STAT 1151
Introduction to Probability

Lecture 2 Probability

Xiaomei Tan

Last Lecture

- **Chapter 2. Probability**
  - Foundations
    - **Terminology**
      - Experiment
      - Sample space
      - Sample point
      - Event
    - **Set Theory Symbols**
      - Union
      - Intersection
      - Inclusion
      - Mutually exclusive
      - Complement
    - **Set Properties**
      - Commutative law
      - Associative law
      - Distributive law
      - DeMorgan's law
    - **Venn Diagram**
  - Probability of An Event
    - Sum of the probabilities of all sample points in E
    - Ratio of #elements in E to #elements in S
      - Multiplication rule
      - Permutations
      - Combinations

Counting Rules

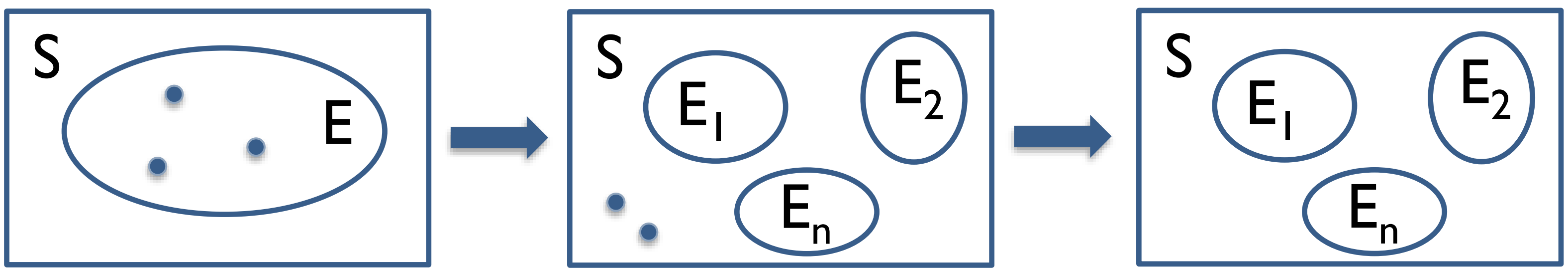
Outline

- Chapter 2 Probability
  - Additive Rules
  - Conditional Probability
  - Independence
  - Product Rule
  - Total Probability
  - Bayes' Rule

Calculating Probability

- **Method 1.** A probability for an event E of a sample space S can be computed by $$P(E) = \frac{\#\text{ elements in event E}}{\#\text{ elements in the sample space S}}$$

- **Method 2.** A probability for an event E of a sample space S is the sum of the probabilities of all sample points in E.



If events $E_1, E_2, \ldots, E_n$ are mutually exclusive, then

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = P(E_1) + P(E_2) + \cdots + P(E_n)$$

* If $E_1, E_2, E_3, \ldots, E_n$ are mutually excusive and $E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n = S$,

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = P(E_1) + P(E_2) + \cdots + P(E_n) = P(S) = 1$$

This is **additive rule**.

Additive Rules

Example:
What is the probability of getting a total of 7 or 11 when a pair of 6-sided fair dice is tossed?

Additive Rules

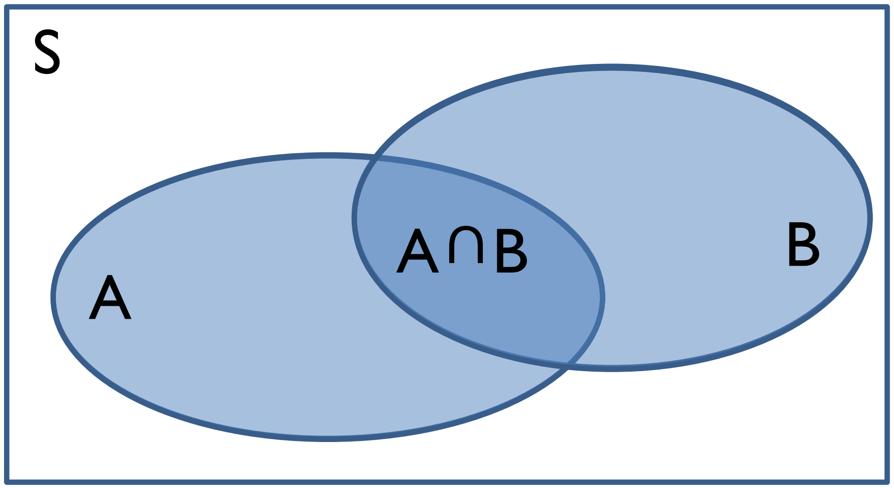
Example:

If the probabilities are, respectively, 0.09, 0.15, 0.21, and 0.23 that a person purchasing a new car will choose the color green, white, red, or blue, what is the probability that a given buyer will purchase a new car that comes in one of those colors?

