Lecture 10 – Transform and Image Compression

This lecture will cover:

– Other Transform

- Discrete Cosine Transform（余弦变换）
- Walsh Transform (沃尔什变换)
- Discrete Wavelet Transform （小波变换）

– Image compression

Cosine Transform (余弦变换)

Discrete Cosine Transform (DCT):

$$F(0) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x) \qquad F(u) = \sqrt{\frac{2}{N}} \sum_{x=0}^{N-1} f(x) \cos\frac{(2x+1)u\pi}{2N}$$

Inverse Discrete Cosine Transform (IDCT):

$$f(x) = \frac{1}{\sqrt{N}} F(0) + \sqrt{\frac{2}{N}} \sum_{u=1}^{N-1} F(u) \cos\frac{(2x+1)u\pi}{2N}$$

Analytic form:

$$\begin{cases} F(0) = 0.500f(0) + 0.500f(1) + 0.500f(2) + 0.500f(3) \\ F(1) = 0.653f(0) + 0.271f(1) - 0.271f(2) - 0.653f(3) \\ F(0) = 0.500f(0) - 0.500f(1) - 0.500f(2) + 0.500f(3) \\ F(0) = 0.271f(0) - 0.653f(1) + 0.653f(2) - 0.271f(3) \end{cases}$$

Matrix Form:

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} 0.500 & 0.500 & 0.500 & 0.500 \\ 0.653 & 0.271 & -0.271 & -0.653 \\ 0.500 & -0.500 & -0.500 & 0.500 \\ 0.271 & -0.653 & 0.653 & -0.271 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$$

Forward Matrix form:

$$[F(u)] = [A][f(x)]$$

Inverse Matrix form:

$$[f(x)] = [A]^{\mathrm{T}}[F(u)]$$

2D DCT and IDCT

Forward Transform:

$$F(0,0) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y)$$

$$F(u,0) = \frac{\sqrt{2}}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \cos\frac{(2x+1)u\pi}{2N}$$

$$F(0,v) = \frac{\sqrt{2}}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \cos\frac{(2y+1)v\pi}{2N}$$

$$F(u,v) = \frac{2}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \cos\frac{(2x+1)u\pi}{2N} \cos\frac{(2y+1)v\pi}{2N}$$

DCT

Inverse Transform:

$$f(x,y) = \frac{1}{N} F(0,0)$$

$$+ \frac{\sqrt{2}}{N} \sum_{u=1}^{N-1} F(u,0) \cos\frac{(2x+1)u\pi}{2N}$$

$$+ \frac{\sqrt{2}}{N} \sum_{v=1}^{N-1} F(0,v) \cos\frac{(2y+1)v\pi}{2N}$$

$$+ \frac{2}{N} \sum_{x=1}^{N-1} \sum_{y=1}^{N-1} F(u,v) \cos\frac{(2x+1)u\pi}{2N} \cos\frac{(2y+1)v\pi}{2N}$$

Forward Matrix form: $[\boldsymbol{F}(\boldsymbol{u},\boldsymbol{v})] = [\boldsymbol{A}][\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})][\boldsymbol{A}]^{\boldsymbol{T}}$

Inverse Matrix form: $[\boldsymbol{f}(\boldsymbol{x},\mathbf{y})] = [\boldsymbol{A}]^{\boldsymbol{T}}[\boldsymbol{F}(\boldsymbol{u},\mathbf{v})][\boldsymbol{A}]$

Calculate DCT(IDCT) by DFT(IDFT)

DCT:

$$F(0) = \frac{1}{\sqrt{N}} \sum_{x=0}^{2N-1} f_e(x)$$

$$F(u) = \frac{2}{N} \sum_{x=0}^{N-1} f(x) \cos\frac{(2x+1)u\pi}{2N}$$

$$= \sqrt{\frac{2}{N}} \sum_{x=0}^{2N-1} f_e(x) \cos\frac{2(x+1)u\pi}{2N}$$

$$= \sqrt{\frac{2}{N}} \mathrm{Re}\left\{ \sum_{x=0}^{2N-1} f_e(x) e^{-j\frac{(2x+1)u\pi}{2N}} \right\}$$

$$= \sqrt{\frac{2}{N}} \mathrm{Re}\left\{ e^{-j\frac{u\pi}{2N}} \sum_{x=0}^{2N-1} f_e(x) e^{-j\frac{2\pi ux}{2N}} \right\}$$

$$= \sqrt{\frac{2}{N}} \mathrm{Re}\left\{ e^{-j\frac{u\pi}{2N}} \mathrm{DFT}[f_e(x)] \right\}$$

