Team Number: <u>213</u>

PUMaC 2024 Power Round Cover Sheet

In years past, this page was for hand-turned in submissions. Now it is for reference for you guys! So that you know how many points each problem is worth.

Problem Number	Points	Attempted?
2.0.1	5	Yes
2.0.2	5	Yes
2.0.3	5	Yes
2.0.4	10	Yes
2.0.5	5	Yes
2.1.1	15	Yes
2.1.2	5	Yes
2.1.3	10	Yes
2.1.4	15	Yes
2.1.5	10	Yes
2.1.6	5	Yes
2.1.7	10	Yes
2.1.8	10	Yes
2.1.9	5	Yes
4.0.1	15	Yes
4.0.2	10	Yes
4.0.3	15	Yes
4.0.4	10	Yes
4.1.1	5	Yes
4.1.2	5	Yes
4.1.3	5	Yes
4.1.4	15	Yes
4.1.5	10	Yes

Problem Number	Points	Attempted?
4.2.1	10	Yes
4.2.2	10	Yes
4.2.3	5	Yes
4.2.4	5	Yes
4.2.5	20	Yes
4.2.6	15	Yes
5.0.1	10	Yes
5.0.2	5	Yes
5.0.3	5	Yes
5.0.4	10	Yes
5.0.5	25	Yes
5.0.6	40	Yes
5.0.7	50	

Contents

1   Problem 2.0.1

Proof:

1. This is a σ-algebra, because

   (a) $\emptyset \in \Sigma$,

   (b) $X \setminus \emptyset = X \in \Sigma, X \setminus X = \emptyset \in \Sigma$,

   (c) $X \cup \emptyset = X \in \Sigma$,

   (d) $X \cap \emptyset = \emptyset \in \Sigma$.

2. This is **not** a σ-algebra, because $1 \in \Sigma$, but $1 \notin \mathcal{P}(X)$, which implies $\Sigma \not\subset \mathcal{P}(X)$ since $\mathcal{P}(X)$ only contains subsets of X as members, and 1 is not a subset of X.

3. This is a σ-algebra, because $\{0, 1, 2, 3, 4\} = X$, $\{0, 1\}$ and $\{2, 3, 4\}$ are complements, $\emptyset$ and X are complements. Therefore rules 1 and 2 are satisfied. Unions with X are $X \in \Sigma$, $\{0, 1\} \cup \{2, 3, 4\} = X \in \Sigma$. Intersections with $\emptyset$ are $\emptyset \in \Sigma$, and $\{0, 1\} \cap \{2, 3, 4\} = \emptyset \in \Sigma$.

4. This is **not** a σ-algebra, because $\{1, 2\} \in \Sigma$, but $X \setminus \{1, 2\} = \{0, 3, 4\} \notin \Sigma$.

2   Problem 2.0.2

Proof:

1. $\Sigma = \mathcal{P}(X)$ is a σ-algebra because:

   (a) $\emptyset \in \Sigma = \mathcal{P}(X)$,

   (b) $X \setminus E_i \subset X$, so $X \setminus E_i \in \mathcal{P}(X)$,

   (c) $\bigcup_{i=1}^{+\infty} E_i \subset X$, so $\bigcup_{i=1}^{+\infty} E_i \in \mathcal{P}(X)$,

   (d) $\bigcap_{i=1}^{+\infty} E_i \subset X$, so $\bigcap_{i=1}^{+\infty} E_i \in \mathcal{P}(X)$,

   In short, $\Sigma = \mathcal{P}(X)$ is closed under complement, intersection and union, and therefore a σ-algebra.

2. μ is a measure because $\mu : \Sigma \to \overline{\mathbb{R}}$ satisfies that:

   (a) $\mu(\emptyset) = \#\emptyset = 0$,

   (b) for all $E \in \mathcal{P}(X), \mu(E) = \#E \geq 0$,

   (c) let $\{E_i\}_{i=0}^{+\infty}$ be pairwise disjoint,

$$\mu\left(\bigcup_{i=1}^{+\infty} E_i\right) = \#\bigcup_{i=1}^{+\infty} E_i$$
$$= \sum_{i=1}^{+\infty} \#E_i$$
$$= \sum_{i=1}^{+\infty} \mu(E_i),$$

   the second equal sign being true since all the sets are pairwise disjoint.

3   Problem 2.0.3

Proof: $A \subset B$ means that $B = A \cup B \setminus A$, and $B \setminus A = B \cap X \setminus A$.

$X \setminus A \in \Sigma$ by Axiom 2 for a σ-algebra, so $B \cap X \setminus A = B \setminus A \in \Sigma$ by Axiom 4.

Clearly, $A \cap B \setminus A = \emptyset$, so by Axiom 3 for a measure,

$$\mu(B) = \mu(A \cup B \setminus A) = \mu(A) + \mu(B \setminus A).$$

By Axiom 2 for a measure, $\mu(B \setminus A) \geq 0$, which means $\mu(B) \geq \mu(A)$ as desired.

4   Problem 2.0.4

Proof: We define a series of sets $\{D_i\}_{i=1}^{+\infty}$ as follows:

- $D_1 = E_1$,

- $D_i = E_i \setminus \bigcup_{k=1}^{i-1} E_k$.

  Clearly $D_i \subset E_i$. Let $i \neq j$, and W.L.O.G. let $i < j$, where $i, j \in \mathbb{N}$. We have

$$D_i \subset E_i \subset \bigcup_{k=1}^{j-1} E_k, D_j = E_j \setminus \bigcup_{k=1}^{j-1} E_k,$$

which implies $D_i \cap D_j = \emptyset$. This means that $\{D_i\}_{i=1}^{+\infty}$ are pairwise disjoint.

Let $F = \bigcup_{i=1}^{+\infty} D_i$. We claim that

$$F = \bigcup_{i=1}^{+\infty} E_i.$$

- $\bigcup_{i=1}^{+\infty} E_i \subset F$. For any $a \in \bigcup_{i=1}^{\infty} E_i$, we must have $a \in E_j$ for some natural number j. Let k be the minimal of such j by well-ordering principle, i.e.

$$a \in E_k, a \notin \bigcup_{i=1}^{k-1} E_i.$$

  This implies that $a \in E_k \setminus \bigcup_{i=1}^{n-1} E_i = D_k$, which therefore means $a \in \bigcup_{i=1}^{+\infty} D_i = F$, and therefore

$$\bigcup_{i=1}^{+\infty} E_i \subset F.$$

- $F \subset \bigcup_{i=1}^{+\infty} E_i$. For any $b \in F = \bigcup_{i=1}^{+\infty} D_i$, b must be a member of D_k for some $k \in \mathbb{N}$. Since $b \in D_k \subset E_k \subset \bigcup_{i=1}^{+\infty} E_i$, this means that $b \in F$, and therefore

$$F \subset \bigcup_{i=1}^{+\infty} E_i.$$

Now, we can show that

$$\mu\left(\bigcup_{i=1}^{+\infty} E_i\right) = \mu(F) = \mu\left(\bigcup_{i=1}^{+\infty} D_i\right)$$

$$= \sum_{i=1}^{+\infty} \mu(D_i)$$

$$\leq \sum_{i=1}^{\infty} \mu(E_i),$$

as desired.

Note the equal sign on the second line arises because D_i are pairwise disjoint, and the less or equal sign on the third line arises because $D_i \subset E_i$.

5   Problem 2.0.5

Proof: Yes, (X, Σ, μ) is a measure space.

We have verified before that $\mathcal{P}(X)$ is a σ-algebra over X. Now we verify that μ is a measure for such σ algebra.

1. $\mu(\emptyset) = 0$ because $x_0 \notin \emptyset$,

2. $\mu(E) = 0$ or 1, so $\mu(E) \geq 0$ for every $E \in \Sigma$,

3. let $\{E_i\}_{i=1}^{+\infty}$ be a collection of pairwise disjoint sets.

   - **Case 1.** There exists such $k \in \mathbb{N}$, $x_0 \in E_k$. Then by definition $\mu(E_k) = 1$. Since E_i are disjoint, for all $n \neq k, n \in \mathbb{N}$, we must have $x_0 \notin E_n$, and therefore $\mu(E_n) = 0$.
   Also, $x_0 \in E_k \subset \bigcup_{i=1}^{+\infty} E_i$, so

   $$\mu\left(\bigcup_{i=1}^{+\infty} E_i\right) = 1 = \sum_{i=1}^{+\infty} \mu(E_i),$$

   and the third axiom for measures holds in this case.

   - **Case 2.** For all $k \in \mathbb{N}$, $x_0 \notin E_k$. Then by definition $\mu(E_k) = 0$ for all k, and the sum $\sum_{i=1}^{+\infty} \mu(E_i) = 0$.
   Also, we will have $x_0 \notin \bigcup_{i=1}^{+\infty} E_i$, which means

   $$\mu\left(\bigcup_{i=1}^{+\infty} E_i\right) = 0 = \sum_{i=1}^{+\infty} \mu(E_i),$$

   and the third axiom for measures holds in this case.

Therefore all axioms for the measure hold, so μ is a measure on Σ, and so (X, Σ, μ) is a measure space.

6   Problem 2.1.1

Proof: Let E_0 be the base set E, and let

$$E_{i+1} = E_i \cup \bigcup_{a,b \in E_i} \overline{ab}, \quad E_\infty = \bigcup_{i=0}^{+\infty} E_i.$$

E_∞ convex: if $a, b \in E_\infty$, then $a \in E_i, b \in E_j$ for some $i, j \in \mathbb{N}$, and therefore $a, b \in E_{\max\{i,j\}}$ and so W.L.O.G. we simply denote this $a, b \in E_i$ for some $i \in \mathbb{N}$, then we will have $\overline{ab} \in E_{i+1}$.

We first show that $|E_{i+1}| \le |E_i|$.

$$
\begin{aligned}
|E_{i+1}| &= \sup_{p_1, p_2 \in E_{i+1}} |p_1 - p_2| \\
&\le \sup_{\substack{a_1, a_2, b_1, b_2 \in E_i, \\ p_1 \in \overline{a_1 b_1}, \\ p_2 \in \overline{a_2 b_2}}} |p_1 - p_2| \\
&\le \sup_{\substack{a_1, a_2, b_1, b_2 \in E_i, \\ p_1 \in \overline{a_1 b_1}}} \max(|p_1 - a_2|, |p_1 - b_2|) \\
&\le \sup_{a_1, a_2, b_1, b_2 \in E_i} \max(|a_1 - a_2|, |b_1 - a_2|, |a_1 - b_2|, |b_1 - b_2|) \\
&\le \sup_{x, y \in E_i} |x - y| \\
&= |E_i|,
\end{aligned}
$$

where the first and second inequalities are due to the fact that for a triangle ABC,

$$\sup_{P \in \overline{BC}} |A - P| \le \max(|A - B|, |A - C|).$$

Hence $|E_0| \le |E_1| \le \ldots \le |E_\infty|$.

Now let

$$D = \bigcup_{p \in E_\infty} B_p\left(\frac{\varepsilon}{2}\right).$$

D is open as it's a union of open balls.

By construction of D, let $a \in D$ be in $B_{p_a}\left(\frac{\varepsilon}{2}\right)$ for $p_a \in E_\infty$.

We show that $|D| \leq \varepsilon + |E|$.

$$
\begin{aligned}
|D| &= \sup_{a,b \in D} |a - b| \\
&= \sup_{a,b \in D} |(a - p_a) + (p_a - p_b) + (p_b - b)| \\
&\leq \sup_{a,b \in D} (|a - p_a| + |p_a - p_b| + |p_b - b|) \\
&\leq \sup_{a,b \in D} \left(\frac{\varepsilon}{2} + |p_a - p_b| + \frac{\varepsilon}{2} \right) \\
&\leq \varepsilon + \sup_{a,b \in D} |p_a - p_b| \\
&\leq \varepsilon + \sup_{a,b \in E_\infty} |a - b| \\
&\leq \varepsilon + |E_\infty| \\
&\leq \varepsilon + |E_0| \\
&= \varepsilon + |E|,
\end{aligned}
$$

as required. (The fourth inequality is from the fact that $p_i \in E_\infty$.)

Now we show D is convex. Let $a, b \in D, p \in \overline{ab}$, let q be the closest point on $\overline{p_a p_b}$ to p. We have $|p - q| \leq \max(|p_a - a|, |p_b - b|)$ by geometry, and $q \in \overline{p_a p_b}$ by convexity of E_∞, so

$$
\begin{aligned}
|p - q| &\leq \max(|p_a - a|, |p_b - b|) \\
&\leq \frac{\varepsilon}{2},
\end{aligned}
$$

and therefore $p \in B_q \left(\frac{\varepsilon}{2} \right) \subseteq D$ by construction of D. Thus all the properties of $U = D$ are satisfied.

7   Problem 2.1.2

Proof: None of them are δ-covers.

1. $\frac{1}{3}$ isn't in any of the intervals, but it's in $[0,1]$, and we want $[0,1] \subset \bigcap_i U_i$.

2. We want $|U_i| < \delta = \frac{1}{2}$ for all i, but $U_1 = [0, \frac{1}{1}] = [0,1]$, and

$$|[0,1]| = \sup_{x,y \in [0,1]} |y - x| \geq |1 - 0| = 1 > \frac{1}{2}.$$

3. We want $|U_i| < \delta = \frac{1}{2}$ for all i, but

$$\left|[0, \frac{1}{2}]\right| = \sup_{x,y \in [0,\frac{1}{2}]} |y - x| \geq \left|\frac{1}{2} - 0\right| = \frac{1}{2} = \delta$$

so it's invalid.

8 Problem 2.1.3

Proof: Let

$$M_\delta(F) = \{\{U_i\}_i : \{U_i\}_i \text{ is a } \delta\text{-cover of } F\}.$$

Claim. $M_{\delta_1}(F) \subset M_{\delta_2}(F)$ for $0 < \delta_1 < \delta_2$.

Proof. Take $\{U_i\}_i \in M_{\delta_1(F)}$. Then $\{U_i\}_i$ is a δ_2 cover of F.

Therefore, $F \subset \bigcup_i U_i$ and $|U_i| < \delta_1$.

$\delta_1 < \delta_2$ so $|U_i| < \delta_1 < \delta_2$.

Thus $\{U_i\}_i$ is a δ_2-cover of F, and thus $\{U_i\}_i \in M_{\delta_2}(F)$. $\qquad\square$

Let

$$R_\delta(F) = \left\{ \sum_{i=1}^\infty |U_i|^s : \{U_i\}_i \in M_\delta(F) \right\}.$$

$$
\begin{aligned}
M_{\delta_1}(F) \subset M_{\delta_2}(F) &\implies R_{\delta_1}(F) \subset R_{\delta_2}(F) \\
&\implies \inf(R_{\delta_1}(F)) \geq \inf(R_{\delta_2}(F)) \\
&\implies \mathcal{H}^s_{\delta_1}(F) \geq \mathcal{H}^s_{\delta_2}(F)
\end{aligned}
$$

as desired.

