Interview Questions

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Contents

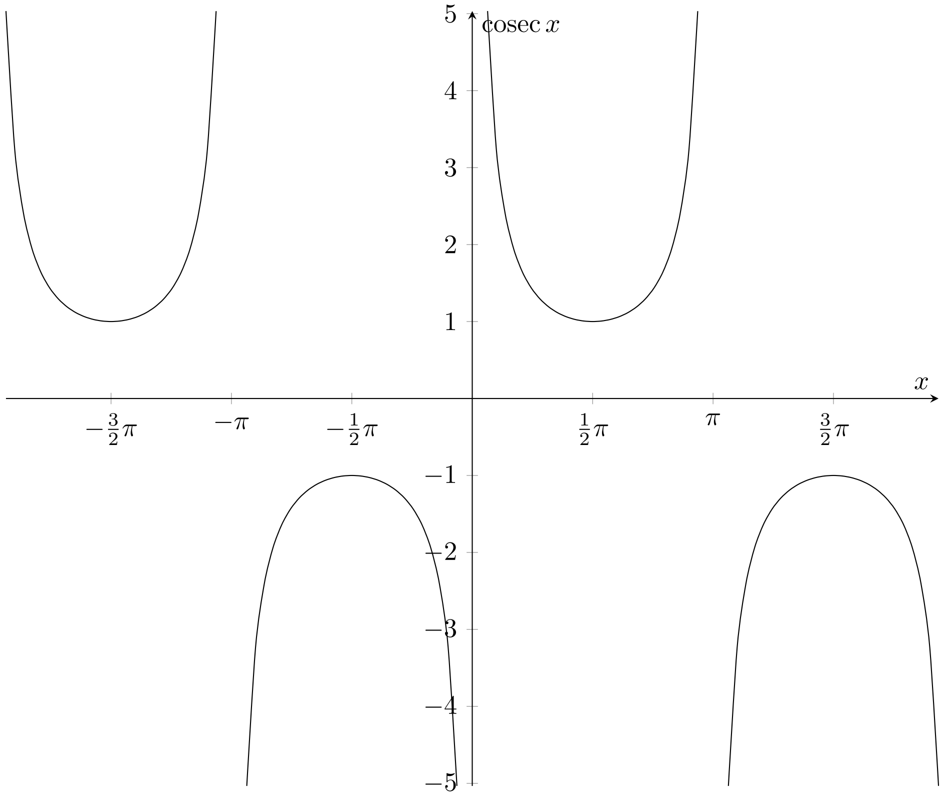
1 Graph Sketching

1.1 Explicit Functions

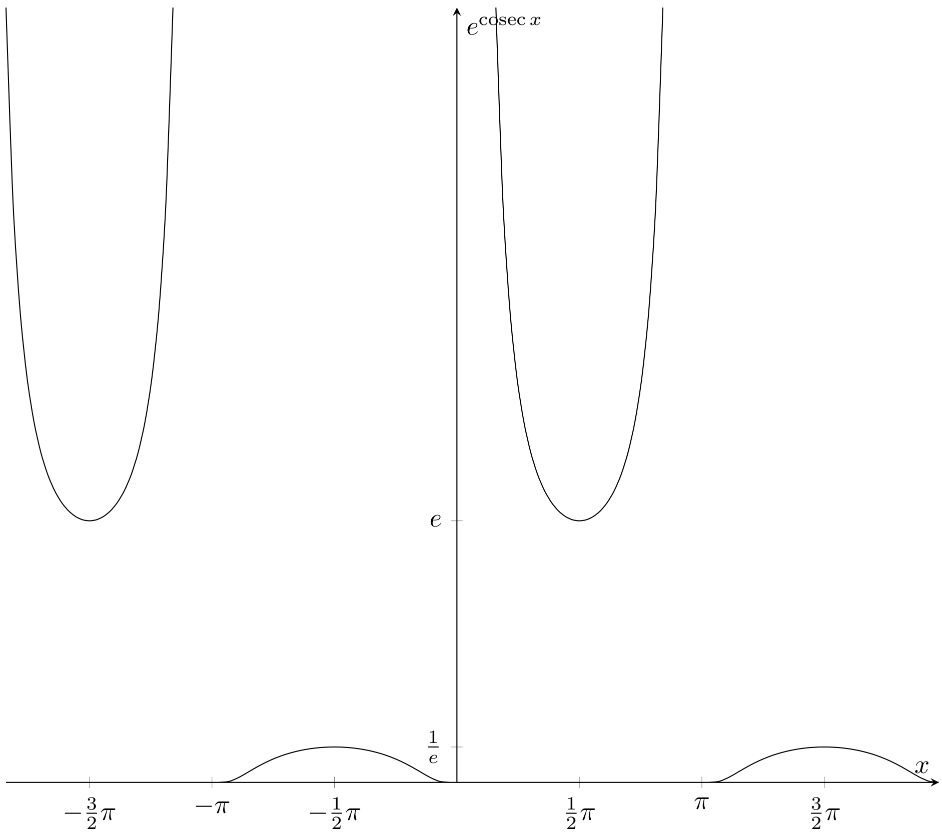
Problem 1.1

Sketch $e^{\operatorname{cosec} x}$.

Solution. It would be helpful to sketch the graph of $\operatorname{cosec} x$, the inner function.



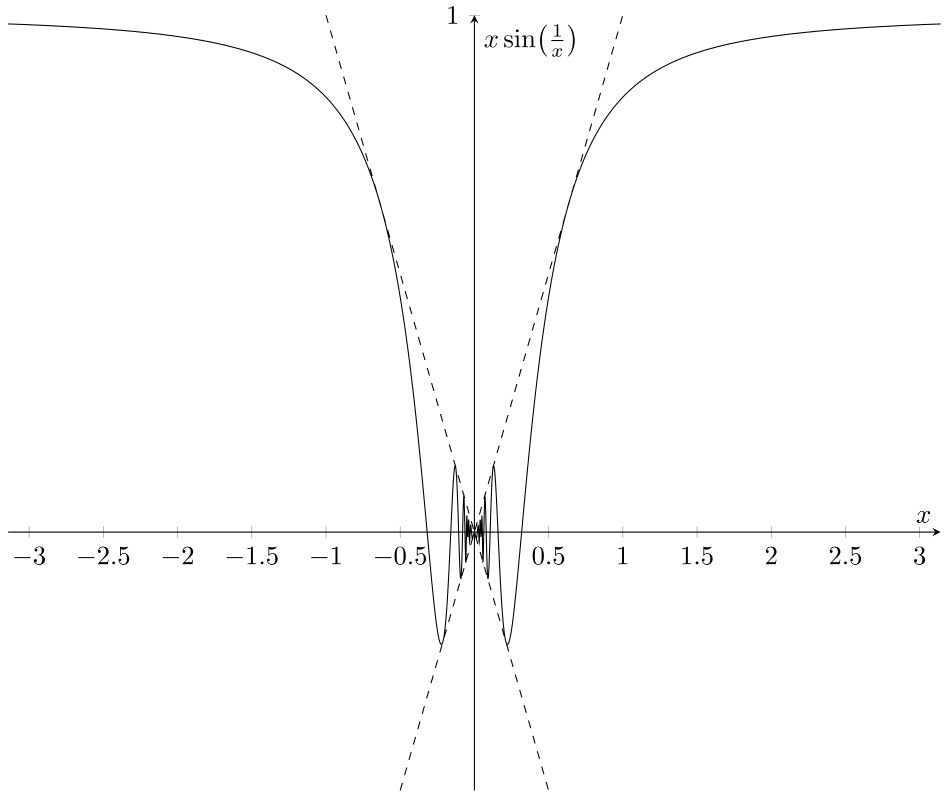
Hence, the graph of $e^{\operatorname{cosec} x}$ is as follows.



