User Guide

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Abstract

A user guide for beginning master students to fill in some gaps between master's courses and basic research. The main content of this guide is about some aspects of gravity and basics of AdS/CFT which are hardly covered in standard textbooks but used a lot in research.

Contents

Part I
Physics

1 Preliminary Knowledge

Some preliminary knowledge is required to work through this guide. When people start doing their research, they often find out that there's a lot they do not know. Therefore, the ability to find resources and knowledge yourself is necessary. You are strongly encouraged to google it, discuss with your friends, ask senior students or staffs.

1.1 coursework

The main targets of this user guide are master students at University of Amsterdam in the track of theoretical physics that will work on AdS/CFT (mainly on gravity side). Therefore, a certain level of prior knowledge in quantum field theory, general relativity and math is required to work on this user guide. In terms of courses offered at UvA, you need to have grasped relevant topics covered by:

- Quantum field theory 1, 2 and 3
- Advanced quantum field theory
- General relativity

These courses are still very helpful for you to work on this user guide although not essential:

- Group theory
- String theory 1 and 2
- Cosmology
- Advanced Cosmology
- Mathematical methods in physics

Of course, as a student in theoretical physics, the more, the better.

1.2 Advanced topic in general relativity

Some master students have only basic understanding of general relativity. Let's say, chapter 1-5 of [1]. In order to proceed, you need to read at least chapter 6, 9 and all the appendix. You may still need to know about basics of asymptotic flat spacetime, such as how to use Komar integral to calculate the total mass of the spacetime. On the math side, you need to be familiar with differential form. This will be very convenient, as these objects often appear in the literature. More advanced choice of GR text book would be [2] [3].

1.3 Black holes

This user guide will mainly play with black holes. Therefore, you need to know it well in advance. There are two notes that covers almost the same contents [4] [5]. The former one is newer and uses differential forms language, while the 2nd one contains valuable exercises at the end. My suggestion is to mainly use the first one, and to use the second one as a reference while trying to think about all of the exercises.

1.4 Basics of AdS/CFT

To work with this guide, you need to know basics of AdS/CFT. It is hard to "understand" AdS/CFT by simply reading a review or lecture note. One reason is that concrete examples of the duality often lies in super string theory or supergravity theory, which is hard to learn in a short time. Therefore, I suggest you to have a hand-waving idea about AdS/CFT firstly and being able to do simple calculations on this duality. Then you can follow this guide to know about more details on certain aspects. After that you might be able to go back to some more advanced lectures on AdS/CFT and super string theory. Learning something back and forth is very common in research.

There are a lot of lecture notes and reviews on AdS/CFT, but I find most of them hard to use for the lack of concrete examples or being too advanced. My suggestions is to follow the open course [6]. This open course provide motivation of holography, derivation of AdS/CFT from D-brane and some properties and applications of AdS/CFT in an intuitive manner. If you want to work out the calculations, there are corresponding problem sets. Another student-friendly note on this topic is [7]. There are a lot of overlaps between these two, and you can also find some contents of this user guide covered in them. but I still suggest you to work on both of them. and come back to my guide. You don't have to know all the details but at least you need to spend time on some of the problems provided in the notes.

2 Structure of This Guide

Before we start the journey, I would like to tell you the structure of this guide.

The section 3 is about the boundary stress tensor for AdS gravity. First we will review the Gibbons–Hawking–York (GHY) boundary term in the Lagrangian formalism of general relativity from Dirichlet boundary condition. Then we will release this boundary condition and define the boundary stress tensor. In the end we will introduce the necessary subtraction term and introduce some useful formulas to compute conserved quantities.

In the beginning of every section, I will tell you some important reference on which the guide is based. It is important to read the original literature in the research, thus I strongly recommend you to read the corresponding chapters in the original papers before working through the sections of this guide.

3 Boundary Stress Tensor and Its Application

When we use Lagrangian formulation of general relativity, we usually set the boundary conditions of a variational problem as Dirichlet boundary condition, which is

$$\delta g_{ab}|_{\partial M} = 0, \tag{3.1}$$

while we say nothing about $\nabla_a g_{bc}$ on the boundary of the manifold. However, if we let the variation of the metric be free on the boundary, we could get a very useful quantity - the boundary stress tensor. Before working on this section, please review the Lagrangian formalism of general relativity by reading [2] or [3] where the GHY term is discussed. The second topic, boundary stress tensor for AdS gravity, is based on the paper [8]. You should also try to read the whole paper and get some ideas about this topic.

3.1 Lagrangian formulation and GHY term

Problem 1: Variation with Dirichlet boundary condition

Consider the action on a manifold M with a fixed boundary:

$$S = \frac{1}{2\kappa}\int_M d^{n+1}x\sqrt{-g}(R - 2\Lambda). \tag{3.2}$$

Calculate the variation with Dirichlet boundary condition $\delta g_{ab}|_{\partial M} = 0$. You will get

$$\delta S_1 = \int_M d^{n+1}x\sqrt{-g}(G_{\mu\nu} + \Lambda g_{\mu\nu})\delta g^{\mu\nu} + \int_{\partial M} d^n\sqrt{-h}\; n^a v_a, \tag{3.3}$$

where $v_a = \nabla^b \delta g_{ab} - g^{cd}\nabla_a \delta g_{cd}$, h is the induced metric on the boundary and n^a is the normalized normal vector of the boundary.

Here you see the interesting point: terms with derivative of δg_{ab} do not vanish. In order to have a well defined variational problem (Dirichlet boundary condition) that reproduces the correct EOM, we need some other terms to cancel it. The extra term should be a pure boundary term so it wouldn't affect the equations of motion (EOM) in the bulk. This term is called the Gibbons–Hawking–York boundary term:

$$S_{GHY} = \frac{1}{\kappa}\int_{\partial M} d^n x\sqrt{-h}K. \tag{3.4}$$

Problem 2: Variation of action with GHY term

Calculate the variation of eq. (3.4), then verify:

$$\delta S = \frac{1}{2\kappa}(\int_M d^{n+1}x\sqrt{-g}(R-2\Lambda) + 2\int_{\partial M} d^n x\sqrt{-h}K) = \int_M d^{n+1}x\sqrt{-g}(G_{ab}+\Lambda g_{ab})\delta g^{ab}. \quad (3.5)$$

hint: you need to calculate the variation of the trace of the extrinsic curvature K. Writing the parameter of the family of fields out explicitly:

$$K(\lambda) = h^a{}_b(\lambda)\nabla_a(\lambda)n^b(\lambda) = h^a{}_b(\lambda)\left[\nabla_a n^b(\lambda) + C^b{}_{ac}(\lambda)\; n^c(\lambda)\right]. \quad (3.6)$$

The variation is:

$$\delta K = \delta h^a{}_b \nabla_a n^b + h^a{}_b(\nabla_a \delta n^b + \delta C^b{}_{ac}\; n^c) = \delta h^a{}_b \nabla_a n^b + h^a{}_b \nabla_a \delta n^b + \frac{1}{2} n^c h^{ad}\nabla_c \delta g_{ad}. \quad (3.7)$$

Now we establish a well-defined variational problem for general relativity. A natural question to ask is what happens if we release the boundary condition and check what is $\frac{\delta S}{\delta h^{ab}}$.