IDCT:

$$f(x) = \frac{1}{\sqrt{N}} F(0) + \sqrt{\frac{2}{N}} \sum_{u=0}^{N-1} F(u) \cos\frac{(2x+1)u\pi}{2N}$$

$$= \frac{1}{\sqrt{N}} F_e(0) + \sqrt{\frac{2}{N}} \sum_{u=1}^{2N-1} F_e(u) \cos\frac{(2x+1)u\pi}{2N}$$

$$= \frac{1}{\sqrt{N}} F_e(0) + \sqrt{\frac{2}{N}} \sum_{u=1}^{2N-1} \mathrm{Re}\left\{ F_e(u) e^{j\frac{(2x+1)u\pi}{2N}} \right\}$$

$$= \frac{1}{\sqrt{N}} F_e(0) + \sqrt{\frac{2}{N}} \sum_{u=1}^{2N-1} \mathrm{Re}\left\{ F_e(u) e^{j\frac{u\pi}{2N}} e^{j\frac{2\pi ux}{2N}} \right\}$$

$$= \left(\frac{1}{\sqrt{N}} - \sqrt{\frac{2}{N}} \right) F_e(0) + \sqrt{\frac{2}{N}} \mathrm{Re}\left\{ \sum_{u=0}^{2N-1} \left\{ F_e(u) e^{j\frac{u\pi}{2N}} \right\} e^{j\frac{2\pi ux}{2N}} \right\}$$

$$= \left(\frac{1}{\sqrt{N}} - \sqrt{\frac{2}{N}} \right) F_e(0) + \sqrt{\frac{2}{N}} \mathrm{Re}\left\{ \mathrm{IDFT}\left[F_e(u) e^{j\frac{u\pi}{2N}} \right] \right\}$$

Where

$$f_e(x) = \begin{cases} f(x), & x = 0,1,2,\cdots,N-1 \\ 0, & x = N, N+1, N+2, \cdots, 2N-1 \end{cases}$$

$$F_e(u) = \begin{cases} F(u), & u = 0,1,2,\cdots,N-1 \\ 0, & u = N, N+1, N+2, \cdots, 2N-1 \end{cases}$$

Walsh Transform

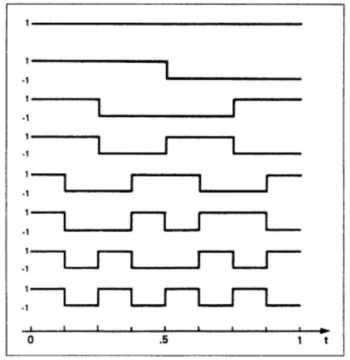
- **Consist of ±1 arranged in a checkerboard pattern**
- **Transform:**

$$W(i) = \frac{1}{N}\sum_{t=0}^{N-1} f(t)\cdot \mathrm{Wal}(i,t)$$

$$f(t) = \sum_{i=0}^{N-1} W(i)\cdot \mathrm{Wal}(i,t)$$

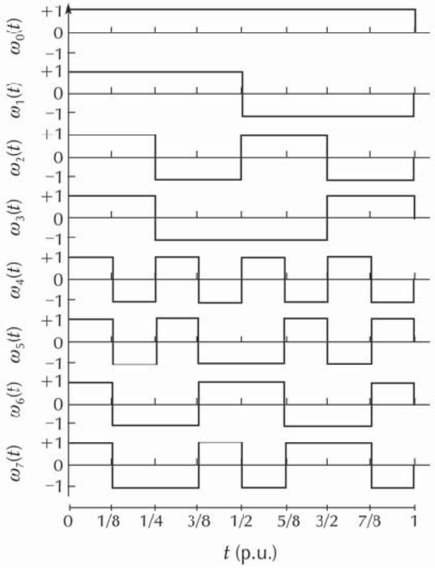
- **Types of** $\mathrm{Wal}(i,t)$

  - Walsh Ordering (沃尔什定序)



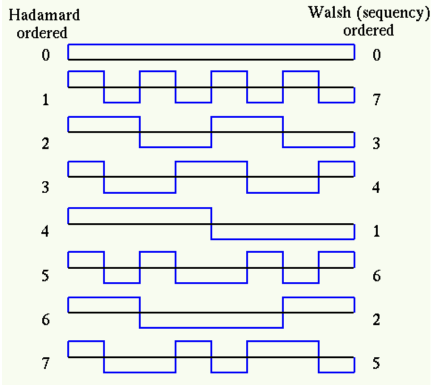
$$W_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

- Paley Ordering （佩利定序）



$$\boldsymbol{W_8} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

- Hadamard Matrix Ordering（哈达玛矩阵定序）



$$V_8 = \begin{pmatrix} W_4 & W_4 \\ W_4 & -W_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

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Relationship Between Ordering

Walsh ordering (*Signal processing*)	**Paley ordering** (*Control Engineering*)	**Hadamard ordering** (*Mathematics*)	**W(m,n)**
$Wal_w(0,t)$	$Wal_P(0,t)$	$Wal_H(0,t)$	[1 1 1 1 1 1 1 1]
$Wal_w(1,t)$	$Wal_P(1,t)$	$Wal_H(4,t)$	[1 1 1 1 -1 -1 -1 -1]
$Wal_w(2,t)$	$Wal_P(3,t)$	$Wal_H(6,t)$	[1 1 -1 -1 -1 -1 1 1]
$Wal_w(3,t)$	$Wal_P(2,t)$	$Wal_H(2,t)$	[1 1 -1 -1 1 1 -1 -1]
$Wal_w(4,t)$	$Wal_P(6,t)$	$Wal_H(3,t)$	[1 -1 -1 1 1 -1 -1 1]
$Wal_w(5,t)$	$Wal_P(7,t)$	$Wal_H(7,t)$	[1 -1 -1 1 -1 1 1 -1]
$Wal_w(6,t)$	$Wal_P(5,t)$	$Wal_H(5,t)$	[1 -1 1 -1 -1 1 -1 1]
$Wal_w(7,t)$	$Wal_P(4,t)$	$Wal_H(1,t)$	[1 -1 1 -1 1 -1 1 -1]

Basic Function for DCT and WHT

$$T(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)r(x,y,u,v)$$

FIGURE 8.23 Discrete-cosine basis functions for $n = 4$. The origin of each block is at its top left.

