CS 0441
Lecture 3: Predicate Logic

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Overview

Predicate logic

- In previous sections, we can see that the usage of a propositional logic can help us better clarify the logics in natural languages and even allows us to complete the following inference.
- \- Socrates is a man.

  \- If Socrates is a man, then Socrates is mortal.

  \- Therefore, Socrates is mortal.

- This is an application of the inference rule called **modus ponens**, which says that from p and $p \to q$ you can deduce q. The first two statements are premises, and the last is the conclusion of the argument.
- However, if we want to extend the above inference rule for ourselves, we simply cannot do it because the second statement only refers to Socrates not every humanity.
- Thus, we will introduce a more powerful type of logic called **predicate logic** in this lecture.

Predicates and variables

- To extend the propositional logic, we need to extend our language to allow formulas that involve variables. Thus we will let x, y, z, *etc.* stand for any element of our **universe of discourse** or simply **domain** - essentially whatever things we happen to be talking about at the moment.
- We can now write statements like:

  - " x is greater than 3."

  - " x is the parent of y."

  - " $x + 2 = x^2$."

- Taking the first sentence for example, the first part, the variable x, is the subject of the statement.
- The second part—**the predicate**, "is greater than 3"—refers to a property that the subject of the statement can have.

- We can denote the statement " x is greater than 3 " by $P(x)$, where where P denotes the predicate "is greater than 3 " and x is the variable. That is, $P(x) =$ "x is greater than 3".
- The statement $P(x)$ is also said to be the value of the propositional function P at x. If we fill in specific values for the variable x, we have a proposition again, and can talk about the truth value of the proposition. For example, $P(1)$ is is false.

Example 1

Example

Let $P(x) =$ "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

Solution

We obtain the statement $P(4)$ *by setting* $x = 4$ *in the statement "*$x > 3$*." Hence,* $P(4)$*, which is the statement "*$4 > 3$*," is true. However,* $P(2)$*, which is the statement "*$2 > 3$*," is false.*

Example 2

Example

Let $A(x) =$ "Computer x is under attack by an intruder." Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are truth values of $A(CS1), A(CS2)$, and $A(MATH1)$?

Solution

We obtain the statement $A(CS1)$ *by setting* $x = CS1$ *in the statement "Computer* x *is under attack by an intruder." Because CS1 is not on the list of computers currently under attack, we conclude that* $A(CS1)$ *is false. Similarly, because CS2 and MATH1 are on the list of computers under attack, we know that* $A(CS2)$ *and* $A(MATH1)$ *are true.*

Quantifiers

- Next we introduce quantification, that is, to create a proposition from a propositional function.
- We bind the variables using **quantifiers**, which state whether the claim we are making applies to all values of the variable (**universal quantification**), or whether it may only apply to some (**existential quantification**).

Universal quantification

- The universal quantifier $\forall$ (pronounced "for all") says that a statement must be true for all values of a variable within some universe of allowed values (which is often implicit).
- For example, "all humans are mortal" could be written $\forall x : \text{Human}(x) \rightarrow \text{Mortal}(x)$ and "if x is positive then $x + 1$ is positive" could be written $\forall x (x > 0 \rightarrow x + 1 > 0)$.

Notations

- The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the universal quantifier.
- We read $\forall x P(x)$ as "for all $xP(x)$ " or "for every $xP(x)$." An element for which $P(x)$ is false is called a counterexample of $\forall x P(x)$.

Binding

- The term **binding** refers to that the quantifier is used on the variable. For example, in the statement $\exists x(x+y=1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free because it is not bound by a quantifier and no value is assigned to this variable.

Example 3

Example

Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Solution

Solution: Because $P(x)$ is true for all $x \in \mathbb{R}$, the quantification $\forall x P(x)$ is true.

Remark

- Generally, an implicit assumption is made that all domains of discourse for quantifiers are **nonempty**. Note that if the domain is empty, then $\forall x P(x)$ is true for any propositional function $P(x)$ because there are no elements x in the domain for which $P(x)$ is false.
- Normally we avoid using "for any x " because it is often ambiguous as to whether "any" means "every" or "some." In some cases, "any" is unambiguous, such as when it is used in negatives, for example, "there is not any reason to avoid studying."

