Lecture 8 – Frequency Domain Transform （频率域变换）

This lecture will cover:

- Image transform
- 2D Fourier Transform (二维傅里叶变换)
  - Sampling Theorem (二维取样定理)
  - Discrete Fourier Transform (离散傅里叶变换)
  - Spectrum and Phase angle (频谱和相角)

Image Transform

- **The general form of image transform**

$$T(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)r(x,y,u,v) \qquad f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} T(u,v)s(x,y,u,v)$$

Where $r(x,y,u,v)$: forward transformation kernel

$s(x,y,u,v)$: inverse transformation kernel

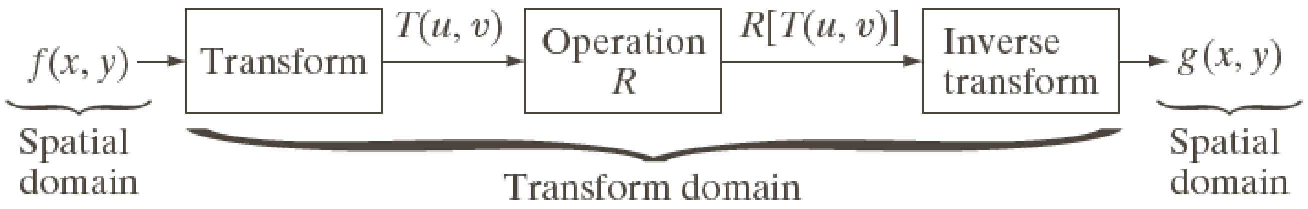
- **Matrix form**

  - **Forward Transform :** $\boldsymbol{T}=\boldsymbol{AFA}$
  - **Inverse Transform :** $\boldsymbol{F}=\boldsymbol{BTB}$ if $\boldsymbol{B}=\boldsymbol{A}^{-1}$; $\widehat{\boldsymbol{F}}=\boldsymbol{BAFAB}$ otherwise

Image Transform

- **Properties**
  - Separable : $r(x,y,u,v) = r_1(x,u)r_2(y,v)$
  - Symmetry : $r(x,y,u,v) = r_1(x,u)r_1(y,v)$

- **The general image operation in the linear transform domain**



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2D Continuous Fourier Transform

2D Fourier Transform

$$F(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + \nu z)} dt dz$$

2D Inverse Fourier Transform

$$f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, \nu) e^{j2\pi(\mu t + \nu z)} d\mu d\nu$$

- (t, z): spatial variables
- (μ, ν): frequency variables, defines the continuous frequency domain

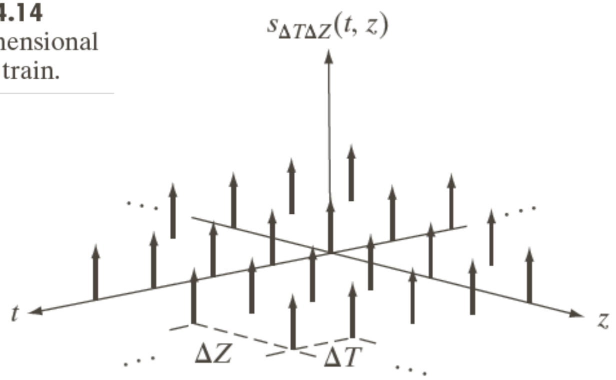
2D Sampling

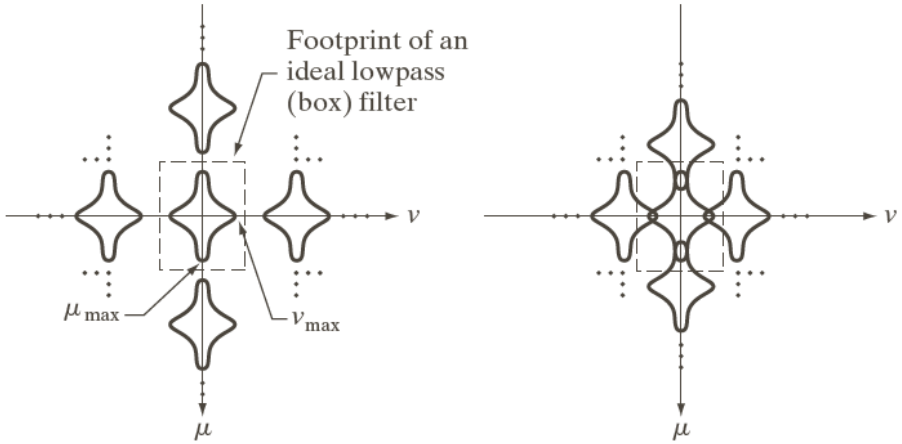
2D Sampling function (二维取样函数)

$$s_{\Delta T \Delta Z}(t,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

- $f(t,z)$ is band-limited (带限函数) if $F(\mu,\nu) = 0,\ |\mu| \geq \mu_{\max}$ and $|\nu| \geq \nu_{\max}$
- The sampling rate: $\frac{1}{\Delta T} > 2\mu_{\max},\ \frac{1}{\Delta Z} > 2\nu_{\max}$ (2D Sampling Theorem, 二维取样定理)

FIGURE 4.14 Two-dimensional impulse train.





a b

FIGURE 4.15 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.

