$

即证 $M+N\ge 4$.

$$
\begin{aligned}
M: \quad & \left[\sum_{\text{cyc}}\frac{a}{b^2(4-ab)}\right]\left[\sum_{\text{cyc}}\frac{4-ab}{a}\right] \\
\ge \quad & \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2,
\end{aligned}
$$

即

$$M\ge\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{4\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-3}\ge 1.$$

$$
\begin{aligned}
N: \quad & \left[\sum_{\text{cyc}}\frac{3b^2}{ab^2(4-ab)}\right]\left[\sum_{\text{cyc}}\frac{4-ab}{a}\right] \\
\ge \quad & \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2,
\end{aligned}
$$

即

$$N \geq 3.$$

于是有 $M + N \geq 4$, 得证.