Example 4

Example

Let $Q(x)$ be the statement " $x < 2$." What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Solution

$Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus $\forall x Q(x)$ is false.

- The statement $\forall x : P(x)$ is equivalent to a very large AND. $\forall x P(x)$ could be rewritten (if you had an infinite amount of paper) as $P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge \ldots$ Normal first-order logic (the logic we learned in this course) doesn't allow infinite expressions like this, but it may help in visualizing what $\forall x P(x)$ actually means.
- For a finite number of elements in a domain to be listed, say, $x_1, x_2, \ldots, x_n$-it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n).$$

Example 5

Example

What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement " $x^2 < 10$ " and the domain consists of the positive integers not exceeding 4?

Solution

The statement $\forall x P(x)$ *is the same as the conjunction*

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4),$$

because the domain consists of the integers $1, 2, 3$, *and 4. Because* $P(4)$, *which is the statement "*$4^2 < 10$*," is false, it follows that* $\forall x P(x)$ *is false.*

Example 6

Example

What is the truth value of $\forall x \left(x^2 \geqslant x\right)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

Solution

The universal quantification $\forall x \in \mathbb{R}, \left(x^2 \geqslant x\right)$ *is false. For example,* $\left(\frac{1}{2}\right)^2 \geqslant \frac{1}{2}$*. Note that* $x^2 \geqslant x$ *if and only if* $x^2 - x = x(x-1) \geqslant 0$*. Consequently,* $x^2 \geqslant x$ *if and only if* $x \leqslant 0$ *or* $x \geqslant 1$*. It follows that* $\forall x \left(x^2 \geqslant x\right)$ *is false if the domain consists of all real numbers (because the inequality is false for all real numbers* x *with* $0 < x < 1$*). However, if the domain consists of the integers,* $\forall x \left(x^2 \geqslant x\right)$ *is true, because there are no integers* x *with* $0 < x < 1$*.*

Existential quantification

- The existential quantifier $\exists$ (pronounced "there exists") says that a statement must be true for at least one value of the variable.
- We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.
- A domain must always be specified when a statement $\exists x P(x)$ is used.

- In addition to the phrase "there exists," we can also express existential quantification in many other ways, such as by using the words "for some," "for at least one," or "there is."
- The existential quantification $\exists x P(x)$ is read as "There is an x such that $P(x)$," "There is at least one x such that $P(x)$, or "For some $xP(x)$."

Example 7

Example

Let $P(x)$ denote the statement " $x > 3$." What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

Solution

Because "$x > 3$" is sometimes true-for instance, when $x = 4$ - the existential quantification of $P(x)$, which is $\exists x P(x)$, is true.

Remark

- When all elements in the domain can be listed-say, $x_1, x_2, \ldots, x_n$ - the existential quantification $\exists x P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n).$$

Example 7

Example

What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement "$x^2 > 10$ " and the universe of discourse consists of the positive integers not exceeding 4?

Solution

Because the domain is $\{1, 2, 3, 4\}$*, the proposition* $\exists x P(x)$ *is the same as the disjunction*

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

Because $P(4)$*, which is the statement "* $4^2 > 10$*," is true, it follows that* $\exists x P(x)$ *is true.*

Other quantifiers

- Note in addition there is no limitation on the number of different quantifiers we can define, such as "there are exactly two," "there are no more than three," "there are at least 100," and so on. Of these other quantifiers, the one that is most often seen is the uniqueness quantifier, denoted by $\exists\,!$ or $\exists_1$.

Quantifiers with restricted domains

- Sometimes we want to limit the universe over which we quantify to some restricted set, e.g., all positive integers or all baseball teams. We've previously seen how to do this using set-membership notation, but can also do this for more general predicates either explicitly using implication:

  $$\forall x : x > 0 \to x - 1 \geqslant 0$$

  or in abbreviated form by including the restriction in the quantifier expression itself:

  $$\forall x > 0 : x - 1 \geqslant 0.$$

- Similarly
$$\exists x : x > 0 \wedge x^2 = 81$$
can be written as
$$\exists x > 0 : x^2 = 81.$$
Note that constraints on $\exists$ get expressed using AND rather than implication.

