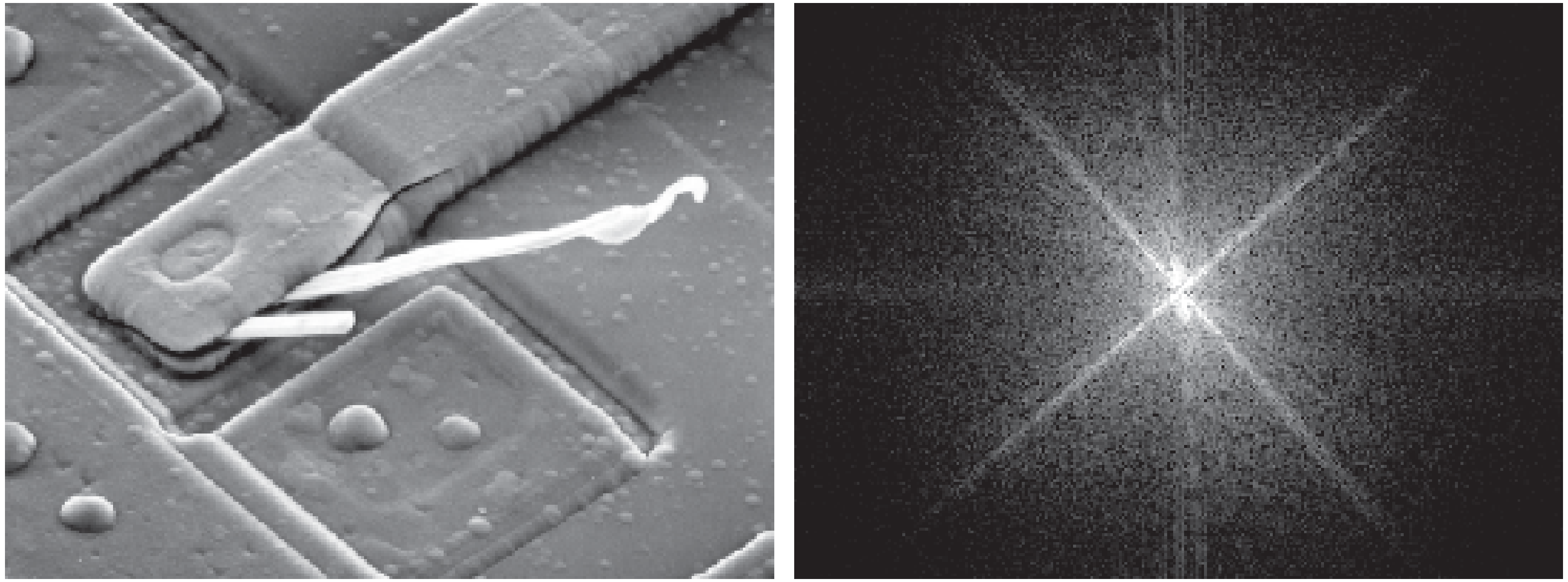
Lecture 9 – Frequency Domain Filtering

This lecture will cover:

- Basic Filtering form
- Steps of Frequency Domain Filtering
- Frequency Domain Filtering
  - Lowpass Filtering (低通滤波器)
  - Highpass Filtering (高通滤波器)
  - Selective Filtering (选择性滤波)

Fourier Spectrum

Frequency spectrum is corresponding to the image intensity including average, smooth variation and fast changes.



a b

FIGURE 4.28 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

Frequency Domain Filtering

Basic Filtering form: $g(x,y) = real\{\mathcal{F}^{-1}[H(u,v)F(u,v)]\}$

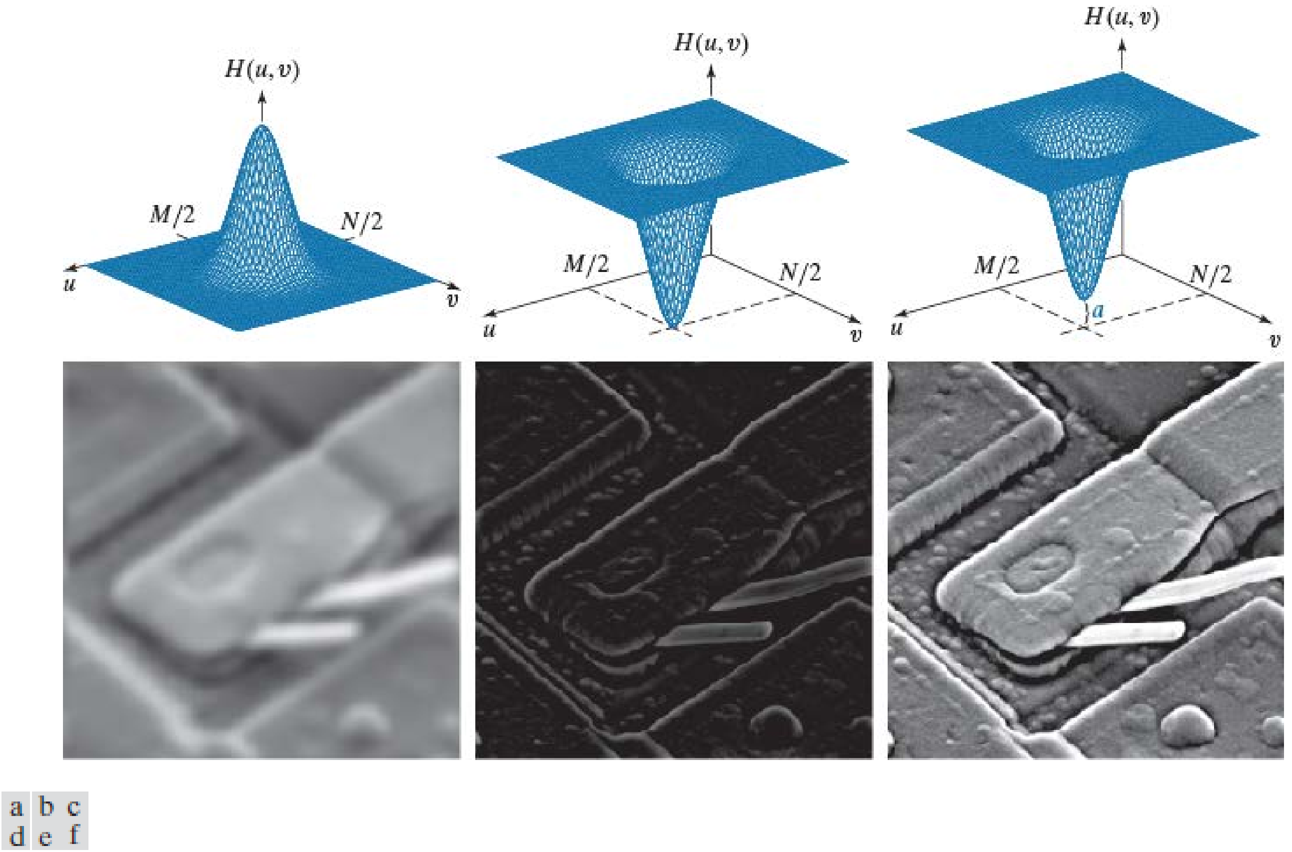
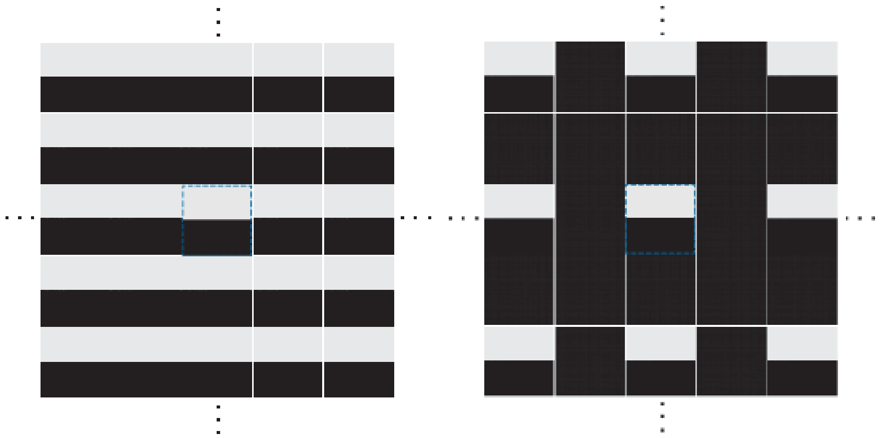


FIGURE 4.30 Top row: Frequency domain filter transfer functions of (a) a lowpass filter, (b) a highpass filter, and (c) an offset highpass filter. Bottom row: Corresponding filtered images obtained using Eq. (4-104). The offset in (c) is $a = 0.85$, and the height of $H(u,v)$ is 1. Compare (f) with Fig. 4.28(a).

Padding (填充)

$$f_p(x,y) = \begin{cases} f(x,y), 0 \le x \le A-1, 0 \le y \le B-1 \\ 0, \quad A \le x \le P, B \le y \le Q \end{cases} \qquad h_p(x,y) = \begin{cases} h(x,y), 0 \le x \le C-1, 0 \le y \le D-1 \\ 0, \quad C \le x \le P, D \le y \le Q \end{cases}$$

Where $P \ge A + C - 1, Q \ge B + D - 1$



a b

FIGURE 4.32 (a) Image periodicity without image padding. (b) Periodicity after padding with 0's (black). The dashed areas in the center correspond to the image in Fig. 4.31(a). Periodicity is inherent when using the DFT. (The thin white lines in both images are superimposed for clarity; they are not part of the data.)

a b c

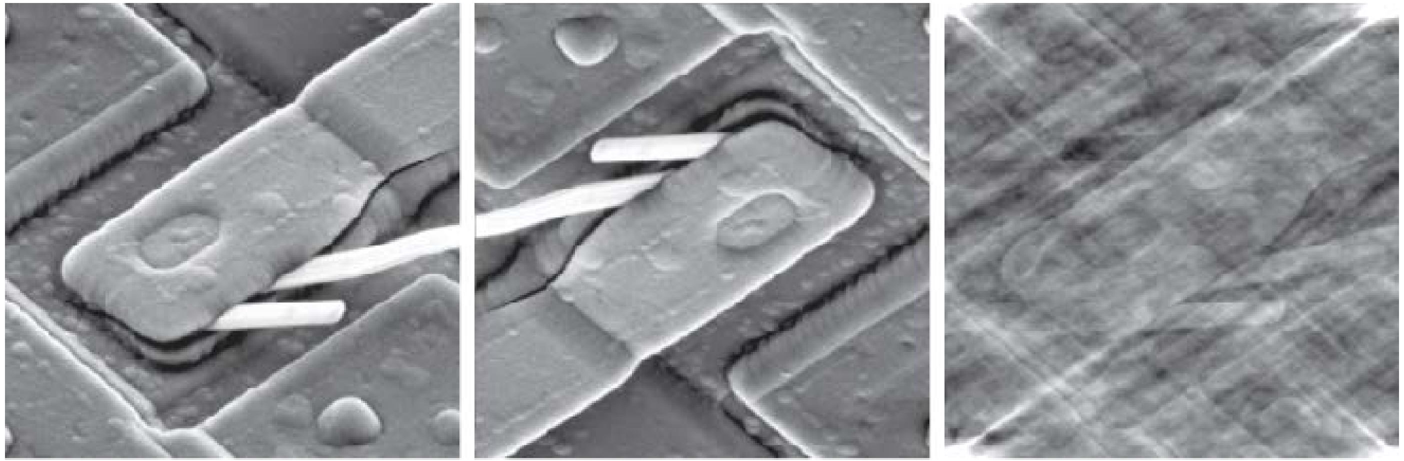
FIGURE 4.31 (a) A simple image. (b) Result of blurring with a Gaussian lowpass filter without padding. (c) Result of lowpass filtering with zero padding. Compare the vertical edges in (b) and (c).