Problem 1.2

Sketch $x\sin\left(\frac{1}{x}\right)$ and $x\cos\left(\frac{1}{x}\right)$.
Explain the behaviour as $x \to 0$ and $x \to \infty$.

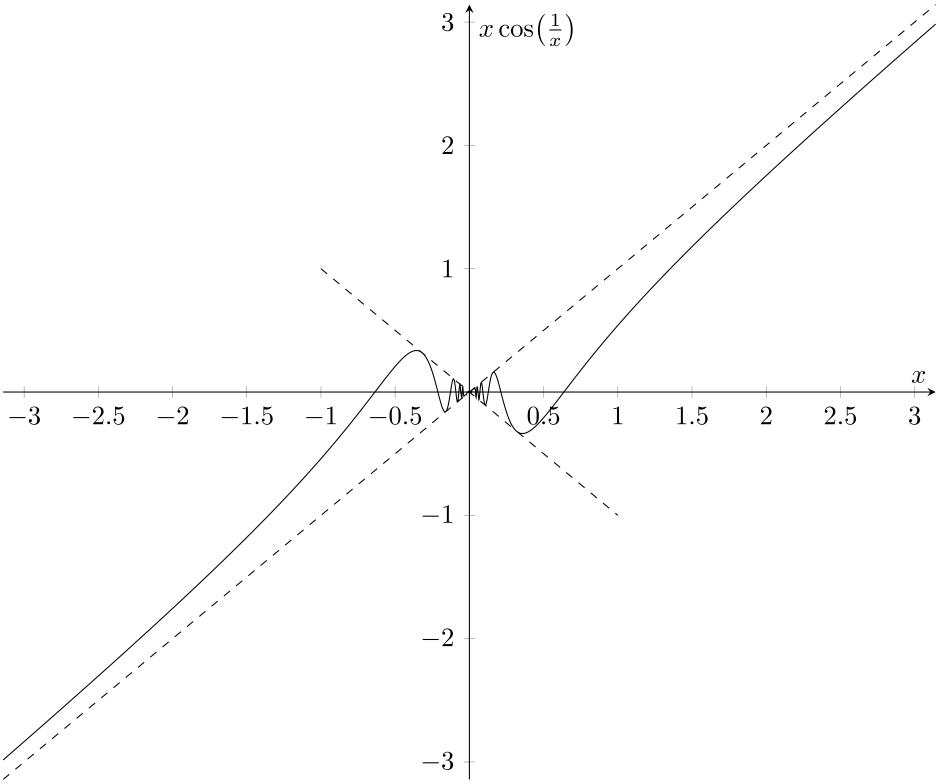
Solution. The sketch of $y = x\sin\left(\frac{1}{x}\right)$ is as follows. There are envelopes of $y = \pm x$ and the coordinates of the intersections can be found.



As $x \to 0$, $\sin\left(\frac{1}{x}\right)$ is bounded (between -1 and 1), and hence $x\sin\left(\frac{1}{x}\right) \to 0$.

As $x \to \infty$, $\frac{1}{x} \to 0$, so $\sin\left(\frac{1}{x}\right) \to \frac{1}{x}$, and hence $x\sin\left(\frac{1}{x}\right) \to 1$.

The sketch of $y = x\cos\left(\frac{1}{x}\right)$ is as follows. There are envelopes of $y = \pm x$ and the coordinates of the intersections can be found.



As $x \to 0$, by similar argument, $x\cos\left(\frac{1}{x}\right) \to 0$.

As $x \to \infty$, $\frac{1}{x} \to 0$, so $\cos\left(\frac{1}{x}\right) \to 1$, and hence $x\cos\left(\frac{1}{x}\right) \to x$.

Problem 1.3

For $n \in \mathbb{N}$,

$$f_n(x) = ||1 - x^n| - |1 + x^n||$$

Sketch $f_1(x)$ and $f_2(x)$. What is

$$f(x) = \lim_{n\to\infty} ||1 - x^n| - |1 + x^n|| = \lim_{n\to\infty} f_n(x)?$$

Explain the behaviour near $x = 1$.
Express f in terms of the Heaviside Step Function, $H(x)$, where

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

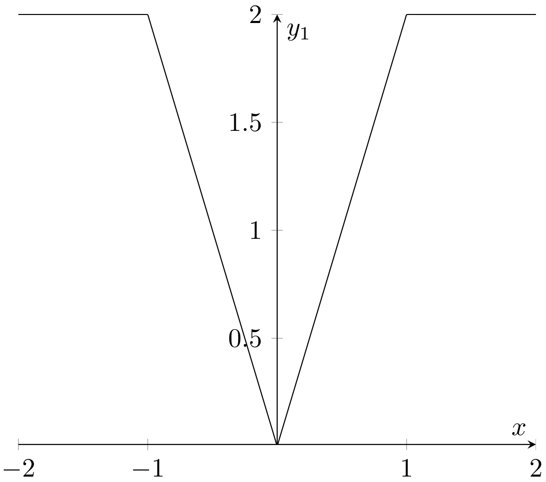
satisfying $y = 0$ when $x = 0$, and y is continuous.
Find f'.

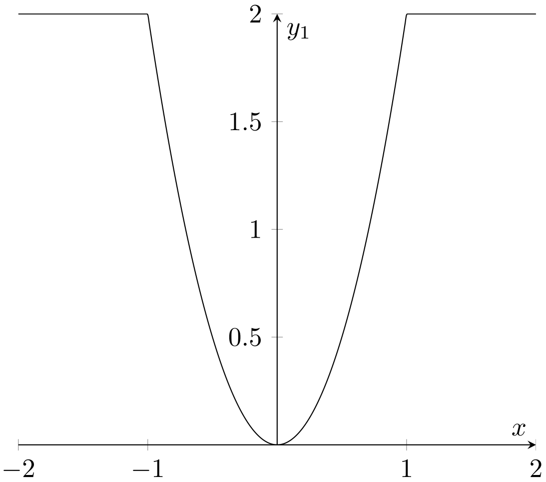
Solution. When $n = 1$,

$$y_1 = ||1 - x| - |1 + x||,$$

and when $n = 2$,

$$y_2 = \left|\left|1 - x^2\right| - \left|1 + x^2\right|\right|,$$





We have

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 2, & x \leq -1 \text{ or } x \geq 1, \\ 0, & -1 < x < 1. \end{cases}$$

Note here we cannot put the limit in for x^n which would diverge for $|x| > 1$. The behaviour of this limit is what is called a 'pointwise convergence' rather than a 'uniform convergence' for $|x| < 1$.