Additive Rules

What if events $E_1, E_2, E_3, \ldots$ are NOT mutually exclusive?

Let's look at a special case for two events A and B:



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

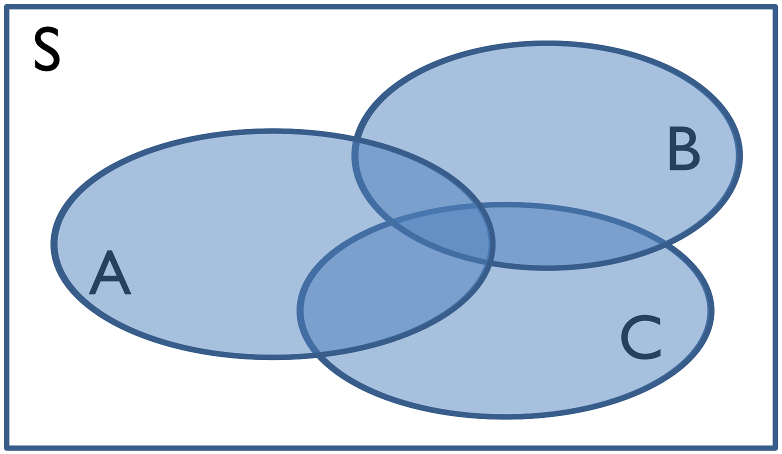
*When events A and B are mutually exclusive, $A \cap B = \phi, P(A \cap B) = 0$, then $P(A \cup B) = P(A) + P(B)$

Calculating Probability

Example:
John is going to graduate from an IE department in a university by the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company A is 0.8, and his probability of getting an offer from company B is 0.6. If he believes that the probability that he will get offers from both companies is 0.5, what is the probability that he will get at least one offer from these two companies?

Additive Rules

Let's look at a special case for three events A, B, and C:



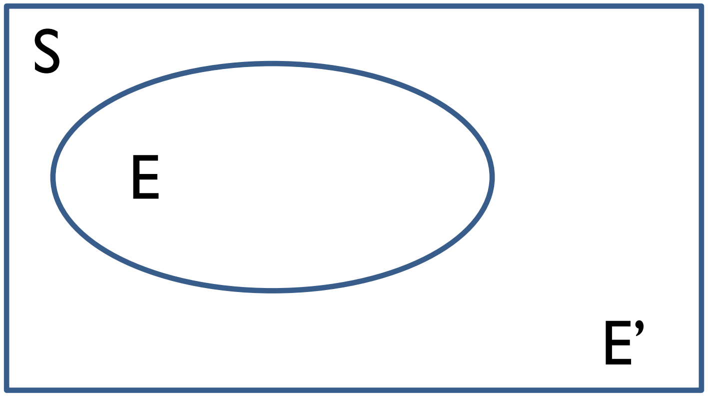
$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$-P(A \cap B) - P(A \cap C) - P(B \cap C)$$
$$+P(A \cap B \cap C)$$

*General Inclusion-Exclusion formula (容斥原理):

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) - \sum_{i<j}\sum P(E_i E_j) + \sum_{i<j<k}\sum\sum P(E_i E_j E_k) - \cdots + (-1)^{n+1} P(E_1 E_2 \cdots E_n)$$

Additive Rules

Sometimes it is easier to calculate the probability that an event does not occur than it is to calculate the probability that the event occurs.



If events E and E' are complementary events (互斥事件), then

$$P(E) + P(E') = P(S) = 1$$

Additive Rules

Example:
If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

Additive Rules

Example:

Suppose the manufacturer's specifications for the length of a certain type of computer cable are 2000 ± 10 millimeters. In this industry, it is known that small cable is just as likely to be defective (not meeting specifications) as large cable. That is, the probability of randomly producing a cable with length exceeding 2010 millimeters is equal to the probability of producing a cable with length smaller than 1990 millimeters. The probability that the production procedure meets specifications is known to be 0.99.

