CH 11 Vibration

outline

- **11.1 Simple harmonic vibration**
- **11.2 Damped vibration**
- **11.3 Impounded vibration and resonance**

vibrate

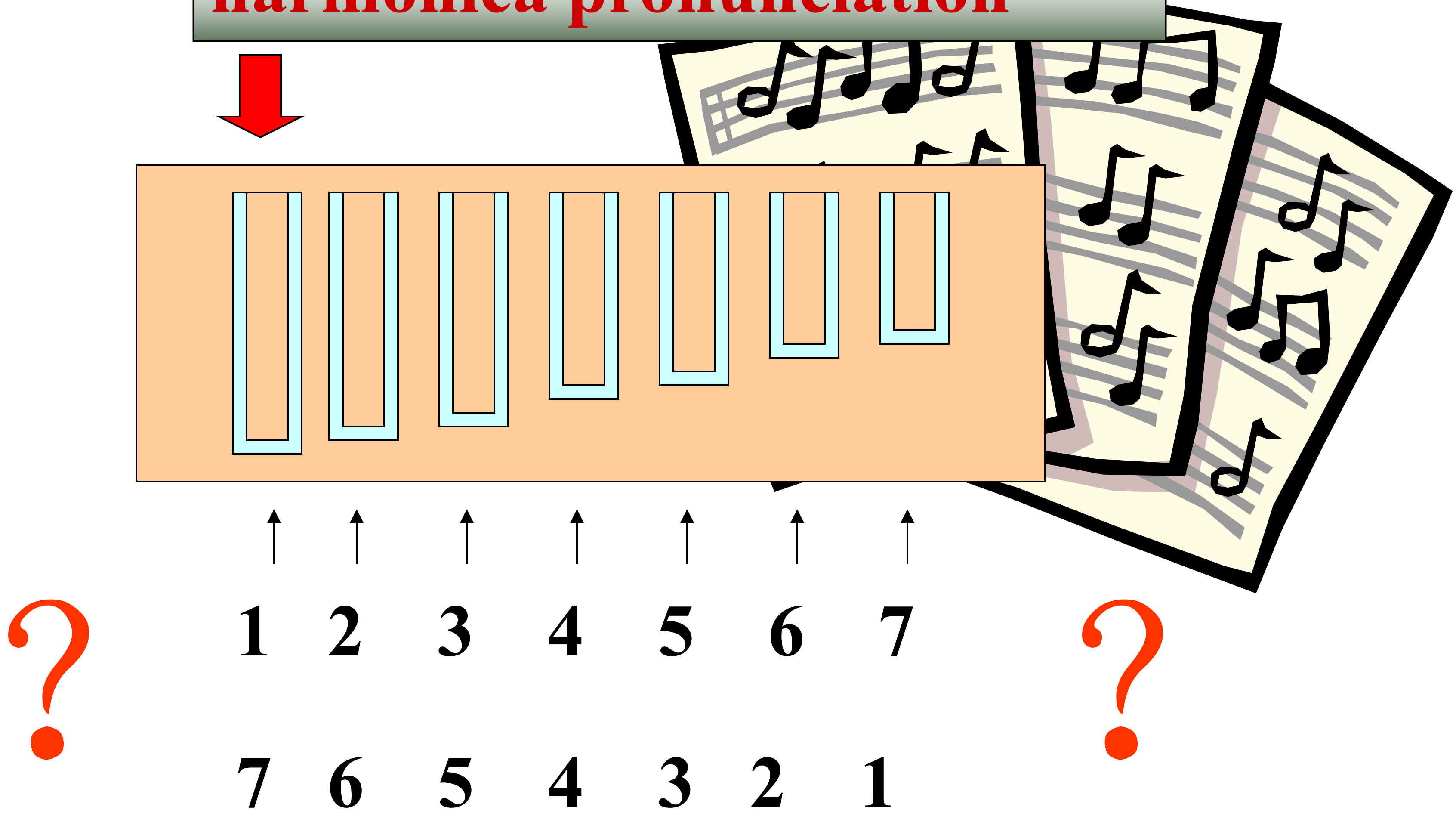
Vibration in a broad sense: a motion that is repetitive or reciprocating in time

Physically, the physical quantity describing the state of motion of matter, changes periodically in a certain value

