Section 2.3

6. a) The domain is $\mathbf{Z}^+ \times \mathbf{Z}^+$ and the range is $\mathbf{Z}^+$.

b) Since the largest decimal digit of a strictly positive integer cannot be 0, we have domain $\mathbf{Z}^+$ and range $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

c) The domain is the set of all bit strings. The number of 1's minus number of 0's can be any positive or negative integer or 0, so the range is $\mathbf{Z}$.

d) The domain is given as $\mathbf{Z}^+$. Clearly the range is $\mathbf{Z}^+$ as well.

e) The domain is the set of bit strings. The range is the set of strings of 1's, i.e., $\{\lambda, 1, 11, 111, \ldots\}$, where λ is the empty string (containing no symbols).

12. a) This is one-to-one, since if $n_1 - 1 = n_2 - 1$, then $n_1 = n_2$.

b) This is not one-to-one, since, for example, $f(3) = f(-3) = 10$.

c) This is one-to-one, since if $n_1^3 = n_2^3$, then $n_1 = n_2$ (take the cube root of each side).

d) This is not one-to-one, since, for example, $f(3) = f(4) = 2$.

14. a) This is clearly onto, since $f(0, -n) = n$ for every integer n.

b) This is not onto, since, for example, 2 is not in the range. To see this, if $m^2 - n^2 = (m - n)(m + n) = 2$, then m and n must have same parity (both even or both odd). In either case, both $m - n$ and $m + n$ are then even, so this expression is divisible by 4 and hence cannot equal 2.

c) This is clearly onto, since $f(0, n - 1) = n$ for every integer n.

d) This is onto. To achieve negative values we set $m = 0$, and to achieve nonnegative values we set $n = 0$.

e) This is not onto, for the same reason as in part **(b)**. In fact, the range here is clearly a subset of the range in that part.

21. Obviously there are an infinite number of correct answers to each part. The problem asked for a "formula." Parts **(a)** and **(c)** seem harder here, since we somehow have to fold the negative integers into the positive ones without overlap. Therefore we probably want to treat the negative integers differently from the positive integers. One way to do this with a formula is to make it a two-part formula. If one objects that this is not "a formula," we can counter as follows. Consider the function $g(x) = \lfloor 2^x \rfloor / 2^x$. Clearly if $x \geq 0$, then 2^x is a positive integer, so $g(x) = 2^x/2^x = 1$. If $x < 0$, then 2^x is a number between 0 and 1, so $g(x) = 0/2^x = 0$. If we want to define a function that has the value $f_1(x)$ when $x \geq 0$ and $f_2(x)$ when $x < 0$, then we can use the formula $g(x) \cdot f_1(x) + (1 - g(x)) \cdot f_2(x)$.

a) We could map the positive integers (and 0) into the positive multiples of 3, say, and the negative integers into numbers that are 1 greater than a multiple of 3, in a one-to-one manner. This will give us a function that leaves some elements out of the range. So let us define our function as follows:

$$f(x) = \begin{cases} 3x + 3 & \text{if } x \geq 0 \\ 3|x| + 1 & \text{if } x < 0 \end{cases}$$

The values of f on the inputs 0 through 4 are then $3, 6, 9, 12, 15$; and the values on the inputs -1 to -4 are $4, 7, 10, 13$. Clearly this function is one-to-one, but it is not onto since, for example, 2 is not in the range.

b) This is easier. We can just take $f(x) = |x| + 1$. It is clearly onto, but $f(n)$ and $f(-n)$ have the same value for every positive integer n, so f is not one-to-one.

c) This is similar to part **(a)**, except that we have to be careful to hit all values. Mapping the nonnegative integers to the odds and the negative integers to the evens will do the trick:

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \geq 0 \\ 2|x| & \text{if } x < 0 \end{cases}$$

d) Here we can use a trivial example like $f(x) = 17$ or a simple nontrivial one like $f(x) = x^2 + 1$. Clearly these are neither one-to-one nor onto.

