The Vector Space and $\mathcal{R}^n$

Vector Space

> *Def.* **Vector Space**
>
> A set of elements V is said to be a **vector space** over a scalar field S if:
>
> 1. An addition operation is defined between any two elements in V.
> 2. A scalar multiplication operation is defined between any elements of S and any vector in V.
> 3. For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and any $a, b \in S$, these 10 properties must hold.
>   1. Closure Properties:
>   1. $(\mathbf{u}+\mathbf{v}) \in V$.
>   2. $a\mathbf{v} \in V$.
>   2. Properties of addition:
>   1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
>   2. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$.
>   3. $\exists\, \theta \in V,\ s.t.\ \forall\, \mathbf{v} \in V,\ \mathbf{v}+\theta=\mathbf{v}$.
>   4. $\forall\, \mathbf{v} \in V, \exists\, -\mathbf{v} \in V, s.t.\ \mathbf{v}+(-\mathbf{v})=\theta$.
>   3. Properties of scalar multiplication:
>   1. $a(b\mathbf{v})=(ab)\mathbf{v}$.
>   2. $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$.
>   3. $(a+b)\mathbf{v}=a\mathbf{v}+b\mathbf{v}$.
>   4. $\forall \mathbf{v} \in V,\ 1\mathbf{v}=\mathbf{v}$.
>
> *Note.* When the set of scalars S is the set of real numbers, V is called a **real vector space**.

> *Th.* **Cancellation Laws for Vector Addition**
>
> Let V be a vector space, and let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in V:
>
> 1. If $\mathbf{u}+\mathbf{v}=\mathbf{u}+\mathbf{w}$, then $\mathbf{v}=\mathbf{w}$.
> 2. If \$\mathbf{v} + \mathbf{u} = \mathbf{w} + \mathbf{u}\$, then $\mathbf{v}=\mathbf{w}$.

> *Th.* If V is a vector space, then:

> 1. The zero vector, θ, is unique.
> 2. For each $\mathbf{v}$, the additive inverse $-\mathbf{v}$ is unique.
> 3. $\forall\ \mathbf{v} \in V,\ 0\mathbf{v} = \theta$, where 0 is the zero scalar.
> 4. $a\theta = \theta$ for every scalar a.
> 5. If $a\mathbf{v} = \theta$, then $a = 0$ or $\mathbf{v} = \theta$.
> 6. $(-1)\mathbf{v} = -\mathbf{v}$.

Subspace

> *Def.* **Subspace**
>
> Let W be a subset of a vector space V. Then W is a **subspace** of V if and only if:
>
> 1. The zero vector θ, of V is in W. (or W is not empty).
> 2. $\mathbf{u} + \mathbf{v} \in W$ if $\mathbf{u} \in W$ and $\mathbf{v} \in W$.
> 3. $\forall\ a \in S \text{ and } \forall\ \mathbf{u} \in W \text{ we have } a\mathbf{u} \in W$.

> *Def.* **Span**
>
> If $Q = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$ is a set of vectors in a vector space V, then the **span** of Q, denoted $Sp(Q)$, is the set of all linear combinations of vectors in Q, or:
>
> $$Sp(Q) = \{\mathbf{v} | \mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m\} \tag{1}$$

> *Def.* **Spanning Set**
>
> Let V be a vector space, and let $Q = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$ be a set of vectors in V. If every vector $\mathbf{v} \in V$ is a linear is a linear combination of vectors in Q, i.e.:
>
> $$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_m\mathbf{v}_m, \tag{2}$$
>
> then we say that Q is a **spanning set** for V.

> *Th.* If V is a vector space and $Q = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is a set of vectors in V, then $Sp(Q)$ is a subspace of V. And $Sp(Q)$ is also called the **subspace spanned by Q**.

> *Def.* **Minimal Spanning Sets**
>
> A linearly independent spanning set is a minimal spanning set.

> *Def.* **Null Space of a Matrix**
>
> Let A be an $(m \times n)$ matrix. The **null space** of A [denoted $\mathcal{N}(A)$] is the set of vectors in R^n defined by:
>
> $$\mathcal{N}(A) = \{\mathbf{x} | A\mathbf{x} = \theta,\ \mathbf{x} \in R^n\} \tag{3}$$
>
> *Th.* If A is an $(m \times n)$ matrix, then $\mathcal{N}(A)$ is a subspace of R^n.