(a) What is the probability that a cable selected randomly is too large?

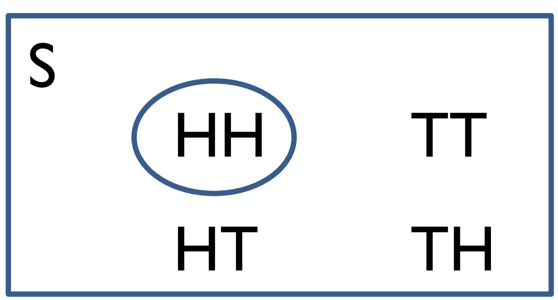
(b) What is the probability that a randomly selected cable is larger than 1990 millimeters?

Conditional Probability

We begin with a simple example to illustrate the difference between <u>probability</u> and <u>conditional probability</u>.

Example:
We toss two coins in succession:

(a) What is the probability of getting both Heads?
(b) What is the probability of getting both Heads given that the first toss shows Head?



Probability

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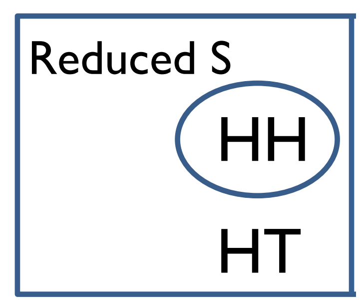

Conditional probability

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Conditional Probability

Definition:
The probability of an event B occurring when it is known that some event A has occurred is called a **conditional probability**, denoted by $P(B|A)$.

*It is usually read "the probability that B occurs given that A occurs" or "the probability of B, given A"

Example:
Among 5 motors, one is defective. Two are to be selected at random for use on a particular day. Find the probability that the second motor selected is non-defective, given that the first one is non-defective.

Conditional Probability

Revisit example:
A six-sided die is constructed so that the even numbers are twice as likely to occur as the odd numbers. Consider the event B of getting a perfect square (完全平方数) when the die is tossed.

(a) What is the probability of B?
(b) What is the probability of B suppose that it is known that the toss of the die resulted in a number greater than 3?

Conditional Probability

The conditional probability of B, given A, denoted by P(B|A), is defined by

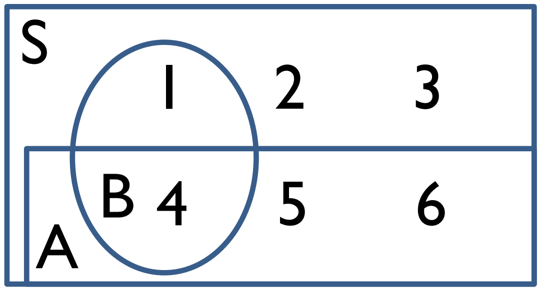
$$P(B|A) = \frac{\#\ Favorable\ outcomes\ for\ (A\ and\ B)}{Total\ \#\ possibe\ outcomes\ for\ A}$$

$$= \frac{P(A \cap B)}{P(A)}, \qquad provided\ P(A) > 0$$

where $P(A \cap B)$ and $P(A)$ are found from the original sample space S.

Previous example:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{2\omega}{5\omega} = \frac{2}{5}$$



Conditional Probability

Example:
Suppose that the sample space is the population of adults in a small town who have obtained a college degree. They are categorized according to gender and employment status as shown in Table. One of these individuals is to be selected at random for a tour throughout the country to publicize the advantages of establishing new industries in the town. Find the probability that a man is chosen, given that the one chosen is employed.

	Employed	**Unemployed**	**Total**
Male	460	40	500
Female	140	260	400
Total	600	300	900

Conditional Probability

Example:
The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane

(a) arrives on time, given that it departed on time,

(b) departed on time, given that it has arrived on time,

(c) arrives on time, given that it did not depart on time.

Independence

Revisit example:

Consider the event B of getting a perfect square (完全平方数) when a die is tossed. The die is constructed so that the even numbers are twice as likely to occur as the odd numbers.

(a) What is the probability of B?

(b) What is the probability of B suppose that it is known that the toss of the die resulted in a number greater than 3?