Problem 3: Variation with Released Boundary Condition
Calculate eq. (3.5) again with no restriction on the boundary. Concentrate on how to evaluate eq. (3.6) in this situation. The result should be:

$$\delta S = \int_{\partial M} dx^n\, \sqrt{-h}\left(K h^{ab} - K^{ab}\right)\delta g_{ab}. \quad (3.8)$$

hint: This is not an easy computation. You need to be aware of what will vary and what will not. In this situation, the boundary is fixed and determined by some hypersurface $f(x) = const$. So the normal vector and covector will vary. Also, because $g_{ab} = h_{ab} + n_a n_b$, the variation of g_{ab} and h_{ab} is related. After all, this computation is done in [9].

eq. (3.8) is very useful for the derivation of boundary stress tensor in the next subsection.

3.2 Definition of boundary stress tensor and conserved quantity in AdS spacetime

In AdS_{d+1} spacetime, let's say in Poincare patch:

$$ds^2 = -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}d\vec{x}^2. \quad (3.9)$$

We can do a (d-1)+1+1 decomposition, where $1+1$ are the r and t directions and $(d-1)$ are the constant r and t surfaces. Now take the bulk as the spacetime region within some constant surface r and the induced metric on the boundary (certain constant r surface) is γ_{ab}. We have the action:

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\left(R + \frac{d(d-1)}{\ell^2}\right) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}K, \tag{3.10}$$

where $\Lambda = -d(d-1)/2\ell^2$. Now we want to use the result from the previous subsection.

Problem 4: Raw Boundary Stress Tensor

Recall the definition of the usual stress tensor in GR is :

$$T^{ab} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{ab}}. \tag{3.11}$$

This definition is good, since the variation of the action contains no boundary term. However, when the boundary condition is released as discussed in the previous subsection, we can define a so called boundary stress tensor:

$$T^{ab} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta \gamma_{ab}}. \tag{3.12}$$

Compute the boundary stress tensor, you will find the result:

$$T^{ab} = -\frac{1}{8\pi G}\left[K^{ab} - K\gamma^{ab}\right]. \tag{3.13}$$

Now, what is the usefulness of this boundary stress tensor? Recall that when we define conserved quantities, such as the mass, of the spacetime, only data of tensor fields near the boundary matters. For example, the Komar integral for 4d asymptotically flat spacetime is:

$$M_{\mathrm{K}} := -\frac{1}{8\pi}\int_S \varepsilon_{abcd}\nabla^c\xi^d. \tag{3.14}$$

So only the value of the tensor fields g_{ab} and ξ_a on the boundary S will contribute to the integration. Therefore, it is natural to guess how the boundary stress tensor could be used to define conserved quantities of the spacetime.

We can define the conserved quantity associated with a killing vector ξ_a of the spacetime as:

$$Q_\xi = \int_\Sigma d^{d-1}x\sqrt{\sigma}\left(u^\mu T_{\mu\nu}\xi^\nu\right). \tag{3.15}$$

Where Σ is a co-dimension 2 submanifold. Here we are considering a constant r, t surface and σ is the induced metric on it. u^a is the timelike unit normal. If you have doubt on this definition, go to chapter 8 of Hartman's note [7]. As a check, we could compute the total mass of AdS_3 spacetime, which should be 0 since there is no matter field in empty AdS spacetime.

Problem 5: Boundary Stress Tensor for Empty AdS_3 **Spacetime**

Compute the boundary stress tensor of pure AdS_3 spacetime, using Poincare patch. Then compute the total mass. You should get the result:

$$\begin{aligned} 8\pi G T_{tt} &= -\frac{r^2}{\ell^3}, \\ 8\pi G T_{xx} &= \frac{r^2}{\ell^3}, \\ 8\pi G T_{tx} &= 0. \end{aligned} \tag{3.16}$$

So the total mass diverges as $r \to \infty$. Even worse, the "energy density" T_{tt} itself diverges. Therefore, we hope to get a finite answer by adding a counterterm that is intrinsic to the boundary to cancel the divergence. According to [8], the counterterm for AdS_3 spacetime is:

$$S_{ct} = \int_{\partial \mathcal{M}_r} \frac{1}{\ell}\sqrt{-\gamma}. \tag{3.17}$$

Problem 6: Full Boundary Stress Tensor

Adding the counterterm, verify that the boundary stress tensor takes the form:

$$T^{\mu\nu} = -\frac{1}{8\pi G}\left[K^{\mu\nu} - K\gamma^{\mu\nu} + \frac{1}{\ell}\gamma^{\mu\nu}\right], \tag{3.18}$$

and its components vanish for empty AdS_3 spacetime. This expression is very useful: it enables us to reproduce correct conserved quantities in AdS_3 spacetime, such as the angular momentum and the mass of the black hole. We will do this when studying the BTZ black hole. [8] gives the correct counterterms in various dimensions and for example, you could verify that it reproduces the correct mass for the AdS Schwarzschild black hole and the AdS-RN black hole in 4 dimension.

