CS 0441
Lecture 6: Set theory

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Overview

Sets

- In this lecture, we introduce the most basic concept in mathematics, *i.e.*, sets.
- A set is a collection of distinct elements. The objects in the set are called elements.
- $\emptyset$ represents the **empty set** (also called the null set), namely, a set with no element. A set with one element is called a **singleton set**.
- $\in$ means "belongs to" and $\notin$ means "not belong to". $x \in S$ means x is an element of S and $x \notin S$ means x is not an element of S.

Roaster method

- It is common for sets to be denoted using uppercase letters. Lowercase letters are usually used to denote elements of sets.
- We can describe a set using different ways. First, the **roster method** is to list elements in a set explicitly.
- For example, the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ and the set V of all vowels in the English alphabet $V = \{a, e, i, o, u\}$.

Set builder method

- Another way to describe a set is to use set builder notation. We list the elements in set using the common characteristics.
- For example, the set of rational numbers $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$ and the set of complex numbers $\mathbb{C} = \{a + bi \mid a, b \text{ are real numbers}\}$.

Defined by words

- We can also define a set by words.
- For example, $\mathbb{R}$ is the set of real numbers, which consists of rational and irrational numbers; $\mathbf{R}^+$ is the set of positive real numbers and $\mathbf{Z}^+$ is the set of positive integers.

Interval notations

- When elements are real numbers, we can use special interval notations. When a and b are real numbers with $a < b$, we write

$$
\begin{aligned}
[a, b] &= \{x \mid a \leqslant x \leqslant b\}, \\
[a, b) &= \{x \mid a \leqslant x < b\}, \\
(a, b] &= \{x \mid a < x \leqslant b\}, \\
(a, b) &= \{x \mid a < x < b\}.
\end{aligned}
$$

- Note that $[a, b]$ is called the closed interval from a to b and (a, b) is called the open interval from a to b.

Relations of sets

- Next, we introduce some important relations between sets.

Subsets

- The set A is a subset of B if and only if every element of A is also an element of B, that is, $\forall x(x \in A \rightarrow x \in B)$. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.
- Note to show A is a subset of B, we need to state every element in A is an element in B. However, to disprove this, we simply need to find an element in A, which is not in B.

- Next, we present an important

Theorem

For every set S, (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.

Proof.

First we prove (i). To show that $\emptyset \subseteq S$, we must show that $\forall x(x \in \emptyset \rightarrow x \in S)$ is true. Because the empty set contains no elements, it follows that $x \in \emptyset$ is always false. It follows that the conditional statement $x \in \emptyset \rightarrow x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true, which is a vacuous proof. Therefore, $\forall x(x \in \emptyset \rightarrow x \in S)$ is true. ■

Proof.

Next, for (ii), we will show $\forall x(x \in S \rightarrow x \in S)$ is true. Let x be an arbitrary element in S. Clearly, we have $x \in S \rightarrow x \in S$ since S is the same set. Hence, we demonstrate that $\forall x(x \in \emptyset \rightarrow x \in S)$ is true. ∎

Example 1

Example

Let S be the set of all integers that are multiples of 6 and let T be the set of all even integers. Then S is a subset of T.

Solution

We will show that S is a subset of T by showing that $\forall x(x \in S \to x \in T)$. Let $x \in S$. (Note: The use of the word "let" is often an indication that the we are choosing an arbitrary element.) This means that x is a multiple of 6. Therefore, there exists an integer m such that

$$x = 6m.$$

Since $6 = 2 \cdot 3$, this equation can be written in the form

$$x = 2(3m).$$

Note that $3\mathrm{~m}$ is an integer. Hence, this last equation proves that x must be even. Therefore, we have shown that if x is an element of S, then x is an element of T, and hence that $S \subseteq T$.

Proper subsets

- When we wish to emphasize that a set A is a subset of a set B but that $A \neq B$, we write $A \subset B$ and say that A is a proper subset of B.
- For $A \subset B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A. That is, A is a proper subset of B if and only if

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A).$$

Equal

- Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

- To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Example 2

Example

Let

$$S = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \middle| \, c_1, c_2, c_3 \in \mathbb{R} \right\},$$

$$T = \left\{ d_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| \, d_1, d_2 \in \mathbb{R} \right\}.$$

Prove that $S = T$.

