CS 0441
Lecture 4: Rules of inference

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Overview

Inference

- We will introduce the inference rule and proof in this lecture. First, we present the following definitions.
- Inferences are steps in reasoning, moving from premises to logical consequences.
- **An argument** is a sequence of statements that end with a conclusion
- An argument is **valid** if and only if the conclusion is true following from the truth value of the premises, namely, it is impossible to derive a false conclusion from true premises.

Valid arguments in propositional logic

- Consider the following argument involving propositions (which, by definition, is a sequence of propositions):

  "If you have a current password, then you can log onto the network."

  "You have a current password."

  Therefore,

  "You can log onto the network."

- We would like to examine the truth values of the above propositions.
- Let $p =$ "You have a current password" and $q =$ "You can log onto the network."
- Then the above argument has the form

$$\begin{array}{l} \quad p \to q \\ \quad \underline{p \qquad\quad} \\ \therefore q \end{array} \tag{1}$$

where $\therefore$ is the symbol that denotes "therefore."

- The argument (1) is in the **argument form**. By argument form, we mean a sequence of compound propositions involving propositional variables.
- We know that when p and q are propositional variables, the statement $((p \to q) \wedge p) \to q$ is a tautology.

Rules of inference

- With the argument form, we are able to demonstrate the validity of an argument via showing the validity of the corresponding argument form.
- Let us present the following rules of inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore\ q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore\ \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore\ p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore\ q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore\ p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore\ p$	$(p \wedge q) \to p$	Simplification
p q $\therefore\ p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore\ q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

Example 1

Example

State which rule of inference is the basis of the following argument: ''It is below freezing now. Therefore, it is either below freezing or raining now.''

Solution

Let $p =$ *"It is below freezing now" and* $q =$ *"It is raining now." Then this argument is of the form*

$$\frac{p}{\therefore p \vee q}$$

This is an argument that uses the addition rule.

Example 2

Example

State which rule of inference is used in the argument: If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Solution

Let $p =$ *"It is raining today," let* $q =$ *"We will not have a barbecue today," and let* $r =$ *"We will have a barbecue tomorrow." Then this argument is of the form*

$$\begin{array}{r} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Using rules of inference to build arguments

- Note the proof of validity of an argument form using a truth table is somehow tedious. In what follows, we introduce the inference rules to complete the proof. The proof is built on small blocks called **rules of inference**, based on which we can complete the validity proof of complicated arguments.

Example 3

Example

Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Solution

Let p be the proposition "It is sunny this afternoon," q the proposition "It is colder than yesterday," r the proposition "We will go swimming," s the proposition "We will take a canoe trip," and t the proposition "We will be home by sunset." Then the premises become $\neg p \wedge q, r \rightarrow p, \neg r \rightarrow s$, and $s \rightarrow t$. The conclusion is simply t. We need to give a valid argument with premises $\neg p \wedge q, r \rightarrow p, \neg r \rightarrow s$, and $s \rightarrow t$ and conclusion t.

Solution

We construct an argument to show that our premises lead to the desired conclusion as follows.

Step	Reason
1. $\neg p \wedge q$	*Premise*
2. $\neg p$	*Simplification using (1)*
3. $r \to p$	*Premise*
4. $\neg r$	*Modus tollens using (2) and (3)*
5. $\neg r \to s$	*Premise*
6. s	*Modus ponens using (4) and (5)*
7. $s \to t$	*Premise*
8. t	*Modus ponens using (6) and (7)*

Remark

- The above proof is presented in the Hilbert style.
- Note we can also present the proof in the Fitch style. Here, we briefly describe an alternative notation for natural deduction derivations, called the "**Fitch style**" notation. It is more linear, in the sense that a proof is essentially a list of formulas, one on each line, and formulas on later lines are meant to be consequences of formulas on earlier lines. Instead of listing premises in all lines, the Fitch style notation uses indentation to indicate a subderivation using a temporary hypotheses.

