From Brachistochrone Curve to Calculus of Variations

How Understanding of Physics Evolve with Mathematics

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3. Noether: Symmetry (Group Theory and Modern Algebra). e.g., QFT, CPT, QED, QCD.

Physics at Newton's Time

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- This makes thermodynamics highly inaccurate and unable to solve certain questions.

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What is the **Brachistochrone Curve**, or the curve of fastest descent, which is the one lying on the plane between a point A and a lower point B (where B is not directly below A) on which a bead slides frictionlessly under the influence of a uniform gravitational field along the curve from A to B in the shortest **time**?

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Please guess what the curve could be before we continue.

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- Newton, 1697: Found the solution because he stayed up for a whole night and posted to Bernoulli.

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Our task is to find y that it minimises such T.

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Theorem (Euler-Lagrange Equation)

A functional $\mathcal{J}$ takes in a path y with the boundary conditions $y(a) = A$ and $y(b) = B$:

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where L is some expression.
$\mathcal{J}$ takes some extremum if and only if the equation

$$\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial y'} = 0$$

is satisfied.

Example (Shortest Path)

The length of a path $y = y(x)$ between two points (a, A) and (b, B) is

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We will realise that

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which means it is a straight line!

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(If you trust my differentiation skills) We will have

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and plugging it back into the equation, mathematicians will tell us the solution is a cycloid.

Visuallisation

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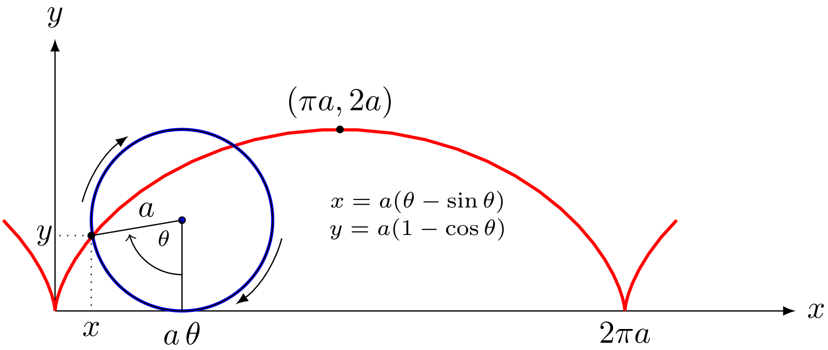

Different curves:

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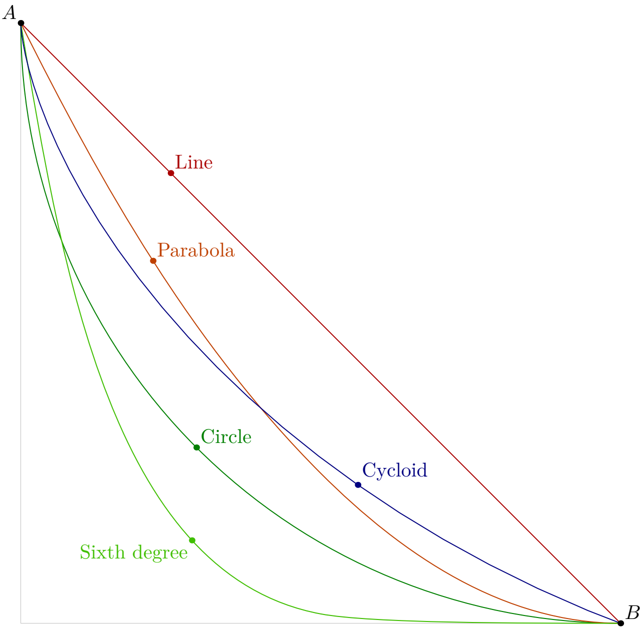

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The idea of variation of a function is by doing $y(x) = y_0(x) + \epsilon\eta(x)$ where $\eta(x)$ is an arbitary fucntion and $\epsilon \to 0$.

Lagrangian and Hamiltonian Mechanics

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&+ i\bar{\psi}\not{D}\psi + h.c. \\
&+ \bar{\psi}_i y_{ij}\psi_j\phi + h.c. \\
&+ |D_\mu\phi|^2 - V(\phi)
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is the Lagrangian of the standard model.

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These will be covered next week by Dara Daneshvar - come if you are interested!

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It provides a unified way to solve all kinds of different physical problems.

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- We can also apply variation to maximise entropy - or minimise free energy - in thermodynamics, and they lead to the same result.
- We can also apply variation on the Lagrangian density of an electromagnetic field to deduce Maxwell's Equations (and this even works for QED).
- We can use calculus of variations to deduce Einstein's field equations based on Einstein-Hilbert action in general relativity.

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What is action, what is entropy, what is free energy?
This is very much less intuitive than forces and speed in Newtonian Mechanics.

Symmetry, Group Theory, and Noether's Theorem

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This is how Physics in the recent century developed.

The chief forms of beauty are order and symmetry and definiteness, which the mathematical sciences demonstrate in a special degree.

– Aristotle

Solving the Differential Equation

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