From Brachistochrone Curve to Calculus of Variations

How Understanding of Physics Evolve with Mathematics

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History of Mathematics and Physics

Mathematics and Physics are highly inter-correlated (but at the same time fundamentally different).

Their relationship can be split into three stages:

0. Aristotle, Archimedes, ...: No particular relation. Physics was highly experimental and experience-based.
1. Newton: Vector Analysis (and Real Analysis/Calculus).
2. Euler, Lagrange: Functional Analysis (Calculus of Variations).
3. Noether: Symmetry (Group Theory and Modern Algebra). e.g., QFT, CPT, QED, QCD.

Physics at Newton's Time

Newton's work:

- *Philosophiæ Naturalis Principia Mathematica*, 1687, gives rise to Classical Mechanics.
- One of the first Mathematical Formulations of Physics.
- Relies on the geometry of our space (being flat).
- Kepler's Laws, elementary electromagnetism, and even elementary thermal physics can be described using forces and motion.

Restrictions:

- Highly relies on a straight-line motion, which restricts the degree of freedom.
- Uses local linear behaviour to analyse complicated paths.
- This makes thermodynamics highly inaccurate and unable to solve certain questions.

The Brachistochrone Curve Problem

This question is related to the ***Tautochrone Curve Problem*** *(which has the same solution).*

Brachistochrone Curve Problem (Galilei, 1638 and Bernoulli, 1696)

What is the **Brachistochrone Curve**, or the curve of fastest descent, which is the one lying on the plane between a point A and a lower point B (where B is not directly below A) on which a bead slides frictionlessly under the influence of a uniform gravitational field along the curve from A to B in the shortest **time**?

Please guess what the curve could be before we continue.

History of the Problem

- Galilei, 1638: Concluded that the arc of a circle is faster than any number of its chords and concluded incorrectly that the arc is the quickest - but warned of possible fallacies and the need for a higher science.
- Johann Bernoulli and Jakob Bernoulli, 1696: Derived the same conclusion (but Johann was incorrect). They also realised this has the same solution as the Tautochrone Problem (Huygens).
- Newton, 1697: Found the solution because he stayed up for a whole night and posted to Bernoulli.

Abstraction and Formulation

To simplify the problem, we build a coordinate system where the origin is the starting point, the y axis points down, and the x axis points to the right. Also, assume the ending point is (a, b).
If we denote the curve as $y = y(x)$, we must have boundary conditions $y(0) = 0$ and $y(a) = b$.
By conservation of energy, we will have the speed v at a general point (x, y) satisfies that

$$\frac{1}{2}mv^2 = mgy \implies v = \sqrt{2gy}.$$

Also, since $v = \frac{\mathrm{d}s}{\mathrm{d}t}$, and by Pythagerous Theorem we have

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \mathrm{d}x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1 + (y')^2}\mathrm{d}x.$$

This means that

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{1 + (y')^2}\frac{\mathrm{d}x}{\mathrm{d}t},$$

and hence

$$\mathrm{d}t = \sqrt{\frac{1 + (y')^2}{2gy}}\mathrm{d}x.$$

Therefore, the total time taken, T, will satisfy that

$$T = \int \mathrm{d}t = \int_0^a \sqrt{\frac{1 + (y')^2}{2gy}}\mathrm{d}x = \frac{1}{\sqrt{2g}}\int_0^a \sqrt{\frac{1 + (y')^2}{y}}\mathrm{d}x.$$

Our task is to find y that it minimises such T.

Functionals

Definition (Functional)

A function $\mathcal{F}$ is called a functional if it takes in some function as its parameter and outputs a scalar value as its result.

Examples

- $\mathcal{I}[f] = f(0)$ is a functional since it takes in a function f and outputs its value at 0.
- $\mathcal{J}[f] = \int_0^1 f(x)\mathrm{d}x$ is a functional. It outputs the area under the curve $f(x)$ from $x = 0$ to $x = 1$.
- $\mathcal{K}[f] = \frac{\mathrm{d}f}{\mathrm{d}x}$ is not a functional. It does not output a scalar value.

Euler-Lagrange Equation

Theorem (Euler-Lagrange Equation)

A functional $\mathcal{J}$ takes in a path y with the boundary conditions $y(a) = A$ and $y(b) = B$:

$$\mathcal{J}[y] = \int_a^b L(x, y, y')\mathrm{d}x,$$

where L is some expression.
$\mathcal{J}$ takes some extremum if and only if the equation

$$\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial y'} = 0$$

is satisfied.

