Calculus without Limits
The Formation of Nonstandard Analysis

Eason Shao
St Paul's School

History of Infinitesimals

- c. 250 B.C., Archimedes: Infinities and infinitesimals.

Theorem (Archimedes Property of the Real Numbers $\mathbb{R}$)**.** *For any two real numbers $x, y \in \mathbb{R}$, $x, y > 0$, there exists a natural number $n \in \mathbb{N}$, such that $nx > y$.*
In other words, for any two positive real numbers, some multiple of one must be greater than the other.

Essentially, no infinities or infinitesimals exist in the real number set $\mathbb{R}$.
In other words, y is an infinity if any multiple of a real number $x \in \mathbb{R}$ is less than y. Infinitesimals can be defined similarly.

- c. 1660, Newton, Leibnitz: Formulation of Non-rigorous Calculus with Infinitesimals.

Example. They argue that the derivative of $f(x) = x^2$ is $f'(x) = 2x$, since

$$\frac{f(x+o) - f(x)}{o} = \frac{(x+o)^2 - x^2}{o} = \frac{2x\cancel{o} + o^{\cancel{2}}}{\cancel{o}} = 2x + o = 2x.$$

We can cancel out the o on the top and bottom since it is non-zero.
But since o is infinitesimal, it is 0, which gives $2x$.
It is inconsistent (and caused the second mathematical crisis) since sometimes $o \neq 0$ and sometimes $o = 0$.

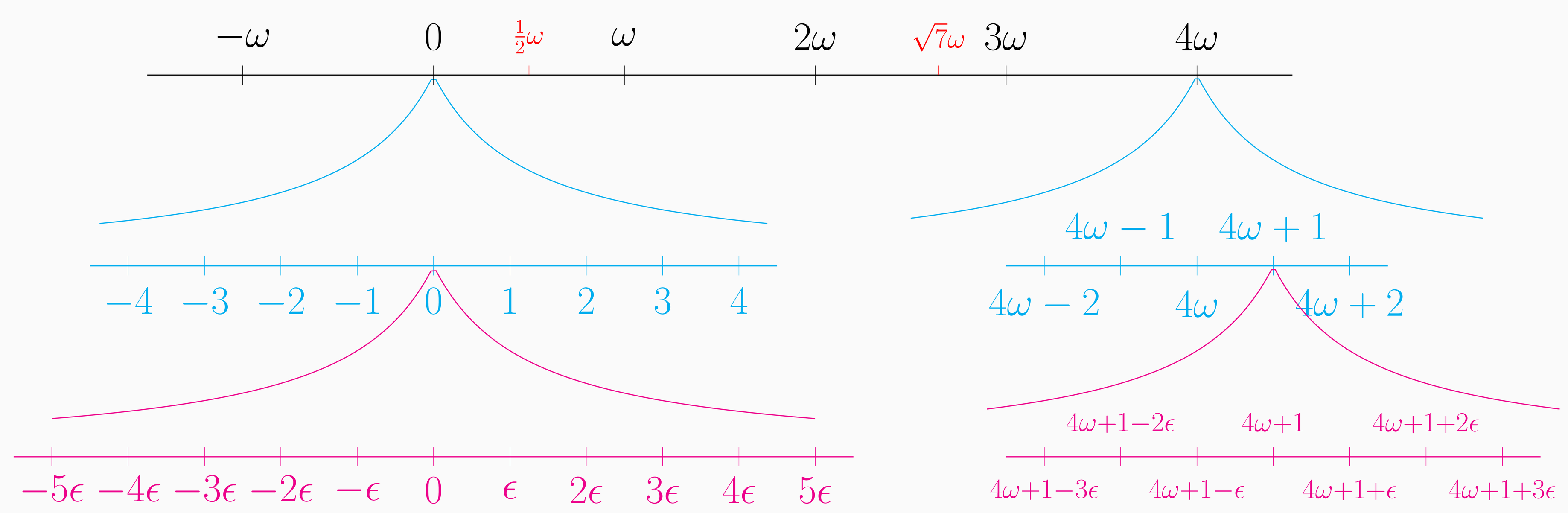
- c. 1820, Cauchy, Weierstrass: Formal definition of a limit ($\epsilon - \delta$).

Definition (The $\epsilon - \delta$ definition of a limit)**.** The limit of $f(x)$ as x tends to c is A, or $\lim_{x \to c} f(x) = A$, if and only if, for all $\epsilon > 0$, there exists a $\delta > 0$, such that for all $0 < |x - c| < \delta$, we have $|f(x) - A| < \epsilon$.

It is very formal and rigorous but less intuitive to understand.

- 1934, Skolem: First development of non-standard analysis.
- 1948, E. Hewitt and 1955, J. Los: Formation of non-standard analysis.
- 1961, A. Robinson: A mathematical (rigorous) interpretation of continuity and infinitesimals (hyperreals) based on work by Hewitt and Los.

Visuallization of Hyperreals



Transfer Principle

To develop a number system that includes infinities and infinitesimals and to make it useful (and intuitive) at the same time, we really would like as many properties to preserve from $\mathbb{R}$ to this number set, **hyperreal numbers** ${}^*\mathbb{R}$ as possible.