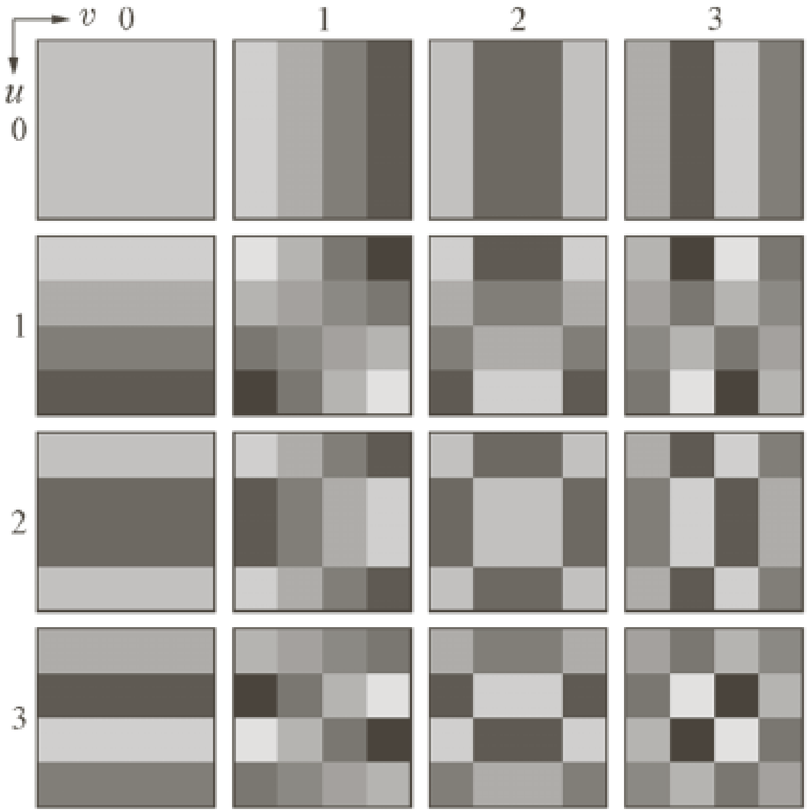
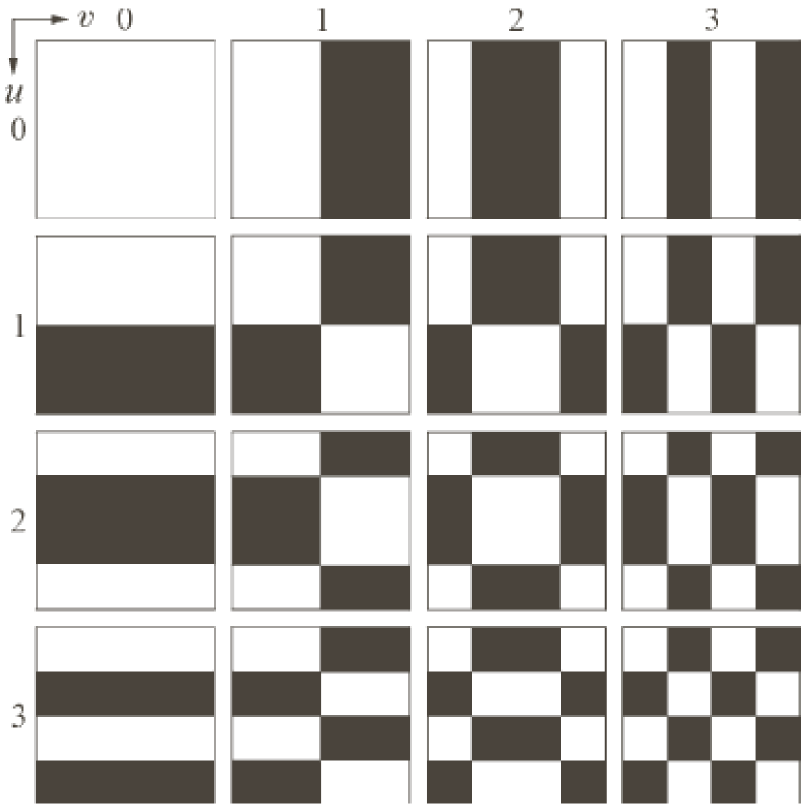


FIGURE 8.22 Walsh-Hadamard basis functions for $n = 4$. The origin of each block is at its top left.

