CS0441

Discrete Structures Section 2

KP. Wang **Assignment #4** **Solutions**

1. Section 2.2 # 23, 24, 47, 50
2. Section 2.3 # 7, 20, 23, 30, 51, 54, 72

1. *Solution:* We construct the following membership table and note that the fifth and eighth columns are identical.

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

2. *Solution:* First suppose x is in the left-hand side. Then x must be in A but in neither B nor C. Thus $x \in A - C$, but $x \notin B - C$, so x is in the right-hand side. Next suppose that x is in the right-hand side. Thus x must be in $A - C$ and not in $B - C$. The first of these implies that $x \in A$ and $x \notin C$. But now it must also be the case that $x \notin B$, since otherwise we would have $x \in B - C$. Thus we have shown that x is in A but in neither B nor C, which implies that x is in the left-hand side.

3. *Solution:* a) The union of these sets is the set of elements that appear in at least one of them. In this case the sets are "increasing": $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$. Therefore every element in any of the sets is in A_n, so the union is $A_n = \{1, 2, \ldots, n\}$.

   b) The intersection of these sets is the set of elements that appear in all of them. Since $A_1 = \{1\}$, only the number 1 has a chance to be in the intersection. In fact 1 is in the intersection, since it is in all of the sets. Therefore the intersection is $A_1 = \{1\}$.

4. a) As i increases, the sets get smaller: $\cdots \subset A_3 \subset A_2 \subset A_1$. All the sets are subsets of A_1, which is the set of positive integers, $\mathbb{Z}^+$. It follows that $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}^+$. Every positive integer is excluded from at least one of the sets (in fact from infinitely many), so $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

   b) All the sets are subsets of the set of natural numbers N (the nonnegative integers). The number 0 is in each of the sets, and every positive integer is in exactly one of the sets, so $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$ and $\bigcap_{i=1}^{\infty} A_i = \{0\}$.

   c) As i increases, the sets get larger: $A_1 \subset A_2 \subset A_3 \cdots$. All the sets are subsets of the set of positive real numbers $\mathbb{R}^+$, and every positive real number is included eventually, so $\bigcup_{i=1}^{\infty} A_i = \mathbb{R}^+$. Because A_1 is a subset of each of the others, $\bigcap_{i=1}^{\infty} A_i = A_1 = (0, 1)$ (the interval of all real numbers between 0 and 1 , exclusive).

d) This time, as in part (a), the sets are getting smaller as i increases: $\cdots \subset A_3 \subset A_2 \subset A_1$. Because A_1 includes all the others, $\bigcup_{i=1}^{\infty} A_1 = (1, \infty)$ (all real numbers greater than 1). Every number eventually gets excluded as i increases, so $\bigcap_{i=1}^{\infty} A_i = \emptyset$. Notice that ∞ is not a real number, so we cannot write $\bigcap_{i=1}^{\infty} A_i = \{\infty\}$

5. *Solution:* In each case, the domain is the set of possible inputs for which the function is defined, and the range is the set of all possible outputs on these inputs.

   a) The domain is $\mathbb{Z}^+ \times \mathbb{Z}^+$, since we are told that the function operates on pairs of positive integers (the word "pair" in mathematics is usually understood to mean ordered pair). Since the maximum is again a positive integer, and all positive integers are possible maximums (by letting the two elements of the pair be the same), the range is $\mathbb{Z}^+$.

   b) We are told that the domain is $\mathbb{Z}^+$. Since the decimal representation of an integer has to have at least one digit, at most nine digits do not appear, and of course the number of missing digits could be any number less than 9 . Thus the range is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

   c) We are told that the domain is the set of bit strings. The block 11 could appear no times, or it could appear any positive number of times, so the range is $\mathbb{N}$.

   d) We are told that the domain is the set of bit strings. Since the first 1 can be anywhere in the string, its position can be $1, 2, 3, \ldots$. If the bit string contains no 1 's, the value is 0 by definition. Therefore the range is $\mathbb{N}$,

6. *Solution:* a) $f(n) = n + 17$

   b) $f(n) = \lceil n/2 \rceil$

   c) We let $f(n) = n - 1$ for even values of n, and $f(n) = n + 1$ for odd values of n. Thus we have $f(1) = 2$, $f(2) = 1$, $f(3) = 4$, $f(4) = 3$, and so on. Note that this is just one function, even though its definition used two formulae, depending on the the parity of n.