9   Problem 2.1.4

Proof:

1. By definition,

$$\mathcal{H}^s_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\}.$$

   $\sum_{i=1}^{\infty} |U_i|^s \geq 0$ as $|U_i| \geq 0$, so $\mathcal{H}^s_\delta(E) \geq 0 \implies \mathcal{H}^s \geq 0$ (as if the limit approaches $L < 0$, there must be infinitely many terms in the ball with radius $-L$ around L, but there can't be any as all this terms in this ball are less than 0, while $\mathcal{H}^s_\delta(E) \geq 0$ for all δ).

2. By definition,

$$\mathcal{H}^s_\delta(\emptyset) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } \emptyset \right\}.$$

   But $\emptyset$ is a δ-cover as $\emptyset \subset \bigcup_{S \in \emptyset} S = \emptyset$, and trivially $|U_i| < \delta$ for all $U_i \in \emptyset$ (as there are none).

   $\sum_i |U_i|^s = 0$ as this is an empty sum, hence by definition of the inf operator, $\mathcal{H}^s_\delta(\emptyset) \leq 0$. By part 1, $\mathcal{H}^s_\delta(E) \geq 0$ so $\mathcal{H}^s_\delta(\emptyset) = 0$.

   Hence $\mathcal{H}^s(\emptyset) = 0$ as the limit of a sequence which is identically 0 is 0 (for any ε-ball around 0 all the terms are always in this ball).

3. Consider a countable set of $\{E_j\}_{j=0}^{+\infty}$;

$$\sum_{j=1}^{+\infty} \mathcal{H}^s_d(E_j) = \sum_{j=1}^{+\infty} \inf \left\{ \sum_{i=1}^{+\infty} |(U_j)_i|^s : (U_j)_i \text{ is a } \delta\text{-cover of } E_j \right\}$$

$$= \inf \left\{ \sum_{j=1}^{+\infty} \sum_{i=1}^{+\infty} |(U_j)_i|^s : (U_j)_i \text{ is a } \delta\text{-cover of } E_j \right\}.$$

   Now, $\bigcup_{j=1}^{+\infty} \{(U_j)_i\}_i$ is a δ-cover of $\bigcup_{j=1}^{+\infty} E_j$ (where $\{(U_j)_i\}_{i=1}^{+\infty}$ is a δ-cover of E_j): each $\{(U_j)_i\}_i$ has elements with diameter $< \delta$ so their union does as well, and each point in $\bigcup_{j=1}^{+\infty} E_j$ is in one of the E_js so is covered in one of the $(U_j)_i$s (as they are covers of E_j).

   Thus,

$$\left\{ \sum_{i,j} |(U_j)_i|^s : (U_j)_i \text{ is any } \delta\text{-cover of } E_j \right\} \geq \sum_k |U_k|^s : U_k \text{ is some } \delta\text{-cover of } \bigcup_j E_j;$$

   by problem 2.0.3;

13

Hence

$$
\inf\left\{\sum_{i,j}|(U_j)_i|^s : (U_j)_i \text{ is a } \delta\text{-cover of } E_j\right\}
$$

$$
\geq \inf\left\{\sum_{k}|U_k|^s : U_k \text{ is a } \delta\text{-cover of } \bigcup_{j}E_j\right\}
$$

$$
= \mathcal{H}_d^s\left(\bigcup_{j}E_j\right).
$$

Thus we have $\sum_{j}\mathcal{H}_d^s(E_j) \geq \mathcal{H}_d^s(\bigcup_{j}E_j)$ for all $\delta > 0$, so it also holds in the limit as $\delta \to 0$.

10 Problem 2.1.5

Proof: For $\delta > 0$ and $A \subset \mathbb{R}^n$, let

$$M_\delta(A) = \{\{U_i\} : \{U_i\} \text{ is a } \delta\text{-cover of } A\}.$$

For $A \subset \mathbb{R}^n$, $c \in \mathbb{R}$ and $c > 0$. Define

$$cA = \{cx : x \in A\}.$$

Note that $E' = cE$.
Note that

$$
\begin{aligned}
|cA| &= \sup_{x',y' \in cA} |x' - y'| \\
&= \sup_{x,y \in A} |cx - cy| \\
&= \sup_{x,y \in A} c|x - y| \\
&= c \sup_{x,y \in A} |x - y| \\
&= cA.
\end{aligned}
$$

We claim that

$$\{U_i\} \in M_\delta(E) \iff \{cU_i\} \in M_{c\delta}(E').$$

- $\implies$ **direction.** Let $\{U_i\} \in M_\delta(E)$. Therefore $E \subset \bigcup_i U_i$ and $|U_i| < \delta$.

  Consider $\{cU_i\}$. For any $x' \in E'$, there exists $x \in E$ such that $x' = cx$ by definition.

$$
\begin{aligned}
E \subset \bigcup_i U_i &\implies \text{There exists } j \in \mathbb{N} \text{ such that } x \in U_j \\
&\implies x' = cx \in cU_j \\
&\implies x' \in \{cU_i\}.
\end{aligned}
$$

  Thus $E' \subset \bigcup_i cU_i$.

  Also, $|cU_i| = c|U_i| < c\delta$, and therefore $|cU_i| < c\delta$, and $E' \subset \bigcup_i cU_i$.

  This means $\{cU_i\}$ is a $c\delta$-cover of E', and therefore $\{cU_i\} \in M_{c\delta}(E')$.

- $\impliedby$ **direction.** By definition, we also have $E = \frac{1}{c}E'$, where $\frac{1}{c} \in \mathbb{R}$ and $\frac{1}{c} > 0$.

  Therefore, by the $\implies$ direction, we also have

$$\{cU_i\} \in M_{c\delta}(E') \implies \left\{ \frac{1}{c} \cdot c \cdot U_i \right\} \in M_{\frac{1}{c} \cdot c\delta}\left(\frac{1}{c}E' \right) \implies \{U_i\} \in M_\delta(E).$$

Therefore we have
$$\{U_i\} \in M_\delta(E) \iff \{cU_i\} \in M_{c\delta}(E')$$

.

Now, we have

$$
\begin{aligned}
\mathcal{H}^s_{c\delta}(E') &= \inf\left\{\sum_{i=1}^{+\infty} |cU_i|^s : \{cU_i\} \in M_{c\delta}(E')\right\} \\
&= \inf\left\{\sum_{i=1}^{+\infty} (c \cdot |U_i|)^s : \{U_i\} \in M_\delta(E)\right\} \\
&= \inf\left\{c^s \sum_{i=1}^{+\infty} |U_i|^s : \{U_i\} \in M_\delta(E)\right\} \\
&= c^s \cdot \mathcal{H}^s_\delta(E).
\end{aligned}
$$

So for any $\delta > 0$, we will have

$$\mathcal{H}^s_{c\delta}(E') = c^s \mathcal{H}^s_\delta(E).$$

Therefore,

$$
\begin{aligned}
\mathcal{H}^s(E') &= \lim_{c\delta \to 0} \mathcal{H}^s_{c\delta}(E') \\
&= \lim_{c\delta \to 0} c^s \mathcal{H}^s_\delta(E) \\
&= c^s \lim_{\delta \to 0} \mathcal{H}^s_{delta}(E) \\
&= c^s \mathcal{H}^s(E),
\end{aligned}
$$

where $\lim_{c\delta \to 0}$ is the same as $\lim_{\delta \to 0}$ since c is a positive constant.

11   Problem 2.1.6

Proof: For $\delta > 0$, $A \subset \mathbb{R}^n$, let

$$M_\delta(A) = \{\{U_i\} : \{U_i\} \text{ is a } \delta\text{-cover of } A\}.$$

For any $A \subset \mathbb{R}^n, x \in \mathbb{R}^n$, we define $A + x = \{a + x : a \in A\}$.
Note that

$$
\begin{aligned}
|A + x| &= \sup_{y', z' \in A+x} |y' - z'| \\
&= \sup_{y, z \in A} |(y + x) - (z + x)| \\
&= \sup_{y, z \in A} |y - z| \\
&= |A|.
\end{aligned}
$$

Claim.
$$\{U_i\} \in M_\delta(E) \iff \{U_i + x\} \in M_\delta(E').$$

Proof.

- $\implies$ **direction.** Let $\{U_i\} \in M_\delta(E)$, which implies $E \subset \bigcup_i U_i$ and $|U_i| < \delta$.

  Consider $\{U_i + x\}$. We have $|U_i + x| = |U_i| < \delta$. Also, let $y' \in E'$. Then there must exist some $y \in E$ such that $y' = y + x$.

  $$
  \begin{aligned}
  y \in E \subset \bigcup_i U_i &\implies y \in U_j \text{ for some } j \in \mathbb{N} \\
  &\implies y' = y + x \in (U_j + x) \\
  &\implies y' \in \{U_j + x\}.
  \end{aligned}
  $$

  So $E' \subset \bigcup_i (U_i + x)$, and therefore $\{U_i + x\}$ is a δ-cover of E', and so $\{U_i + x\} \in M_\delta(E')$.

- $\impliedby$ **direction.** Note that $E = E' + (-x)$ and $(-x) \in \mathbb{R}^n$. So therefore by the forward direction,

  $$\{U_i + x\} \in M_\delta(E') \implies \{U_i + x + (-x)\} \in M_\delta(E' + (-x)) \implies \{U_i\} \in M_\delta(E).$$

This finishes our proof of the claim. $\hfill\square$

Now we have

$$
\begin{aligned}
\mathcal{H}_\delta^s(E') &= \inf\left\{\sum_{i=1}^{+\infty} |U_i + x|^s : \{U_i + x\} \in M_\delta(E')\right\} \\
&= \inf\left\{\sum_{i=1}^{+\infty} |U_i|^s : \{U_i\} \in M_\delta(E)\right\} \\
&= \mathcal{H}_\delta^s(E).
\end{aligned}
$$

Since

$$\mathcal{H}^s_\delta(E') = \mathcal{H}^s_\delta(E)$$

is true for all $\delta > 0$, we will have

$$\mathcal{H}^s(E') = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E') = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \mathcal{H}^s(E).$$

12   Problem 2.1.7

Proof:

1. Let $0 < \delta < 1$, and let $\{U_i\}$ be a δ-cover of E.

$$0 \leq |U_i| < \delta < 1, r < s$$
$$\implies |U_i|^s \leq |U_i|^r$$
$$\implies \sum_{i=1}^{+\infty} |U_i|^s \leq \sum_{i=1}^{+\infty} |U_i|^r.$$

By definition,

$$\mathcal{H}_\delta^s(E) = \inf \left\{ \sum_{i=1}^{+\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\}$$

$$\implies \mathcal{H}_\delta^s(E) \leq \sum_{i=1}^{+\infty} |U_i|^s \leq \sum_{i=1}^{+\infty} |U_i|^r \text{ where } \{U_i\} \text{ is a } \delta\text{-cover of } E,$$

$$\implies \mathcal{H}_\delta^s(E) \leq \inf \left\{ \sum_{i=1}^{+\infty} |U_i|^r : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\} = \mathcal{H}_\delta^r(E)$$

2.

$$|U_i| \geq 0, s > 0 \implies \sum_{i=1}^{+\infty} |U_i|^s \geq 0$$

$$\implies 0 \leq \mathcal{H}_\delta^s(E) = \inf \left\{ \sum_{i=1}^{+\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\}.$$

So $\mathcal{H}_\delta^s(E) \geq 0$ for all $\delta > 0$.

We will show that $\mathcal{H}_\delta^s(E)$ gets arbitrarily close to 0 as $\delta \to 0$, which then implies $\mathcal{H}^s(E) = 0$.

Let $b = s - r > 0$ since $r < s$. Let $0 < \delta < 1$. Let $\{U_i\}$ be a δ-coverage of F, and therefore $|U_i| < \delta < 1$.

$$|U_i|^s = |U_i|^r \cdot |U_i|^b$$
$$\implies |U_i|^s \leq |U_i|^r \delta^b$$
$$\implies \sum_{i=1}^{+\infty} |U_i|^s \leq \sum_{i=1}^{+\infty} |U_i|^r \delta^b = \delta^b \sum_{i=1}^{+\infty} |U_i|^r.$$

19

$$\mathcal{H}_\delta^r(E) = \inf \left\{ \sum_{i=1}^{+\infty} |U_i|^r : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\}$$

$$\implies \text{ For all } \varepsilon > 0, \text{ there exists a } \delta\text{-cover } \{U_i\} \text{ such that } \mathcal{H}_\delta^r(E) + \varepsilon > \sum_{i=1}^{+\infty} |U_i|^r$$

$$\implies \delta^b(\mathcal{H}_\delta^r(E) + \varepsilon) > \delta^b \sum_{i=1}^{+\infty} |U_i|^r \geq \sum_{i=1}^{+\infty} |U_i|^s,$$

and thus $\mathcal{H}_\delta^s(E) = \inf \left\{ \sum_{i=1}^{+\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\} < \delta^b(\mathcal{H}_\delta^r(E) + \varepsilon)$.

Setting $\varepsilon = 1$, we get $\mathcal{H}_\delta^s(E) < \delta^b(\mathcal{H}_\delta^r(E) + 1)$.

From problem 2.1.3, we know that $\mathcal{H}_\delta^r(E)$ weakly increases as δ decreases towards 0, and therefore

$$\mathcal{H}^r(E) = \lim_{\delta \to 0} \mathcal{H}_\delta^r(E) \geq \mathcal{H}_\delta^r(E),$$

so for any $\delta > 0$, we have $\mathcal{H}_\delta^s(E) < \delta^b(\mathcal{H}^r(E) + 1)$.

By our assumption, $\mathcal{H}^r(E) + 1$ is fixed and $< \infty$, so as $\delta \to 0$, $\delta^b(\mathcal{H}^r(E) + 1)$ gets arbitrarily close to 0. It is decreasing as $\delta \to 0$, therefore the limit is defined as $\inf(\lim_{\delta \to 0} \delta^b(\mathcal{H}^r(E) + 1))$, which is 0. Therefore,

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(E) = 0,$$

so $\mathcal{H}^s(E) = 0$ as desired.

13 Problem 2.1.8

Proof: Let $E \subset \mathbb{R}^n, n \in \mathbb{N}$. Let A be the n-dimensional unit open hypercube centred at the origin, i.e., $A = (-0.5, 0.5)^n$.

Let $s > n$. We prove the following claim:

Claim. $\mathcal{H}^s(A) = 0$.

Proof. Let $\delta > 0$. We can partition A into closed n-dimensional hypercubes of side length $\frac{\delta}{2\sqrt{n}}$.

