Exploring Graph Coloring and Chromatic Polynomial: Analyzing Graphs for Maximum Chromatic Vectors

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Graph Coloring

Graph coloring is a process of assigning colors to the vertices of a graph $G = (V, E)$ such that no two adjacent vertices share the same color.

Definition

A coloring of a graph G is a function $c : V \to C$, where V is the set of vertices in G, C is the set of colors, and $c(v)$ is the color assigned to vertex v.
A valid coloring of a graph G must satisfy the following condition:

- For every edge $(u, v) \in E$, the colors of vertices u and v must be distinct, i.e., $c(u) \neq c(v)$.

A graph that can be assigned a λ-coloring is considered λ-colorable.

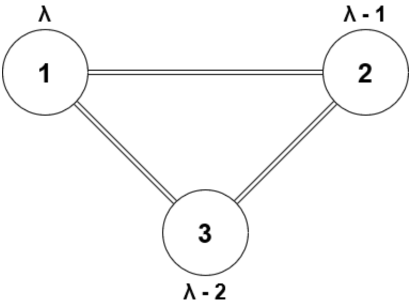
Chromatic Polynomial

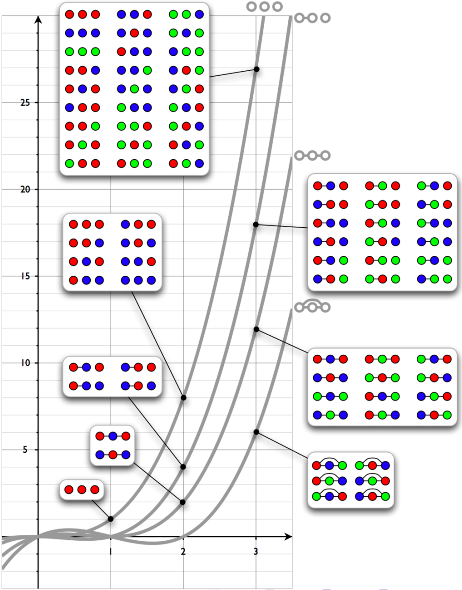
The chromatic polynomial counts the number of ways a graph can be colored using some of a given number of colors.

Definition

For a graph G, $P(G, \lambda)$ counts the number of its proper λ-colorings. There is a unique polynomial $P(G, x)$ which evaluated at any integer $k \geq 0$ coincides with $P(G, \lambda)$; it is called the chromatic polynomial of G.

Chromatic Polynomial - Illustrations

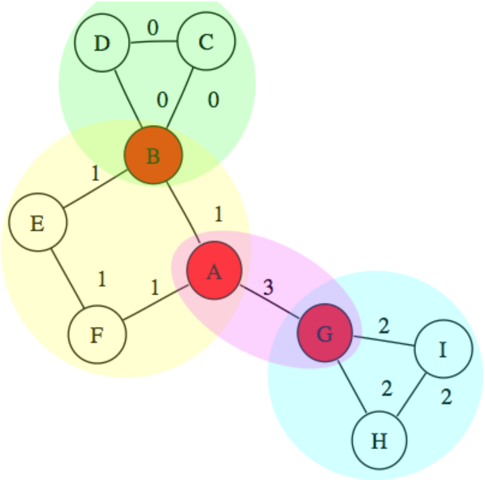




Biconnected Component

Definition

A biconnected component in a graph is a maximal subgraph that remains connected after the removal of any single vertex, except for articulation points.



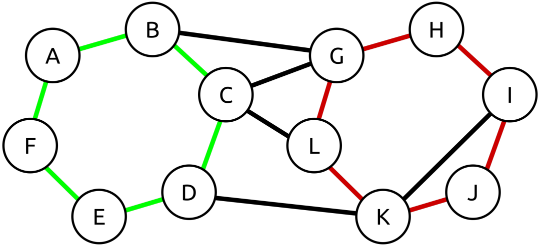
Cycle & Induced Cycle

Cycle Definition

A cycle of a graph G is a subset of the edge set of G that forms a path such that the first node of the path corresponds to the last.