23. **a)** One way to determine whether a function is a bijection is to try to construct its inverse. This function is a bijection, since its inverse (obtained by solving $y = 2x + 1$ for x) is the function $g(y) = (y - 1)/2$. Alternatively, we can argue directly. To show that the function is one-to-one, note that if $2x + 1 = 2x' + 1$, then $x = x'$. To show that the function is onto, note that $2((y - 1)/2) + 1 = y$, so every number is in the range.

b) This function is not a bijection, since its range is the set of real numbers greater than or equal to 1 (which is sometimes written $[1, \infty)$), not all of $\mathbf{R}$. (It is not injective either.)

c) This function is a bijection, since it has an inverse function, namely the function $f(y) = y^{1/3}$ (obtained by solving $y = x^3$ for x).

d) This function is not a bijection. It is easy to see that it is not injective, since x and $-x$ have the same image, for all real numbers x. A little work shows that the range is only $\{ y \mid 0.5 \le y < 1 \} = [0.5, 1)$.

31. In each case, we need to compute the values of $f(x)$ for each $x \in S$.

a) Note that $f(\pm 2) = \lfloor (\pm 2)^2/3 \rfloor = \lfloor 4/3 \rfloor = 1$, $f(\pm 1) = \lfloor (\pm 1)^2/3 \rfloor = \lfloor 1/3 \rfloor = 0$, $f(0) = \lfloor 0^2/3 \rfloor = \lfloor 0 \rfloor = 0$, and $f(3) = \lfloor 3^2/3 \rfloor = \lfloor 3 \rfloor = 3$. Therefore $f(S) = \{0, 1, 3\}$.

b) In addition to the values we computed above, we note that $f(4) = 5$ and $f(5) = 8$. Therefore $f(S) = \{0, 1, 3, 5, 8\}$.

c) Note this time also that $f(7) = 16$ and $f(11) = 40$, so $f(S) = \{0, 8, 16, 40\}$.

d) $\{f(2), f(6), f(10), f(14)\} = \{1, 12, 33, 65\}$

40. **a)** This really has two parts. First suppose that b is in $f(S \cup T)$. Thus $b = f(a)$ for some $a \in S \cup T$. Either $a \in S$, in which case $b \in f(S)$, or $a \in T$, in which case $b \in f(T)$. Thus in either case $b \in f(S) \cup f(T)$. This shows that $f(S \cup T) \subseteq f(S) \cup f(T)$. Conversely, suppose $b \in f(S) \cup f(T)$. Then either $b \in f(S)$ or $b \in f(T)$. This means either that $b = f(a)$ for some $a \in S$ or that $b = f(a)$ for some $a \in T$. In either case, $b = f(a)$ for some $a \in S \cup T$, so $b \in f(S \cup T)$. This shows that $f(S) \cup f(T) \subseteq f(S \cup T)$, and our proof is complete.

b) Suppose $b \in f(S \cap T)$. Then $b = f(a)$ for some $a \in S \cap T$. This implies that $a \in S$ and $a \in T$, so we have $b \in f(S)$ and $b \in f(T)$. Therefore $b \in f(S) \cap f(T)$, as desired.

44. **a)** We need to prove two things. First suppose $x \in f^{-1}(S \cup T)$. This means that $f(x) \in S \cup T$. Therefore either $f(x) \in S$ or $f(x) \in T$. In the first case $x \in f^{-1}(S)$, and in the second case $x \in f^{-1}(T)$. In either case, then, $x \in f^{-1}(S) \cup f^{-1}(T)$. Thus we have shown that $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$. Conversely, suppose that $x \in f^{-1}(S) \cup f^{-1}(T)$. Then either $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$, so either $f(x) \in S$ or $f(x) \in T$. Thus we know that $f(x) \in S \cup T$, so by definition $x \in f^{-1}(S \cup T)$. This shows that $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$, as desired.

b) This is similar to part **(a)**. We have $x \in f^{-1}(S \cap T)$ if and only if $f(x) \in S \cap T$, if and only if $f(x) \in S$ and $f(x) \in T$, if and only if $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$, if and only if $x \in f^{-1}(S) \cap f^{-1}(T)$.

