CS 0441
Lecture 7: Functions and sequences

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Overview

Functions

- In the previous lecture, we have introduced the sets. Now we can relate with elements between two sets and define

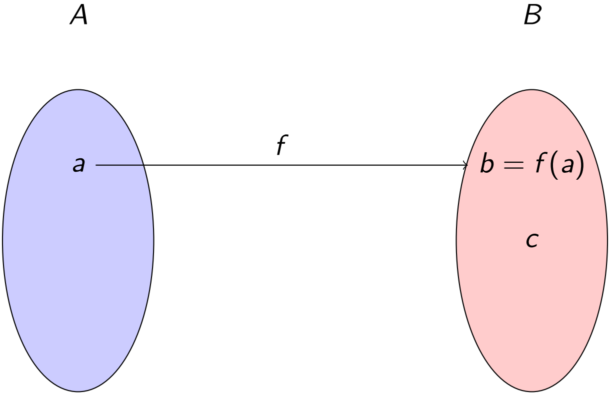
Definition

Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

- We also call functions as mappings or transformations.

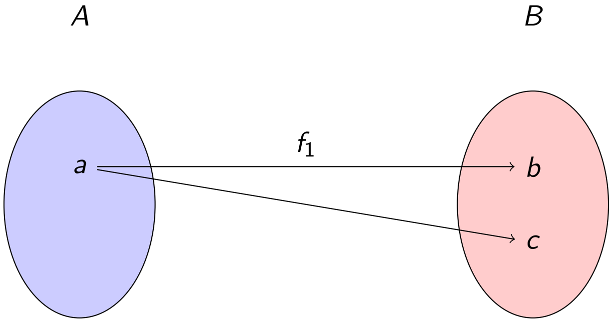
- We can represent a function using an arrow diagram as shown in Page 5.
- Also, we can use algebraic representations such as $y = x + 1$ if we specify the relation.

An arrow diagram



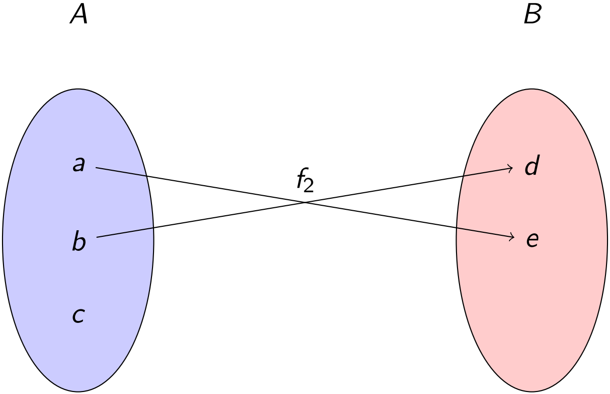
- If f is a function from A to B, we say that A is the **domain** of f and B is the **codomain** of f. If $f(a) = b$, we say that b is the **image** of a and a is a **preimage** of b. The **range**, or image, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f maps A to B.

Counterexamples



The above f_1 is not a function since it is a *one-to-many* type.

Counterexamples



f_2 is not a function either since the third element c does not have a correspondence in B.

Equal functions

- Two functions are equal when they have the same domain, have the same codomain, and the mapping relation f are also identical.

Example 1

Example

Let $f : \mathbb{Z} \to \mathbb{Z}$ assign the square of an integer to this integer. Then, $f(x) = x^2$, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, \ldots\}$.

Operations on functions

- A function is called real-valued if its codomain is the set of real numbers, and it is called integer-valued if its codomain is the set of integers. Two real-valued functions or two integervalued functions with the same domain can be added, as well as multiplied.

- Let f_1 and f_2 be functions from A to $\mathbb{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$ defined for all $x \in A$ by

$$
\begin{gathered}
(f_1 + f_2)(x) = f_1(x) + f_2(x), \\
(f_1 f_2)(x) = f_1(x) f_2(x).
\end{gathered}
$$

Example 2

Example

Let f_1 and f_2 be functions from $\mathbf{R}$ to $\mathbf{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution

From the definition of the sum and product of functions, it follows that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + \left(x - x^2\right) = x$$

and

$$(f_1 f_2)(x) = x^2 \left(x - x^2\right) = x^3 - x^4.$$

- When f is a function from A to B, the imalge of a subset of A can also be defined.
- Let f be a function from A to B and let S be a subset of A. The image of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S(t = f(s))\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

Example 3

Example

Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1$, and $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

One-to-one functions

- A function f is said to be one-to-one, or an injunction, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be injective if it is one-to-one.
- Note that a function f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$.