Problem 7: Mass and Momentum For $AAdS_3$

In this problem we derive general formula to compute total mass and momentum for asymptotic AdS_3 ($AAdS_3$) spacetime. We make the following ansatz for the metric:

$$ds^2 = \frac{\ell^2}{r^2}dr^2 + \frac{r^2}{\ell^2}\left(-dt^2 + dx^2\right) + \delta g_{MN} dx^M dx^N, \tag{3.19}$$

where δg_{MN} has a block diagonal form, with nonzero components $(r), (t, x)$. Compute the boudnary stress tensor, you should get the result:

$$\begin{aligned} 8\pi G T_{tt} &= \frac{r^4}{2\ell^5}\delta g_{rr} + \frac{\delta g_{xx}}{\ell} - \frac{r}{2\ell}\partial_r \delta g_{xx}, \\ 8\pi G T_{xx} &= \frac{\delta g_{tt}}{\ell} - \frac{r}{2\ell}\partial_r \delta g_{tt} - \frac{r^4}{2\ell^5}\delta g_{rr}, \\ 8\pi G T_{tx} &= \frac{1}{\ell}\delta g_{tx} - \frac{r}{2\ell}\partial_r \delta g_{tx}. \end{aligned} \tag{3.20}$$

Then via eq. (3.15) we get the final result:

$$
\begin{aligned}
M &= \frac{1}{8\pi G}\int dx \left[\frac{r^4}{2\ell^5}\delta g_{rr} + \frac{\delta g_{xx}}{\ell} - \frac{r}{2\ell}\partial_r \delta g_{xx}\right], \\
P_x &= -\frac{1}{8\pi G}\int dx \left[\frac{1}{\ell}\delta g_{tx} - \frac{r}{2\ell}\partial_r \delta g_{tx}\right].
\end{aligned}
\tag{3.21}
$$

hint: you can compute the stress tensor via mathematica packages.

4 Basics of $\mathrm{AdS}_3/\mathrm{CFT}_2$

Usually one needs supersymmetry for concrete realization of AdS/CFT. However, for $\mathrm{AdS}_3/\mathrm{CFT}_2$ we have abundant symmetries from the 2D conformal group. Therefore, we don't necessarily need supersymmetry for this duality. In this section we will learn about the central charge of AdS_3 gravity and the computation of black hole entropy via CFT method.

4.1 Central charge of AdS_3 gravity

We start by focusing on one important feature of AdS spacetime: its asymptotic symmetry group. Why is it important? As a spoiler, I can tell you about this in 2 different ways. The first one is about the transformation of the boundary stress tensor. Remember that in CFT, the quantum stress tensors transform in a nontrivial manner: there is an extra term called central charge. Since we already found that the boundary stress tensor is somehow a "real" stress tensor, we might be wondering about what kind of transformation in the bulk resembles conformal transformation on the boundary? On the other hand, we want to inspect what kind of real symmetry could exist in AdS gravity: the transformation that brings one configuration of spacetime to an inequivalent one, let's say, with different conserved quantities, such as the mass.

Many contents of this section is based on [10]. Please read the corresponding contents before working on this guide.

4.1.1 Asymptotic AdS_3 Spacetime

The most rigorous definition of asymptotic AdS_3 (AAdS_3) spacetime is advanced and not studied here. Generally speaking, there are three criteria for AAdS_3 spacetimes as solution to 3d Einstein equation. They must contain known solutions, lead to finite and integrable surface charges and the symmetry group must contain at least SO(2,2), the exact symmetry of Global AdS_3. Based on these conditions, there is the Fefferman-Graham theorem: Any AAdS spacetime can be written in the neighborhood of the boundary as :

$$
\begin{aligned}
ds^2 &= \ell^2 d\rho^2 + g_{ab}dx^a dx^b \\
&= \ell^2 d\rho^2 + e^{2\rho}(g_{(0)ab} + \mathcal{O}\left(e^{-\rho}\right))dx^a dx^b \\
&= \ell^2 \frac{dr^2}{r^2} + \frac{r^2}{\ell^2}(g_{(0)ab} + \mathcal{O}(\frac{1}{r}))dx^a dx^b \quad (r = \ell e^{\rho})\,.
\end{aligned}
\tag{4.1}
$$

Solving Einstein equations order by order, The FG expansion takes the form:

$$\begin{aligned} ds^2 &= \ell^2 \frac{dr^2}{r^2} + \frac{r^2}{\ell^2}\left(g_{(0)ab} + \frac{\ell^2}{r^2} g_{(2)ab} + \frac{\ell^4}{r^4} g_{(4)ab}\right) dx^a dx^b \\ &= \ell^2 \frac{dr^2}{r^2} - \left(r dx^+ - \ell^2 \frac{L_-\left(x^-\right)}{r} dx^-\right)\left(r dx^- - \ell^2 \frac{L_+\left(x^+\right)}{r} dx^+\right), \end{aligned} \tag{4.2}$$

where $g_{(0)ab}dx^a dx^b = -dt^2 + \ell^2 d\phi^2$ and $x^\pm = \frac{t}{\ell} \pm \phi$.

4.1.2 Asymptotic Killing vector

Recall that killing vectors are those vectors that preserve the metric with the action of Lie derivative. Since we want to translate spacetime between different states, we might release the condition such that the killing vector preserve the metric up to some small quantities near the boundary. We will call such a vector an asymptotic killing vector (AKV) and the corresponding group asymptotic symmetry group (ASG). The exact definition for AKV that preserve eq. (4.1) is:

$$\begin{aligned} &\mathcal{L}_\xi g_{\rho\rho} = 0, \\ &\mathcal{L}_\xi g_{\rho a} = 0, \\ &\mathcal{L}_\xi g_{ab} = O(e^\rho). \end{aligned} \tag{4.3}$$

Problem 1: Asymptotic Killing Vector
Solve the AKV in eq. (4.3), you will get:

$$\begin{aligned} \xi^{(+)} &= V^+\left(x^+\right)\partial_+ - \frac{1}{2}\partial_+ V^+ \partial_\rho + \frac{1}{2}\int d\rho g^{+-}\partial_-\partial_- V^- \partial_+, \\ \xi^{(-)} &= V^-\left(x^-\right)\partial_- - \frac{1}{2}\partial_- V^- \partial_\rho + \frac{1}{2}\int d\rho g^{+-}\partial_+\partial_+ V^+ \partial_-. \end{aligned} \tag{4.4}$$

Do the coordinate transformation $\rho \to r$, you will get the result:

$$\begin{aligned} \xi^{(+)} &= V^+\left(x^+\right)\partial_+ + \frac{\ell^2}{2r^2}\partial_-\partial_- V^- \partial_+ - \frac{r}{2}\partial_+ V^+ \partial_r, \\ \xi^{(-)} &= V^-\left(x^-\right)\partial_- + \frac{\ell^2}{2r^2}\partial_+\partial_+ V^+ \partial_- - \frac{r}{2}\partial_- V^- \partial_r. \end{aligned} \tag{4.5}$$

hint: This is not an easy task. Read chapter 2.2.3 of [10] and fill in the all the middle steps of his calculation. Be aware of the possible typos or mistakes in the reference.