Phase Angle

Let : $F(u,v) = R(u,v) + jI(u,v)$

$$\boldsymbol{g}(x,y) = \mathcal{F}^{-1}[H(u,v)R(u,v) + jH(u,v)I(u,v)]$$

$H(u,v)$: **zero-phase-shift filter（零相移滤波器）**



a b c

FIGURE 4.34 (a) Original image. (b) Image obtained by multiplying the phase angle array by –1 in Eq. (4-86) and computing the IDFT. (c) Result of multiplying the phase angle by 0.25 and computing the IDFT. The magnitude of the transform, $|F(u,v)|$, used in (b) and (c) was the same.

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Steps of Frequency Domain Filtering

a b c
d e f
g h

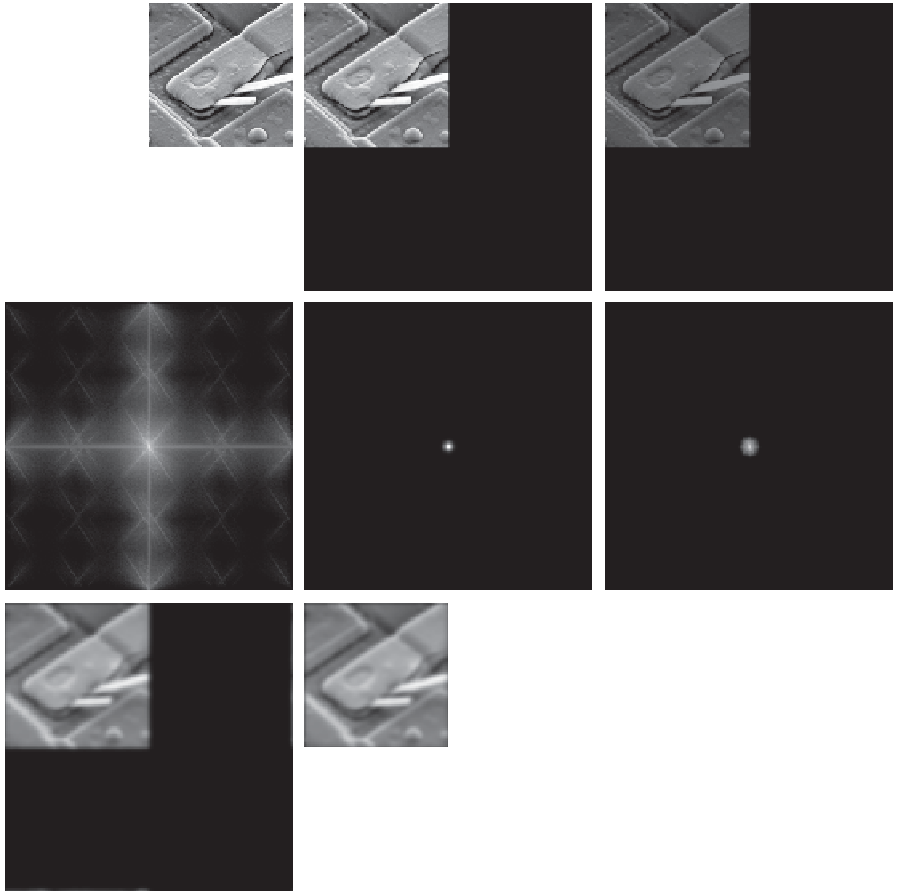


FIGURE 4.35 (a) An $M \times N$ image, f. (b) Padded image, f_p of size $P \times Q$. (c) Result of multiplying f_p by $(-1)^{x+y}$. (d) Spectrum of F. (e) Centered Gaussian lowpass filter transfer function, H, of size $P \times Q$. (f) Spectrum of the product HF. (g) Image g_p, the real part of the IDFT of HF, multiplied by $(-1)^{x+y}$. (h) Final result, g, obtained by extracting the first M rows and N columns of g_p.

1. Zero-padding input image $f_p(x, y)$ to 2M*2N

2. $f_p(x, y)(-1)^{(x+y)}$ to center its transform

3. Compute DFT

4. $\boldsymbol{G}(u, v) = H(u, v)F(u, v)$

5. $\boldsymbol{g_p}(x, y) = \{Real[\mathcal{F}^{-1}(G(u, v))]\}\,(-1)^{(x+y)}$

6. Obtain $g(x, y)$ from top-left quadrant

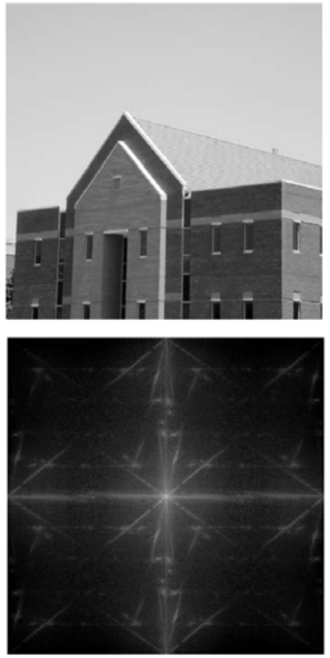
Filtering in Spatial and Frequency Domains

➢ **Frequency filters ⇒ Spatial filter** $H(u,v) \Longrightarrow h(x,y)$

Design filter in frequency domain → Compute IDFT → Full size spatial filter → Smaller spatial masks

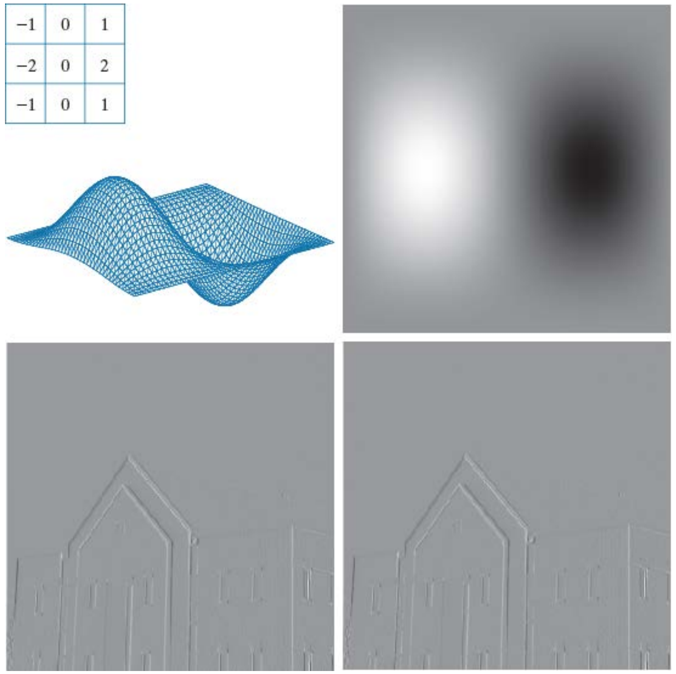
a b

FIGURE 4.37 (a) Image of a building, and (b) its Fourier spectrum.



a b
c d

FIGURE 4.38 (a) A spatial kernel and perspective plot of its corresponding frequency domain filter transfer function. (b) Transfer function shown as an image. (c) Result of filtering Fig. 4.37(a) in the frequency domain with the transfer function in (b). (d) Result of filtering the same image in the spatial domain with the kernel in (a). The results are identical.



Lowpass Filtering

- Ideal Lowpass Filter（理想低通滤波器）
- Butterworth Lowpass Filter（布特沃斯低通滤波器）
- Gaussian Lowpass Filter （高斯低通滤波器）

Lowpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D(u,v)/D_0]^{2n}}$	$H(u,v) = e^{-D^2(u,v)/2D_0^2}$

Ideal Lowpass Filter（理想低通滤波器）

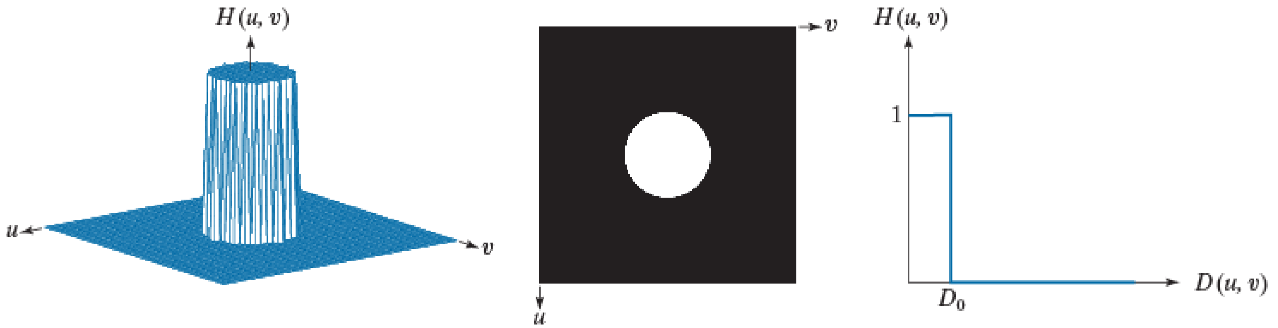
Ideal Lowpass Filter (ILPF):

$$H(u,v) = \begin{cases} 1, & D(u,v) \le D_0 \\ 0, & D(u,v) > D_0 \end{cases}$$

Where

$$D(u,v) = \left[(u-\tfrac{P}{2})^2 + (v-\tfrac{Q}{2})^2\right]^{1/2}$$

D_0 : Cutoff Frequency (截止频率)

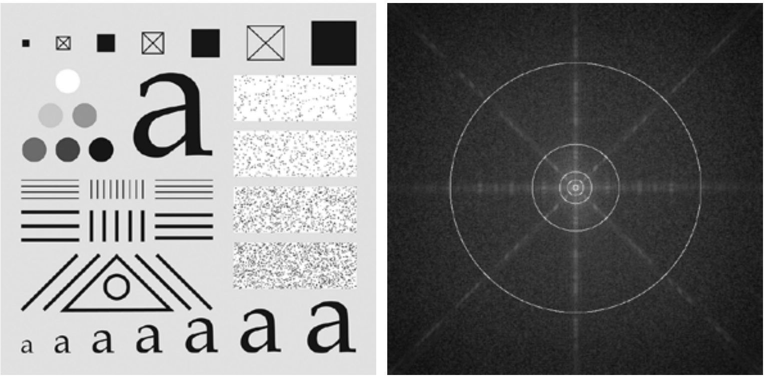


a b c

FIGURE 4.39 (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Function displayed as an image. (c) Radial cross section.

