Lecture 13 – Image Reconstruction （图像重建）

This lecture will cover:

– Reconstruction modalities（重建模式）

– Reconstruction from projection（投影重建算法）

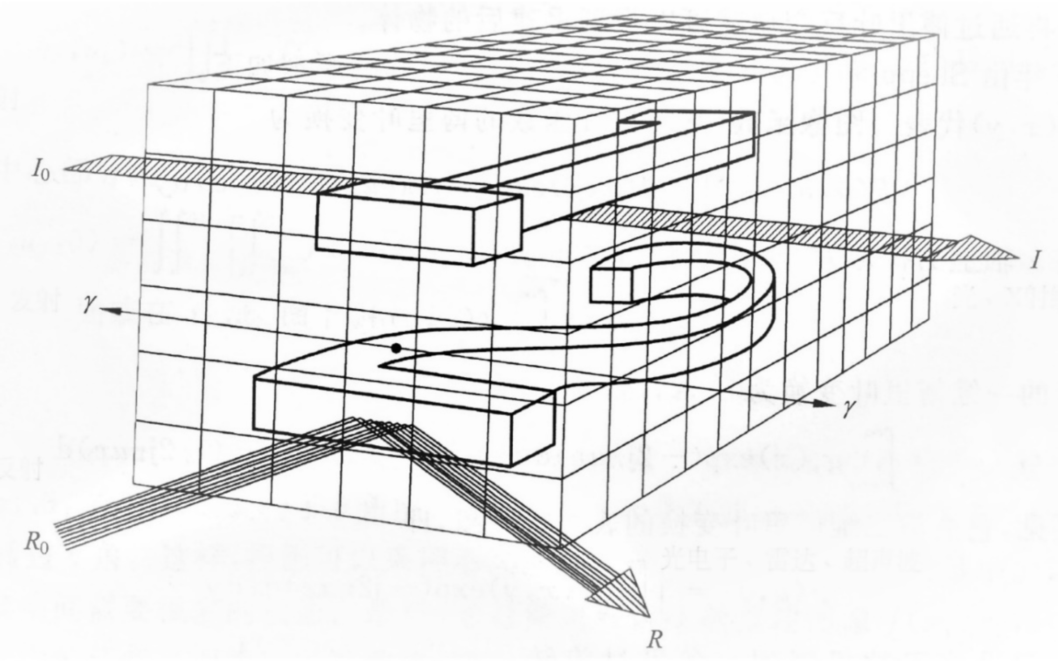
- Computed Tomography (计算机断层成像)
- Radon transform（雷登变换）
- The Fourier-Slice Theorem（傅里叶切片定理）
- Parallel-Beam Filtered Backprojections（平行射线束滤波反投影）
- Fan-Beam Filtered Backprojections（扇形射线束滤波反投影）

– Reflection imaging

- Time of flight
- Born Approximation and Inverse theory（玻恩近似与反演理论）

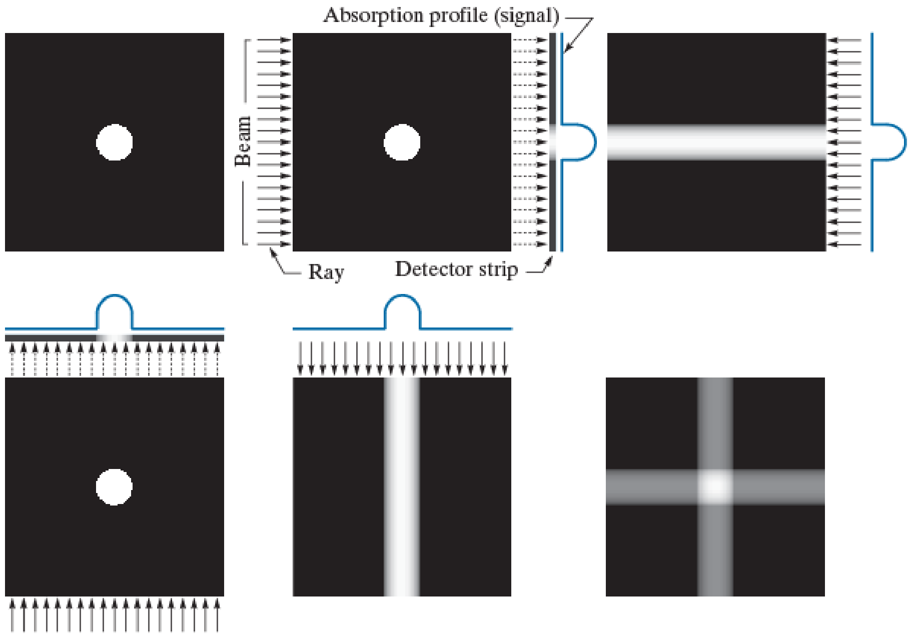
2D Reconstruction Modalities

- **Transmission -** back-projection
  - Computed tomography
- **Emission -** physically located
  - Gamma camera: Anger position network
  - PET: Annihilation coincidence detection
  - MRI: gradient coils
- **Reflection**
  - B-mode ultrasound: time of flight
  - Wave equation based reconstruction : migration & inverse problem



Back Projection

- To explore the cross-sectional of a 3D region
- Project the 1D signal back in the opposite direction from beam incidence.



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d e f

FIGURE 5.32 (a) Flat region with a single object. (b) Parallel beam, detector strip, and profile of sensed 1-D absorption signal. (c) Result of back-projecting the absorption profile. (d) Beam and detectors rotated by 90°. (e) Backprojection. (f) The sum of (c) and (e), intensity-scaled. The intensity where the backprojections intersect is twice the intensity of the individual back-projections.