Usually after finding the killing vectors, we want to obtain the corresponding symmetry group by checking the commutation relationships of the vectors (the Lie algebra) and making comparison with the known ones, such as $SU(2), SL(2), SO(1,3)$ and so on. But before doing this, we need to aware that the linear combination of killing vectors are still killing vectors. Therefore, it would be clever if we could check the commutation relationship under certain basis.

Problem 2: Asymptotic Symmetry Group

Let's do the calculation under the basis: $V^+ = e^{imx^+}, V^- = e^{imx^-}$. Define:

$$\begin{aligned} \zeta_m^{(+)} &= \xi^{(+)}\left(V^+ = e^{imx^+}\right), \\ \zeta_m^{(-)} &= \xi^{(-)}\left(V^- = e^{imx^-}\right). \end{aligned} \tag{4.6}$$

Compute the commutators of $\zeta_m^{(+)}$ and $\zeta_m^{(-)}$, you need to find the result:

$$\begin{aligned} i\left[\xi_m^{(+)}, \xi_n^{(+)}\right] &= (m-n)\xi_{m+n}^{(+)}, \\ i\left[\xi_m^{(+)}, \xi_n^{(-)}\right] &= 0. \end{aligned} \tag{4.7}$$

What algebra is it? Actually you should have seen it when learning conformal field theory: Witt algebra, which is the algebra of generators of 2d conformal transformations. You should also recall that this group contains the subgroup of global 2d conformal transformation, which is also the isometric group of AdS_3 spacetime: $SO(2,2)$. If you search online, you will find such relation ship: $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}) \simeq \mathrm{so}(2,1) \oplus so(2,1) \simeq \mathrm{so}(2,2)$. Indeed, you can check:

$$i\left[\xi_1^{(+)}, \xi_0^{(+)}\right] = \xi_1^{(+)}; \quad i\left[\xi_1^{(+)}, \xi_{-1}^{(+)}\right] = 2\xi_0^{(+)}; \quad i\left[\xi_0^{(+)}, \xi_{-1}^{(+)}\right] = \xi_{-1}^{(+)}, \tag{4.8}$$

which forms the group $sl(2,\mathbb{R})$.

Then we discuss the central charge of AdS_3 gravity. In some sense, the central charge is the key result of a quantum conformal field theory, which arises from the central extension of Witt algebra. Sometimes physicists will also call it "anomaly". If we believe the duality of AdS/CFT, we need to reproduce the central charge from the gravity side. However, I find the subject "central extension" is missing from many textbooks and courses, so I will spend some words to explain it in an intuitive, physical manner.

4.1.3 Central charge of a quantum theory

When we quantize a known classic theory with certain symmetries, we need to find a representation of the classical symmetry group: a homomorphism from the group to a Hilbert space. For example, when we try to quantize a rotationally invariant theory, in the end we will get the representation of $SU(2)$ group, which will contain discrete angular momentum: spins. However, the word “representation” is not rigorous. Recall that a quantum state $|\Psi\rangle$ transform under a symmetry operator $U(\tau)$ as:

$$|\Psi\rangle \rightarrow |\Psi'\rangle = U(\tau)|\Psi\rangle. \tag{4.9}$$

So the successive action of 2 symmetry operators should result in:

$$U(\tau_1)U(\tau_2)\,|\Psi\rangle \sim U(\tau_1\tau_2)\,|\Psi\rangle\,. \tag{4.10}$$

Here I use $\sim$ instead of $=$ because we only need two states that represent the same physical state: they can differ by a phase. Therefore, we have:

$$U\left(\tau_2\tau_1\right) = e^{i\omega(\tau_2,\tau_1)}U\left(\tau_2\right)U\left(\tau_1\right) \tag{4.11}$$

The “representation” is actually called “Projective Representation”. As a result, the classic lie algebra:

$$[A_1, A_2] = b_1 A_3 \tag{4.12}$$

will become:

$$[A_1, A_2] = b_1 A_3 + b_2 I, \tag{4.13}$$

where I is the identity operator and bs are constant numbers. This is called a central extension of an algebra and b_2 is called the central charge. For example, “c” in CFT and the mass in non-relativistic quantum mechanics are both central charges. See [11] [12] [13] for more details on central charge.

4.1.4 Central Charge from asymptotic killing vector

Now we want to derive central charge of AdS gravity. In order to do this, let’s recall how the central charges manifest in CFT and check whether there is an analogy on the gravity side. Firstly, the TT OPE. However, we have no idea about how the OPE is defined for the boundary stress tensor. Secondly, the Virasoro algebra. The generators in Virasoro algebra are the Laurent expansion coefficients of the stress tensor, but to compute the commutation relationship, we still need to define the Lie bracket from the gravity side, or know the OPE. The last one is the “anomaly term” of the transformation law of the stress tensor:

$$\begin{aligned}\delta_\epsilon T(z) &= T'(z) - T(z)\\ &= -\frac{1}{12}c\partial_z^3\epsilon(z) - 2\partial_z\epsilon(z)T(z) - \epsilon(z)\partial_z T(z).\end{aligned} \tag{4.14}$$

Luckily, we have just derived a transformation, the asymptotic isometric transformation. We see that the symmetry algebra is Witt algebra, the same as that of a 2D conformal transformation.

Therefore, we might expect to be able to perform the asymptotic isometric transformation for the boundary stress tensor. It will be most obvious to see the correct transformation law of the stress tensor if the original boundary stress tensor is 0.

Problem 3: Central Charge of AdS_3 Gravity

From empty AdS_3 spacetime:

$$ds^2 = \frac{\ell^2}{r^2}dr^2 - r^2 dx^+ dx^-. \tag{4.15}$$

Do the asymptotic isometric transformation, get the new $AAdS_3$ spacetime. Compute the boundary stress tensor, read the central charge.The result should be:

$$\begin{gathered} T_{++} = +\frac{\ell}{16\pi G}\partial_+^3 V^+, \quad T_{--} = +\frac{\ell}{16\pi G}\partial_-^3 V^- \\ c = \frac{3\ell}{2G}. \end{gathered} \tag{4.16}$$

hint: split the 2 generators in eq. (4.5) into transformations of 3 different coordinates.

