Basic Concepts of Linear Algebra

Linear Systems

> *Def.* **Equivalent linear systems**
>
> Two systems of linear equations on u unknowns are **equivalent** provided that they have the same set of solutions

> *Def.* **Raw Operations**
>
> The following operations, performed on the rows of a matrix, are called **elementary row operations**:
>
> 1. Interchange two rows.
> 2. Multiply a row by a **nonzero** scalar.
> 3. Add a constant multiple of one row to another.

> *Def.* **Raw Equivalent**
>
> We say two matrices with same shape, B and C, are **raw equivalent** if one can be obtained from the other by a sequence of elementary row operations.
>
> *Corollary.* Suppose $[A|\vec{b}]$ and $[C|\vec{d}]$ are augmented matrices with same shape. If these two augmented matrices are row equivalent matrices, then the two systems are also equivalent.

> *Def.* **Echelon Form**
>
> An $m \times n$ matrix B is in **Echelon Form** if:
>
> 1. All rows whose elements are all zero are at the bottom of the matrix.
> 2. In every nonzero row, the first nonzero entry is 1.
> 3. If the $(i+1)$-st row contains nonzero entries, then the 1-st non-zero entry is in a column to the right of the first nonzero entry in the i-th row.

> *Def.* **Reduced Echelon Form**
>
> A matrix that is in echelon form is also in **reduced echelon form** provided that the first nonzero entry in any row is the **only** nonzero entry in its **column**.
>
> *Note*: The reduced echelon form of a linear system is unique.

Def. **Inconsistent System**

If a linear system whose augmented matrix $[A|\vec{b}]$ is in reduced echelon form and the last nonzero row of it has its leading 1 in the last column, then the linear system is inconsistent and has no solution.

Th. Let $A = [C|\vec{d}]$ be an matrix whose shape is [m, n+1] in reduced echelon form, where $[C|\vec{d}]$ represents a consistent system. Let A have $r \leq m$ nonzero rows. Then $r \leq n$ and in the solution of the system there are $n - r$ variables that can be assigned arbitrary values.

Corollary. Consider an $(m \times n)$ system of linear equations. If $m < n$, then either the system is inconsistent or it has infinitely many solutions.

Def. **Homogeneous System**

A $(m \times n)$ system of linear equation given below in called a **homogeneous system** of linear equations:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= 0 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= 0 \\ \cdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= 0 \end{aligned} \tag{13}$$

Remark. Homogeneous system always has a solution $x_i = 0 \ \forall i$ and this solution is called **trivial solution** or zeros solution, and other solution is called a **nontrivial solution**.

Th. A homogeneous system of linear equations with shape $m \times n$ always has infinitely many **nontrivial solutions** when $m < n$. [From the corollary above]

Application of Homogeneous System: Conic Sections and Quadric Surfaces

An interesting application of homogeneous equations involves the quadratic equation in two variables.

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{1}$$

If the equation above has real solutions, then the graph is a curve in the xy-plane. If at least one of a, b and c is nonzero, the resulting graph is known as a **conic section**.

Matrix

Def. **Matrix Product**

Scalar Definition

Let $A = (a_{i,j})$ be an $(m \times n)$ matrix, and let $B = (b_{i,j})$ be an $(r \times s)$ matrix. If $n = r$, then the product AB is the matrix $(m \times s)$ matrix defined by:

$$(AB)_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j} \tag{2}$$

Vector Definition

Lemma. Let $A = [\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n]$, where for each $j,\ 1 \le j \le n$, $\mathbf{A}_j$ denotes the j-th column vector of A with shape $(m \times 1)$. And let vector $\mathbf{x}$ with shape $(n \times 1)$, we have:

$$A\mathrm{x} = x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_n\mathbf{A}_n \tag{3}$$

Let $A = [\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n]$ with shape $(m \times n)$ and $B = [\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_s]$ with shape $(n \times s)$, we have

$$AB = A[\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_s] = [A\mathbf{B}_1, A\mathbf{B}_2, \ldots, A\mathbf{B}_s] \tag{4}$$

Dot Product Definition

Let $A = [\alpha_1^T; \alpha_2^T; \cdots; \alpha_m^T]$ be an $(m \times n)$ matrix with row vector α_i^T . And $B = [\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_s]$ with shape $(n \times s)$, the product AB is matrix:

$$(AB)_{i,j} = \alpha_i^T \mathbf{B}_j \tag{5}$$

Matrix Definition

Let $A = [\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n]$ and $B = [\beta_1^T; \beta_2^T; \cdots; \beta_n^T]$, we have:

$$\begin{aligned} AB &= \sum_{k=1}^{n} \mathbf{A}_k \beta_k^T \\ (\mathbf{A}_k \beta_k^T)_{i,j} &= A_{i,k} B_{k,j} \end{aligned} \tag{6}$$

Prop. **Properties of Matrix Operations**

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $\exists$ unique matrix O with shape $(m \times n)$ that $A + O = A$ for any matrix A with shape $(m \times n)$.
4. Given an $(m \times n)$ matrix A, there exists a unique matrix P with shape $(m \times n)$ s.t. $A + P = O$
5. Given matrices A, B, C with shape $(m \times n), (n \times p), (p \times q)$, we have $(AB)C = A(BC)$
6. r, s are scalars, we have $r(sA) = (rs)A$
7. $r(AB) = (rA)B = A(rB)$
8. With valid shape, we have $(A + B)C = AC + BC$
9. With valid shape, we have $A(B + C) = AB + AC$
10. $(r + s)A = rA + sA$
11. $r(A + B) = rA + rB$

Def. **Transpose of a Matrix**

If $A = (a_{i,j})$ is an $(m \times n)$ matrix, then the **transpose** of A, denoted as A^T, is the $(n \times m)$ matrix $A^T = (b_{i,j})$ and $b_{j,i} = a_{i,j}$.