We have

$$f(x) = 2[H(x-1) + H(-x-1)],$$

and hence

$$f'(x) = 2[\delta(x-1) - \delta(-x-1)],$$

where δ is the Dirac Delta Function.

The differential equation solves to

$$F(x) = 2[R(x-1) - R(-x-1)] = \begin{cases} 2(x-1), & x \leq 1, \\ 0, & -1 < x < 1, \\ 2(x+1), & x \leq -1, \end{cases}$$

where R is the ramp function,

$$R(x) = \begin{cases} x, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

1.2 Implicit Functions

1.3 Polar Coordinates

1.4 Graph Transformation

2 Analysis

2.1 Series

2.2 Differentiation

Problem 2.1

> Differentiate x^x and x^{x^x}. Differentiate $f(x)^{g(x)}$.

Solution. The issue with this question is it is not of the form of a standard elementary function, i.e. not monomial, exponential, logarithmic, trigonometric and inverse trigonometric. We might prefer to re-write this in terms of those.

Note $x^x = e^{x \ln x}$. Hence, we have

$$\begin{aligned}\frac{\mathrm{d}x^x}{\mathrm{d}x} &= \frac{\mathrm{d}e^{x \ln x}}{\mathrm{d}x} \\ &= e^{x \ln x} \cdot \frac{\mathrm{d}x \ln x}{\mathrm{d}x} \\ &= x^x(\ln x + 1).\end{aligned}$$

Note that exponentials are right-associative, so

$$x^{x^x} = x^{(x^x)}.$$

Similarly, note $x^{x^x} = e^{x^x \ln x}$, and hence

$$\begin{aligned}\frac{\mathrm{d}x^{x^x}}{\mathrm{d}x} &= \frac{\mathrm{d}e^{x^x \ln x}}{\mathrm{d}x} \\ &= e^{x^x \ln x} \cdot \frac{\mathrm{d}x^x \ln x}{\mathrm{d}x} \\ &= x^{x^x}\left[x^x \cdot \frac{1}{x} + x^x(\ln x + 1)\ln x\right] \\ &= x^{x^x + x - 1}\left[x(\ln x)^2 + x \ln x + 1\right].\end{aligned}$$

In general, we have $f^g = \exp(g \ln f)$, and hence

$$\begin{aligned}\frac{\mathrm{d}f^g}{\mathrm{d}x} &= \frac{\mathrm{d}\exp(g \ln f)}{\mathrm{d}x} \\ &= \exp(g \ln f) \cdot \frac{\mathrm{d}g \ln f}{\mathrm{d}x} \\ &= f^g\left(g' \ln f + \frac{g}{f} f'\right).\end{aligned}$$

Problem 2.2

> Differentiate $\sin x$ with respect to $\cos x$.

Solution. This problem is best done using the chain rule.

$$\begin{aligned}\frac{\mathrm{d}\sin x}{\mathrm{d}\cos x} &= \frac{\mathrm{d}\sin x}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}\cos x} \\ &= \frac{\mathrm{d}\sin x}{\mathrm{d}x} \cdot \left(\frac{\mathrm{d}\cos x}{\mathrm{d}x}\right)^{-1} \\ &= -\frac{\cos x}{\sin x} \\ &= -\cot x.\end{aligned}$$

Alternative, let $u = \cos x$, and $\sin x = \pm\sqrt{1-u^2}$. Hence,

$$\begin{aligned}\frac{\mathrm{d}\sin x}{\mathrm{d}\cos x} &= \pm\frac{\mathrm{d}\sqrt{1-u^2}}{\mathrm{d}u} \\ &= \pm\frac{1}{2}\cdot(-2u)\cdot\frac{1}{\sqrt{1-u^2}} \\ &= -\frac{u}{\pm\sqrt{1-u^2}} \\ &= -\frac{\cos x}{\sin x} \\ &= -\cot x.\end{aligned}$$

2.3 Integration

2.4 Differential Equation

Problem 2.3

Solve the differential equation

$$y' - y = e^{ux}$$

for $u \neq 1$, and show it can be written in terms of

$$y = Ae^x + \frac{e^{ux} - e^x}{u-1}.$$

Using this, and taking the limit as $u \to 1$, solve the differential equation when $u = 1$.

Solution. The original differential equation can be solved using integrating factor of e^{-x}, and hence

$$\begin{aligned}y &= \left[\frac{e^{(u-1)x}}{u-1} + C\right]\cdot e^x \\ &= \left[\frac{e^{(u-1)x}-1}{u-1} + \left(C + \frac{1}{u-1}\right)\right]\cdot e^x \\ &= \frac{e^{ux}-e^x}{u-1} + Ae^x\end{aligned}$$

for

$$A = C + \frac{1}{u-1}.$$

As $u \to 1$, by L'Hôpital's Rule, we have

$$\begin{aligned}\lim_{u\to1}\frac{e^{ux}-e^x}{u-1} &= \lim_{u\to1}\frac{xe^x}{1} \\ &= xe^x,\end{aligned}$$

and hence the general solution becomes

$$y = xe^x + Ae^x.$$

Problem 2.4

Find all solutions of the equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x} - x = 0$$

and give a sketch showing the solutions.

By considering a suitable substitution, sketch

$$(\ln u)^2 - 2x\ln u = C,$$

drawing first the lines to which $y = \pm x$ are mapped.

