**CS0441**

**Discrete Structures Section 2**

**KP. Wang** **Assignment #2** **Solutions**

1. Section 1.4 # 39, 50, 51, 60
2. Section 1.5 # 15, 23, 25, 33
3. Section 1.6 # 9, 34(only a) and e)), 35
4. Prove the following equivalences in Fitch style natural deduction.
   (a) $\exists x P(x) \vee \exists x Q(x) \equiv \exists x(P(x) \vee Q(x))$,
   (b) $\neg A \wedge \neg B \vdash \neg(A \vee B)$.

1. *Solution:*
   (a) If there is a printer that is both out of service and busy, then some job has been lost.
   (b) If every printer is busy, then there is a job in the queue.
   (c) If there is a job that is both queued and lost, then some printer is out of service.
   (d) If every printer is busy and every job is queued, then some job is lost.

2. *Solution:* It is enough to find a counterexample. It is intuitively clear that the first proposition is asserting much more than the second. It is saying that one of the two predicates, $P$ or $Q$, is universally true; whereas the second proposition is simply saying that for every $x$ either $P(x)$ or $Q(x)$ holds, but which it is may well depend on $x$. As a simple counterexample, let $P(x)$ be the statement that $x$ is odd, and let $Q(x)$ be the statement that $x$ is even. Let the domain of discourse be the positive integers. The second proposition is true, since every positive integer is either odd or even. But the first proposition is false, since it is neither the case that all positive integers are odd nor the case that all of them are even.

3. *Solution:* We can show that these are not logically equivalent by giving an example in which one is true and the other is false. Let $P(x)$ be the statement "$x$ is odd" applied to positive integers. Similarly let $Q(x)$ be "$x$ is even." Then since there exist odd numbers and there exist even numbers, the statement $\exists x P(x) \wedge \exists x Q(x)$ is true. On the other hand, no number is both odd and even, thus $\exists x(P(x) \wedge Q(x))$ is false.

4. *Solution:*
   (a) $\forall x(P(x) \rightarrow Q(x))$,
   (b) $\exists x(R(x) \wedge \neg Q(x))$,
   (c) $\exists x(R(x) \wedge \neg P(x))$,
   (d) Yes. The unsatisfactory excuse guaranteed by part (b) cannot be a clear explanation by part (a).

5. *Solution:* The answers presented here are not the only ones possible; other answers can be obtained using different predicates and different variables, or by varying the domain (universe of discourse).

   (a) $\forall x N$(x, discrete mathematics), where $N(x, y)$ is "$x$ needs a course in $y$ " and the domain for $x$ is computer science students and the domain for $y$ is academic subjects.

   (b) $\exists x O$(x, personal computer), where $O(x, y)$ is "$x$ owns $y$, " and the domain for $x$ is students in this class.

   (c) $\forall x \exists y P(x, y)$, where $P(x, y)$ is "$x$ has taken $y$ "; $x$ ranges over students in this class, and $y$ ranges over computer science courses.

   (d) $\exists x \exists y P(x, y)$, with the environment of part (c) (i.e., the same definition of $P$ and the same domain).

   (e) $\forall x \forall y P(x, y)$, where $P(x, y)$ is "$x$ has been in $y$ "; $x$ ranges over students in this class, and $y$ ranges over buildings on campus.

   (f) $\exists x \exists y \forall z (P(z, y) \rightarrow Q(x, z))$, where $P(z, y)$ is "$z$ is in $y$ " and $Q(x, z)$ is "$x$ has been in $z$ "; ranges over students in this class, $y$ ranges over buildings on campus, and $z$ ranges over rooms.

   (g) $\forall x \forall y \exists z (P(z, y) \wedge Q(x, z))$, with the environment of part (f).

6. *Solution:*

   (a) $\forall x \forall y ((x < 0) \wedge (y < 0) \rightarrow (xy > 0))$.

   (b) $\forall x (x - x = 0)$.

   (c) To say that there are exactly two objects that meet some condition, we must have two existentially quantified variables to represent the two objects, we must say that they are different, and then we must say that an object meets the conditions if and only if it is one of those two. In this case we have $\forall x \exists a \exists b \left(a \neq b \wedge \forall c \left(c^2 = x \leftrightarrow (c = a \vee c = b)\right)\right)$.