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Spatial Aliasing（空间混淆）

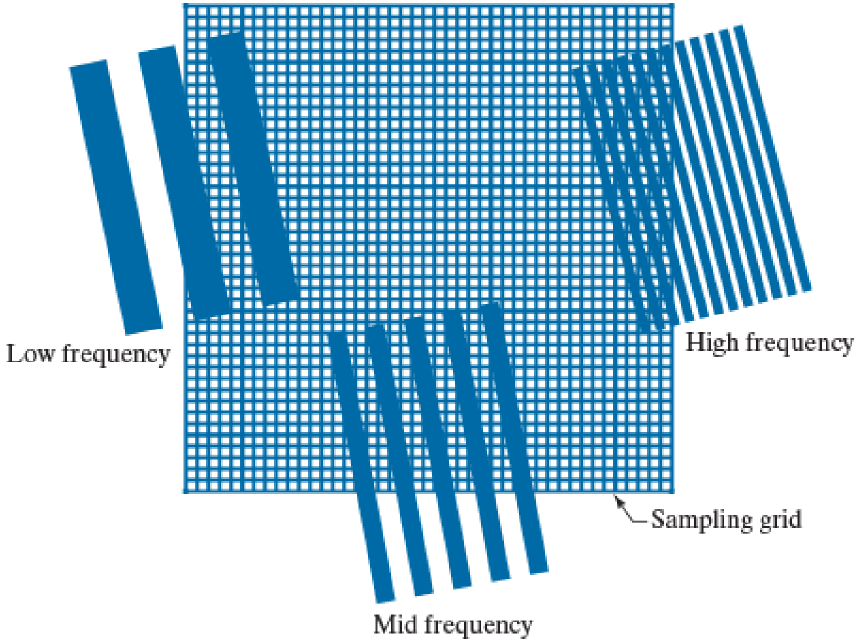
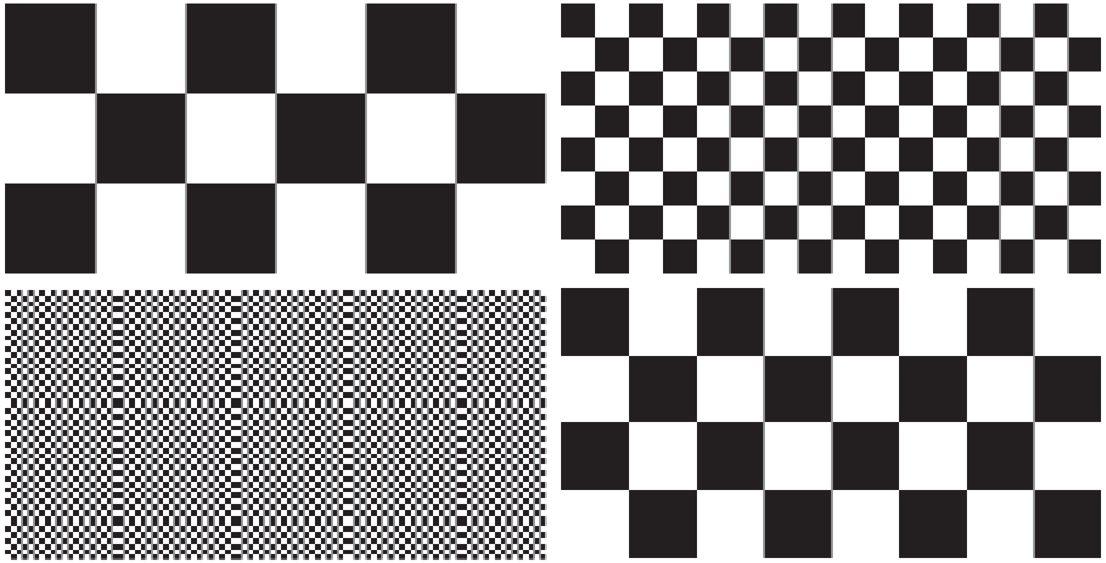


FIGURE 4.17 Various aliasing effects resulting from the interaction between the frequency of 2-D signals and the sampling rate used to digitize them. The regions outside the sampling grid are continuous and free of aliasing.

a b
c d

FIGURE 4.18 Aliasing. In (a) and (b) the squares are of sizes 16 and 6 pixels on the side. In (c) and (d) the squares are of sizes 0.95 and 0.48 pixels, respectively. Each small square in (c) is one pixel. Both (c) and (d) are aliased. Note how (d) masquerades as a "normal" image.



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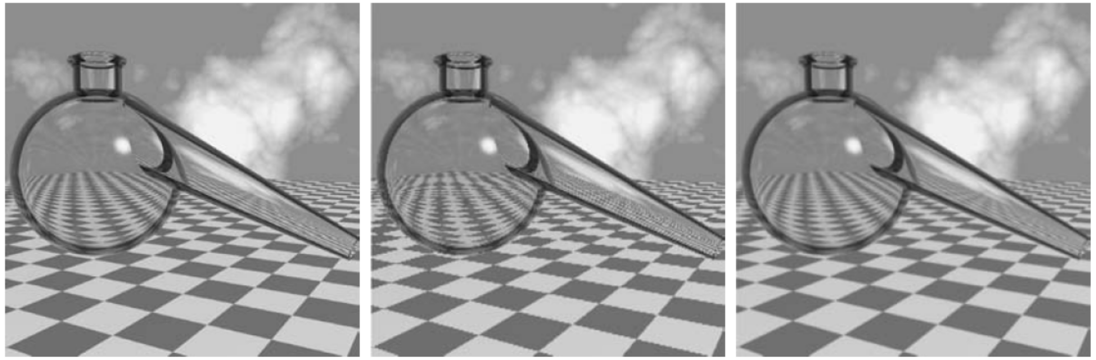
Zoom （图像缩放）

- **Image resampling and interpolation**

a b c

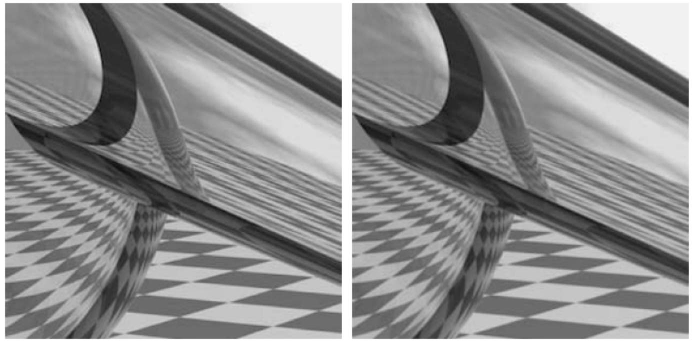
FIGURE 4.19 Illustration of aliasing on resampled natural images. (a) A digital image of size 772 × 548 pixels with visually negligible aliasing. (b) Result of resizing the image to 33% of its original size by pixel deletion and then restoring it to its original size by pixel replication. Aliasing is clearly visible. (c) Result of blurring the image in (a) with an averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