71. We can prove all of these identities by showing that the left-hand side is equal to the right-hand side for all possible values of x. In each instance (except part **(c)**, in which there are only two cases), there are four cases to consider, depending on whether x is in A and/or B.

a) If x is in both A and B, then $f_{A \cap B}(x) = 1$; and the right-hand side is $1 \cdot 1 = 1$ as well. Otherwise $x \notin A \cap B$, so the left-hand side is 0, and the right-hand side is either $0 \cdot 1$ or $1 \cdot 0$ or $0 \cdot 0$, all of which are also 0.

b) If x is in both A and B, then $f_{A \cup B}(x) = 1$; and the right-hand side is $1 + 1 - 1 \cdot 1 = 1$ as well. If x is in A but not B, then $x \in A \cup B$, so the left-hand side is still 1, and the right-hand side is $1 + 0 - 1 \cdot 0 = 1$, as desired. The case in which x is in B but not A is similar. Finally, if x is in neither A nor B, then the left-hand side is 0, and the right-hand side is $0 + 0 - 0 \cdot 0 = 0$ as well.

c) If $x \in A$, then $x \notin \overline{A}$, so $f_{\overline{A}}(x) = 0$. The right-hand side equals $1 - 1 = 0$ in this case, as well. On the other hand, if $x \notin A$, then $x \in \overline{A}$, so the left-hand side is 1, and the right-hand side is $1 - 0 = 1$ as well.

d) If x is in both A and B, then $x \notin A \oplus B$, so $f_{A \oplus B}(x) = 0$. The right-hand side is $1 + 1 - 2 \cdot 1 \cdot 1 = 0$ as well. Next, if $x \in A$ but $x \notin B$, then $x \in A \oplus B$, so the left-hand side is 1. The right-hand side is $1 + 0 - 2 \cdot 1 \cdot 0 = 1$ as well. The case $x \in B \wedge x \notin A$ is similar. Finally, if x is in neither A nor B, then $x \notin A \oplus B$, so the left-hand side is 0; and the right-hand side is also $0 + 0 - 2 \cdot 0 \cdot 0 = 0$.

73. **a)** This is true. Since $\lfloor x \rfloor$ is already an integer, $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$.

b) A little experimentation shows that this is not always true. To disprove it we need only produce a counterexample, such as $x = \frac{1}{2}$. In that case the left-hand side is $\lfloor 1 \rfloor = 1$, while the right-hand side is $2 \cdot 0 = 0$.

c) This is true. We prove it by cases. If x is an integer, then by identity (4b) in Table 1, we know that $\lceil x+y \rceil = x + \lceil y \rceil$, and it follows that the difference is 0. Similarly, if y is an integer. The remaining case is that $x = n + \epsilon$ and $y = m + \delta$, where n and m are integers and ϵ and δ are positive real numbers less than 1. Then $x+y$ will be greater than $m+n$ but less than $m+n+2$, so $\lceil x+y \rceil$ will be either $m+n+1$ or $m+n+2$. Therefore the given expression will be either $(n+1)+(m+1)-(m+n+1) = 1$ or $(n+1)+(m+1)-(m+n+2) = 0$, as desired.

d) This is clearly false, as we can find with a little experimentation. Take, for example, $x = 1/10$ and $y = 3$. Then the left-hand side is $\lceil 3/10 \rceil = 1$, but the right-hand side is $1 \cdot 3 = 3$.

e) Again a little trial and error will produce a counterexample. Take $x = 1/2$. Then the left-hand side is 1 while the right-hand side is 0.