   (d) $\forall x \left((x < 0) \rightarrow \neg \exists y \left(x = y^2\right)\right)$ where the domain (universe of discourse) consists of all real numbers.

7. *Solution:*

   (a) This says that there exists a real number $x$ such that for every real number $y$, the product $xy$ equals $y$. That is, there is a multiplicative identity for the real numbers. This is a true statement, since $x = 1$ is the identity.

   (b) The product of two negative real numbers is always a positive real number.

   (c) There exist real numbers $x$ and $y$ such that $x^2$ exceeds $y$ but $x$ is less than $y$. This is true, since we can take $x = 2$ and $y = 3$, for instance.

   (d) This says that for every pair of real numbers $x$ and $y$, there exists a real number $z$ that is their sum. In other words, the real numbers are closed under the operation of addition, another true fact.

8. *Solution:*

(a) $\neg\forall x\forall y P(x,y) = \exists x\neg\forall y P(x,y) = \exists x\exists y\neg P(x,y)$.

(b) $\neg\forall y\exists x P(x,y) = \exists y\neg\exists x P(x,y) = \exists y\forall x\neg P(x,y)$.

(c)
$$
\begin{aligned}
&\neg\forall y\forall x(P(x,y)\vee Q(x,y))\\
=&\exists y\neg\forall x(P(x,y)\vee Q(x,y))\\
=&\exists y\exists x\neg(P(x,y)\vee Q(x,y))\\
=&\exists y\exists x(\neg P(x,y)\wedge\neg Q(x,y)).
\end{aligned}
$$

(d) First we apply De Morgan's Law and then follow the procedures in (a) and (b). Hence, we obtain

$$
\begin{aligned}
&\neg(\exists x\exists y\neg P(x,y)\wedge\forall x\forall y Q(x,y))\\
=&\neg(\exists x\exists y\neg P(x,y))\vee\neg(\forall x\forall y Q(x,y))\\
=&(\forall x\forall y P(x,y))\vee(\exists x\exists y\neg Q(x,y)).
\end{aligned}
$$

(e) Similar with (d) we have

$$
\begin{aligned}
&\neg\forall x(\exists y\forall z P(x,y,z)\wedge\exists z\forall y P(x,y,z))\\
=&\exists x\neg(\exists y\forall z P(x,y,z)\wedge\exists z\forall y P(x,y,z))\\
=&\exists x(\neg((\exists y\forall z P(x,y,z))\vee(\exists z\forall y P(x,y,z)))\\
=&\exists x(\forall y\exists z\neg P(x,y,z)\vee\forall z\exists y\neg P(x,y,z)).
\end{aligned}
$$

9. *Solution:*

(a) Let $T(x)$, $R(x)$ and $S(x)$ denote "I take the $x$th day in a week off", "It rained on $x$th day in a week" and "It snowed on $x$th day in a week", respectively. Note $x$ represents the days in a week, from Sunday to Saturday (In English culture, Sunday is viewed as the 1st day in a week) Before we proceed, we need to make necessary assumption, that is, the sunny day simply means that it is not rainy or it is not snowy.

Note we are provided with the following premises: $\forall x(T(x)\rightarrow R(x)\vee S(x))$, $T(3)\vee T(5)$, $\neg(R(3)\wedge S(3))$ and $\neg S(5)$.

| Step | Reason |
|---|---|
| 1. $\forall x(T(x)\rightarrow R(x)\vee S(x))$ | Premise |
| 2. $T(3)\rightarrow R(3)\vee S(3)$ | UI from (1) |
| 3. $\neg(R(3)\wedge S(3))$ | Premise |
| 4. $\neg T(3)$ | MT from (2) and (3) |
| 5. $T(3)\vee T(5)$ | Premise |
| 6. $T(5)$ | DS from (4) and (5) |
| 7. $\forall x(T(x)\rightarrow R(x)\vee S(x))$ | Premise |
| 8. $T(5)\rightarrow R(5)\vee S(5)$ | UI from (7) |
| 9. $R(5)\vee S(5)$ | MP from (6) and (8) |
| 10. $\neg S(5)$ | Premise |
| 11. $R(4)$ | DS from (9) and (10) |

The conclusion is that it rained on Thursday.