- **Illustration of jaggies (锯齿现象)**



a b c

FIGURE 4.18 Illustration of jaggies. (a) A 1024 × 1024 digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a 5 × 5 averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)



a b

FIGURE 4.19 Image zooming. (a) A 1024 × 1024 digital image generated by pixel replication from a 256 × 256 image extracted from the middle of Fig. 4.18(a). (b) Image generated using bi-linear interpolation, showing a significant reduction in jaggies.

Moire Patterns（莫尔模式）

- Arise routinely when sampling media or images with periodic components whose spacing is comparable to the spacing between samples

a b c
d e f

FIGURE 4.20 Examples of the moiré effect. These are vector drawings, not digitized patterns. Superimposing one pattern on the other is analogous to multiplying the patterns.

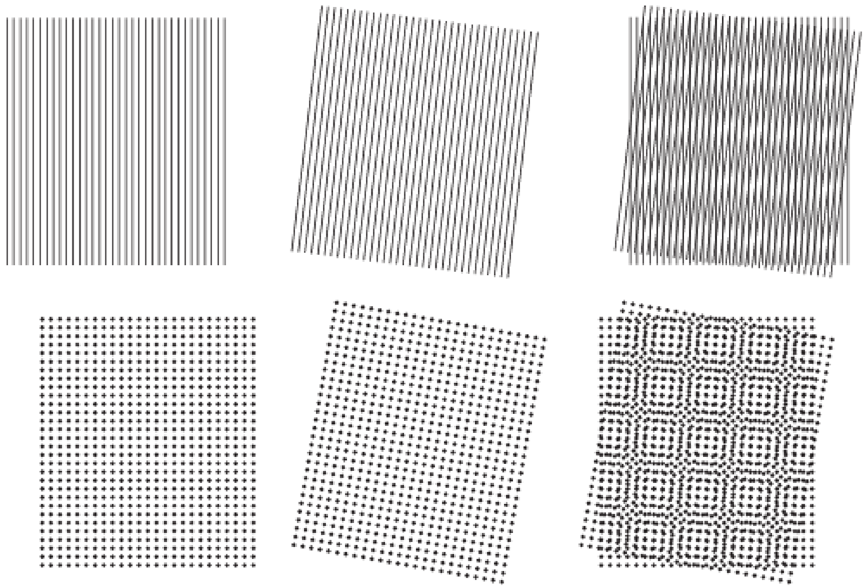
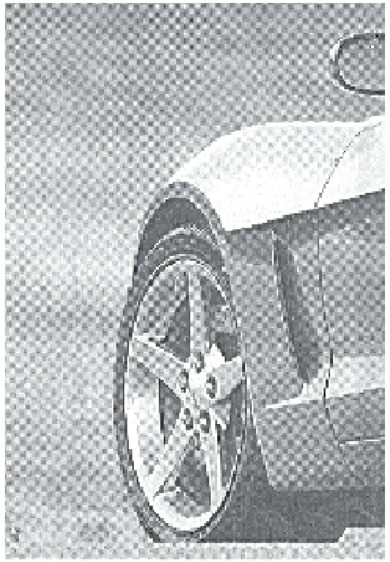


FIGURE 4.21 A newspaper image digitized at 75 dpi. Note the moiré-like pattern resulting from the interaction between the ±45° orientation of the half-tone dots and the north-south orientation of the sampling elements used to digitized the image.



Discrete Fourier Transform （离散傅里叶变换）

2D Discrete Fourier Transform (DFT)

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

2D Inverse Discrete Fourier Transform (IDFT)

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

- $f(x,y)$: M∗N input image

Properties of 2D DFT

- Spatial and frequency intervals （空间和频率间隔）
- Translation （平移）
- Periodicity（周期性）
- Rotation（旋转）
- Separability (可分性)
- Symmetry（对称性）
- Spectrum and Phase angle（频谱和相角）
- 2D Convolution theorem（卷积定理）

Spatial & Frequency Intervals (空间和频率间隔)

$f(x,y)$: a $M \times N$ digital image sampled from a countinous 2D function $f(t,z)$

$\Delta T, \Delta Z$: sampling interval in spatial domain

$\Delta u, \Delta v$: sampling interval in frequency domain

$$\Delta u = \frac{\frac{1}{\Delta T}}{M} = \frac{1}{M\Delta T} \qquad \Delta v = \frac{\frac{1}{\Delta Z}}{N} = \frac{1}{N\Delta Z}$$

Translation and Periodicity

- **Translation (平移)**

$$f(x,y)e^{j2\pi\left(\frac{u_0 x}{M}+\frac{v_0 y}{N}\right)} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi\left(\frac{u x_0}{M}+\frac{v y_0}{N}\right)}$$