4.2 Reproduce Black Hole Entropy Conformal Field Theory

With central charge at hand, we can compute entropy of a CFT at high temperature without knowing the details of the theory by "Cardy's formula". Some recent research [14] derive the entropy via this method. Before start working on this section, I strongly suggest you to read section 1 and 2 of [15].

4.2.1 Cardy's formula

Now we will derive Cardy's formula. There is some online literature for this, such as [16]. You should read section 2.5 of it before working on this.

Problem 1: Cardy's formula

Consider the CFT on the torus with the partition function:

$$Z(\beta,\theta) = \mathrm{Tr}\, e^{-\beta H - i\theta J} = \mathrm{Tr}\, q^{(L_0 - \frac{c}{24})}\bar{q}^{(\bar{L}_0 - \frac{\bar{c}}{24})} = \mathrm{Tr}\, e^{2\pi i\tau(L_0 - \frac{c}{24})}e^{-2\pi i\bar{\tau}(\bar{L}_0 - \frac{\bar{c}}{24})}, \tag{4.17}$$

where the chemical potential θ and inverse temperature β are related to the modular parameter as:

$$q = e^{2\pi i\tau}, \quad 2\pi\tau = \theta + i\beta \tag{4.18}$$

Use modular invariance $Z(\tau,\bar{\tau}) = Z(-1/\tau,-1/\bar{\tau})$ and take the high temperature limit $\mathrm{Im}(\tau) \to 0^+$, show that the partition function can be written as:

$$\ln Z(\tau,\bar{\tau}) = \frac{i\pi}{12}\left(\frac{c}{\tau} - \frac{\tilde{c}}{\bar{\tau}}\right). \tag{4.19}$$

In this way we sort out the vacuum state (the state with lowest energy). We have accomplished half of the goal. Then we need to revoke another form of the partition function:

$$Z(\tau,\bar{\tau}) = \sum \rho(\Delta,\bar{\Delta}) e^{2\pi i \Delta\tau} e^{-2\pi i \bar{\Delta}\bar{\tau}}. \tag{4.20}$$

Then use Laplace transform (or Legendre transformation, or residue theorem in some sense), you should get the result:

$$\rho(\Delta,\bar{\Delta}) \approx \exp\left\{2\pi\sqrt{\frac{c\Delta}{6}} + a.h\right\}, \tag{4.21}$$

with the saddle point $\tau \approx i\sqrt{\frac{c}{24\Delta}}$. a.h. means anti hermitian part. Then the entropy is:

$$S = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)} + 2\pi\sqrt{\frac{\tilde{c}}{6}\left(\tilde{L}_0 - \frac{\tilde{c}}{24}\right)}. \tag{4.22}$$

hint: if you do not know what is saddle point approximation, see saddle point approximation. .

Now we have Cardy's formula eq. (4.22) or eq. (4.21). We could compute the entropy of a 2d CFT with modular invariance in the high temperature limit once we know central charge and L_0.

4.2.2 Basic properties of BTZ black hole

Then we will put Cardy's formula into use by computing the entropy of the BTZ black hole, a 3D asymptotic AdS_3 rotating black hole. Before working on this subsection, you need to know what is a BTZ black hole and its basic properties. See [15] and [17]. In my user's guide I will adopt this convention for the metric:

$$\begin{gathered} ds^2 = -\frac{\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right)}{l^2 r^2} dt^2 + \frac{l^2 r^2 dr^2}{\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right)} + r^2\left(d\phi - \frac{r_+ r_-}{l r^2} dt\right)^2 \\ M = \frac{r_+^2 + r_-^2}{8Gl^2}, \quad J = \frac{r_+ r_-}{4Gl^2}, \end{gathered} \tag{4.23}$$

where M and J are the physical mass and angular momentum, as you will derive in this subsection.

Problem 2: Surface Gravity of BTZ

Compute the surface gravity at outer horizon $r = r_+$. You should get $\left(r_+^2 - r_-^2\right)/\ell^2 r_+$.
There is an easy way to compute the surface gravity. It is:

$$\kappa^2 = -\lim_{\lambda \to 0}\left[\frac{1}{4\lambda}\nabla_a \lambda \nabla^a \lambda\right], \tag{4.24}$$

where $\lambda = g_{ab}\xi^a\xi^b = g_{tt} + \Omega_H^2 g_{\phi\phi} + 2g_{t\phi}\Omega_H$ and $\xi = \partial_t + \Omega_H \partial_\phi$. The definition of Ω_H is similar to the one of Kerr black hole and you need to find it yourself.

Problem 3: Physical Mass and Angular Momentum of BTZ
Compute the total mass and angular momentum. There are several ways to do it, for instance, eq. (3.21).

Problem 4: Thermodynamics of BTZ
Compute the entropy and temperature of the black hole. Then verify the second law of thermodynamics.
hint: when compute the second law, note there are 2 independent variables. You can take them as r_+ and r_-

4.2.3 Reproduce the entropy via CFT

Now it is time to reproduce the black hole entropy via Cardy's formula. Therefore, we need to know about $L_0 - \frac{c}{24}$ and $\tilde{L}_0 - \frac{\tilde{c}}{24}$.

Problem 5: Virasoro Generator on a Torus
Try to understand:

$$\begin{aligned} L_n - \frac{c}{24}\delta_{n,0} &= \oint dw e^{-inw} T_{ww}, \\ \tilde{L}_n - \frac{\tilde{c}}{24}\delta_{n,0} &= \oint d\bar{w} e^{in\bar{w}} T_{\overline{ww}}, \end{aligned} \tag{4.25}$$

over the normal CFT relationship:

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z), \tag{4.26}$$

where w is the complex coordinate ($w = \phi + it/\ell$) of the BTZ black hole.
hint: The CFT of BTZ black hole is defined on a torus.

Problem 6: Entropy of BTZ via Cardy's Formula

Compute the boundary stress tensor for the BTZ black hole, then find the Virasoro generator:

$$L_0 - \frac{c}{24} = \frac{1}{2}(M\ell - J), \quad \tilde{L}_0 - \frac{\tilde{c}}{24} = \frac{1}{2}(M\ell + J), \tag{4.27}$$

together with the central charge $c = \bar{c} = \frac{3\ell}{2G}$, compute the entropy and compare the result with the Hawking-Bekenstein entropy $S = \frac{A_{hor}}{4\pi}$. This is a remarkable result in $\mathrm{AdS}_3/\mathrm{CFT}_2$.

