# Pascal's Triangle

Eason Shao

24/05/2024

Try and give combinatorial explanations of the identities where possible, as well as from algebraic methods (including definition, binomial expansion and generating functions).

Please experiment with the identities using small examples.

1. Find and prove the formula for ${}^nC_k$, often also denoted as $\binom{n}{k}$, which stands for the number of choosing $k$ from $n$ distinct objects. This is called the **Choose Coefficient** or the **Binomial Coefficient**.

2. Write down the values of $\binom{n}{k}$ where $0 \le k \le n \le 5$. What pattern did you spot, about the relationship between $\binom{n}{k}$ and $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$

3. Prove the **Pascal's Identity**, which is the pattern spotted above.

4. Find and explain the values of $\binom{n}{0}$ and $\binom{n}{n}$.

5. Prove the following symmetry of the binomial coefficients
$$\binom{n}{k} = \binom{n}{n-k}.$$

6. Find $11^n$ where $0 \le n \le 4$. What pattern did you spot? Does it hold for $n \ge 5$? Why?

7. Write down the expansion of $(a+b)^n$, where $a, b$ are arbitary real constants, and $0 \le n \le 5$. What pattern did you spot between the coefficients of each term and the entries in Pascal's Triangle?

8. Prove the **Binomial Expansion**, which is the pattern spotted above.

9. Find a formula and prove for the following sum:
$$\sum_{t=0}^{n} \binom{n}{t}.$$

10. Find a formula and prove for the following sum:
$$\sum_{t=0}^{n} t\binom{n}{t}.$$

11. Prove the following identity:
$$\sum_{t=0}^{n} (-1)^t \binom{n}{t} = 0,$$
i.e. the odd indexed terms in a row of Pascal's Triangle sum up to the even indexed terms.

12. Prove the following identity:
$$\sum_{t=k}^{n} \binom{n}{t}\binom{t}{k} = 2^{n-k}\binom{n}{k}.$$

13. Prove the following identity:
$$\sum_{t=0}^{n} \binom{n}{t}^2 = \binom{2n}{n}.$$

14. Prove the **Hockey-Stick Identity**, stated as follows:
$$\sum_{t=m}^{n} \binom{t}{m} = \binom{n+1}{m+1}.$$

15. Prove the **Chu-Vandermonde Identity**, stated as follows:
$$\sum_{t=0}^{k} \binom{m}{t}\binom{n-m}{k-t} = \binom{n}{k}.$$