It is clear that these cubes have diameter equal to the length of the diagonal that goes from $(-0.5, -0.5, \ldots, -0.5)$ to $(0.5, 0.5, \ldots, 0.5)$. And therefore, by Pythagoras theorem, the diameter is $\sqrt{\left(\frac{\delta}{2\sqrt{n}}\right)^2 \cdot n} = \frac{\delta}{2} < \delta$.

To cover one edge of A, we need $\frac{1}{\left(\frac{\delta}{2\sqrt{n}}\right)} = \frac{2\sqrt{n}}{\delta}$ cubes, so to cover all of A, we will need $\left(\frac{2\sqrt{n}}{\delta}\right)^n = \frac{(2\sqrt{n})^n}{\delta^n}$ cubes.

Therefore,

$$\sum_{i=1}^{\frac{(2\sqrt{n})^n}{\delta^n}} |\text{small cube}|^s$$

$$= \frac{(2\sqrt{n})^n}{\delta^n} \cdot \left(\frac{\delta}{2}\right)^s$$

$$= \delta^{s-n} \cdot \left(\frac{(2\sqrt{n})^n}{2^s}\right).$$

Since these cubes cover A, and each have diameter $< \delta$, they are a δ-cover of A, and therefore, by definition,

$$H_\delta^s(A) \le \delta^{s-n} \left(\frac{(2\sqrt{n})^n}{2^s}\right).$$

$s - n > 0$ since $s > n$, and thus as $\delta \to 0$, $\delta^{s-n} \left(\frac{(2\sqrt{n})^n}{2^s}\right)$ decreases. This means

$$\lim_{\delta \to 0} \delta^{s-n} \left(\frac{(2\sqrt{n})^n}{2^s}\right) = \inf \left\{\delta^{s-n} \left(\frac{(2\sqrt{n})^n}{2^s}\right)\right\} = 0 \implies \mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(A) \le 0.$$

By part 1 of problem 2.1.4, we know $\mathcal{H}^s(A) \ge 0$, and therefore $\mathcal{H}^s(A) = 0$. $\square$

Now consider $cA = \{cx : x \in A\}$ for $c > 0$. By problem 2.1.5, $\mathcal{H}^s(cA) = c^s \mathcal{H}^s(A) = 0$. $cA = \left(-\frac{c}{2}, \frac{c}{2}\right)^n$. Let

$$B = \bigcup_{i \in \mathbb{N}} iA.$$

Claim. $B = \mathbb{R}^n$.

Proof. $A \subset \mathbb{R}^n \implies iA \subset \mathbb{R}^n \implies B \subset \mathbb{R}^n$.

Take some $x \in \mathbb{R}^n$ and let $x = (x_1, x_2, \ldots, x_n)$. Let

$$m = \lfloor \max_{1 \le i \le n} |x_i| \rfloor + 1 \implies |x_i| < m \text{ for all } 1 \le i \le n.$$

21

Consider $2mA = (-m, m)^n$. Clearly $x = (x_1, \ldots, x_n) \in 2mA$ which in turn implies that $x \in B = \bigcup_{i \in \mathbb{N}} iA$, and therefore $\mathbb{R}^n \subset B$.

Therefore $B = \mathbb{R}^n$. $\square$

By part 3 of problem 2.1.4, we must have

$$\mathcal{H}^s(\mathbb{R}^n) \leq \sum_{i=1}^{+\infty} \mathcal{H}^s(iA) = 0,$$

but by part 1 of problem 2.1.4, $\mathcal{H}^s(\mathbb{R}^n) \geq 0$. This means that $\mathcal{H}^s(\mathbb{R}^n) = 0$.

By problem 2.0.3 and theorem 2.1.1, for $E \subset \mathbb{R}^n$, we must have

$$\mathcal{H}^s(E) \leq \mathcal{H}^s(\mathbb{R}_n) = 0,$$

but also by part 1 of 2.1.4, we have $\mathcal{H}^s(E) \geq 0 \implies \mathcal{H}^s(E) = 0$.

Therefore, for all $s > n$, we have $\mathcal{H}^s(E) = 0$.

B.W.O.C. assume that the Hausdorff dimension of $E = d > n$. Therefore, for all $r < d$, we have $\mathcal{H}^r(E) = +\infty$ by definition 2.1.2 and part 1 of problem 2.1.7.

Since $d > n$, we may choose $n < s < d$. $s > n \implies \mathcal{H}^s(E) = 0$, but $s < d \implies \mathcal{H}^s(E) = +\infty$, which is a contradiction.

Therefore, the Hausdorff dimension of $E \leq n$, which finishes our proof of the question.

14   Problem 2.1.9

Proof: Let $E \in \mathbb{R}^n$, we consider two cases on E.

- **E is finite**. Let $x_1, \ldots, x_m$ be the elements of E, where $x_i \neq x_j$ for $i \neq j$, i.e.,

$$E = \{x_1, x_2, \ldots, x_m\},$$

  where

$$\#E = m.$$

  Let $1 \leq k \leq m, k \in \mathbb{N}$, let $\delta > 0$. $\{\{x_k\}\}$ is a δ-cover for $\{x_k\}$ since $x_k \in \{x_k\}$ and $|\{x_k\}| = |x_k - x_k| = 0 < \delta$. Therefore,

$$\mathcal{H}_\delta^0(\{x_k\}) = \inf \left\{ \sum_{i=1}^{+\infty} |U_i|^0 : \{U_i\} \text{ is a } \delta\text{-cover of } \{x_k\} \right\}$$
$$\leq \sum_{i=1}^{1} 0^0$$
$$= 1.$$

  Notice that any cover of x_k must contain a set, since $x_k \notin \emptyset$. This means $\sum_{i=1}^{+\infty} |U_i|^0 \geq 1$. So $\mathcal{H}_\delta^0(\{x_k\}) = 1$ for all $\delta > 0$, which then implies $\mathcal{H}^0(\{x_k\}) = \lim_{\delta \to 0} \mathcal{H}_\delta^0(\{x_k\}) = 1$.

  By theorem 2.1.1, we must have

$$\mathcal{H}^0(E) = \mathcal{H}^0(\{x_1\}) + \mathcal{H}^0(\{x_2\}) + \ldots + \mathcal{H}^0(\{x_m\})$$
$$= 1 + 1 + \ldots + 1$$
$$= m$$
$$= \#E,$$

  so if E is finite, $\mathcal{H}^0(E)$ is simply the counting measure.

- **E is infinite.** As shown in the first case, for any point $x \in \mathbb{R}^n$, we have $\mathcal{H}^0(\{x\}) = 1$. B.W.O.C. suppose that $\mathcal{H}^0(E)$ is finite, and say it is equal to some $a \in \mathbb{R}$.

  We choose $x_1, x_2, \ldots, x_{\lfloor a \rfloor + 1} \in E$ which are pairwise distinct. This is always possible since E is infinite. Let $A = \{x_1, x_2, \ldots, x_{\lfloor a \rfloor + 1}\} \subset E$. By the first case, $\mathcal{H}^0(E) = \#A = \lfloor a \rfloor + 1 > a = \mathcal{H}^0(E)$. But by theorem 2.1.1 and problem 2.0.3, $A \subset E \implies \mathcal{H}^0(A) \leq \mathcal{H}^0(E)$ which gives a contradiction. So $\mathcal{H}^0(E)$ cannot be finite and must be infinity. And also $\#E = \infty$ by the definition of the counting measure.

Therefore, $\mathcal{H}^0$ is simply the counting measure.

15   Problem 4.0.1

Proof:

1. Let $k = \lceil \log_c(\varepsilon/|x - y|) \rceil + 1$ is finite. Notice that

$$
\begin{aligned}
|f^{(k)}(x) - f^{(k)}(y)| &= c|f^{(k-1)}(x) - f^{(k-1)}(y)| \\
&= c^2|f^{(k-2)}(x) - f^{(k-2)}(y)| \\
&= \cdots \\
&= c^{k-1}|f(x) - f(y)| \\
&= c^k|x - y| \\
&\leq c^{\log_c(\varepsilon/|x-y|)+1}|x - y| \\
&= c\varepsilon \\
&< \varepsilon.
\end{aligned}
$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ s.t.

$$
f(x) = \frac{x - \sqrt{x^2 + 4}}{2}.
$$

(This is a branch of the hyperbola with $y = x$ and $y = 0$ as asymptotes, the branch below both of them.)

Notice that f is increasing and concave. This is because

$$
\begin{aligned}
f'(x) &= \frac{1}{2} - \frac{2x}{4\sqrt{x^2 + 4}} \\
&= \frac{1}{2} - \frac{x}{2\sqrt{x^2 + 4}} \\
&= \frac{\sqrt{x^2 + 4} - x}{2\sqrt{x^2 + 4}} \\
&> \frac{\sqrt{x^2} - x}{2\sqrt{x^2 + 4}} \\
&= \frac{|x| - x}{2\sqrt{x^2 + 4}} \\
&\geq 0
\end{aligned}
$$

and so $f'(x) > 0$, and

$$
\begin{aligned}
f''(x) &= -\frac{1}{2} \cdot \frac{\sqrt{x^2+4} \cdot 1 - x \cdot \frac{1}{2} \cdot 2x \cdot \frac{1}{\sqrt{x^2+4}}}{x^2+4} \\
&= -\frac{\sqrt{x^2+4} - \frac{x^2}{\sqrt{x^2+4}}}{2(x^2+4)} \\
&= -\frac{(x^2+4) - x^2}{2(x^2+4)\sqrt{x^2+4}} \\
&= -\frac{2}{(x^2+4)\sqrt{x^2+4}} \\
&< 0,
\end{aligned}
$$

and so $f''(x) < 0$.

Also, notice that

$$
\begin{aligned}
f'(x) - 1 &= \frac{\sqrt{x^2+4} - x}{2\sqrt{x^2+4}} - 1 \\
&= -\frac{\sqrt{x^2+4} + x}{2\sqrt{x^2+4}} \\
&< 0,
\end{aligned}
$$

so we can show that $f'(x) < 1$.

W.L.O.G. let $x > y$, since $f''(y) < 0$, we will know that

$$
f(x) - f(y) < (x - y)f'(y) < (x - y)
$$

and so $|f(x) - f(y)| < |x - y|$.

Notice at the same time,

$$
f(x) - x = -\frac{x + \sqrt{x^2+4}}{2} < 0
$$

so there does not exist x such that $f(x) = x$.

16   Problem 4.0.2

Proof: To show that $f(K)$ is compact given f is a contraction and K is compact, we show that $f(K)$ is closed and bounded.

- $f(K)$ **is bounded.** Since K is bounded, there exists some $r > 0$ such that $K \subset B_{(0,\dots,0)}(r)$. In other words, for all $x \in K$, $|x - (0,\dots,0)| < r$.

  Since f is a contraction, set $y = (0,\dots,0)$, we will see that $|f(x) - f((0,\dots,0))| = c|x - (0,\dots,0)| < cr$.

  By the triangular inequality, we have

  $$|f(x) - (0,\dots,0)| \leq |f(x) - f((0,\dots,0))| + |f((0,\dots,0)) - (0,\dots,0)|$$
  $$< cr + |f((0,\dots,0)) - (0,\dots,0)|.$$

  Therefore, $f(K) \subset B_{(0,\dots,0)}(cr + |f((0,\dots,0)) - (0,\dots,0)|)$, which shows $f(K)$ is bounded.

- $f(K)$ **is closed.** We would like to first prove the lemma that a set is closed if and only if all its limit points (a point that can be reached using a limit of a sequence of points within the set) is contained in the set.

  - **If direction.** Suppose a set C contains all its limit points. Now suppose $x \in \mathbb{R}^n \setminus C$. Then x must not be a limit point of C. By the $\varepsilon - \delta$ definition of a limit, it must be true that there is an open neighbourhood of x (an open ball centred at x) that does not intersect with C. (It is not a limit point means it is not arbitrarily close to C.) For every $x \in \mathbb{R}^n \setminus C$, say such ball is $B_x(r_x)$.
    Define that
    $$S = \bigcup_{x \in \mathbb{R}^n \setminus C} B_x(r_x).$$

    Obviously $S \subset \mathbb{R}^n \setminus C$ by how we defined r_x. At the same time $\mathbb{R}^n \setminus C \subset S$ since $x \in B_x(r_x) \subset S$. Therefore, $S = \mathbb{R}^n \setminus C$ and we have shown that it is a union of open balls, therefore open, and therefore C being closed.

  - **Only if direction.** Suppose C is closed. Then $\mathbb{R}^n \setminus C$ is open. Suppose there is a limit point $x \notin C$, and therefore $x \in \mathbb{R}^n \setminus C$.
    By the definition of an open set being a union of open balls, x must be within some open ball (say $B_p(s)$ which is itself a subset of $\mathbb{R}^n \setminus C$. It is not difficult to see that $s > |x - p|$ by definition of an open ball, and from the triangular inequality that $B_x(s - |x - p|) \subset B_p(s) \subset \mathbb{R}^n \setminus C$: for some point y s.t. $|y - x| < s - |x - p|$,

    $$|y - p| \leq |y - x| + |x - p| < s - |x - p| + |x - p| < s.$$

    Therefore for $r = s - |x - p|$, $B_x(r) \subset \mathbb{R}^n \setminus C$, and $B_x(r) \cap C = \emptyset$, which contradicts with given that x is a limit point of C by considering $\varepsilon \leq r$ in the $\varepsilon - \delta$-definition of a limit. Therefore C contains all its limit points.

26

We would also like to show that f is continuous. But this is straightforward since

$$\lim_{x \to y} |f(x) - f(y)| = c |\lim_{x \to y} x - y| = 0$$

which then implies that $\lim_{x \to y} f(x) = f(y)$ which means f is continuous.

Now we are ready to start the proof. We would like to show that $f(K)$ is closed by showing it contains all its limit points. Consider a limit point formed by $\{f(x_i)\}_{i=1}^{+\infty}$ where $x_i \in K$. We must have

$$\lim_{i \to +\infty} f(x_i) = f \left(\lim_{i \to +\infty} x_i \right).$$

Since K is closed, K contains all its limit points, and therefore $\lim_{i \to +\infty} x_i \in K$, and therefore $\lim_{i \to +\infty} f(x_i) = f(\lim_{i \to +\infty} x_i) \in f(K)$, which shows that $f(K)$ contains all its limit points, hence closed.

We have shown that given K is closed, $f(K)$ is closed. We have also shown that given K is bounded, $f(K)$ is bounded. Therefore, given K is compact, $f(K)$ is compact.

17   Problem 4.0.3

Proof: We first prove that C is closed if and only if it contains all points x such that for all $r > 0$, $B_x(r) \cap C \neq \emptyset$.

- **Only if direction.** We would like to prove that if C is closed, then for x s.t. for all $r > 0$, $B_x(r) \cap C \neq \emptyset$, we have $x \in C$.