(b) Let $p =$ " I eat spicy foods", $q =$ "I have strange dreams" and $r =$ "there is thunder while I sleep".

We are given premises: $p \to q$, $r \to q$ and $\neg q$.

| Step | Reason |
|---|---|
| 1. $p \to q$ | Premise |
| 2. $\neg q$ | Premise |
| 3. $\neg p$ | MT from (1) and (2) |
| 4. $r \to q$ | Premise |
| 5. $\neg r$ | MT from (4) and (2) |
| 6. $\neg p \wedge \neg r$ | Conjunction from (3) and (5) |

We conclude that it I did not eat spicy food and there was no thunder when I was in sleep.

(c) Let $p$ be "I am clever", $q$ be "I am lucky" and $r$ be "I win the lottery".

The premises are $p \vee q$, $\neg q$ and $q \to r$. It follows from the disjunctive syllogism that $p \vee q$ and $\neg q$ give rise to $p$. Note $q \to r$ is useless. Hence, we conclude that I am clever.

(d) Let $C(x)$ and $H(x)$ denote "$x$ majors in Computer Science" and $x$ has a personal computer, where $x$ is a student in the university.

We know the premises are $\forall x(C(x) \to H(x))$, $\neg H(Ralph)$ and $H(Ann)$.

| Step | Reason |
|---|---|
| 1. $\forall x(C(x) \to H(x))$ | Premise |
| 2. $C(Ralph) \to H(Ralph)$ | UI from (1) |
| 3. $\neg H(Ralph)$ | Premise |
| 4. $\neg C(Ralph)$ | MT from (2) and (3) |

For Ralph we conclude that he does not major in CS. But we can not draw any conclusion for Ann.

(e) Denote $C(x) =$ "$x$ is good for corporations", $U(x) =$ "$x$ is good for the US", $Y(x) =$ "$x$ is good for you", where $x$ stands for certain things or actions.

Hence, we are given $\forall x(C(x) \to U(x))$, $\forall x(U(x) \to Y(x))$ and $C$(to buy lots of stuff).

| Step | Reason |
|---|---|
| 1. $C$(to buy lots of stuff) | Premise |
| 2. $\forall x(C(x) \to U(x))$ | Premise |
| 3. $C$(to buy lots of stuff) $\to U$(to buy lots of stuff) | UI from (2) |
| 4. $U$(to buy lots of stuff) | MP from (1) and (3) |
| 5. $\forall x(U(x) \to Y(x))$ | Premise |
| 6. $U$(to buy lots of stuff) $\to Y$(to buy lots of stuff) | UI from (5) |
| 7. $Y$(to buy lots of stuff) | MP from (4) and (6) |

Thus, we conclude to buy lots of stuff is good for the US and good for you. (**Note:** The conclusion is the opinion of the textbook author. We don't pass or convey any of value judgement on this.)

(f) We label "$x$ is a rodent" by $R(x)$, "$x$ gnaw food" by $G(x)$, where $x$ is an animal. Then we are provided with $\forall x(R(x) \to G(x))$, $R(Mice)$, $\neg G(Rabbit)$ and $\neg R(Bat)$.

| Step | Reason |
|---|---|
| 1. $\forall x(R(x) \to G(x))$ | Premise |
| 2. $R(Mice)$ | Premise |
| 3. $R(Mice) \to G(Mice)$ | UI from (1) |
| 3. $G(Mice)$ | MP from (2) and (3) |
| 4. $\neg G(Rabbit)$ | Premise |
| 5. $R(Rabbit) \to G(Rabbit)$ | UI from (1) |
| 6. $\neg R(Rabbit)$ | MT from (4) and (5) |
| 7. $G(Mice) \land \neg R(Rabbit)$ | Conjunction from (3) and (6) |

Thus, we conclude that mice do not gnaw food and rabbits are not rodents. We cannot arrive at any conclusion about bats.

10. *Solution:* Let us use the following letters to stand for the relevant propositions: $d$ for "logic is difficult"; $s$ for "many students like logic"; and $e$ for "mathematics is easy." Then the assumptions are $d \lor \neg s$ and $e \to \neg d$. Note that the first of these is equivalent to $s \to d$, since both forms are false if and only if $s$ is true and $d$ is false. In addition, let us note that the second assumption is equivalent to its contrapositive, $d \to \neg e$. And finally, by combining these two conditional statements, we see that $s \to \neg e$ also follows from our assumptions.