Independence

Revisit example:

Consider the event B of getting a perfect square (完全平方数) when a die is tossed. ~~The die is constructed so that the even numbers are twice as likely to occur as the odd numbers.~~

(a) What is the probability of B?

(b) What is the probability of B suppose that it is known that the toss of the die resulted in a number greater than 3?

Independence

Example:
Consider an experiment in which 2 cards are drawn in succession from an ordinary deck, with replacement.
The events are defined as

A: the first card is an ace,

B: the second card is a spade.

Independence

Probabilities are usually sensitive to the conditioning information. However, sometimes a probability does not change even if we have different conditions. We define such events are **independent**.

Definition:
Two events A and B are said to be **independent** if and only if

$$P(B|A) = P(B) \text{ or } P(A|B) = P(A),$$

assuming the existences of the conditional probabilities.

Otherwise, A and B are **dependent**.

Product Rule

Conditional probability:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \qquad provided\ P(A) > 0$$

Multiply the formula by P(A), we obtain the **<u>multiplicative rule</u>,** or **<u>product rule</u> (乘法法则).**

$$P(A \cap B) = P(A)P(B|A), \qquad provided\ P(A) > 0$$

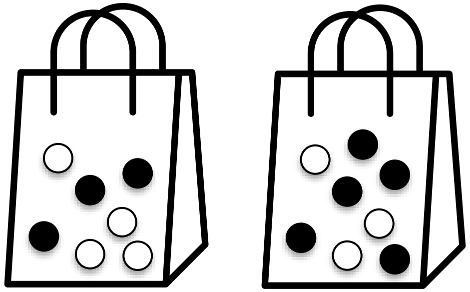
$$P(B \cap A) = P(B)P(A|B), \qquad provided\ P(B) > 0$$

It does not matter which event is referred to as *A* and which event is referred to as *B*.

Product Rule

Example:
Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Product Rule



Example:

One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Product Rule & Independence

Product rule:

$$P(A \cap B) = P(A)P(B|A), \qquad provided\ P(A) > 0$$

Independence: $P(B) = P(B|A)$

$$P(A \cap B) = P(A)P(B)$$

$$\boxed{P(AB) = P(A)P(B) \Leftrightarrow P(A \mid B) = P(A) \Leftrightarrow P(B \mid A) = P(B)}$$

Product Rule & Independence

Example:

Suppose that a foreman must select one worker from a pool of four available workers (numbered 1,2,3,4) for a special job. He selects the worker by mixing the four names and randomly selecting one. Events are defined:

- Event A: Worker 1 or 2 is selected,
- Event B: Worker 1 or 3 is selected,
- Event C: Worker 1 is selected.

Are A and B independent? Are A and C independent?

Product Rule & Independence

The product rule can be extended to more than two-event situations:

If, in an experiment, events $A_1, A_2, \ldots, A_k$ can occur, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k)$$
$$= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \ldots P(A_K|A_1 \cap A_2 \cap \cdots \cap A_{k-1})$$

$$P(A_2) \qquad \frac{P[A_3 \cap (A_1 \cap A_2)]}{P(A_1 \cap A_2)} = \frac{P(A_3)P(A_1 \cap A_2)}{P(A_1 \cap A_2)} = P(A_3)$$

If events $A_1, A_2, \ldots, A_k$ are independent, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)\,P(A_2) \ldots P(A_k)$$

Product Rule & Independence

Let us consider some simple examples in which independence is assumed.

Example:
Each person has blood type A, B, AB, or O. Type O represents the absence of a factor and is recessive to factors A and B. Thus, a person with type A blood may be either "homozygous" (AA, 纯合子) or "heterozygous" (AO, 杂合子). Similarly type B may be (BB) or (BO). Type AB occurs if a person is given A factor by one parent and a B factor by the other parent. To have type O, a person must be homozygous (OO).

Suppose a couple is preparing to have a child. One parent has blood type AB and the other is heterozygous B (BO). What are the possible blood types the child will have? What is the probability if the child has blood type B?