  Since C is closed, $\mathbb{R}^n \setminus C$ is open. B.W.O.C. suppose that $x \notin C$, so $x \in \mathbb{R}^n \setminus C$. By the definition of an open set being a union of open balls, x must be within some open ball (say $B_p(s)$ which is itself a subset of $\mathbb{R}^n \setminus C$. It is not difficult to see that $s > |x - p|$ by definition of an open ball, and from the triangular inequality that $B_x(s - |x - p|) \subset B_p(s) \subset \mathbb{R}^n \setminus C$: for some point y s.t. $|y - x| < s - |x - p|$,

  $$|y - p| \leq |y - x| + |x - p| < s - |x - p| + |x - p| < s.$$

  Therefore for $r = s - |x - p|$, $B_x(r) \subset \mathbb{R}^n \setminus C$, and $B_x(r) \cap C = \emptyset$, which contradicts with given that $B_x(r) \cap C \neq \emptyset$.

- **If direction.** We would like to prove that if x is s.t. for all $r > 0$, $B_x(r) \cap C \neq \emptyset$, then $x \in C$, then C is closed.

  To show that C is closed, we show that $\mathbb{R}^n \setminus C$ is open. From the given condition, we know that for all $x \in \mathbb{R}^n \setminus C$, there exists some $r_x > 0$ s.t. $B_x(r_x) \subset \mathbb{R}^n \setminus C$.

  (If not, then for all $r > 0$, $B_x(r_x) \not\subset \mathbb{R}^n \setminus C$, which implies $B_x(r_x) \cap C \neq \emptyset$ which implies $x \in C$ which contradicts with $x \in \mathbb{R}^n \setminus C$.)

  Define that

  $$S = \bigcup_{x \in \mathbb{R}^n \setminus C} B_x(r_x).$$

  Obviously $S \subset \mathbb{R}^n \setminus C$ by how we defined r_x. At the same time $\mathbb{R}^n \setminus C \subset S$ since $x \in B_x(r_x) \subset S$. Therefore, $S = \mathbb{R}^n \setminus C$ and we have shown that it is a union of open balls, therefore open, and therefore C being closed.

Now we show the original statement. Assume F_1, F_2 are both attractors of an IFS $(f_1, \ldots, f_m)$ and F_1, F_2 are non-empty and compact. We would like to show that $F_1 = F_2$. Define $f(X) = \bigcup_i f_i(X)$, we have $f(F_1) = F_1, f(F_2) = F_2$.

Let $\mathcal{C}$ be the set of all non-empty compact subsets of $\mathbb{R}^n$. We consider defining a metric d on $\mathcal{C}$ for $A, B \in \mathcal{C} \subset \mathbb{R}^n$ by follows.

We first define the distance of a point $x \in X$ to the set Y as

$$d(x, Y) = \inf_{y \in Y} |x - y|,$$

which is the 'shortest distance from x to any point $y \in Y$'. Now, the distance between two sets (to let it satisfy properties of metrics), could then be defined as follows:

$$d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\},$$

as 'the furthest the shortest distance can be'.

It is a metric since for $A, B, C \in \mathcal{C}$:

- $d(A, B) \geq 0$ is true since $|x - y| \geq 0$.

  If $d(A, B) = 0$, then it must be true that $\sup_{a \in A} d(a, B) = \sup_{b \in B} d(b, A) = 0$, which means $\inf_{b \in B} |a - b| = 0$ for all $a \in A$ and $\inf_{a \in A} |a - b| = 0$ for all $b \in B$.

  This means that for all $a \in A$, for any $\varepsilon > 0$, there exists a $b \in B$ such that $a \in B_b(\varepsilon)$. So b satisfies that for all $\varepsilon > 0$, $B_b(\varepsilon) \cap A \supset \{a\}$ and therefore $B_b(\varepsilon) \cap A \neq \emptyset$. But since A is compact, it is closed, and therefore, it contains all points x such that for $r > 0, B_x(r) \cap C \neq \emptyset$, and therefore $b \in A$ for all $b \in B$, therefore $B \subset A$. A symmetrical argument can be made to show $A \subset B$ and therefore $A = B$.

- $d(A, B) = d(B, A)$ is true by symmetry of the definition;

- $d(A, B) + d(B, C) \geq d(A, C)$ inherits from the fact that $|a - b|$ is a metric. Specifically, $d(a, C) \leq |a - c| \leq |a - b| + |b - c|$ by definition and by triangular inequality for any $a \in A, b \in B, c \in C$. Taking $\inf_{c \in C}$ will therefore then give us $d(a, C) \leq |a - b| + d(B, C)$, for which if we take $\inf_{b \in B}$ will therefore be $d(a, C) \leq d(a, B) + d(B, C)$, but since $d(a, B) \leq d(A, B)$ we must have $d(a, C) \leq d(A, B) + d(B, C)$, and therefore $\sup_{a \in A} d(a, C) \leq d(A, B) + d(B, C)$. A similar argument can be shown on $\sup_{c \in C} d(A, c) \leq d(A, B) + d(B, C)$, and therefore

$$d(A, C) = \max \left\{ \sup_{a \in A} d(a, C), \sup_{c \in C} d(b, A) \right\} \leq d(A, B) + d(B, C)$$

  as required.

Now, we show that f has only one stationary point on $\mathcal{C}$. By definition, for any $A, B \in \mathcal{C}$, we must have

$$d(f(A), f(B)) = d \left(\bigcup_{i=1}^{m} f_i(A), \bigcup_{i=1}^{m} f_i(B) \right),$$

Let

$$M = \max_{1 \leq i \leq m} d(f_i(A), f_i(B)),$$

we must have

$$\sup_{a \in f_i(A)} d(a, f_i(B)) \leq M$$

for all $1 \leq i \leq M$ by definition, and therefore that

$$\sup_{a \in \bigcup_{i=1}^{m} f_i(A)} d \left(a, \bigcup_{i=1}^{m} f_i(B) \right) \leq M,$$

and similarly

$$\sup_{a \in \bigcup_{i=1}^{m} f_i(B)} d \left(a, \bigcup_{i=1}^{m} f_i(A) \right) \leq M.$$

This then shows that

$$d(f(A), f(B)) \leq M = \max_{1 \leq i \leq m} d(f_i(A), f_i(B)) \leq \left(\max_{1 \leq i \leq m} c_i \right) d(A, B).$$

Therefore, if $F_1, F_2 \in \mathcal{C}$ are bot attractors for the IFS $(f_1, f_2, \ldots, f_m)$, i.e., $f(F_1) = F_1, f(F_2) = F_2$, we must have

$$d(f(F_1), f(F_2)) \leq d(F_1, F_2) \leq \left(\max_{1 \leq i \leq m} c_i \right) d(F_1, F_2)$$

since we must have $0 < \max_{1 \leq i \leq m} c_i < 1$, this implies that we must have $d(F_1, F_2) = 0$, and therefore, $F_1 = F_2$, which shows the non-empty compact attractor is unique, as desired.

18   Problem 4.0.4

Proof: For this question, due to the constructive proof of the existence of a attractor for any IFS, we just have to show that the IFS's union process is equivalent to the iterative process of the iterative definition of the fractal, and that the starting point is compact.

- **Sierpinski Carpet.** An equivalent definition of the Sierpinski Carpet is by starting with $[0,1] \times [0,1]$, and each time shrinking the size of the whole shape with scale factor $\frac{1}{3}$ and creating 8 copies of that. Therefore, we can define the IFS $(f_1, \ldots, f_8)$ on $\mathbb{R}^2 \to \mathbb{R}^2$:

$$f_1((x,y)) = \left(\frac{1}{3}x, \frac{1}{3}y\right),$$

$$f_2((x,y)) = \left(\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y\right),$$

$$f_3((x,y)) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y\right),$$

$$f_4((x,y)) = \left(\frac{1}{3}x, \frac{1}{3}y + \frac{1}{3}\right),$$

$$f_5((x,y)) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{1}{3}\right),$$

$$f_6((x,y)) = \left(\frac{1}{3}x, \frac{1}{3}y + \frac{2}{3}\right),$$

$$f_7((x,y)) = \left(\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{2}{3}\right),$$

$$f_8((x,y)) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3}\right).$$

  Here, f_1 to f_8 generates the bottom-left, bottom, bottom-right, left, right, top-left, top, top-right $\frac{1}{3}$ by $\frac{1}{3}$ respectively. We can verify this simply by plugging two values for the first iteration (say, $(0,0)$ and $(1,0)$) since they are affine. This is an IFS since f_1 to f_8 are all shrinking, in fact with contraction ratio $\frac{1}{3}$

  It is not difficult to see that $f(X) = \bigcup_{i=1}^{8} f_i(X)$ maps $\square_k$ to $\square_{k+1}$ for $k \in \mathbb{N}$ since this is exactly the definition. Therefore, the attractor of this IFS is

$$\square = \bigcap_{i \in \mathbb{N}} f^{(n)}(\square_0)$$

  since $\square_0 = [0,1]^2$ is compact. This is exactly the Sierpinski Carpet $\square$.

- **Koch Curve.** We can see that the iterative process of the Koch curve can be split into four pieces: the two pieces that got shrunken with scale factor $\frac{1}{3}$ and gets copied four times, two of which are then rotated into the correct position. Therefore, define

the IFS (f_1, f_2, f_3, f_4) on $\mathbb{R}^2 \to \mathbb{R}^2$:

$$f_1((x, y)) = \left(\frac{1}{3}x, \frac{1}{3}y \right),$$

$$f_2((x, y)) = \left(\frac{1}{6}x - \frac{\sqrt{3}}{6}y + \frac{1}{3}, \frac{\sqrt{3}}{6}x + \frac{1}{6}y \right),$$

$$f_3((x, y)) = \left(\frac{1}{6}x + \frac{\sqrt{3}}{6}y + \frac{1}{2}, -\frac{\sqrt{3}}{6}x + \frac{1}{6}y + \frac{\sqrt{3}}{6} \right),$$

$$f_4((x, y)) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y \right).$$

Here, f_1 generates the left part, f_4 generates the right part, f_2 generates the part tilted counter clockwise by $\frac{\pi}{3}$, and f_3 generates the part tilted clockwise by $\frac{\pi}{3}$. This is an IFS since f_1 to f_4 are all shrinking, in fact with contraction ratio $\frac{1}{3}$.

It is not difficult to see that $f(X) = \bigcup_{i=1}^{4} f_i(X)$ maps K_k to K_{k+1} for $k \in \mathbb{N}$ since this is exactly the definition. Therefore, the attractor of this IFS is

$$K = \bigcap_{i \in \mathbb{N}} f^{(n)}(K_0)$$

since $K_0 = [0, 1]$ is compact. This is exactly the Koch Curve K.

- **Minkowski Sausage.** We can see that the iterative process of the Minkowski Sausage can be split into 8 pieces, all shrunken with scale factor $\frac{1}{4}$ and affine transformed into the correct position. Therefore, define the IFS $(f_1, \ldots, f_8)$ on $\mathbb{R}^2 \to \mathbb{R}^2$:

$$f_1((x, y)) = \left(\frac{1}{4}x, \frac{1}{4}y \right),$$

$$f_2((x, y)) = \left(-\frac{1}{4}y + \frac{1}{4}, \frac{1}{4}x \right),$$

$$f_3((x, y)) = \left(\frac{1}{4}x + \frac{1}{4}, \frac{1}{4}y + \frac{1}{4} \right),$$

$$f_4((x, y)) = \left(\frac{1}{4}y + \frac{1}{2}, -\frac{1}{4}x + \frac{1}{4} \right),$$

$$f_5((x, y)) = \left(\frac{1}{4}y + \frac{1}{2}, -\frac{1}{4}x - \frac{1}{4} \right),$$

$$f_6((x, y)) = \left(\frac{1}{4}x + \frac{1}{2}, \frac{1}{4}y - \frac{1}{4} \right),$$

$$f_7((x, y)) = \left(-\frac{1}{4}y + \frac{3}{4}, \frac{1}{4}x - \frac{1}{4} \right),$$

$$f_8((x, y)) = \left(\frac{1}{4}x + \frac{3}{4}, \frac{1}{4}y \right).$$

Here, f_1, f_3, f_6, f_8 generates the four horizontal parts, f_2, f_7 generates the two counter-clockwise rotations, and f_4, f_5 generates the two clockwise rotations. This is an IFS since f_1 to f_8 are all shrinking, in fact with contraction ratio $\frac{1}{4}$.

It is not difficult to see that $f(X) = \bigcup_{i=1}^{8} f_i(X)$ maps M_k to M_{k+1} for $k \in \mathbb{N}$ since this is exactly the definition. Therefore, the attractor of this IFS is

$$M = \bigcap_{i \in \mathbb{N}} f^{(n)}(M_0)$$

since $M_0 = [0, 1]$ is compact. This is exactly the Minkowski Sausage M.

19   Problem 4.1.1

Proof: Consider the open set $U = (0,1) \times \left(0, \frac{\sqrt{3}}{2}\right)$.

We have

$$f_1(U) = \left(0, \frac{1}{2}\right) \times \left(0, \frac{\sqrt{3}}{4}\right),$$

$$f_2(U) = \left(\frac{1}{2}, 1\right) \times \left(0, \frac{\sqrt{3}}{4}\right),$$

$$f_3(U) = \left(\frac{1}{4}, \frac{3}{4}\right) \times \left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right).$$

It is not difficult to see that $f_1(U) \cap f_2(U) = f_1(U) \cap f_3(U) = f_2(U) \cap f_3(U) = \emptyset$ due to obviously separated x or y intervals, and they are all contained within U. So this U is an open set, and this IFS satisfies the open set condition.

Notice that f_1, f_2, f_3 all have contraction ratio $c_1 = c_2 = c_3 = \frac{1}{2}$ as shown in the definition. Solving

$$\sum_{i=1}^{3} \frac{1}{2^s} = 1 \iff 3 = 2^s \iff s = \frac{\log 3}{\log 2},$$

which means the Hausdorff dimension of this IFS's attractor is $\frac{\log 3}{\log 2}$.

20   Problem 4.1.2

Proof: If we recall the IFS of the Cantor Set is (f_1, f_2) with

$$f_1(x) = \frac{1}{3}x,$$
$$f_2(x) = \frac{1}{3}x + \frac{2}{3},$$

we notice this satisfies the OSC by taking $U = (0, 1)$ (the open interval, which is an open set in $\mathbb{R}$), and $f_1(U) = \left(0, \frac{1}{3}\right)$, $f_2(U) = \left(\frac{2}{3}, 1\right)$. The contraction ratios are $c_1 = c_2 = \frac{1}{3}$ from the equations.

Therefore, solving

$$\sum_{i=1}^{2} \frac{1}{3^s} = 1 \iff 2 = 3^s \iff s = \frac{\log 2}{\log 3},$$

which means the Hausdorff dimension of the Cantor Set is $\frac{\log 2}{\log 3}$ as desired.