5 Near Extremal Black Hole and AdS_2 Spacetime

After playing with $\mathrm{AdS}_3/\mathrm{CFT}_2$, let's have a look at a more realistic system: black holes in 4D spacetime. We will see that AdS_2 arises from the near horizon region of near extremal black holes. In most of this section, We will use dyonic (electrically and magnetically charged) RN black hole as example. First let's fix the convention. The Einstein-Maxwell action is: ($G = 1$)

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}R - \frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}. \tag{5.1}$$

The EOMs (equation of motion) are:

$$\begin{aligned} &R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi\left(F^c_a F_{bc} - \frac{1}{4}g_{ab}F^{cd}F_{cd}\right) = 8\pi T_{ab}, \\ &\nabla^b F_{ab} = 0, \quad dF = 0. \end{aligned} \tag{5.2}$$

The solution is:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2, \\ A &= -\frac{Q}{r}dt - P\cos\theta d\phi \quad e = \sqrt{Q^2 + P^2}, \\ F &= -\frac{Q}{r^2}dt\wedge dr + P\sin\theta d\theta\wedge d\phi. \end{aligned} \tag{5.3}$$

We need to exclude the bare singularity at $r = 0$. Therefore, $r^2 - 2Mr + e^2 = 0$ must have at least one root, which gives the following inequality:

$$M^2 \geqslant e^2. \tag{5.4}$$

We call the black hole an extremal one if $M^2 = e^2$ while a near extremal one if $\frac{M^2}{e^2} = 1 + \epsilon$ where $\epsilon \ll 1$. I will call the two roots of the equation $r^2 - 2Mr + e^2 = (r - r_+)(r - r_-) = 0$ r_+, r_-, which are the inner and outer horizons of the black hole.

5.1 Near horizon region of near extremal black hole

We will explore how the spacetime looks like in the near horizon region of the near extremal black hole (near-NHEB). First let us do an exercise for the Schwarzschild black hole.

Problem 1: Near Horizon Region of the Schwarzschild Black Holes

The metric of the Schwarzschild solution is:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \quad f(r) = 1 - \frac{2M}{r}. \tag{5.5}$$

We zoom into the near horizon region using this coordinate transformation: $r = 2M(1+\epsilon^2)$, and taking the limit $\epsilon \to 0$. Derive the metric in the near horizon region:

$$ds^2 = -\epsilon^2 dt^2 + 16M^2 d\epsilon^2 + 4M^2 d\Omega_2^2. \tag{5.6}$$

So the spacetime is Rindler $\times$ S^2. Is this spacetime well defined? Why?
hint: The metric should be a solution to the Einstein equations to be physically allowed.

Then let's investigate the near horizon region of a non-extremal black hole with two horizons.

Problem 2: Near Horizon Region of non-Extremal RN Black Holes
Do such coordinate transformation for eq. (5.3) ($r_+ > r_-$):

$$r = r_+ + \lambda\rho, \quad t = \frac{T}{\lambda}, \quad \lambda \to 0. \tag{5.7}$$

You should again get a Rindler $\times S^2$ spacetime. This spacetime is still not well defined.
hint: You may not get a standard Rindler spacetime. Then you might need to do another coordinate transformation.

Then we will see that we could get a well defined spacetime by taking the near horizon limit of an extremal black hole.

Problem 3: Near Horizon Region of Extremal RN Black Holes
For simplicity, we use electrically charged black holes ($P = 0$). At extremality, $M = Q$, and $r+ = r_- = r_0 = Q$. We perform the following coordinate transformation to go to the near horizon region:

$$r = Q + \lambda\rho, \quad t = \frac{T}{\lambda}, \quad \lambda \to 0. \tag{5.8}$$

Show that the solution becomes:

$$\begin{gathered} ds^2 = -\frac{\rho^2}{Q^2} dT^2 + \frac{Q^2}{\rho^2} d\rho^2 + Q^2 d\Omega_2^2, \\ F = \frac{1}{Q} dT \wedge d\rho. \end{gathered} \tag{5.9}$$

The new line element does not contain the infinitesimal parameter λ explicitly, thus we could expect this solution to be well-defined, satisfying the Einstein-Maxwell equation. Please verify this.

Now we have found the beautiful property of extremal black holes, we will investigate the near horizon region of near extremal black holes. We still use electrically charged black holes. But now the black hole deviates from extremality a little bit:

$$r_\pm = Q \pm \lambda\delta.$$

Then we need to find suitable coordinate transformation and limit to get near-NHEB.

Problem 4: Near Horizon Region of Near Extremal RN Black Holes

First try:

$$t = \frac{T}{\lambda}, \quad r = Q + \lambda\delta + \lambda\rho, \tag{5.10}$$

and find

$$ds^2 = -\frac{\rho^2 + 2\delta\rho}{Q^2}dT^2 + \frac{Q^2}{\rho^2 + 2\delta\rho}d\rho^2 + Q^2 d\Omega_2^2. \tag{5.11}$$

Then try

$$r_\pm = Q \pm \lambda\delta, \quad t = \frac{T}{\lambda}, \quad r = Q + \lambda\rho, \tag{5.12}$$

and find

$$ds^2 = -\frac{\rho^2 - \delta^2}{Q^2}dT^2 + \frac{Q^2}{\rho^2 - \delta^2}d\rho^2 + Q^2 d\Omega_2^2. \tag{5.13}$$

The results are $\mathrm{AAdS}_2 \times S^2$ spacetime, written in different coordinate patches. When $\delta = 0$ we recover $\mathrm{AdS}_2 \times S^2$. There is another choice to reproduce the $AAdS_2$ spacetime in FG gauge:

$$r_\pm = Q \pm \lambda\delta, \quad t = \frac{T}{\lambda}, \quad r = Q + \lambda\rho + \frac{\lambda\delta^2}{4\rho}. \tag{5.14}$$

You need to show the result:

$$ds^2 = -\frac{\rho^2 - \delta^2}{Q^2}dT^2 + \frac{Q^2}{\rho^2 - \delta^2}d\rho^2 + Q^2 d\Omega_2^2. \tag{5.15}$$

Problem 5: NHEB is Well Defined

Verify that the NHEB and the nearly-NHEB metric satisfy Einstein equation.

hint: You also need to derive the gauge field near the horizon.

