Determinant

If A is an $(n \times n)$ matrix, the determinant of A, denoted $det(A)$, is a number that we associate with A. Determinants are usually defined either in terms of *cofactors* or in terms of *permutations*, and we elect to use the cofactor definition here.

Definition

> *Def.* **Cofactor**
>
> Let $A = (a_{i,j})$ be an $(n \times n)$ matrix, and let $M_{r,s}$ denote the $[(n-1) \times (n-1)]$ matrix obtained by deleting the rth row and sth column of A. Then $M_{r,s}$ is called a **minor matrix** of A, and the number $det(M_{r,s})$ is the minor of the (r, s)th entry, $a_{r,s}$. In addition, the numbers:
>
> $$A_{i,j} = (-1)^{i+j} det(M_{i,j}) \tag{10}$$
>
> are called **cofactors** (or **signed minors**).

> *Def.* **Determinant(cofactor)**
>
> Let $A = (a_{i,j})$ be an $(n \times n)$ matrix. Then the **determinant** of A is:
>
> $$det(A) = a_{1,1}A_{1,1} + a_{1,2}A_{1,2} + \cdots + a_{1,n}A_{1,n} \tag{1}$$
>
> where $A_{i,j}$ is the cofactor of $a_{i,j}$

> *Def.* **Permutation**
>
> A **permutation** $(j_1, j_2, \cdots, j_n)$ of the set $S = \{1, 2, \cdots, n\}$ is just a rearrangement of the numbers in S. In **inversion** of this permutation occurs whenever a number j_r is followed by a smaller number j_s, or $j_r > j_s$ but $r < s$. A permutation of S is called *odd* or *even* if it has an odd or even number of inversions.

> *Def.* **Determinant(Permutation)**
>
> It can be shown that $det(A)$ is the sum of all possible terms of the form $\pm a_{1,j_1} a_{2,j_2} \cdots a_{n,j_n}$, where the sign is taken as $+$ or $-$, depending on whether the permutation is even or odd.
>
> Let $p = (j_1, j_2, \cdots, j_n)$ be any permutation, we have:

> $$det(A) = \sum_{(j_1,j_2,\cdots,j_n)} (-1)^{\Gamma(j_1,j_2,\cdots,j_n)} \prod_{i=1}^{n} a_{i,j_i} \tag{2}$$

> *Th.* Let $T = (t_{i,j})$ be an $(n \times n)$ lower-triangular matrix. Then
>
> $$det(T) = \prod_{i=1}^{n} t_{i,i} \tag{3}$$

Elementary Operations

> *Th.* If A is an $(n \times n)$ matrix, then $det(A) = det(A^T)$

> *Th.* Let $A = [\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n]$ be an $(n \times n)$ matrix. If B is obtained from A by interchanging two rows (or columns) of A, then $det(B) = -det(A)$.

> *Th.* If A is an $(n \times n)$ matrix, and if B is the $(n \times n)$ matrix resulting from multiplying the k th column (or row) of A by a scalar c, then $det(B) = c \cdot det(A)$.
>
> *Corollary.* $det(cA) = c^n A$

> *Th.* If A, B and C are $(n \times n)$ matrices that are equal except that the sth column (or row) if A is equal to the sum of the sth column (or row) of B and C, then $det(A) = det(B) + det(C)$
>
> *Note.* $det(A + B) \neq det(A) + det(B)$

> *Th.* Let A be an $(n \times n)$ matrix. If the jth column (or row) of A is a multiple of the kth column (or row) of A, then $det(A) = 0$.

> *Th.* If A is an $(n \times n)$ matrix, and if a multiple of the kth column (or row) is added to the jth column (or row), then the determinant is not changed.

Cramer's Rule

> *Lemma.* Let $A = [\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n]$ be an $(n \times n)$ matrix, and let $\mathbf{b}$ be any vector in R^n. For each $i,\ 1 \leq i \leq n$, let B_i be the $(n \times n)$ matrix:
>
> $$B_i = [\mathbf{A}_1, \ldots, \mathbf{A}_{i-1}, \mathbf{b}, \mathbf{A}_{i+1}, \ldots, \mathbf{A}_n]. \tag{4}$$
>
> If the system of equations $A\mathbf{x} = \mathbf{b}$ is consistent and x_i is the ith component of a solution, then:
>
> $$x_i det(A) = det(B_i) \tag{5}$$

> *Th.* If A is an $(n \times n)$ **singular** matrix, then $det(A) = 0$.
>
> *Proof.* A is singular matrix, $A\mathbf{x} = \mathbf{b}$ has nontrivial solution. Choose i s.t. $x_i \neq 0$, we have $x_i det(A) = det(B_i)$. But $det(B_i) = 0$, and $x_i \neq 0$, so $det(A) = 0$.

> *Th.* If A and B are $(n \times n)$ matrices, then
>
> $$det(AB) = det(A)det(B) \tag{6}$$

> *Lemma*. Let A and B be $(n \times n)$ matrices, and let $C = AB$. Let $\hat{C}$ denote the result of applying an elementary column operation to C and let $\hat{B}$ denote the result of applying the same column operation to B. Then $\hat{C} = A\hat{B}$

> *Th.* If the $(n \times n)$ matrix A is nonsingular, then $det(A) \neq 0$. Moreover, $det(A^{-1}) = 1/det(A)$.

> *Th.* **Cramer's Rule**
>
> Let $A = [\mathbf{A}_1, \cdots, \mathbf{A}_n]$ be a nonsingular $(n \times n)$ matrix, and let $\mathbf{b}$ be any vector in R^n. For each $i,\ 1 \leq i \leq n$, let B_i be the matrix $B_i = [\mathbf{A}_1, \ldots, \mathbf{A}_{i-1}, \mathbf{b}, \mathbf{A}_{i+1}, \ldots, \mathbf{A}_n]$. Then the ith component, x_i of the solution of $A\mathbf{x} = \mathbf{b}$ is given by:
>
> $$x_i = \frac{det(B_i)}{det(A)}. \tag{7}$$
>
> *Proof.* From the first lemma in this section.
>
> *Note.* Cramer's rule is a valuable theoretical tool instead of a practical tool for real computation.

Applications of Determinants

Inverses

> *Lemma.* Let A be an $(n \times n)$ matrix. Then there is a nonsingular matrix $Q_{n \times n}$ such that $AQ = L$, where L is lower triangular. Moreover, $det(Q^T) = det(Q)$.
>
> This Lemma can give the proof for the fact that $det(A) = det(A^T)$

> *Th.* Let $A = (a_{i,j})$ be an $(n \times n)$ matrix, Then:
>
> $$\begin{aligned} det(A) &= a_{i,1}A_{i,1} + a_{i,2}A_{i,2} + \cdots + a_{i,n}A_{i,n} \\ det(A) &= a_{1,j}A_{1,j} + a_{2,j}A_{2,j} + \cdots + a_{n,j}A_{n,j} \end{aligned} \tag{8}$$

> *Lemma.* If A is an $(n \times n)$ matrix and if $i \neq k$, then $\sum_{j=1}^{n} a_{i,j}A_{k,j} = 0$.

> *Def.* **Adjoint Matrix**
>
> Let A be an $(n \times n)$ matrix, and let C denote the matrix of cofactors; $C = (c_{i,j})$ is $(n \times n)$, and $c_{i,j} = A_{i,j}$. The **adjoint matrix** of A, denoted $\mathrm{Adj}(A)$, is equal to C^T.

> *Th.* If A is an $(n \times n)$ **nonsingular** matrix, then
>
> $$A^{-1} = \frac{1}{\det(A)} \mathrm{Adj}(A). \tag{9}$$

Elementary Matrices

The result of applying a sequence of elementary column operations to a matrix A can be represented in matrix terms as multiplication of A by a sequence of elementary matrices.

> *Def.* Let I denote the $(n \times n)$ identity matrix, and let E be the matrix that results when an elementary column operation is applied to I. Such a matrix E is called an **elementary matrix**.

> *Th.* Let E be the $(n \times n)$ elementary matrix that results from performing a certain column operation on the $(n \times n)$ identity matrix. If A is any $(n \times n)$ matrix, then AE is the matrix that results when this same column operation is performed on A.