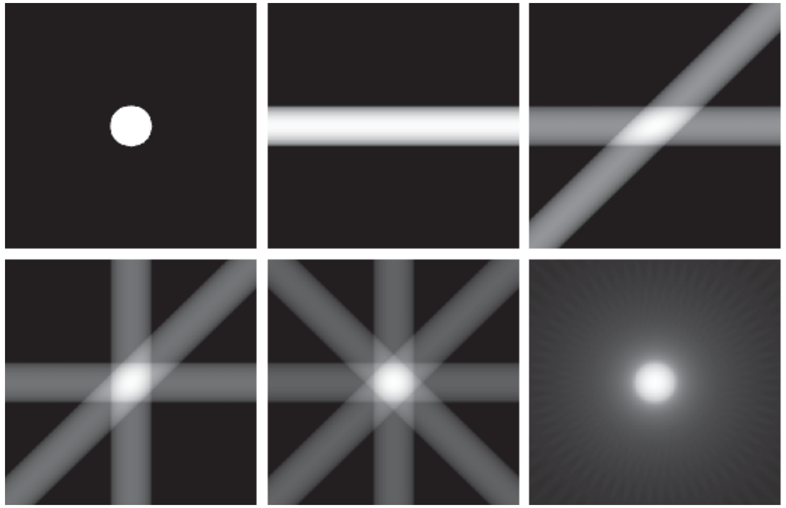

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FIGURE 5.33 (a) Same as Fig. 5.32(a). (b)-(e) Reconstruction using 1, 2, 3, and 4 back-projections 45° apart. (f) Reconstruction with 32 backprojections 5.625° apart (note the blurring).

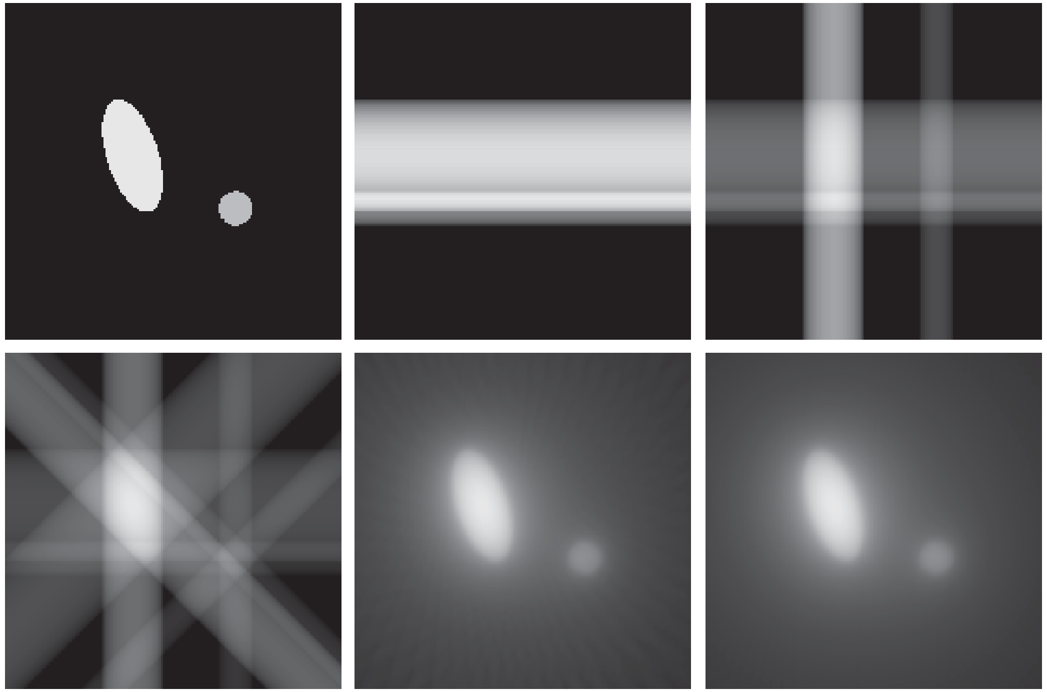

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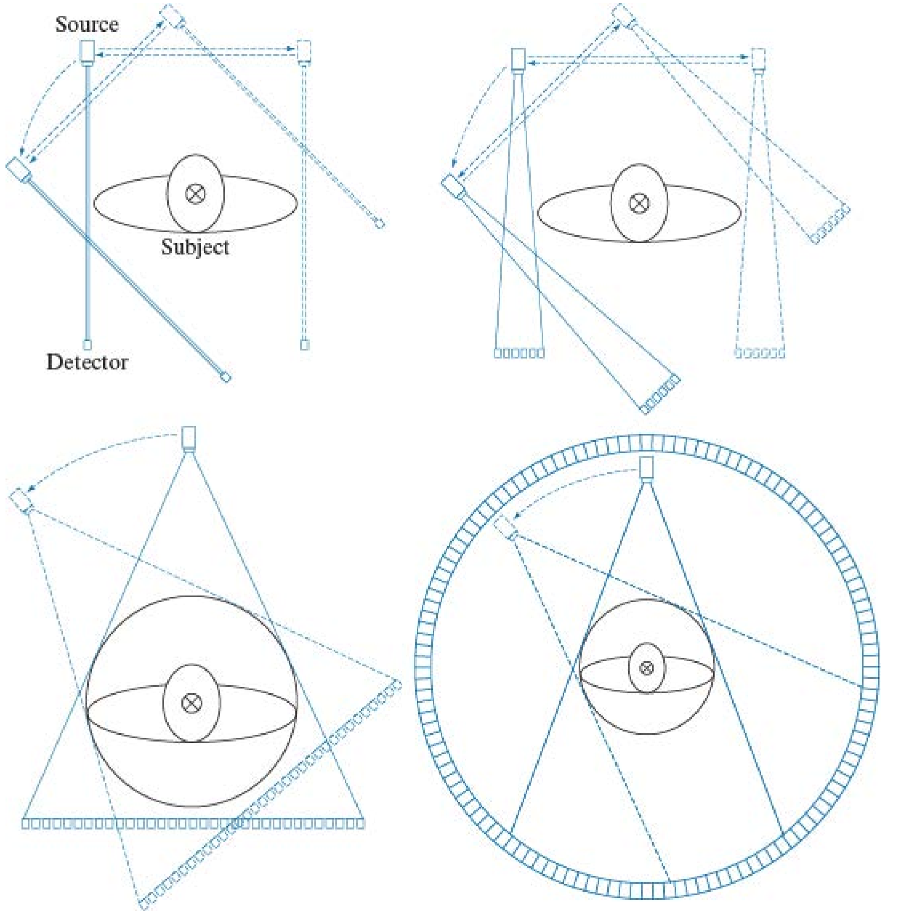
FIGURE 5.34 (a) Two objects with different absorption characteristics. (b)–(d) Reconstruction using 1, 2, and 4 backprojections, 45° apart. (e) Reconstruction with 32 backprojections, 5.625° apart. (f) Reconstruction with 64 backprojections, 2.8125° apart.