Show that this is the solution to

$$(\ln u - x)\frac{\mathrm{d}u}{\mathrm{d}x} - u \ln u = 0.$$

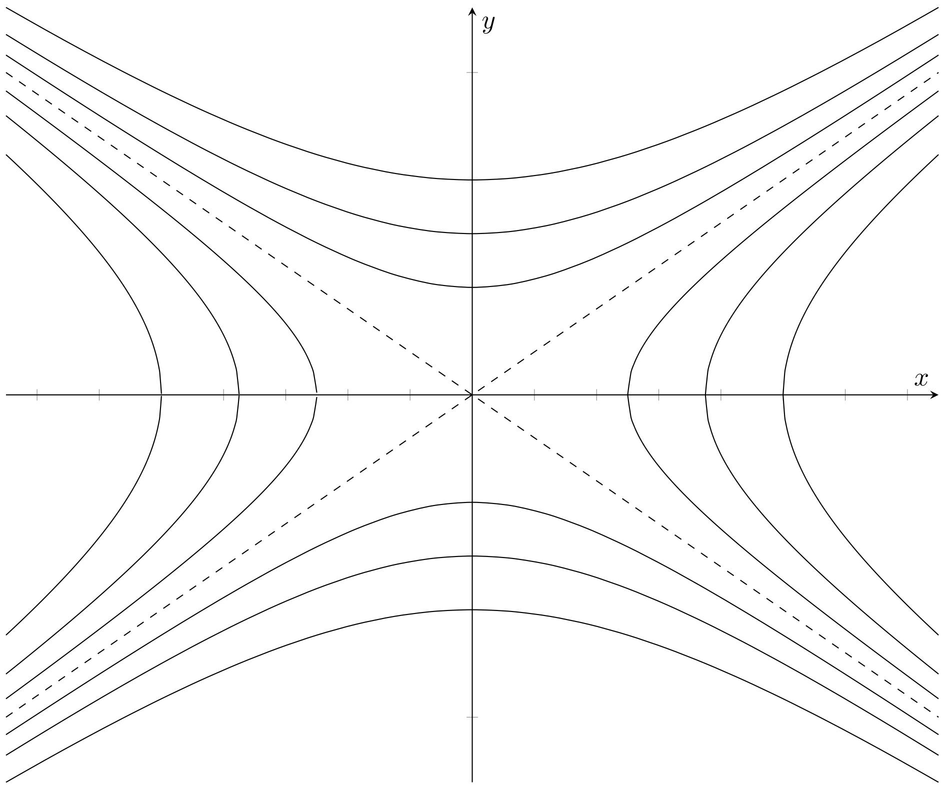
Solution. By separation of variables, the differential equation solves to

$$y^2 - x^2 = C$$

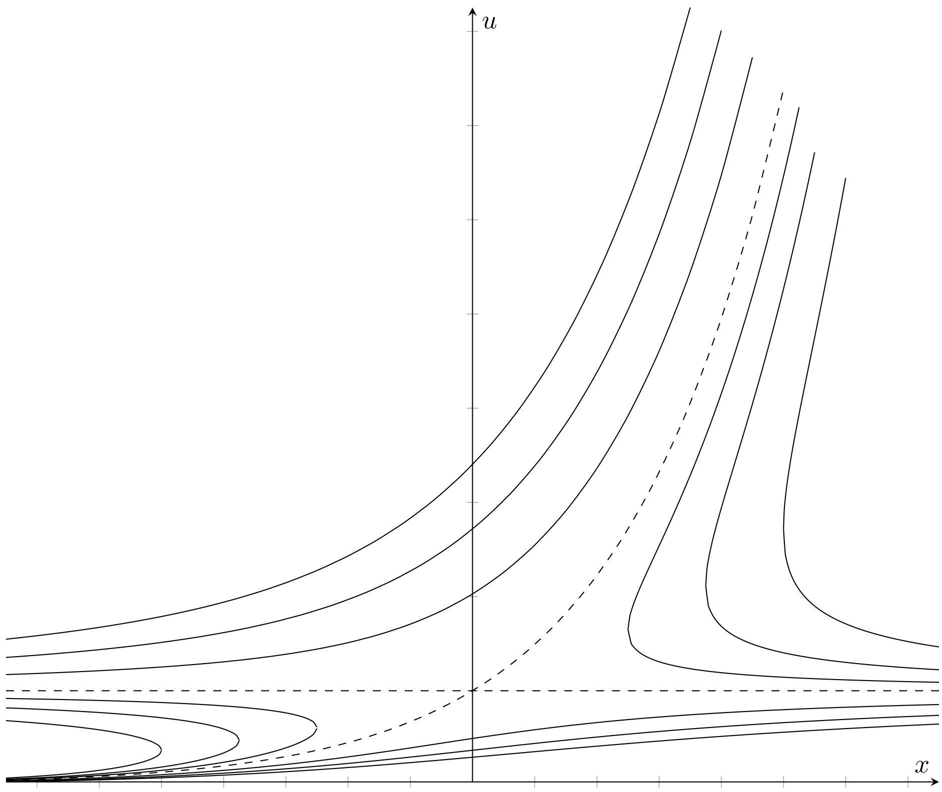
for constants $C \in \mathbb{R}$.

If $C = 0$, the solutions are $y = \pm x$, the pair of straight lines.

If $C \neq 0$ this gives hyperbolas.



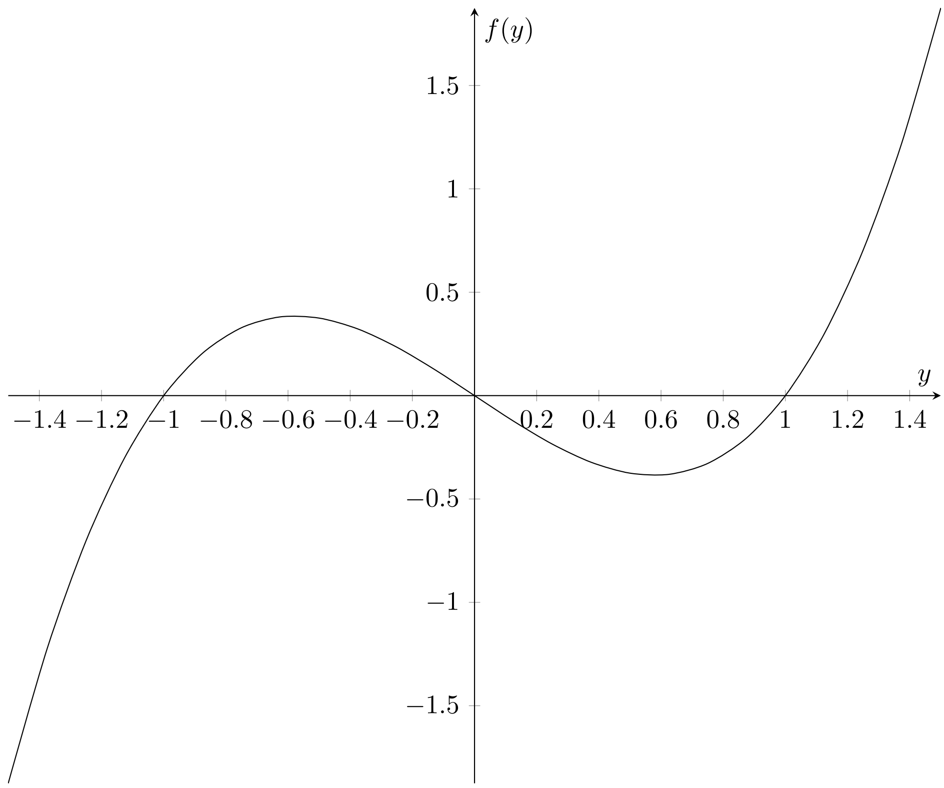
Under the transformation $y = \ln u - x$, the line $y = x$ gets mapped to $u = e^{2x}$, and the line $y = -x$ gets mapped to $u = 1$



