Calculus without Limits

The Formation of Nonstandard Analysis

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History of Infinitesimals

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Theorem (Archimedes Property of the Real Numbers $\mathbb{R}$)

For any two real numbers $x, y \in \mathbb{R}$, $x, y > 0$, there exists a natural number $n \in \mathbb{N}$, s.t.

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In other words, for any two positive real numbers, some multiple of one must be greater than the other.

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Essentially, no infinities or infinitesimals exist in the real number set $\mathbb{R}$.
In other words, y is an infinity if such $n \in \mathbb{N}$ and $x \in \mathbb{R}$ does not exist, and infinitesimals can be defined similarly.

- c. 1660, Newton, Leibnitz: Formulation of Non-rigorous Calculus with Infinitesimals.

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Example

As an example, they argue that the derivative of $f(x) = x^2$ is $f'(x) = 2x$, since

$$\frac{f(x+o) - f(x)}{o} = \frac{(x+o)^2 - x^2}{o} = \frac{2xo + o^2}{o}.$$

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We can cancel out the o on the top and bottom since it is infinitesimal (so non-zero), which gives $2x + o$.
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It is inconsistent (and caused the second mathematical crisis) since sometimes $o \neq 0$ and sometimes $o = 0$.

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Definition (The $\epsilon - \delta$ definition of a limit)

The limit of $f(x)$ as x tends to c is A, or

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if and only if, for all $\epsilon > 0$, there exists a $\delta > 0$, such that for all $0 < |x - c| < \delta$, we have $|f(x) - A| < \epsilon$.

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It is very formal and rigorous but less intuitive to understand.

- 1934, Skolem: First development of non-standard analysis.

- 1934, Skolem: First development of non-standard analysis.
- 1948, E. Hewitt and 1955, J. Łoś: Formation of non-standard analysis.

- 1934, Skolem: First development of non-standard analysis.
- 1948, E. Hewitt and 1955, J. Łoś: Formation of non-standard analysis.
- 1961, A. Robinson: A mathematical (rigorous) interpretation of continuity and infinitesimals (hyperreals) based on work by Hewitt and Łoś.

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Transfer Principle

To develop a number system that includes infinities and infinitesimals and to make it useful (and intuitive) at the same time, we really would like as many properties to preserve from $\mathbb{R}$ to this number set, **hyperreal numbers** $^*\mathbb{R}$ as possible.

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Remark

Statements on sets and functions might not hold since they might depend on specific properties of real numbers.

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We define ω as an infinity. $\omega > x$ for all $x \in \mathbb{R}$.
We define ϵ as an infinitesimal: $\epsilon = \frac{1}{\omega}$. For all $x \in \mathbb{R}_+$, $0 < \epsilon < x$.

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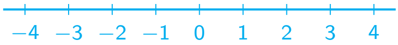
   This also holds for hyperreal numbers $x, y, z \in {}^*\mathbb{R}$ satisfying $y, z \neq 0$. We will have, for example,

$$\frac{1}{1+\epsilon} = \frac{1-\epsilon}{1-\epsilon^2}.$$

Visuallization of Hyperreals



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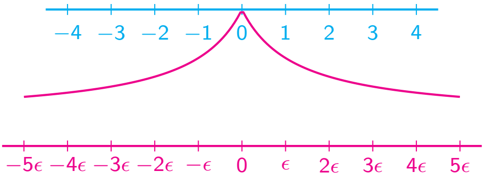

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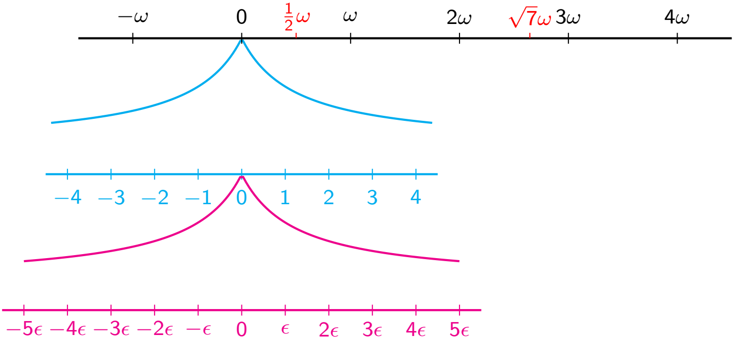