Section 2.5

4. **a)** This set is countable. The integers in the set are ± 1, ± 2, ± 4, ± 5, ± 7, and so on. We can list these numbers in the order $1, -1, 2, -2, 4, -4, 5, -5, 7, -7, \ldots$, thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow 1$, $2 \leftrightarrow -1$, $3 \leftrightarrow 2$, $4 \leftrightarrow -2$, $5 \leftrightarrow 4$, and so on.

b) This is similar to part **(a)**; we can simply list the elements of the set in order of increasing absolute value, listing each positive term before its corresponding negative: $5, -5, 10, -10, 15, -15, 20, -20, 25, -25, 30, -30, 40, -40, 45, -45, 50, -50, \ldots$.

c) This set is countable but a little tricky. We can arrange the numbers in a 2-dimensional table as follows:

$$\begin{array}{llllllll}
.\overline{1} & .1 & .11 & .111 & .1111 & .11111 & .111111 & \ldots \\
1.\overline{1} & 1 & 1.1 & 1.11 & 1.111 & 1.1111 & 1.11111 & \ldots \\
11.\overline{1} & 11 & 11.1 & 11.11 & 11.111 & 11.1111 & 11.11111 & \ldots \\
111.\overline{1} & 111 & 111.1 & 111.11 & 111.111 & 111.1111 & 111.11111 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Thus we have shown that our set is the countable union of countable sets (each of the countable sets is one row of this table). Therefore by Exercise 27, the entire set is countable. For an explicit correspondence with the positive integers, we can zigzag along the positive-sloping diagonals as in Figure 3: $1 \leftrightarrow .\overline{1}$, $2 \leftrightarrow 1.\overline{1}$, $3 \leftrightarrow .1$, $4 \leftrightarrow .11$, $5 \leftrightarrow 1$, and so on.

d) This set is not countable. We can prove it by the same diagonalization argument as was used to prove that the set of all reals is uncountable in Example 5. All we need to do is choose $d_i = 1$ when $d_{ii} = 9$ and choose $d_i = 9$ when $d_{ii} = 1$ or d_{ii} is blank (if the decimal expansion is finite).

10. In each case, let us take A to be the set of real numbers.

a) We can let B be the set of real numbers as well; then $A - B = \emptyset$, which is finite.

b) We can let B be the set of real numbers that are not positive integers; in symbols, $B = A - \mathbf{Z}^+$. Then $A - B = \mathbf{Z}^+$, which is countably infinite.

c) We can let B be the set of positive real numbers. Then $A - B$ is the set of negative real numbers and 0, which is certainly uncountable.

11. In each case, we can make the intersection what we want it to be, and then put additional elements into A and into B (with no overlap) to make them uncountable.

a) The simplest solution would be to make $A \cap B = \emptyset$. So, for example, take A to be the interval $(1, 2)$ of real numbers, and take B to be the interval $(3, 4)$.

b) Take the example from part **(a)** and adjoin the positive integers. Thus, let $A = (1, 2) \cup \mathbf{Z}^+$ and let $B = (3, 4) \cup \mathbf{Z}^+$.

c) Let $A = (1, 3)$ and $B = (2, 4)$.

18. The hypothesis gives us a one-to-one and onto function f from A to B. By Exercise 16e in the supplementary exercises for this chapter, the function S_f from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ defined by $S_f(X) = f(X)$ for all $X \subseteq A$ is one-to-one and onto. Therefore $\mathcal{P}(A)$ and $\mathcal{P}(B)$ have the same cardinality.

19. By what we are given, we know that there are bijections f from A to B and g from C to D. Then we can define a bijection from $A \times C$ to $B \times D$ by sending (a, c) to $(f(a), g(c))$. This is clearly one-to-one and onto, so we have shown that $A \times C$ and $B \times D$ have the same cardinality.

28. We can think of $\mathbf{Z}^+ \times \mathbf{Z}^+$ as the countable union of countable sets, where the i^{th} set in the collection, for $i \in \mathbf{Z}^+$, is $\{(i, n) \mid n \in \mathbf{Z}^+\}$. The statement now follows from Exercise 27.