> *Def.* **Range of a Matrix (or Column Space of a Matrix)**
>
> Let A be an $(m \times n)$ matrix. The **range** of A [denoted $\mathcal{R}(A)$] is the set of vectors in R^m defined by:
>
> $$\mathcal{R}(A) = \{\mathbf{y} | \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } R^n\} \tag{4}$$
>
> In a words, the range of A consists of the set of all vectors $\mathbf{y} \in R^m$ s.t. the linear system $A\mathbf{x} = \mathbf{y}$ is consistent.
>
> *Note.* $\mathcal{R}(A) = Sp(\{\mathbf{A}_1, \cdots, \mathbf{A}_n\})$.
>
> *Th.* If A is an $(m \times n)$ matrix and if $\mathcal{R}(A)$ is the range of A, then $\mathcal{R}(A)$ is a subspace of R^m.

> *Def.* **Row Space of a Matrix**
>
> Let A be an $(m \times n)$ matrix. The **row space** of A is defined to be $\mathcal{R}(A^T)$
>
> *Th.* Let A be an $(m \times n)$ matrix, and suppose that A is row equivalent to the $(m \times n)$ matrix B. Then A and B have the same row space.
>
> *Th.* If the nonzero matrix A is row equivalent to the matrix B in echelon form, then the nonzero rows of B form a basis for the row space of A.

Basis(Bases)

> *Def.* **Basis**
>
> Let V be a vector space, and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ be a spanning set for V. If B is linearly independent, then B is a **basis** for V.
>
> i.e.: Basis is minimal spanning set.
>
> *Th.* All bases of same vector space have same number of vectors.

> *Th.* Let W be a vector space, and let $B = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p\}$ be a spanning set for W containing p vectors. Then any set of $p+1$ or more vectors in W is linearly dependent.

> *Def.* **Dimension**
>
> Let V be a vector space.
>
> 1. If V has a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ of n vectors, then V has **dimension** n, and we write $dim(V) = n$. If $V = \{\theta\}$, then $dim(V) = 0$.
> 2. If V is nontrivial and does not have a basis containing a finite number of vectors, then V is an **infinite-dimensional** vector space.

> *Th.* The V be a vector space, and let $Q = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\}$ be a spanning set for V. Then there is a subset Q' of Q that is a basis of V.

> *Th.* Let V be a finite-dimensional vector space with $dim(V) = p$.

1. Any set of $p+1$ or more vectors in V is linearly dependent.
2. Any spanning set for V must contain at least p vectors.
3. Any set of p linearly independent vectors in V is a basis for V.
4. Any set of p vectors that spans V is a basis for V.

Def. **Coordinate**

Given a vector space V and its basis $B = \{ \mathbf{v}_{1}, \mathbf{v}_{2}, \cdots, \mathbf{v}_{p} \}$, for any vector $\mathbf{v} \in V$, we have

$$\mathbf{v} = w_1 \mathbf{v}_1 + w_2 \mathbf{v}_2 + \cdots + w_p \mathbf{v}_p \tag{5}$$

or

$$[\mathbf{v}]_B = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_p \end{bmatrix} \tag{6}$$

We will call the unique scalars $w_1, w_2, \cdots, w_p$ the **coordinates** of $\mathbf{v}$ with respect to the basis B, and $[\mathbf{v}]_B$ the **coordinate vector** of $\mathbf{v}$ with respect to B.

Lemma. Let V be a vector space with basis $B = \{ \mathbf{v}_{1}, \mathbf{v}_{2}, \cdots, \mathbf{v}_{p} \}$. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in V and if c if a scalar, then the following hold:

1. $[\mathbf{u} + \mathbf{v}]_B = [\mathbf{u}]_B + [\mathbf{v}]_B$.
2. $[c\mathbf{u}]_B = c[\mathbf{u}]_B$.

Th. Let V be a vector space with basis $B = \{ \mathbf{v}_{1}, \mathbf{v}_{2}, \cdots, \mathbf{v}_{p} \}$. Let $S = \{ \mathbf{u}_{1}, \mathbf{u}_{2}, \cdots, \mathbf{u}_{m} \}$ be a subset of V, let $T = \{[\mathbf{u}_1]_B, [\mathbf{u}_2]_B, \cdots, [\mathbf{u}_m]_B\}$.

