HW9

Q1

(a) $\nabla f = \langle f_x, f_y \rangle = \langle 2x+2y, 2x+2y+2 \rangle$

$\because$ no condition that $2x+2y = 2x+2y+2 = 0$

$\therefore$ no stationary point

(b) $\nabla f = \langle 6xy-12x, 3x^2+3y^2-12y \rangle$

solve $\begin{cases} 6xy-12x=0 \\ 3x^2+3y^2-12y=0 \end{cases}$

$\therefore$ 4 points $(0,0), (0,4), (-2,2), (2,2)$

$f_x = 6xy-12x \qquad f_{xx} = 6y-12 \qquad f_{xy} = 6x$

$f_y = 3x^2+3y^2-12y \qquad f_{yy} = 6y-12$

$\therefore D = f_{xx}f_{yy} - (f_{xy})^2 = (6y-12)^2 - (6x)^2$

$= 36y^2 - 144y + 144 - 36x^2$

$\because D(0,0) = 144 > 0, \quad f_{xx}(0,0) = -12 < 0$

$(0,0)$ is a local maximum

$\because D(0,4) = 36\times16 - 36\times4\times4 + 144 = 144$

$f(x,4) = 12 > 0$

$(0,4)$ is a local minimum

$\because D(-2,2) = -144 < 0 \quad D(2,2) = -144 < 0$

$(-2,2)$ and $(2,2)$ are saddle points

(c) $\nabla f = \langle e^x \cos y, -e^x \sin y \rangle$

let $\begin{cases} e^x \cos y = 0 \\ -e^x \sin y = 0 \end{cases}$

$\because$ no condition that $e^x \cos y = -e^x \sin y = 0$

$\therefore$ function not exist

(d) $f(x,y) = (x^2+y^2)e^{y^2-x^2}$

$f_x = 2xe^{y^2-x^2} + (x^2+y^2)(-2xe^{y^2-x^2})$

$= (2x - 2x^3 - 2xy^2)e^{y^2-x^2}$

$f_y = 2ye^{y^2-x^2} + (x^2+y^2) \cdot 2y \cdot e^{y^2-x^2}$

$= (2y + 2x^2y + 2y^3) \cdot e^{y^2-x^2}$

$\therefore \nabla f = \langle 2xe^{y^2-x^2}(1-x^2-y^2), 2ye^{y^2-x^2}(1+x^2+y^2) \rangle$

solve $\begin{cases} 2xe^{y^2-x^2}(1-x^2-y^2) = 0 \\ 2ye^{y^2-x^2}(1+x^2+y^2) = 0 \end{cases}$

we can get $(0,0)$ $(1,0)$

$\therefore f_{xx} = 2e^{y^2-x^2}(2x^2y^2 + 2x^4 - 5x^2 - y^2 + 1)$

$f_{yy} = 2e^{y^2-x^2}(2x^2y^2 + 2y^4 + 5y^2 + x^2 + 1)$

$f_{xy} = -4xy(x^2+y^2)e^{y^2-x^2}$

$\therefore D = f_{xx}f_{yy} - (f_{xy})^2$

$= (2e^{y^2-x^2})(2x^2y^2 + 2x^4 - 5x^2 - y^2 + 1)(2x^2y^2 + 2y^4 + 5y^2 + x^2 + 1)$
$- (-4xy(x^2+y^2)e^{y^2-x^2})^2$

at $(0,0)$, $D=2>0$, $f_{xx}(0,0)=2>0$

$\therefore$ $(0,0)$ is a local minimum

at $(1,0)$, $D=-\frac{16}{e^2}<0$, $f_{xx}(1,0)=-\frac{4}{e}<0$

$\therefore$ $(1,0)$ is not extrema

Q2

(a) $\nabla f = \langle 2x+2xy, 2y+x^2 \rangle$

solve $\begin{cases} 2x+2xy=0 \\ 2y+x^2=0 \end{cases}$

$\therefore$ we can get $(0,0)$, $(-\sqrt{2},-1)$ $(\sqrt{2},-1)$

$f_{xx}=2+2y$ $f_{yy}=2$, $f_{xy}=2x$

$D=4y+4-4x^2$

$(0,0)$, $D=4>0$ $f_{xx}(0,0)=2>0$

$(-\sqrt{2},-1)$, $D=-8<0$

$(\sqrt{2},-1)$, $D=-8<0$

$\therefore$ only $(0,0)$ is Crital point

$f(0,0)=4$

L_1: $f(x,-1)=5$

L_2: $f(1,y)=y^2+y+5$

max: $f(1,1)=7$

min: $f(1,\frac{1}{2})=\frac{19}{4}$

L_3: $f(x,1)=2x^2+5$

max: $f(1,1)=7$

min: $f(0,1)=5$

L_4: $f(-1,y)=y^2+y+5$

max: $f(-1,1)=7$

min: $f(-1,-\frac{1}{2})=\frac{19}{4}$

$\therefore$ Absolute maximum value $= 7$

absolute minimum value $= 4$

(b) $\nabla f = \langle 2x-2, 2y \rangle$

solve $\begin{cases} 2x-2=0 \\ 2y=0 \end{cases} \Rightarrow (1,0)$

$f_{xx}=2, \ f_{yy}=2, \ f_{xy}=0$

$D=4>0$

$f_{xx}(1,0)=2>0$

$\therefore (1,0)$ is the local minhm point

$\because D=\{(x,y) \mid x^2+y^2 \leq 1\}$

let $x=\sin t, \ y=\cos t$

$f(t)=\sin^2 t+\cos^2 t-2\sin t$

$=1-2\sin t \in [-1, 3]$

$\therefore f_{min}=-1$

$f_{max}=3$

Q3 let $2x, 2y, 2z$ represent the length 3 edge of the box

$V=8xyz \quad \sqrt{x^2+y^2+z^2}=r$

$\therefore V=8xy\sqrt{r^2-x^2-y^2}$

let $f(x,y)=V^2=64x^2y^2\sqrt{r^2-x^2-y^2}$

let $\begin{cases} f_x = 128xy^2r^2 - 256x^3y^2 - 128xy^4 = 0 \\ f_y = 128x^2yr^2 - 128x^4y - 256x^2y^3 = 0 \end{cases}$

wet get $(0,0)$, $\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right)$

hence $z = \sqrt{r^2 - x^2 - y^2} = \frac{r}{\sqrt{3}}$

$\therefore$ Volumn $= 8xyz = \frac{8r^3}{3\sqrt{3}}$