Example (Shortest Path)

The length of a path $y = y(x)$ between two points (a, A) and (b, B) is

$$s = \int_a^b \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_a^b \sqrt{1 + (y')^2}\mathrm{d}x.$$

The L in the equation is $L(x, y, y') = \sqrt{1 + (y')^2}$.
We will realise that

$$\frac{\partial L}{\partial y} = 0, \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}}.$$

Example (Shortest Path)

$$\frac{\partial L}{\partial y} = 0, \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}.$$

If we substitute this into the equation, we will get that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{y'}{\sqrt{1+(y')^2}} = 0,$$

and therefore

$$\frac{y'}{\sqrt{1+(y')^2}} = \text{const}.$$

This means that

$$y' = \frac{\text{const}}{\sqrt{1-\text{const}^2}} = \text{another const},$$

which means it is a straight line!

Solution of Problem

In the original problem, we would like to minimise

$$T = \frac{1}{\sqrt{2g}} \int_0^a \sqrt{\frac{1 + (y')^2}{y}} \mathrm{d}x.$$

Our L here is

$$L(x, y, y') = \sqrt{\frac{1 + (y')^2}{y}}.$$

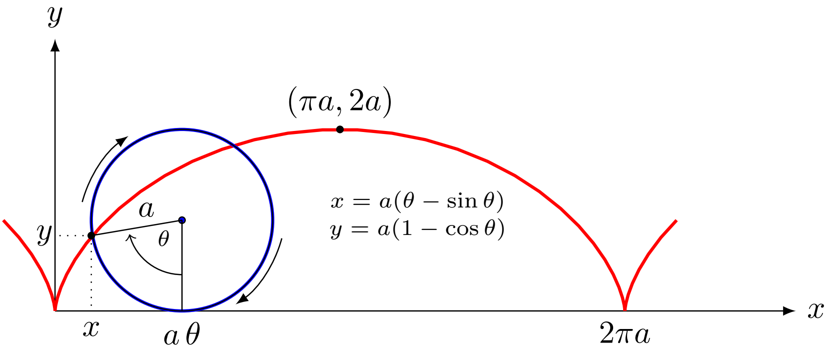
(If you trust my differentiation skills) We will have

$$\frac{\partial L}{\partial y} = -\frac{1}{2}\sqrt{\frac{1 + y'^2}{y^3}}, \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{y(1 + y')^2}},$$

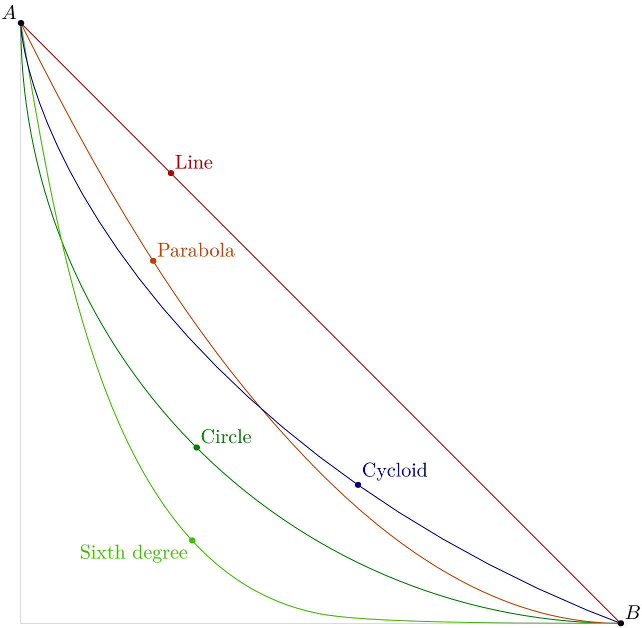
and plugging it back into the equation, mathematicians will tell us the solution is a cycloid.

Visuallisation

A cycloid:



Different curves:



Calculus of Variations

Calculus of Variations is the study of how we can minimise functionals by varying functions.
The idea is similar:

- To minimise $y = f(x)$, we solve for $f'(x) = 0$.
- To minimise $t = f(x, y, z)$, we solve for $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$.
- To minimise $\mathcal{L}[f]$, we solve for the Euler-Lagrange equation, which generalises to Euler-Poisson Equation and others.