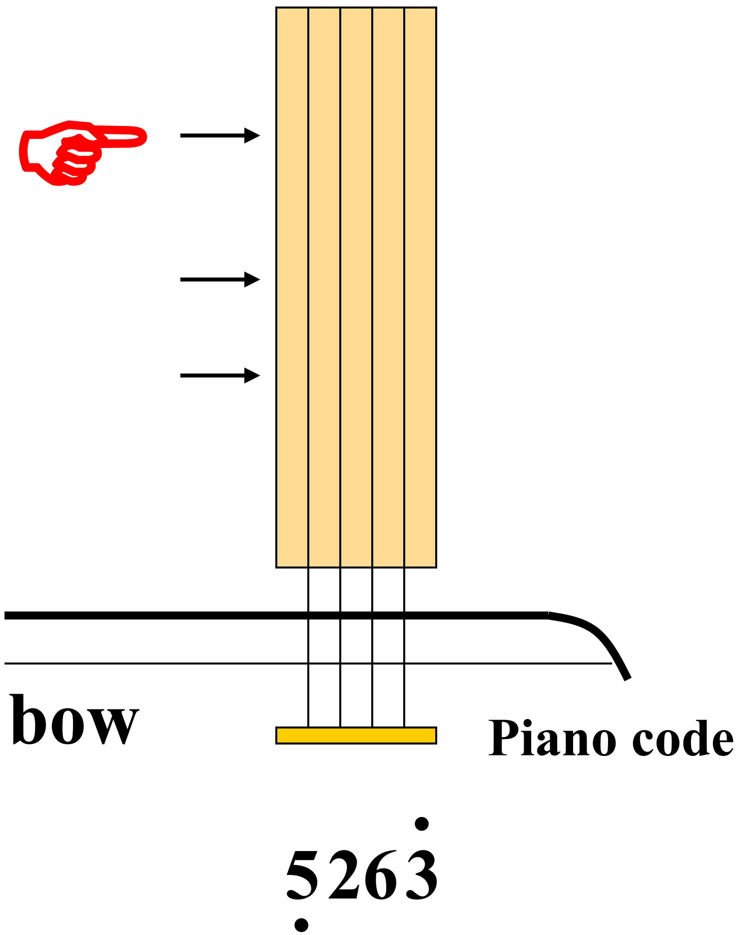
mechanical vibration

- **The repeated movement of an object near a certain position in the same path is called mechanical vibration**
- **Such as the heart beating, pendulum, Musical Instruments, earthquake and so on**

The mechanism of the harmonica pronunciation



Vibration of the violin strings



Mechanical vibration classification

According to the law of vibration points:
simple harmonic, non-simple harmonic, random vibration

According to the cause of vibration:
free, forced, self-excitation, variable vibration

According to the degree of freedom:
single degree of freedom system, multi-degree of freedom system vibration

According to the vibration displacement points:
angular vibration, line vibration

According to the system parameter characteristics:
linear, nonlinear vibration

Among them, simple harmonic vibration is the most fundamental and exists in many physical phenomena

Complex vibrations can all be decomposed into a superposition of some simple harmonic vibrations

