CS 0441
Lecture 2: Propositional Equivalences

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Course information

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Textbook Discrete Mathematics and Its Applications (7th Edition), by Kenneth H. Rosen.

Overview

1. Section 1.3
2. Propositional equivalences
3. Classifications of compound propositions
4. The logical equivalence of implication

Overview

5. Conjuctive normal form (CNF)

6. Logical equivalence identities

7. The extended De Morgan's Laws

8. Satisfiability

Propositional equivalences

- Note it is important to find an alternative expression or form in mathematics. In logics, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments.
- We call the procedure making the propositions with the same truth value **propositional equivalences**.

Classifications of compound propositions

- A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a **tautology**.
- A compound proposition that is always false is called a **contradiction**.
- A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Example 1

Example

For example, $p \vee \neg p$ is always true, it is a tautology. Because $p \wedge \neg p$ is always false, it is a contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical equivalences

- Next we present the definition of *logically equivalent*:

Definition

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- **Remark:** The symbol $\equiv$ is not a logical connective, and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology. The symbol $\Leftrightarrow$ is sometimes used instead of $\equiv$ to denote logical equivalence.

- Based on the above definition, we can use the truth table to determine if two compound propositions are equivalent.
- In particular, the compound propositions p and q are equivalent if and only if the columns giving their truth values agree.

De Morgan's laws

- One of the most important result in logical equivalences is **De Morgan's laws**, which are presented as

$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Table: De Morgan's laws.

Example 2

Example

Let us prove the second De Morgan's law $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

Proof.

We will complete the proof via the following truth table.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Because the truth values of the compound propositions $\neg(p \vee q)$ and $\neg p \wedge \neg q$ agree for all possible combinations of the truth values of p and q, it follows that $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology and that these compound propositions are logically equivalent. ■

The logical equivalence of implication

- Next, we introduce an important identity as the propositional equivalence for the implication, that is, $p \to q$ and $\neg p \vee q$ are logically equivalent.

- The identity can also be demonstrated via a truth table as follows

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

- Therefore, we show that $p \rightarrow q$ and $\neg p \vee q$ are equivalent.

Conjuctive normal form (CNF)

- A compound proposition is in **conjunctive normal form** (CNF for short) if it is obtained by ANDing together ORs of one or more variables or their negations (an OR of one variable is just the variable itself).
- The above implication equivalence is a good example of CNFs.
- The CNF are particularly useful because they support **resolution**, which we will introduce in Section 1.6.

Discussions on the rows of the truth table

- In the above discussions, we can see the truth tables involve 2^2 rows given with 2 variables in a compound proposition.
- Given with 3 variables, we need to present eight combinations TTT, TTF, TFT, TFF, FTT, FTF, FFT and FFF to tabulate the entire truth table, namely, the corresponding truth table consists of 2^3 rows.
- In general, 2^n rows are required if a compound proposition involves n propositional variables.

Logical equivalence identities

- The logical equivalence identities are presented as

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$ $P \oplus F \equiv P$ $P \oplus T \equiv \neg P$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws

$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \land q) \equiv \neg p \lor \neg q$ $\neg(p \lor q) \equiv \neg p \land \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv T$ $p \land \neg p \equiv F$	Negation laws

- Note that $\land, \lor, \oplus$, and $\leftrightarrow$ are all commutative.

Equivalences involving implication

- We here display some useful equivalences for compound propositions involving conditional statements.

$$
\begin{aligned}
&p \to q \equiv \neg p \vee q \\
&p \to q \equiv \neg q \to \neg p \\
&p \vee q \equiv \neg p \to q \\
&p \wedge q \equiv \neg(p \to \neg q) \\
&\neg(p \to q) \equiv p \wedge \neg q \\
&(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r) \\
&(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r \\
&(p \to q) \vee (p \to r) \equiv p \to (q \vee r) \\
&(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r
\end{aligned}
$$

- The proof of the equivalences will be a good exercise for you.
- Now let us rewrite $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$ using the equivalence laws.

$$
\begin{aligned}
& (p \to r) \vee (q \to r) \\
\equiv\ & (\neg p \vee r) \vee (\neg q \vee r) && [\text{Using } p \to q \equiv \neg p \vee q \text{ twice}] \\
\equiv\ & \neg p \vee \neg q \vee r \vee r && [\text{Associativity and commutativity of } \vee] \\
\equiv\ & \neg p \vee \neg q \vee r && [p \equiv p \vee p] \\
\equiv\ & \neg(p \wedge q) \vee r && [\text{De Morgan's law}] \\
\equiv\ & (p \wedge q) \to r. && [p \to q \equiv \neg p \vee q]
\end{aligned}
$$

Remark

- This last equivalence is a little surprising. It shows, for example, that if somebody says "It is either the case that if you study you will graduate from Yale with distinction, or that if you join the right secret society you will graduate from Yale with distinction", then this statement (assuming we treat the or as $\vee$) is logically equivalent to "If you study and join the right secret society, then you will graduate from Yale with distinction."
- It is simple to get tangled up in trying to parse the first of these two propositions; translating to logical notation and simplifying using logical equivalence is a good way to simplify it.