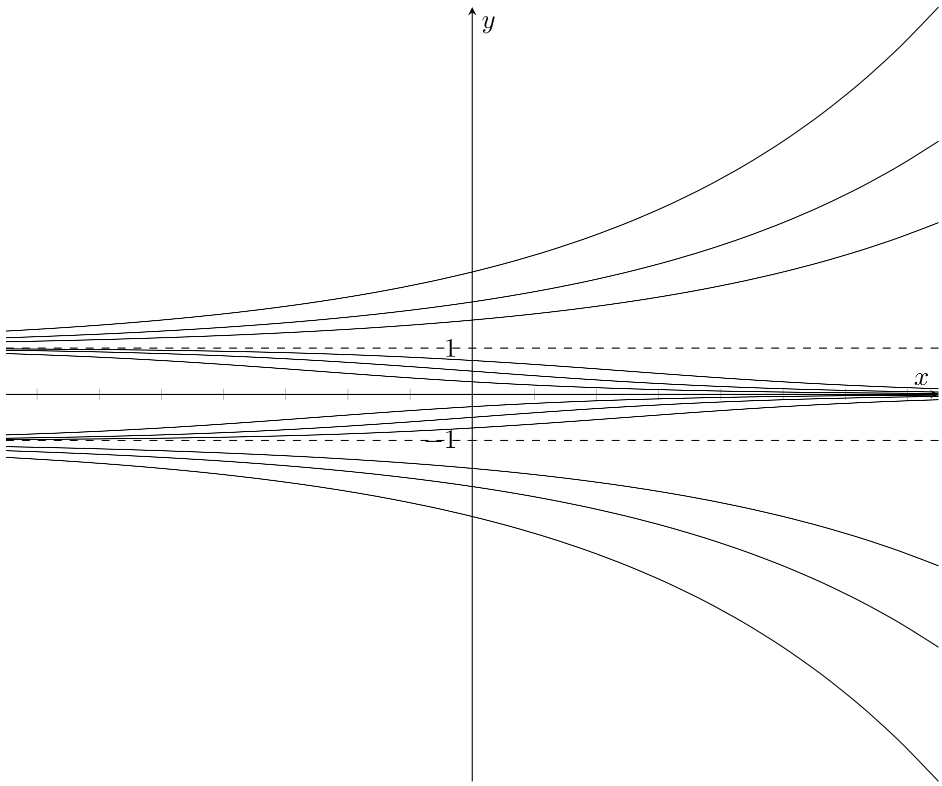
Problem 2.5

Sketch the solution curves to the differential equation $y' = (y+1)y(y-1)$. Sketch the solution curves to the differential equation $y' = (y+1)^2 y(y-1)^2$. It would be helpful considering constant solutions in both cases, since solutions to differential equations cannot cross.

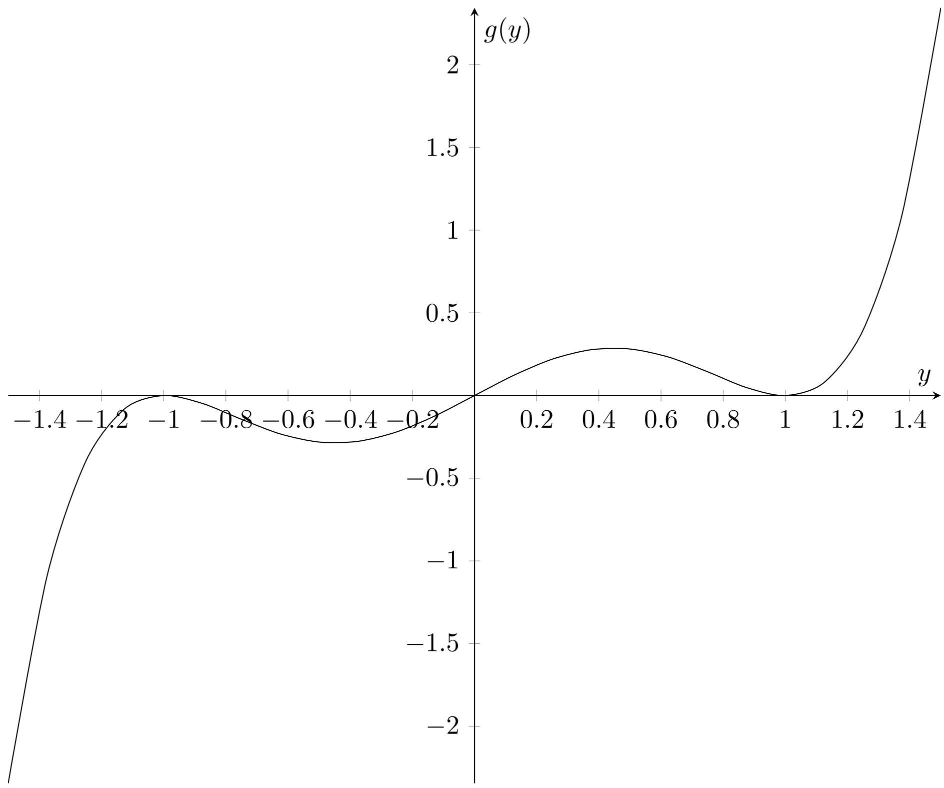
Solution. Let $f(y) = (y+1)y(y-1)$. We first sketch the graph of $f(y)$.



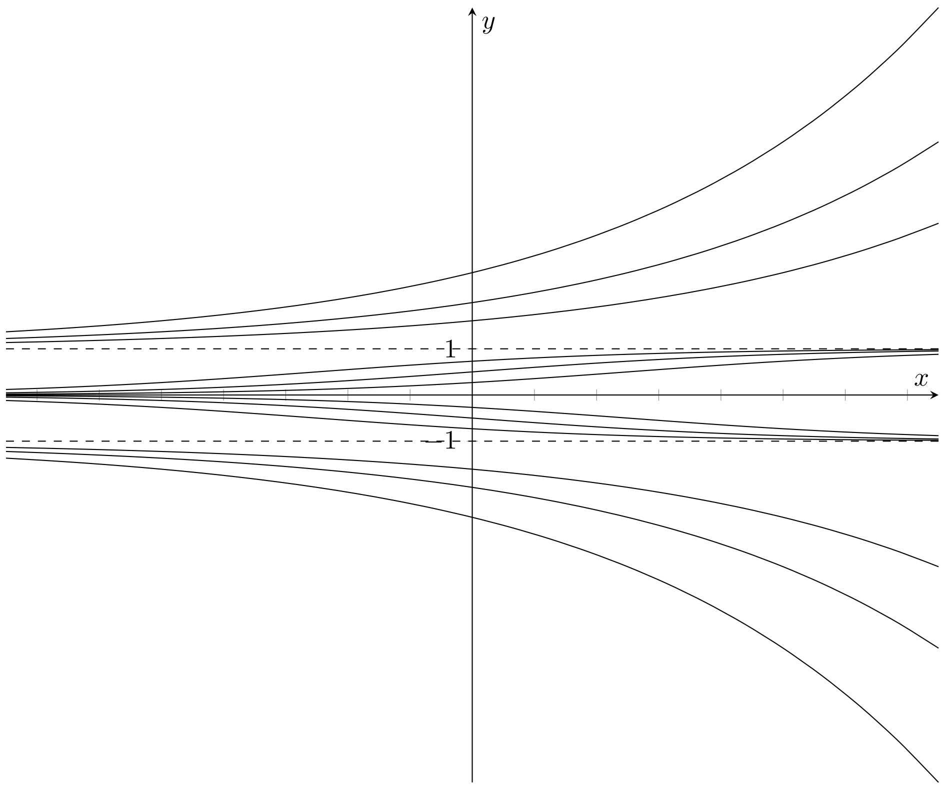
Therefore, constant solutions of y are $y = 0, \pm 1$, and y is increasing if $-1 < y < 0$ or $y > 1$, and y is decreasing if $0 < y < 1$ or $y < -1$.