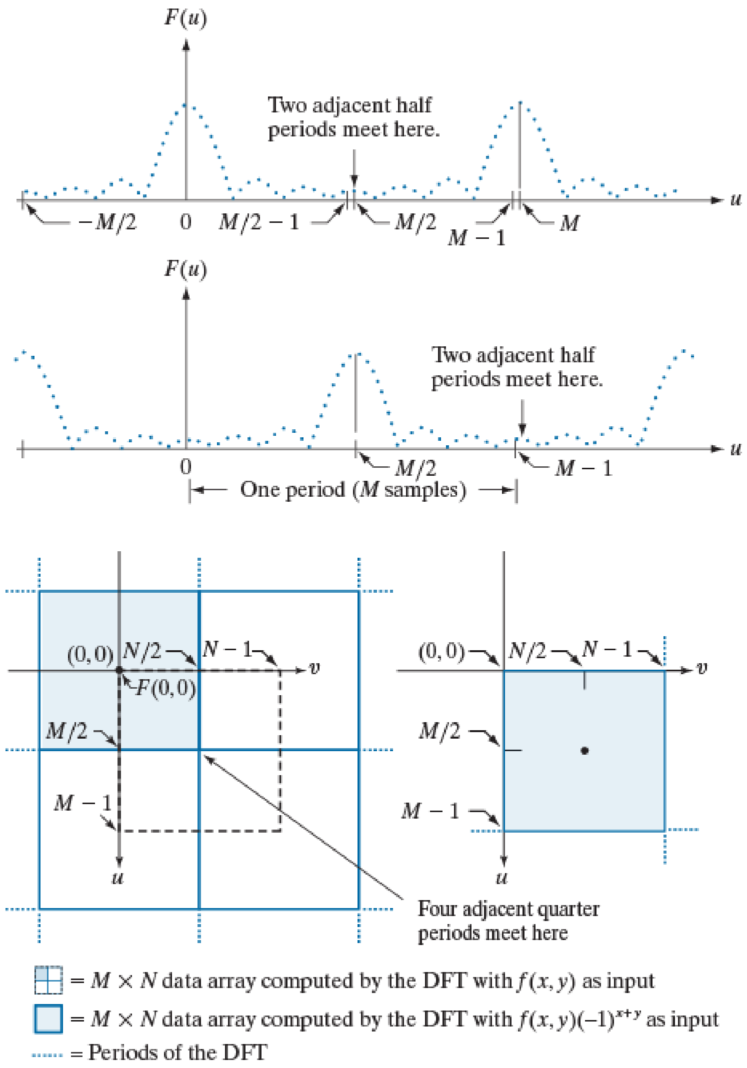
When $u_0=\frac{M}{2}, v_0=\frac{N}{2}$

$$F\left(u-\frac{M}{2}, v-\frac{N}{2}\right) \Leftrightarrow f(x,y)e^{j\pi(x+y)} = f(x,y)(-1)^{(x+y)}$$

- **Periodicity（周期性）**

$$f(x,y)=f(x+k_1M,y)=f(x,y+k_2N)=f(x+k_1M,y+k_2N)$$

$$F(u,v)=F(u+k_1M,v)=F(u,v+k_2N)=F(u+k_1M,v+k_2N)$$



a
b
c d

FIGURE 4.22 Centering the Fourier transform. (a) A 1-D DFT showing an infinite number of periods. (b) Shifted DFT obtained by multiplying $f(x)$ by $(-1)^x$ before computing $F(u)$. (c) A 2-D DFT showing an infinite number of periods. The area within the dashed rectangle is the data array, $F(u,v)$, obtained with Eq. (4-67) with an image $f(x,y)$ as the input. This array consists of four quarter periods. (d) Shifted array obtained by multiplying $f(x,y)$ by $(-1)^{x+y}$ before computing $F(u,v)$. The data now contains one complete, centered period, as in (b).

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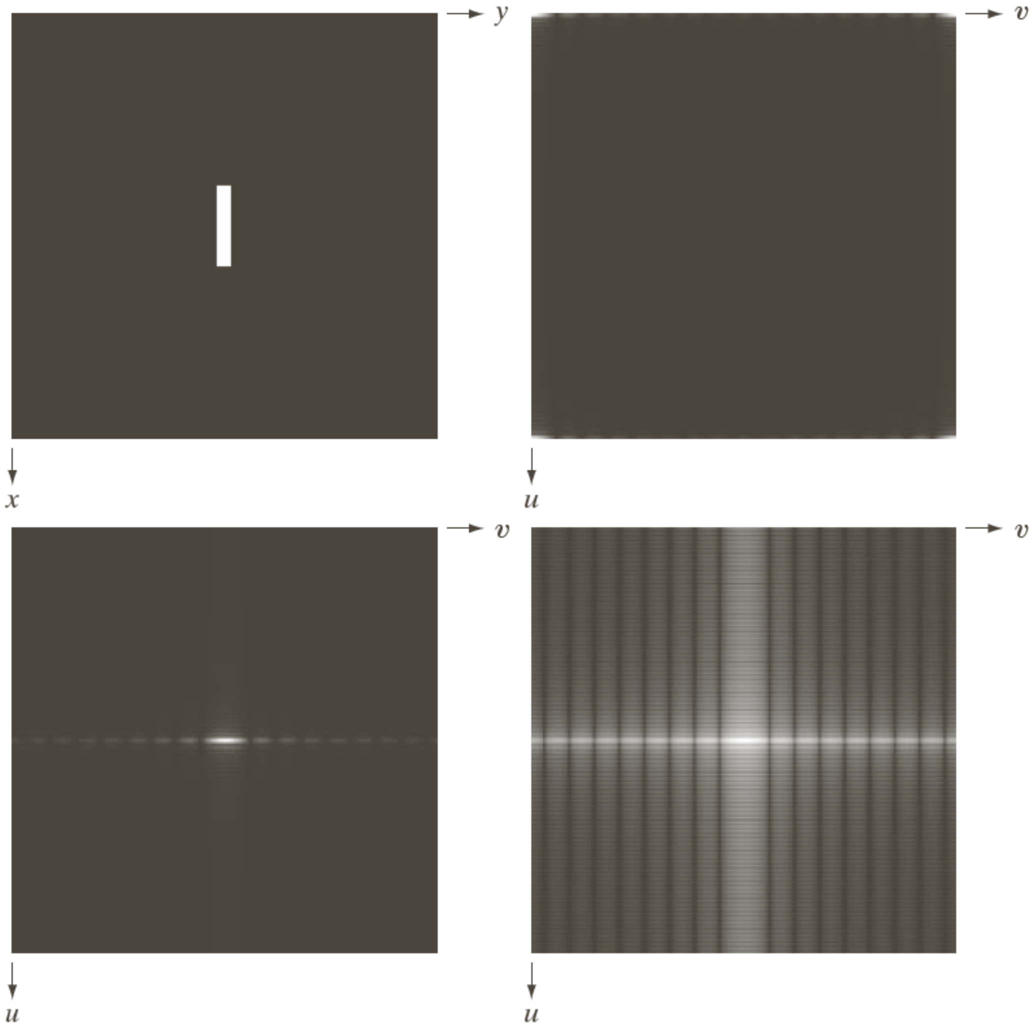
Frequency spectrum （频谱）

a b
c d

FIGURE 4.24 (a) Image. (b) Spectrum showing bright spots in the four corners. (c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.



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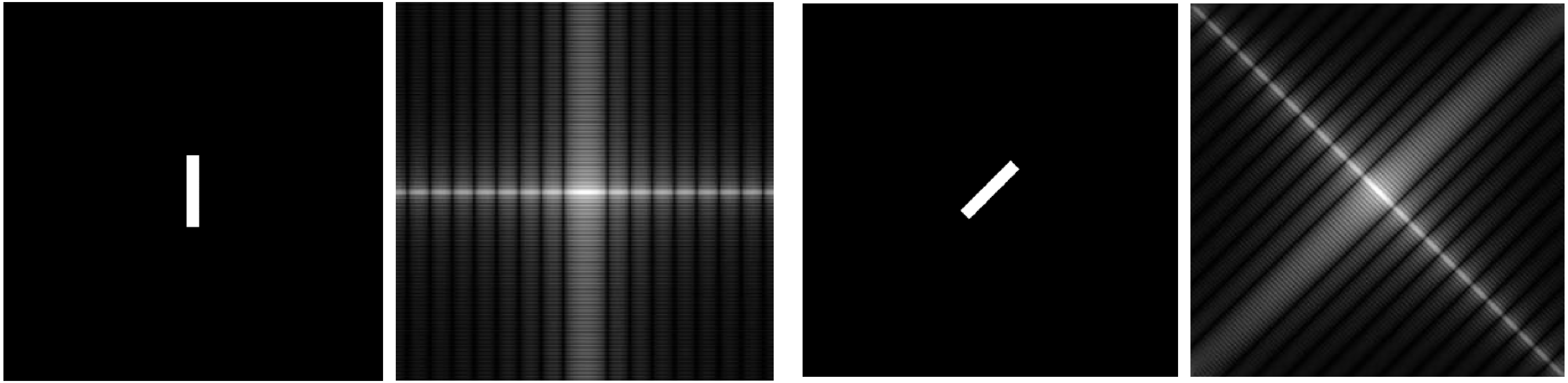
Rotation（旋转）

- **Rotation**

$$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$$

Where $x = r\cos\theta$, $y = r\sin\theta$, $u = \omega\cos\varphi$, $v = \omega\sin\varphi$



Separability（可分性）

- **2D DFT to 1D DFT**

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}=\sum_{x=0}^{M-1} e^{-j2\pi\frac{ux}{M}}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\frac{vy}{N}}=\mathcal{F}_x\{\mathcal{F}_y\{f(x,y)\}\}$$

- **Calculate IDFT by DFT**

$$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

$$MNf^*(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^*(u,v)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

Symmetry（对称性）

- **Even Function (偶函数)**

$$w_e(x,y) = w_e(-x,-y)$$

$$w_e(x,y) = w_e(M-x,N-y)$$

- **Odd Function (奇函数)**

$$w_o(x,y) = -w_o(-x,-y)$$

$$w_o(x,y) = -w_o(M-x,N-y)$$

- **Conjugate symmetric (共轭对称)**

$$F^*(u,v) = F(-u,-v)$$

$$F^*(u,v) = F(M-u,N-v)$$

- **Conjugate antisymmetric (共轭反对称)**

$$F^*(u,v) = -F(-u,-v)$$

$$F^*(u,v) = -F(M-u,N-v)$$

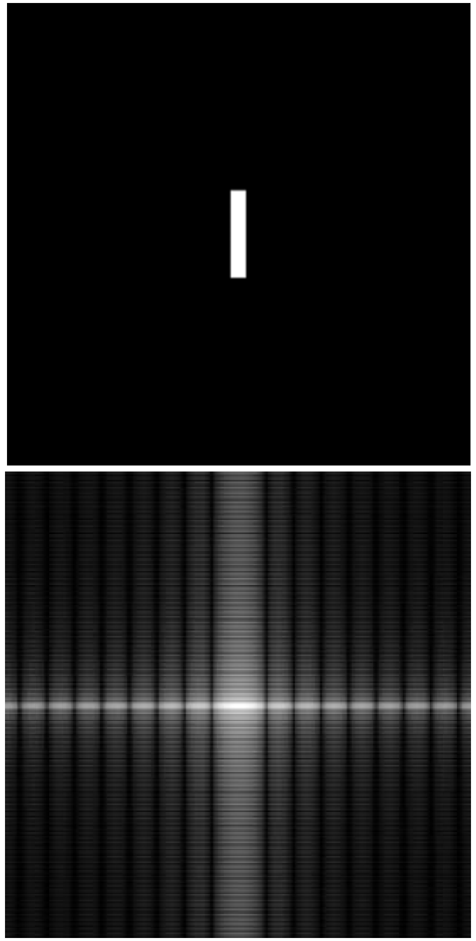
	Spatial Domain†		Frequency Domain†
1)	$f(x, y)$ real	$\Leftrightarrow$	$F^*(u, v) = F(-u, -v)$
2)	$f(x, y)$ imaginary	$\Leftrightarrow$	$F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow$	$R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow$	$R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow$	$F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow$	$F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow$	$F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow$	$F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow$	$F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow$	$F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow$	$F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow$	$F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow$	$F(u, v)$ complex and odd

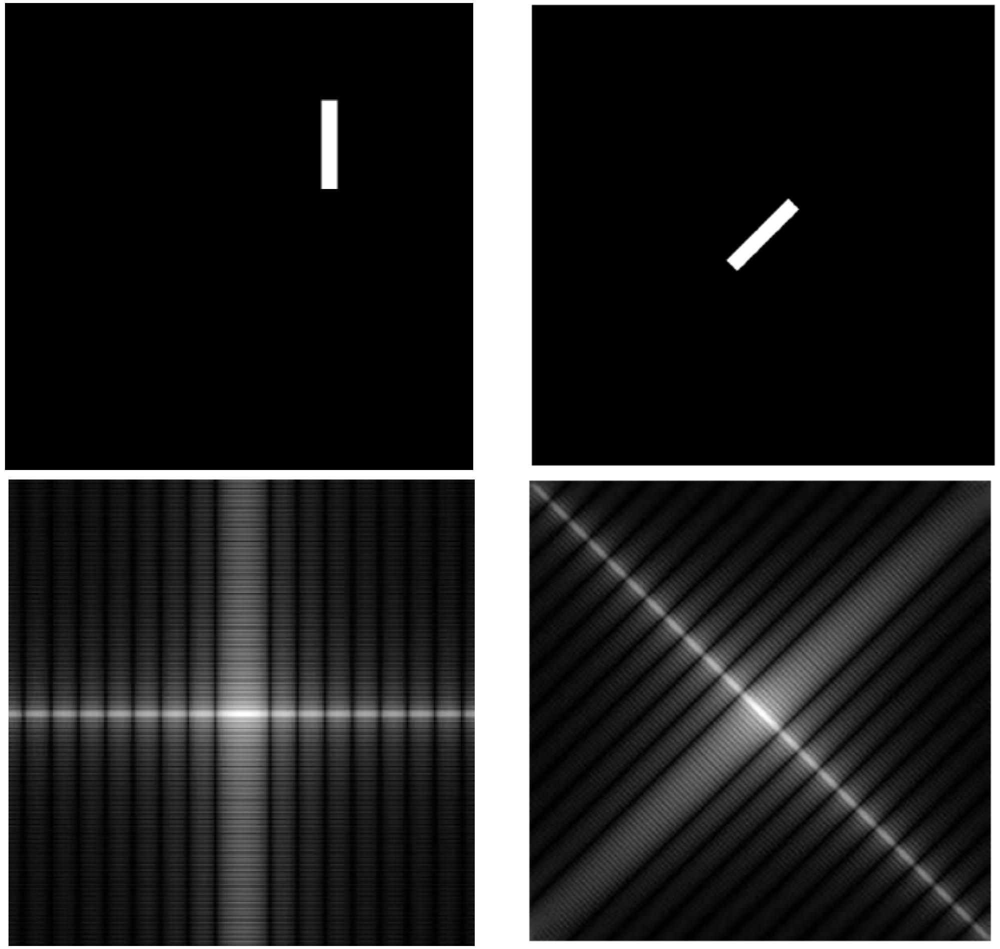
Spectrum and Phase angle（频谱和相角）

2D DFT in polar form: $F(u,v) = |F(u,v)|e^{-j\Phi(u,v)}$ **, then**

- **Fourier spectrum（频谱）:** $|F(u,v)| = [R^2(u,v) + I^2(u,v)]^{\frac{1}{2}}$
- **Phase angle（相角）:** $\Phi(u,v) = \arctan\frac{I(u,v)}{R(u,v)}$
- **Power spectrum(功率谱):** $P(u,v) = |F(u,v)|^2$
- **DC component(直流分量):** $F(0,0) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) = MN\overline{f(x,y)}$

Fourier Spectrum （频谱）

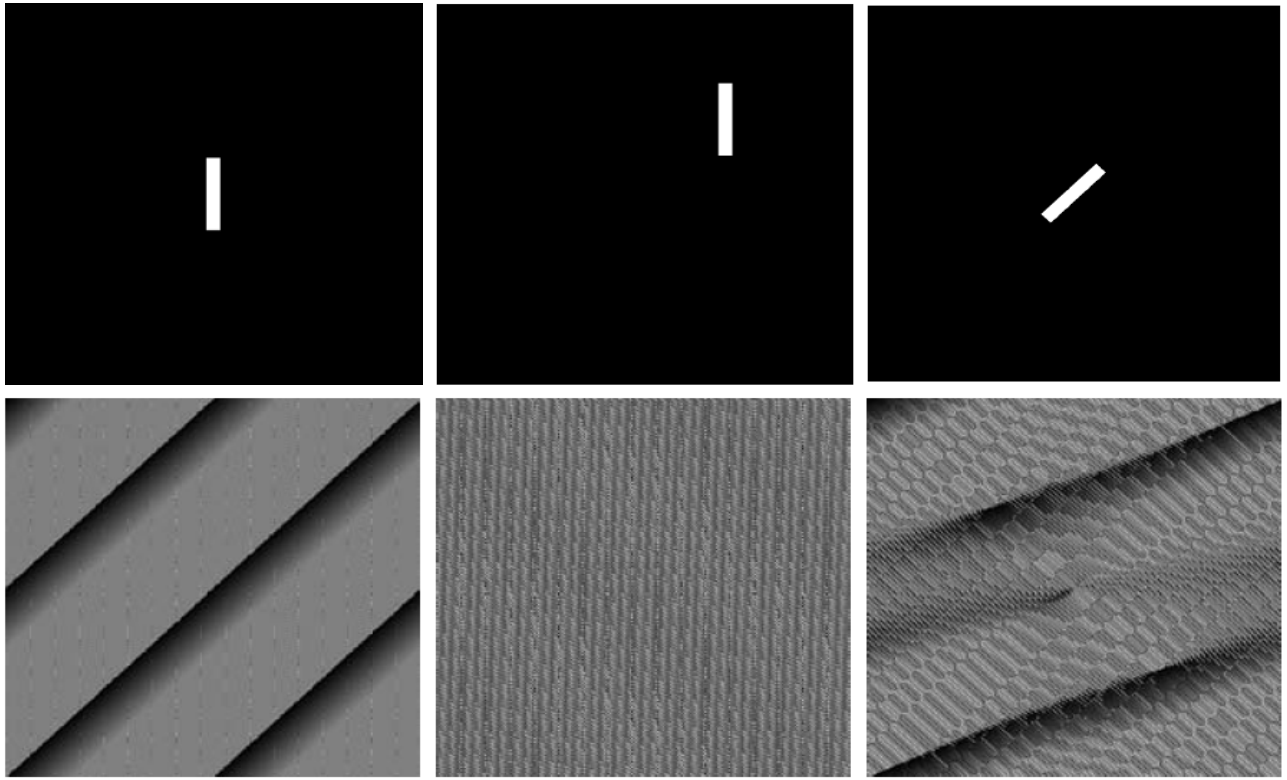




a b
c d

FIGURE 4.24
(a) The rectangle in Fig. 4.23(a) translated.
(b) Corresponding spectrum.
(c) Rotated rectangle.
(d) Corresponding spectrum.
The spectrum of the translated rectangle is identical to the spectrum of the original image in Fig. 4.23(a).

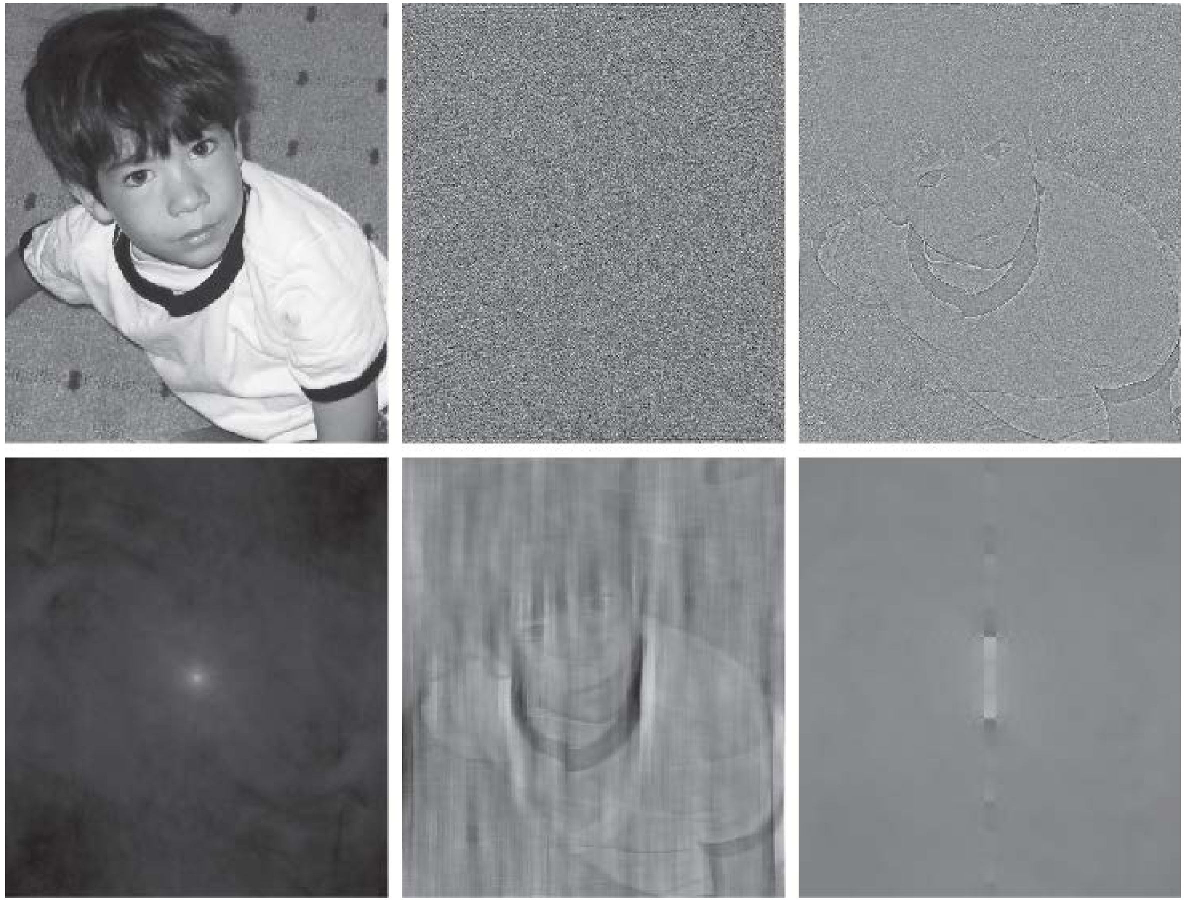
Phase angle（相角）



a b c

FIGURE 4.25 Phase angle images of (a) centered, (b) translated, and (c) rotated rectangles.

Spectrum and Phase angle （频谱和相角）



a b c
d e f

FIGURE 4.26 (a) Boy image. (b) Phase angle. (c) Boy image reconstructed using only its phase angle (all shape features are there, but the intensity information is missing because the spectrum was not used in the reconstruction). (d) Boy image reconstructed using only its spectrum. (e) Boy image reconstructed using its phase angle and the spectrum of the rectangle in Fig. 4.23(a). (f) Rectangle image reconstructed using its phase and the spectrum of the boy's image.

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2D Convolution theorem（卷积定理）

- **Convolution theorem**

$$f(x,y) \star h(x,y) \Leftrightarrow F(u,v)H(u,v) \text{ or } f(x,y)\,h(x,y) \Leftrightarrow F(u,v) \star H(u,v)$$

- **Zero padding (零填充)**

$$f_p(x,y) = \begin{cases} f(x,y), & 0 \le x \le A-1, 0 \le y \le B-1 \\ 0, & A \le x \le P, B \le y \le Q \end{cases}$$

$$h_p(x,y) = \begin{cases} h(x,y), & 0 \le x \le C-1, 0 \le y \le D-1 \\ 0, & C \le x \le P, D \le y \le Q \end{cases}$$

Where $f(x,y)$: $A \times B$ image; $h(x,y)$: $C \times D$ image; $P \ge A + C - 1$; $Q \ge B + D - 1$

Summary of DFT

TABLE 4.3
Summary of DFT definitions and corresponding expressions.

Name	Expression(s)
1) Discrete Fourier transform (DFT) of $f(x,y)$	$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
2) Inverse discrete Fourier transform (IDFT) of $F(u,v)$	$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
3) Spectrum	$\lvert F(u,v)\rvert=\left[R^2(u,v)+I^2(u,v)\right]^{1/2} \quad R=\text{Real}(F);\ I=\text{Imag}(F)$
4) Phase angle	$\phi(u,v)=\tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
5) Polar representation	$F(u,v)=\lvert F(u,v)\rvert e^{j\phi(u,v)}$
6) Power spectrum	$P(u,v)=\lvert F(u,v)\rvert^2$
7) Average value	$\bar{f}=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)=\frac{1}{MN}F(0,0)$
8) Periodicity (k_1 and k_2 are integers)	$F(u,v)=F(u+k_1M,v)=F(u,v+k_2N)$ $=F(u+k_1,v+k_2N)$ $f(x,y)=f(x+k_1M,y)=f(x,y+k_2N)$ $=f(x+k_1M,y+k_2N)$
9) Convolution	$(f\star h)(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n)$
10) Correlation	$(f\,☆\,h)(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f^*(m,n)h(x+m,y+n)$
11) Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.
12) Obtaining the IDFT using a DFT algorithm	$MNf^*(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^*(u,v)e^{-j2\pi(ux/M+vy/N)}$ This equation indicates that inputting $F^*(u,v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x,y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.

TABLE 4.4
Summmary of DFT pairs. The closed-form expressions in 12 and 13 are valid only for continuous variables. They can be used with discrete variables by sampling the continuous expressions.

Name	DFT Pairs
1) Symmetry properties	See Table 4.1
2) Linearity	$af_1(x,y)+bf_2(x,y)\Leftrightarrow aF_1(u,v)+bF_2(u,v)$
3) Translation (general)	$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)}\Leftrightarrow F(u-u_0,v-v_0)$ $f(x-x_0,y-y_0)\Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$
4) Translation to center of the frequency rectangle, $(M/2,N/2)$	$f(x,y)(-1)^{x+y}\Leftrightarrow F(u-M/2,v-N/2)$ $f(x-M/2,y-N/2)\Leftrightarrow F(u,v)(-1)^{u+v}$
5) Rotation	$f(r,\theta+\theta_0)\Leftrightarrow F(\omega,\varphi+\theta_0)$ $r=\sqrt{x^2+y^2} \quad \theta=\tan^{-1}(y/x) \quad \omega=\sqrt{u^2+v^2} \quad \varphi=\tan^{-1}(v/u)$
6) Convolution theorem†	$f\star h)(x,y)\Leftrightarrow (F\bullet H)(u,v)$ $(f\bullet h)(x,y)\Leftrightarrow (1/MN)[(F\star H)(u,v)]$
7) Correlation theorem†	$(f\,☆\,h)(x,y)\Leftrightarrow (F^*\bullet H)(u,v)$ $(f^*\bullet h)(x,y)\Leftrightarrow (1/MN)[(F\,☆\,H)(u,v)]$
8) Discrete unit impulse	$\delta(x,y)\Leftrightarrow 1$ $1\Leftrightarrow MN\delta(u,v)$
9) Rectangle	$\text{rec}[a,b]\Leftrightarrow ab\frac{\sin(\pi ua)}{(\pi ua)}\frac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
10) Sine	$\sin(2\pi u_0x/M+2\pi v_0y/N)\Leftrightarrow\frac{jMN}{2}[\delta(u+u_0,v+v_0)-\delta(u-u_0,v-v_0)]$
11) Cosine	$\cos(2\pi u_0x/M+2\pi v_0y/N)\Leftrightarrow\frac{1}{2}[\delta(u+u_0,v+v_0)+\delta(u-u_0,v-v_0)]$
The following Fourier transform pairs are derivable only for continuous variables, denoted as before by t and z for spatial variables and by μ and ν for frequency variables. These results can be used for DFT work by sampling the continuous forms.	
12) Differentiation (the expressions on the right assume that $f(\pm\infty,\pm\infty)=0$.	$\left(\frac{\partial}{\partial t}\right)^m\left(\frac{\partial}{\partial z}\right)^n f(t,z)\Leftrightarrow (j2\pi\mu)^m(j2\pi\nu)^n F(\mu,\nu)$ $\frac{\partial^m f(t,z)}{\partial t^m}\Leftrightarrow (j2\pi\mu)^m F(\mu,\nu);\ \frac{\partial^n f(t,z)}{\partial z^m}\Leftrightarrow (j2\pi\nu)^n F(\mu,\nu)$
13) Gaussian	$A2\pi\sigma^2e^{-2\pi^2\sigma^2(t^2+z^2)}\Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2}$ (A is a constant)

† Assumes that $f(x,y)$ and $h(x,y)$ have been properly padded. Convolution is associative, commutative, and distributive. Correlation is distributive (see Table 3.5). The products are elementwise products (see Section 2.6).