Example 4

Example

Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution

The function $f(x) = x^2$ *is not one-to-one because, for instance,* $f(1) = f(-1) = 1$, *but* $1 \neq -1$.

Monotonicity

- A function f whose domain and codomain are subsets of the set of real numbers is called increasing if $f(x) \leqslant f(y)$, and strictly increasing if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f.
- Similarly, f is called decreasing if $f(x) \geqslant f(y)$, and strictly decreasing if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f. (The word strictly in this definition indicates a strict inequality.)
- Note a function that is either strictly increasing or strictly decreasing must be one-to-one.

Onto functions

- A function f from A to B is called onto, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called surjective if it is onto.

Example 5

Example

Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution

The function f is not onto because there is no integer x with $x^2 = -1$, for instance.

One-to-one correspondence

- The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto. We also say that such a function is bijective.

Example 6

Example

Let A be a set. The identity function on A is the function $\iota_A : A \to A$, where

$$\iota_A(x) = x$$

for all $x \in A$. In other words, the identity function ι_A is the function that assigns each element to itself. The function ι_A is one-to-one and onto, so it is a bijection. (Note that ι is the Greek letter iota.)

- To show the onto and one-to-one properties of a function $f : A \to B$, we need
  - To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.
  - To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
  - To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
  - To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Inverse functions

- Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$.
- The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.
- Note $f^{-1}(x) \neq \frac{1}{f(x)}$.

Example 7

Example

Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ with $f(x) = x^2$. Is f invertible?

Solution

Solution: Because $f(-2) = f(2) = 4, f$ *is not one-to-one. If an inverse function were defined, it would have to assign two elements to* 4 *. Hence,* f *is not invertible. (Note we can also show that* f *is not invertible because it is not onto.)*

Example 8

Example

Show that if we restrict the function $f(x) = x^2$ in Example 7 to a function from the set of all nonnegative real numbers to the set of all nonnegative real numbers, then f is invertible.

Solution

The function $f(x) = x^2$ *from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one. To see this, note that if* $f(x) = f(y)$*, then* $x^2 = y^2$*, so* $x^2 - y^2 = (x+y)(x-y) = 0$*. This means that* $x + y = 0$ *or* $x - y = 0$*, so* $x = -y$ *or* $x = y$*. Because both* x *and* y *are nonnegative, we must have* $x = y$*. So, this function is one-to-one.*

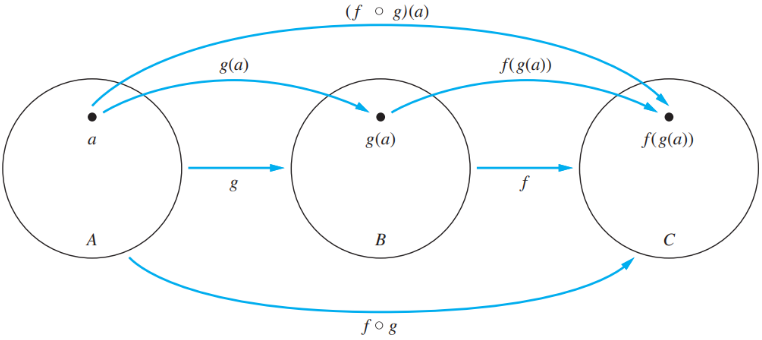
Solution

Furthermore, $f(x) = x^2$ *is onto when the codomain is the set of all nonnegative real numbers, because each nonnegative real number has a square root. That is, if* y *is a nonnegative real number, there exists a nonnegative real number* x *such that* $x = \sqrt{y}$*, which means that* $x^2 = y$*. Because the function* $f(x) = x^2$ *from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one and onto, it is invertible. Its inverse is given by the rule* $f^{-1}(y) = \sqrt{y}$*.*

Composition of functions

- Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a)).$$



Example 9

Example

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g ? What is the composition of g and f ?