5.2 Thermodynamics of near extremal black hole

In this subsection we investigate the thermodynamics of near extremal black holes. We will know that there is a universal feature of thermodynamics for near extremal black holes. Many problems in this section are adapted from [18]

Problem 1: Thermodynamics of Near Extremal BTZ Black Holes

In this problem we investigate the thermodynamics of near extremal BTZ black holes eq. (4.23). Compute mass M_0 and entropy S_0 at extremality $r_+ = r_- = r_0$. Then consider the case of near extremality by keeping r_0 fixed while varying $r_+ - r_- = a,\ a \ll r_0$. Express your result in the form:

$$\begin{aligned} S &= S_0 + 2M_{gap}^{-1}T, \\ M &= M_0 + M_{gap}^{-1}T^2, \end{aligned} \tag{5.16}$$

and determine the value of M_{gap}.

hint: temperature is linear in a.

Problem 2: Thermodynamics of Near Extremal AdS_{n+1} RN Black Holes

Now we investigate the thermodynamics of near extremal RN black holes in $n+1$ dimensional AdS spacetime. The metric is given by:

$$\begin{aligned} ds^2 &= -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_{n-1}^2, \\ V(r) &= 1 - \frac{m}{r^{n-2}} + \frac{q^2}{r^{2n-4}} + \frac{r^2}{l^2}. \end{aligned} \tag{5.17}$$

And the thermodynamics quantities are:

$$\begin{aligned} M &= \frac{(n-1)w_{n-1}}{16\pi G}m, \\ Q &= \frac{\sqrt{2(n-1)(n-2)}w_{n-1}}{8\pi G}q, \\ T &= \frac{V'(r_+)}{4\pi} = \frac{nr_+^{2n-2} + (n-2)l^2 r_+^{2n-4} - (n-2)q^2 l^2}{4\pi l^2 r_+^{2n-3}}, \\ S &= \frac{w_{n-1} r_+^{n-1}}{4G}. \end{aligned} \tag{5.18}$$

Again, find the M_{gap} in eq. (5.16).

hint: First in this problem we could keep r_0 fixed while change r_+: $r_+ = r_0 + \delta$. Secondly, we need to relate $m(r_+)$ with r_+ by $V(r_+) = 0$.

Problem 3: Thermodynamics of Near Extremal RN Black Holes

As a last check, we investigate the thermodynamics of near extremal RN black holes in n dimensional Minkowski spacetime. The metric is:

$$\begin{aligned} ds^2 &= -f(r) + f^{-1}(r)dr^2 + r^2 d\Omega_{n-1}^2, \\ f(r) &= 1 - \frac{m}{r^{n-2}} + \frac{q^2}{r^{2n-4}}. \end{aligned} \tag{5.19}$$

Again find the M_{gap}.
hint: The physical mass is: $M = \frac{(n-1)w_{n-1}}{16\pi G} m$.

By solving the problems above, we confirmed the universal feature of the thermodynamics for near extremal black holes eq. (5.16). Remember we also have the universal geometry $AAdS_2$, we want to know whether we could reproduce the thermodynamics by $AAdS_2$ spacetime.

5.3 Failure of black hole thermodynamics in $AAdS_2$

The biggest problem is that we don't have a non-zero total mass in $AAdS_2$ spacetime.

Problem 1: Boundary Stress Tensor for AdS_2 Spacetime
Compute the boundary stress tensor for AdS_2 spacetime:

$$ds^2 = \left(\frac{l^2}{r^2} + \delta g_{rr}\right) dr^2 + \left(-\frac{r^2}{l^2} + \delta g_{tt}\right) dt^2, \tag{5.20}$$

and show that it is 0:

$$T_{\mu\nu} = 0. \tag{5.21}$$

There is also an explanation from the CFT side. In 1d CFT, we only have the T_{tt} component, which is the Hamiltonian. And since 1d theory has no curvature, the trace anomaly is always 0, thus we have $H = 0$. It seems it is not possible to reproduce the black hole thermodynamics in the near horizon region. However, a possible way to do it [19] [20] is via JT gravity (to be discussed briefly in the next section), which is beyond the scope of this user guide.

6 Dimensional Reduction to 2D Gravity

JT gravity arises from dimensional reduction of 4D Einstein-Hilbert action. Although we will not study this theory in detail, it is still very important to know about dimensional reduction.

6.1 dimensional reduction on S^2

We assume to have a S^2 factor in the metric and that all the fields are independent of it. Then a question arises: is it possible to reproduce all the dynamics via the 2d spacetime? The answer is yes, and in this subsection we will explore this problem. We use the Einstein-Maxwell theory as example. Let's start with an Euclidean Einstein-Maxwell theory in Euclidean signature: (We set $G_N = 1$ here)

$$S = -\frac{1}{16\pi}\int d^4x\sqrt{\hat{g}}\hat{R} - \frac{1}{8\pi}\int d^3x\sqrt{\hat{\gamma}}K^{(3)} + \frac{1}{4}\int d^4x\sqrt{\hat{g}}F^2. \tag{6.1}$$

After dimensional reduction, we will have a 2d effective action whose EOMs will agree with EOMs from original 4d action. But how could we get this 2d effective action? We could try to solve the original EOMs and substitute some of the fields with their on-shell value over the S^2. Moreover, since we need to make comparison between the EOMs, we need to evaluate the 4d EOMs. We make such ansatz for metric: (for later convenience, define $m = \Phi^2$)

$$ds^2 = g_{\alpha\beta}(t,r)dx^\alpha dx^\beta + \Phi^2(t,r)d\Omega_2^2. \tag{6.2}$$

We name the original metric $\hat{g}$, with indicies $a, b, c, ...$, and $g; \mu, \nu, ...$ the 2d base space metric, and $h; i, j, ...$ the sphere.

Problem 1: Solution for EM field
Derive the EOM for the gauge field, show that

$$\begin{aligned} F_{ab} &= \frac{-iQ}{|m|}\epsilon_{ab}, \\ F_{ij} &= P\epsilon_{ij} = P\sin\theta d\theta\wedge d\phi \end{aligned} \tag{6.3}$$

(where Q is real) is a solution. ϵ_{ab} and ϵ_{ij} are the volume forms for the corresponding 2d manifold. The first one is an electrically charged solution while the second is a magnetically charged one. Think about why "i" is necessary for the electrically charged solution. Then compute the stress tensor for electric and magnetic fields separately:

$$\hat{T}^P_{\mu\nu} = -\frac{g_{\mu\nu}}{2m^2}P^2, \quad \hat{T}^P_{ij} = \frac{h_{ij}}{2m}P^2, \tag{6.4}$$

$$\hat{T}^Q_{\mu\nu} = -\frac{g_{\mu\nu}}{2m^2}Q^2, \quad \hat{T}^Q_{ij} = \frac{h_{ij}}{2m}Q^2. \tag{6.5}$$

Then show that they are both traceless.