Precedence of quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.
- For example, $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$. In other words, it means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x (P(x) \vee Q(x))$.

Logical equivalences involving quantifiers

- Logical equivalences involving quantifiers are defined in a similar way as we defined the equivalences without quantifiers.

Definition

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Example 8

Example

Show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent (where the same domain is used throughout). This logical equivalence shows that we can distribute a universal quantifier over a conjunction. Furthermore, we can also distribute an existential quantifier over a disjunction. However, we cannot distribute a universal quantifier over a disjunction, nor can we distribute an existential quantifier over a conjunction.

Solution

We can use the following truth table to demonstrate.

Negation and quantifiers

- The following equivalences hold:

$$\neg \forall x P(x) \equiv \exists x \neg P(x).$$
$$\neg \exists x P(x) \equiv \forall x \neg P(x).$$

- These are essentially the quantifier version of De Morgan's laws: the first says that if you want to show that not all humans are mortal, it's equivalent to finding some human that is not mortal. The second says that to show that no human is mortal, you have to show that all humans are not mortal.

Example 9

Example

What are the negations of the statements "There is an honest politician" and "All Americans eat cheeseburgers"?

Solution

Let $H(x) =$ "x is honest". Then the statement "There is an honest politician" is represented by $\exists x H(x)$, where the domain consists of all politicians. The negation of this statement is $\neg\exists x H(x)$, which is equivalent to $\forall x \neg H(x)$. Translated into natural English, we have "Not all politicians are honest".

Solution

Let $C(x) =$ *"x eats cheeseburgers." Then the statement "All Americans eat cheeseburgers" is represented by* $\forall x C(x)$*, where the domain consists of all Americans. The negation of this statement is* $\neg \forall x C(x)$*, which is equivalent to* $\exists x \neg C(x)$*. This negation can be expressed "Some Americans do not eat cheeseburgers".*

Example 10

Example

Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

Solution

By De Morgan's law for universal quantifiers, $\neg\forall x(P(x) \rightarrow Q(x))$ *can be expressed as* $\exists x(\neg(P(x) \rightarrow Q(x)))$*. By the equivalence identity of the implication, we know*

$$\begin{aligned} \neg(P(x) \rightarrow Q(x)) &\equiv \neg(\neg P(x) \vee Q(x)) \\ &\equiv P(x) \wedge \neg Q(x) \\ &\equiv P(x) \wedge \neg Q(x). \end{aligned}$$

It follows that $\neg\forall x(P(x) \rightarrow Q(x))$ *and* $\exists x(P(x) \wedge \neg Q(x))$ *are logically equivalent.*

Solution

We introduce $C(x) =$ *" x has studied calculus." Consequently, if the domain for x consists of the students in the class, we can translate our statement as* $\forall x C(x)$*.*

Solution

Alternatively, if we change the domain to consist of all people, we will need to express our statement as "For every person x, if person x is a student in this class then x has studied calculus." Denote $S(x) =$ "person x is in this class", we see that our statement can be expressed as $\forall x(S(x) \to C(x))$.

Solution

Finally, when we are interested in the background of people in subjects besides calculus, we may prefer to use the two-variable quantifier $Q(x, y)$ for the statement "student x has studied subject y." Then we would replace $C(x)$ by $Q(x, \text{calculus})$ in both approaches to obtain $\forall x Q(x, \text{calculus})$ or $\forall x(S(x) \rightarrow Q(x, \text{calculus}))$.

Example 11

Example

Use predicates and quantifiers to express the system specifications "Every mail message larger than one megabyte will be compressed" and "If a user is active, at least one network link will be available."

Solution

Solution: Let $S(m, y)$ be "Mail message m is larger than y megabytes," where the variable x has the domain of all mail messages and the variable y is a positive real number, and let $C(m)$ denote "Mail message m will be compressed." Then the specification "Every mail message larger than one megabyte will be compressed" can be represented as $\forall m(S(m, 1) \rightarrow C(m))$.

Solution

Let $A(u)$ represent "User u is active," where the variable u has the domain of all users, let $S(n, x)$ denote "Network link n is in state x," where n has the domain of all network links and x has the domain of all possible states for a network link. Then the specification "If a user is active, at least one network link will be available" can be represented by $\exists u A(u) \rightarrow \exists n S(n, \text{available})$.