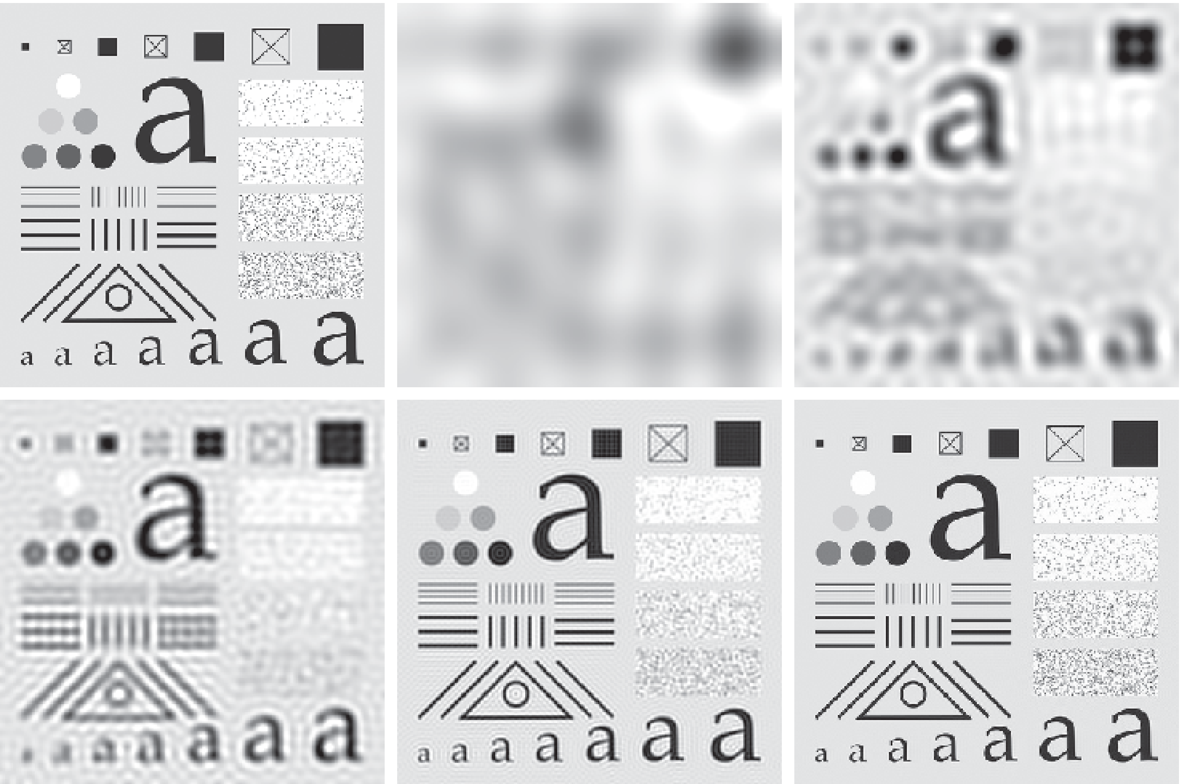
- **Power:** $P(u,v) = |F(u,v)|^2$
- **Total image Power:** $P_T = \sum_{u=0}^{P-1}\sum_{v=0}^{Q-1} P(u,v)$
- **The power in a circle of radius D_0:** $\alpha = 100\left[\frac{\sum_u \sum_v P(u,v)}{P_T}\right]$

Cutoff frequency and Ringing (振铃效应)



a b

FIGURE 4.40 (a) Test pattern of size 688 × 688 pixels, and (b) its spectrum. The spectrum is double the image size as a result of padding, but is shown half size to fit. The circles have radii of 10, 30, 60, 160, and 460 pixels with respect to the full-size spectrum. The radii enclose 86.9, 92.8, 95.1, 97.6, and 99.4% of the padded image power, respectively.

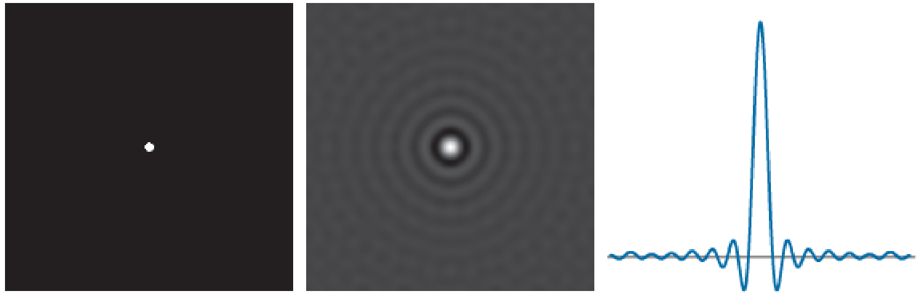


a b c
d e f

FIGURE 4.41 (a) Original image of size 688 × 688 pixels. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.40(b). The power removed by these filters was 13.1, 7.2, 4.9, 2.4, and 0.6% of the total, respectively. We used mirror padding to avoid the black borders characteristic of zero padding, as illustrated in Fig. 4.31(c).

a b c

FIGURE 4.42 (a) Frequency domain ILPF transfer function. (b) Corresponding spatial domain kernel function. (c) Intensity profile of a horizontal line through the center of (b).



Butterworth Lowpass Filter（布特沃斯）

$$H(u,v) = \frac{1}{1 + \left[\frac{D(u,v)}{D_0}\right]^{2n}}$$

Where $D(u,v) = \left[(u - \frac{P}{2})^2 + (v - \frac{Q}{2})^2\right]^{1/2}$, and $H(u,v) = 0.5$ when $D(u,v) = D_0$

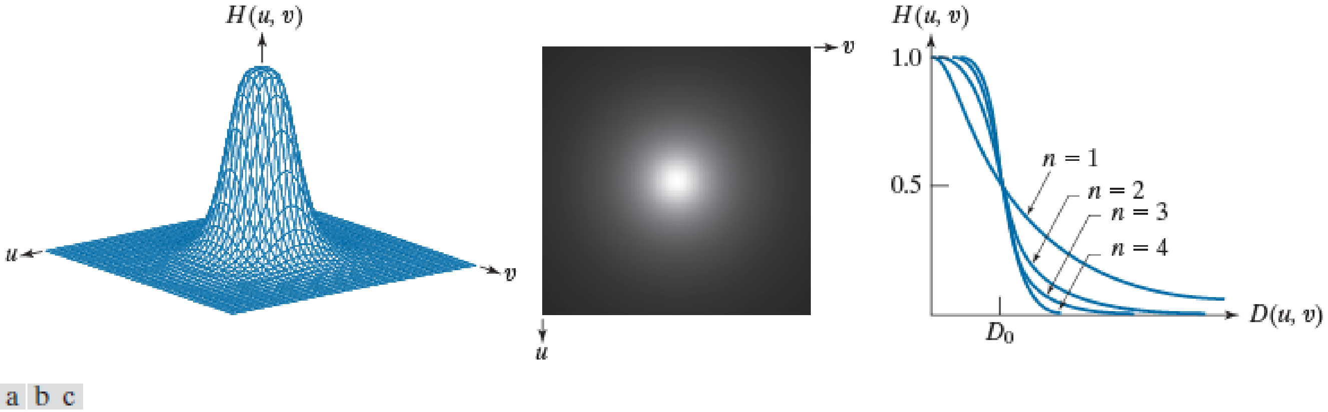
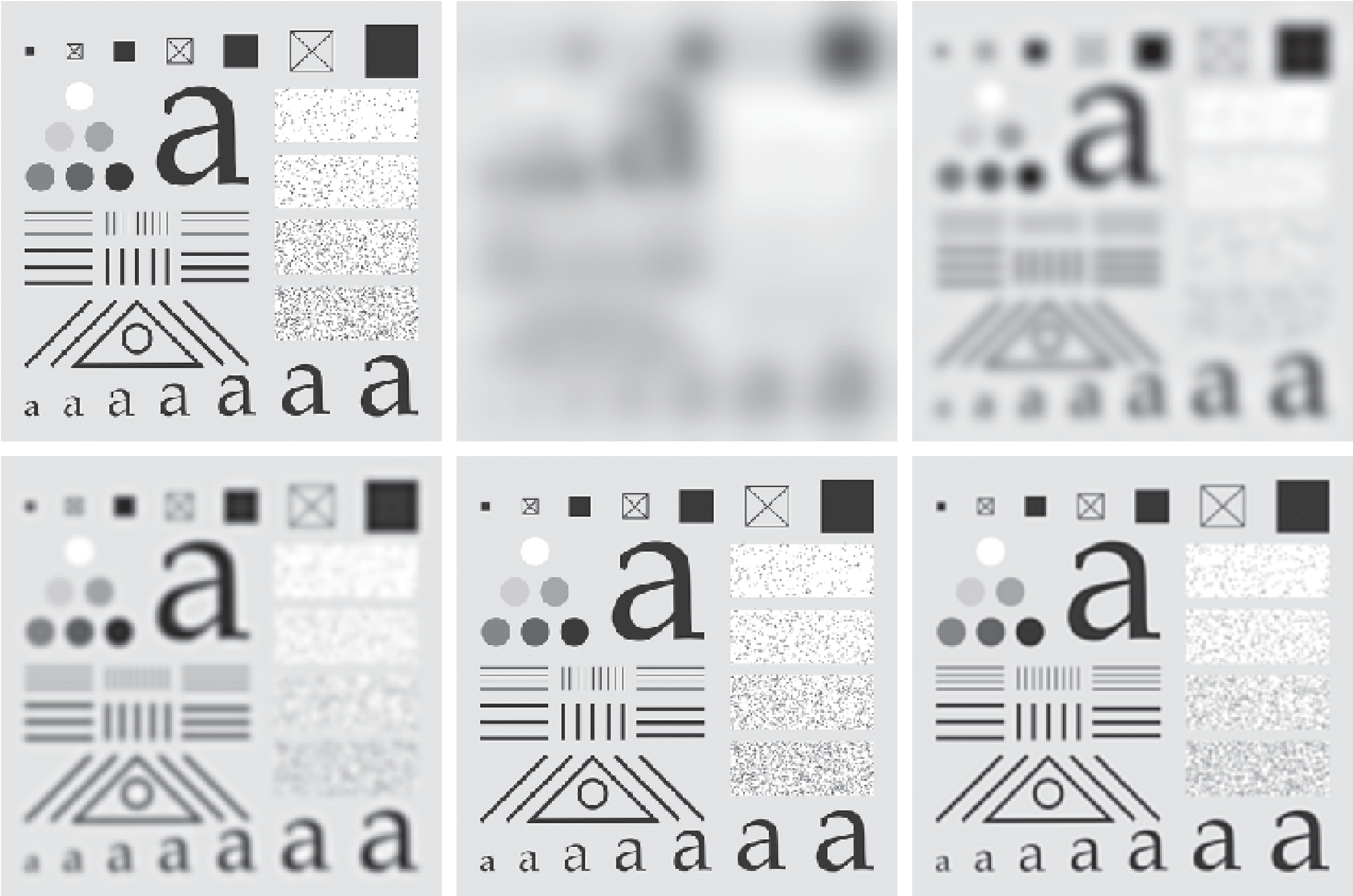


FIGURE 4.45 (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Function displayed as an image. (c) Radial cross sections of BLPFs of orders 1 through 4.