The idea of variation of a function is by doing $y(x) = y_0(x) + \epsilon\eta(x)$ where $\eta(x)$ is an arbitary fucntion and $\epsilon \to 0$.

Lagrangian and Hamiltonian Mechanics

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ i\bar{\psi}\not{D}\psi + h.c. \\ &+ \bar{\psi}_i y_{ij}\psi_j\phi + h.c. \\ &+ |D_\mu\phi|^2 - V(\phi)\end{aligned}$$

is the Lagrangian of the standard model.
Doing something similar to (but much more complicated than) the Euler-Lagrange equation and solving it will enable you to explain everything apart from gravity.
Hamilton, 1833 provides an alternative formulation using the Hamiltonian $\mathcal{H}$.
These will be covered next week by Dara Daneshvar - come if you are interested!

Benefits

If we look back at our derivation of the solution:

- we used energy rather than forces,
- we used path rather than points,

and we are considering an action along the path.

- It does not assume any geometry of the space, and does not ignore interior structures of objects, does not use simple linear approximations.
- It converts all problems to finding the extremum of some quantity - and applying variations to such quantity will enable us to form a sophisticated but self-consistent system.

It provides a unified way to solve all kinds of different physical problems.

Applications

- Direct applications in mechanics include the Principle of Least Action and Hamilton's Principle.
- We can also apply variation to maximise entropy - or minimise free energy - in thermodynamics, and they lead to the same result.
- We can also apply variation on the Lagrangian density of an electromagnetic field to deduce Maxwell's Equations (and this even works for QED).
- We can use calculus of variations to deduce Einstein's field equations based on Einstein-Hilbert action in general relativity.

Restrictions of this Mathematical Formulation

Mathematics should not rely on intuition, but Physics should, or at least an intuitive explanation should exist alongside an abstract formulation. So what is

$$L(x, y, y') = \sqrt{\frac{1 + (y')^2}{y}},$$

and what is

$$\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial y'} = 0?$$

What is action, what is entropy, what is free energy?
This is very much less intuitive than forces and speed in Newtonian Mechanics.

Symmetry, Group Theory, and Noether's Theorem

Theorem (Noether's Theorem)

Every continuous symmetry of the action of a physical system with conservative forces has a corresponding conservation law.

Examples

- Energy conserves since physical laws have time symmetry (i.e., physical laws do not change over time).
- Momentum conserves since physical laws have space symmetry (i.e., physical laws do not change under translation).

Under this formulation, the Lagrangian we see can simply take the mathematical structure of a group, specifically the group

$$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$$

This is how Physics in the recent century developed.

The chief forms of beauty are order and symmetry and definiteness, which the mathematical sciences demonstrate in a special degree.

– Aristotle

Solving the Differential Equation

If we recall

$$\frac{\partial L}{\partial y} = -\frac{1}{2}\sqrt{\frac{1+y'^2}{y^3}}, \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{y(1+y')^2}},$$

we can simplify the equation to

$$\frac{1}{2}(1+y'^2) + yy'' = 0.$$

If we multiply $2y'$ on both sides,

$$y'(1+y'^2) + 2yy'y'' = 0 \implies \frac{\mathrm{d}}{\mathrm{d}x}\left[y(1+y'^2)\right] = 0.$$

This means that for some constant k, we have

$$y(1+y'^2) = k \implies y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{k-y}{y}}.$$

If we let $y = k\sin^2\theta = \frac{1}{2}k(1-\cos 2\theta)$, we notice that

$$
\begin{aligned}
x &= \int \sqrt{\frac{k-y}{y}}\mathrm{d}y \\
&= \int \sqrt{\frac{k\sin^2\theta}{k-k\sin^2\theta}}\mathrm{d}(k\sin^2\theta) \\
&= \int \frac{\sin\theta}{\cos\theta} 2ksin\theta\cos\theta\mathrm{d}\theta \\
&= \int 2k\sin^2\theta\mathrm{d}\theta \\
&= k\int(1-\cos 2\theta)\mathrm{d}\theta \\
&= k\theta - \frac{1}{2}k\sin 2\theta.
\end{aligned}
$$

Now we let $a = \frac{1}{2}k, t = 2\theta$, we will get the parametric equation of

$$(x, y) = (a(t-\sin t), a(1-\cos t)).$$