Prop.

1. $(A + B)^T = A^T + B^T$
2. $(AB)^T = B^T A^T$
3. $(A^T)^T = A$

Def. **Symmetric Matrix**

A matrix A is symmetric if $A = A^T$

Prop. Q^TQ is **always** a symmetric matrix.

Def. **Skew Symmetric Matrix**

A **square matrix**, $A = (a_{i,j})$, is called **skew symmetric** if $A^T = -A$.

Linear Independence and Non-singular Matrix

Def. **Linear Combination**

If $A = [\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n]$, we call $x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_n\mathbf{A}_n$ a linear combination of the vectors $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n$.

Th. System of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ if a linear combination of the columns of A.

Note. Here $\mathbf{x}$ is variable and unknown.

Remark. Homogeneous system $A\mathbf{x} = \theta$ always has a trivial solution $\mathbf{x} = \theta$.

Def. **Linearly Independence**

A set of m-dimensional vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ is said to be **linearly independent** if the **only** solution to the vector equation

$$\begin{gathered} a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p = \theta \\ \text{or} \\ V\mathbf{a} = \theta \text{ where } V = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p] \end{gathered} \tag{7}$$

is the trivial solution $a_1 = a_2 = \cdots = a_p = 0$.

The set of vectors is said to be **linearly dependent** if it is not linearly independent. That is, the set is linearly dependent if we can find a non-trivial solution to the equation above.

Remark. If a set of vectors has zero vector, then this set of vector must be linearly dependent.

Th. Let $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ be a set of vectors in $\mathcal{R}^m$. If $p > m$, then this set is linearly dependent. Else if $p \leq m$, this set of vectors may be either linearly independent or linearly dependent.

Def. **Non-singular Matrix**

An $(n \times n)$ matrix A is **nonsingular** if the only solution to $A\mathbf{x} = \theta$ is its trivial solution. Otherwise A is singular matrix.

> *Corollary.* The $(n \times n)$ matrix $A = [\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n]$ is nonsingular matrix if and only if the set of vector $\{\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n\}$ is linearly independent.

> *Th.* Let A be an $(n \times n)$ matrix. The equation $A\mathbf{x} = \mathbf{b}$ has **unique** solution for any $(n \times 1)$ column vector $\mathbf{b}$ if and only if A is nonsingular.

Matrix Inverse and Properties

> *Def.* **Matrix Inverse**
>
> Let A be an $(n \times n)$ matrix. We say that A is **invertible** if we can find an $(n \times n)$ matrix A^{-1} s.t.:
>
> $$A^{-1}A = AA^{-1} = I. \tag{8}$$
>
> The matrix A^{-1} is called an inverse for A.
>
> *Note.* If A is invertible, then A^{-1} is unique.

The Existence of Inverses

> *Lemma.* Let P, Q and R be $(n \times n)$ square matrix s.t. $PQ = R$. We have
>
> $$(P \text{ is singular}) \vee (Q \text{ is singular}) \to (R \text{ is singular}) \tag{9}$$

> *Th.* Let A be an $(n \times n)$ matrix. Then A is invertible if and only if A is nonsingular.

> *Th.* Let A be an $(n \times n)$ matrix. Then A is nonsingular if and only if A is row equivalent to I.

> *Prop.* **Properties of Matrix Inverse**
>
> Let A and B be $(n \times n)$ matrices, each of which has an inverse. Then:
>
> 1. A^{-1} has an inverse and $(A^{-1})^{-1} = A$.
> 2. AB has an inverse, and $(AB)^{-1} = B^{-1}A^{-1}$.
> 3. If $k \neq 0$, then kA has an inverse and $(kA)^{-1} = (1/k)A^{-1}$.
> 4. A^T has an inverse and $(A^T)^{-1} = (A^{-1})^T$.

> *Th.* Let A be an $(n \times n)$ matrix. The following are equivalent:
>
> 1. A is nonsingular.
> 2. The column vectors of A are linearly independent.
> 3. $A\mathbf{x} = \mathbf{b}$ always has a unique solution.
> 4. A is invertible.
> 5. A is row equivalent to I.

Ill-Conditioned Matrix

In applications the equation $A\mathbf{x} = \mathbf{b}$ often serves as a mathematical model for a physical problem. In these cases it is important to know whether solutions to this system are sensitive to small changes in the right-hand side $\mathbf{b}$. If small changes in $\mathbf{b}$ can lead to relatively large changes in the solution $\mathbf{x}$, then the matrix A is called **ill-conditioned**.

Ill-conditioned matrix usually has large scale inverse.

Partitioned Matrix (Block Matrix)

A matrix A is a (2 x 2) **block matrix** if it is represented in the form:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \tag{10}$$

Note that the matrix A need not be a square matrix. The partition can have any shape.

Now suppose matrix B is also a (2 x 2) block matrix given by:

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \tag{11}$$

with all submatrix products valid, we have:

$$AB = \begin{bmatrix} A_1 B_1 + A_2 B_3 & A_1 B_2 + A_2 B_4 \\ A_3 B_1 + A_4 B_3 & A_3 B_2 + A_4 B_4 \end{bmatrix} \tag{12}$$