Let $g(y) = (y+1)^2 y(y-1)^2$. We first sketch the graph of $g(y)$.



Therefore, constant solutions are the same, but y is increasing for $0 < y < 1$ or $y > 1$, and y is decreasing if $-1 < y < 0$ or $y < -1$.


y
x
1
−1

3 Elementary Number Theory

3.1 Primes and Divisibility

Problem 3.1

Define what is meant by a prime. Prove that there are infinitely primes. By considering $4p_1p_2\cdots p_n - 1$, prove that there are infinitely primes of the form $4n+3$. Can we use the same proof for the primes of the form $4n+1$?

Solution. For contradiction, let $p_1, p_2, \ldots, p_n$ be a list of finitely many primes. Consider $n = p_1p_2\cdots p_n + 1$ does not have any prime factors, contradicting with the fundamental theorem of arithmetic.

A number of the form $4n+3$ must have a prime of the form $4n+3$ as its prime factor, and a similar argument follows.

This does not work precisely in this form since a number of the form $4n+1$ does not necessarily have a prime factor of the form $4n+1$.

Problem 3.2

Is there a block of 100 integers, none of which are prime? What about precisely 2 of which are prime?

Solution. Consider $101!+2$, $101!+3$, $\ldots$, $101!+101$. Note they are all not prime.

Consider $\{1, 2, \ldots, 100\}$ has 25 primes.

If $\{a+1, \ldots, a+100\}$ has n_a primes, then $\{(a+1)+1, \ldots, (a+100)+1\}$ has n_a or $n_a \pm 1$ primes.

This means from $\{1, 2, \ldots, 100\}$ having 25 primes to $\{101!+2, \ldots, 101!+101\}$ having 0 primes, it must have been through a block with only 2 primes.

Problem 3.3

Find all positive integers n for which n does not divide $(n-1)!$.

Solution. We can quickly convince ourselves that if $n = ab$ for some $a, b \in \mathbb{N}$, $1 < a, b < n$ and $a \neq b$, then n divides $(n-1)!$.

What remains is $n = p$ or $n = p^2$ for p a prime. The first one does not work. For the second one, it is possible for $p > 2$ since $1 \leq p < 2p \leq p^2 - 1$.

Problem 3.4

Let $\varphi(n)$ be the number of integers m where $1 \leq m \leq n$ such that $\gcd(m, n) = 1$. What is $\varphi(6)$? What is $\varphi(p)$ where p is a prime? By considering the fractions

$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n},$$

prove that

$$n = \sum_{d|n} \varphi(d).$$

Solution. We have

$$\varphi(6) = |\{1, 5\}| = 2,$$

and for primes p,

$$\varphi(p) = p - 1.$$

There are n fractions in this list, in this form.

On the other hand, if we simply the fractions to the simple form (where the numerator and the denominators are coprime), then the denominator is a factor of n, say $d \mid n$. There are $\varphi(d)$ of such fractions since the numerator is coprime. This gives the identity as desired.

3.2 Modular Arithmetic

Problem 3.5

> What is the last digit of 3^{2022}? What is the remainder of 2022^{2022} when divided by 7?

Solution . Note the final digit of 3^n is periodic:

$$3, 9, 7, 1, 3, 9, 7, 1, \ldots$$

and since $2022 \equiv 2 \pmod 4$, so the final digit is 9.

Alternatively, the period 4 can be seen from $\gcd(3, 10) = 1$, and that $\varphi(10) = 4$, by Euler-Fermat.

Note that $2022 \equiv -1 \pmod 7$, so

$$2022^{2022} \equiv (-1)^{2022} \equiv 1 \pmod 7.$$

Problem 3.6

> Are there any squares of the form $4n + 2$? What about $4n + 3$? Consider the equation $a^2 + b^2 = c^2$ where a, b, c are positive integers. What can be said about the parity of a, b and c? What about Pythagorean quadruples or quintuples?

Solution . No, since quadratic residues modulo 3 are 0 and 1.

$(a, b, c) \equiv (0, 0, 0), (0, 1, 1), (1, 0, 1) \pmod 2$ but not $(1, 1, 0)$ by modulo 4.

For a, b, c, d, it is possible to have a, b, c all even, one odd, but not two odds or three odds by modulo 4.

For a, b, c, d, e, it is possible to have a, b, c all even, one odd, or all odds.

Problem 3.7

> Is 123454321 a multiple of 9? What about 12321? Show that a positive integer n is a multiple of 9 if and only if the sum of its digits is a multiple of 0. The number 2^{29} is a nine-digit number with all digits distinct. Which digit is missing?

Solution . No for 123454321, since $1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 = 25$ is not a multiple of 9.

Yes for 12321, since $1 + 2 + 3 + 2 + 1 = 9$ is a multiple of 9.

A number $m = \overline{x_n x_{n-1} \cdots x_1 x_0}$ satisfies

$$\begin{aligned} m &\equiv \sum_{k=0}^{n} 10^k x_k \\ &\equiv \sum_{k=0}^{n} 1^k x_k \\ &\equiv \sum_{k=0}^{n} x_k \pmod 9 \end{aligned}$$

and this proves the statement.

We notice that

$$\begin{aligned} 2^{29} &\equiv \left(2^3\right)^9 \cdot 2^2 \\ &\equiv (-1)^9 \cdot 4 \\ &\equiv -4 \pmod 9 \end{aligned}$$

and hence the digit missing is 4.

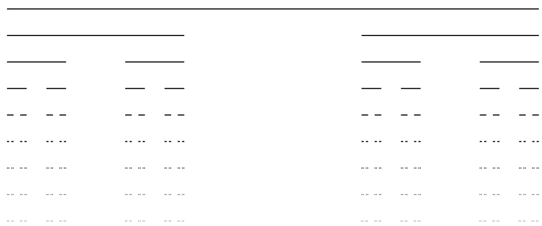
3.3 Number Base

Problem 3.8

What is 100_{10} in base 3? What is 10201_{11} in base 3? What is $\left(\frac{1}{2}\right)_{10}$ in base 3? Describe the Cantor set in base 3.