- Let us see how we use the predicate logic for the aforementioned modus ponens with the following example.

Example 12

Example

Consider these statements, of which the first three are premises and the fourth is a valid conclusion.
"All hummingbirds are richly colored."
"No large birds live on honey."
"Birds that do not live on honey are dull in color."
"Hummingbirds are small."
Let $P(x), Q(x), R(x)$, and $S(x)$ be the statements "x is a hummingbird," "x is large," "x lives on honey," and "x is richly colored," respectively. Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and $P(x), Q(x), R(x)$, and $S(x)$.

Solution

We can express the statements in the argument as

$$\forall x(P(x) \to S(x)).$$
$$\neg \exists x(Q(x) \wedge R(x)).$$
$$\forall x(\neg R(x) \to \neg S(x)).$$
$$\forall x(P(x) \to \neg Q(x)).$$

Nested quantifiers

- It is possible to **nest quantifiers**, meaning that the statement bound by a quantifier itself contains quantifiers. For example, the statement "there is no largest prime number" could be written as

  $$\neg \exists x : (\text{Prime}(x) \wedge \forall y : y > x \to \neg \, \text{Prime}(y)),$$

  i.e., "there does not exist an x that is prime and any y greater than x is not prime." Note $\text{Prime}(\cdot)$ denotes a prime number.

- Or in a shorter (though not strictly equivalent) form:

$$\forall x \exists y : y > x \wedge \mathrm{Prime}(y),$$

which we can read as "for any x there is a bigger y that is prime."

Order of the nested quantifiers

- To read a statement on Page 56, treat it as a game between the $\forall$ player and the $\exists$ player. Because the $\forall$ comes first in this statement, the for-all player gets to pick any x it likes. The exists player then picks a y to make the rest of the statement true. The statement as a whole is true if the $\exists$ player always wins the game.

- As in many two-player games, it makes a difference who goes first. If we write likes (x, y) for the predicate that x likes y, the statements

$$\forall x \exists y : \text{likes}(x, y)$$

and

$$\exists y \forall x : \text{likes}(x, y)$$

mean very different things.

- The first says that for every person, there is somebody that that person likes: we live in a world with no complete misanthropes. The second says that there is some single person who is so immensely popular that everybody in the world likes them.
- The nesting of the quantifiers is what makes the difference: in $\forall x \exists y : \text{likes}(x, y)$, we are saying that no matter who we pick for x, $\exists y : \text{likes}(x, y)$ is a true statement; while in $\exists y \forall x : \text{likes}(x, y)$, we are saying that there is some y that makes $\forall x : \text{likes}(x, y)$ true.

- As a summary, we can tabulate the following table.

Statement	When True?	When False?
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$	There is a pair x, y for which $\exists y \exists x P(x, y)$	$P(x, y)$ is true. pair x, y.

- Naturally, such games can go on for more than two steps, or allow the same player more than one move in a row. For example
$$\forall x \forall y \exists z : x^2 + y^2 = z^2$$
is a kind of two-person challenge version of the Pythagorean theorem where the universal player gets to pick x and y and the existential player has to respond with a winning z.

Remark

- One thing to note about nested quantifiers is that we can switch the order of two universal quantifiers or two existential quantifiers, but we can't swap a universal quantifier for an existential quantifier or vice versa.
- So for example $\forall x \forall y : (x = y \to x + 1 = y + 1)$ is logically equivalent to $\forall y \forall x : (x = y \to y + 1 = x + 1)$, but $\forall x \exists y : y < x$ is not logically equivalent $\exists y \forall x : y < x$.

Example 13

Example

Translate the statement "The sum of two positive integers is always positive" into a logical expression.

Solution

We first rewrite it so that the implied quantifiers and a domain are shown: "For every two integers, if these integers are both positive, then the sum of these integers is positive."
Next, we introduce the variables x and y to obtain "For all positive integers x and y, $x + y$ is positive." Consequently, we can express this statement as

$$\forall x \forall y((x > 0) \land (y > 0) \rightarrow (x + y > 0)).$$

Solution

Using the positive integers as the domain, then the statement "The sum of two positive integers is always positive" becomes "For every two positive integers, the sum of these integers is positive. We can express this as

$$\forall x \forall y (x + y > 0),$$

where the domain for both variables consists of all positive integers.