Generation of Computed Tomography

- **G1 –** A “pencil” X-ray beam and a single detector; translation-rotation
- **G2 –** fan beam with multiple detectors
- **G3 –** a bank of detectors which can cover the entire FOV;
- **G4 –** circular ring of detectors; only source rotates;
- **G5 –** electron beam CT; electron beams controlled electromagnetically to avoid all mechanical motion;
- **G6 –** helical CT; continuously rotate through 360 degree.
- **G7 –** multislice CT



a b
c d

FIGURE 5.35 Four generations of CT scanners. The dotted arrow lines indicate incremental linear motion. The dotted arrow arcs indicate incremental rotation. The cross-mark on the subject’s head indicates linear motion perpendicular to the plane of the paper. The double arrows in (a) and (b) indicate that the source/detector unit is translated and then brought back into its original position.

Radon Transform (雷登变换)

Normal representation for a line:

$$x\cos\theta + y\sin\theta = \rho$$

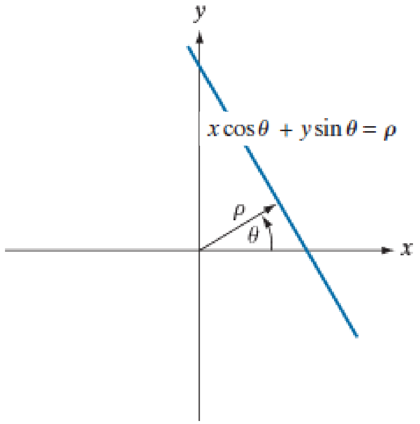


FIGURE 5.36 Normal representation of a line.

The projection of $f(x,y)$ along an arbitrary line in the xy-plane (Radon transform):

$$g(\rho,\theta) = \sum_{x=0}^{M-1}\sum_{y-0}^{N-1} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho)$$

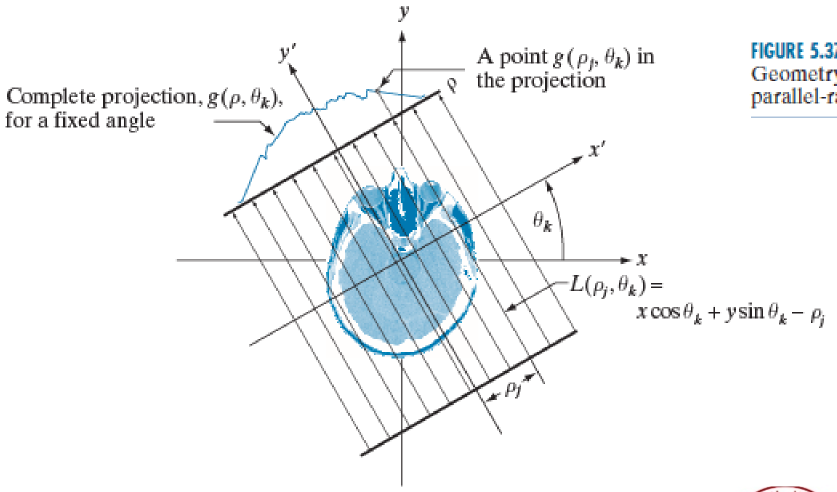
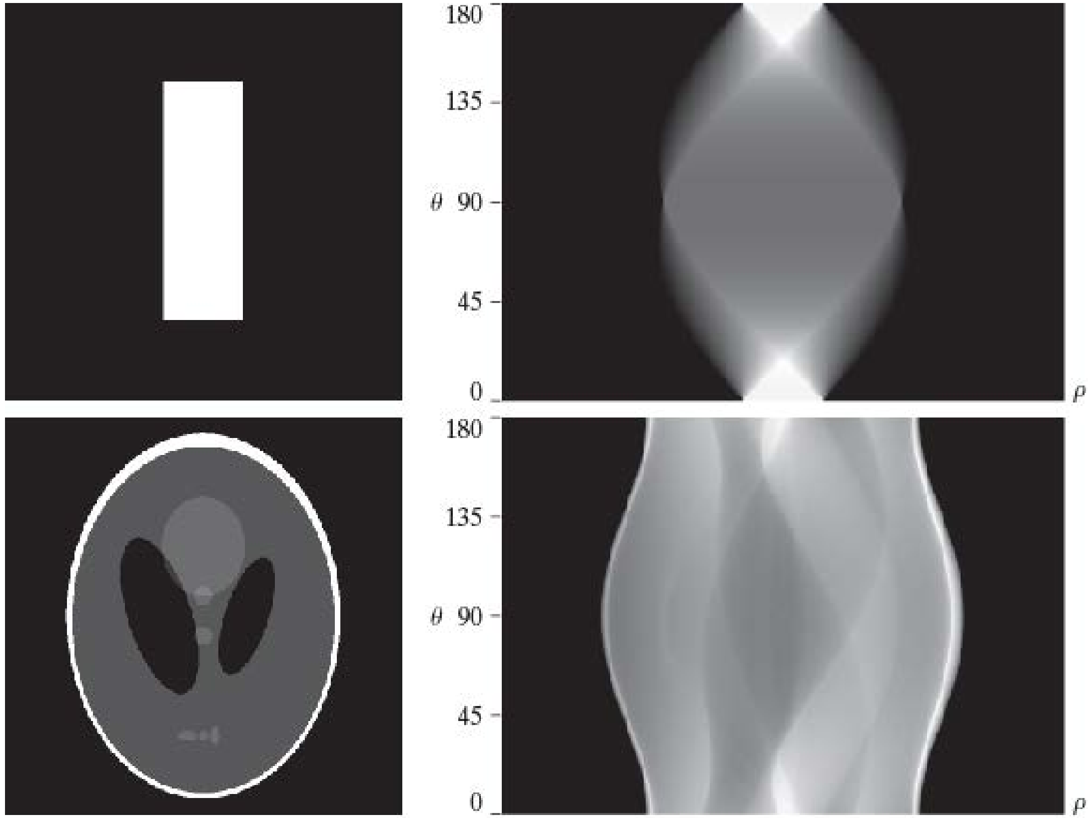


FIGURE 5.37 Geometry of a parallel-ray beam.