Definition 3.1 (*Cantor Set*)

The Cantor set is defined as (the limit of) the following fractal, where each line segment is an open interval, and the first represents the closed interval $(0, 1)$.



Every time the middle $\frac{1}{3}$ of a line segment is removed.

Solution . $100_{10} = (10_{10})^2 = (101_3)^2 = 10201_3$.

Alternatively, $100 = 81 + 2 \times 9 + 1 = 3^4 + 2 \times 3^2 + 1$ so $100_{10} = 10201_3$.

$121_{11} = (11_{11})^2 = (12_{10})^2 = (110_3)^2 = 12100_3$.

We notice that

$$\begin{aligned} \frac{1}{2} &= \frac{1}{3} \cdot \frac{1}{1 - \frac{1}{3}} \\ &= \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots \end{aligned}$$

hence

$$\left(\frac{1}{2}\right)_{10} = 0.111\ldots_3 .$$

The Cantor set contains precisely the numbers in $[0, 1]$ without any 11 in their ternary expansion. If we define the line segments to be closed intervals, then it contains precisely the numbers in $[0, 1]$ with finitely many 1s in their ternary expansion.

4 Combinatorics

4.1 Counting and Probability

Problem 4.1

How many subsets of $\{1, 2, 3, \ldots, n\}$ are there? How many of those have even size?

Solution. There are 2^n subsets of $\{1, 2, 3, \ldots, n\}$ of this, and this is simply by observing that there are two choices for each element, either it is in the subset or not. Alternatively,

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = (1 + 1)^n = 2^n.$$

There are 2^{n-1} of those which have even size. This is because when you decide whether the first $(n - 1)$ elements are in the subset, whether n is in the subset is determined for the size of the subset to be even. Alternatively,

$$\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2\lfloor \frac{n}{2} \rfloor} = \frac{1}{2}[(1 + 1)^n + (1 - 1)^n] = \frac{1}{2} \cdot 2^n = 2^{n-1}.$$

Problem 4.2

Let $p(n)$ be the number of partitions of $\{1, 2, \ldots, n\}$. What is $p(3)$?

Show that

$$2^{n-1} \le p(n) \le 2^{\frac{n(n-1)}{2}}.$$

Improve the upper bound to

$$p(n) \le n!.$$

Solution. For $\{1, 2, 3\}$, the partitions are

- $\{1, 2, 3\}$;
- $\{1, 2\}, \{3\}$; $\{1, 3\}, \{2\}$; $\{2, 3\}, \{1\}$;
- $\{1\}, \{2\}, \{3\}$.

Hence, $p(3) = 7$.

For the lower bound, we see that each element $m = 2, 3, \ldots, n$ has at least two choices, to belong in the same partition as 1, or to not. In other words, we choose a subset $P \subseteq \{2, 3, \ldots, n\}$, and the partition $\{1\} \cup P, \{2, 3, \ldots, n\} \setminus P$ is a partition. This gives a lower bound of 2^{n-1}.

For the upper bound, we see that for each partition, this gives a unique indication on whether two elements are in the same set in the partition or not. There are $\binom{n}{2} = \frac{n(n-1)}{2}$ such pairs, so there are at most $2^{\frac{n(n-1)}{2}}$ partitions.

To get the better upper bound, we construct a partition by putting in elements one by one. For 1, there is only one choice, to be in the first set. For 2, it could choose to go in the same set as 1, or to go into a new set on its own, giving 2 choices. Similarly, for k, it could go into the same set as 1, or 2, $\ldots$, or $(k - 1)$, or on its own (although some choices are equivalent). This gives an upper bound of $n!$.

Problem 4.3

Prove that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Show that

$$\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1},$$

and hence or otherwise, show that

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Solution. For the first identity, we notice that on one hand, the coefficient in $(1+x)^n(1+x)^n = (1+x)^{2n}$ is simply $\binom{2n}{n}$.

On the other hand, if we take the term x^k in the first $(1+x)^n$, the coefficient is $\binom{n}{k}$, and we take the term x^{n-k} in the second $(1+x)^n$, the coefficient is $\binom{n}{n-k} = \binom{n}{k}$ as well.

Summing the product of those two over $k = 0$ to $k = n$ gives the identity as desired.

Alternatively, we could think these as choosing n elements out of the set $\{1, 2, \ldots, 2n\}$, and we take k out of $\{1, 2, \ldots, n\}$ and $(n-k)$ out of $\{n+1, n+2, \ldots, 2n\}$.

Pascal's Identity can be considered by considering the coefficient of x^{m+1} in $(1+x)(1+x)^n = (1+x)^{n+1}$ as $\binom{n+1}{m+1}$.

On the other hand,

$$(1+x)(1+x)^n = (1+x)^n + x(1+x)^n,$$

and the coefficient of x^{m+1} in this is

$$\binom{n}{m+1} + \binom{n}{m}.$$

This gives the identity as desired.

Alternatively, we choose $(m+1)$ elements out of the set $\{1, 2, \ldots, n+1\}$. If we choose $(n+1)$, then we have $\binom{n}{m}$ ways to choose the remaining m from $\{1, 2, \ldots, n\}$. If not, we need to choose $(m+1)$ from $\{1, 2, \ldots, n\}$.

Using telescoping, we notice

$$\binom{m}{k} = \binom{m+1}{k+1} - \binom{m}{k+1},$$

and hence

$$\begin{aligned}\sum_{m=k}^{n} \binom{m}{k} &= \sum_{m=k}^{n} \binom{m+1}{k+1} - \sum_{m=k}^{n} \binom{m}{k+1} \\ &= \binom{n+1}{k+1} - \binom{k}{k+1} \\ &= \binom{n+1}{k+1},\end{aligned}$$

as desired.

Alternatively, a combinatorial argument may be as follows. If we choose $(k+1)$ elements out of the set $\{1, 2, \ldots, n+1\}$, if the final element we choose is $m+1$, then we have to choose k elements out of $\{1, 2, \ldots, m\}$. The final element we choose is at least $k+1$, therefore $k \leq m \leq n$, and hence

$$\binom{n+1}{k+1} = \sum_{m=k}^{n} \binom{m}{k}$$

as desired.

Alternatively, we think of a generating function. Notice that

$$\begin{aligned}(1+x)^k + (1+x)^{k+1} + \cdots + (1+x)^n &= \frac{(1+x)^{n+1} - (1+x)^k}{(1+x) - 1} \\ &= \frac{(1+x)^{n+1} - (1+x)^k}{x}.\end{aligned}$$

The coefficient of x^k on the left-hand side is

$$\sum_{m=k}^{n} \binom{m}{k},$$

and the coefficient of x^k on the right-hand side is the coefficient of x^{k+1} on the numerator, which is

$$\binom{n+1}{k+1}$$

as desired.