    (a) Here we are asked whether we can conclude that $s \to \neg e$. As we noted above, the answer is yes, this conclusion is valid.

    e) This sentence says $\neg s \to (\neg e \lor \neg d)$. The only case in which this is false is when $s$ is false and both $e$ and $d$ are true. But in this case, our assumption $e \to \neg d$ is also violated. Therefore, in all cases in which the assumptions hold, this statement holds as well, so it is a valid conclusion.

11. *Solution:* This argument is valid. We argue by contradiction. Assume that Superman does exist. Then he is not impotent, and he is not malevolent (this follows from the fourth sentence). Therefore by (the contrapositives of) the two parts of the second sentence, we conclude that he is able to prevent evil, and he is willing to prevent evil. By the first sentence, we therefore know that Superman does prevent evil. This contradicts the third sentence. Since we have arrived at a contradiction, our original assumption must have been false, so we conclude finally that Superman does not exist.

12. *Solution:* Note we need to show both directions in the equivalence. First we prove $\exists(P(x) \lor Q(x)) \vdash \exists x P(x) \lor \exists x Q(x)$.

| | | |
|---|---|---|
| 1 | $\exists x(P(x) \lor Q(x))$ | |
| 2 | $u \quad P(u) \lor Q(u)$ | |
| 3 | $\quad\quad P(u)$ | |
| 4 | $\quad\quad \exists x P(x)$ | $\exists$I, 3 |
| 5 | $\quad\quad \exists x P(x) \lor \exists x Q(x)$ | $\lor$I, 4 |
| 6 | $\quad\quad Q(u)$ | |
| 7 | $\quad\quad \exists x Q(x)$ | $\exists$I, 6 |
| 8 | $\quad\quad \exists x P(x) \lor \exists x Q(x)$ | $\lor$I, 7 |
| 9 | $\quad \exists x P(x) \lor \exists x Q(x)$ | $\lor$E, 2, 3–5, 6–8 |
| 10 | $\exists x P(x) \lor \exists x Q(x)$ | $\exists$E, 1, 2–9 |

In the other direction, we need to prove $\exists x P(x) \lor \exists x Q(x) \vdash \exists(P(x) \lor Q(x))$.

| | | |
|---|---|---|
| 1 | $\exists x P(x) \lor \exists x Q(x))$ | |
| 2 | $\quad \exists x P(x)$ | |
| 3 | $\quad u \quad P(u)$ | |
| 4 | $\quad\quad P(u) \lor Q(u)$ | $\lor$I, 3 |
| 5 | $\quad\quad \exists x(P(x) \lor Q(x))$ | $\exists$I, 4 |
| 6 | $\quad \exists x(P(x) \lor Q(x))$ | $\exists$E, 3, 4–5 |
| 7 | $\quad \exists x Q(x)$ | |
| 8 | $\quad u \quad Q(u)$ | |
| 9 | $\quad\quad P(u) \lor Q(u)$ | $\lor$I, 8 |
| 10 | $\quad\quad \exists x(P(x) \lor Q(x))$ | $\exists$I, 9 |
| 11 | $\quad \exists x(P(x) \lor Q(x))$ | $\exists$E, 8, 9–10 |
| 12 | $\exists x(P(x) \lor Q(x))$ | $\lor$E, 1, 2–6, 7–11 |

13. *Solution:* We present the proof as follows.

| | | |
|---|---|---|
| 1 | $\neg A \land \neg B$ | |
| 2 | $\quad A \lor B$ | |
| 3 | $\quad\quad A$ | |
| 4 | $\quad\quad \neg A$ | $\land$E, 1 |
| 5 | $\quad\quad \bot$ | $\neg$E, 3, 4 |
| 6 | $\quad\quad B$ | |
| 7 | $\quad\quad \neg B$ | $\land$E, 1 |
| 8 | $\quad\quad \bot$ | $\neg$E, 3, 4 |
| 9 | $\quad \bot$ | $\lor$E, 2, 3–5, 6–8 |
| 10 | $\neg(A \lor B)$ | $\neg$I, 2–9 |