Solution

Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$$(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2)+3 = 6x+7$$

and

$$(g \circ f)(x) = g(f(x)) = g(2x+3) = 3(2x+3)+2 = 6x+11.$$

Composition of bijection and its inverse

- Suppose that f is a one-to-one correspondence from the set A to the set B. Then the inverse function f^{-1} exists and is a one-to-one correspondence from B to A. The inverse function reverses the correspondence of the original function, so $f^{-1}(b) = a$ when $f(a) = b$, and $f(a) = b$ when $f^{-1}(b) = a$.

- Hence,
$$\left(f^{-1} \circ f\right)(a) = f^{-1}(f(a)) = f^{-1}(b) = a,$$
and
$$\left(f \circ f^{-1}\right)(b) = f\left(f^{-1}(b)\right) = f(a) = b.$$
- Consequently, $f^{-1} \circ f = \iota_A$ and $f \circ f^{-1} = \iota_B$, where ι_A and ι_B are the identity functions on the sets A and B, respectively. That is, $\left(f^{-1}\right)^{-1} = f$.

Graph of a function

- Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs

$$\{(a, b) \mid a \in A \text{ and } f(a) = b\}.$$

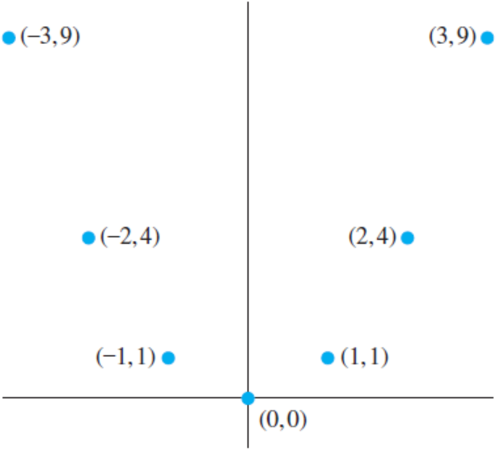
Example 10

Example

Display the graph of the function $f(x) = x^2$ from the set of integers to the set of integers.

Solution

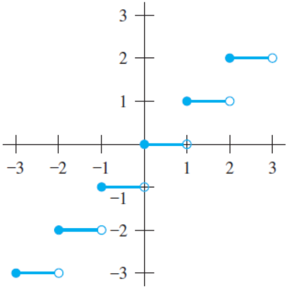
We plot as follows.



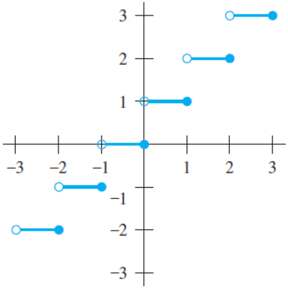
Floor and ceiling functions

- The floor function assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$.
- The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

The graphs of floor and ceiling functions are as follows.



(a) $y = [x]$



(b) $y = [x]$

- We list the properties of the floor and ceiling functions in the following table, where x is a real number and n is an integer.

$\lfloor x \rfloor = n$ if and only if $n \leqslant x < n+1$
$\lceil x \rceil = n$ if and only if $n-1 < x \leqslant n$
$\lfloor x \rfloor = n$ if and only if $x-1 < n \leqslant x$
$\lceil x \rceil = n$ if and only if $x \leqslant n < x+1$
$x-1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x+1$
$\lfloor -x \rfloor = -\lceil x \rceil$
$\lceil -x \rceil = -\lfloor x \rfloor$
$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
$\lceil x+n \rceil = \lceil x \rceil + n$

Example 11

Example

Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$.

Proof.

To prove this statement we let $x = n + \epsilon$, where n is an integer and $0 \leqslant \epsilon < 1$. There are two cases to consider, depending on whether ϵ is less than, or greater than or equal to $\frac{1}{2}$. (The reason we choose these two cases will be made clear in the proof.) ■

Proof.

