BOUNDARY ELEMENT METHOD

Bioelectric Fundamentals

BEM approximation

The task consists on using some Matlab scripts based on the BEM method for different surfaces, in order to answer different questions.

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Lab.a

Sea un cuadrado de 6 cm de lado en el que las condiciones de contorno se definen como:

$$\boldsymbol{V(x=0)=300\,mV;\qquad V(x=6)=0\,mV}$$
$$\boldsymbol{I(y=0)=0\,A;\qquad I(x=6)=0\,mV}$$

Se pide calcular el voltaje para todos los puntos del interior del cuadrado.

1. Resuelva el problema sobre papel.

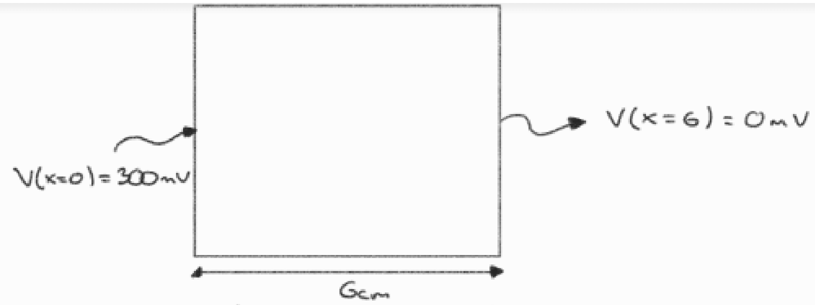

Figure 1: Square representation with the conditions given

Using the Laplace equation, and considering (as the exercise gave to us) that the potential only depends on the x-variable:

$$\nabla^2 V(x)=0 \Rightarrow \frac{\partial^2 V(x)}{\partial x^2}=0 \qquad (1)$$

Solving the differential equation, a solution can be obtained as follows:

$$V(x)=Ax+B$$

Then, with the conditions given by the exercise the following computations can be made:

- Computation of constant B, taking into account the condition $V(x=0)$:

$$V(x=0)=300\,mV \Rightarrow 300=A\cdot 0+B \Rightarrow B=300\,mV$$

- Computation of constant A taking into account the condition $V(x=6)$ and B:

$$V(x=6)=0\,mV \Rightarrow 0=A\cdot 6+B|_{B=300} \Rightarrow 0=6A+300 \Rightarrow A=-50\,mV$$

Finally, an equation for the voltage for all interior points is gotten:

$$V(x)=-50x+300 \qquad (2)$$

2. Compruebe la solución mediante el script llamado *elementalIBEM.m.*

In order to compare the results obtained in the previous section with the mentioned script, the following graphs (obtained for the Matlab script) are taken into account).

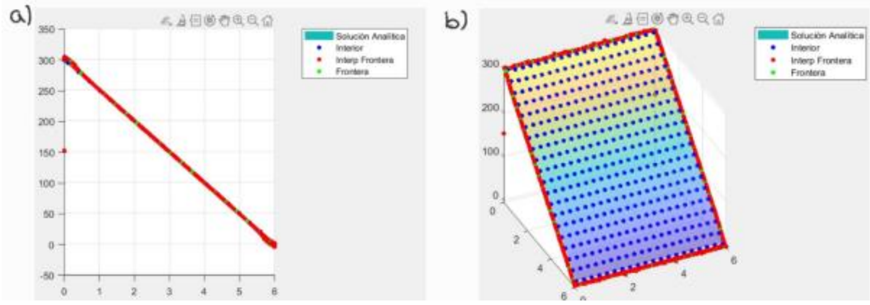

Figure 2: Graph a): Lienal regression of the 2D representation of $V(x)$, depending only on the x-variable. Graph b): Field form by the 3D representation of $V(x)$. It is needed to be remarked that the values used for the interior and boundary points are: $nb=20$ and $ni=20$.

Then, focusing on graph a), the lineal regression of $V(x)$, and on equation (2), the results obtained by analytically computing the problem, it can be said that they both perfectly fit, as the equation gotten represents the graph plotted by Matlab.