Induced Cycle Definition

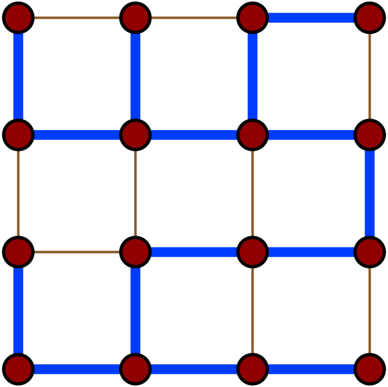
An induced cycle of a graph G is a cycle such that each two adjacent vertices in the sequence are connected by an edge in G, and each two nonadjacent vertices in the sequence are not connected by any edge in G.



Spanning Tree

Spanning Tree Definition

A spanning tree T of an undirected graph G is a subgraph that is a tree which includes all of the vertices of G.



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Motivation

- Dr. Scofield had once used the Patriot Cluster Supercomputer to calculate the chromatic vector of all connected graphs up to order 9.
- The computation took several days to finish, and it yielded result for the chromatic vector of 273191 connected graphs.
- There exists proof that identifies graph that would yield the smallest chromatic vector
- However, there are none for graph that would yield the largest chromatic vector.
- The big question - Given a graph's chromatic vector and various graph properties, could we find any relationship between them?

Interpreting Coefficients

Let $|E| = m$ and let $C_n, C_{n-1}, C_{n-2}, ..., C_2, C_1$ denote the coefficients of the terms of the chromatic polynomial
$P(G,x) = C_n x^n + C_{n-1}x^{n-1} + C_{n-1}x^{n-2} + ... + C_2x^2 + C_1x^1$

- $C_n = 1$
- $C_{n-1} = -m$
- $C_{n-2} = \binom{m}{2} -$ (Number of 3-cycle)
- $C_{n-3} = \binom{m}{3} - (m-2) \cdot$ (Number of 3-cycle) $-$ (Number of induced 4-cycle) $+ 2 \cdot$ (Number of 4-cycle)
- $C_{n-4} =$ It's complicated...

The coefficients of lower order terms are increasingly complex. Is there an easier way to analyze the relationship between a graph and its chromatic vector?

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Technologies & Tools

I used Python in Jupyter Notebook, along with JetBrains DataSpell as IDE

Packages Used

- NumPy: for data manipulation and matrix operations.
- Pandas: for data manipulation and analysis.
- NetworkX: for graph theory and graph operations.
- SciPy: for statistical computations.
- MatPlotLib: for graph visualization.

Calculated Graph Properties

Most of my work was focusing on identifying which graph properties that could be easily calculated and add to the dataset using existing libraries. The following graph properties were calculated:

- Number of spanning tree
- Number of biconnected component
- Number of 3-cycle
- Number of 4-cycle
- Number of 5-cycle
- Number of 6-cycle
- Degree sequence
- Degree variance
- Degree deviation

Statistical Analysis

- Out of 273191 connected graphs, I identified 172 graphs that yield the maximum chromatic vector within their group.
- Excluding all groups of graphs that has less than 30 elements (central limit theorem) and all groups that is all tree graphs.
- Narrowed down to 57 graphs that maximizes chromatic vector.
- Computed the percentile rank of each applicable graph properties for all 57 graphs that maximizes chromatic vector and compare it against other non-maximum graphs in the same group.

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Results

After performing statistical analysis on the computational result, we discovered graphs that yield large chromatic vectors have these properties:

- Biconnected, meaning the number of biconnected component is 1
- Number of spanning tree is within the top 10%
- Number of 3-cycle is within the bottom 5%

Other graph properties did not show a strong enough statistical significance. This further emphasizes how complicated chromatic coefficients can get.

Future Investigations

Include even more graph properties pertaining to connectivity and robustness of a graph and re-analyze for statistical significance.

- Induced cycles
- Vertex Connectivity
- Edge Connectivity
- And a lot more...

Start working toward a mathematical proof that concretely identifies what makes a graph yields large chromatic vector.