We first consider the case when $0 \leqslant \epsilon < \frac{1}{2}$. In this case, $2x = 2n + 2\epsilon$ and $\lfloor 2x \rfloor = 2n$ because $0 \leqslant 2\epsilon < 1$. Similarly, $x + \frac{1}{2} = n + \left(\frac{1}{2} + \epsilon\right)$, so $\left\lfloor x + \frac{1}{2} \right\rfloor = n$, because $0 < \frac{1}{2} + \epsilon < 1$. Consequently, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = n + n = 2n$.

■

Proof.

Next, we consider the case when $\frac{1}{2} \leqslant \epsilon < 1$. In this case, $2x = 2n + 2\epsilon = (2n+1) + (2\epsilon - 1)$. Because $0 \leqslant 2\epsilon - 1 < 1$, it follows that $\lfloor 2x \rfloor = 2n + 1$. Because $\left\lfloor x + \frac{1}{2} \right\rfloor = \left\lfloor n + \left(\frac{1}{2} + \epsilon\right) \right\rfloor = \left\lfloor n + 1 + \left(\epsilon - \frac{1}{2}\right) \right\rfloor$ and $0 \leqslant \epsilon - \frac{1}{2} < 1$, it follows that $\left\lfloor x + \frac{1}{2} \right\rfloor = n + 1$. Consequently, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = n + (n+1) = 2n + 1$. This concludes the proof. ■

Example 12

Example

Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$ for all real numbers x and y.

Solution

Note the statement is false. A counterexample is supplied by $x = \frac{1}{2}$ *and* $y = \frac{1}{2}$*. With these values we find that* $\lceil x + y \rceil = \left\lceil \frac{1}{2} + \frac{1}{2} \right\rceil = \lceil 1 \rceil = 1$*, but* $\lceil x \rceil + \lceil y \rceil = \left\lceil \frac{1}{2} \right\rceil + \left\lceil \frac{1}{2} \right\rceil = 1 + 1 = 2$*.*

Factorial functions

- Another important function is the factorial function $f : \mathbb{N} \to \mathbb{Z}^+$, denoted by $f(n) = n\,!$. The value of $f(n) = n\,!$ is the product of the first n positive integers, so $f(n) = 1 \cdot 2 \cdots (n-1) \cdot n$.
- Note $f(1) = 1! = 1$ and we normally regulate $f(0) = 0! = 1$.
- We can approximate the factorial via the Stirling's formula, that is,
$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Partial functions

- In the last, we define the partial functions. The idea is introduced, for example, when we want to consider a big set as the domain of the function while the function only takes values from a subset of the domain.
- More formally, a partial function f from a set A to a set B is an assignment to each element a in a subset of A, called the domain of definition of f, of a unique element b in B. The sets A and B are called the domain and codomain of f, respectively. We say that f is undefined for elements in A that are not in the domain of definition of f. When the domain of definition of f equals A, we say that f is a total function.

- For example, the function $f : \mathbb{Z} \to \mathbb{R}$ where $f(n) = \sqrt{n}$ is a partial function from $\mathbb{Z}$ to $\mathbb{R}$ where the domain of definition is the set of nonnegative integers. Note that f is undefined for negative integers.

Sequences

- A sequence is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, \ldots\}$ or the set $\{1, 2, 3, \ldots\}$) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.

Example 13

Example

Consider the sequence $\{a_n\}$, where

$$a_n = \frac{1}{n}$$

The list of the terms of this sequence, beginning with a_1, namely,

$$a_1, a_2, a_3, a_4, \ldots$$

starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

- We review the following two special sequences.
- A **geometric progression** is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots,$$

where the initial term a and the common ratio r are real numbers.

- An **arithmetic progression** is a sequence of the form

$$a, a+d, a+2d, \ldots, a+nd, \ldots,$$

where the initial term a and the common difference d are real numbers.

Strings

- Sequences of the form $a_1, a_2, \ldots, a_n$ are often used in computer science. These finite sequences are also called strings. This string is also denoted by $a_1 a_2 \ldots a_n$.
- The length of a string is the number of terms in this string.
- The empty string, denoted by λ, is the string that has no terms. The empty string has length zero.