1. A vector $\mathbf{u} \in V$ is in $Sp(S)$ if and only if $[\mathbf{u}]_B \in T$.
2. The set S is linearly independent in V if and only if the set T is linearly independent in R^p.

Corollary. S is a basis for V if and only if T is a basis for R^p.

Def. **Nullity of a Matrix**

For a $(m \times n)$ matrix A, the dimension of the null space is called the the **nullity** of A.

Def. **Rank of a Matrix**

For a $(m \times n)$ matrix A, the dimension of the range of A is called the **rank** of A. i.e. the rank of matrix A is the dimension of column space of A:

$$\mathrm{rank}(A) = \dim(Sp(\{\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n\})) = \dim(\mathcal{R}(A)) \tag{7}$$

Th. If A is an $(m \times n)$ matrix, we have $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.

> *Corollary.*: If A is an $(m \times n)$ matrix, then the row space and the column space of A have the same dimension.
>
> *Th.* If A is an $(m \times n)$ matrix, then $n = \mathrm{rank}(A) + \mathrm{nullity}(A)$.

> *Th.* An $(m \times n)$ system of linear equations, $A\mathbf{x} = \mathbf{b}$, is consistent if and only if $\mathrm{rank}(A) = \mathrm{rank}([A|\mathbf{b}])$.

> *Th.* An $(n \times n)$ matrix A is nonsingular if and only if $\mathrm{rank}(A) = n$.

Inner Product Space

> *Def.* **Inner Product**
>
> An **Inner Product** on a real vector space V is a function that assigns a real number, $\langle \mathbf{u}, \mathbf{v} \rangle$, to each pair of vectors $\mathbf{v}$ and $\mathbf{u}$ is V, and that s.t. these properties:
>
> 1. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0 \Leftrightarrow \mathbf{u} = \theta$.
> 2. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.
> 3. $\langle a\mathbf{u}, \mathbf{v} \rangle = a\langle \mathbf{u}, \mathbf{v} \rangle$.
> 4. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.

> *Th.* The operation $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A\mathbf{v}$ is an valid inner product for $\mathcal{R}^n$ if and only if A is symmetric positive-definite matrix.

> *Def.* **Inner Product Space**
>
> We call a **vector space** with an inner product is an **inner-product space**.

> *Def.* **Norm**
>
> If V is an inner-product space, then for each $\mathbf{v} \in V$ we define $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ the **norm** of $\mathbf{v}$.

> *Def.* **Orthogonal**
>
> If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner-product space V, we say these two vectors are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

> *Def.* **Orthogonal Set**
>
> Vector set $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ is an **orthogonal set** in inner-product space V if $\forall i \neq j,\ \mathbf{v}_i \in B,\ \mathbf{v}_j \in B,\ s.t.\ \langle \mathbf{v}_j, \mathbf{v}_i \rangle = 0$.

> *Def.* **Orthogonal Basis**

> If an orthogonal set of vectors B is a basis for inner-product space V, we call B an **orthogonal basis** for V.
>
> Furthermore, if $\forall \mathbf{u} \in B,\ \|\mathbf{u}\| = 1$, then B is said to be an **orthonormal basis** for V.

> *Th.* In $\mathcal{R}^n$, let $S = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\}$ be a set of vectors in $\mathcal{R}^n$ and $\forall\ i,\ \mathbf{u}_i \neq \theta$. If S is an orthogonal set of vectors, then S is a linearly independent set of vectors.
>
> *Proof.*
>
> Suppose $\sum_i c_i \mathbf{u}_i = \theta$, then for any j, we have $\sum_i c_i \mathbf{u}_j^T \mathbf{u}_i = c_j \mathbf{u}_j^T \mathbf{u}_j = \theta$. So $c_j = 0$ because $\mathbf{u_j} \neq \theta$.
>
> *Corollary.* Let W be a subspace of $\mathcal{R}^n$, where $\dim(W) = p$. If S is an orthogonal set of p **nonzero** vectors and is also a subset of W, then S is an orthogonal basis for W.

> *Th.* **Coordinate**
>
> Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be an orthogonal basis for an inner-product space V. If $\mathbf{u}$ is any vector in V, we have:
>
> $$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle}\mathbf{v}_n \tag{8}$$

> *Th.* **Gram-Schmidt Orthogonalization**
>
> [*Used for constructing an orthogonal basis*]
>
> Let V be an inner-product space, and let $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\}$ be any basis for V. Let $\mathbf{v}_1 = \mathbf{u}_1$, and other vectors defined by:
>
> $$\mathbf{v}_k = \mathbf{u}_k - \sum_{j=1}^{k-1} \frac{\langle \mathbf{u}_k, \mathbf{v}_j \rangle}{\langle \mathbf{v}_j, \mathbf{v}_j \rangle}\mathbf{v}_j \tag{9}$$
>
> Then $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ is an orthogonal basis for V.