4.2 Games

5 Algebra

5.1 Coordinate Geometry

Problem 5.1

> For what values of a, b, c does the circle $x^2 - 2ax + y^2 - 2by + c = 0$ contain the origin?

Solution. Note that the original equation to be simplified as

$$(x-a)^2 + (y-b)^2 = a^2 + b^2 - c,$$

and for this to contain the point (x, y), we need to have

$$(x-a)^2 + (y-b)^2 < a^2 + b^2 - c.$$

Putting in $(x, y) = (0, 0)$ gives

$$a^2 + b^2 < a^2 + b^2 - c$$

which gives $c < 0$.

5.2 Equations

Problem 5.2

> For what values of a does the equation $ax^2 - x + 1 = 0$ have two real roots? Explain the behaviour of the roots as $a \to 0_\pm$.

Solution. When $a = 0$, this equation has one root $x = 1$.

When $a \neq 0$, this equation is a quadratic indeed, and we have

$$\Delta = 1 - 4a > 0$$

so $a < \frac{1}{4}$.

We recall the quadratic formula for $ax^2 + bx + c = 0$ as

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for $a \neq 0$.

Alternatively, we can transform the original equation to $c\left(\frac{1}{x}\right)^2 + b\left(\frac{1}{x}\right) + a = 0$ by dividing both sides by x^2, and for $c \neq 0$, this solves to

$$x_{1,2} = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}.$$

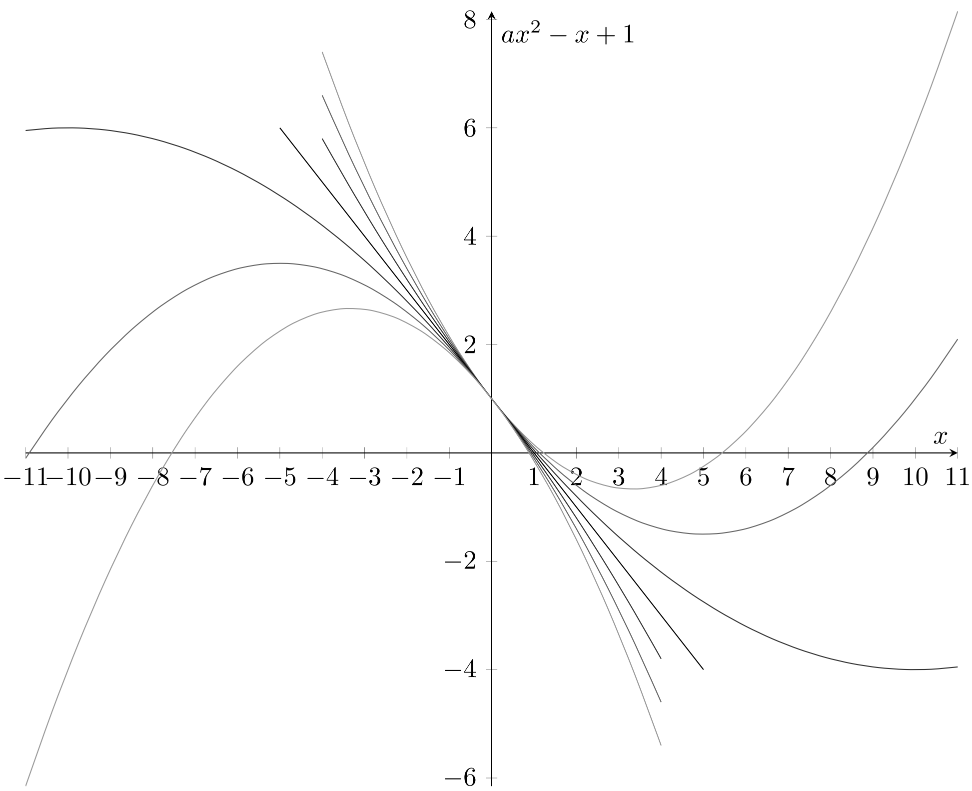
If $a \to 0_+$, note the root

$$x = \frac{2}{1 - \sqrt{1 - 4a}} \to +\infty,$$

and if $a \to 0_-$, note the root

$$x = \frac{2}{1 - \sqrt{1 - 4a}} \to -\infty.$$

From here we can see a single line is indeed a degenerate case of a parabola.



Problem 5.3

> Find all real solutions of $\left(4x^3 - 3x\right)^{2\sin \pi x} = 1$.

Solution. We may utilise the fact that $a^b = 1$ if and only if $a = 1$, or $a \neq 0$ and $b = 0$, or $a = -1$ and b is even.

We consider the following casework.

- $4x^3 - 3x = 1$. Notice $x = 1$ is a solution, and hence

$$4x^3 - 3x - 1 = (x-1)\left(4x^2 + 4x + 1\right) = (x-1)(2x+1)^2$$

  so $4x^3 - 3x = 1$ solves to $x = 1$ or $x = -\frac{1}{2}$.
- $4x^3 - 3x \neq 0$ *and* $2\sin \pi x = 0$. Note $\sin \pi x = 0$ if and only if $x \in \mathbb{Z}$, and $4x^3 - 3x = 0$ if and only if $x = 0$ or $x = \pm\frac{\sqrt{3}}{2}$. Hence, this solves to $x \in \mathbb{Z}, x \neq 0$.
- $4x^3 - 3x = -1$ *and* $2\sin \pi x$ *is even.* Note $x = -1$ is a solution to $4x^3 - 3x = -1$, and hence

$$4x^3 - 3x + 1 = (x+1)4x^2 - 4x + 1 = (x-1)(2x-1)^2$$

  so $4x^3 - 3x = -1$ solves to $x = 1$ or $x = \frac{1}{2}$.
  Note that they both satisfy $2\sin \pi x$ is even.

Therefore, the solutions are $x \in \mathbb{Z}$ and $x \neq 0$, or $x = \pm\frac{1}{2}$.

5.3 Inequalities

Problem 5.4

> Suppose that we have some positive integers (not necessarily distinct) whose sum is 100. How large can their product be?

Solution. First, we notice that having 4 in the optimal case is equivalent to having two 2s, so we break them down into 2s.

If we have anything at least 5 in the optimal case, breaking it down into two integers would produce a greater product.

If we have a 1 in the optimal case, it cannot be on its own, then combining it with any other integer n to get $(n+1)$ would give a greater product.

Hence, the optimal product must (essentially) only contain 2s and 3s. But notice $3^2 = 9 > 8 = 2^3$, so there has to be at most two 2s in the optimal case (since if we have at least three 2s, converting it into 3s would produce a bigger product).

Since $100 = 3 \times 32 + 2 \times 2$, the greatest product must be

$$2^2 \times 3^{32}.$$