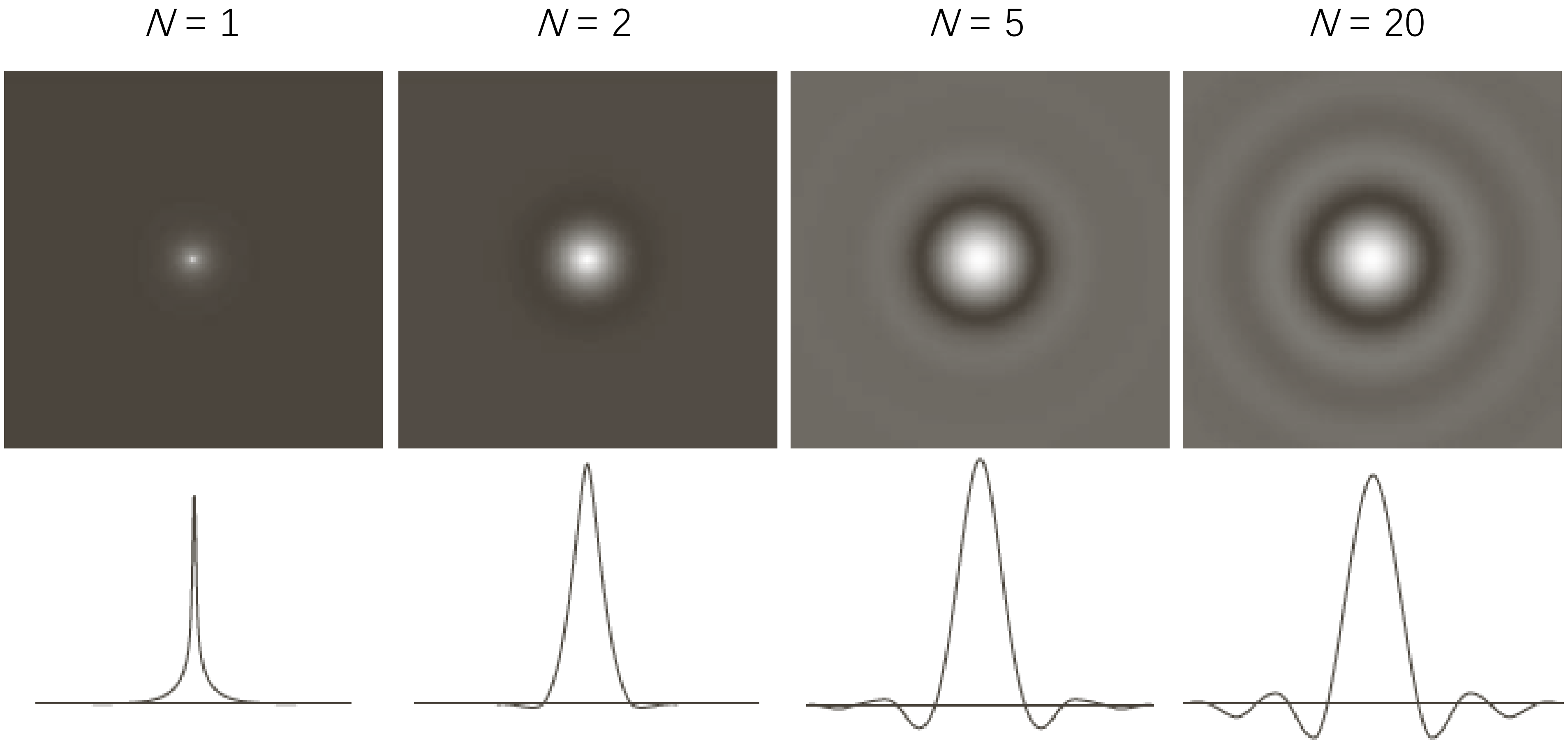
Butterworth Lowpass Filter（布特沃斯）



a b c
d e f

FIGURE 4.46 (a) Original image of size 688 × 688 pixels. (b)–(f) Results of filtering using BLPFs with cutoff frequencies at the radii shown in Fig. 4.40 and $n = 2.25$. Compare with Figs. 4.41 and 4.44. We used mirror padding to avoid the black borders characteristic of zero padding.

*n*th Butterworth Filter



Gaussian Lowpass Filter (高斯滤波器)

$$H(u,v) = e^{-\frac{D(u,v)^2}{2D_0^2}},$$ where $H(u,v) = 0.607$ when $D(u,v) = D_0$

- **Inverse is Gaussian and reciprocal:** $H(u) = Ae^{-\frac{u^2}{2\sigma^2}} \Leftrightarrow h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2x^2}$
- **Linear:** $H(u) = Ae^{-\frac{u^2}{2\sigma_1^2}} - Be^{-\frac{u^2}{2\sigma_2^2}} \Leftrightarrow h(x) = \sqrt{2\pi}\sigma_1 A e^{-2\pi^2\sigma_1^2x^2} - \sqrt{2\pi}\sigma_2 B e^{-2\pi^2\sigma_2^2x^2}$
- **In 2D:** $H(u,v) = Ae^{-\frac{u^2+v^2}{2\sigma^2}} \Leftrightarrow h(x,y) = A2\pi\sigma^2 e^{-2\pi^2\sigma^2(x^2+y^2)}$

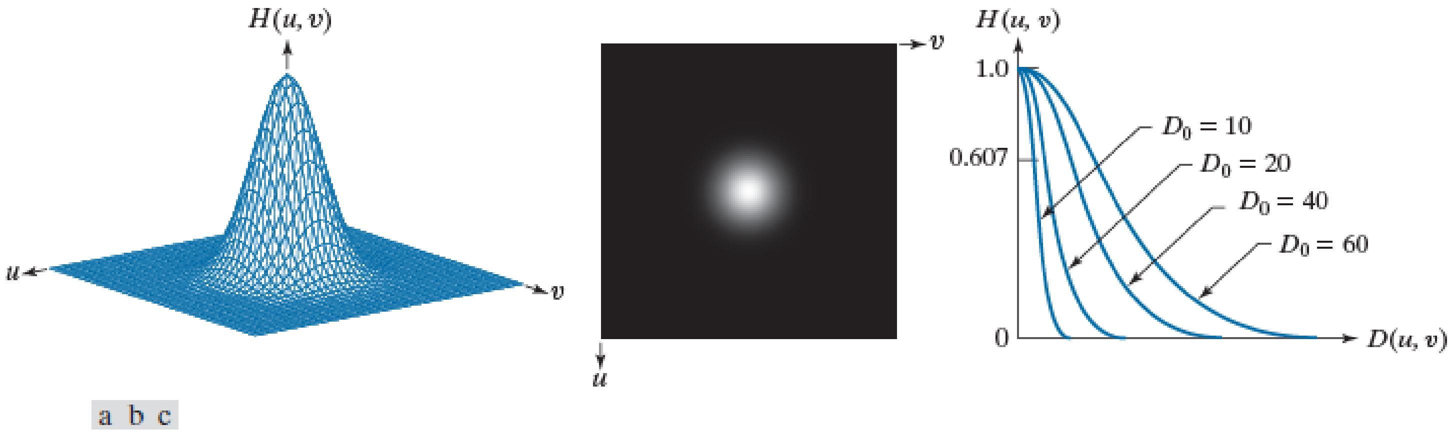
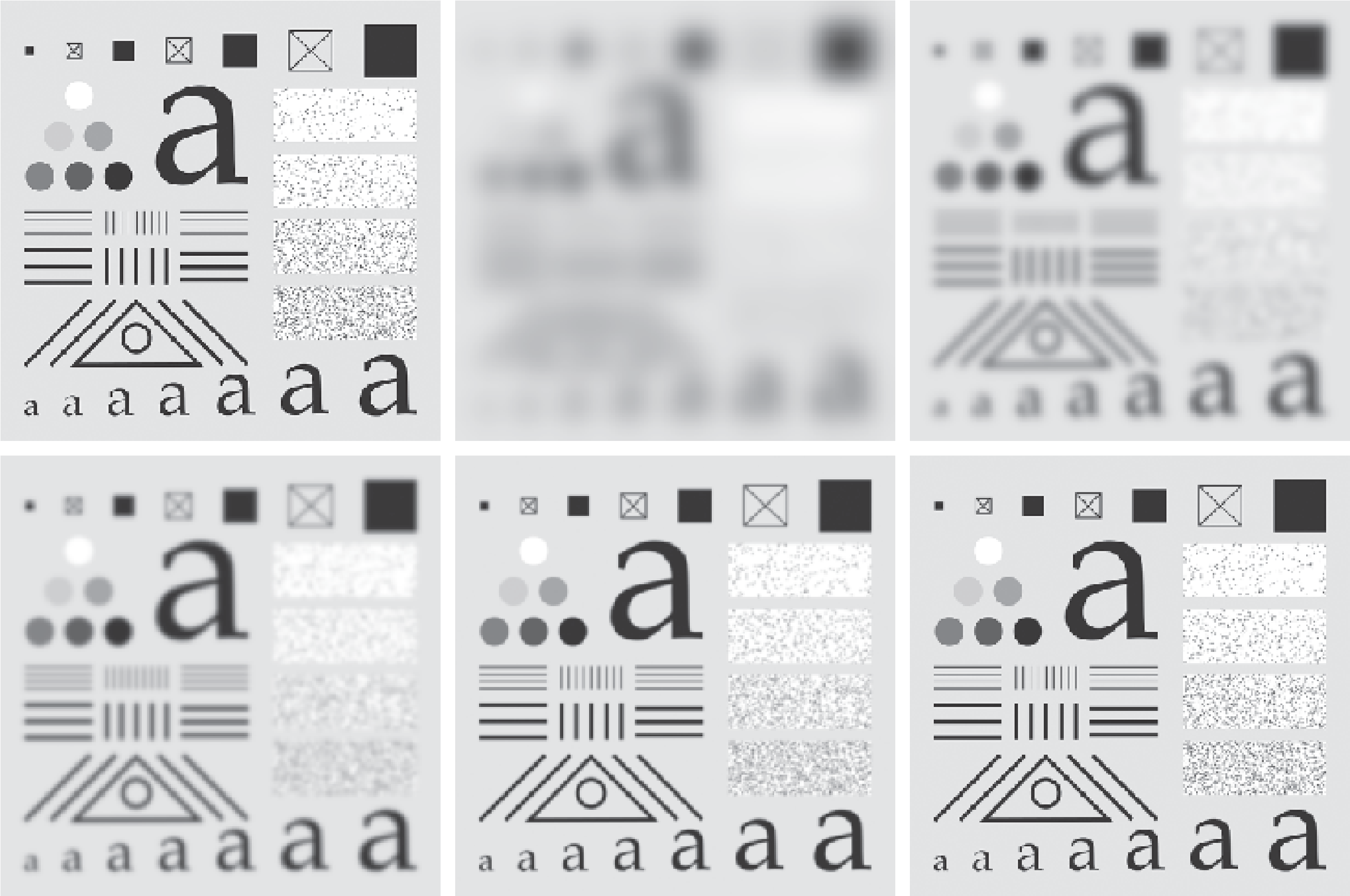


FIGURE 4.43 (a) Perspective plot of a GLPF transfer function. (b) Function displayed as an image. (c) Radial cross sections for various values of D_0.

Gaussian Lowpass Filter (高斯滤波器)



a b c
d e f

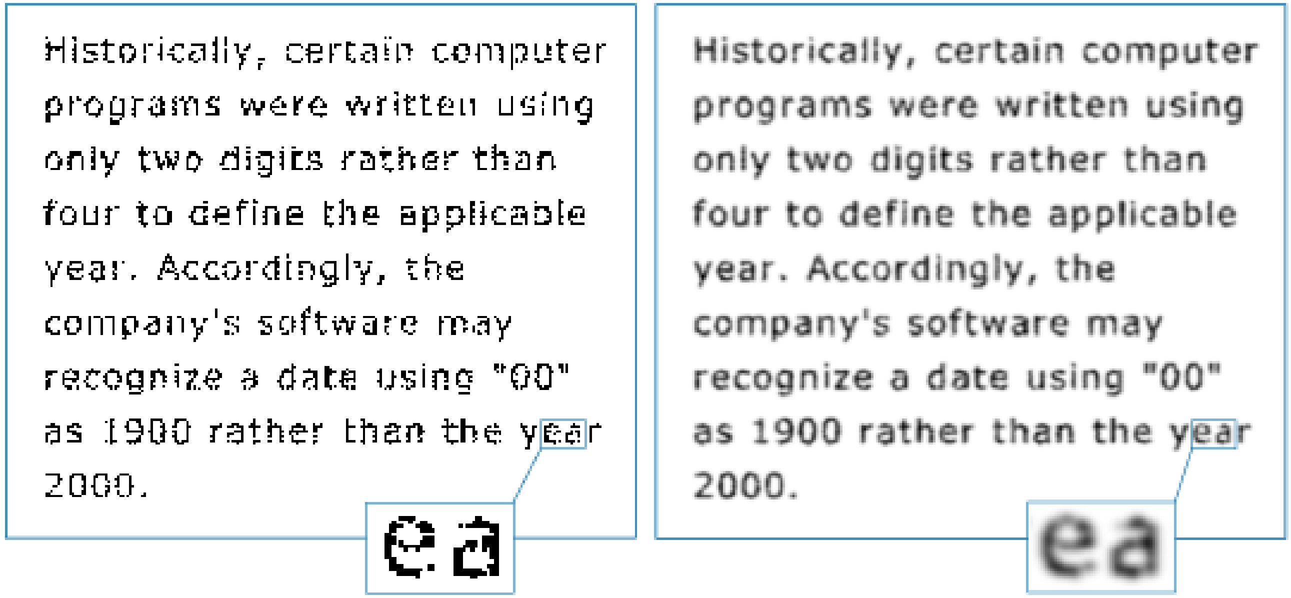
FIGURE 4.44 (a) Original image of size 688 × 688 pixels. (b)–(f) Results of filtering using GLPFs with cutoff frequencies at the radii shown in Fig. 4.40. Compare with Fig. 4.41. We used mirror padding to avoid the black borders characteristic of zero padding.

Application of Lowpass Filters

- **Character Recognition**

a b

FIGURE 4.48
(a) Sample text of low resolution (note the broken characters in the magnified view). (b) Result of filtering with a GLPF, showing that gaps in the broken characters were joined.



Application of Lowpass Filters

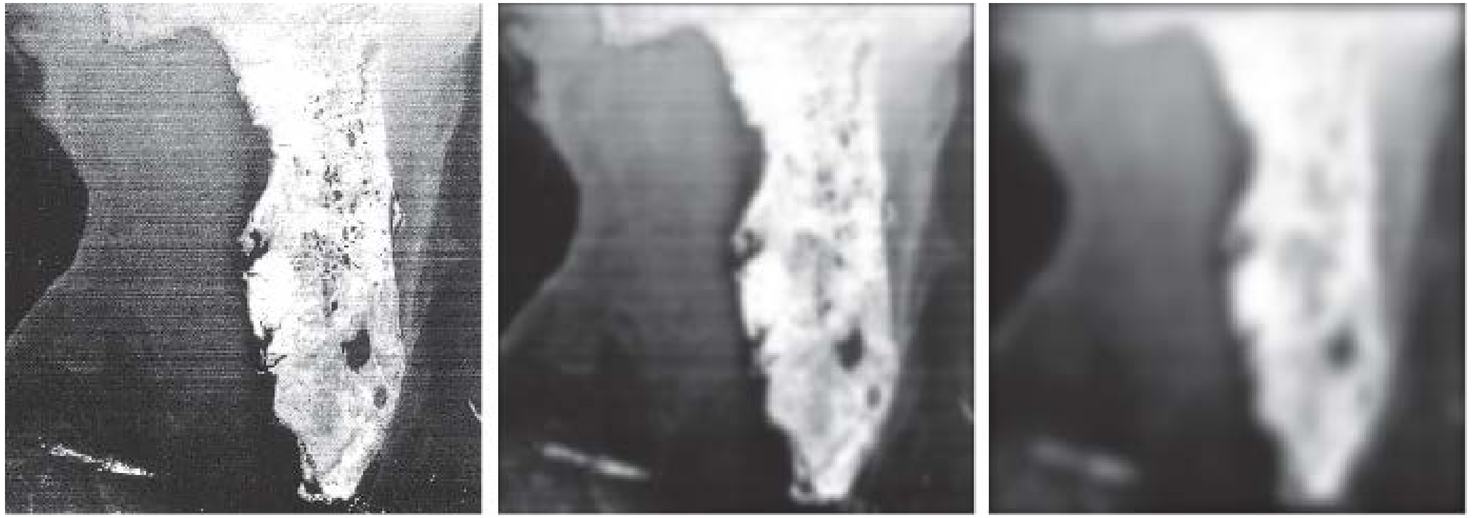
- “Cosmetic” processing

a b c

FIGURE 4.49 (a) Original 785 × 732 image. (b) Result of filtering using a GLPF with $D_0 = 150$. (c) Result of filtering using a GLPF with $D_0 = 130$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).

Application of Lowpass Filters

- **Satellite and Aerial Images**



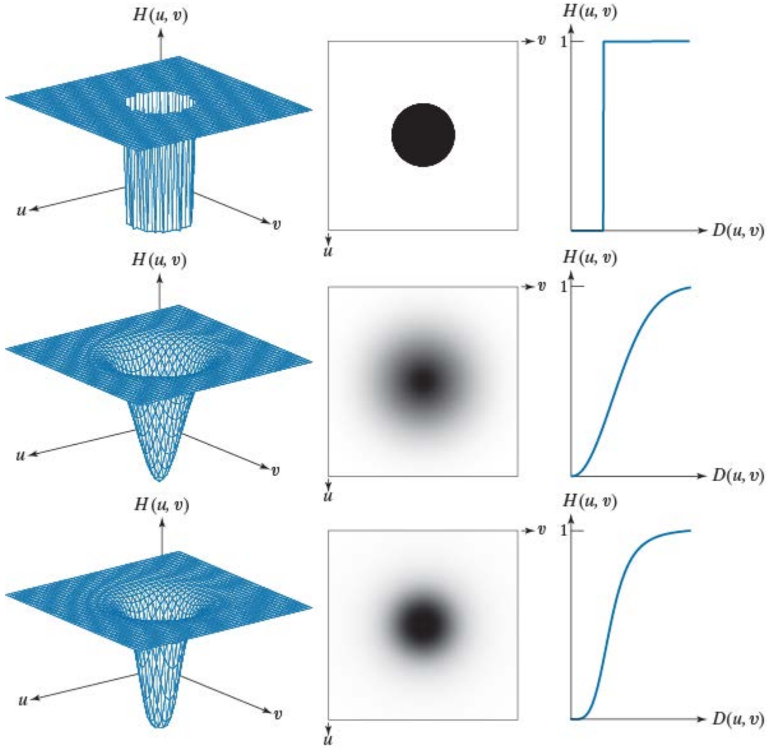
a b c

FIGURE 4.50 (a) 808 × 754 satellite image showing prominent horizontal scan lines. (b) Result of filtering using a GLPF with $D_0 = 50$. (c) Result of using a GLPF with $D_0 = 20$. (Original image courtesy of NOAA.)

Highpass Filtering

- Ideal Highpass Filter（理想滤波器）
- Butterworth Highpass Filter (布特沃斯滤波器)
- Gaussian Highpass Filter （高斯滤波器）

$$H_{\mathrm{HP}}(u,v) = 1 - H_{\mathrm{LP}}(u,v)$$



a b c
d e f
g h i

FIGURE 4.51 Top row: Perspective plot, image, and, radial cross section of an IHPF transfer function. Middle and bottom rows: The same sequence for GHPF and BHPF transfer functions. (The thin image borders were added for clarity. They are not part of the data.)

Highpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$	$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$

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Highpass Filter in Spatial Domain

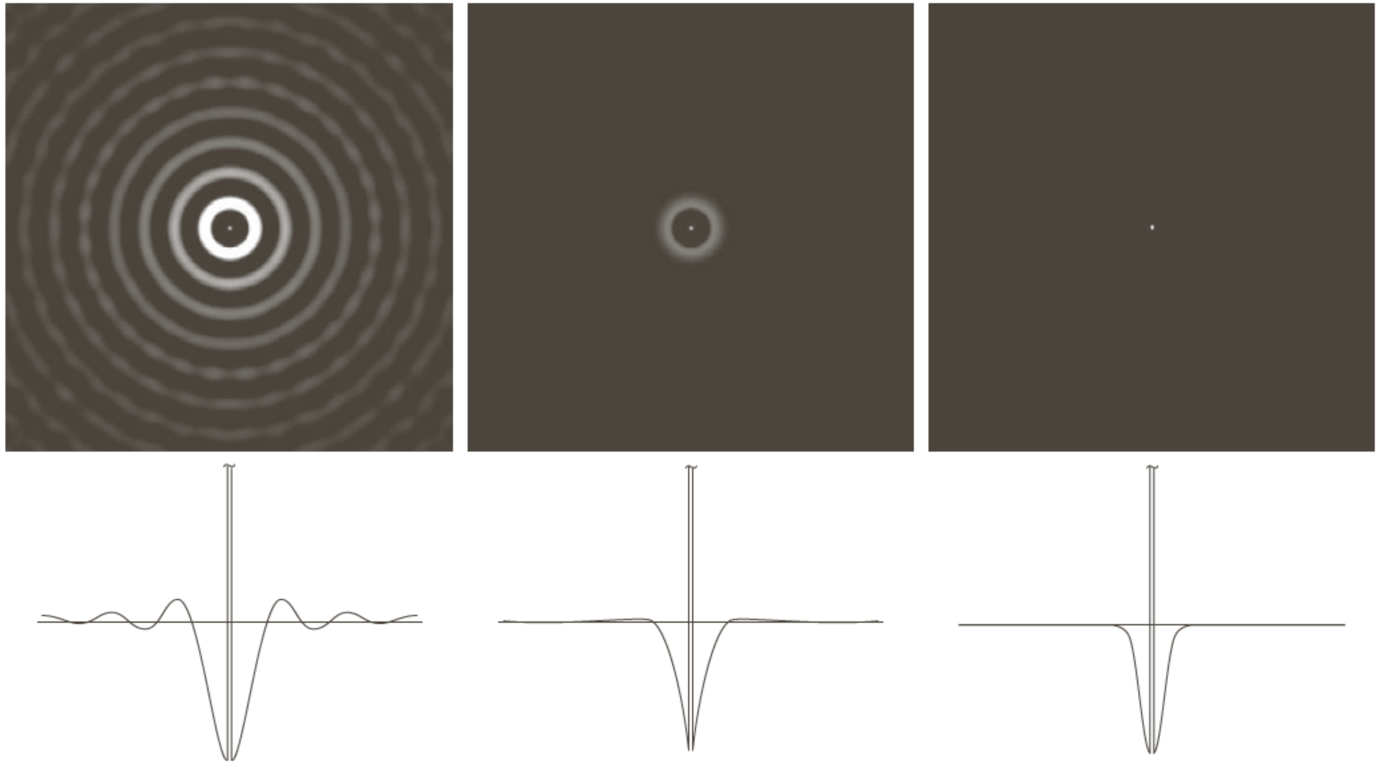


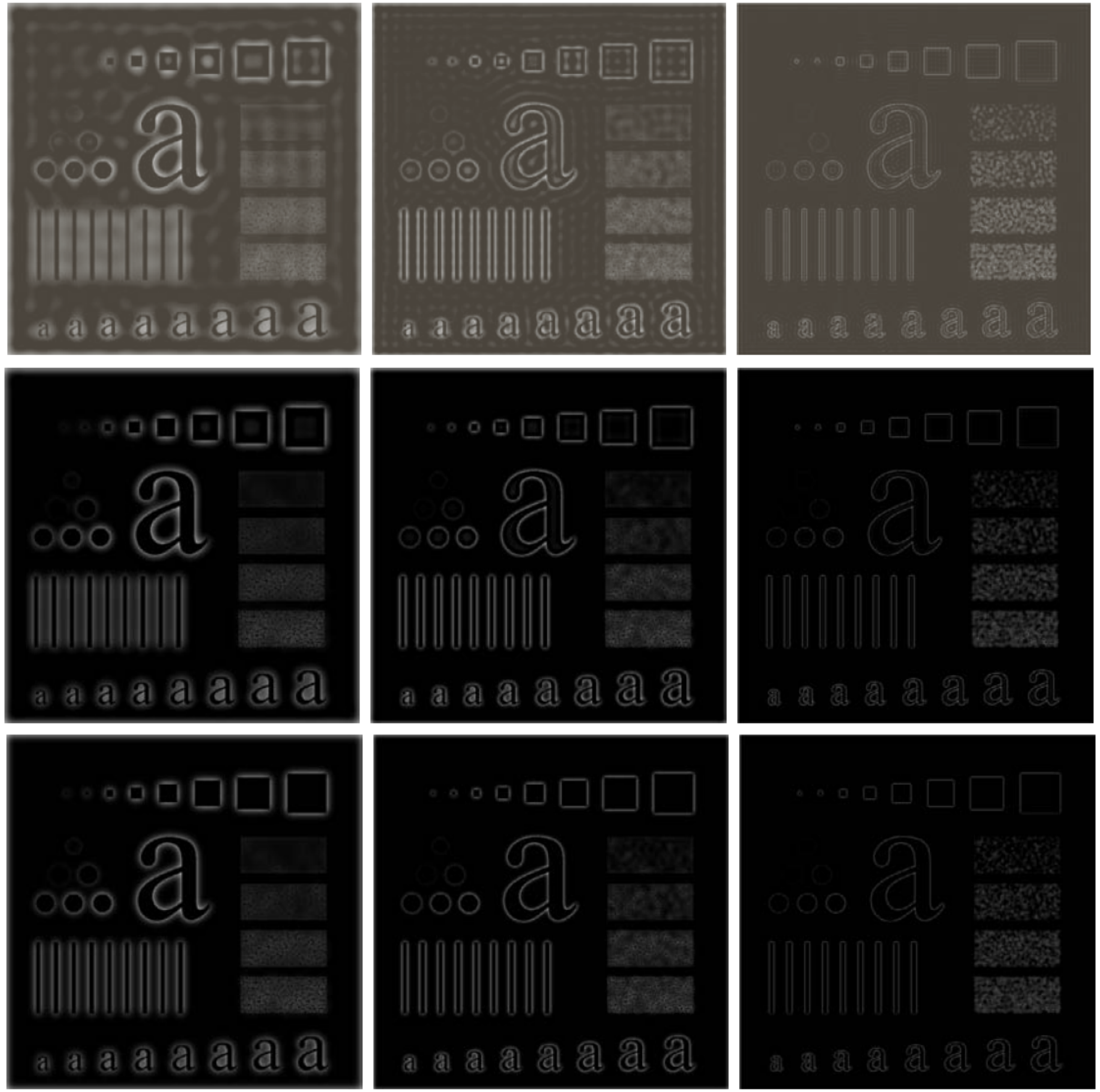
FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

$D_0=30$ $D_0=60$ $D_0=160$

IHPF

BHPF

GHPF



Highpass Filtering and Thresholding

FIGURE 4.55 (a) Smudged thumbprint. (b) Result of highpass filtering (a). (c) Result of thresholding (b). (Original image courtesy of the U.S. National Institute of Standards and Technology.)

Other Highpass Filtering

- **Laplacian（拉普拉斯算子）**
- **Unsharp Mask（钝化模板）**
- **Homomorphic Filtering（同态滤波）**

Laplacian（拉普拉斯算子）

$$\nabla^2 f(x,y) = \mathcal{F}^{-1}\{H(u,v)F(u,v) \quad \text{where } H(u,v) = -4\pi^2 D^2(u,v)$$

a b

FIGURE 4.56
(a) Original, blurry image. (b) Image enhanced using the Laplacian in the frequency domain. Compare with Fig. 3.52(d). (Original image courtesy of NASA.)

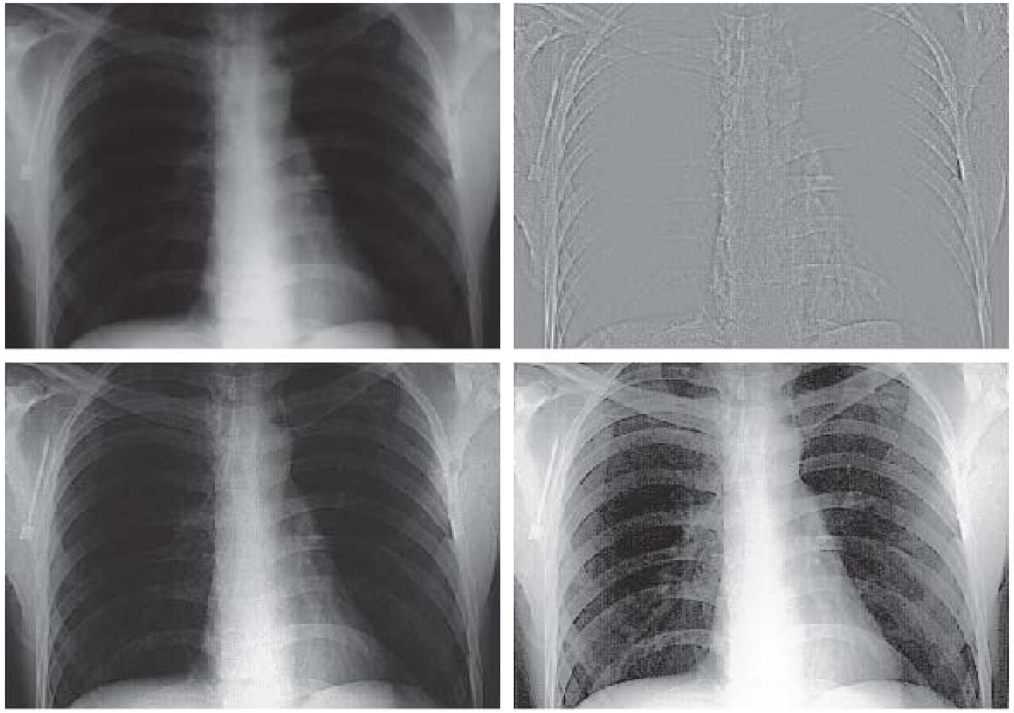
Unsharp Mask

$$g_{\text{mask}}(x,y) = f(x,y) - \overline{f(x,y)}$$
$$= f(x,y) - f_{LP}(x,y)$$

$$g(x,y) = f(x,y) + k * g_{\text{mask}}(x,y)$$
$$= \mathcal{F}^{-1}\{[1 + k * H_{HP}(u,v)]F(u,v)\}$$