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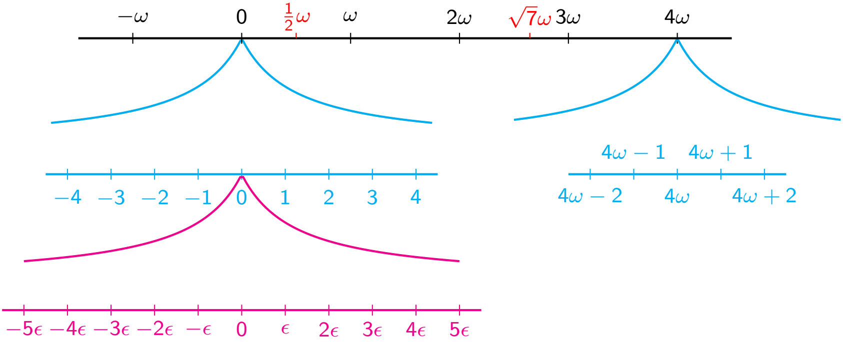

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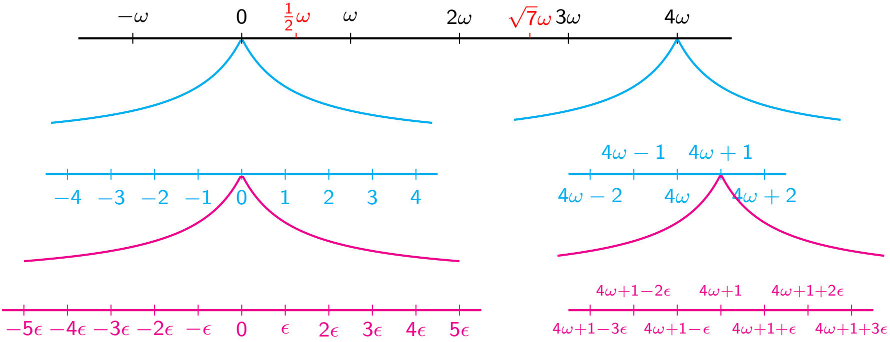

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1. $\mathrm{st}(2 + \epsilon) = 2,$
2. $\mathrm{st}(3 - \epsilon + \epsilon^4) = 3.$

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Derivative

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Here, h is just an infinitesimal, and so we can define the derivative using hyperreals as follows:

Definition (Derivative from Hyperreals)

$$f'(x) = \mathrm{st}\left(\frac{f(x+\epsilon) - f(x)}{\epsilon}\right).$$

Example

Let us use this to find the derivative of $f(x) = x^n$.

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where $\square$ is something that is not infinity.
So $\square\epsilon$ must be an infinitesimal, and the standard part of this must be nx^{n-1}.
This is consistent with what we learned in class.

Limits

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Definition (Limits)

We define

$$\lim_{x \to +\infty} f(x) = \mathrm{st}(f(\omega)), \lim_{x \to -\infty} f(x) = \mathrm{st}(f(-\omega)).$$

and

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It is soon apparent that this and the First Principle give the exact definition of the derivative.

If we recall the definition of e:

Definition (e)

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x} = \operatorname{st}\left(\left(1 + \frac{1}{\omega}\right)^{\omega}\right) = \operatorname{st}\left((1 + \epsilon)^{\omega}\right),$$

we try to find the derivative of $g(x) = e^{x}$.

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It is famous since it is not continuous at any point (and the limit of this function at any point does not exist).

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Remark

There is a way in which hyperreal numbers can be formalized, using hypernatural and hyperrational numbers. However, at this point, we might as well just go back to the $\epsilon - \delta$ definition of the limit.

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