Least-Squares Solutions to Inconsistent Systems

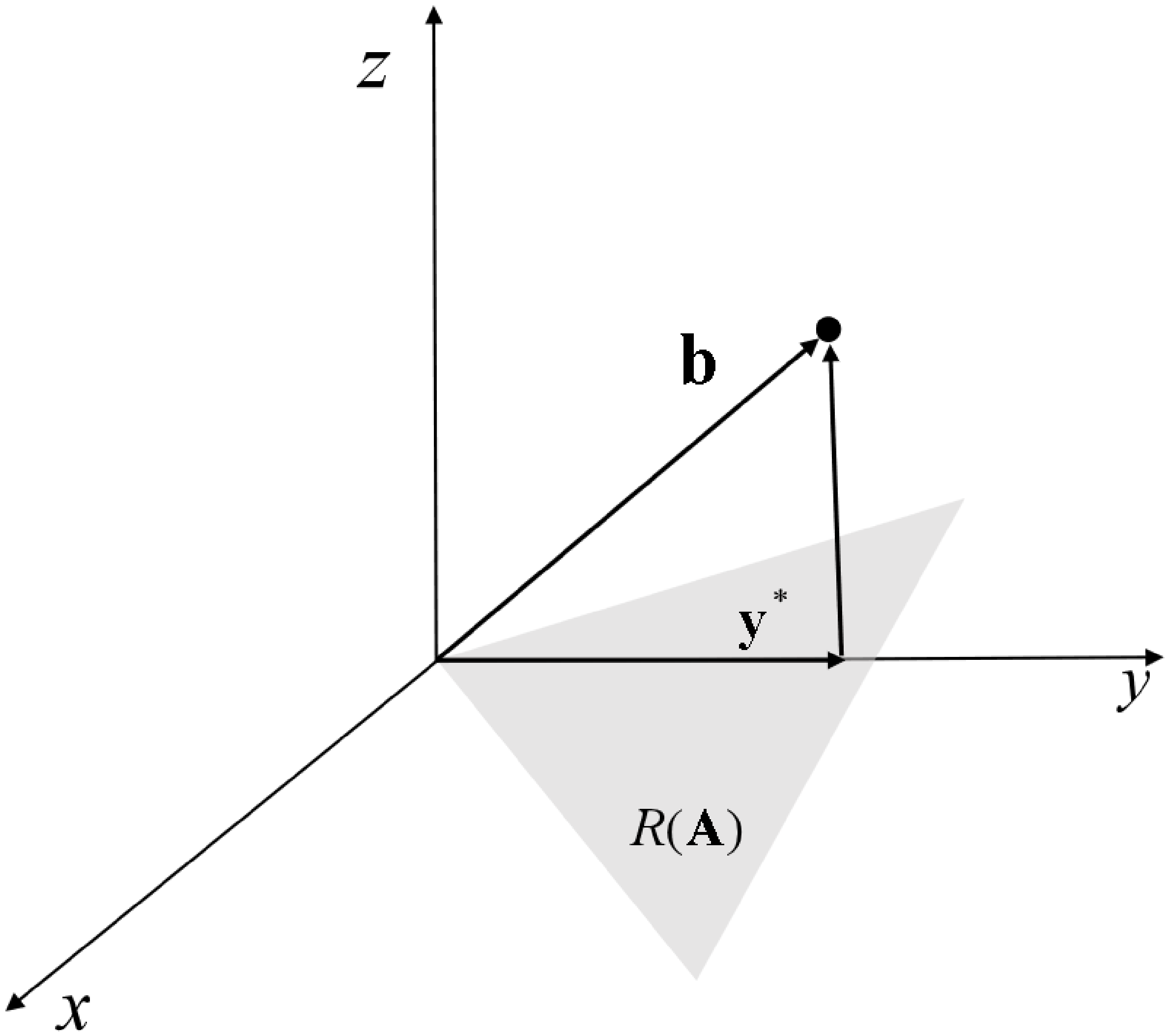
If $A\mathbf{x} = \mathbf{b}$ is inconsistent. We want to find a approximating solution.