Solution

First, we verify $S \subseteq T$ by showing for any vector $\vec{x} \in S$, $\vec{x} \in T$. Hence, suppose $\vec{x} \in S$, then there exist $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$\begin{aligned}
\vec{x} &= c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&= c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_3 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \\
&= (c_1 + c_3) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (c_2 + c_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$

from which it follows that $\vec{x} \in T$ and hence $S \subseteq T$.

Solution

We now complete the other direction $T \subseteq S$ by demonstrating that if $\vec{y} \in T$ then $\vec{y} \in S$. Taking arbitrary $\vec{y} \in T$, there exists $d_1, d_2 \in \mathbb{R}$ such that

$$\begin{aligned} \vec{y} &= d_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= d_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{aligned}$$

which implies that $\vec{y} \in S$ (take $c_1 = d_1$, $c_2 = d_2$ and $c_3 = 0$) and hence $T \subseteq S$. Since $S \subseteq T$ and $T \subseteq S$, we conclude that $S = T$.

The size of a set

- Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.
- Note that $|\emptyset| = 0$ and $|\mathbb{R}| = \infty$. We call a set whose cardinality is infinite an **infinite** set.

Power sets

- Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.
- Note that if a set has n elements, then its power set has 2^n elements.

Example 3

Example

What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

Solution

The empty set has exactly one subset, namely, itself. Consequently,

$$\mathcal{P}(\emptyset) = \{\emptyset\}.$$

The set $\{\emptyset\}$ *has exactly two subsets, namely,* $\emptyset$ *and the set* $\{\emptyset\}$ *itself. Therefore,*

$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$

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Cartesian products

- Here we introduce a different structure from sets. A set is unordered. The **ordered n-tuple** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its nth element.
- We say that two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal. In other words, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$, for $i = 1, 2, \ldots, n$.
- In particular, ordered 2-tuples are called ordered pairs. The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$. Note that (a, b) and (b, a) are not equal unless $a = b$.

- With the ordered sets defined, we introduce the definition of Cartesian product.

Definition

Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

- Note that when $A = B = \emptyset$, the Cartesian product $A \times B = \emptyset$.
- In general, the Cartesian products $A \times B$ and $B \times A$ are not equal, unless $A = \emptyset$ or $B = \emptyset$ or $A = B$.

Example 4

Example

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$? Then show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$.

Solution

The Cartesian product $A \times B$ is

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

Solution

Solution: The Cartesian product $B \times A$ *is*

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\},$$

which is not equal to $A \times B$.

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Remark

- More generally, we can have an n-dimensional Cartesian product.

Definition

The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

- We use the notation A^2 to denote $A \times A$, the Cartesian product of the set A with itself. Similarly, $A^3 = A \times A \times A, A^4 = A \times A \times A \times A$, and so on. More generally,

$$A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}.$$

Example 5

Example

What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}, B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

Solution

The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. Hence,

$$\begin{aligned} A \times B \times C = & \{(0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1),(0,2,2), \\ & (1,1,0),(1,1,1),(1,1,2),(1,2,0),(1,2,1),(1,2,2)\}. \end{aligned}$$

Remark

- Note that when A, B, and C are sets, $(A \times B) \times C$ is not the same as $A \times B \times C$.
- Take Example 5 for illustration, an element in $(A \times B) \times C$, for example, is $((0, 1), 0)$.

Relation

- A subset R of the Cartesian product $A \times B$ is called a relation from the set A to the set B. The elements of R are ordered pairs, where the first element belongs to A and the second to B.
- For example,

$$R = \{(a,0),(a,1),(a,3),(b,1),(b,2),(c,0),(c,3)\}$$

is a relation from the set $\{a, b, c\}$ to the set $\{0, 1, 2, 3\}$. A relation from a set A to itself is called a relation on A. More details will be given in future lectures.

Sets with quantifiers

- Now we can restrict the predicate logic with quantifiers with clear domains. $\forall x \in S(P(x))$ denotes the universal quantification of $P(x)$ over all elements in the set S. In other words, $\forall x \in S(P(x))$ is shorthand for $\forall x(x \in S \rightarrow P(x))$.

Truth sets

- We will now tie together concepts from set theory and from predicate logic. Given a predicate P, and a domain D, we define the truth set of P to be the set of elements x in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$.
- Note that $\forall x P(x)$ is true over the domain U if and only if the truth set of P is the set U. Likewise, $\exists x P(x)$ is true over the domain U if and only if the truth set of P is nonempty.

Example 6

Example

What are the truth sets of the predicates $P(x)$ and $R(x)$, where the domain is the set of integers and $P(x)$ is "$|x| = 1$ and $R(x)$ is "$|x| = x$."

Solution

The truth set of $P, \{x \in \mathbb{Z} \mid |x| = 1\}$*, is the set of integers for which* $|x| = 1$*. Because* $|x| = 1$ *when* $x = 1$ *or* $x = -1$*, and for no other integers* x*, we see that the truth set of* P *is the set* $\{-1, 1\}$*.*
The truth set of $R, \{x \in \mathbb{Z} \mid |x| = x\}$*, is the set of integers for which* $|x| = x$*. Because* $|x| = x$ *if and only if* $x \geqslant 0$*, it follows that the truth set of* R *is* $\mathbb{N}$*, the set of nonnegative integers.*

Set operations

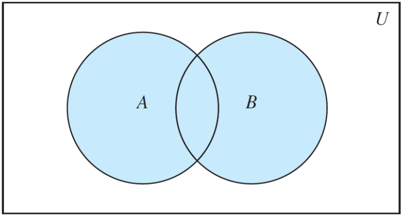
- We will introduce basic operations between sets in what follows.

Unions

- Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both. That is,

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

- We can also present using the following Venn diagram.



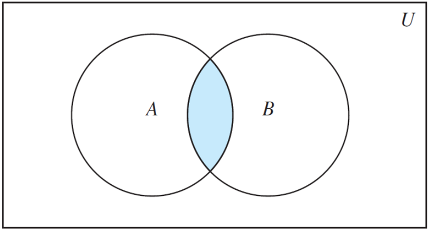
$A \cup B$ is shaded.

Intersections

- Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B. Namely,

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$

- The corresponding Venn representation is



$A \cap B$ is shaded.

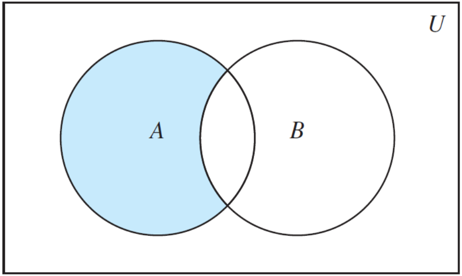
- Two sets are called **disjoint** if their intersection is the empty set. For example, Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$, A and B are disjoint.
- The cardinality of the union set $A \cup B$ can be calculated via the inclusion-exclusion principle

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Differences

- Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A. In terms of the set builder notation, we have

$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$



$A - B$ is shaded.

Remark

- Note we can also denote the difference by $A \backslash B$.

Example 6

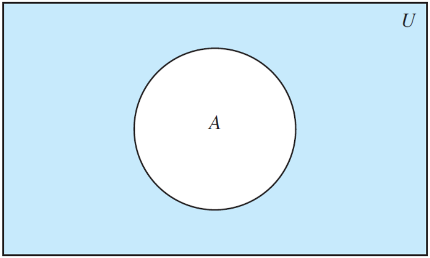
Example

The difference of $\{1,3,5\}$ and $\{1,2,3\}$ is the set $\{5\}$; that is, $\{1,3,5\} - \{1,2,3\} = \{5\}$ while the difference of $\{1,2,3\}$ and $\{1,3,5\}$ is the set $\{2\}$.

Complements

- Let U be the universal set. The complement of the set A, denoted by $\bar{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$. Thus,

$$\bar{A} = \{x \in U \mid x \notin A\}.$$



$\overline{A}$ is shaded.

- You can also denote the complement by A^c.

Example 7

Example

Let $A = \{a, e, i, o, u\}$ (where the universal set is the set of letters of the English alphabet). Then
$\bar{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$

Set identities

- Here, we present an identity given by

Theorem

For any sets A and B, $A - B = A \cap \bar{B}$.

Proof.

We must show $A - B \subseteq A \cap \bar{B}$ and $A \cap \bar{B} \subseteq A - B$. WLOG, we can assume that there exists a universal set U, of which A and B are both subsets. First, we show that $A - B \subseteq A \cap \bar{B}$. Let $x \in A - B$. By definition of set difference, $x \in A$ and $x \notin B$. By definition of complement, $x \notin B$ implies that $x \in \bar{B}$. Hence, it is true that both, $x \in A$ and $x \in \bar{B}$. By definition of intersection, $x \in A \cap \bar{B}$. ∎

Proof.

Now we show that $A \cap \bar{B} \subseteq A - B$. Let $x \in A \cap \bar{B}$. By definition of intersection, $x \in A$ and $x \in \bar{B}$. By definition of complement, $x \in \bar{B}$ implies that $x \notin B$. Hence, $x \in A$ and $x \notin B$. By definition of set difference, $x \in A - B$. Thus, $A - B = A \cap \bar{B}$. ■

More set identities can be given in the following table.

Identity	Name
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\bar{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \bar{A} \cup \bar{B}$ $\overline{A \cup B} = \bar{A} \cap \bar{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \bar{A} = U$ $A \cap \bar{A} = \emptyset$	Complement laws

Example 8

- Let us prove several of the above identities.

Example

Show $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

Proof.

We have proved the identity $A - B = A \cap \bar{B}$ by mutual subsets. Here let us show how to complete the proof using the set builder notation. We can prove this identity with the following steps.

Solution: We can prove this identity with the following steps.

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} \\
&= \{x \mid \neg(x \in (A \cap B))\} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} \\
&= \{x \mid x \notin A \vee x \notin B\} \\
&= \{x \mid x \in \bar{A} \vee x \in \bar{B}\} \\
&= \{x \mid x \in \bar{A} \cup \bar{B}\} \\
&= \bar{A} \cup \bar{B}.
\end{aligned}
$$

∎

Example 9

Example

Prove the second distributive law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A, B, and C.

Proof.

Let us prove this identity by mutual subsets, that is, $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

First, we demonstrate that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Suppose that $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. By the definition of union, it follows that $x \in A$, and $x \in B$ or $x \in C$ (or both). In other words, we know that the compound proposition $(x \in A) \wedge ((x \in B) \vee (x \in C))$ is true. By the distributive law for conjunction over disjunction, it follows that $((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))$. We conclude that either $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. By the definition of intersection, it follows that $x \in A \cap B$ or $x \in A \cap C$. Using the definition of union, we conclude that $x \in (A \cap B) \cup (A \cap C)$. Hence, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. ■

Proof.

Next, we show $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. Now suppose that $x \in (A \cap B) \cup (A \cap C)$. Then, by the definition of union, $x \in A \cap B$ or $x \in A \cap C$. By the definition of intersection, it follows that $x \in A$ and $x \in B$ or that $x \in A$ and $x \in C$. From this we see that $x \in A$, and $x \in B$ or $x \in C$. Consequently, by the definition of union we see that $x \in A$ and $x \in B \cup C$. Furthermore, by the definition of intersection, it follows that $x \in A \cap (B \cup C)$. We conclude that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. This completes the proof of the identity. ■

Membership tables

- Let us present $A \cup (B \cap C)$ using tables. We use 1 to denote the presence of some element x and 0 to denote its absence.

A	B	C	$B \cap C$	$A \cup (B \cap C)$
1	1	1	1	1
1	1	0	0	1
1	0	1	0	1
1	0	0	0	1
0	1	1	1	1
0	1	0	0	0
0	0	1	0	0
0	0	0	0	0

- This is an example of a membership table. The equality of two sets can be demonstrated by examining if the two sets have the identical inputs in the corresponding columns. In the above, we can see $B \cap C \neq A \cup (B \cap C)$.

Example 10

Example

We can also prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A, B, and C via the membership table.

Proof.

The membership table shall include $2^3 = 8$ rows.

A	B	C	$B \cup C$	$\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C})$	$A \cap B$	$A \cap C$	$(\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$
1	1	1	1	**1**	1	1	**1**
1	1	0	1	**1**	1	0	**1**
1	0	1	1	**1**	0	1	**1**
1	0	0	0	**0**	0	0	**0**
0	1	1	1	**0**	0	0	**0**
0	1	0	1	**0**	0	0	**0**
0	0	1	1	**0**	0	0	**0**
0	0	0	0	**0**	0	0	**0**

Note the highlighted columns $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ have the identical columns. Hence, we prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. ■

Generalized unions and intersections

- The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection, that is,

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i.$$

- The intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection, namely,

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i.$$

- For example, with $i = 1, 2, \ldots$, let $A_i = \{i, i+1, i+2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\},$$

and

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n.$$

- More generally, when I is a set, the notations $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$ are used to denote the intersection and union of the sets A_i for $i \in I$, respectively. Note that we have $\bigcap_{i \in I} A_i = \{x \mid \forall i \in I\, (x \in A_i)\}$ and $\bigcup_{i \in I} A_i = \{x \mid \exists i \in I\, (x \in A_i)\}$.
- For example, when $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Then,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1, 2, 3, \ldots\} = \mathbb{Z}^+$$

and

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1\}.$$