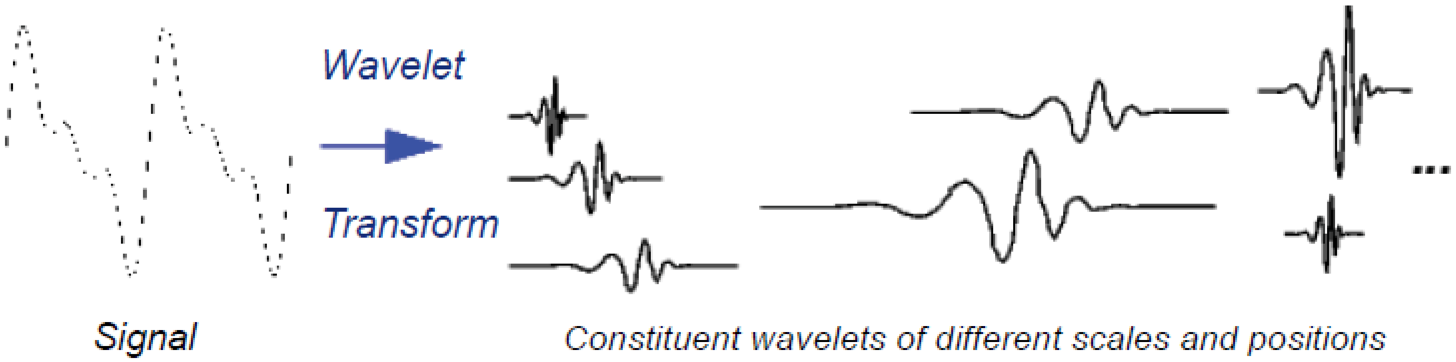


Block Transform

FIGURE 8.24 Approximations of Fig. 8.9(a) using the (a) Fourier, (b) Walsh-Hadamard, and (c) cosine transforms, together with the corresponding scaled error images in (d)–(f).

Wavelet Transform (小波变换)

- **Based on small waves called Wavelets –** 1) limited; 2) oscillation
- **Mother wavelet（母小波）:** Translation & Scaling
- **Varying frequency and limited duration**
- **localized in both time and frequency**



Continuous Wavelet Transform (连续小波变换)

- **Continuous Wavelet Transform (CWT)**

$$W_\psi(s,\tau)=\int_{-\infty}^{\infty} f(x)\psi_{s,\tau}(x)dx$$

Where $\psi_{s,\tau}(x)=\frac{1}{\sqrt{s}}\psi\left(\frac{x-\tau}{s}\right)$

s : scale parameter (尺度参数) τ : translation parameter (平移参数)

- **Inverse Continuous Wavelet Transform (ICWT)**

$$f(x)=\frac{1}{C_\psi}\int_0^{\infty}\int_{-\infty}^{\infty} W_\psi(s,\tau)\frac{\psi_{s,\tau}(x)}{s}d\tau\, ds$$

Where $C_\psi=\int_{-\infty}^{\infty}\frac{|\Psi(\mu)|^2}{|\mu|}d\mu$, $\Psi(\mu)$ is Fourier transform of $\psi(x)$

Discrete Wavelet Transform (离散小波变换)

- **Discrete Wavelet Transform (DWT)**

$$W_\varphi(j_0,k)=\frac{1}{\sqrt{M}}\sum_n f(n)\,\varphi_{j_0,k}(n)$$

$$W_\psi(j,k)=\frac{1}{\sqrt{M}}\sum_n f(n)\,\psi_{j,k}(n)\qquad j\geq j_0$$

- **Inverse Continuous Wavelet Transform (ICWT)**

$$f(n)=\frac{1}{\sqrt{M}}\sum_k W_\varphi(j_0,k)\,\varphi_{j_0,k}(n)+\frac{1}{\sqrt{M}}\sum_{j=j_0}^{\infty}\sum_k W_\psi(j,k)\,\psi_{j,k}(n)$$

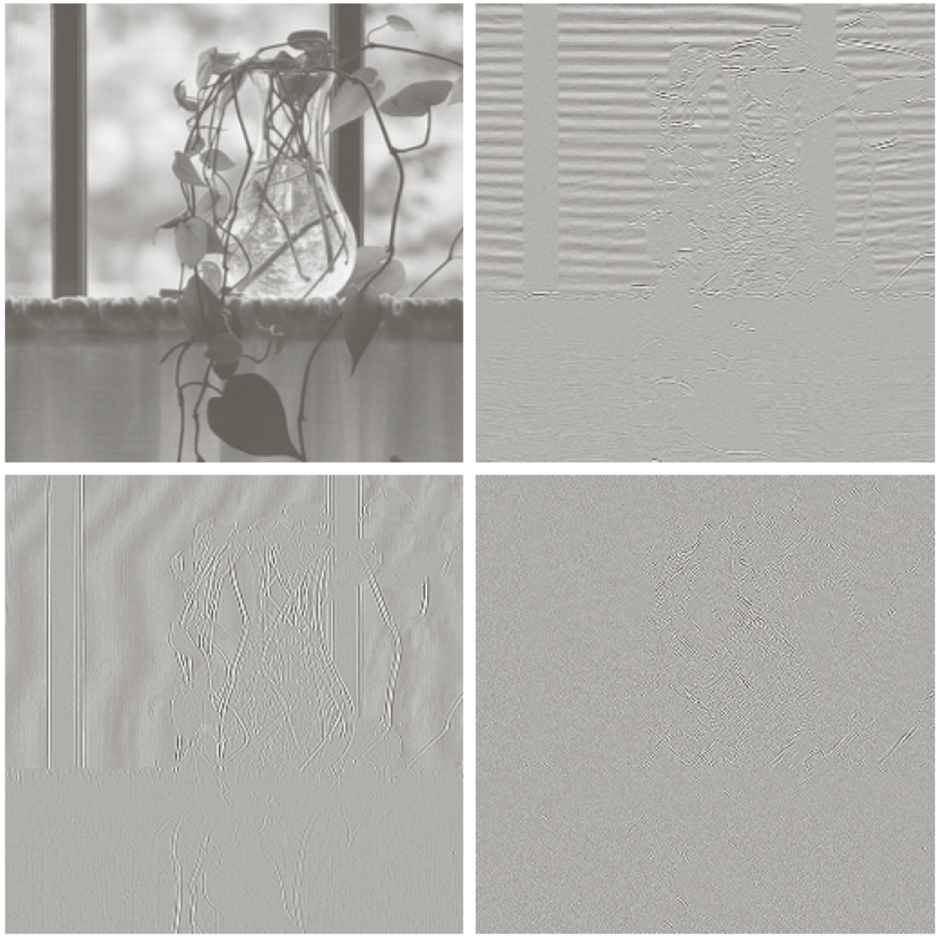
Where

$\varphi_{j_0,k}(n)$: scaling function (尺度函数)