> *Def.* **Overdetermined Systems**
>
> A linear system with more equations than unknowns is called **overdetermined systems**.

> *Def.* **Residual Vector and Least-square Solution**
>
> Consider the linear system $A\mathbf{x} = \mathbf{b}$ where A is $(m \times n)$. If x is a vector in R^n, then the vector $\mathbf{r} = A\mathbf{x} - \mathbf{b}$ is called a **residual vector**. A vector $\mathbf{x}^* \in R^n$ that yields the smallest possible residual vector is called a least-squares solution to $A\mathbf{x} = \mathbf{b}$. Or $\mathbf{x}^*$ is a **least-squares solution** to $A\mathbf{x} = \mathbf{b}$ if

$$\|A\mathbf{x}^* - \mathbf{b}\| \leq \|A\mathbf{x} - \mathbf{b}\|, \quad \text{for all } \mathbf{x} \in R^n \tag{10}$$

Consider a special case of an inconsistent (3×2) system $A\mathbf{x} = \mathbf{b}$ suggests how we can calculate least-squares solutions. In particular, consider figure below with illustrates a vector $\mathbf{b}$ that is not in $\mathcal{R}(A)$.



Let the vector $\mathbf{y}^*$ in $\mathcal{R}(A)$ be the closest vector in $\mathcal{R}(A)$ to $\mathbf{b}$, we have:

$$\|\mathbf{y}^* - \mathbf{b}\| \leq \|\mathbf{y} - \mathbf{b}\|, \quad \text{for all } \mathbf{y} \in \mathcal{R}(A) \tag{11}$$

Geometry suggests that the vector $\mathbf{y}^* - \mathbf{b}$ is orthogonal to any vector in $\mathcal{R}(A)$. I.e.: for any column vector $\mathbf{A}_i$ of A. we have:

$$\mathbf{A}_i^T(\mathbf{y}^* - \mathbf{b}) = 0 \tag{12}$$

Or in matrix-vector terms,

$$A^T(\mathbf{y}^* - \mathbf{b}) = \theta \tag{13}$$

Since $\mathbf{y}^* = A\mathbf{x}^*$, we have:

$$A^T A\mathbf{x}^* = A^T\mathbf{b} \tag{14}$$

> *Th.* Consider the $(m \times n)$ system $A\mathbf{x} = \mathbf{b}$:

> 1. The associated system $A^TA\mathbf{x} = A^T\mathbf{b}$, which is called the **normal equations**, is always consistent.
> 2. The least-squares solutions of $A\mathbf{x} = \mathbf{b}$ are precisely the solutions of $A^TA\mathbf{x} = A^T\mathbf{b}$.
> 3. The least-squares solution is unique if and only if A has rank n. If $rank(A) < n$, we say that A is **rank deficient** and may have infinitely many solutions.

Least-Squares Fits to Data

> *Def.* **Least-squares Criterion**
>
> 1. **Best Least-squares Linear Fit**: Find m and c to minimize $\sum_{i=0}^{n}[(mt_i + c) - y_i]^2$.
> 2. **Best Least-squares Quadratic Fit**: Find a, b, c to minimize $\sum_{i=0}^{n}[(at_i^2 + bt_i + c) - y_i]^2$.

Consider the following table of data:

$\mathbf{t}$	t_0	t_1	t_2	$\cdots$	t_m
$\mathbf{y}$	y_0	y_1	y_2	$\cdots$	y_m

Suppose we decide to fit these data with an n-th degree polynomial:

$$p(t) = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \tag{15}$$

We want to minimize

$$Q(a_0, a_1, \cdots, a_n) = \sum_{i=0}^{m}[p(t_i) - y_i]^2 \tag{16}$$

Let

$$A = \begin{bmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^n \\ 1 & t_1 & t_1^2 & \cdots & t_1^n \\ \vdots & & & & \vdots \\ 1 & t_m & t_m^2 & \cdots & t_m^n \end{bmatrix}, \mathbf{x} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{bmatrix}. \tag{17}$$

We have

$$Q(a_0, a_1, \cdots, a_n) = \|A\mathbf{x} - \mathbf{b}\|^2 \tag{18}$$

As before, we can minimize this by solving $A^TA\mathbf{x} = A^T\mathbf{b}$.