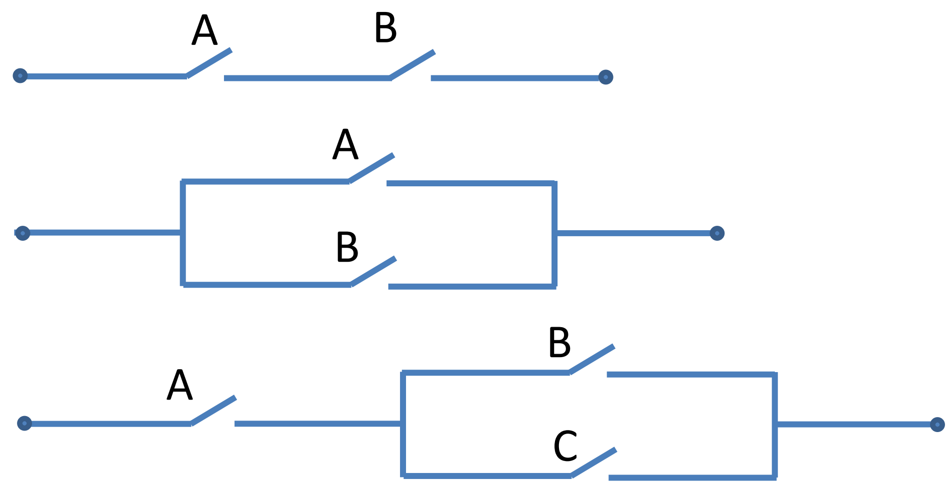
Product Rule & Independence

Example:
Switches in electrical circuits are often assumed to work (or fail) independently of each other. These switches may be set up as follows:

In series:

In parallel:

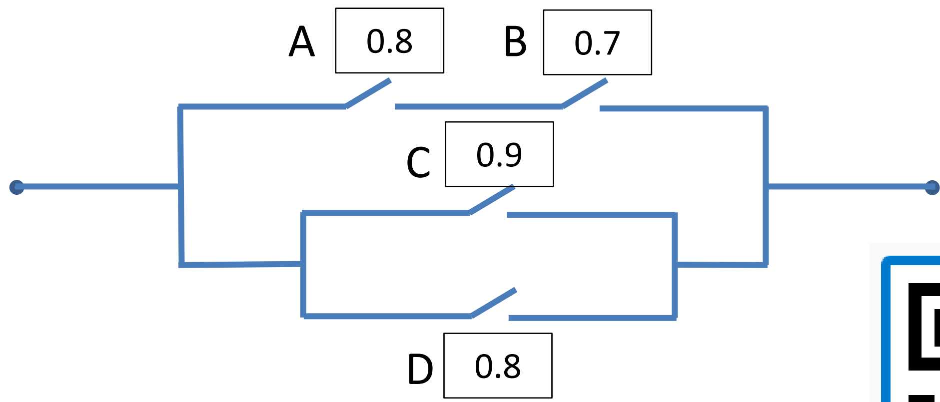
In combination:



When a switch is flipped, it will close with a probability 0.8. Suppose that all the switches are in open status, find the probability that the current will flow through when all the switches are flipped.

Product Rule & Independence

In-class exercise:
An electrical circuit is displayed below. The switches operate independently of each other, and the probability that each switch closes when it is flipped is displayed in the figure. What is the probability that current will flow through when the switches are flipped?



Put your answers here: PollEv.com/xtan166

Total Probability

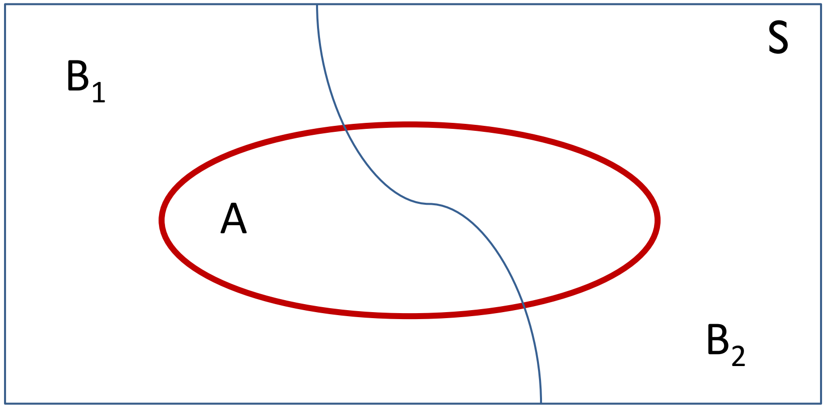
Revisit example:

A resident is being selected at random from the adults of a small town for a tour throughout the country to publicize the advantages of establishing new industries in the town. Suppose that we are now given the additional information that 36 of those employed and 12 of those unemployed are members of the Rotary Club. Find the probability of the event A that the individual selected is a member of the Rotary Club.