Recurrence relation

- We can also represent a sequence using the recurrence relation. A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geqslant n_0$, where n_0 is a nonnegative integer.
- A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Example 14

Example

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$, and suppose that $a_0 = 2$. What are a_1, a_2, and a_3 ?

Solution

We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$*. It then follows that* $a_2 = 5 + 3 = 8$ *and* $a_3 = 8 + 3 = 11$*.*

Remark

- We normally call the first given value in a recurrence relation an initial condition.
- One of the famous recurrence relations is the Fibonacci sequence. More specifically, the Fibonacci sequence, $f_0, f_1, f_2, \ldots$, is defined by the initial conditions $f_0 = 0, f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2}, \quad \text{for } n = 2, 3, 4, \ldots.$$

- We can find from the recurrence relation an explicit formula for the sequence, which we call the **closed (form) formula**.

Example 15

Example

Solve the recurrence relation and initial condition in Example 14.

Solution

We can successively apply the recurrence relation in Example 14, starting with the initial condition $a_1 = 2$, and working upward until we reach a_n to deduce a closed formula for the sequence. We see that

$$
\begin{aligned}
a_2 &= 2 + 3 \\
a_3 &= (2+3) + 3 = 2 + 3 \cdot 2 \\
a_4 &= (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3 \\
&\vdots \\
a_n &= a_{n-1} + 3 = (2 + 3 \cdot (n-2)) + 3 = 2 + 3(n-1).
\end{aligned}
$$

This is called a forward substitution.

Solution

Alternatively, we can start with the term a_n *and working downward until we reach the initial condition* $a_1 = 2$ *to deduce this same formula. The steps are*

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&\vdots \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1).
\end{aligned}
$$

This approach is called a backward substitution.

- Note the sequences are widely applied in the filed of computer science, finance, *etc*. In the following example, we see how a sequence is applied in calculating interest.

Example 16

Example

Suppose that a person deposits $\$10,000$ in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Solution

To solve this problem, let P_n denote the amount in the account after n years. Because the amount in the account after n years equals the amount in the account after $n-1$ years plus interest for the nth year, we see that the sequence $\{P_n\}$ satisfies the recurrence relation

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}.$$

The initial condition is $P_0 = 10,000$. We can use an iterative approach to find a formula for P_n. Note that

$$\begin{aligned}
P_1 &= (1.11)P_0 \\
P_2 &= (1.11)P_1 = (1.11)^2 P_0 \\
P_3 &= (1.11)P_2 = (1.11)^3 P_0 \\
&\vdots \\
P_n &= (1.11)P_{n-1} = (1.11)^n P_0.
\end{aligned}$$

Solution

When we insert the initial condition $P_0 = 10,000$*, the formula* $P_n = (1.11)^n 10,000$ *is obtained. Inserting* $n = 30$ *into the formula* $P_n = (1.11)^n 10,000$ *shows that after 30 years the account contains*

$$P_{30} = (1.11)^{30} 10,000 = \$228,922.97.$$

Patterns of a sequence

- When a sequence is generated with patterns, we are able to deduce the following terms based upon some first given terms. Let us see the following examples.

Example 17

Example

How can we produce the terms of a sequence if the first 10 terms are $1, 2, 2, 3, 3, 3, 4, 4, 4, 4$?

Solution

In this sequence, the integer 1 appears once, the integer 2 appears twice, the integer 3 appears three times, and the integer 4 appears four times. A reasonable rule for generating this sequence is that the integer n appears exactly n times, so the next five terms of the sequence would all be 5, the following six terms would all be 6, and so on. The sequence generated this way is a possible match.

Example 18

Example

How can we produce the terms of a sequence if the first 10 terms are $1, 3, 4, 7, 11, 18, 29, 47, 76, 123$?

Solution

Observe that each successive term of this sequence, starting with the third term, is the sum of the two previous terms. That is, $4 = 3 + 1, 7 = 4 + 3, 11 = 7 + 4$, *and so on. Consequently, if* L_n *is the nth term of this sequence, we guess that the sequence is determined by the recurrence relation* $L_n = L_{n-1} + L_{n-2}$ *with initial conditions* $L_1 = 1$ *and* $L_2 = 3$. *This sequence is known as the Lucas sequence.*

Summations

- We can represent the sum of numbers in a sequence by the summation notation. We begin by describing the notation used to express the sum of the terms

$$a_m, a_{m+1}, \ldots, a_n$$

from the sequence $\{a_n\}$.

