HW 4 Key

1. (15pt) A coin is biased such that a head is three times as likely to occur as a tail. Find the expected number of tails when this coin is tossed twice.

   Assigning wrights of 3*w* and *w* for a head and tail, respectively. We obtain P(H) = ¾ and P(T) = ¼. The sample space for the experiment is S = {HH, HT, TH, TT}. Now if X represents the number of tails that occur in two tosses of the coin, we have

   P(X=0) = P(HH) = (3/4)(3/4) = 9/16
   P(X=1) = P(HT) + P(TH) = (2)(3/4)(1/4) = 3/8
   P(X=2) = P(TT) = (1/4)(1/4) = 1/16

   We get E(X) = (0)(9/16)+(1)(3/8)+(2)(1/16) = 1/2

2. (15pt) The density function of the continuous random variable X, the total number of hours, in units of 100 hours, that a family runs a vacuum cleaner over a period of one year, is

$$f(x) = \begin{cases} x, & 0 < x < 1, \\ 2 - x, & 1 \le x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

   Find the average number of hours per year that families run their vacuum cleaners.

$$E(X) = \int_0^1 x^2 dx + \int_1^2 x(2-x)dx = 1$$

   Thus, the average number of hours per year is (1)(100) = 100 hours.

3. (15pt) The hospitalization period, in days, for patients following treatment for a certain type of kidney disorder is a random variable Y = X + 4, where X has the density function

$$f(x) = \begin{cases} \frac{32}{(x+4)^3}, & x > 0, \\ 0, & elsewhere. \end{cases}$$

   Find the average number of days that a person is hospitalized following treatment for this disorder.

$$E(Y) = E(X+4) = \int_0^\infty (x+4)\left[\frac{32}{(x+4)^3}\right] dx = 8\ days$$

4. (15pt) The cost of a certain vehicle diagnostic test depends on the number of cylinders X in the vehicle's engine. The probability distribution of X is shown as the table below. Suppose the cost function is given by $h(X) = 20 + 3X + 0.5X^2$.

x	4	6	8
p(x)	0.5	0.3	0.2

   Find the expected cost.

Let the random variable Y = h(X). The probability distribution of Y is as follows:

Y	40	56	76
p(y)	0.5	0.3	0.2

$$E(Y) = E[h(X)] = \sum yp(y) = (40)(0.5) + (56)(0.3) + (76)(0.2) = 52$$

OR

$E(X) = (4)(0.5) + (6)(0.3) + (8)(0.2) = 5.4$

$E(X^2) = (16)(0.5) + (36)(0.3) + (64)(0.2) = 31.6$

$E[h(X)] = 20 + 3E(X) + 0.5E(X^2) = 52$

5. (20pt) Evaluate $E(2XY^2 - X^2Y)$ for the joint probability distribution shown as follows.

$f(x,y)$		$x=0$	$x=1$	$x=2$	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$$E(2XY^2 - X^2Y) = 2E(XY^2) - E(X^2Y)$$

$$E(XY^2) = \sum_0^2 \sum_0^2 xy^2 f(x,y) = \frac{3}{14}$$

$$E(X^2Y) = \sum_0^2 \sum_0^2 x^2 y f(x,y) = \frac{3}{14}$$

Thus, $E(2XY^2 - X^2Y) = \frac{3}{14}$

6. (20pt) If the joint density function of X and Y is given by

$$f(x,y) = \begin{cases} \frac{2}{7}(x+2y), & 0 < x < 1, 1 < y < 2, \\ 0, & elsewhere, \end{cases}$$

find the expected value of $g(X,Y) = \frac{X}{Y^3} + X^2Y$.

$$E[g(X,Y)] = E\left(\frac{X}{Y^3} + X^2Y\right) = E\left(\frac{X}{Y^3}\right) + E(X^2Y)$$

$$E\left(\frac{X}{Y^3}\right) = \int_1^2 \int_0^1 \frac{2x(x+2y)}{7y^3} dxdy = \frac{15}{84}$$

$$E(X^2Y) = \int_1^2 \int_0^1 \frac{2x^2y(x+2y)}{7} dxdy = \frac{139}{252}$$

Thus, $E[g(X,Y)] = \frac{46}{63}$