Sinogram (正弦图)

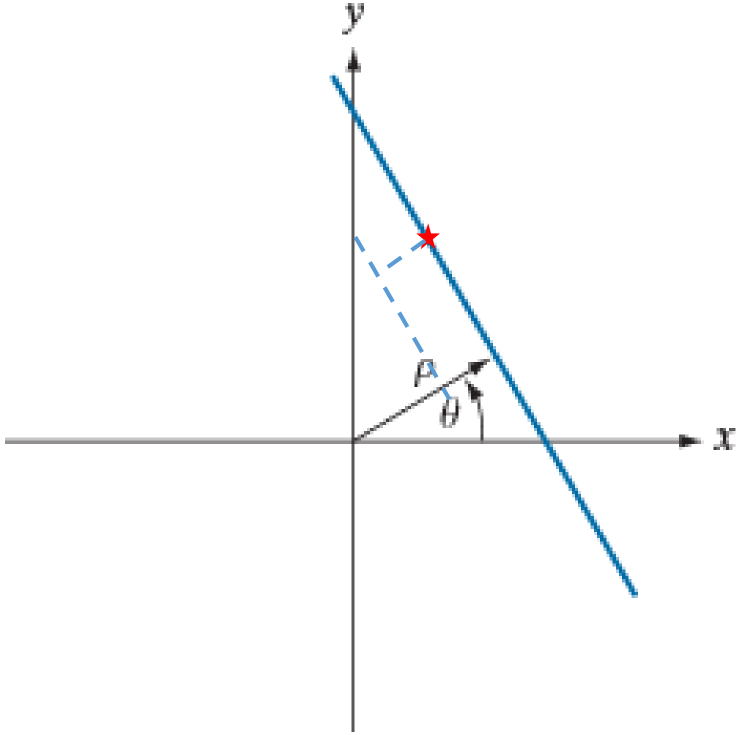
- The Radon transform $g(\rho, \theta)$ is displayed as an image with ρ and θ as rectilinear coordinates



a b
c d

FIGURE 5.39 Two images and their sinograms (Radon transforms). Each row of a sinogram is a projection along the corresponding angle on the vertical axis. (Note that the horizontal axis of the sinograms are values of ρ.) Image (c) is called the *Shepp-Logan phantom*. In its original form, the contrast of the phantom is quite low. It is shown enhanced here to facilitate viewing.

Back Projection from Radon Transform



For a fixed value of rotation θ_k:

$$f_{\theta_k}(x,y) = g(\rho,\theta_k) = g(x\cos\theta_k + y\sin\theta_k\,,\theta_k)$$

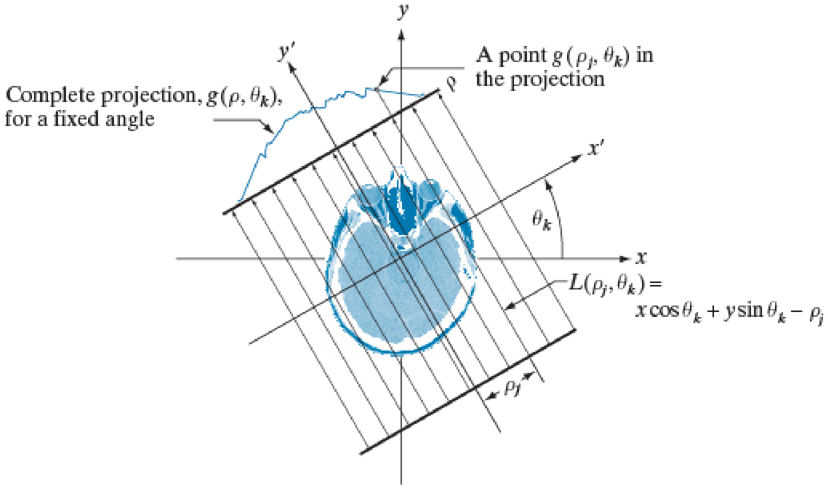
Then a single backprojection obtained at an angle θ :

$$f_\theta(x,y) = g(x\cos\theta + y\sin\theta\,,\theta)$$

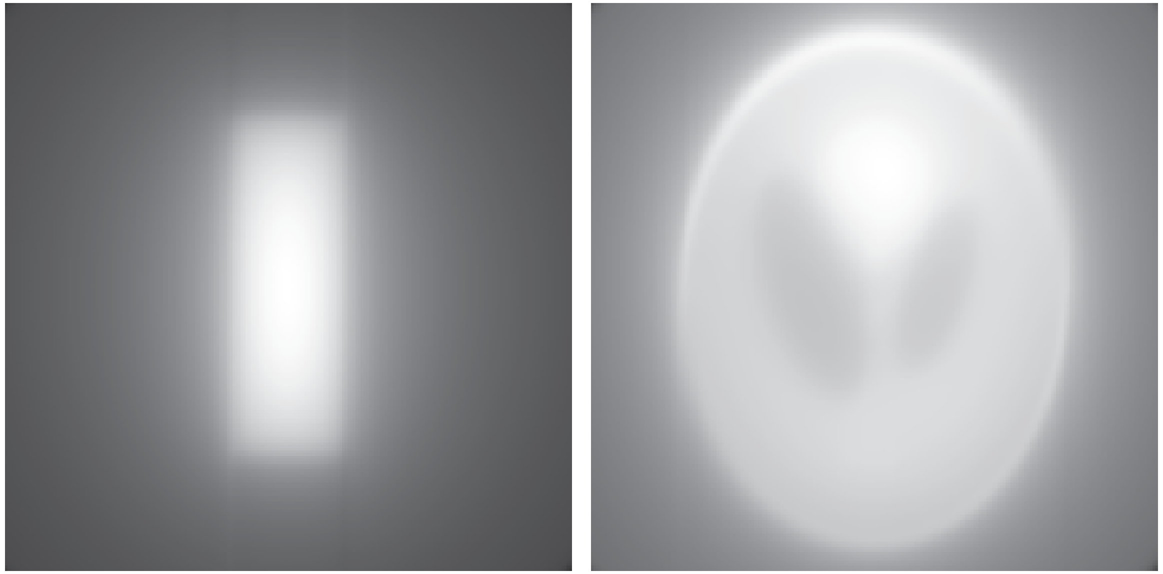
Where $g(\rho,\theta)$ is the projection value.

The final image by summing over all the back-projected images

$$f(x,y) = \sum_{\theta=0}^{\pi} f_\theta(x,y)$$



Back Projection from Radon Transform



a b

FIGURE 5.40 Backprojections of the sinograms in Fig. 5.39.