If we recall the IFS of the Koch Curve is (f_1, f_2, f_3, f_4) with

$$f_1((x, y)) = \left(\frac{1}{3}x, \frac{1}{3}y\right),$$

$$f_2((x, y)) = \left(\frac{1}{6}x - \frac{\sqrt{3}}{6}y + \frac{1}{3}, \frac{\sqrt{3}}{6}x + \frac{1}{6}y\right),$$

$$f_3((x, y)) = \left(\frac{1}{6}x + \frac{\sqrt{3}}{6}y + \frac{1}{2}, -\frac{\sqrt{3}}{6}x + \frac{1}{6}y + \frac{\sqrt{3}}{6}\right),$$

$$f_4((x, y)) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y\right).$$

we notice that this satisfies the OSC by taking U to be the interior of the triangle with vertices at $(0, 0)$, $(1, 0)$ and $\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$. We denote the interior triangle formed by vertices A, B and C to be $\Delta(A, B, C)$. Therefore, $U = \Delta\left((0, 0), (1, 0), \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)\right)$,

$$f_1(U) = \Delta\left((0, 0), \left(\frac{1}{3}, 0\right), \left(\frac{1}{6}, \frac{\sqrt{3}}{18}\right)\right),$$

$$f_2(U) = \Delta\left(\left(\frac{1}{3}, 0\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right), \left(\frac{1}{3}, \frac{\sqrt{3}}{9}\right)\right),$$

$$f_3(U) = \Delta\left(\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right), \left(\frac{2}{3}, 0\right), \left(\frac{2}{3}, \frac{\sqrt{3}}{9}\right)\right),$$

$$f_4(U) = \Delta\left(\left(\frac{2}{3}, 0\right), (1, 0), \left(\frac{5}{6}, \frac{\sqrt{3}}{18}\right)\right),$$

which looks like the following on a diagram. It can be clearly seen that they do not intersect each other, and they are all subsets of U. So this IFS satisfies the OFC.
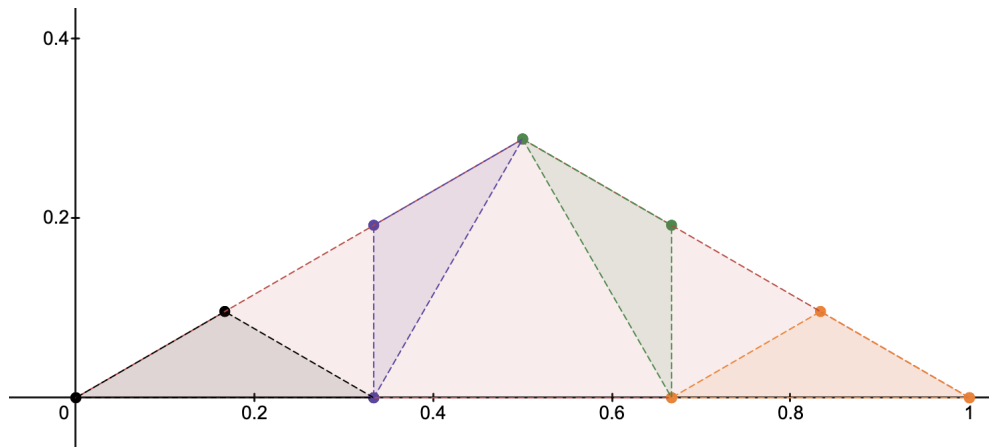
The contraction ratios are $c_1 = c_2 = c_3 = c_4 = \frac{1}{3}$ from the definition of a Koch curve. Therefore, solving

$$\sum_{i=1}^{4} \frac{1}{3^s} = 1 \iff 4 = 3^s \iff s = \frac{\log 4}{\log 3},$$

which means the Hausdorff dimension of the Koch curve is $\frac{\log 4}{\log 3}$.

21   Problem 4.1.3

Proof: B.W.O.C. assume the Cantor Set C was countable. By definition of a countable set, we could list the points as $\{a_i\}_{i=1}^{+\infty}$. Recall that for $s, \delta > 0$,

$$\mathcal{H}_\delta^s(C) = \inf \left\{ \sum_i |U_i|^s : \{U_i\}_i \text{ is a } \delta\text{-cover of } C \right\}.$$

Consider the cover $\{A_i\}_i(\varepsilon)$ for $\varepsilon > 0$, where A_i is the open interval of size $\frac{\varepsilon}{2^i}$ centred at a_i, i.e., $A_i = B_{a_i}\left(\frac{\varepsilon}{2^i}\right)$. Since each point has an open interval covering it, this is clearly a cover of a_i; and we can make the first set have diameter $|A_1| < \delta$ by taking ε to be arbitrarily small, so the it satisfies the condition, and since the set sizes monotonically decrease the rest of them also satisfy the condition. Thus the cover is a δ-cover, and therefore

$$\mathcal{H}_\delta^s(C) \leq \sum_{i=0}^{\infty} \left(\frac{\varepsilon}{2^n}\right)^s$$

$$= \varepsilon^s \sum_{i=0}^{\infty} (2^{-s})^n$$

$$= \frac{\varepsilon^s}{1 - 2^{-s}}.$$

Since $s > 0$, this converges, and taking ε arbitrarily small we can make this arbitrarily close to 0. This is true for each δ, so $\mathcal{H}^s(C) \leq 0$. By 2.1.4.1, $\mathcal{H}^s(C) \geq 0$, so $\mathcal{H}^s(C) = 0$. Hence the Hausdorff dimension of C is $\leq s$ for all $s > 0$. But taking $s \ll \frac{\log(2)}{\log(3)}$, we know that the Hausdorff dimension of $C \ll \frac{\log(2)}{\log(3)}$. However by Theorem 4.1.2 $\frac{\log(2)}{\log(3)}$ was the unique Hausdorff dimension so this leads to a contradiction, and C cannot be countable.

22   Problem 4.1.4

Proof: We aim to construct an IFS in $\mathbb{R}^k$ where $k = \lceil \frac{m}{n} \rceil$, and will be concerning the open set condition with the open unit cube $(0,1)^k \subset \mathbb{R}^k$.

We let the contraction ratio be $c = 2^{-n}$, and we let the number of contractions be $p = 2^m$. Now, if there exists an IFS such that the OSC is satisfied, we will have the dimension s satisfies that

$$p \cdot c^s = 1 \iff 2^m \cdot \left(2^{-n}\right)^s = 1 \iff 2^{m-ns} = 1 \iff s = \frac{m}{n}.$$

Now the task remaining is to find some IFS that satisfies the OSC. But this is straightforward: for each contraction in the IFS, we first apply $x \mapsto 2^{-n}x$ to the input, which means the unit cube $(0,1)^k$ will be mapped to $(0, 2^{-n})^k$.

It is well-known that one could place $(2^n)^k = 2^{nk}$ cubes of size congruent to $(0, 2^{-n})^k$ in the unit cube $(0,1)^k$ without overlapping, simply by splitting the unit cube into small cubes with side length 2^{-n}. Each side will be divided into 2^n pieces, which gives the result number of cubes 2^{nk} as we desired. We only have 2^m contractions, and since we have $k = \lceil \frac{m}{n} \rceil \geq \frac{m}{n}$, this means $m \leq nk$ and therefore the placement of 2^m contractions is possible in this case, such that the image $(0,1)^k$ of any two contractions do not overlap, and is still a subset of $(0,1)^k$.

Therefore for some IFS, $(0,1)^k$ satisfies the OSC with the desired number of contractions and contraction ratio. meaning the dimension of the IFS can take any rational number. And since every IFS has an attractor $E \subset \mathbb{R}^n$, we know that there must exist some set $E \subset \mathbb{R}^n$ that has a Hausdorff dimension of $\frac{m}{n}$.

But since postive rational numbers $\mathbb{Q}_+$ is dense in positive real numbers $\mathbb{R}_+$, we can now conclude that the Hausdorff dimension can be arbitrarily close to any positive real number $r > 0, r \in \mathbb{R}$.

23   Problem 4.1.5

Proof: Consider the IFS (f_1, f_2) defined on $\mathbb{R} \to \mathbb{R}$ as follows:

- $f_1(x) = \frac{2}{3}x,$

- $f_2(x) = \frac{2}{3}x + \frac{1}{3}.$

Note that they have contraction ratios $c_1 = c_2 = \frac{2}{3}$.
Notice that $D = [0, 1]$ is an attractor of this IFS, since

$$f_1([0, 1]) = \left[0, \frac{2}{3}\right],$$

$$f_2([0, 1]) = \left[\frac{1}{3}, 1\right],$$

and therefore $f_1(D) \cup f_2(D) = [0, 1] = D$.

The Hausdorff dimension of $[0, 1]$ is $s = 1$. This can be shown by considering the IFS (g_1, g_2) on $\mathbb{R}$:

- $g_1(x) = \frac{1}{2}x,$

- $g_2(x) = \frac{1}{2}x + \frac{1}{2}.$

Both g_1 and g_2 have contraction ratios $\frac{1}{2}$, and its is easy to see it satisfies the OSC by taking $(0, 1)$ as the open set, and we can see as well that $[0, 1]$ is the attractor. Therefore, the Hausdorff dimension of $[0, 1]$ s' must satisfy that $2 \cdot \frac{1}{2}^s = 1$, which gives $s = 1$.

However, if the OSC holds, then the dimension should also satisfy that

$$\sum_{i=1}^{2} \left(\frac{2}{3}\right)^s = 1,$$

which is clearly not the case. Therefore, the OSC does not hold for this IFS.

24   Problem 4.2.1

Proof:

- **The Hausdorff dimension of $\mathbb{R}$ is 1.** The Hausdorff dimension of $[0,1]$ is $s = 1$. This can be shown by considering the IFS (f_1, f_2) on $\mathbb{R}$:

  - $f_1(x) = \frac{1}{2}x$,
  - $f_2(x) = \frac{1}{2}x + \frac{1}{2}$.

  Both f_1 and f_2 have contraction ratios $\frac{1}{2}$, and its is easy to see it satisfies the OSC by taking $(0,1)$ as the open set, and we can see as well that $[0,1]$ is the attractor. Therefore, the Hausdorff dimension of $[0,1]$ s' must satisfy that $2 \cdot \frac{1}{2}^{s'} = 1$, which gives $s' = 1$.

  By problem 2.1.8, we know that $\mathbb{R} \subset \mathbb{R}^1$ so the Hausdorff dimension of $\mathbb{R}$ is ≤ 1.

  By theorem 2.1.1, the Hausdorff measure is a measure on $\mathbb{R}^n$ over the Borel σ-algebra, which contains $[0,1]$ and $\mathbb{R}$, and so since $[0,1] \subset \mathbb{R}$, we know that for all $s > 0$, we know that
  $$\mathcal{H}^s([0,1]) \leq \mathcal{H}^s(\mathbb{R}).$$

  From problem 2.1.7, we know
  $$\text{Hausdorff dimension of } E = \{\inf_{s \geq 0} \mathcal{H}^s(E) = 0\} = \{\sup_{s \geq 0} \mathcal{H}^s(E) = +\infty\}.$$

  The Hausdorff dimension of $[0,1]$ being 1 implies that for all $s < 1$,
  $$+\infty = \mathcal{H}^s([0,1]) \leq \mathcal{H}^s(\mathbb{R}),$$

  and therefore
  $$\text{Hausdorff dimension of } \mathbb{R} = \sup_{s \geq 0} \mathcal{H}^s(\mathbb{R}) = +\infty \geq 1.$$

  Since the Hausdorff dimension of $\mathbb{R}$ is ≥ 1 and ≤ 1, we conclude that the Hausdorff dimension of $\mathbb{R} = 1$.

- **The 1-dimensional Hausdorff measure of $\mathbb{R}$ is $+\infty$.** We first show that
  $$\mathcal{H}^1([0,1]) = 1.$$

  - **Upper bound:** $\mathcal{H}^1([0,1]) \leq 1$. To show this, we just have to show that for every δ, we have some δ-cover $\{U_i\}$ that satisfies $\sum_i |\{U_i\}| = 1$. We just consider the case where $0 < \delta < 1$. Consider the following construction for $\{U_i\}$ where $0 < \varepsilon < \delta$:
    $$\{[0, (\delta - \varepsilon)], [(\delta - \varepsilon), 2(\delta - \varepsilon)], \dots, [(n-1)(\delta - \varepsilon), n(\delta - \varepsilon)], [n(\delta - \varepsilon), 1]\},$$

where $n = \lfloor \frac{1}{\delta - \varepsilon} \rfloor$. It is not difficult to see for this construction,

$$\sum_i |U_i|$$
$$= [(\delta - \varepsilon) - 0] + \ldots + [n(\delta - \varepsilon) - (n-1)(\delta - \varepsilon)] + [1 - n(\delta - \varepsilon)]$$
$$= 1.$$

which by definition implies that $\mathcal{H}_\delta^1([0,1]) \leq 1$ for any $0 < \delta < 1$, and hence $\mathcal{H}^1([0,1]) \leq 1$.

– **Lower bound:** $\mathcal{H}^1([0,1]) \geq 1$. We consider the following definition of a measure μ on the Borel σ-algebra restricted to $[0,1]$ which is a compact set:

   * For Borel Sets $E \in [0,1]$ which take the form of the interval

$$E = \left[\frac{k-1}{2^n}, \frac{k}{2^n} \right]$$

   for some $n, k \in N$, $1 \leq k \leq 2^n$, simply define

$$\mu(E_{n,k}) = \frac{1}{2^n}.$$

   The countable additivity is very easy to verify, following from directly the construction.

   * For the other Borel Sets $E \subset [0,1]$, if we define

$$S_n = \left\{ \left[\frac{k-1}{2^n}, \frac{k}{2^n} \right] : k \in \mathbb{N}, 1 \leq k \leq 2^n, \left[\frac{k-1}{2^n}, \frac{k}{2^n} \right] \subset E \right\},$$

   and define
$$\mu(E) = \lim_{n \to \infty} \frac{\#S_n}{2^n}.$$

This is indeed a mass distribution since $\mu([0,1]) = 1$ is finite. Now we can check that this satisfies the hypothesis of the mass distribution principle, since $U \subset [\inf U, \sup U]$ and $|U| = \sup U - \inf U$, and

$$\mu(U) \leq \mu([\inf U, \sup U]) = \sup U - \inf U,$$

so it simply satisfies the hypothesis of mass distribution principle for $c = 1$.

From theorem 4.2.2 the mass distribution principle, we can see that

$$\mathcal{H}^1([0,1]) \geq \frac{\mu([0,1])}{1} = 1.$$

Therefore $\mathcal{H}^1([0,1]) = 1$.

By problem 2.1.6 on the translation invariance property of the s-dimensional Hausdorff measure, we know that $\mathcal{H}^1([n, n+1]) = 1$ for all $n \in \mathbb{Z}$.

