Intuition Mathematics v.s. Deduction Mathematics

A discussion involving Infinities and Infinitesimals

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St Paul's School

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Calculus of Infinitesimals

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Definition (The $\epsilon - \delta$ definition of a limit)

The limit of $f(x)$ as x tends to c is A, or

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if and only if, for all $\epsilon > 0$, there exists a $\delta > 0$, such that for all $0 < |x - c| < \delta$, we have $|f(x) - A| < \epsilon$.

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$$\epsilon\omega = 1 \iff \frac{1}{\omega} = \epsilon$$

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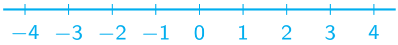

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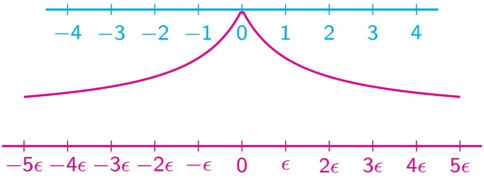

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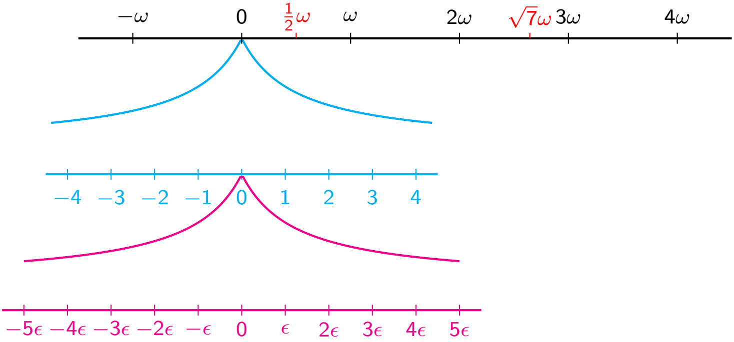

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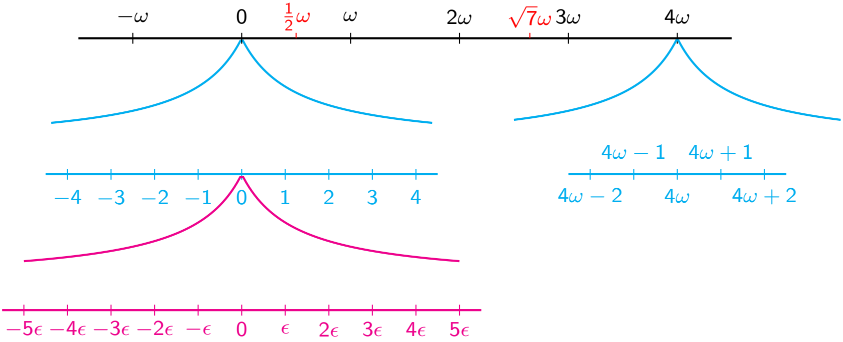

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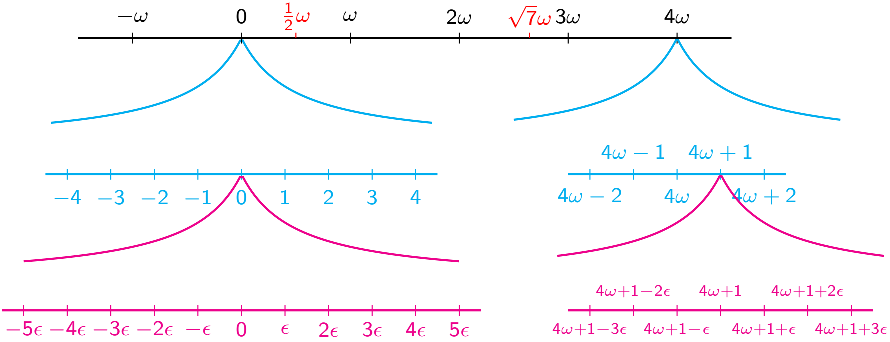

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2. $\text{st}(3 - \epsilon + \epsilon^4) = 3.$

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Here, h is just an infinitesimal, and so we can define the derivative using hyperreals as follows:

Definition (Derivative from Hyperreals)

$$f'(x) = \mathrm{st}\left(\frac{f(x+\epsilon) - f(x)}{\epsilon}\right).$$

Example

Let us use this to find the derivative of $f(x) = x^n$.

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This is consistent with what we learned in class.

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Definition (Limits)

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$$\lim_{x \to +\infty} f(x) = \mathrm{st}(f(\omega)), \lim_{x \to -\infty} f(x) = \mathrm{st}(f(-\omega)).$$

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It is soon apparent that this and the First Principle give the exact definition of the derivative.

If we recall the definition of e:

Definition (e)

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \mathrm{st}\left(\left(1+\frac{1}{\omega}\right)^{\omega}\right) = \mathrm{st}\left((1+\epsilon)^{\omega}\right),$$

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Do improper integrals still work? (How to integrate to ω? Or 2ω?)
Why was it okay for us to assume that we can introduce new numbers that fits well into ordering of real numbers $\mathbb{R}$? (Complex Numbers!)
Why was it okay for us to say $1 + \epsilon - 1 = \epsilon + 1 - 1 = \epsilon$? (Matrix Multiplication!)
Will special properties on real numbers $\mathbb{R}$, such as the intermediate value theorem, still hold on hyperreals ${}^*\mathbb{R}$? (Rationals!)

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How can we just take this for granted?

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