Then we need to compute the 4D Einstein equation. Because the stress tensor is traceless, the spacetime is Ricci-flat. The Einstein equation simplifies to be $\hat{R}_{ab} = 8\pi T_{ab}$. To make comparison with the 2D version, we need to split 4D geometric quantities into 2D ones.

Problem 2: 4D Geometric Quantities into 2D Ones

Verify the relationship:

$$\begin{aligned}
\hat{g} &= m^2 \sin^2\theta g, \\
\hat{R}_{\mu\nu} &= R_{\mu\nu} - \frac{\nabla_\mu \nabla_\nu m}{m} + \frac{\nabla_\mu m \nabla_\nu m}{2m^2}, \\
\hat{R}_{\mu i} &= 0, \\
\hat{R}_{ij} &= R_{ij} - \frac{h_{ij}\nabla^2 m}{2}, \\
\hat{R} &= R + \frac{2}{m} - 2\frac{\nabla^2 m}{m} + \frac{(\nabla m)^2}{2m^2} = R + \frac{2}{\Phi^2} - 2\frac{(\nabla\Phi)^2}{\Phi^2} - 4\frac{\nabla^2\Phi}{\Phi}.
\end{aligned} \tag{6.6}$$

hint: You can verify them via Mathematica. To compute them by yourself, non-coordinate basis (einbeins) method will be easier than normal method (coordinate basis and Christoffel symbols).

Problem 3: 4D EOM into 2D Components

Split Einstein equation $R_{ab} = 8\pi T_{ab}$ into trace part:

$$\begin{aligned}
R - \frac{\nabla^2 m}{m} + \frac{(\nabla m)^2}{2m^2} &= -\frac{8\pi G P^2}{m^2}, \\
2 - \nabla^2 m &= \frac{8\pi G P^2}{m^2},
\end{aligned} \tag{6.7}$$

and traceless part:

$$\nabla_\alpha \nabla_\beta m - \frac{\nabla_\alpha m \nabla_\beta m}{2m} - \frac{1}{2}g_{\alpha\beta}\left(\nabla^2 m - \frac{(\nabla m)^2}{2m}\right) = 0. \tag{6.8}$$

Then we need to get the 2D action and then 2D EOMs.

Problem 3: 2D Effective Action

Using the previous relationship, substitute the solution of gauge fields back to the 4D action, then integrate over the angular coordinates, and show that the 2D action is:

$$
\begin{aligned}
S = &-\frac{1}{4}\int d^2x\sqrt{g}\left[2+\Phi^2(R-2\hat{\Lambda})+2(\nabla\Phi)^2\right]+2\pi P^2\int d^2x\sqrt{g}\frac{1}{\Phi^2} \\
&-\frac{1}{2}\int\sqrt{\gamma}\Phi^2 K.
\end{aligned}
\tag{6.9}
$$

Then derive EOMs and show that they agree with the 4D EOMs for the pure magnetically charged case. For the pure electrically charged case, verify that we need to add

$$
2\pi\int d^2x\sqrt{g}mF_{ab}\epsilon^{ab}iQ
$$

to the 2D EOMs to match the 4D EOMs. Think about why.

hint: Is it correct to substitute the solution of the fields back to the action?

6.2 Basic Feature of 2D Gravity

We have seen that after dimensional reduction we have a 2D gravity theory eq. (6.9). In this section we will see several basic aspects of 2D gravity theory. First note that eq. (6.9) is not an Einstein-Hilbert action in 2D. Then you might be interested in what 2D Einstein gravity looks like.

Problem 1: 2D Ricci tensor

Show that in 2D the relationship

$$
R_{ab} = \frac{R\,g_{ab}}{2} \tag{6.10}
$$

holds.

hint:$R_{abcd} = \frac{R}{2}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$ holds for constant curvature spacetime.

From eq. (6.10) we know that Einstein tensor G_{ab} is identically 0 in 2D. Therefore, 2D spacetime can not support matter with non-zero stress tensor. And sometimes we refer AdS spacetime to be a solution to vacuum Einstein with negative cosmological constant. But in 2D this is no longer valid. We have to call AdS_2 spacetime to be the manifold with negative constant curvature.

Problem 2: Conformally flat metric

2D metric is locally conformally flat, which means that by a coordinate transformation any metric could be put into the form:

$$
ds^2 = f(t,x)(-dt^2+dx^2). \tag{6.11}
$$

Intuitively this is correct, because there are 3 independent components for the 2D metric, and there are 2 independent coordinate transformations. So in the end we have only 1 degree of freedom. Now find such coordinate transformation explicitly.

Therefore in some sense 2D Einstein gravity is boring. A more interesting one is JT gravity, which is beyond the scope of this user guide.

Part II

Introduction to Mathematica GR packages

Usually the computation in real research is involved, especially for gravity theory. The tensor structure makes calculation in general relativity very tedious sometimes. Luckily we could use software to help with the calculations. The most popular one for formal calculation is Mathematica and here we will introduce two packages for gravity calculation.

The reader should have basic knowledge about Mathematica in order to keep working on this user guide. Also, the following content is only for introductory purpose targeted at master student and one should go for other resources for more advance tutorial.

7 Diffgeo

General relativity is a diffeomorphism invariant theory, which means one can use any reasonable coordinate to do computations. However, there are usually coordinates that adapt well with geometry and most computations in research are done with given coordinate. In this case one only needs to input the data for tensors and then evaluate equations. The package diffgeo works well. You should read the manual document diffgeoManual.nb for basic knowledge about this package. Then I will show you some examples of this package. For technical reason I will append the PDF printed from MMA notebook to the end of this user guide.

8 xAct

Having introduced a package, there is another powerful package xAct that makes the diffeomorphism invariant property manifest. It is good at dealing with calculations in abstract index notation, without reference to a coordinate system. Personally I think xAct is very good at deriving equations of motion from a given Lagrangian and doing perturbation over the equations. The notebooks from the official website are enough for the pedagogical purpose and examples are not provided in this guide.

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