Theory of Least Squares

> *Def.* **The Least-Squares Problem in R^n**
>
> Let W be a p-dim subspace of R^n. Given a vector $\mathbf{v} \in R^n$, find a vector $\mathbf{w}^* \in W,\ s.t.$:
>
> $$\|\mathbf{v} - \mathbf{w}^*\| \leq \|\mathbf{v} - \mathbf{w}\|,\ \forall\ \mathbf{w} \in W. \tag{19}$$
>
> The vector $\mathbf{w}^*$ is called the **best least-squares approximation** of $\mathbf{v}$.

> *Th.* Let W be a p-dim subspace of R^n, and let $\mathbf{v}$ be a vector in R^n. Suppose there is a vector $\mathbf{w}^*$ such that $(\mathbf{v}-\mathbf{w}^*)^T\mathbf{w}=0$ for every vector $\mathbf{w}$ in W. Then $\mathbf{w}^*$ is the best least-squares approximation of $\mathbf{v}$.

> *Th.* Let W be a p-dim subspace of R^n, and let $\{ \mathbf{u}_{1}, \mathbf{u}_{2}, \cdots , \mathbf{u}_{p} \}$ be a basis for W. Let $\mathbf{v}$ be a vector in R^n. Then $(\mathbf{v}-\mathbf{w}^*)^T\mathbf{w}=0$ for all $\mathbf{w}\in W$ if and only if:
>
> $$(\mathbf{v}-\mathbf{w}^*)^T\mathbf{u}_i=0,\ 1\le i\le p. \tag{20}$$

> *Th.* Let W be a p-dim subspace of R^n, and let $\mathbf{v}$ be a vector in R^n. Then there is **one and only one** best least-squares approximation in W to $\mathbf{v}$. And if we have an orthogonal basis $\{ \mathbf{u}_{1}, \mathbf{u}_{2}, \cdots , \mathbf{u}_{p} \}$ for W, then:
>
> $$\mathbf{w}^*=\sum_{i=1}^{p}\frac{\mathbf{v}^T\mathbf{u}_i}{\mathbf{u}_i^T\mathbf{u}_i}\mathbf{u}_i \tag{21}$$

Orthogonal Projections

> *Def.* **Projections**
>
> Let V be an inner-product space and let W be a subspace of V. Given a vector $\mathbf{v}$ in V, if there exist a vector $\mathbf{w}^*\in W\ s.t.\ \|\mathbf{v}-\mathbf{w}^*\|\le\|\mathbf{v}-\mathbf{w}\|\ \forall\ \mathbf{w}\in W$. We say that $\mathbf{w}^*$ is the **projection** of $\mathbf{v}$ onto W, or (frequently) **the best least-squares approximation** to $\mathbf{v}$.

> *Th.* Let V be an inner-product space and let W be a subspace of V. Let $\mathbf{v}$ be a vector in V, and suppose $\mathbf{w}^*$ is a vector in W such that:
>
> $$\langle\mathbf{v}-\mathbf{w}^*,\mathbf{w}\rangle=0,\ \forall\ \mathbf{w}\in W. \tag{22}$$
>
> Then we have $\|\mathbf{v}-\mathbf{w}^*\|\le\|\mathbf{v}-\mathbf{w}\|\ \forall\ \mathbf{w}\in W$ with equality holding only for $\mathbf{w}=\mathbf{w}^*$.

> *Th.* Let V be an inner-product space, and let $\mathbf{v}$ be a vector in V. Let W be an n-dimensional subspace of V, and let $\{\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_n\}$ be an **orthogonal** basis for W. Then
>
> $$\|\mathbf{v}-\mathbf{w}^*\|\le\|\mathbf{v}-\mathbf{w}\|,\ \forall\ \mathbf{w}\in W. \tag{23}$$
>
> hold if and only if
>
> $$\mathbf{w}^*=\frac{\langle\mathbf{v},\mathbf{u}_1\rangle}{\langle\mathbf{u}_1,\mathbf{u}_1\rangle}\mathbf{u}_1+\frac{\langle\mathbf{v},\mathbf{u}_2\rangle}{\langle\mathbf{u}_2,\mathbf{u}_2\rangle}\mathbf{u}_2+\cdots+\frac{\langle\mathbf{v},\mathbf{u}_n\rangle}{\langle\mathbf{u}_n,\mathbf{u}_n\rangle}\mathbf{u}_n \tag{25}$$