Fitch style proof notations

- In its simplest form, a Fitch style natural deduction is just a list of numbered lines, each containing a formula, such that each formula is either a premise (separated from the rest of the proof by a horizontal line), or else follows from previous formulas (indicated by a rule name and line numbers of relevant formulas). The very last line in the derivation contains the conclusion.

- Here is an example of a derivation of $A \to B, B \to C, A \vdash C$, where $\vdash$ is called turnstile, namely, the RHS can be derived from the LHS, and often called "proves" and "yields".

$$\begin{array}{r|ll}
1 & A \to B & \\
2 & B \to C & \\
3 & A & \\
\cline{2-2}
4 & B & MP, 1, 3 \\
5 & C & MP, 2, 4
\end{array}$$

- Note the vertical line here means implication or yield. Reading the line 1 and the line 5, we can see $A \to B, B \to C, A \vdash C$.

Remark

- We employ the modus ponens in the above Fitch style deduction, which is actually an improper example, since the Fitch style deduction is based on its own basic rules. Hence, **when using different styles, USE the corresponding given rules and DO NOT misuse the basic rules**.

- Sometimes a derivation contains a subderivation that depends on a hypothetical, or temporary, assumption. Such subderivations are indented and marked with another vertical line.

Example 4

- For example, here is a derivation of $A \to B, B \to C \vdash A \to C$:

1	$A \to B$	
2	$B \to C$	
3	$\quad A$	
4	$\quad B$	$\to$E, 1, 3
5	$\quad C$	$\to$E, 2, 4
6	$A \to C$	$\to$I, 3–5

- On line 3, we "temporarily" assume A. Lines 4 and 5 are consequences. The subderivation ends on line 5 with conclusion C; therefore, one has proved $A \to C$ at the "next level up" on line 6 .

- Note in Example 4, we prove using different reasons, such as Implication Elimination ($\rightarrow$ E) and Implication Introduction ($\rightarrow$ I). This is another difference between the Hilber style and the Fitch style, where they base on different rules of inference. Note **the Fitch style is based on more fundamental rules** rather than the inference rules on Page 8 in the slides, which we list with the file "Fitch style proof handout".

- Then we can rework on Example 3 with the Fitch style natural deduction.

Solution

1	$\neg p \wedge q$	
2	$r \to p$	
3	$\neg r \to s$	
4	$s \to t$	
5	$\quad r$	
6	$\quad p$	$\to E$, 2, 4
7	$\quad \neg p$	$\wedge E$, 1, 5
8	$\quad \bot$	$\neg E$, 5, 6
9	$\neg r$	$\wedge E$, 4, 7
10	s	$\to E$, 3, 9
11	t	$\to E$, 4, 10

Fallacies

- A fallacy is the use of invalid or otherwise faulty reasoning in the construction of an argument that may appear to be well-reasoned if unnoticed.
- Let us discuss the following two types of fallacies.

Fallacy of affirming the conclusion

- The first type is $((p \to q) \land q) \to p$, called **fallacy of affirming the conclusion**. Note this proposition is not a tautology because it is false when p is false and q is true.

Example 5

Example

Is the following argument valid?

If you do every problem in this book, then you will learn discrete mathematics.

You learned discrete mathematics.

Therefore, you did every problem in this book.

Solution

Let $p =$ *"You did every problem in this book" and* $q =$ *"You learned discrete mathematics." Then this argument is of the form: if* $p \to q$ *and* q*, then* p*, namely,* $((p \to q) \wedge q) \to p$*. This is an example of an incorrect argument using the fallacy of affirming the conclusion. Indeed, it is possible for you to learn discrete mathematics in some way other than by doing every problem in this book. (You may learn discrete mathematics by reading, listening to lectures, doing some, but not all, the problems in this book, and so on.)*

Fallacy of denying the hypothesis

- The other type $((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$, called **fallacy of denying the hypothesis**, is not a tautology, because it is false when p is false and q is true. Many incorrect arguments use this incorrectly as a rule of inference.

Example 6

Example

Following Example 5, is it correct to assume that you did not learn discrete mathematics if you did not do every problem in the book, assuming that if you do every problem in this book, then you will learn discrete mathematics?