The Fourier-Slice Theorem (傅里叶切片定理)

The 1D FT of a projection with respect of ρ:

$$G(\omega,\theta) = \int_{-\infty}^{\infty} g(\rho,\theta)e^{-j2\pi\omega\rho}d\rho$$

where projection $g(\rho,\theta)$ is

$$g(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho)dxdy$$

then

$$G(\omega,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi\omega(x\cos\theta + y\sin\theta)}dxdy$$

$$= \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy\right]_{u=\omega\cos\theta; v=\omega\sin\theta}$$

Therefore $G(\omega,\theta) = F(\omega\cos\theta, \omega\sin\theta)$

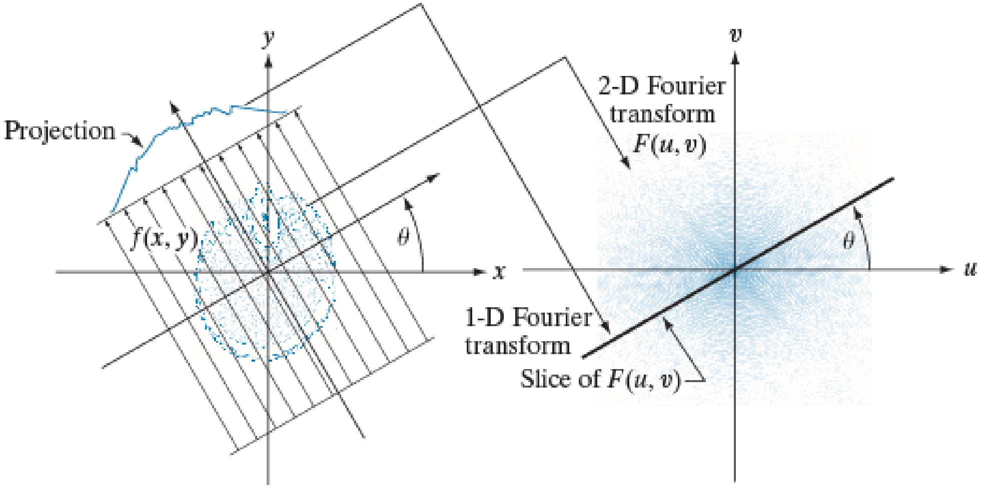


FIGURE 5.41 Illustration of the Fourier-slice theorem. The 1-D Fourier transform of a projection is a slice of the 2-D Fourier transform of the region from which the projection was obtained. Note the correspondence of the angle θ in the two figures.

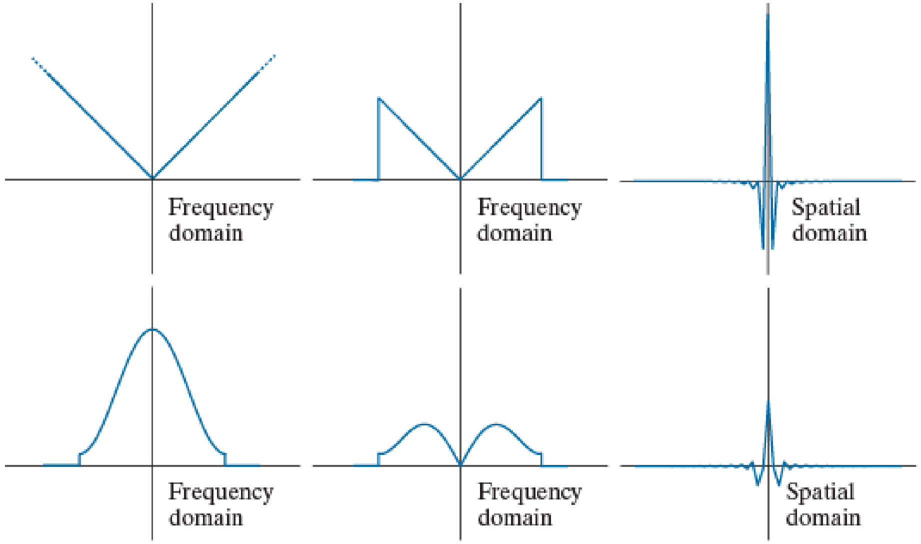
Parallel-Beam Filtered Backprojections

The 2D IFT of $F(u,v)$ with Fourier-slice theorem:

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)e^{j2\pi(ux+vy)}dudv$$

$$=\int_{0}^{2\pi}\int_{0}^{\infty}G(\omega,\theta)e^{j2\pi\omega(x\cos\theta+y\sin\theta)}\omega d\omega d\theta$$

$$=\int_{0}^{\pi}\left[\int_{-\infty}^{\infty}|\omega|G(\omega,\theta)e^{j2\pi\omega\rho}d\omega\right]_{\rho=x\cos\theta+y\sin\theta}d\theta$$



a b c
d e f

FIGURE 5.42 (a) Frequency domain ramp filter transfer function. (b) Function after band-limiting it with a box filter. (c) Spatial domain representation. (d) Hamming windowing function. (e) Windowed ramp filter, formed as the product of (b) and (d). (f) Spatial representation of the product. (Note the decrease in ringing.)

Convolution backprojection

$$f(x,y)=\int_{0}^{\pi}[s(\rho)\bigstar g(\rho,\theta)\,]_{\rho=x\cos\theta+y\sin\theta}d\theta$$

Where $s(\rho)=\mathrm{IFT}(|\omega|)$, $g(\rho,\theta)=\mathrm{IFT}[G(\omega,\theta)]$

Parallel-Beam Filtered Backprojections

The complete, backprojected image $f(x, y)$ is obtained as follows:

1. Compute the 1-D Fourier transform of each projection.
2. Multiply each 1-D Fourier transform by the filter transfer function $|\omega|$ which, as explained above, has been multiplied by a suitable (e.g., hamming) window.
3. Obtain the inverse 1-D Fourier transform of each resulting filtered transform.
4. Integrate (sum) all the 1-D inverse transforms from Step 3.

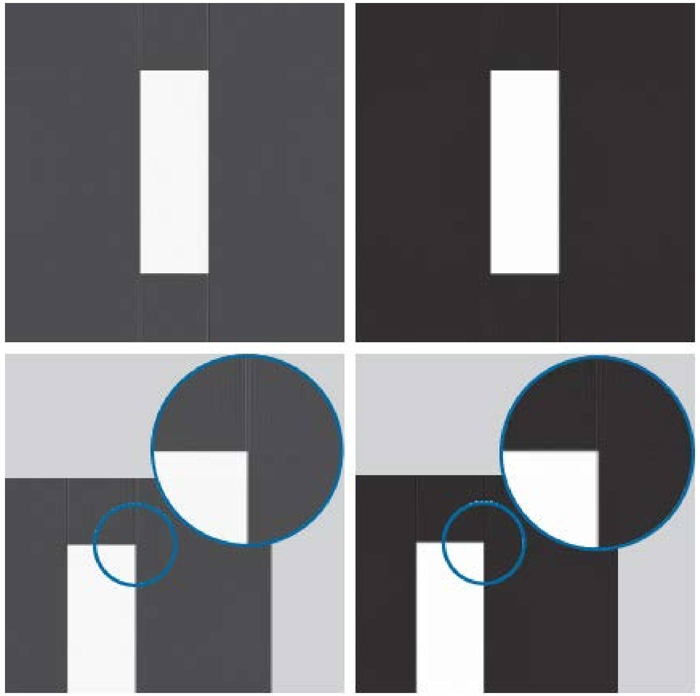


FIGURE 5.43 Filtered backprojections of the rectangle using (a) a ramp filter, and (b) a Hamming windowed ramp filter. The second row shows zoomed details of the images in the first row. Compare with Fig. 5.40(a).

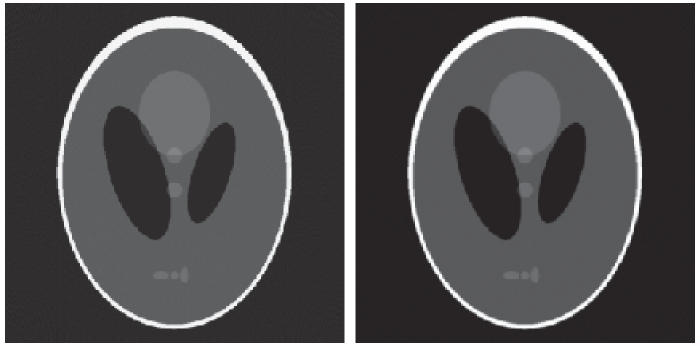


FIGURE 5.44 Filtered backprojections of the head phantom using (a) a ramp filter, and (b) a Hamming windowed ramp filter. Compare with Fig. 5.40(b)

Fan-Beam Filtered Backprojections

Fundamental fan-beam reconstruction based on filtered backprojection:

$$f(r,\varphi) = \int_0^{2\pi} \frac{1}{R^2}\left[\int_{-\alpha_m}^{\alpha_m} q(\alpha,\beta)h(\alpha'-\alpha)\mathrm{d}\alpha\right]\mathrm{d}\beta$$

Where $h(\alpha) = \frac{1}{2}\left(\frac{\alpha}{\sin\alpha}\right)^2 s(\alpha)$, $q(\alpha,\beta) = p(\alpha,\beta)D\cos\alpha$, $p(\alpha,\beta) = g(\rho,\theta) = g(D\sin\alpha\,,\alpha+\beta)$

FIGURE 5.45 Basic fan-beam geometry. The line passing through the center of the source and the origin (assumed here to be the center of rotation of the source) is called the *center ray*.

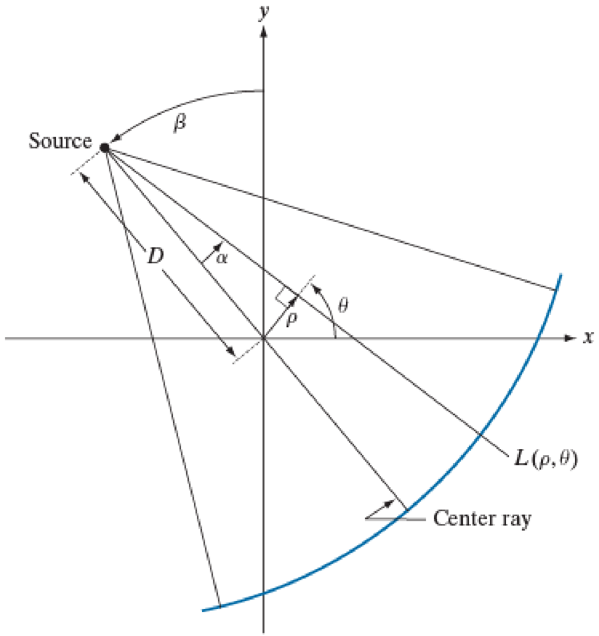


FIGURE 5.46 Maximum value of α needed to encompass a region of interest.

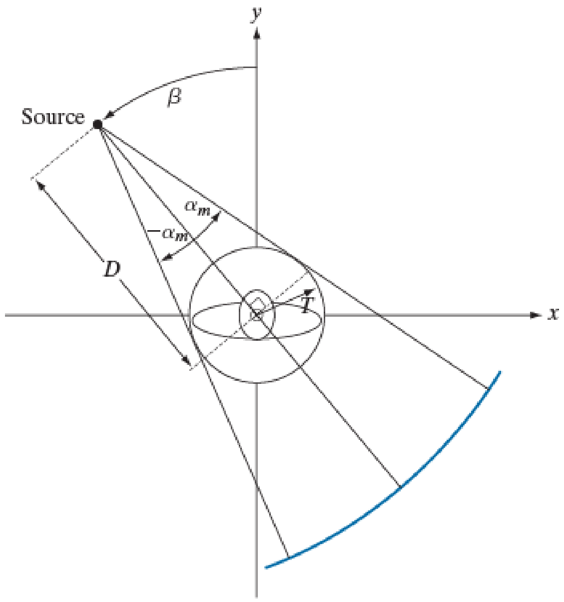
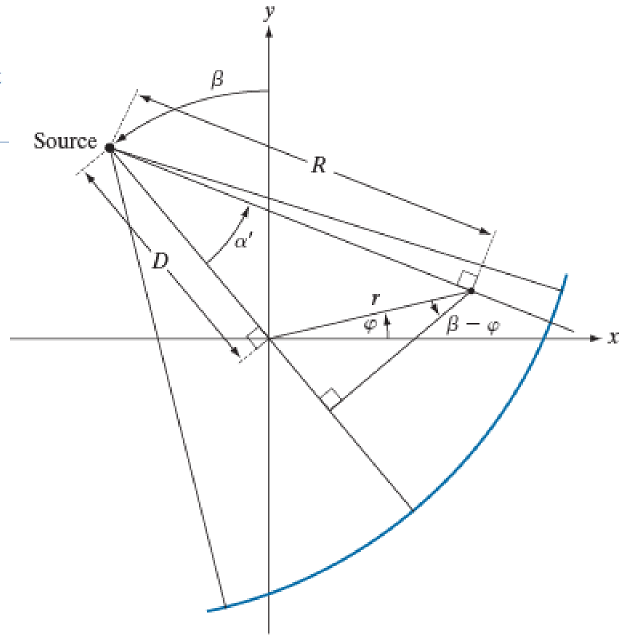


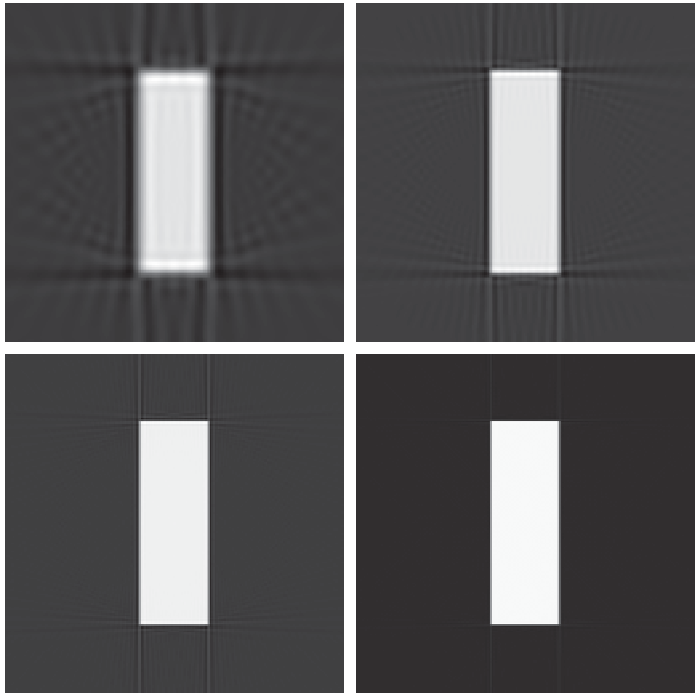
FIGURE 5.47 Polar representation of an arbitrary point on a ray of a fan beam.



Fan-Beam Filtered Backprojections

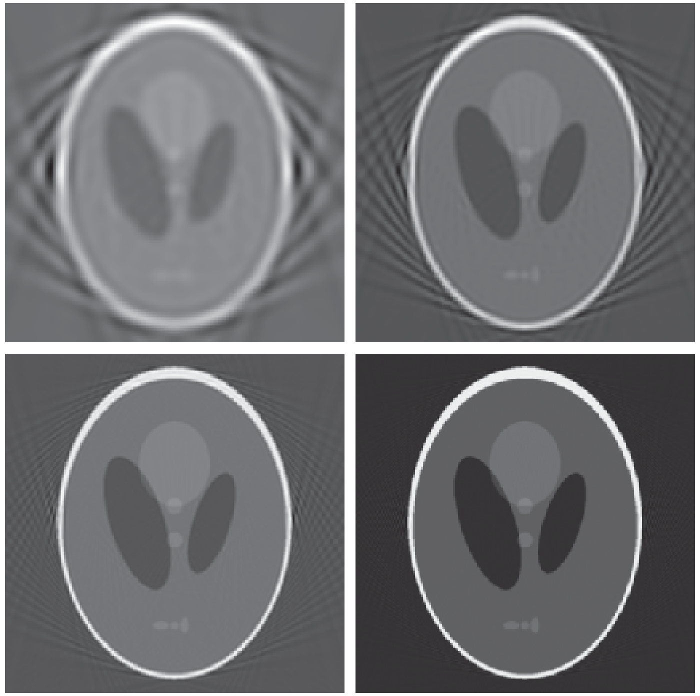
a b
c d

FIGURE 5.48 Reconstruction of the rectangle image from filtered fan backprojections. (a) 1° increments of α and β. (b) 0.5° increments. (c) 0.25° increments. (d) 0.125° increments. Compare (d) with Fig. 5.43(b).



a b
c d

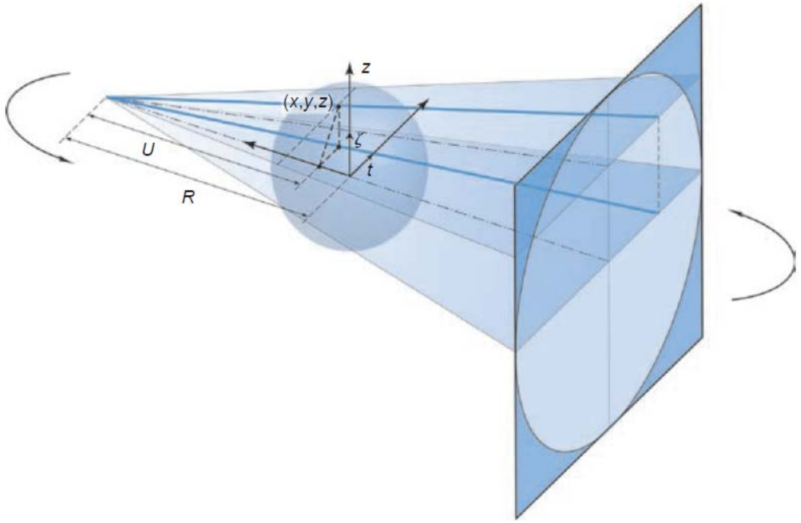
FIGURE 5.49 Reconstruction of the head phantom image from filtered fan backprojections. (a) 1° increments of α and β. (b) 0.5° increments. (c) 0.25° increments. (d) 0.125° increments. Compare (d) with Fig. 5.44(b).