Since

$$\mathbb{R} \supset \bigcup_{n=-\infty}^{+\infty} [2n, 2n+1]$$

where the right-hand-side is a disjoint union and since s-dimensional Hausdorff measure is a measure on the Borel σ-algebra by theorem 2.1.1, it will satisfy Axiom 3 for a measure (for all the sets concerning below, since they are all Borel sets), and hence

$$\mathcal{H}^1(\mathbb{R}) \geq \sum_{n=-\infty}^{+\infty} \mathcal{H}^1([2n, 2n+1])$$

$$= \sum_{n=-\infty}^{+\infty} 1$$

$$= +\infty$$

which then implies $\mathcal{H}^1(\mathbb{R}) = +\infty$ which finishes our proof.

25   Problem 4.2.2

Proof: We alter slightly the definition of the natural measure. Specifically, we assign an uneven amount of measure on the left interval created and the right interval created each step. Let $0 < p < 1$ such that $p \cdot 3^s > 1$ where $s = \log 2 / \log 3$, i.e., $p > \frac{1}{2}$. Let the IFS of the Cantor set be (f_1, f_2) where $f_1(x) = \frac{1}{3}x$ and $f_2(x) = \frac{1}{3}x + \frac{2}{3}$, and we denote $f_{i_1, i_2, \ldots, i_m}(E)$ to be $f_{i_1}(f_{i_2}(\ldots(f_{i_m}(E))))$, where each $i_k \in \{1, 2\}$. Let $t_1 = p, t_2 = 1 - p$, and define the measure μ such that

$$\mu\left(f_{i_1, i_2, \ldots, i_m}(C)\right) = \mu\left(f_{i_1, i_2, \ldots, i_m}([0, 1]) \cap C\right) = \prod_{j=1}^{m} t_{i_j} = t_{i_1} t_{i_2} \ldots t_{i_m}.$$

This means every time a left interval is created, it takes up p of the measure of the original interval, with the right interval taking up $1 - p$ of the measure.

Now, we can generalise this for any $U \subset C$ simply using the similar concept as before, by approximating it with smaller and smaller intervals $\cap C$:

$$\mu(U) = \inf\left\{\sum_{i=1}^{\mu(E_i)} : U \subset \bigcup_i E_i, E_i \text{ is an interval in the construction}\right\}.$$

It is not difficult to see the non-negativity and the measure of the empty set being zero from the definition. The countable additivity is also true: consider disjoint Borel subsets $\{E_i\}$ of C, we will have $\mu(\bigcup E_i) = \sum \mu(E_i)$ since the intervals are either disjoint or contains one-another (and we obviously don't want them to contain one-another to get closer to the inf since simply removing the smaller one won't change the validity of the cover), and so they are disjoint, and so this holds.

Now consider $\frac{\mu(U)}{|U|^s}$ for $U = f_{1, 1, \ldots, 1}(C)$ where there are n 1s. It's not difficult to see that

$$|U| = \left(\frac{1}{3}\right)^n,$$

and that

$$\mu(U) = p^n.$$

Notice that

$$\frac{\mu(U)}{|U|^s} = \frac{p^n}{3^{-sn}} = (2p)^n.$$

Notice that this value decreases as p increases. For any $\varepsilon > 0$, we can find n that is sufficiently big enough such that $|U| = 3^{-n} > \varepsilon$, and for any $c > 0$, we can further increase n to be big enough such that $\mu(U)/|U|^s = (2p)^n > c$, since $2p > 1$ and $(2p)^n$ is increasing with respect to n. This means that for all $c, \varepsilon > 0$, we can find some $U \subset C, |U| \leq \varepsilon$, such that $\mu(U) > c|U|^s$ for this measure μ that we just constructed, as desired.

26   Problem 4.2.3

Proof: First of all, we need to determine the Hausdorff Dimension of the Cantor Dust $D = C \times C$. We will do this by finding an IFS where it is the attractor, proving OSC, then using 4.1.2 to determine the Hausdorff Dimension.

An IFS for this set is (f_1, f_2, f_3, f_4) with

$$f_1((x, y)) = \left(\frac{1}{3}x, \frac{1}{3}y \right),$$

$$f_2((x, y)) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y \right),$$

$$f_3((x, y)) = \left(\frac{1}{3}x, \frac{1}{3}y + \frac{2}{3} \right),$$

$$f_4((x, y)) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3} \right).$$

These all have contraction ratio $\frac{1}{3}$.

This satisfies the OSC: take the open interval $I = (0, 1)^2$ as our open set, then

$$f_1(I) = \left(0, \frac{1}{3} \right)^2,$$

$$f_2(I) = \left(\frac{2}{3}, 1 \right) \times \left(0, \frac{1}{3} \right)$$

$$f_3(I) = \left(0, \frac{1}{3} \right) \times \left(\frac{2}{3}, 1 \right)$$

$$f_4(I) = \left(\frac{2}{3}, 1 \right)^2.$$

These are all disjoint and subsets of $(0, 1)^2$, so the OSC is satisfied. The Cantor Dust is an attractor of this IFS, and by problem 4.0.2 it is the attractor. By theorem 4.1.2, if the Hausdorff dimension equals s then

$$\sum_{i=0}^{4} c_i^s = 1 \implies \frac{4}{3^s} = 1 \implies s = \frac{\log(4)}{\log(3)}.$$

Now let's determine a mass distribution on D - by theorem 4.2.4 we can pick the natural measure. Since $c_i = \frac{1}{3}$ for all i, $c_i^s = \frac{1}{3^{\left(\frac{\log(4)}{\log(3)} \right)}} = \frac{1}{4}$, so $\mu(I_k) = 4^{-k}$ (where I_k is a square constructed in the kth iteration of the Cantor Dust).

Now let $U \subset D, |U| < \sqrt{2}$: there exists some k such that

$$3^{-(k+1)} \le |U| < 3^{-k}.$$

Then $U \subset I_k \cap D$ for I_k defined earlier – since by this size restriction it can't be in multiple I_ks. Hence by problem 2.0.3,

$$\begin{aligned}
\mu(U) \leq \mu(I_k) &= 4^{-k} \\
&= (3^s)^{-k} \\
&= (3^{-k})^s \leq (3|U|)^s,
\end{aligned}$$

so by theorem 4.2.2 $\mathcal{H}^s(D) \geq \mu(D) * 3^{-s} = \frac{1}{4} > 0$, as required.

27 Problem 4.2.4

Proof: Consider the set $V = \left(-\frac{1}{4}, \frac{5}{4}\right) = \left(-\frac{3}{12}, \frac{15}{12}\right)$. For the IFS (f_1, f_2) for the Cantor set C, we have

$$f_1(x) = \frac{x}{3},$$
$$f_2(x) = \frac{x}{3} + \frac{2}{3}.$$

Therefore, we have

$$f_1(V) = \left(-\frac{1}{12}, \frac{5}{12}\right),$$
$$f_2(V) = \left(\frac{9}{12}, \frac{13}{12}\right).$$

Apparently $f_1(V) \subset V, f_2(V) \subset V, f_1(V) \cap f_2(V) = \emptyset$. $V \supset [0,1] \supset C$ so not only does V satisfy the strong open set condition that $C \cap V = C \neq \emptyset$, V also satisfies that $V \supset C$ that it contains C completely.

For the Sierpinski Carpet, it is possible for $V = (0,1)^2$ to satisfy the SOSC. Recall that the IFS for the Sierpinski Carpet satisfies that

$$g_1((x,y)) = \left(\frac{x}{3}, \frac{y}{3}\right),$$
$$g_2((x,y)) = \left(\frac{x}{3}, \frac{y}{3} + \frac{1}{3}\right),$$
$$g_3((x,y)) = \left(\frac{x}{3}, \frac{y}{3} + \frac{2}{3}\right),$$
$$g_4((x,y)) = \left(\frac{x}{3} + \frac{1}{3}, \frac{y}{3}\right),$$
$$g_5((x,y)) = \left(\frac{x}{3} + \frac{1}{3}, \frac{y}{3} + \frac{2}{3}\right),$$
$$g_6((x,y)) = \left(\frac{x}{3} + \frac{2}{3}, \frac{y}{3}\right),$$
$$g_7((x,y)) = \left(\frac{x}{3} + \frac{2}{3}, \frac{y}{3} + \frac{1}{3}\right),$$
$$g_8((x,y)) = \left(\frac{x}{3} + \frac{2}{3}, \frac{y}{3} + \frac{2}{3}\right),$$

and therefore

$$g_1(V) = \left(0, \frac{1}{3}\right) \times \left(0, \frac{1}{3}\right),$$

$$g_2(V) = \left(0, \frac{1}{3}\right) \times \left(\frac{1}{3}, \frac{2}{3}\right),$$

$$g_3(V) = \left(0, \frac{1}{3}\right) \times \left(\frac{2}{3}, 1\right),$$

$$g_4(V) = \left(\frac{1}{3}, \frac{2}{3}\right) \times \left(0, \frac{1}{3}\right),$$

$$g_5(V) = \left(\frac{1}{3}, \frac{2}{3}\right) \times \left(\frac{2}{3}, 1\right),$$

$$g_6(V) = \left(\frac{2}{3}, 1\right) \times \left(0, \frac{1}{3}\right),$$

$$g_7(V) = \left(\frac{2}{3}, 1\right) \times \left(\frac{1}{3}, \frac{2}{3}\right),$$

$$g_8(V) = \left(\frac{2}{3}, 1\right) \times \left(\frac{2}{3}, 1\right),$$

and we can see they are pairwise disjoint and all subsets of V. Furthermore, let $g(E) = \bigcup_{i=1}^{8} g_i(E)$, note that the point $\left(\frac{1}{3}, \frac{1}{3}\right) \in V$, and also note that

$$g_8((1,1)) = (1,1) \implies (1,1) \in g^{(k)}([0,1]^2),$$

$$g_1((1,1)) = \left(\frac{1}{3}, \frac{1}{3}\right),$$

and therefore

$$\left(\frac{1}{3}, \frac{1}{3} \in g^{(k)}([0,1]^2)\right),$$

and therefore $\left(\frac{1}{3}, \frac{1}{3}\right) \in \square$, the Sierpinski Carpet. But $\left(\frac{1}{3}, \frac{1}{3}\right) \in V$ as well, so $V \cap C \supset \left\{\left(\frac{1}{3}, \frac{1}{3}\right)\right\}$, and therefore $V \cap C \neq \emptyset$.

However, it is impossible for such $V \supset C$ to satisfy the SOSC. This is because if $V \supset C$, then $(0,1) \in V, (1,1) \in V$. But notice

$$g_4((0,1)) = \left(\frac{1}{3}, \frac{1}{3}\right) = g_1((1,1)),$$

and therefore $g_1(V) \cap g_4(V) \supset \left\{\left(\frac{1}{3}, \frac{1}{3}\right)\right\}$, and $g_1(V) \cap g_4(V) \neq \emptyset$ which violates the pairwise disjoint condition of the OSC, hence the second part is impossible for the Sierpinski carpet.

28   Problem 4.2.5

Proof: Consider the following 2^{n-1} contractions each with contraction ration 2^{-n} forming an IFS for some $n \in N, n \geq 2$:

$$(f_n)_1(x) = \frac{1}{2^n}(x),$$

$$(f_n)_2(x) = \frac{1}{2^n}(x) + \frac{1}{2^n},$$

$$(f_n)_3(x) = \frac{1}{2^n}(x) + \frac{2}{2^n},$$

$$\vdots = \vdots$$

$$(f_n)_{2^{n-1}}(x) = \frac{1}{2^n}(x) + \frac{2^{n-1}-1}{2^n}.$$

This satisfies the open set condition for $U = (0,1)$, since

$$(f_n)_1(U) = \left(0, \frac{1}{2^n}\right),$$

$$(f_n)_2(U) = \left(\frac{1}{2^n}, \frac{2}{2^n}\right),$$

$$(f_n)_3(U) = \left(\frac{2}{2^n}, \frac{3}{2^n}\right),$$

$$\vdots = \vdots$$

$$(f_n)_{2^{n-1}}(U) = \left(\frac{2^{n-1}-1}{2^n}, \frac{2^{n-1}}{2^n}\right),$$

which are obviously pairwise disjoint and all subsets of U.

Let its attractor be E_n. We will see that its dimension s_n satisfies that

$$2^{n-1} \cdot \left(2^{-n}\right)^s = 1 \iff 2^{n-1-sn} = 1 \iff s = \frac{n-1}{n}.$$

Now, consider the set

$$E = \bigcup_{i=2}^{n} E_n \subset \mathbb{R}$$

Since $E \subset \mathbb{R} = \mathbb{R}^1$, we know the Hausdorff dimension of $E \leq 1$.
Consider $\mathcal{H}^1(E)$. By the countable sub-additivity property of a measure, we can see

$$\mathcal{H}^1(E) = \mathcal{H}^1\left(\bigcup_{i=2}^{+\infty} E_n\right)$$

$$\leq \sum_{i=2}^{+\infty} \mathcal{H}^1(E_n).$$

But since each E_n has a Hausdorff dimension of $\frac{n-1}{n} < 1$, by problem 2.1.7 we must have $\mathcal{H}^1(E_n) = 0$.

Therefore, $\mathcal{H}^1(E) = 0$, which shows the Hausdorff dimension of E must be ≥ 1.

This shows that E has Hausdorff dimension 1 but have $\mathcal{H}^1(E) = 0$.

29   Problem 4.2.6

Proof: Recall from 4.0.4 that the IFS for the Sierpinski carpet $(f_1, \ldots, f_8)$ on $\mathbb{R}^2 \to \mathbb{R}^2$:

$$f_1((x,y)) = \left(\frac{1}{3}x, \frac{1}{3}y\right),$$

$$f_2((x,y)) = \left(\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y\right),$$

$$f_3((x,y)) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y\right),$$

$$f_4((x,y)) = \left(\frac{1}{3}x, \frac{1}{3}y + \frac{1}{3}\right),$$

$$f_5((x,y)) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{1}{3}\right),$$

$$f_6((x,y)) = \left(\frac{1}{3}x, \frac{1}{3}y + \frac{2}{3}\right),$$

$$f_7((x,y)) = \left(\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{2}{3}\right),$$

$$f_8((x,y)) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3}\right).$$

Denote $f(X) = \bigcup_{i=1}^{8} f_i(X)$, $\square_0 = [0,1]^2$ and $\square_k = f^{(k)}(\square_0)$, and the Sierpinski carpet as $\square$.