Equivalences involving bi-implication

- Equivalences for biconditional statements are also given in what follows.

$$
\begin{aligned}
&p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \\
&p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q \\
&p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q) \\
&\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q
\end{aligned}
$$

- Now we see the logical equivalence allows us to construct additional logical equivalences. The follow-up two examples will illustrate.

Example 3

Example

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution

$$
\begin{aligned}
& \neg(p \vee (\neg p \wedge q)) & \\
\equiv\ & \neg p \wedge \neg(\neg p \wedge q) & \textit{[Second De Morgan's law]} \\
\equiv\ & \neg p \wedge [\neg(\neg p) \vee \neg q] & \textit{[First De Morgan's law]} \\
\equiv\ & \neg p \wedge (p \vee \neg q) & \textit{[Double negation law]} \\
\equiv\ & (\neg p \wedge p) \vee (\neg p \wedge \neg q) & \textit{[Second distributive law for disjunction]} \\
\equiv\ & F \vee (\neg p \wedge \neg q) & [\neg p \wedge p \equiv F] \\
\equiv\ & (\neg p \wedge \neg q) \vee F & \textit{[Commutative law]} \\
\equiv\ & \neg p \wedge \neg q & \textit{[Identity law for } F]
\end{aligned}
$$

Example 4

Example

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution

$$
\begin{aligned}
& (p \wedge q) \rightarrow (p \vee q) & \\
\equiv\ & \neg(p \wedge q) \vee (p \vee q) & \textit{[Equivalence for implication]} \\
\equiv\ & (\neg p \vee \neg q) \vee (p \vee q) & \textit{[First De Morgan's law]} \\
\equiv\ & (\neg p \vee p) \vee (\neg q \vee q) & \textit{[Associative and commutative laws]} \\
\equiv\ & T \vee T & \textit{[Negation and commuative laws]} \\
\equiv\ & T & \textit{[Domination law]}
\end{aligned}
$$

The extended De Morgan's Laws

- With more propositions, we can further extend De Morgan's laws to

$$\neg\,(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$$

  and

$$\neg\,(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n)\,.$$

- Note that disjunctions $\vee$ and conjunctions $\wedge$ are associative and commutative. Hence, the above notations of the extended De Morgan's laws are well-defined.

- We will sometimes use the notation $\bigvee_{j=1}^{n} p_j$ for $p_1 \vee p_2 \vee \cdots \vee p_n$ and $\bigwedge_{j=1}^{n} p_j$ for $p_1 \wedge p_2 \wedge \cdots \wedge p_n$.
- Using this notation, the extended version of De Morgan's laws can be written concisely as $\neg\left(\bigvee_{j=1}^{n} p_j\right) \equiv \bigwedge_{j=1}^{n} \neg p_j$ and $\neg\left(\bigwedge_{j=1}^{n} p_j\right) \equiv \bigvee_{j=1}^{n} \neg p_j$.

Example 5

- Let us see more examples of De Morgan's laws.

Example

Use De Morgan's laws to express the negations of the two compound propositions "Miguel has a cellphone and he has a laptop computer" and "Heather will go to the concert or Steve will go to the concert."

Solution

Let p be "Miguel has a cellphone" and q be "Miguel has a laptop computer." Then "Miguel has a cellphone and he has a laptop computer" can be represented by $p \wedge q$. By the first of De Morgan's laws, $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$. Consequently, we can express the negation of our original statement as "Miguel does not have a cellphone or he does not have a laptop computer."

Solution

Let r be "Heather will go to the concert" and s be "Steve will go to the concert." Then "Heather will go to the concert or Steve will go to the concert" can be represented by $r \vee s$. By the second of De Morgan's laws, $\neg(r \vee s)$ is equivalent to $\neg r \wedge \neg s$. Consequently, we can express the negation of our original statement as "Heather will not go to the concert and Steve will not go to the concert."

Satisfiability

- Let us end this lecture with the tangential discussion of satisfiability.
- We first introduce a concept truth assignment to simplify our discussion. Let p be a proposition. A truth assignment corresponds to one row of the truth table of p, that is, it can tell us either p is True or False.

- If the truth value of a compound proposition under truth assignment is true, we say that the truth assignment **satisfies** P, otherwise we say that the truth assignment **falsifies** P.
- A compound proposition p is **satisfiable** if there is a truth assignment that satisfies p; that is, at least one entry of its truth table is true. For example, the conditional statement is satisfiable.
- A compound proposition P is **unsatisfiable** if it is not satisfiable; that is, all entries of its truth table are false. A contradiction must be unsatisfiable.