   d) $f(n) = 17$

7. *Solution:*

   a) To show that the function is one-to-one, note that if $f(x) = f(y)$, then $2x+1 = 2y+1$, then $x = y$. Hence, the function is injective. To show that the function is onto, note that if $f(x) = z$, then $2x + 1 = z$ and $x = \dfrac{z-1}{2}$, which is in $\mathbb{R}$. Hence, the function is surjective.

   b) This function is not a bijection, since its range is the set of real numbers greater than or equal to 1 (which is sometimes written $[1, \infty)$), not all of $\mathbb{R}$. (It is not injective either.)

   c) This function is a bijection. Note if $f(x) = f(y)$, then $x^3 = y^3$. Thus, $x = y$, which means that the function is injective. Next, when $f(x) = z$, we have $x^3 = z$ and $x = \sqrt[3]{z}$, which is in $\mathbb{R}$. Then the function is onto. Therefore, the function is bijective.

   d) This function is not a bijection. It is easy to see that it is not injective, since x and $-x$ have the same image, for all real numbers x. A little work shows that the range is only $\{y \mid 0.5 \leqslant y < 1\} = [0.5, 1)$.

8. *Solution:* In all parts, we simply need to compute the values $f(-1), f(0), f(2), f(4)$, and $f(7)$ and collect the values into a set.

   a) $\{1\}$ (all five values are the same)

   b) $\{-1, 1, 5, 8, 15\}$

   c) $\{0, 1, 2\}$

   d) $\{0, 1, 5, 16\}$

9. *Solution:* a) One direction (the "only if" part) is obvious: If $x < n$, then since $\lfloor x \rfloor \leqslant x$ it follows that $\lfloor x \rfloor < n$. We will prove the other direction (the "if" part) indirectly (we will prove its contrapositive). Suppose that $x \geqslant n$. Then "the greatest integer not exceeding x " must be at least n, since n is an integer not exceeding x. That is, $\lfloor x \rfloor \geqslant n$.

   b) One direction (the "only if" part) is obvious: If $n < x$, then since $x \leqslant \lceil x \rceil$ it follows that $n < \lceil x \rceil$. We will prove the other direction (the "if" part) indirectly (we will prove its contrapositive). Suppose that $n \geqslant x$. Then "the smallest integer not less than x " must be no greater than n, since n is an integer not less than x. That is, $\lceil x \rceil \leqslant n$.

10. *Solution:* To prove the first equality, write $x = n - \epsilon$, where n is an integer and $0 \leqslant \epsilon < 1$; thus $\lceil x \rceil = n$. Therefore, $\lfloor -x \rfloor = \lfloor -n + \epsilon \rfloor = -n = -\lceil x \rceil$. The second equality is proved in the same manner, writing $x = n + \epsilon$, where n is an integer and $0 \leqslant \epsilon < 1$. This time $\lfloor x \rfloor = n$, and $\lceil -x \rceil = \lceil -n - \epsilon \rceil = -n = -\lfloor x \rfloor$.

11. *Solution:* This follows immediately from the definition. We want to show that

$$\left((f \circ g) \circ \left(g^{-1} \circ f^{-1}\right)\right)(z) = z$$

   for all $z \in Z$ and that $\left(\left(g^{-1} \circ f^{-1}\right) \circ (f \circ g)\right)(x) = x$ for all $x \in X$. For the first we have

$$\begin{aligned}
\left((f \circ g) \circ \left(g^{-1} \circ f^{-1}\right)\right)(z) &= (f \circ g)\left(\left(g^{-1} \circ f^{-1}\right)(z)\right) \\
&= (f \circ g)\left(g^{-1}\left(f^{-1}(z)\right)\right) \\
&= f\left(g\left(g^{-1}\left(f^{-1}(z)\right)\right)\right) \\
&= f\left(f^{-1}(z)\right) = z
\end{aligned}$$

   The second equality can be proved as follows

$$\begin{aligned}
\left(\left(g^{-1} \circ f^{-1}\right) \circ (f \circ g)\right)(x) &= \left(g^{-1} \circ f^{-1}\right)((f \circ g)(x)) \\
&= \left(g^{-1} \circ f^{-1}\right)(f(g(x))) \\
&= g^{-1}(f^{-1}(f(g(x)))) \\
&= g\left(g^{-1}(x)\right) = x.
\end{aligned}$$