-High Frequency Emphasis Filter （高频强调滤波器）

$$g(x,y) = \mathcal{F}^{-1}\{[k_1 + k_2 * H_{HP}(u,v)]F(u,v)\}$$



a b
c d

FIGURE 4.57 (a) A chest X-ray. (b) Result of filtering with a GHPF function. (c) Result of high-frequency-emphasis filtering using the same GHPF. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

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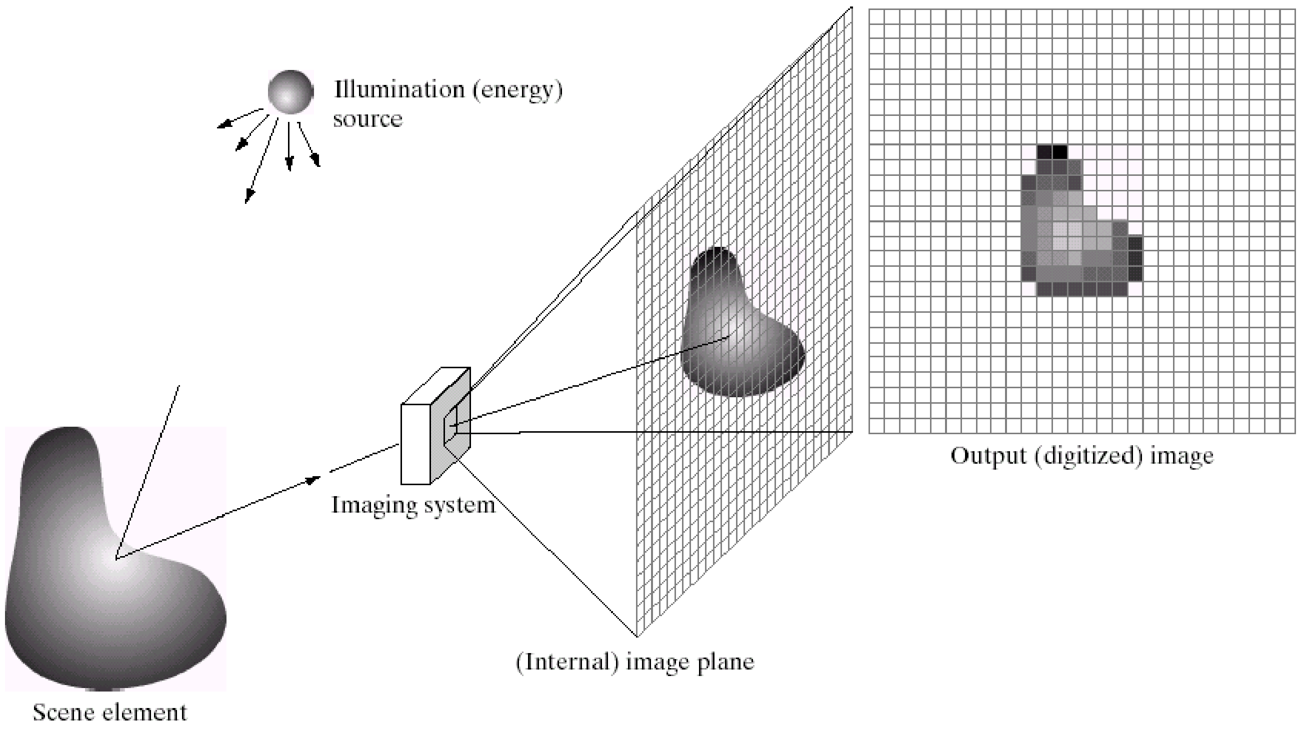
Image Acquisition



$$f(x,y) = i(x,y)r(x,y) \qquad 0 < i(x,y) < \infty, 0 \leq r(x,y) < 1$$

Homomorphic Filtering（同态滤波）

Let $z(x,y) = \ln f(x,y) = \ln i(x,y) + \ln r(x,y)$

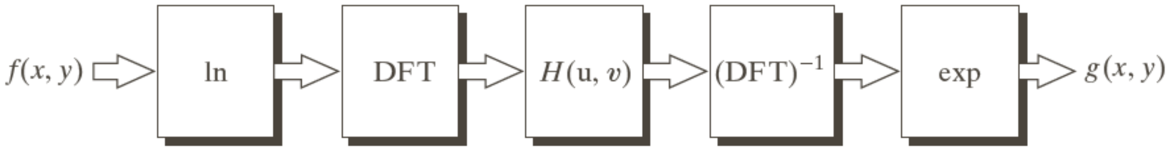
$$\boldsymbol{Z(u,v) = F_i(u,v) + F_r(u,v)}$$

Where
$Z(u,v) = \mathcal{F}[z(x,y)],\ F_i(u,v) = \mathcal{F}[\ln i(x,y)],\ \ F_r(u,v) = \mathcal{F}[\ln r(x,y)]$

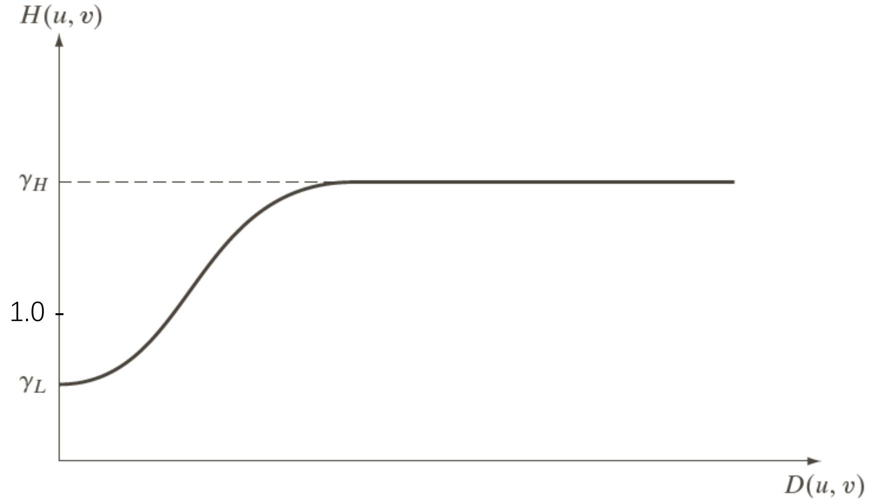
$$s(x,y) = \mathcal{F}^{-1}[H(u,v)Z(u,v)]$$
$$= \mathcal{F}^{-1}[H(u,v)F_i(u,v)] + \mathcal{F}^{-1}[H(u,v)F_r(u,v)]$$

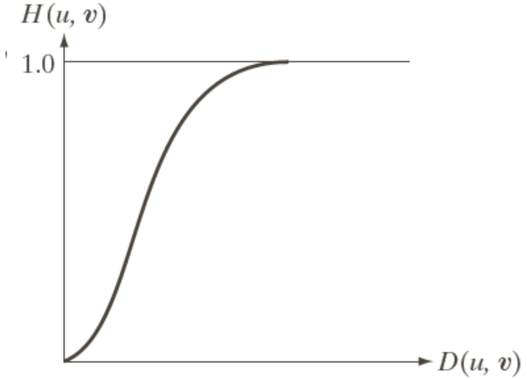
$$\boldsymbol{g(x,y) = e^{s(x,y)} = i_0(x,y)r_0(x,y)}$$

Where
$i_0(x,y) = \exp\{\mathcal{F}^{-1}[H(u,v)F_i(u,v)]\},\ \ r_0(x,y) = \exp\{\mathcal{F}^{-1}[H(u,v)F_r(u,v)]\}$



$$H(u,v) = (\gamma_H - \gamma_L)\left\{1 - e^{-c\left[\frac{D(u,v)}{D_0}\right]^2}\right\} + \gamma_L$$

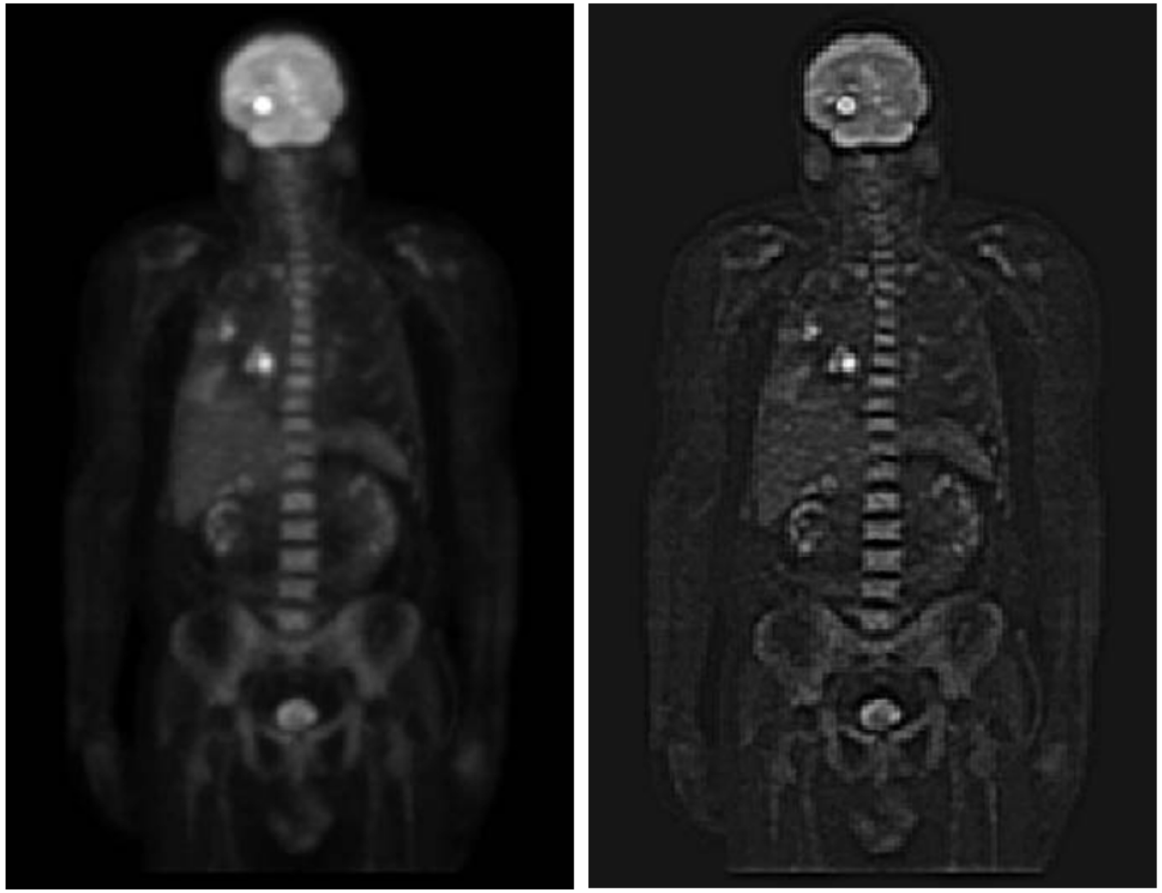




Homomorphic Filtering（同态滤波）

a b

FIGURE 4.60
(a) Full body PET scan. (b) Image enhanced using homomorphic filtering. (Original image courtesy of Dr. Michael E. Casey, CTI Pet Systems.)