The Sierpinski carpet's IFS satisfies the OSC: notice that the open unit square $U = (0,1)^2$ satisfies the OSC by

$$f_1(U) = \left(0, \frac{1}{3}\right) \times \left(0, \frac{1}{3}\right)$$

$$f_2(U) = \left(\frac{1}{3}, \frac{2}{3}\right) \times \left(0, \frac{1}{3}\right)$$

$$f_3(U) = \left(\frac{2}{3}, 1\right) \times \left(0, \frac{1}{3}\right)$$

$$f_4(U) = \left(0, \frac{1}{3}\right) \times \left(\frac{1}{3}, \frac{2}{3}\right)$$

$$f_5(U) = \left(\frac{2}{3}, 1\right) \times \left(\frac{1}{3}, \frac{2}{3}\right)$$

$$f_6(U) = \left(0, \frac{1}{3}\right) \times \left(\frac{2}{3}, 1\right)$$

$$f_7(U) = \left(\frac{1}{3}, \frac{2}{3}\right) \times \left(\frac{2}{3}, 1\right)$$

$$f_8(U) = \left(\frac{2}{3}, 1\right) \times \left(\frac{2}{3}, 1\right)$$

.

and notice they are all disjoint subsets of U. Therefore, the dimension of $\square$, s, must satisfy that

$$8 \cdot 3^{-s} = 1 \implies s = \frac{\log 8}{\log 3}.$$

Let $\square$ be the Sierpinski carpet, and denote S_k as a square generated in the k-th iteration of the process, for example, $S_0 = (0,1)^2$, note that S_k has side length 3^{-k} and there are 8^k such squares in the process. Now, we define the measure μ:

$$\mu(S_k \cap \square) = 8^{-k},$$

and for any $U \subset \square$ a subset of the Sierpinski carpet, we can approximate it using some smaller squares:

$$\mu(U) = \inf \left\{ \sum_{i=1}^{+\infty} \mu(E_i) : U \subset \bigcup_{i=1}^{+\infty} E_i, E_i \text{ is some square in the process} \right\}.$$

Assume that $|U| < \sqrt{1}$. There must therefore exist some k such that $3^{-(k+1)} \leq |U| < 3^{-k}$ for some integer k. Then, $U \subset (S_{k1} \cup S_{k2} \cup S_{k3} \cup S_{k4}) \cap \square$ where S_{ki} is one of the squares in the k-th step of the construction, i.e., U can only be situated in at most 4 squares at the same time. Therefore,

$$\begin{aligned}
\mu(U) &\leq 4 \cdot 8^{-k} \\
&= 4 \cdot (3^s)^{-k} \\
&= 4 \cdot \left(3^{-k} \right)^s \\
&= 4 \cdot 3^s \cdot \left(3^{-k-1} \right)^s \\
&\leq 4 \cdot 8 \cdot |U|^s \\
&= 32 \cdot |U|^s.
\end{aligned}$$

This means the hypothesis for the mass distribution principle is true for $\varepsilon = 1$ and $c = 32$. Therefore, from mass distribution principle, we have

$$\mathcal{H}^s(\square) \geq \frac{1}{32}.$$

30   Problem 5.0.1

Proof: Let $s = \frac{\log 3}{\log 2}$. We restate some definitions first: let $I_n = (i_1, i_2, \ldots, i_n) \in \{1, 2, 3\}^n$ where $i_k \in \{1, 2, 3\}$ for each $1 \le k \le n$, and define $S_{I_n}(\Delta) = S_{i_1}(S_{i_2}(\ldots(S_{i_n}(\Delta))))$ where Δ is the closed equilateral triangle with vertices $(0, 0), (1, 0)$ and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

We define for $E \subset \mathbb{R}^2$ that

$$S(E) = \bigcup_{i=1}^{3} S_i(E).$$

- **Upper Bound:** $\mathcal{H}^s(\blacktriangle) \le 1.$ Recall the definition that

$$\mathcal{H}_\delta^s(\blacktriangle) = \inf_{\{U_i\} \text{ is a } \delta\text{-cover of } \blacktriangle} \sum_{i=1}^{+\infty} |U_i|^s.$$

We can first show that $|\triangle| \ge 1$ since $|(0, 0) - (1, 0)| = 1$ and $|\triangle| \le 1$ since $\triangle \subset \overline{B_{(0,0)}(1)}$. And therefore, we can see that

$$|S_{I_n}(\triangle)| = \frac{1}{2^n}$$

from the definition of a contraction immediately.

By theorem 4.0.2, we can see that

$$\blacktriangle = \bigcap_{k=0}^{+\infty} S^{(k)}(\Delta)$$

since $\blacktriangle$ is the attractor of the IFS.

Consider a certain $k \in \mathbb{Z}, k \ge 0$, and it is not difficult to see from definition that

$$S^{(k)}(\Delta) = \bigcup_{I_n \in \{1,2,3\}^n} S_{I_n}(\Delta)$$

from the definition of $S^{(k)}$. (This can be shown by induction on k.)

Therefore, we can see

$$\blacktriangle \subset \bigcup_{I_n \in \{1,2,3\}^n} S_{I_n}(\Delta),$$

and therefore, $\{S_{I_n}(\Delta)\}$ gives a δ-cover for $\blacktriangle$, when $\delta > \frac{1}{3^n} \iff 3^n > \frac{1}{\delta} \iff n > -\log_3(\delta)$.

Also, notice that $\#\{S_{I_n}(\Delta)\} = 3^n$.

Therefore, for any $0 < \delta < 1$, choose $n = \lfloor -\log_3(\delta) \rfloor + 1 \implies n > -\log_3(\delta)$, and therefore

$$
\mathcal{H}^s_\delta(\blacktriangle) = \inf_{\{U_i\} \text{ is a } \delta\text{-cover of } \blacktriangle} \sum_{i=1}^{+\infty} |U_i|^s
$$

$$
\leq \sum_{i=1}^{3^n} \left(\frac{1}{2^n}\right)^s
$$

$$
= 3^n \cdot \frac{1}{2^{(n \cdot \log_2(3))}}
$$

$$
= 3^n \cdot \frac{1}{3^n}
$$

$$
= 1,
$$

and therefore

$$
\mathcal{H}^s(\blacktriangle) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(\blacktriangle) \leq 1,
$$

which shows the upper bound as desired.

- We show that the IFS of the Sierpinski triangle satisfies the OCS. Consider the open set U which is the open triangle with vertices $(0,0), (1,0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. We notice that

$S_1(U)$ is an open triangle with vertices $(0,0), \left(\frac{1}{2}, 0\right), \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right),$

$S_2(U)$ is an open triangle with vertices $\left(\frac{1}{2}, 0\right), (1,0), \left(\frac{3}{4}, \frac{\sqrt{3}}{4}\right),$

$S_3(U)$ is an open triangle with vertices $\left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right), \left(\frac{3}{4}, \frac{\sqrt{3}}{4}\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$

and therefore they are pairwise disjoint and are all subsets of U. Therefore the IFS satisfies the open set condition.

Now, we show that the natural measure on $\blacktriangle$ satisfies the hypothesis for the mass distribution principle, for $\varepsilon = 1$ and $c = 6$.

Consider some $U \subset \blacktriangle$ such that $|U| < 1$. There must exist some k such that $2 \cdot 2^{-(k+1)} \leq |U| < 2 \cdot 2^{-k}$ for some $k \geq 1$. Then, $U \subset \bigcup_{i=1}^{6} (\Delta_k)_i \cap \blacktriangle$, where Δ_t is a triangle formed in the t's iteration of the IFS. This means that U is in at most 6 triangles in the k-th iteration, as shown in the diagram below:
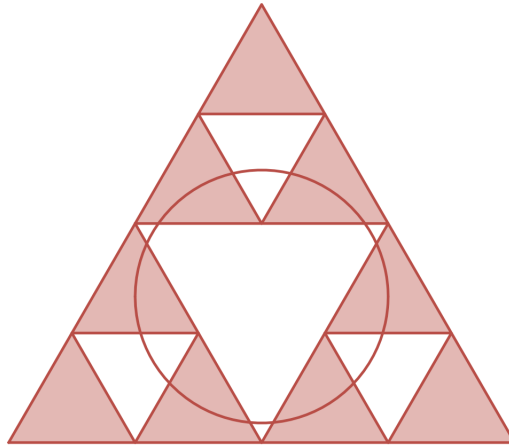
Therefore we will have

$$\mu(U) \leq 2 \cdot 3^{-k}$$
$$= 6 \cdot \left(2^{\log_2^3}\right)^{-(k+1)}$$
$$= 6 \cdot \left(2^{-(k+1)}\right)^{\log_2^3}$$
$$\leq 6 \cdot |U|^s.$$

Notice that $\mu(\blacktriangle) = \mu(\Delta \cap \blacktriangle) = \mu(S_{I_0}(\Delta) \cap \blacktriangle) = 1$ and therefore,

$$\mathcal{H}^s(\blacktriangle) \geq \frac{\mu(\blacktriangle)}{6} = \frac{1}{6}$$

as desired.

31   Problem 5.0.2

Proof: For a_1, there are only 3 possible distinct non-empty coverings: 1 triangle, 2 triangles, and 3 triangles.

1. For 1 triangle, the diameter of the set is $\frac{1}{2}$, the proportion covered is $\frac{1}{3}$, so the value of a_1 is $\frac{(1/2)^s}{(1/3)} = 1$;

2. For 2 triangles, the diameter is 1, while the proportion covered is $\frac{2}{3}$, notably less than 1: thus this will result in a higher value for a_1;

3. For 3 triangles, the diameter is 1, the proportion covered is 1, hence our value is also 1.

Hence $a_1 = 1$.

For a_2, we need to do a similar job – but we can ignore coverings which have points in only one of the 3 largest triangles due to these being equivalent to colourings of a_1. This is clear as if we have a covering of T_n, we can scale it down by scale factor $\frac{1}{2}$ centred at $(0,0)$ to cover one of the three largest triangles of T_{n+1}. This is clearly a covering as these triangles have been scaled down in a similar fashion to the first contraction function in the IFS, thus it's the same as just adding this function to the beginning of $S_{I_n}(\Delta)$. This will have half the diameter, and have $\frac{1}{3}$ of the measure (as there are 3 identical largest triangles that could be filled). Thus letting D be the diameter, and M be the measure, our value for a_{i+1} is $\frac{(D/2)^s}{M/3} = \frac{D^s}{M}$, which is the same value as our covering for a_i. Thus our lemma is proven.

Therefore, there are then 2 cases to consider: if one of the corner triangles is filled, and if one of the corner triangles is not filled.

I claim the best answer is when we pick the 6 triangles closest to the centre empty triangle: this would give an answer of

$$\frac{0.75^{\log(2)/\log(3)}}{\frac{6}{9}} = \frac{3^{\log(3)/\log(2)-1}}{2} = 0.95075.$$

If one of the corner triangles is filled, we may as well fill out the 3 triangles in the sub-triangle of that corner triangle (as we must have one outside this sub-triangle anyway.) If we use a triangle adjacent to this sub-triangle, we may as well also have the one symmetrically opposite (as this doesn't increase the diameter) But this only covers 5 triangles with the same diameter as last time so can't be better.

If we choose one of the other corner triangles, we may as well cover the whole triangle anyway but this would just give us a value of 1. If we don't choose the corner triangles but use all the other triangles, this gives us a diameter of $\frac{\sqrt{13}}{4}$, and a measure of $\frac{7}{9}$: plugging this into the formula we get a value of around 1.09 which is worse than the trivial bound of 1.

The other case is if we don't pick any of the corners. Then there are really only two cases to consider: picking 3 adjacent triangles and picking all 6 triangles. If we pick 3 adjacent triangles, this gives us a diameter of greater than $\frac{1}{2}$ but the same number of triangles as if

we just picked one of the sub-triangles with 3 triangles - thus it's strictly worse than that bound. The other case is the best bound shown earlier - hence we are done and

$$a_2 = \frac{3^{\log(3)/\log(2)-1}}{2}.$$

The diagram below shows the choice of triagnles for a_2.
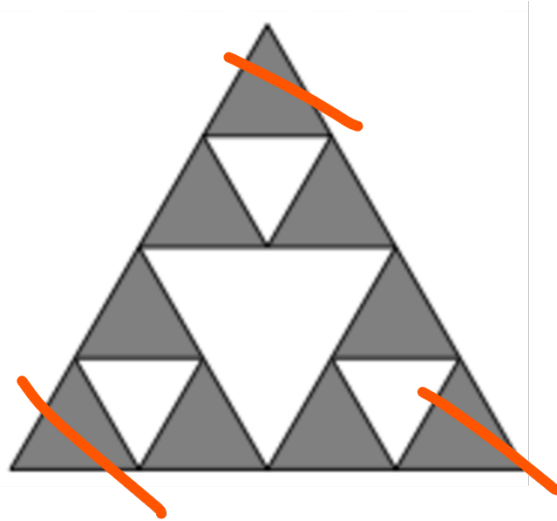
32   Problem 5.0.3

Proof: I will prove that $a_i \geq a_{i+1}$ – if I could show that for any possible value of a_i, there exists a covering of T_{n+1} with the same value, then we would be done (as a_{i+1} is defined as the minimum of these values, there could be one less than these but it is at most a_i).

However this is clear as if we have a covering of T_n, we can scale it down by scale factor $\frac{1}{2}$ centred at $(0,0)$ to cover one of the three largest triangles of T_{n+1}. This is clearly a covering as these triangles have been scaled down in a similar fashion to the first contraction function in the IFS, thus it's the same as just adding this function to the beginning of $S_{I_n}(\Delta)$. This will have half the diameter, and have $\frac{1}{3}$ of the measure (as there are 3 identical largest triangles that could be filled). Thus letting D be the diameter, and M be the measure, our value for a_{i+1} is $\frac{(D/2)^s}{M/3} = \frac{D^s}{M}$, which is the same value as our covering for a_i. Thus our lemma is proven.