Solution

It is possible that you learned discrete mathematics even if you did not do every problem in this book. This incorrect argument is of the form $p \to q$ *and* $\neg p$ *imply* $\neg q$*, which is an example of the fallacy of denying the hypothesis.*

Rules of Inference for Quantified Statements

- Next, let us introduce the rules of inference for quantified statements. Note the Fitch style deduction is more applied in this type of inference.

- The inference rule is introduced in what follows

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existential generalization

- Universal instantiation is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the domain, given the premise $\forall x P(x)$.
- Universal generalization is the rule of inference that states that $\forall x P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain.
- Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which $P(c)$ is true if we know that $\exists x P(x)$ is true.
- Universal instantiation is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the domain, given the premise $\forall x P(x)$. Universal instantiation is used

Example 7

Example

Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Solution

Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam." The premises are $\exists x(C(x) \land \neg B(x))$ and $\forall x(C(x) \to P(x))$. The conclusion is $\exists x(P(x) \land \neg B(x))$. These steps can be used to establish the conclusion from the premises.

Step	*Reason*
1. $\exists x(C(x) \land \neg B(x))$	*Premise*
2. $C(a) \land \neg B(a)$	*Existential instantiation from (1)*
3. $C(a)$	*Simplification from (2)*
4. $\forall x(C(x) \to P(x))$	*Premise*
5. $C(a) \to P(a)$	*Universal instantiation from (4)*
6. $P(a)$	*Modus ponens from (3) and (5)*
7. $\neg B(a)$	*Simplification from (2)*
8. $P(a) \land \neg B(a)$	*Conjunction from (6) and (7)*
9. $\exists x(P(x) \land \neg B(x))$	*Existential generalization from (8)*

- Note we can also complete a deductive argument via the Fitch style notations. You can see the four basic derivation rules in "Fitch style natural deduction handout".

Alternatively, we can solve Example 7 as follows

Solution

1		$\exists(C(x) \wedge \neg B(x)$	
2		$\forall(C(x) \rightarrow P(x))$	
3	u	$C(u) \wedge \neg B(u)$	
4		$\neg B(u)$	$\wedge E$, 3
5		$C(u)$	$\wedge E$, 3
6		$C(u) \rightarrow P(u)$	$\forall E$, 2
7		$P(u)$	$\rightarrow E$, 5, 6
8		$P(u) \wedge \neg B(u)$	$\wedge I$, 4, 6
9		$\exists(P(x) \wedge \neg B(x))$	$\exists I$, 8

Example 8

Example

Show that $\forall x(A(x) \to B(x)) \vdash \exists y A(y) \to \exists y B(y)$ in the Fitch style notations, namely, one direction in quantified De Morgan's Rule.

Solution

1		$\forall x(A(x) \rightarrow B(x))$	
2		$\quad \exists y A(y)$	
3	u	$\quad\quad A(u)$	
4		$\quad\quad \forall x(A(x) \rightarrow B(x))$	R, 1
5		$\quad\quad A(u) \rightarrow B(u)$	$\forall E$, 4
6		$\quad\quad B(u)$	$\rightarrow E$, 3, 5
7		$\quad\quad \exists y B(y)$	$\exists I$, 6
8		$\quad \exists y B(y)$	$\exists E$, 2, 3–7
9		$\exists y A(y) \rightarrow \exists y B(y)$	$\rightarrow I$, 2–8

Hence, we complete the proof.

Example 9

Example

Demonstrate $\forall x P(x, x) \to \forall y \exists z P(y, z)$ in the Fitch style proof.

Solution

1	$\forall x P(x,x)$	
2	$u \quad \forall x P(x,x)$	R, 1
3	$P(u,u)$	$\forall E$, 2
4	$\exists z P(u,z)$	$\exists I$, 3
5	$\forall y \exists z P(y,z)$	$\forall I$, 2–4
6	$\forall x P(x,x) \rightarrow \forall y \exists z P(y,z)$	$\rightarrow I$, 1–5