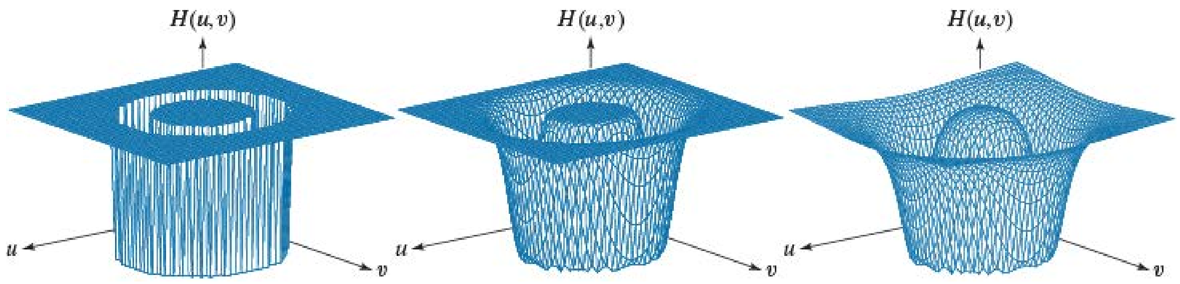


Selective Filtering

- **Bandreject(带阻) and Bandpass(带通) Filters**

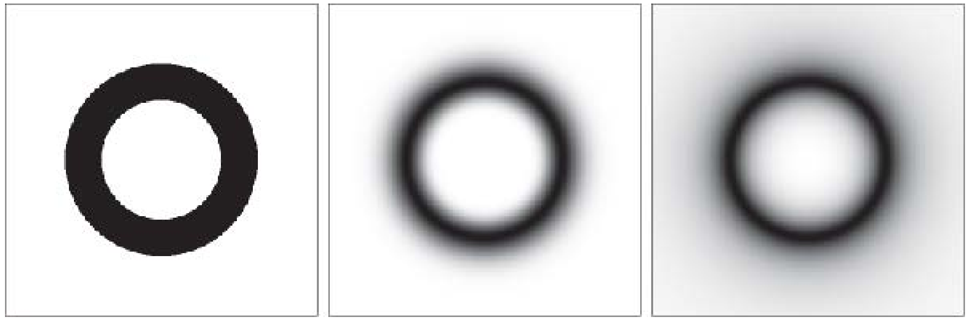
$$H_{\mathrm{BP}}(u,v) = 1 - H_{\mathrm{BR}}(u,v)$$

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 0 & \text{if } D_0 - \frac{W}{2} \le D \le D_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u,v) = \dfrac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2}\right]^{2n}}$	$H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$



a b c

FIGURE 4.62 Perspective plots of (a) ideal, (b) modified Gaussian, and (c) modified Butterworth (of order 1) bandreject filter transfer functions from Table 4.7. All transfer functions are of size 512×512 elements, with $C_0 = 128$ and $W = 60$.



a b c

FIGURE 4.63 (a) The ideal, (b) Gaussian, and (c) Butterworth bandpass transfer functions from Fig. 4.62, shown as images. (The thin border lines are not part of the image data.)

Notch Filter (陷波滤波器)

- Reject or pass frequencies in predefined neighborhood
- Symmetric about the origin for a zero-phase shift filters
- Design of NF
  - Highpass filter

$$H_{\mathrm{NR}}(u,v) = \prod_{k=1}^{Q} H_k(u,v) H_{-k}(u,v) \qquad H_{\mathrm{NP}}(u,v) = 1 - H_{\mathrm{NR}}(u,v)$$

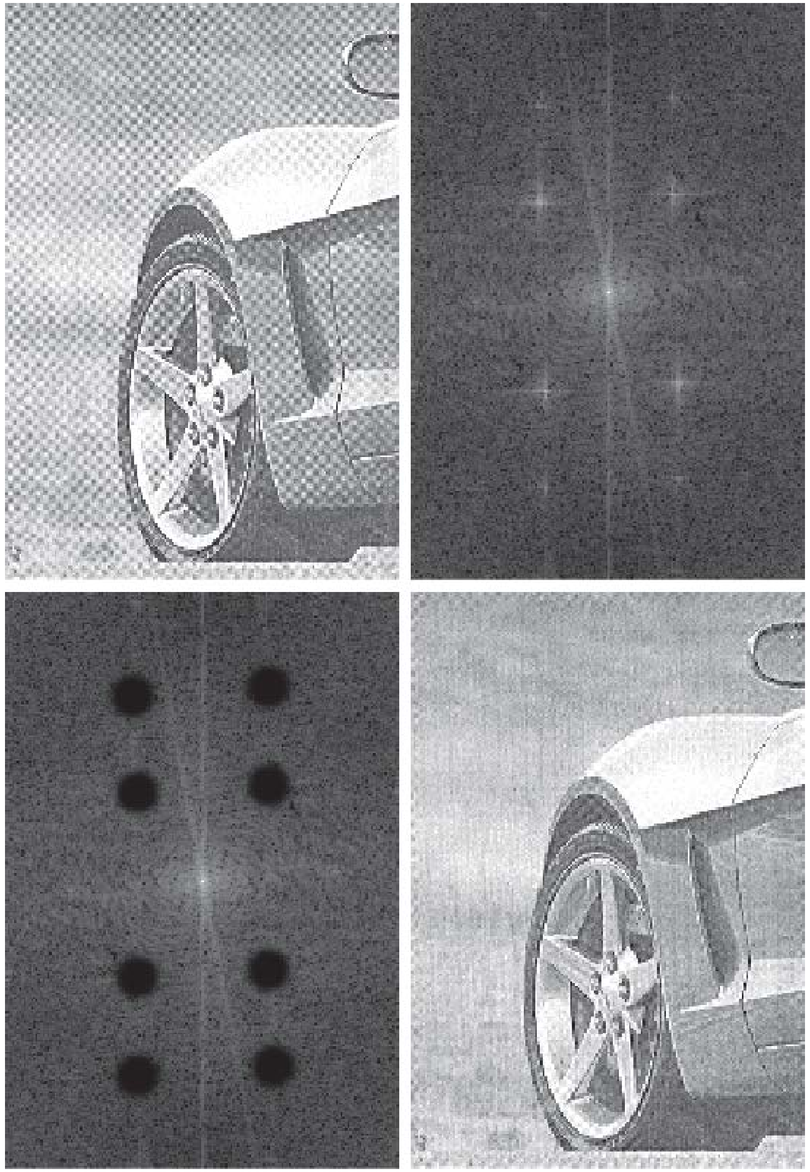
Where $H_k(u,v)$, $H_{-k}(u,v)$ are Highpass filters with center at (u_k, v_k) and (u_{-k}, v_{-k})

  - Selectively modify local regions of the DFT

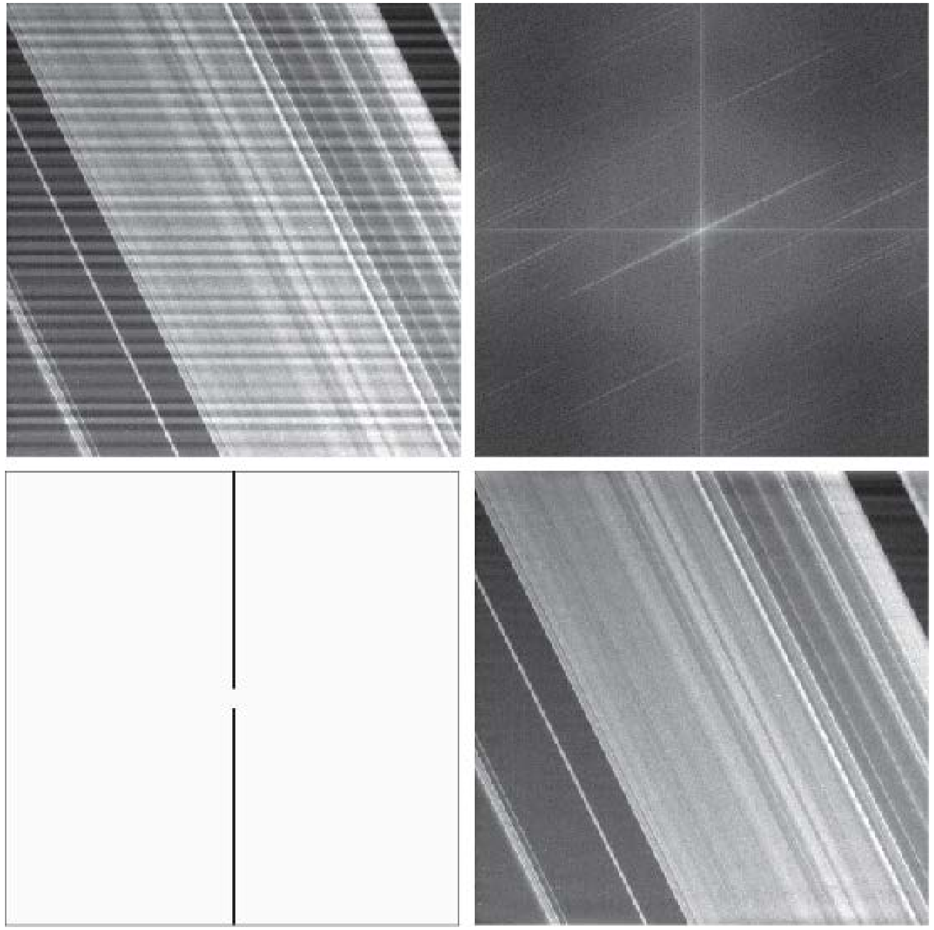
Notch Filter (陷波滤波器)

a b
c d

FIGURE 4.64
(a) Sampled newspaper image showing a moiré pattern.
(b) Spectrum.
(c) Fourier transform multiplied by a Butterworth notch reject filter transfer function.
(d) Filtered image.



Notch Filter (陷波滤波器)



a b
c d

FIGURE 4.65 (a) Image of Saturn rings showing nearly periodic interference. (b) Spectrum. (The bursts of energy in the vertical axis near the origin correspond to the interference pattern). (c) A vertical notch reject filter transfer function. (d) Result of filtering. (The thin black border in (c) is not part of the data.) (Original image courtesy of Dr. Robert A. West, NASA/JPL.)

a b

FIGURE 4.66 (a) Notch pass filter function used to isolate the vertical axis of the DFT of Fig. 4.65(a). (b) Spatial pattern obtained by computing the IDFT of (a).