3. **Como cambia la solución:**
   (a) Si aumentamos el número de puntos de la frontera:

According to the following figures:

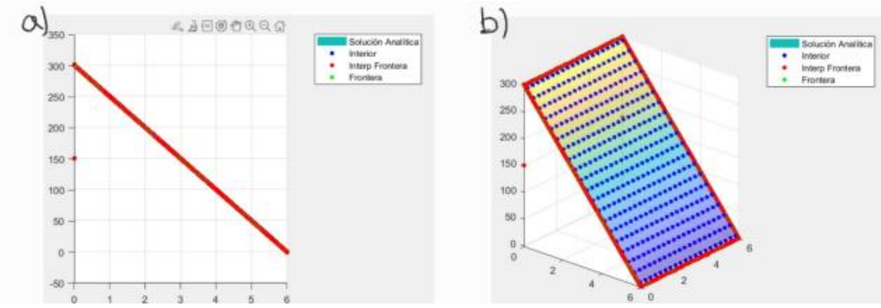

Figure 3: Similar to Figure 2, but with $nb = 100$ *and* $ni = 20$

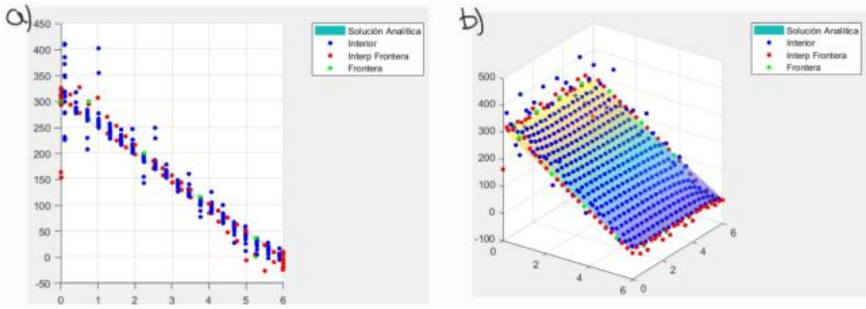

Figure 4: Similar to Figure 2, but with $nb = 5$ *and* $ni = 20$

It can be seen, with the help of both figures, that, when increasing the boundary points the graph resolution improves (until reaching a value where the approximation remains the same). It resembles a linear progression in 2D or a "perfect" field plane in 3D, as shown below:

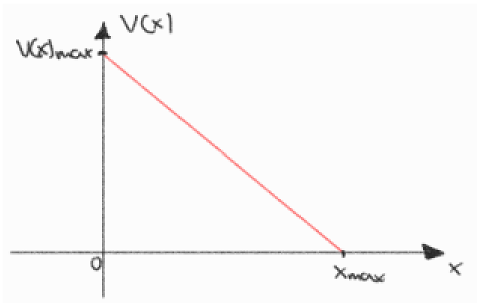

Figure 5: Approximation of the linear progression of $V(x)$ *shown in the graph a) of Figure 3.*

However, when decreasing the boundary points, the graph resolution worsens, as it resembles the following:

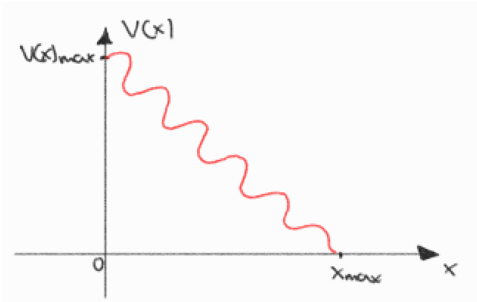

Figure 6: Approximation of the fluctuations shown in the the graph a) of Figure 4.

Which basically shows the fluctuations that will appear if the boundary points given are too low.

(b) Si aumentamos el número de puntos en el interior:

According to the following figures:

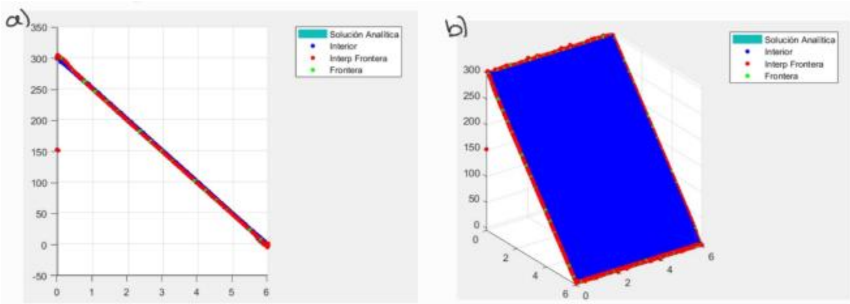

Figure 7: Similar to Figure 2, but with $nb = 20$ *and* $ni = 100$

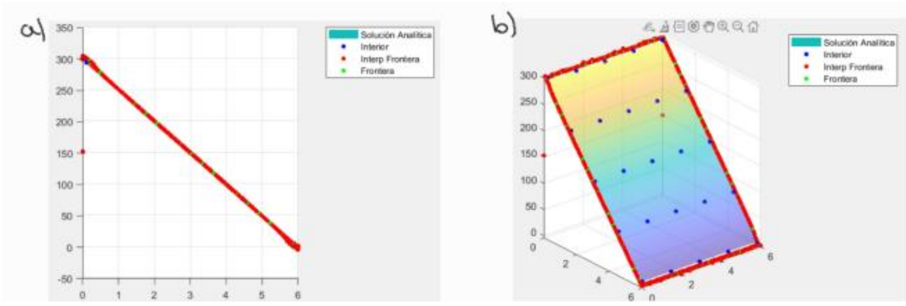

Figure 8: Similar to Figure 2, but with $nb = 20$ *and* $ni = 5$

As it can be seen in both figures, for the graphs that show the linear regression of $V(x)$ in (2D), and for those that show the field in 3D of $V(x)$, it can be said that when increasing or decreasing the interior, nothing changes with their resolutions, concluding that only the boundary points have practical consequences regarding the resolution of the graphs. This makes sense, as the method used is BEM, referring to Boundary Element Method.

The only thing that changed is that it can be seen that the plane is filled with more pints when increasing the interior points, and with less when decreasing those points. However, this does not concern the practical issues.

Lab.b

Sea un rectángulo de $12x6\ cm$ de lado en el que las condiciones de contorno se definen como $V(x=0) = 300\ mV$, $V(x=12) = 0\ mV$, $I(y=0) = 0\ A$ e $I(y=6) = 0\ A$. Dicho rectángulo está compuesto por dos medios homogéneos con $\sigma_1 = 1$ desde $x = 0\ cm$ hasta $x = 6\ cm$ y $\sigma_2 = 2$ desde $x = 6\ cm$ hasta $x = 12\ cm$. Se pide calcular el voltaje para todos los puntos del interior del rectángulo.

1. Resuelve el problema sobre papel.

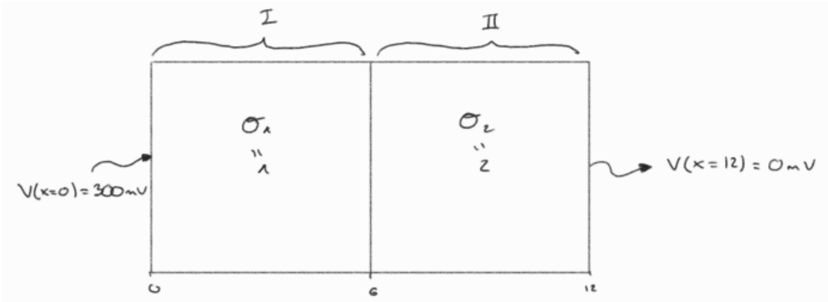

Figure 9: Representation of the rectangle with different conductivities

As it is said in the exercise and seen in *Figure 9*, the rectangle presents different media, having the different equalities fulfilled:

$$V_I = V_{II} \qquad (3)$$

$$\sigma_1 \frac{\partial V_I}{\partial \mathrm{x}} = \sigma_2 \frac{\partial V_{II}}{\partial \mathrm{x}} \qquad (4)$$

Following the same process as in the previous exercise and recalling the Laplace equation:

$$\frac{\partial^2 V(x)}{\partial x^2} = 0 \Rightarrow \nabla^2 V(x) = 0$$

having,

$$V_I(x) = Ax + B \Rightarrow V_I(x=0) = 300 \Rightarrow A \cdot 0 + B = 300 \Rightarrow B = 300$$

$$V_{II}(x) = Cx + D \Rightarrow V_I(x=12) = 0 \Rightarrow C \cdot 12 + D = 0 \Rightarrow 12C + D = 0$$

Then, through the previous assumptions:

- From equation (3):

$$V_I = V_{II} \Rightarrow V_I(x=6) = V_{II}(x=6) \Rightarrow 6A + B = 6C + D$$

- From equation (4):

$$\sigma_1 \frac{\partial V_I}{\partial \mathrm{x}} = \sigma_2 \frac{\partial V_{II}}{\partial \mathrm{x}} \Rightarrow \sigma_1 \frac{\partial (Ax+B)}{\partial \mathrm{x}} = \sigma_2 \frac{\partial (Cx+D)}{\partial \mathrm{x}} \Rightarrow 1 \cdot A = 2 \cdot C \Rightarrow A = 2C$$

Grouping everything in a system of equation:

$$\begin{cases} A = 2C \\ 6A + B = 6C + D \\ 12C + D = 0 \\ B = 300 \end{cases} \Rightarrow \begin{cases} A = -33.3 \\ B = 300 \\ C = -16.67 \\ D = 200 \end{cases}$$

finally obtaining:

$$V_I = -33.3x + 300$$

$$V_{II} = -16.67x + 200$$

2. Compruebe la solución con el script *dobleBEM.m*.

In order to compare the results obtained in the previous section with the ones gotten with the Matlab script, the following figure is taken into account:

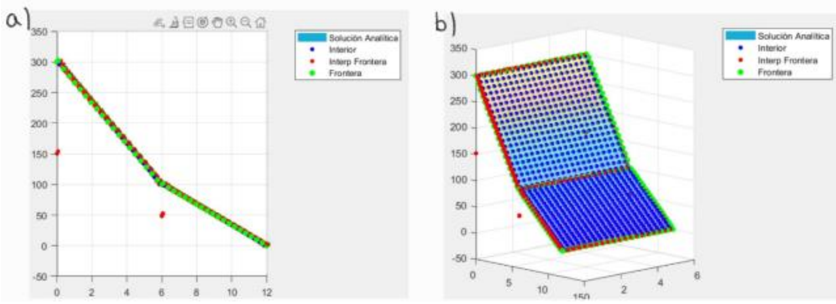

Figure 10: Graph a): Lienal regression of the 2D representation of $V(x)$, depending only on the x-variable. Graph b): Field form by the 3D representation of $V(x)$. It is needed to be remarked that there is a change on the slope in a point (for the 2D representation) or in a line (for the 3D representation), separating the straight lines which represent the surfaces with different homogenous conductivities, having: $\sigma_1 = 1, \sigma_2 = 2$.

Then, focusing on graph a) from *Figure 10*, the lineal regression of $V(x)$, and on the results obtained by analytically computing the problem, it can be said that they both perfectly fit, as the equations gotten represents the graph (both straight lines with different slope) plotted by Matlab.

3. Compare ambos scripts

To compare both scripts, Figure 2 and Figure 10 are taken into account, having:

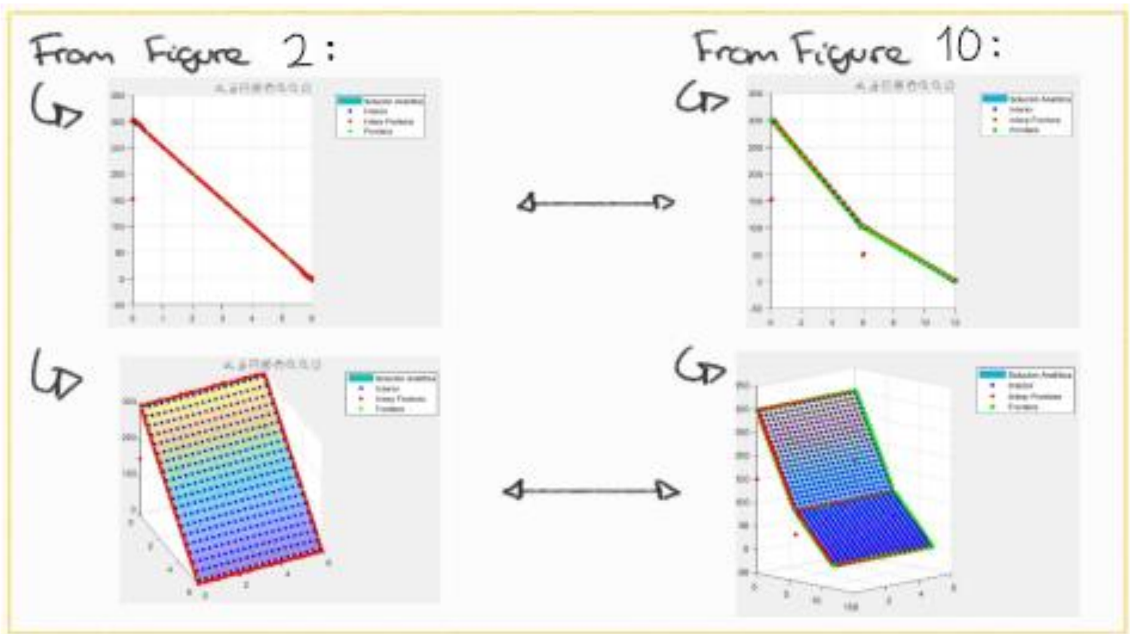

Figure 11: Comparison between scripts

Thanks to Figure 11, the only difference that can be seen between the scripts are the graphs which each of them plot. The graph obtained from *elementalIBEM.m*. basically represents a linear regression in 2D and a "perfect" field plane in 3D, while from *dobleBEM.m*. there is a change in the slope in each plot, given by the change in the conductivity.
As a conclusion, it can be said that the latter (*dobleBEM.m*.) script is used for surfaces with different homogenous conductivities (mainly two), while the former (*elementalIBEM.m*.) is prepared for surfaces presenting a unique homogenous conductivity.

4. Como cambia la solución:

(a) Si aumentamos el número de puntos de la frontera:

The solution of this section will be the same as in section 3a) of Lab.a having:

- If boundary points increase, the graph resolution improves while if they decrease, the resolution worsens.
- If interior points change, the graph resolution do not experiment any change.

The only remarkable aspect with respect a surface with regions with different homogenous conductivities is that when increasing the difference between the conductivities, the angle between the straight lines (representing $V_I(x)$ and $V_{II}(x)$) becomes smaller. Making a comparison between *Figure 6* and the figure below:

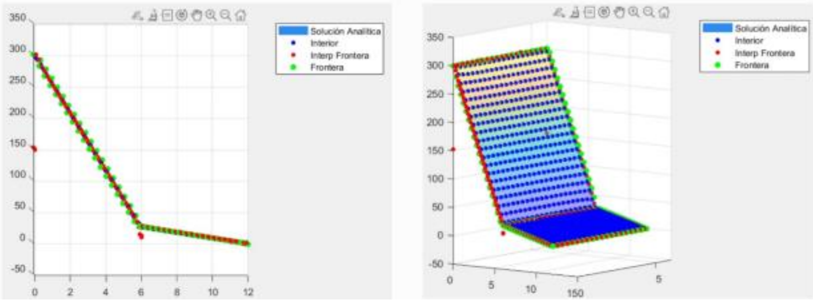

Figure 12: Same as in Figure 10, but having: $\sigma_1 = 1, \sigma_2 = 10$

This happens because when the conductivity of a medium tends to infinity, the potential of that medium will tend to cero, which graphically is represented by the x-axis. Then, if the conductivity of only one region increases, its linear progression will become closer to the x-axis as the slope will grow. It is said that the slope of the linear progression of a $V(x)$ is growing when increasing the conductivity as the slope is initially negative (seen in the computations made in pervious sections and exercises).

Lab.c

Compruebe el funcionamiento del script *realBEM*, donde se realiza el método BEM sobre un corte de un torso volumétrico real.

1. Compare este script con el anterior.

In order to compare the script *realBEM* with the previous scripts used in the other exercises, the following figure (obtained from this new script) is taken into account:

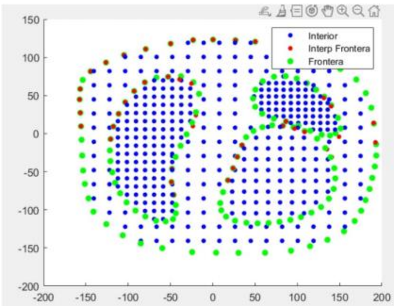

Figure 13: Representation of the torso, the heart, and the lungs using BEM model.

Basically, the first two scripts (*elementalIBEM.m.* and *dobleBEM.m*.) used in Lab.a and Lab.b (respectively) were based on the BEM used for squares and rectangles, ergo, for easy geometric figures to treat. However, the script used in this exercise introduce more complex figures to deal with, as the torso, heart, and lungs are.

2. ¿Qué forma tienen las matrices G y F?

Focusing on the 3D plot of the matrices,

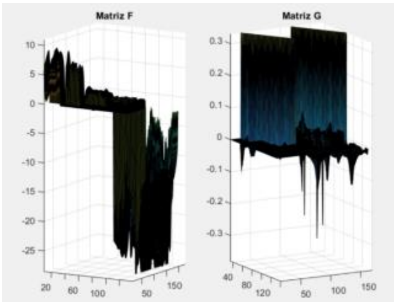

Figure 14: 3D plot of F and G matrices

and the 2D plot,

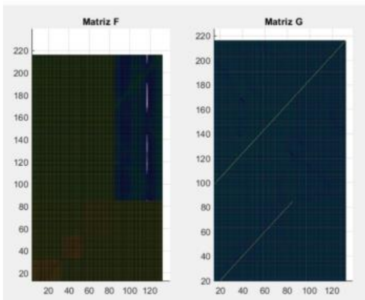

Figure 15: 2D plot of F and G matrices

The form of the matrices F and G are given. It is remarkable that both matrices represent the influence of the same points, being F the one making reference to the potential and G to the currents (derivative of the potential).

Also, taking matrix F, it can be seen that it has different shapes. The orange squares are the ones representing the points that the torso share with heart and both lungs, the blue rectangle the points that it shares with itself, and the rest are cero as are points which are not share with anything.

Lab.d

Utilice ahora el script *EsferasConc*, donde se realizará el método BEM sobre un modelo de dos esferas concéntricas.

1. **Haciendo uso de las funciones proporcionadas, calcule el error cometido al realizar la aproximación BEM frente a la solución analítica proporcionada.**

First of all, the following graphs are shown:

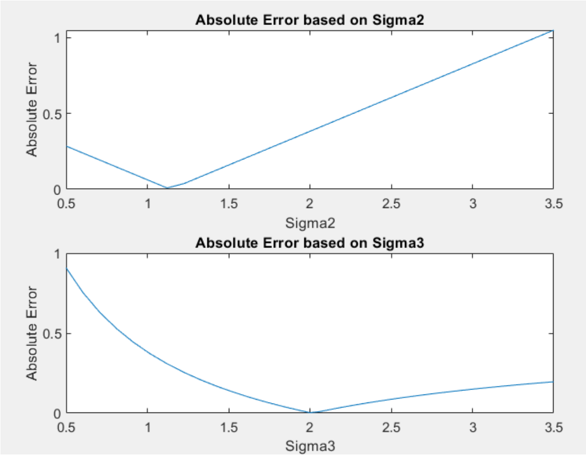

Figure 16: Graphs representing the absolute errors when only changing Sigma2 or Sigma3 respectively.

Each graph represents the variation of the absolute error, with respect the variation of Sigma2 or Sigma3 respectively. On both, it can be seen that there is a threshold value at which the error is minimum. For Sigma2, before the threshold value the error can be considered as small, but when it is surpassed, it will infinitely grow, while, for Sigma3, before the threshold value, the error is large, but is decreasing rapidly, and when it is surpassed, it will infinitely grow, but slowly.

Then, focusing on the solution obtained analytically, the following graphs are taken into account.

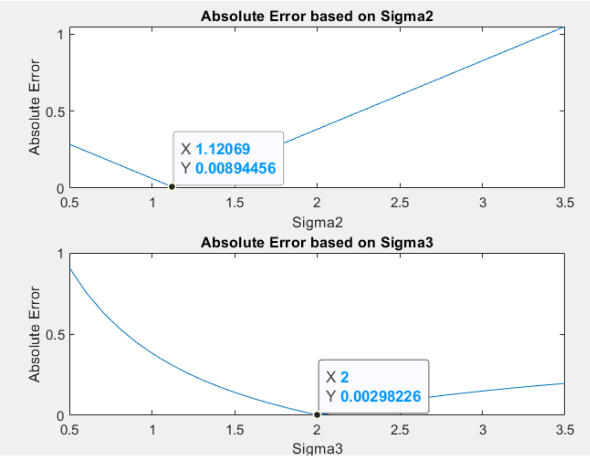

Figure 17: Graphs representing the absolute errors when only changing Sigma2 or Sigma3 respectively. There are also pointed out the minimum of both graphs, giving the values of both conductivities with lowest absolute error.

From *Figure 17*, the values for Sigma2 and Sigma3 with lowest error are given.

- When varying only Sigma2, the minimum error 0.008945 is associated with conductivity Sigma2 1.120690.
- When varying only Sigma3, the minimum error 0.002982 is associated with conductivity Sigma3 2.000000.

This data coincides with that which is printed by the Matlab script, representing the analytical solution of the conductivities (for Sigma2 and Sigma3) with lowest error.

Now, the BEM approximation errors are marked out through the following graphs.

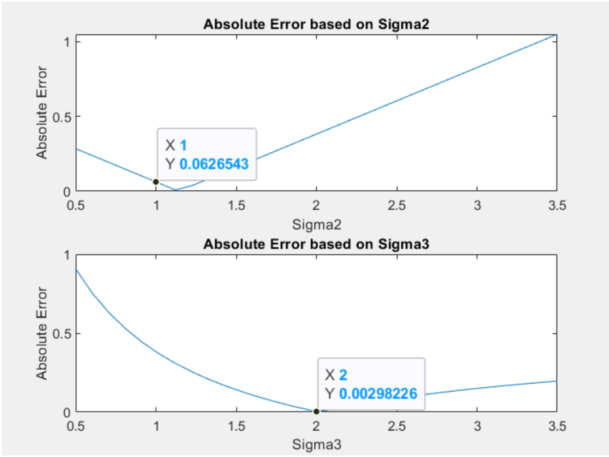

Figure 18: Graphs representing the absolute errors when only changing Sigma2 or Sigma3 respectively. There are also pointed out the conductivities for the analytically computation made by the method anaSolveSphere, with their respective errors, which coincide with the ones given by the BEM approximation.

Form *Figure 18*, the values taken for Sigma2 and Sigma3 are those used for solving analytically the problem with the method *anaSolveSphere*, also obtaining their errors, which correspond with the error produced by the BEM approximation.

From the graph: Absolute Error based on Sigma2

- When having $\sigma_2 = 1$, the error associated with such conductivity is $\varepsilon_{\sigma_2} = 0.06265$.

From the graph: Absolute Error based on Sigma3

- When having $\sigma_3 = 2$, the error associated with such conductivity is $\varepsilon_{\sigma_3} = 0.00298$.

In order to compare the errors produced by both methods (analytical solution and BEM approximation) the results obtained with the help of *Figure 17* and *Figure 18* are marked out, as they are the values concerning the problem.

For Sigma2:

- The absolute error obtained from the analytical solution is lower than the obtained with the BEM approximation, having:

	Analytical Solution		**BEM approximation**	
Sigma2	$\sigma_2 = 1.120690$	$\varepsilon_{\sigma_2} = 0.00895$	$\sigma_2 = 1$	$\varepsilon_{\sigma_2} = 0.0627$

Then, for Sigma2, the analytical solution (as it has lower absolute error) it will be better for practical concerns and computations.

For Sigma3:

- The absolute error obtained from the analytical solution coincides with the obtained with the BEM approximation, having:

	Analytical Solution		**BEM approximation**	
Sigma3	$\sigma_3 = 2$	$\varepsilon_{\sigma_3} = 0.00298$	$\sigma_3 = 2$	$\varepsilon_{\sigma_3} = 0.00298$

Then, for Sigma3, both methods can be considered in order to perform practical computations.

It is needed to be remarked, that the values chosen from the analytical solution (both graphs) to compare with the values of the BEM approximation were those with lowest error as they were the fittest values for practical concerns.

2. Compruebe como afecta al modelo una mala determinación de las conductividades de los medios.

In order to answer the problem, the initial values used for the conductivities to compute the analytical solution are marked out, having:

$$\sigma_1 = 0.5; \quad \sigma_2 = 1; \quad \sigma_3 = 2$$

Also, the following graphs should be recalled:

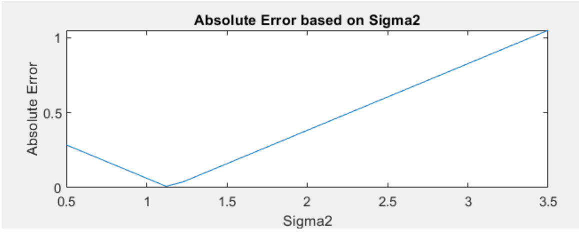

Figure 19: Graph representing the absolute error when only changing Sigma2, with $\sigma_1 = 0.5;\ \sigma_2 = 1;\ \sigma_3 = 2$

Which represents how the absolute error changes with respect the only variation of Sigma2, depending also on the conductivity values which do not vary.

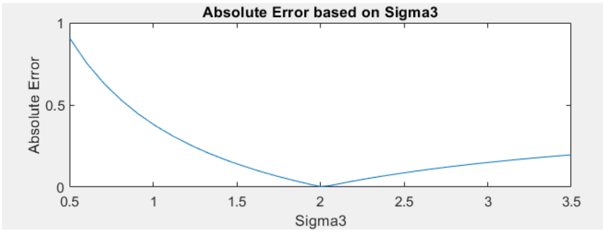

Figure 20: Graph representing the absolute error when only changing Sigma3 $\sigma_1 = 0.5;\ \sigma_2 = 1;\ \sigma_3 = 2$

Which represents how the absolute error changes with respect the only variation of Sigma3, depending also on the conductivity values which do not vary.

This graphs, were the ones used in the previous section, and as previously mentioned that on both, it can be seen that there is a threshold value at which the error is minimum. For Sigma2, before the threshold value the error can be considered as small, but when it is surpassed, it will infinitely grow, while, for Sigma3, before the threshold value, the error is large, but is decreasing rapidly, and when it is surpassed, it will infinitely grow, but slowly. Then, the fittest values for Sigma2 and Sigma3 are the ones with lowest absolute error, having:

- $\sigma_2 = 1.120690 \rightarrow \varepsilon_{\sigma_2} = 0.008945$. This corresponds to the analytical solution (previous section).
- $\sigma_3 = 2.000000 \rightarrow \varepsilon_{\sigma_2} = 0.002982$. This corresponds to the analytical solution and the BEM approximation.

Lab.e

Utilice ahora el script *PlnvTorso*, donde se realizará el método BEM sobre un humano experimental.

1. **Compruebe el funcionamiento del método y represente como varía el resultado obtenido con la variación de las conductividades del sistema.**

   The script provided printed out the following plot:

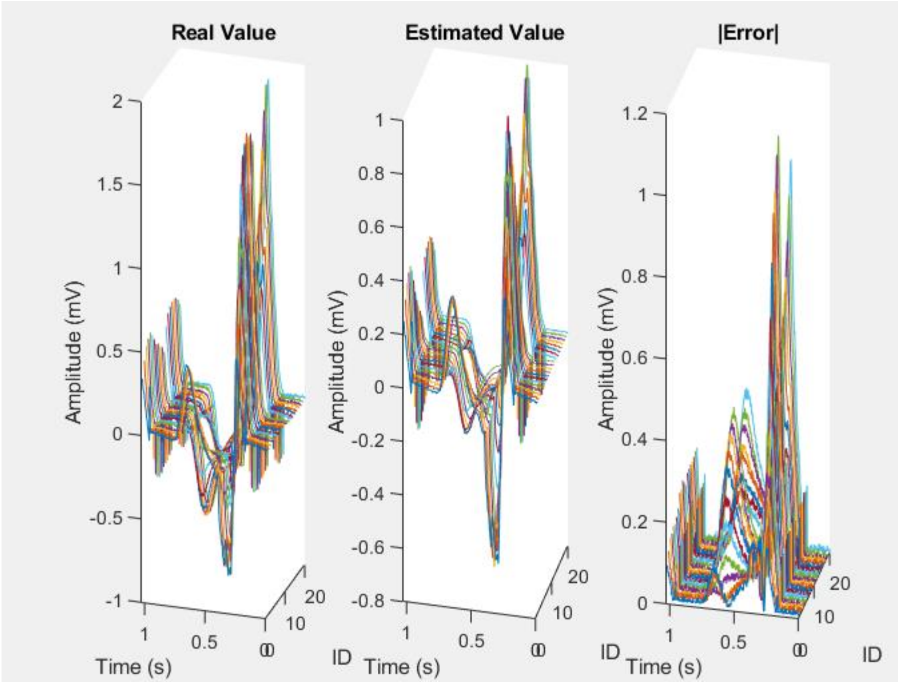

Figure 21: The first graph shows the real value for the potential on a torso. The second graph estimates the value of the torso using the BEM approximation. The third graph show the error the approximation had.

At first look, it can be said that both plots, the real value and estimated value, are really similar, saying that there could only be a minimum error when computing the estimated torso potentials. However, the third graphs shows that there is a considerable error. Then, what the exercise wants is to change the conductivities of the different surfaces in order to minimize the obtained error.

In order to change the conductivities, the code is analysed, finding that the values to change are *tankexactgridgeom.sigma* and *cagegeom.sigma*. These variables represent the conductivities through two vectors, where:

- the second value from the former variable (or vector) and the first value from the latter variable (or vector) corresponds to the conductivity of the same surface.
- the first value from the former variable (or vector) and the second value from the latter variable (or vector) corresponds to the conductivity of the same surface.

Then, it is needed to keep in mind that those values need to be equal. At first, the code had the following:

```
% Parse geometrical data in structures
tankexactgridgeom.pts = (auxTank.torso.node);
tankexactgridgeom.fac = (auxTank.torso.face);
tankexactgridgeom.sigma = [1,2]; %No value provided for inside-outside conductivities

cagegeom.pts = (auxCage.cage.node);
cagegeom.fac = (auxCage.cage.face);
cagegeom.sigma = [2,1]; %No value provided for inside-outside conductivities
```

Figure 22: Part of the code of the script PlnvTorso.m with the initial values

Then, changing the values of the previously mentioned variables, having:

```
% Parse geometrical data in structures
tankexactgridgeom.pts = (auxTank.torso.node);
tankexactgridgeom.fac = (auxTank.torso.face);
tankexactgridgeom.sigma = [1,100]; %No value provided for inside-outside conductivities

cagegeom.pts = (auxCage.cage.node);
cagegeom.fac = (auxCage.cage.face);
cagegeom.sigma = [100,1]; %No value provided for inside-outside conductivities
```

Figure 23: Part of the code of the script PlnvTorso.m with the values changed

The following is obtained:

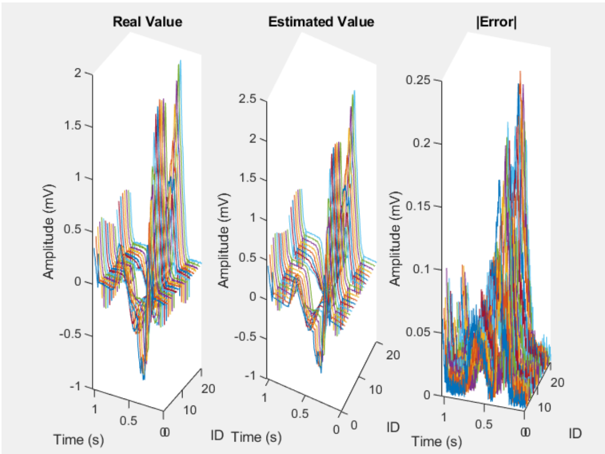

Figure 24: Similar to Figure 21, but with different values, giving a better approximation, as the error is lower.

Finally, a better approximation is obtained, as the error obtained is lower than the initial one.