Theorem (Transfer Principle from $\mathbb{R}$ to ${}^*\mathbb{R}$)**.** *Any statement of the form "for any number x ..." that is true for the reals $x \in \mathbb{R}$ is also true for the hyperreals $x \in {}^*\mathbb{R}$.*

Statements on sets and functions might not hold since they might depend on specific properties of real numbers.

Hyperreal Numbers

Luckily, such formation does indeed exist!

Definition (Hyperreal Numbers)**.** The **hyperreal numbers** ${}^*\mathbb{R}$ are defined to be the real numbers $\mathbb{R}$, together with:

- **Infinitesimals** which have (absolute value) smaller than any real number.
- **Infinities** which have (absolute value) larger than any real number.

We define ω as an infinity, and ϵ as an infinitesimal:

$$\epsilon = \frac{1}{\omega}, \epsilon\omega = 1.$$

Example. 1. For any real number $x, y \in \mathbb{R}$, we have $x + y = y + x$. (Commutativitiy of Addition over $\mathbb{R}$).
This also holds for hyperreal numbers $x, y \in {}^*\mathbb{R}$. We will have $\omega + 1 = 1 + \omega$, and things like $\epsilon^2 + \omega = \omega + \epsilon^2$.

2. For any real number $x, y, z \in \mathbb{R}$, $y, z \neq 0$, we have $\frac{x}{y} = \frac{xz}{yz}$.
This also holds for hyperreal numbers $x, y, z \in {}^*\mathbb{R}$ satisfying $y, z \neq 0$. We will have, for example,

$$\frac{1}{1+\epsilon} = \frac{1-\epsilon}{1-\epsilon^2}.$$

Standard Part

Definition (Standard Part)**.** If some hyperreal number $x \in {}^*\mathbb{R}$ and is finite, then we define the function st, as

$$\mathrm{st}(x) = \text{the closest real number to } x.$$

Example. $\mathrm{st}(2 + \epsilon) = 2$, $\mathrm{st}(3 - \epsilon + \epsilon^4) = 3$.

Derivative

For a function f, we defined its derivative using First Principles:

Definition (First Principles)**.**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Here, h is just an infinitesimal, and so we can define the derivative using hyperreals:

Definition (Derivative from Hyperreals)**.**

$$f'(x) = \mathrm{st}\left(\frac{f(x+\epsilon) - f(x)}{\epsilon}\right).$$

Example. If we set $f(x) = x^n$, we try and find f' using this.

$$\begin{aligned} f'(x) &= \mathrm{st}\left(\frac{(x+\epsilon)^n - x^n}{\epsilon}\right) \\ &= \mathrm{st}\left(\frac{\left(x^n + n\epsilon x^{n-1} + \frac{n(n-1)}{2}\epsilon^2 x^{n-2} + \dots\right) - x^n}{\epsilon}\right) \\ &= \mathrm{st}\left(\frac{n\epsilon x^{n-1} + \frac{n(n-1)}{2}\epsilon^2 x^{n-2} + \dots}{\epsilon}\right) = \mathrm{st}\left(nx^{n-1} + \square\epsilon\right), \end{aligned}$$

where $\square$ is something that is not infinity.
So $\square\epsilon$ must be an infinitesimal, and therefore $f'(x) = nx^{n-1}$.

Limits

Definition (Limits)**.** We define

$$\lim_{x \to +\infty} f(x) = \mathrm{st}(f(\omega)), \lim_{x \to -\infty} f(x) = \mathrm{st}(f(-\omega)).$$

and

$$\lim_{x \to a^+} f(x) = \mathrm{st}(f(a+\epsilon)), \lim_{x \to a^-} f(x) = \mathrm{st}(f(a-\epsilon)).$$

It is soon apparent that this and the First Principle give the exact definition of the derivative. If we recall the definition of e:

Definition (e)**.**

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \mathrm{st}\left(\left(1 + \frac{1}{\omega}\right)^\omega\right) = \mathrm{st}\left((1+\epsilon)^\omega\right),$$

we try to find the derivative of $g(x) = e^x$.

Example.

$$\begin{aligned} g'(x) &= \mathrm{st}\left(\frac{e^{x+\epsilon} - e^x}{\epsilon}\right) = e^x \,\mathrm{st}\left(\frac{e^\epsilon - 1}{\epsilon}\right) \\ &= e^x \,\mathrm{st}\left(\frac{(1+\epsilon)^{\omega\epsilon} - 1}{\epsilon}\right) \\ &= e^x \,\mathrm{st}\left(\frac{1 + \epsilon - 1}{\epsilon}\right) \\ &= e^x \,\mathrm{st}(1) = e^x. \end{aligned}$$

Limitation

There is a famous function called the **Dirichlet Function**, which is not continuous at any point.

Definition (Dirichlet Fucntion)**.** The **Dirichlet Function** $D : \mathbb{R} \to \{0, 1\}$ is defined as

$$D(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

It is difficult to formalize this limit using hyperreals intuitively since we did not classify hyperreals as rational and irrational - things like $D(\epsilon)$ are not defined.
If we use the standard part of x to define $D(x)$, i.e., ${}^*D(x) = D(\mathrm{st}(x))$, then this will give ridiculous results such as D is continuous at every point, which does not make sense.

There is a way in which hyperreal numbers can be formalized, using hypernatural and hyperrational numbers. However, at this point, we might as well just go back to the $\epsilon - \delta$ definition of the limit.