33   Problem 5.0.4

Proof:

This is one of the coverings of T_4. The diameter is clearly $\frac{13}{16}$ - as this is the distance between two opposite points in the figure. It also contains 66 out of 81 triangles - thus its value is $\frac{\frac{13}{16}^s}{\frac{66}{81}}$ which is approximately 0.8831093965, notably less than 0.9. Hence $a_4 \leq$ 0.8831093965 < 0.9 so by theorem 5.0.2 the Hausdorff dimension is $\leq 0.8831093965 < 0.9$, as required.
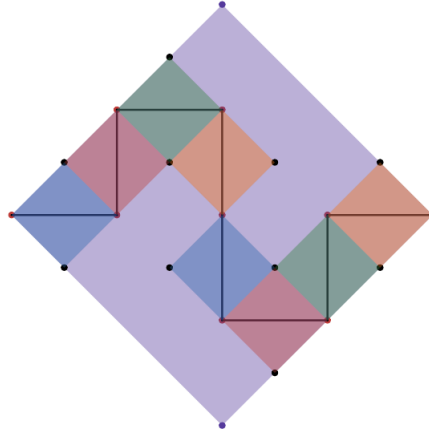
34   Problem 5.0.5

Proof: Recall the IFS $(m_1, m_2, \ldots, m_8)$ for the Minkowski Sausage M is:

$$S_1((x, y)) = \left(\frac{1}{4}x, \frac{1}{4}y\right),$$

$$S_2((x, y)) = \left(-\frac{1}{4}y + \frac{1}{4}, \frac{1}{4}x\right),$$

$$S_3((x, y)) = \left(\frac{1}{4}x + \frac{1}{4}, \frac{1}{4}y + \frac{1}{4}\right),$$

$$S_4((x, y)) = \left(\frac{1}{4}y + \frac{1}{2}, -\frac{1}{4}x + \frac{1}{4}\right),$$

$$S_5((x, y)) = \left(\frac{1}{4}y + \frac{1}{2}, -\frac{1}{4}x - \frac{1}{4}\right),$$

$$S_6((x, y)) = \left(\frac{1}{4}x + \frac{1}{2}, \frac{1}{4}y - \frac{1}{4}\right),$$

$$S_7((x, y)) = \left(-\frac{1}{4}y + \frac{3}{4}, \frac{1}{4}x - \frac{1}{4}\right),$$

$$S_8((x, y)) = \left(\frac{1}{4}x + \frac{3}{4}, \frac{1}{4}y\right).$$

Notice this satisfies the OSC by considering a open square O with vertices at

$$(0, 0), (1, 0), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right),$$

as shown in the following diagram.

Therefore, the dimension of M, m must satisfy that

$$8 \cdot \frac{1}{4^m} = 1 \iff m = \frac{3}{2} = 1.5,$$

by theorem 4.1.2

Let $M_0 = [0,1]$ the closed interval. Let $I_n = (i_1, i_2, \ldots, i_n)$ be a list with each $i_k \in \{1, 2, \ldots, 8\}$, and set

$$S_{I_n}(M_0) = S_{i_1}\left(S_{i_2}\left(\ldots\left(S_{i_n}(M_0)\right)\right)\right),$$

Define the natural measure as

$$\mu(S_{I_n}(M_0) \cap M) = \left(\frac{1}{4}\right)^{mn} = 8^{-n}.$$

Let μ be the natural measure on M, the Minkowski sausage. Define

$$\mathcal{M}_n = \{S_{I_n}(M_0) : I_n \in \{1, 2, \ldots, 8\}^n\}$$

be the collection of all 8^n scaling of size 4^{-n} of the Minkowski Sausage, and let $\{E_i\}_i \subset \mathcal{M}_n$ be a non-empty collection of those iterations of the scaling, define

$$b_n = \min_{\{E_i\}_i \subset \mathcal{M}_n} \left\{ \frac{|\bigcup_i E_i|^m}{\mu(\bigcup_i E_i)} \right\}.$$

Claim. $\mathcal{H}^m(M) \leq b_n$.

Proof. We first show $\mathcal{H}^m_\delta(M) \leq b_n$ for $\mathcal{H}^m_1(M)$.

Since b_n is a minimum of some finite number of possibilities (subsets of the finite set $\mathcal{M}_n$), so there must be a collection of set U with $\#U = k$ that gives the minimum of b_n. Let $\mathcal{U} = \bigcup_{E_i \in U} E_i$, the union of the elements in the collection U.

In the case where $\mathcal{U} = M$ is the Minkowski Sausage, then we see that this must be a cover of the Minkowski Sausage. In this case,

$$b_n = \frac{|M|^m}{\mu(M)} \geq \frac{1^m}{1} = 1,$$

since notice that the Minkowski Sausage contains the points $(0,0)$ and $(1,0)$ and they are distance of 1 apart, and noticing that $S_{I_0}(M) = M$ so $\mu(M) = 1$ naturally.

On the other hand, notice that $\mathcal{H}^m_1(M) \leq 1^m = 1$ since the set containing the closed square with vertices at $(0,0), (1,0), \left(\frac{1}{2}, \frac{1}{2}\right)$ covers M and has diameter 1, and therefore from the definition, $\mathcal{H}^m_1(M) \leq 1$ as desired. Therefore, $\mathcal{H}^m_1(M) \leq 1 \leq b_n$ and satisfies the claim as desired.

Otherwise, let $V = \mathcal{M}_n \setminus U$, all sets that are not contained in the collection U. $\#V = 8^n - k$. If for some $S_{I_n}(M) = E'_m \in V$ that is not in U, we scale down of these, and place it in V in the sense $\mathcal{M}_{2n}$, the $2n$-th level of the construction of M. If U is scaled down to U' and $\mathcal{U}' = \bigcup_{E_i \in U} E_i$, we will see $|\mathcal{U}'| = 4^{-n}|\mathcal{U}|$. Therefore, there will be $8^n - k$ copies of these U'(s), one each for each element in $\mathcal{M}_n$ that U does not hit. However this still does not cover M (since there are still missing pieces in each of parts of V), so we repeatedly create U'', U''', etc.

Therefore, we cover M using 1 copy of $\mathcal{U}$, $(8^n - k)$ copies of $\mathcal{U}'$, $(8^n - k)^2$ copies of $\mathcal{U}''$, and so on. Therefore,

$$
\begin{aligned}
\mathcal{H}_1^m(M) &\leq \sum_{i=0}^{+\infty} (8^n - k)^i |\mathcal{U}^{(i)}|^m \\
&= \sum_{i=0}^{+\infty} (8^n - k)^i (4^{-in} |\mathcal{U}|)^m \\
&= |\mathcal{U}|^m \sum_{i=0}^{+\infty} \left(\frac{8^n - k}{8^n} \right) i \\
&= \frac{|\mathcal{U}|^s}{1 - \frac{8^n - k}{8^n}} \\
&= \frac{|\mathcal{U}|^s}{\frac{k}{8^n}} \\
&= \frac{|\mathcal{U}|^s}{\mu(\mathcal{U})} \\
&= b_n,
\end{aligned}
$$

the first inequality sign arising due to definition, the second due to what we just show, the third by sum properties, the fourth by geometric series, the fifth by trivial algebra, the sixth by the definition of $\mu(\mathcal{U})$ and that $\mathcal{U}$ being the union of k copies of the nth iteration of the Minkowski Sausage which only share at most points which has dimension 0 so zero m-dimension measure, the seventh equal sign simply by definition of b_n.

Now, we move on to $\delta < 1$, and there must exist some $t \in \mathbb{N}$ such that $4^{-t} < \delta$. Therefore, instead with starting the cover U states in the previous part, we simply start with 8^t copies of U scaled down by 4^{-t}, one at each level t iteration of M. Then we simply preform the same process for each scaled down copies of U, and at each stage we get 8^n extra copies of $U^{(n)}$, with a diameter scaled by 4^{-n}, and therefore

$$
\begin{aligned}
\mathcal{H}_\delta^m(M) &\leq 8^t \sum_{i=0}^{+\infty} (8^n - k)^i (4^{-t} |\mathcal{U}^{(n)}|)^m \\
&= 8^t \cdot 4^{-t \cdot \frac{3}{2}} \sum_{i=0}^{+\infty} (8^n - k)^i (|\mathcal{U}^{(n)}|)^s \\
&= \sum_{i=0}^{+\infty} (8^n - k)^i (|\mathcal{U}^{(n)}|)^s \\
&= b_n
\end{aligned}
$$

similar to earlier. The largest size we used in this cover is $4^{-t} < \delta$ is valid. Therefore, if we take the limit $\delta \to \infty$, we will have

$$
\mathcal{H}^m(M) \leq b_n.
$$

Now consider the bounds of b_n when $n = 2$. Consider the following diagram of the second iteration of M, i.e., $\mathcal{M}_2$.
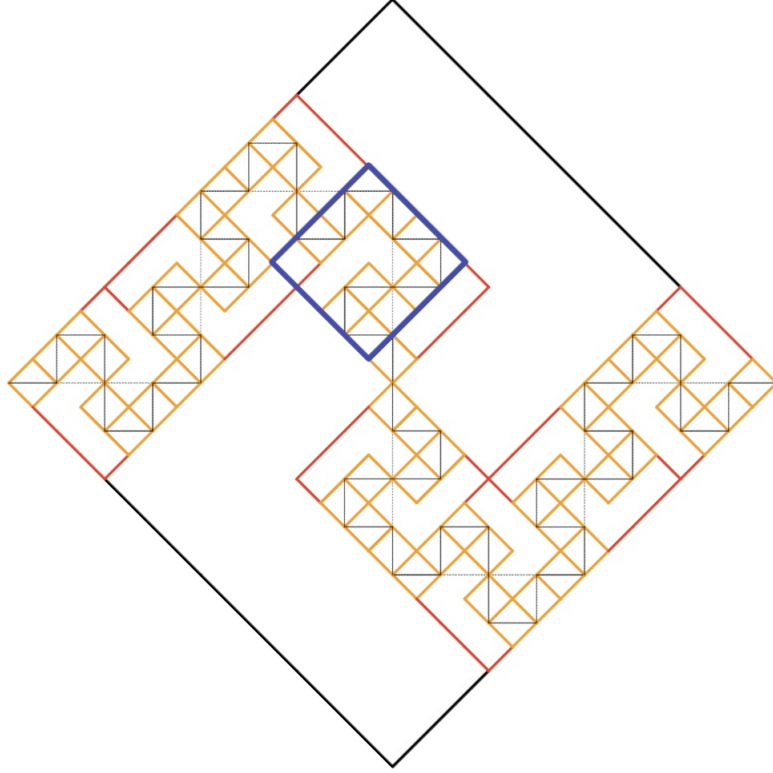
It is worth mentioning why this diagram makes sense and what are the different elements of the diagram. First, note that $M_0 = [0, 1] \subset$ the black square, which is a closed square with vertices at $(0, 0), (1, 0), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right)$. The first iteration of the black square is the eight red squares, which are all subsets of the black square. Therefore, the Minkowski Sausage $M \subset$ the black square, and the first iteration of the Minkowski Sausage, elements in the set $\mathcal{M}_1$, are each contained in one of the red squares. The first iteration of the red squares (the second iteration of the black square) are the orange squares, and elements in the set $\mathcal{M}_2$ are each contained in one of the orange squares.

Now, we investigate the 10 orange squares marked in blue, and let our $U \subset \mathcal{M}_2$ be the set of the 10 elements of $\mathcal{M}_2$ contained in the 10 orange squares. We know that $|\mathcal{U}| \leq$ |blue square| $= \frac{1}{4}$ since $\mathcal{U} \subset$ blue square.

Furthermore, each element in U has measure $8^{-2} = \frac{1}{64}$, and therefore since there are 10

elements in U, $\mathcal{U}$ will have measure $\frac{10}{64} = \frac{5}{32}$. Therefore,

$$
\begin{aligned}
\mathcal{H}^m(M) &\leq b_2 \\
&\leq \frac{|\mathcal{U}|^m}{\mu(\mathcal{U})} \\
&\leq \frac{4^{-\frac{3}{2}}}{\frac{5}{32}} \\
&= \frac{\frac{1}{8}}{\frac{5}{32}} \\
&= \frac{4}{5} \\
&= 0.8
\end{aligned}
$$

as desired.

35   Problem 5.0.6

Proof: We show that a_3 equals 0.910411 – precisely

$$\frac{7^{\frac{\log(3)}{\log(2)}}}{24}.$$

This is done using this program, which brute-forces every combination of triangles to make the cover, finds the diameter, and computes the value of a_3. Then it finds the minimum out of all these values and prints it out. This is extremely naive and would not work for a_4.

```cpp
#include <cmath>
#include <vector>
#include <set>
#include <tuple>
#include <algorithm>
#include <iostream>
#include <limits>

using namespace std;
using Point = pair<double, double>;
using Triangle = vector<Point>;

double root3 = sqrt(3.0);
double s = log(3.0) / log(2.0);

Point div(const Point& pt, double x) {
    return {pt.first / x, pt.second / x};
}

Point add(const Point& pt1, const Point& pt2) {
    return {pt1.first + pt2.first, pt1.second + pt2.second};
}

Point f0(const Point& pt) {
    return div(pt, 2.0);
}

Point f1(const Point& pt) {
    return add({0.5, 0.0}, div(pt, 2.0));
}

Point f2(const Point& pt) {
    return add({0.25, root3 / 4.0}, div(pt, 2.0));
}
```

```cpp
using Func = Point (*)(const Point&);

vector<Func> funcs = {f0, f1, f2};

Triangle f(int n, const Triangle& tri) {
    Triangle result;
    for (const auto& pt : tri) {
        result.push_back(funcs[n](pt));
    }
    return result;
}

vector<Triangle> nexttri(const vector<Triangle>& tri) {
    vector<Triangle> tri1;
    for (const auto& t : tri) {
        for (int i = 0; i < 3; ++i) {
            tri1.push_back(f(i, t));
        }
    }
    return tri1;
}

set<Point> nextpts(const set<Point>& pts) {
    set<Point> pts1;
    for (const auto& pt : pts) {
        for (int i = 0; i < 3; ++i) {
            pts1.insert(funcs[i](pt));
        }
    }
    return pts1;
}

double dist(const Point& pt1, const Point& pt2) {
    return sqrt(pow(pt1.first - pt2.first, 2)
            + pow(pt1.second - pt2.second, 2));
}

double diameter(const set<Point>& pts) {
    double diam = 0.0;
    for (const auto& pt1 : pts) {
        for (const auto& pt2 : pts) {
            diam = max(diam, dist(pt1, pt2));
        }
    }
    return diam;
```

65

```cpp
}

int main() {
    vector<Triangle> tri0 = {{{{0, 0}, {1, 0}, {0.5, root3 / 2}}}};
    set<Point> pts0 = {{0, 0}, {1, 0}, {0.5, root3 / 2}};

    auto tri1 = nexttri(tri0);
    auto pts1 = nextpts(pts0);
    int N = 3;

    for (int i=0;i<N-1;++i){
        tri1 = nexttri(tri1);
        pts1 = nextpts(pts1);
    }



    double mina = numeric_limits<double>::max();

    size_t numTriangles = tri1.size();

    double dd = -1, nn = -1; // store exact values
    for (size_t i = 1; i < (1u << numTriangles); ++i) {

        vector<Triangle> tri;
        set<Point> pts;

        for (size_t j = 0; j < numTriangles; ++j) {
            if (i & (1u << j)) {
                tri.push_back(tri1[j]);
                for (const auto& pt : tri1[j]) {
                    pts.insert(pt);
                }
            }
        }

        double d = diameter(pts);
        double n = static_cast<double>(tri.size());
        double a = pow(d, s) / (n / pow(3.0, N));
        if (a < mina) {
            mina = a;
            dd = d;
            nn = n;
        }
    }
```

```
    cout << mina << endl;
    cout << dd << " " << nn << endl;
    return 0;
}
```

36   Problem 5.0.7

Proof: Unattempted.