Q1

To prove the set M_i forms a group, we need to prove that it satisfies the four properties of a group. The processes are as follows.

- closure

  Let M_a, M_b are two elements of $\{M_i\}$

$$
\begin{aligned}
M_a * M_b &= \begin{bmatrix} R_a & t_a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_b & t_b \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R_a R_b & R_a t_b + t_a \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Because of the feature of orthonormal matrix, $R_a R_B$ is also a orthonormal matrix. And it is easy to know that $R_a t_b + t_a$ is still a 3*1 vector. So it satisfy closure.

- associativity

  Let M_a, M_b, M_c are three elements of $\{M_i\}$

  Because of the feature of matrix multiplication,

  $M_a M_b M_c = M_a(M_b M_c)$.

  So it satisfy associativity.

- identify element

  When R_0 is a 3*3 identity matrix, and t_0=[0 0 0],

  M_0 is an identify element, because $M_0 M_i = M_i M_0$.

- Inverse element

  For $M_i = \begin{bmatrix} R_a & t_a \\ 0 & 1 \end{bmatrix}$,

  the inverse is $\begin{bmatrix} R_a^T & -R_a^T t_a \\ 0 & 1 \end{bmatrix}$.

  This is because

  $\begin{bmatrix} R_a & t_a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_a^T & -R_a^T t_a \\ 0 & 1 \end{bmatrix} = M_0$

Based on all above, $\{M_i\}$ forms a group.

Q2

Solve the partial derivative of G with respect to δ

$$
\begin{aligned}
\frac{\partial G}{\partial \delta} &= \lim_{\triangle\delta \to 0} \frac{G(x,y,\delta+\triangle\delta) - G(x,y,\delta)}{(\delta+\triangle\delta)-\delta} \\
&\approx \frac{G(x,y,k\delta) - G(x,y,\delta)}{k\delta-\delta} \\
&= \frac{DoG}{(k-1)\delta}
\end{aligned}
$$

Or

$$
\begin{aligned}
\frac{\partial G}{\partial \delta} &= \frac{-2\delta^2+x^2+y^2}{2\pi\delta^5} e^{\frac{-2(x^2+y^2)}{2\delta^2}} \\
&= \frac{LoG}{\delta}
\end{aligned}
$$

So

$$
\frac{DoG}{(k-1)\delta} \approx \frac{LoG}{\delta}
$$
$$
DoG \approx (k-1)LoG
$$

So DoG can approximate LoG.

Q3

$A^T A$ is a $n * n$ matrix, so we need to prove $Rank(A^T A) = n$, then $A^T A$ is non-singlar.

That is to say, we need to prove $Rank(A^T A) = Rank(A)$.

Then we need to prove $A^T A x = 0$ and $Ax = 0$ have the same solutions.

1. For $Ax = 0$, we have $A^T (Ax) = 0$, then $A^T A x = 0$. So the solution of $Ax = 0$ is also the solution of $A^T A x = 0$.
2. For $A^T A x = 0$, we have $x^T A^T A x = 0$, Then $(Ax)^T A x = 0$, so $Ax = 0$. Therefore, the solution of $A^T A x = 0$ is also the solution of $Ax = 0$.

So $A^T A x = 0$ and $Ax = 0$ have the same solutions.

Then $Rank(A^T A) = Rank(A) = n$, plus $A^T A$ is $n * n$, so $A^T A$ is non-singlar.