- We use the notation

$$\sum_{j=m}^{n} a_j, \quad \sum_{j=m}^{n} a_j, \quad \text{or} \quad \sum_{m\leqslant j\leqslant n} a_j$$

(read as the sum from $j = m$ to $j = n$ of a_j) to represent

$$a_m + a_{m+1} + \cdots + a_n,$$

where j is called the index of summation and you use other letters such as i or k for the index.

- Note i, j and k are dummy variables, that is,

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k.$$

- We call m the lower limit and n the upper limit. A large uppercase Greek letter sigma, $\sum$, is used to denote summation.

Example 19

Example

Use summation notation to express the sum of the first 100 terms of the sequence $\{a_j\}$, where $a_j = 1/j$ for $j = 1, 2, 3, \ldots$

Solution

The lower limit for the index of summation is 1 and the upper limit is 100. We write this sum as

$$\sum_{j=1}^{100} \frac{1}{j}.$$

- An important result for the sequence summation is the summation of geometric sequence, that is, $a, ar, ar^2, \ldots, ar^n$. We provide

Theorem

If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1, \\ (n+1)a & \text{if } r = 1. \end{cases}$$

Let us prove the above theorem

Proof.

Let

$$S_n = \sum_{j=0}^{n} ar^j.$$

To compute S, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$\begin{aligned} rS_n =& r\sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} ar^{j+1} = \sum_{k=1}^{n+1} ar^k \\ =& \left(\sum_{k=0}^{n} ar^k\right) + \left(ar^{n+1} - a\right) \\ =& S_n + \left(ar^{n+1} - a\right) \end{aligned}$$

From these equalities, we see that

$$rS_n = S_n + \left(ar^{n+1} - a\right).$$

Proof.

Solving for S_n shows that if $r \neq 1$, then

$$S_n = \frac{ar^{n+1} - a}{r - 1}.$$

If $r = 1$, then the $S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n+1)a$. ■

Example 20

Example

Double summations arise in many contexts (as in the analysis of nested loops in computer programs). An example of a double summation is

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij.$$

To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation:

$$\begin{aligned}\sum_{i=1}^{4}\sum_{j=1}^{3} ij &= \sum_{i=1}^{4}(i+2i+3i)\\ &= \sum_{i=1}^{4} 6i\\ &= 6+12+18+24=60.\end{aligned}$$

Summation notation for functions

- We can also use summation notation to add all values of a function, or terms of an indexed set, where the index of summation runs over all values in a set. That is, we write

$$\sum_{s \in S} f(s)$$

to represent the sum of the values $f(s)$, for all members s of S.

Summation formulas

- The following formulas can help some evaluations.

Sum	Closed Form
$\sum_{k=0}^{n} ar^k (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1 - x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1 - x)^2}$

Example 21

Example

Find $\sum_{k=50}^{100} k^2$.

Solution

First note that because $\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2$*, we have*

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$$

Using the formula $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ *, we see that*

$$\sum_{k=50}^{100} k^2 = \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} = 338,350 - 40,425 = 297,925.$$

Series

- When a sequence has infinitely many terms, then we call it a series. For a quick review from Calculus 2. We present the following example.

Example 22

Example

Let x be a real number with $|x| < 1$. Find $\sum_{n=0}^{\infty} x^n$.

Solution

We first compute the partial sums of the geometric sequence from $n = 0$ *to* $n = k$ *with* $a = 1$ *and* $r = x$ *we see that* $\sum_{n=0}^{k} x^n = \frac{x^{k+1} - 1}{x - 1}$. *Because* $|x| < 1, x^{k+1}$ *approaches 0 as* k *approaches infinity. It follows that*

$$\sum_{n=0}^{\infty} x^n = \lim_{k \to \infty} \frac{x^{k+1} - 1}{x - 1} = \frac{0 - 1}{x - 1} = \frac{1}{1 - x}.$$

Note when $|x| \geqslant 1$, *the series diverges.*