$\psi_{j,k}(n)$: Wavelet (小波)

$W_\varphi(j_0,k)$: Approximation coefficients (近似系数)

$W_\psi(j,k)$: detail coefficients (细节系数)



a b
c d

FIGURE 7.9 A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.7. The four subbands that result are the (a) approximation, (b) horizontal detail, (c) vertical detail, and (d) diagonal detail subbands.

2D DWT

Define 2D scale function（二维尺度函数）:

$$\varphi(x,y)=\varphi(x)\varphi(y)$$

"Directionally sensitive" wavelet（"方向敏感" 小波）

$$\psi^H(x,y)=\psi(x)\varphi(y) \qquad \psi^V(x,y)=\varphi(x)\psi(y) \qquad \psi^D(x,y)=\psi(x)\psi(y)$$

2D DWT

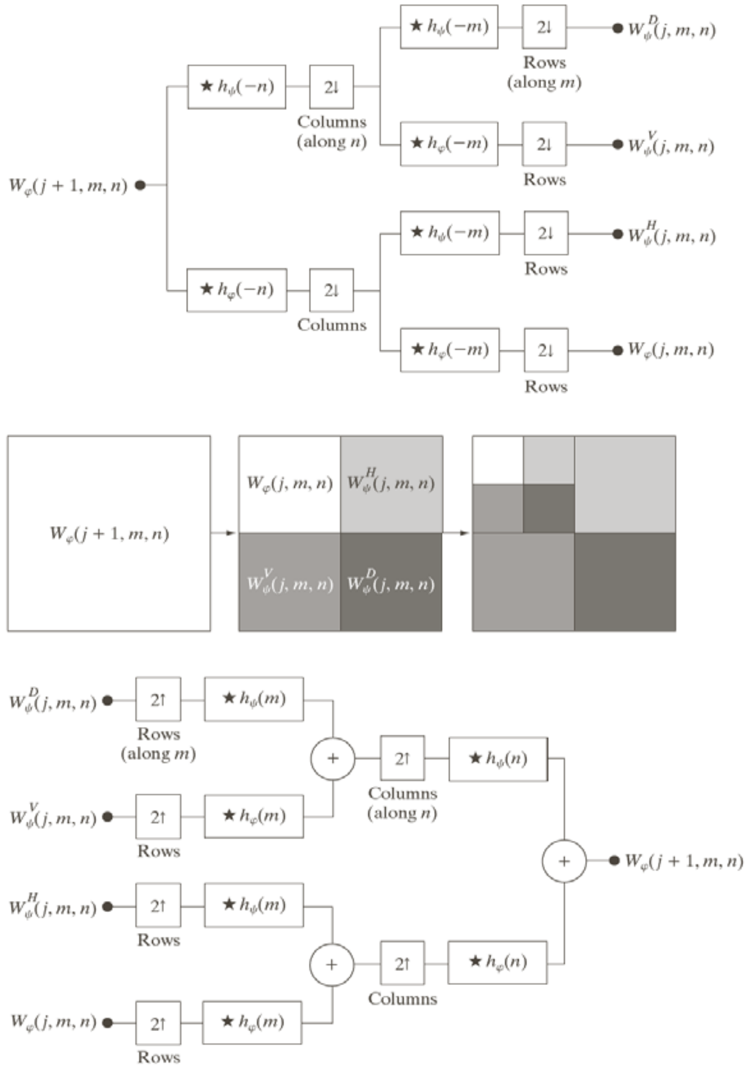
$$W_\varphi(j_0,m,n)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,\varphi_{j_0,m,n}(x,y)$$

$$W_\psi(j,m,n)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,\psi^i_{j,m,n}(x,y) \qquad i=\{H,V,D\}$$

2D IDWT

$$f(x,y)=\frac{1}{\sqrt{MN}}\sum_m\sum_n W_\varphi(j_0,m,n)\,\varphi_{j_0,m,n}(x,y)$$

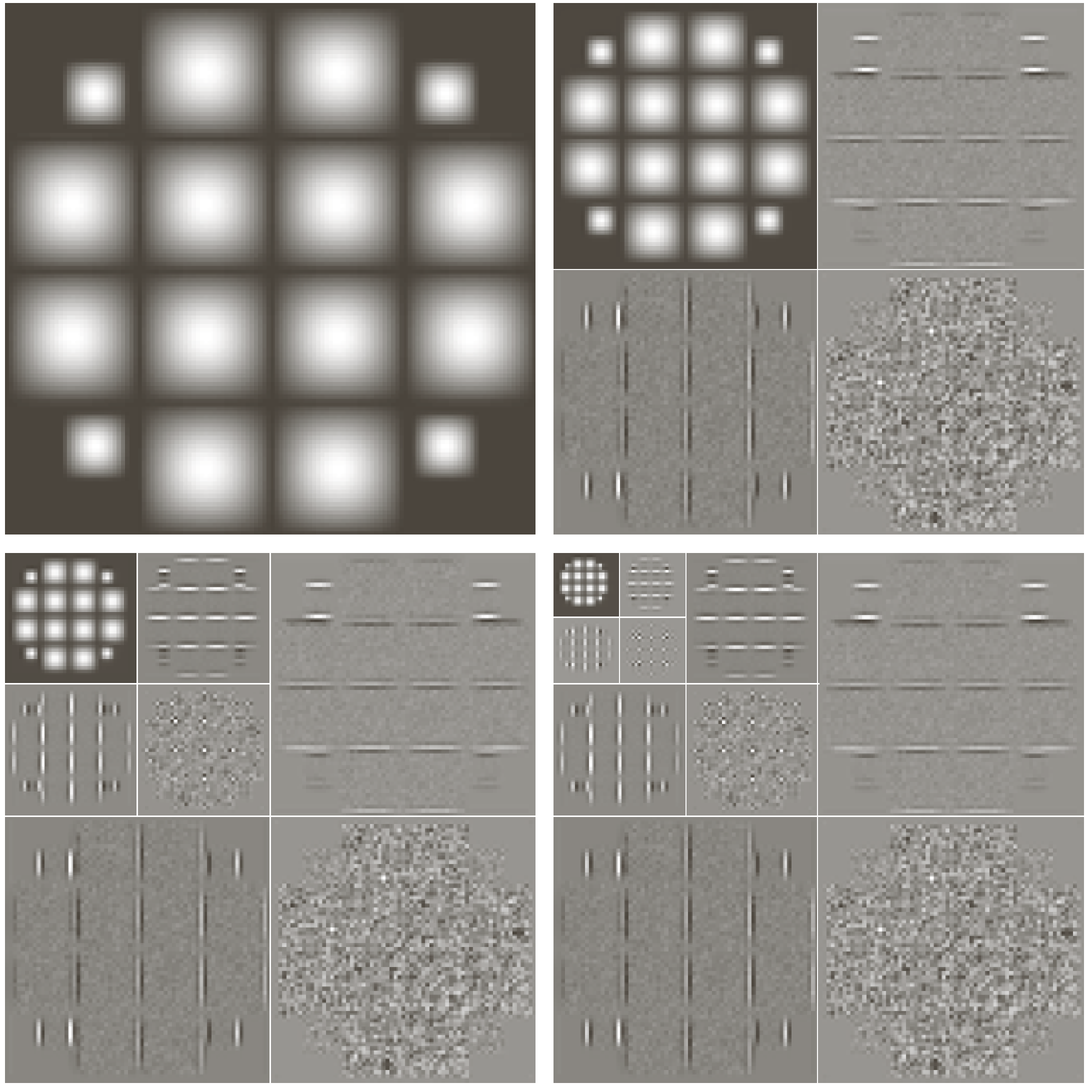
$$+\frac{1}{\sqrt{MN}}\sum_{i=\{H,V,D\}}\sum_{j=j_0}^{\infty}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} W_\psi(j,m,n)\,\psi^i_{j,m,n}(x,y)$$



a
b
c

FIGURE 7.24 The 2-D fast wavelet transform: (a) the analysis filter bank; (b) the resulting decomposition; and (c) the synthesis filter bank.

2D DWT



a b
c d

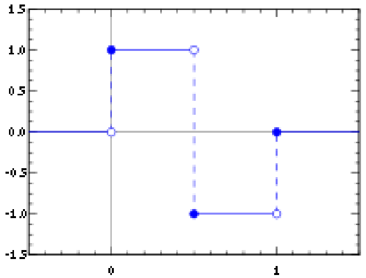
FIGURE 7.25 Computing a 2-D three-scale FWT: (a) the original image; (b) a one-scale FWT; (c) a two-scale FWT; and (d) a three-scale FWT.

Mother Wavelet (母小波)

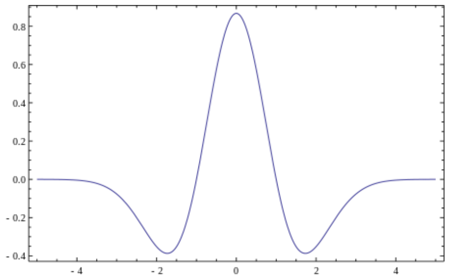
➢ **Mother Wavelet should satisfy**

- $\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$
- $\int_{-\infty}^{\infty} |\psi(t)| dt < \infty$
- $\int_{-\infty}^{\infty} \psi(t) dt = 0$

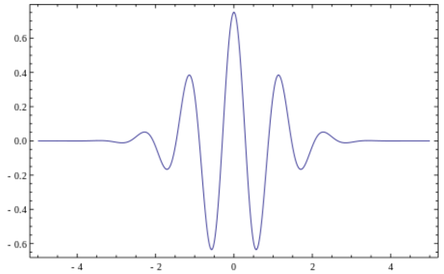
Haar



Mexican Hat



Morlet



Meyer

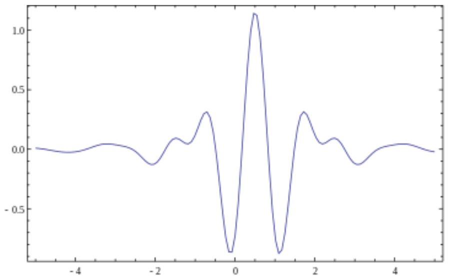


Image Compression (图像压缩)

– Fundamentals (基础知识)

- Coding Redundancy (编码冗余)
- Spatial and Temporal Redundancy（空间和时间冗余）
- Irrelevant Information（不相关信息）

– Measuring Image Information（信息量）

– Fidelity Criteria (保真度准则)

– Image Compression Model（图像压缩模型）

- Source coding (信源编码)
- Channel coding（信道编码）

– Image Formats, Containers and Compression Standards（图像格式、容器和压缩标准）

Fundamentals of Image Compression

- Coding Redundancy (编码冗余)
- Spatial and Temporal Redundancy（空间和时间冗余）
- Irrelevant Information（不相关信息）

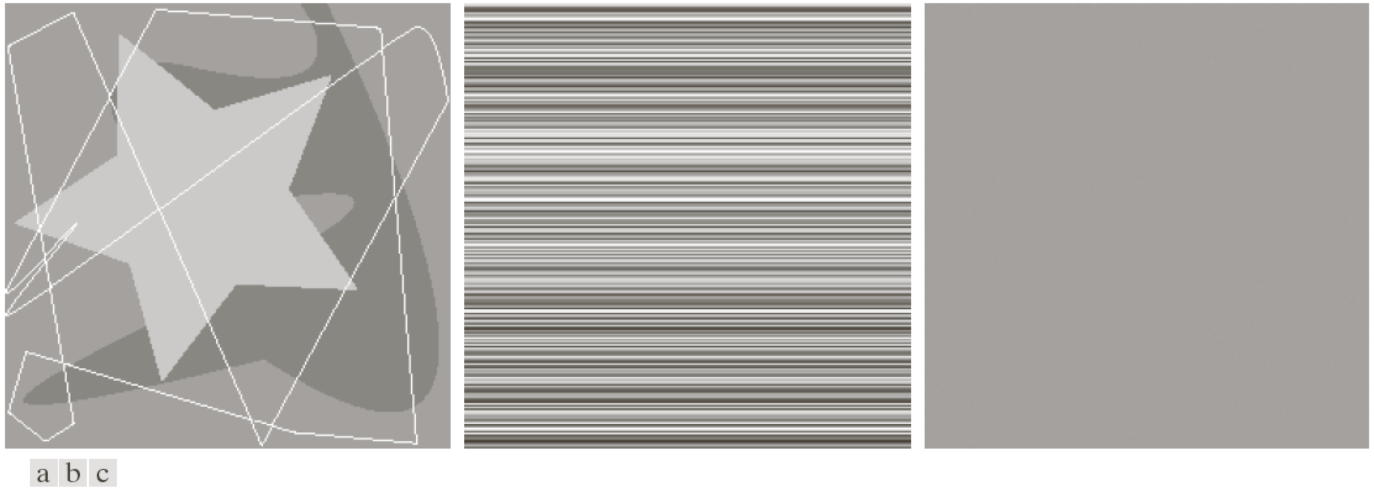


FIGURE 8.1 Computer generated 256 × 256 × 8 bit images with (a) coding redundancy, (b) spatial redundancy, and (c) irrelevant information. (Each was designed to demonstrate one principal redundancy but may exhibit others as well.)

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Measuring Image Information（信息量）

- **Information Unit:**

$$I(E) = \log\frac{1}{P(E)} = -\log P(E)$$

Where $P(E)$ is the probability of a random event E.

- **Entropy (熵)**

$$H = -\sum_{j=1}^{J} P(a_j)\log P(a_j)$$

Calculate from Histogram

$$\tilde{H} = -\sum_{k=0}^{L-1} p_r(r_k)\log_2 p_r(r_k)$$

Fidelity Criteria (保真度准则)

Objective Fidelity Criteria (客观保真度准则)

- **Root Mean Square Error (均方根误差)**

$$e_{\mathrm{rms}} = \left\{ \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left[\hat{f}(x,y) - f(x,y) \right]^2 \right\}^{1/2}$$

Where $f(x,y)$ is the original image, and $\hat{f}(x,y)$ is an approximation.

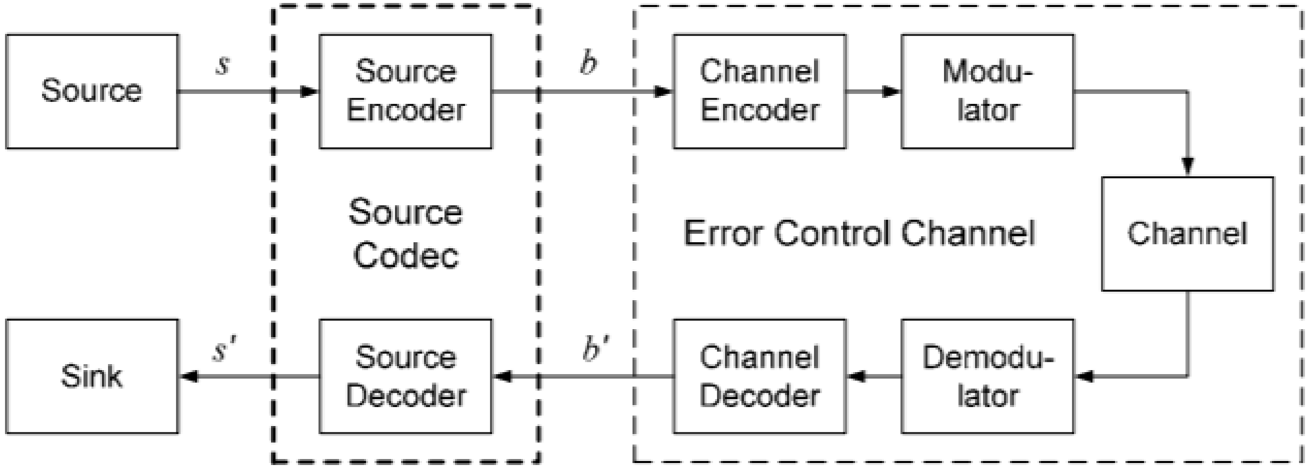
- **Mean-square Signal-to-noise ratio (均方信噪比)**

$$\mathrm{SNR}_{\mathrm{ms}} = \frac{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} [f(x,y)]^2}{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left[\hat{f}(x,y) - f(x,y) \right]^2}$$

Subjective Fidelity Criteria (主观保真度准则)

Value	Rating	Description
1	Excellent	An image of extremely high quality, as good as you could desire.
2	Fine	An image of high quality, providing enjoyable viewing. Interference is not objectionable.
3	Passable	An image of acceptable quality. Interference is not objectionable.
4	Marginal	An image of poor quality; you wish you could improve it. Interference is somewhat objectionable.
5	Inferior	A very poor image, but you could watch it. Objectionable interference is definitely present.
6	Unusable	An image so bad that you could not watch it.

Image Compression Model（图像压缩模型）



Source coding (信源编码)

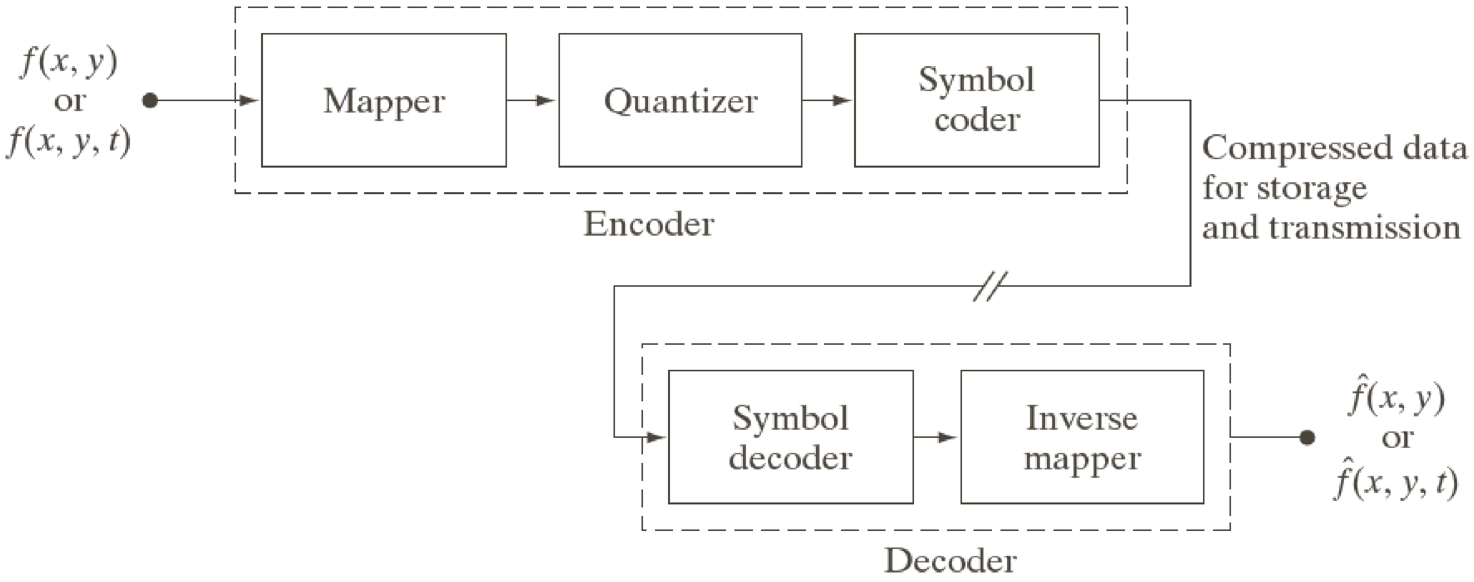


FIGURE 8.5 Functional block diagram of a general image compression system.

Channel coding（信道编码）

Hamming Code

For a 4-bit binary number $b_3 b_2 b_1 b_0$, define the 7-bit code word as

$$h_1 = b_3 \oplus b_2 \oplus b_0 \qquad h_2 = b_3 \oplus b_1 \oplus b_0 \qquad h_4 = b_2 \oplus b_1 \oplus b_0$$

$$h_3 = b_3 \qquad h_5 = b_2 \qquad h_6 = b_1 \qquad h_7 = \oplus b_0$$

Parity (奇偶校验)

$$c_1 = h_1 \oplus h_3 \oplus h_5 \oplus h_7$$

$$c_2 = h_2 \oplus h_3 \oplus h_6 \oplus h_7$$

$$c_4 = h_4 \oplus h_5 \oplus h_6 \oplus h_7$$

Image Formats, Containers and Compression Standards