Other reconstruction algorithms

- **Data parallel re-sorting (数据重排算法):** synthetic parallel projection

- **Volumetric/multi-slice CT (容积CT)**
  - Circular cone-beam reconstruction.
  - Helical cone-beam reconstruction
  - Iterative reconstruction: Bayesian approach
    - ✓ Maximum-likelihood (ML)
    - ✓ Maximum-a-posteriori probability (MAP)



B-mode ultrasound imaging

- Reflection from the interface between different tissues
- Using time of flight (*t*) to determine the distance (*d)* and locate the structures

$$t = \frac{2 * d}{V}$$

- Use gray level to indicate the amplitude (B mode)

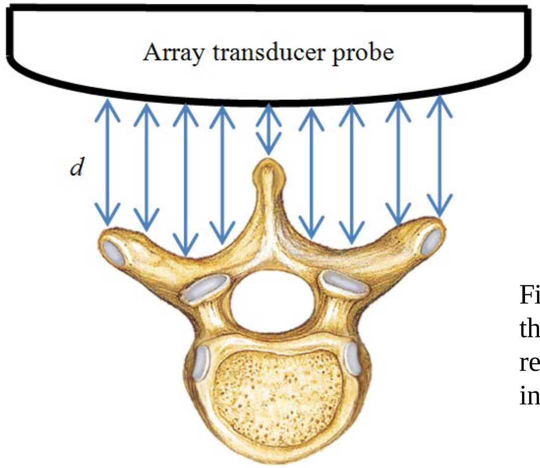


Fig. The schematic of from the vertebra cortex ultrasound reflection waves soft tissue interface.

Wave propagation

Wave equation:

$$\nabla^2 u(\vec{s},\vec{x},t) - \frac{1}{c^2(\vec{x})}\frac{\partial^2 u(\vec{s},\vec{x},t\ )}{\partial t} = -\delta(\vec{x}-\vec{s},t)$$

Based on the perturbation velocity profile assumption:

$$\frac{1}{c^2(\vec{x})} = \frac{1}{c_0^2(\vec{x})} + f(\vec{x})$$

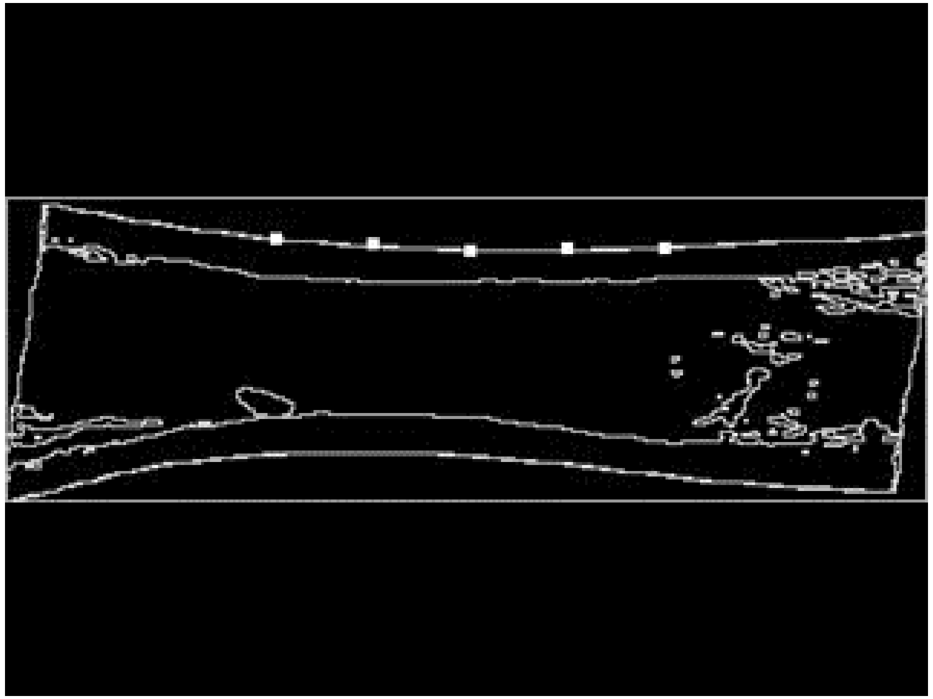
An approximate solution for acoustic wave equation (Born Approximation) is

$$G_0(\vec{x},\omega) = A(\vec{x})e^{-j\omega\tau(\vec{x})}$$

where $G_0(\vec{x},\omega)$: **Green function,**

The data can be expressed as

$$d(\vec{s},\vec{r},\omega) = \omega^2 \int d^3\vec{x}\, G_0(\vec{x},\vec{r},\omega) f(\vec{x}) G_0(\vec{s},\vec{x},\omega)$$



(*Miller et al. 1987, Weglein, 1982; Nemeth et al, 1999*)

Operator & cost function

- Rewrite in operator form: $D(\vec{s},\vec{r}) = L(\vec{s},\vec{r},\vec{x},\omega_i) * F(\vec{x})$
- Cost function $\boldsymbol{J} = \left\|\boldsymbol{D} - \boldsymbol{D}^{obs}\right\|_2^2$,

  where $\boldsymbol{D}$ is the theoretical data, $\boldsymbol{D}^{obs}$ is the observed data
- Inversion is to minimize $\boldsymbol{J}$ to find the best solution of $F(\vec{x})$
  - Adjoint operator: $F(\vec{x}) = L^*(\vec{s},\vec{r},\vec{x},\omega_i) * D(\vec{s},\vec{r})$
  - Minimum norm solution (LS): $J = \|LF - D\|_2^2$
  - Damped minimum norm solution (DLS): $J = \|LF - D\|_2^2 + \mu\|F\|_2^2$
  - Weighted minimum norm solution (WLS): $J = \|LF - D\|_2^2 + \mu\|WF\|_2^2$

Conjugate Gradient Method

Consider a system of linear equations： $\boldsymbol{y} = \boldsymbol{L}\boldsymbol{x}$

Let $\boldsymbol{x} = \boldsymbol{x}_0, \boldsymbol{p}_0 = \boldsymbol{r}_0 = \boldsymbol{y} - \boldsymbol{L}\boldsymbol{x}_0$ and $k = 0$.

The following steps will be repeated until number of iterations or the tolerance limit for convergence is reached.

a) $\alpha_k = \frac{\boldsymbol{r}_k{}^T\boldsymbol{r}_k}{\boldsymbol{p}_k{}^T\boldsymbol{L}\boldsymbol{p}_k}$

b) $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha_k\boldsymbol{p}_k$

c) $\boldsymbol{r}_{k+1} = \boldsymbol{y} - \boldsymbol{L}\boldsymbol{x}_{k+1}$

d) $\beta_k = \frac{\boldsymbol{r}_{k+1}{}^T\boldsymbol{r}_{k+1}}{\boldsymbol{r}_k{}^T\boldsymbol{r}_k}$

e) $\boldsymbol{p}_{k+1} = \boldsymbol{r}_{k+1} + \beta_k\boldsymbol{p}_k$

f) $k = k + 1$