Eigenvalues and Applications

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- Quadratic Forms
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Quadratic Forms

Basic

> *Def.* **Quadratic Form**
>
> In General, a **quadratic form** in the variables $x_{1:n}$ is an expression of the form:
>
> $$q(\mathbf{x}) = q(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{i,j}x_ix_j \tag{5}$$
>
> **Matrix Representation for Quadratic Form**
>
> Define $a_{i,i} = b_{i,i}$ and $a_{i,j} = (b_{i,j} + b_{ji})/2,\ i \neq j$, we have
>
> $$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} \tag{1}$$
>
> where A is a symmetric matrix.

Homogeneous: All terms in this equation(or function, etc.) have the same exponential degree.

> **From Text Book**
>
> The term *form* means homogeneous polynomial; that is, $q(a\mathbf{x}) = a^k q(\mathbf{x})$.

> *Th.* For $\mathbf{x} \in R^n$, let $q(\mathbf{x})$ denote the quadratic form that:
>
> $$q(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{i,j}x_ix_j. \tag{2}$$
>
> Let $A = (a_{i,j})$ be the $(n \times n)$ matrix defined by
>
> $$a_{i,j} = (b_{i,j} + b_{j,i})/2 \tag{3}$$
>
> The matrix A is symmetric and $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. Moreover, there is no other **symmetric** matrix B s.t. $\mathbf{x}^T B\mathbf{x} = q(\mathbf{x})$.
>
> *Note.* But there exists many **nonsymmetric** matrix B s.t. $\mathbf{x}^T B\mathbf{x} = q(\mathbf{x})$.

Diagonalizing Quadratic Forms

For many applications, it is useful top have the even simpler representation described in this subsection. Because a real symmetric matrix can be diagonalized with an orthogonal matrix. That is, there is a square matrix Q s.t.

1. $Q^T Q = I$.
2. $Q^T A Q = D$, where D is diagonal.
3. The diagonal entries of D are the eigenvalues of A.

If we make the substitution $\mathbf{x} = Q\mathbf{y}$, we have:

$$\begin{aligned} q(\mathbf{x}) &= q(Q\mathbf{y}) \\ &= (Q\mathbf{y})^T A (Q\mathbf{y}) \\ &= \mathbf{y}^T Q^T A Q \mathbf{y} \\ &= \mathbf{y}^T D \mathbf{y} \\ &= \sum_{i=1}^{n} \lambda_i y_i^2 \end{aligned}$$

Classifying Quadratic Forms

We can think of a quadratic form as a function from R^n to R. Specifically, if $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ where A is a real symmetric matrix, then we can define a real-valued function with domain R^n where

$$y = q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} \tag{4}$$

The quadratic form is classified as:

1. **Positive Definite**: if $q(\mathbf{x}) > 0$ for all $\mathbf{x} \in R^n$, $\mathbf{x} \neq \theta$.
2. **Positive Semidefinite**: if $q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in R^n$, $\mathbf{x} \neq \theta$.
3. **Negative Definite**: if $q(\mathbf{x}) < 0$ for all $\mathbf{x} \in R^n$, $\mathbf{x} \neq \theta$.
4. **Negative Semidefinite**: if $q(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in R^n$, $\mathbf{x} \neq \theta$.
5. **Indefinite**: if $q(\mathbf{x})$ assumes both positive and negative values.

> *Th.* Let $q(\mathbf{x})$ be a quadratic form with representation $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$, where A is a symmetric $(n \times n)$ matrix. Let the eigenvalues of A be $\lambda_1, \lambda_2, \ldots, \lambda_n$. The quadratic form is:
>
> 1. Positive definite $\Leftrightarrow \forall i,\ \lambda_i > 0$.
> 2. Positive semidefinite \Leftrightarrow \forall\ i,\ \lambda_{i} \geq 0.
> 3. Negative definite \Leftrightarrow \forall\ i,\ \lambda_{i}<0.
> 4. Negative semidefinite $\Leftrightarrow \forall\ i, \lambda_i \leq 0$.
> 5. Indefinite $\Leftrightarrow A$ has both positive and negative eigenvalues.