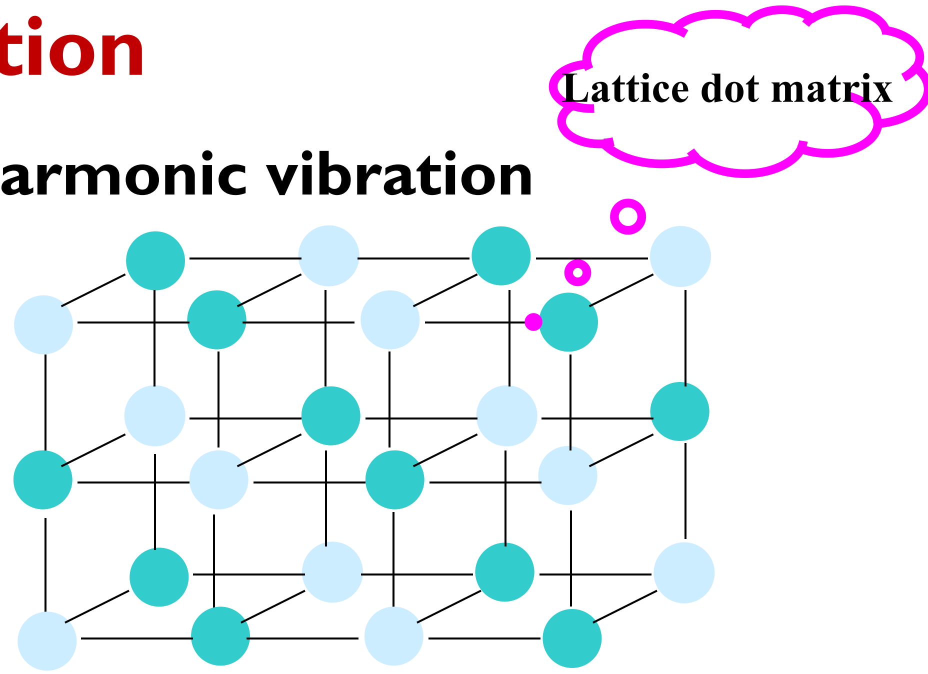
§ 1 Simple harmonic vibration

- **Description of the simple harmonic vibration**

- **Synthesis of simple harmonic vibrations**

Description of the simple harmonic vibration

1. Characteristics of a simple harmonic vibration

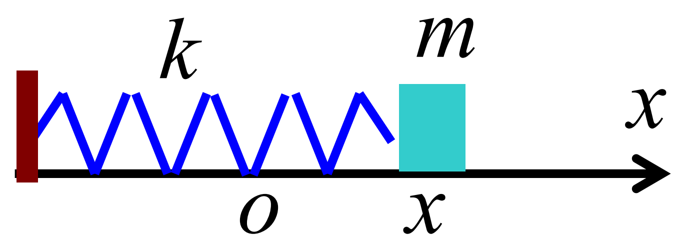


Any stable system that deviates slightly from the equilibrium state can be regarded as a simple harmonic oscillator. For many problems in physics, the harmonic oscillator works as an approximate or rather accurate model

Kinematic equations for simple harmonic vibration

$$m\ddot{x} = -kx$$

elastic force



$$U(x) = \frac{1}{2}kx^2$$

suppose $k = m\omega_0^2 \Rightarrow \ddot{x} + \omega_0^2 x = 0$

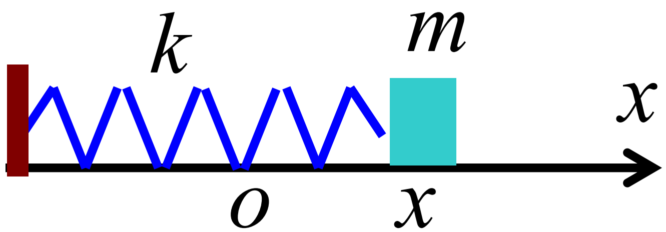
Its solution: $x(t) = A\cos(\omega_0 t + \varphi_0)$

The external force subjected to the particle is proportional and reverse to the displacement of the equilibrium position, or the potential energy of the particle is proportional to the square of the displacement (angular displacement), which is the simple harmonic vibration. This vibrational system is called a harmonic oscillator.

Kinematic description of simple harmonic vibrations

Take the spring oscillator for example

Movement pattern of the system displacement



$$x(t) = A\cos(\omega_0 \cdot t + \varphi_o)$$

ω_0 **determined by the system itself**

Conclusion:

Simple harmonic vibration —— which is a sine or cosine function of time

The motions shown are all simple harmonic vibration

The period and frequency and amplitude of the simple harmonic vibration

$$A\cos(\omega_0 t+\varphi_0)=A\cos(\omega_0 t+\varphi_0+2\pi n)$$
$$=A\cos[\omega_0(t+\frac{2\pi}{\omega_0}n)+\varphi_0]$$
$$=A\cos[\omega_0(t+nT)+\varphi_0]$$

$$T=\frac{2\pi}{\omega_0}$$

Period, the motion is completely repeated every other T time

$$\nu=\frac{1}{T}=\frac{\omega_0}{2\pi}$$

Vibration frequency, the number of vibrations per unit of time

$$\omega_o=\frac{2\pi}{T}=\sqrt{\frac{k}{m}}$$

Angular frequency (or round frequency)
That is, the value of the change in phase per unit of time

Phase, initial phase, and amplitude of the simple harmonic vibration

$$x(t) = A\cos(\omega_0 t + \varphi_0)$$

A **--Maximum displacement in the amplitude vibration**

ω_0 **angular frequency**

$\varphi(t) = \omega_0 t + \varphi_0$ **phase position**

φ_0 **initial phase**

The same state of motion corresponds to an integer multiple with a phase difference of 2π

Simple harmonic vibration can be expressed by sine function in addition to the cosine function form

$$x(t) = A\cos(\omega_0 t + \varphi_0) = A\sin(\omega_0 t + \varphi_0 + \pi/2)$$

$$= A\sin(\omega_0 t + \varphi_0')$$

The velocity of the simple harmonic vibration

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = -A\omega \sin(\omega t + \varphi_0) = A\omega \cos(\omega t + \varphi_0 + \frac{\pi}{2})$$