	Employed	Unemployed	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900

Total Probability

Definition:



$B_1 + B_2 = S$

$$A = (AB_1) \cup (AB_2)$$

where AB_1 and AB_2 are mutually exclusive; therefore we can find

$$P(A) = P(AB_1) + P(AB_2)$$

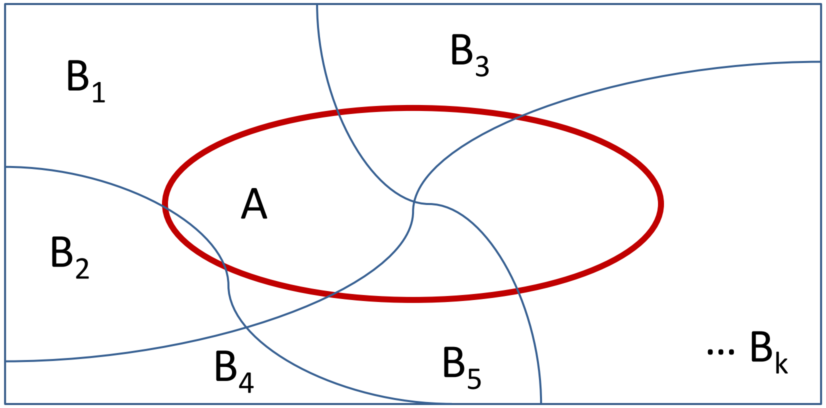
If conditional probabilities $P(A|B_1)$ and $P(A|B_2)$ are known, then

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2)$$

This is known as the theorem of **total probability**.

Total Probability

Generalization:



$B_1 + B_2 + \ldots + B_k = S$

If events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i)$$

Total Probability

Example:
In a certain assembly plant, three machines, B1, B2, and B3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Bayes' Rule

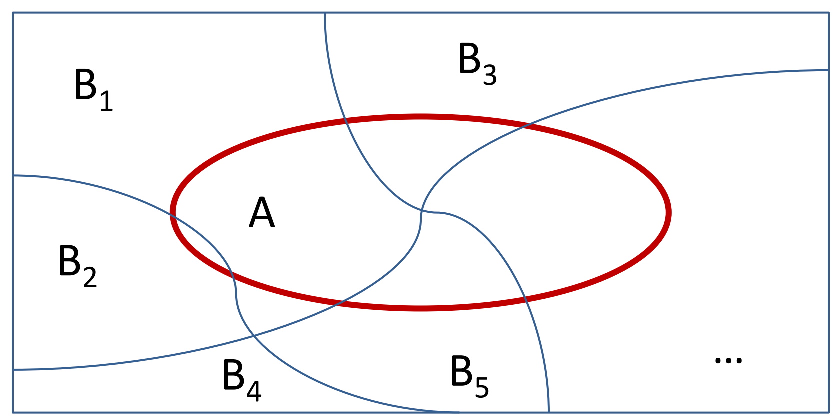
Revisit example:
In a certain assembly plant, three machines, B1, B2, and B3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected and it is defective. What is the probability that this product was made by machine B_1?

Bayes' Rule

Definition of **Bayes' Rule**:
If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) = 0$ for $i = 1, 2, \ldots, k$, then for any event A in S such that $P(A) = 0$,

$$P(B_r|A) = \frac{P(B_r \cap A)}{P(A)} = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(A \cap B_i)} = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}$$

for $r = 1, 2, \ldots, k$,



$B_1 + B_2 + \ldots + B_k = S$

Total Probability

Example:

A manufacturing firm employs three analytical plans for the design and development of a particular product. For cost reasons, all three are used at varying times. In fact, plans 1, 2, and 3 are used for 30%, 20%, and 50% of the products, respectively. The defect rate is different for the three procedures as follows:

$$P(D|P1) = 0.01, P(D|P2) = 0.03, P(D|P3) = 0.02,$$

where $P(D|Pj)$ is the probability of a defective product, given plan j. If a random product was observed and found to be defective, which plan was most likely used and thus responsible?

Homework

- Due: The beginning of next lecture
- Be sure to submit the correct document
- Save the record of the time when the work is finished, just in case