Example 14

Example

Use quantifiers to express the definition of the limit of a real-valued function $f(x)$ of a real variable x at a point a in its domain.

Solution

Recall that the definition of the statement

$$\lim_{x \to a} f(x) = L$$

is: For every real number $\epsilon > 0$ *there exists a real number* $\delta > 0$ *such that* $|f(x) - L| < \epsilon$ *whenever* $0 < |x - a| < \delta$. *This definition of a limit can be phrased in terms of quantifiers by*

$$\forall \epsilon \exists \delta \forall x (0 < |x - a| < \delta \to |f(x) - L| < \epsilon),$$

where the domain for the variables δ *and* ϵ *consists of all positive real numbers and for* x *consists of all real numbers.*

Solution

This definition can also be expressed as

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$$

when the domain for the variables ϵ *and* δ *consists of all real numbers, rather than just the positive real numbers.*

Example 15

Example

Every even number greater than 2 can be expressed as the sum of two primes. Translate it into logic expressions.

Solution

We have

$$\forall x : (\text{Even}(x) \wedge x > 2) \to (\exists p \exists q : \text{Prime}(p) \wedge \text{Prime}(q) \wedge (x = p + q)),$$

where $\text{Even}(\cdot)$ *denotes an even number and* $\text{Prime}(\cdot)$ *is a prime number. Note this is Goldbach's conjecture. The truth value of this statement is currently unknown.*

Example 16

Example

Translate the statement

$$\exists x \forall y \forall z((F(x, y) \land F(x, z) \land (y \neq z)) \to \neg F(y, z))$$

into English, where $F(a, b)$ means a and b are friends and the domain for x, y, and z consists of all students in your school.

Solution

We first examine the expression $(F(x, y) \wedge F(x, z) \wedge (y \neq z)) \rightarrow \neg F(y, z)$. *This expression says that if students* x *and* y *are friends, and students* x *and* z *are friends, and furthermore, if* y *and* z *are not the same student, then* y *and* z *are not friends.*

It follows that the original statement, which is triply quantified, says that there is a student x *such that for all students* y *and all students* z *other than* y, *if* x *and* y *are friends and* x *and* z *are friends, then* y *and* z *are not friends. In other words, there is a student none of whose friends are also friends with each other.*

Example 17

Example

Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution

Solution: Let $P(w, f)$ *be "*w *has taken* f*" and* $Q(f, a)$ *be "*f *is a flight on* a*." We can express the statement as*

$$\exists w \forall a \exists f (P(w, f) \land Q(f, a)),$$

where the domains of discourse for w, f*, and* a *consist of all the women in the world, all airplane flights, and all airlines, respectively.*

Solution

The statement could also be expressed as

$$\exists w \forall a \exists f R(w, f, a),$$

where $R(w, f, a)$ *is "* w *has taken* f *on* a*." Although this is more compact, it somewhat obscures the relationships among the variables. Consequently, the first solution is usually preferable.*

Negating nested quantifiers

- We can also negate nested quantifiers, in which we will negate from the outer to the inner statement. Let us illustrate with what follows.

Example 18

Example

Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution

By Example 17, we know the statement can be expressed as $\neg\exists w \forall a \exists f (P(w,f) \wedge Q(f,a))$*, where* $P(w,f)$ *is " w has taken f " and* $Q(f,a)$ *is " f is a flight on a." By successively applying De Morgan's laws for quantifiers, we find*

$$\begin{aligned}\neg\exists w \forall a \exists f (P(w,f) \wedge Q(f,a)) &\equiv \forall w \neg\forall a \exists f (P(w,f) \wedge Q(f,a)) \\ &\equiv \forall w \exists a \neg\exists f (P(w,f) \wedge Q(f,a)) \\ &\equiv \forall w \exists a \forall f \neg(P(w,f) \wedge Q(f,a)) \\ &\equiv \forall w \exists a \forall f (\neg P(w,f) \vee \neg Q(f,a)).\end{aligned}$$

This last statement states "For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline."