Physics Problem Solving

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Water Tank

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Question. A tank contains water to a depth of 1.0m. Water emerges from a small hole in the vertical side of the tank at 20cm below the surface. Determine:

1. the speed at which the water emerges from the hole
2. the distance from the base of the tank at which the water strikes the floor on which the tank is standing.

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We also had a follow-up question on the envelope of the trajectory of the water (i.e. the boundary of the set of points that the water trajectory can hit).

Question. A tank contains water to a depth of H. Water emerges from a small hole in the vertical side of the tank at h above ground. Determine the envelope formed by the trajectories of the water.

Ideas

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4. A straight line.
5. A hyperbola.

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Desmos Demo

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First, it is not difficult to see that the trajectory of the water satisfies the parametric equation

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$$(x, y) = \left(\sqrt{2g(H-h)}t, h - \frac{gt^2}{2}\right).$$

If we re-arrange the equation to find t in terms of x, we have

$$t = \frac{x}{\sqrt{2g(H-h)}},$$

and plugging this back for y gives us the explicit Cartesian equation

$$y = h - \frac{x^2}{4(H-h)}.$$

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Alternative Approach

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We notice that

$$\begin{aligned} x + y &= -\frac{gt^2}{2} + \sqrt{2g(H-h)}t + h \\ &\leq \frac{-2gh - 2g(H-h)}{-2g} \\ &= H \end{aligned}$$

from basic propertiy of a quadratic curve.

Important Maths

Quadratic Curve. For a quadratic curve with equation

$$y = ax^2 + bx + c, a \neq 0$$

its vertex will be the point with coordinate

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right).$$

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Series Springs

We have learnt in class that the formula for the resultant spring constant k and resultant initial length l of springs with spring constants $k_1, k_2, \ldots, k_n$ and initial lengths $l_1, l_2, \ldots, l_n$ being in series satisfies that

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$$\frac{1}{k} = \sum_{i=1}^{n} \frac{1}{k_i}, l = \sum_{i=1}^{n} l_i.$$

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and therefore the resultant spring constant satisfies

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In some time, we will soon see that the red text is completely useless.

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1. what will happen if they have different initial lengths?
2. what will happen if they not only have different initial lengths, but also different spring constants?

Different l

Question. Say if there are springs with spring constant $k_1, k_2, \ldots, k_n$ and with different initial lengths, $l_1, l_2, \ldots, l_n$ connected together with the previously said setup,

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You may consider that the springs are ideally attached at the same point between the wall and the plastic board, so no rotation has to be taken into account.

Solution

Solution. Say that the new spring has a new initial length l. Then the extension in each spring satisfies that

$$x_i = l - l_i,$$

and therefore the force provided by each spring satisfies that

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Since it is the new initial length, l must satisfy that

$$F = \sum_{i=1}^{n} F_i = 0,$$

which means that

$$\sum_{i=1}^{n} k_i l = \sum_{i=1}^{n} k_i l_i,$$

and that l is the weighted arithmetic mean of the l_is.

Solution

Now consider the new spring constant. If the spring is extended by a new Δx, then the extra force provided by each spring will be equal to

$$\Delta F_i = k_i \Delta x,$$

which results in a resultant change in force of

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This gives the same result of the original formula where they have the same initial lengths.

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I think one of the most important reasons is that $F \propto x$, i.e., the linearity of the Hooke's law gives us a lot of nice properties, including the fact that any resultant combination is in fact obeying the Hooke's law.

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which is the key part of our new derivation.
A nice example might be the fact that non-ohmic resistors typically do not observe the series circuit law of resistances (non-linearality).

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Question 2

The pulley system in the figure consists of two pulleys of radii a and b rigidly fixed together, but free to rotate about a common horizontal axis. The weight W hangs from the axle of a freely suspended pulley P, which can rotate about its axle. If section A of a rough rope is pulled down with velocity V:

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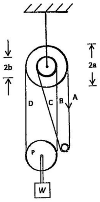

Figure: Pulley System

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Rope D moves upwards with speed V because we assume that the rope is inextensible and ropes A and D are attached.
However, we cannot say that rope B moves downwards with speed V.
Since the two pulleys of radii a and b are rigidly fixed together, they must rotate with the same angular velocity, $\omega_a = \omega_b$.

$$V = \omega_a a \;\Rightarrow\; \omega_a = \frac{V}{a} \quad \text{and} \quad V_B = \omega_b b.$$

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Substituting $\omega_a = \frac{V}{a}$ into the expression for V_B using $\omega_a = \omega_b$,

$$V_B = \omega_b b = \omega_a b = \frac{V}{a} b.$$

Solution

W rises as a result of the difference in speeds of ropes D and B since a greater length of rope D is pulled than length of rope B is pushed in a given time.
Therefore, the centre of P and W rise with speed V_W,

$$V_W = \frac{1}{2}\left(V - \frac{b}{a}V\right)$$

$$V_W = \frac{V}{2}\left(1 - \frac{b}{a}\right)$$

$$V_W = V\left(\frac{a-b}{2a}\right).$$

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The factor of a half is required because length of rope on both sides of W must decrease by a length l for W to rise a length W.