# Intuition Mathematics v.s. Deduction Mathematics

## A discussion involving Infinities and Infinitesimals

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# Calculus of Infinitesimals

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They argue that the derivative of $f(x) = x^2$ is $f'(x) = 2x$, since

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The limit of $f(x)$ as $x$ tends to $c$ is A, or

$$\lim_{x \to c} f(x) = A,$$

if and only if, for all $\epsilon > 0$, there exists a $\delta > 0$, such that for all $0 < |x - c| < \delta$, we have $|f(x) - A| < \epsilon$.

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$$\epsilon\omega = 1 \iff \frac{1}{\omega} = \epsilon$$

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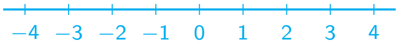

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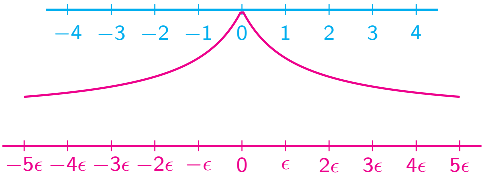

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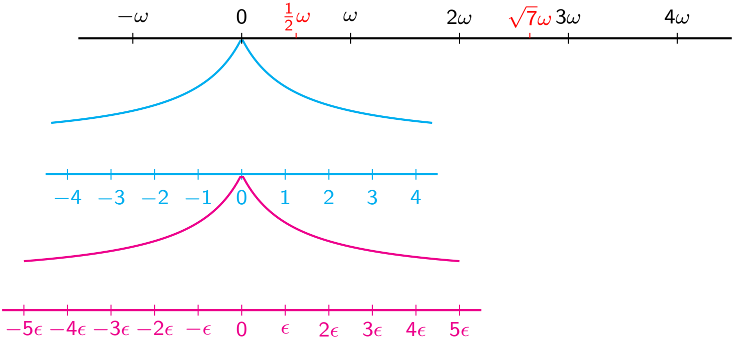

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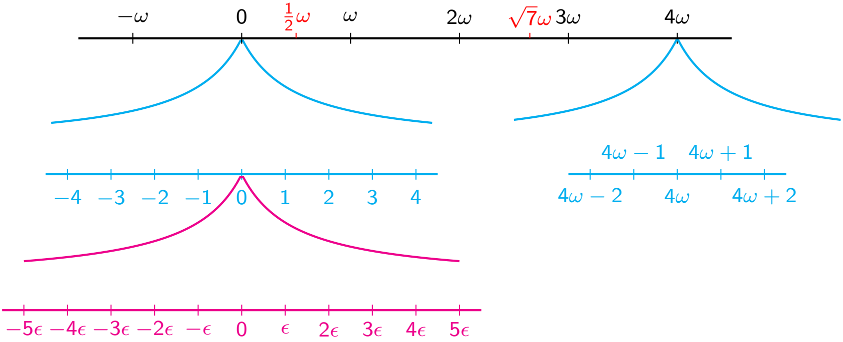

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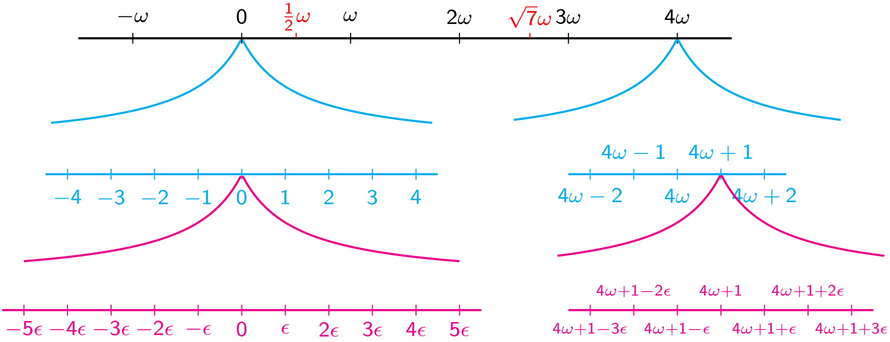

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2. $\mathrm{st}(3 - \epsilon + \epsilon^4) = 3$.

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Here, $h$ is just an infinitesimal, and so we can define the derivative using hyperreals as follows:

**Definition (Derivative from Hyperreals)**

$$f'(x) = \mathrm{st}\left(\frac{f(x+\epsilon) - f(x)}{\epsilon}\right).$$

## Example

Let us use this to find the derivative of $f(x) = x^n$.

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where $\Box$ is something that is not infinity.
So $\Box\epsilon$ must be an infinitesimal, and the standard part of this must be $nx^{n-1}$.
This is consistent with what we learned in class.

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It is soon apparent that this and the First Principle give the exact definition of the derivative.

If we recall the definition of $e$:

**Definition ($e$)**

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \operatorname{st}\left(\left(1+\frac{1}{\omega}\right)^\omega\right) = \operatorname{st}\left((1+\epsilon)^\omega\right),$$

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# How can we just take this for granted?

# References


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