Physics Problem Solving

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Physics Problem Solving Society
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Water Tank

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Question. A tank contains water to a depth of 1.0m. Water emerges from a small hole in the vertical side of the tank at 20cm below the surface. Determine:

1. the speed at which the water emerges from the hole
2. the distance from the base of the tank at which the water strikes the floor on which the tank is standing.

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Question. A tank contains water to a depth of H. Water emerges from a small hole in the vertical side of the tank at h above ground. Determine the envelope formed by the trajectories of the water.

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4. A straight line.
5. A hyperbola.

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Desmos Demo

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$$(x, y) = \left(\sqrt{2g(H-h)}t, h - \frac{gt^2}{2}\right).$$

If we re-arrange the equation to find t in terms of x, we have

$$t = \frac{x}{\sqrt{2g(H-h)}},$$

and plugging this back for y gives us the explicit Cartesian equation

$$y = h - \frac{x^2}{4(H-h)}.$$

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We notice that

$$\begin{aligned} x + y &= -\frac{gt^2}{2} + \sqrt{2g(H-h)}t + h \\ &\leq \frac{-2gh - 2g(H-h)}{-2g} \\ &= H \end{aligned}$$

from basic propertiy of a quadratic curve.

Important Maths

Quadratic Curve. For a quadratic curve with equation

$$y = ax^2 + bx + c, a \neq 0$$

its vertex will be the point with coordinate

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right).$$

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Series Springs

We have learnt in class that the formula for the resultant spring constant k and resultant initial length l of springs with spring constants $k_1, k_2, \ldots, k_n$ and initial lengths $l_1, l_2, \ldots, l_n$ being in series satisfies that

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$$\frac{1}{k} = \sum_{i=1}^{n} \frac{1}{k_i}, l = \sum_{i=1}^{n} l_i.$$

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and therefore the resultant spring constant satisfies

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and therefore

$$k = \frac{F}{x} = \sum_{i=1}^{n} k_i.$$

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Question 2

The pulley system in the figure consists of two pulleys of radii a and b rigidly fixed together, but free to rotate about a common horizontal axis. The weight W hangs from the axle of a freely suspended pulley P, which can rotate about its axle. If section A of a rough rope is pulled down with velocity V:

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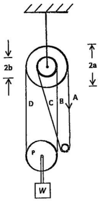

Figure: Pulley System

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Rope D moves upwards with speed V because we assume that the rope is inextensible and ropes A and D are attached.
However, we cannot say that rope B moves downwards with speed V.
Since the two pulleys of radii a and b are rigidly fixed together, they must rotate with the same angular velocity, $\omega_a = \omega_b$.

$$V = \omega_a a \;\Rightarrow\; \omega_a = \frac{V}{a} \quad \text{and} \quad V_B = \omega_b b.$$

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Substituting $\omega_a = \frac{V}{a}$ into the expression for V_B using $\omega_a = \omega_b$,

$$V_B = \omega_b b = \omega_a b = \frac{V}{a} b.$$

Solution

W rises as a result of the difference in speeds of ropes D and B since a greater length of rope D is pulled than length of rope B is pushed in a given time.
Therefore, the centre of P and W rise with speed V_W,

$$V_W = \frac{1}{2}\left(V - \frac{b}{a}V\right)$$

$$V_W = \frac{V}{2}\left(1 - \frac{b}{a}\right)$$

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The factor of a half is required because length of rope on both sides of W must decrease by a length l for W to rise a length W.