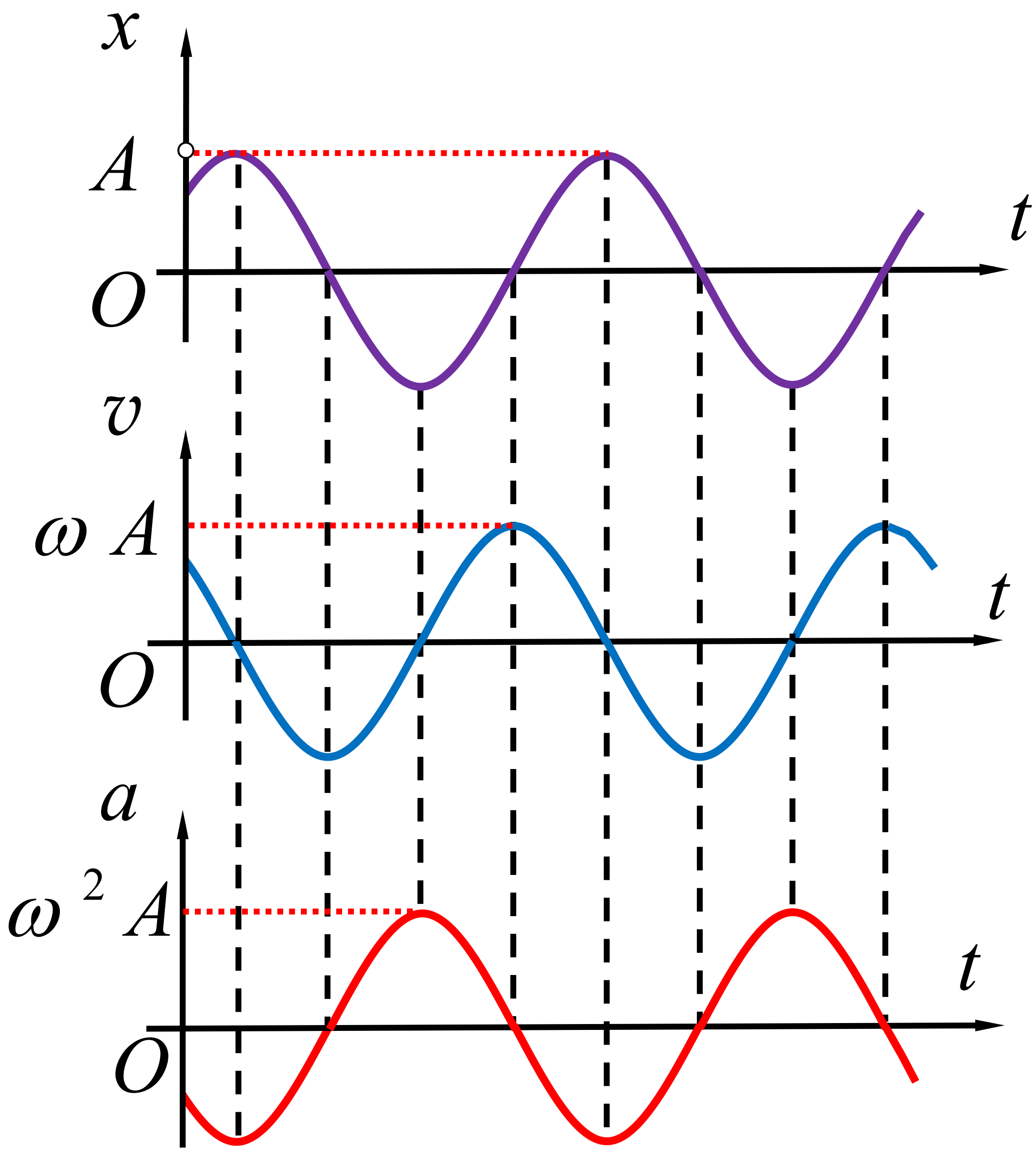
Acceleration of the simple harmonic vibration

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -A\omega^2 \cos(\omega t + \varphi_0) = A\omega^2 \cos(\omega t + \varphi_0 + \pi)$$

The acceleration of the simple harmonic vibration is the variable acceleration

Displacement and acceleration are inverted

$$a = -x\omega^2$$


x
A
t
O
v
ω A
t
O
a
ω 2 A
t
O

A, φ_0 are determined by the initial state

$$x = A\cos(\omega t + \varphi) \qquad v = -A\omega\sin(\omega t + \varphi)$$

$$x_0 = A\cos\varphi_0 \qquad v_0 = -\omega A\sin\varphi_0$$

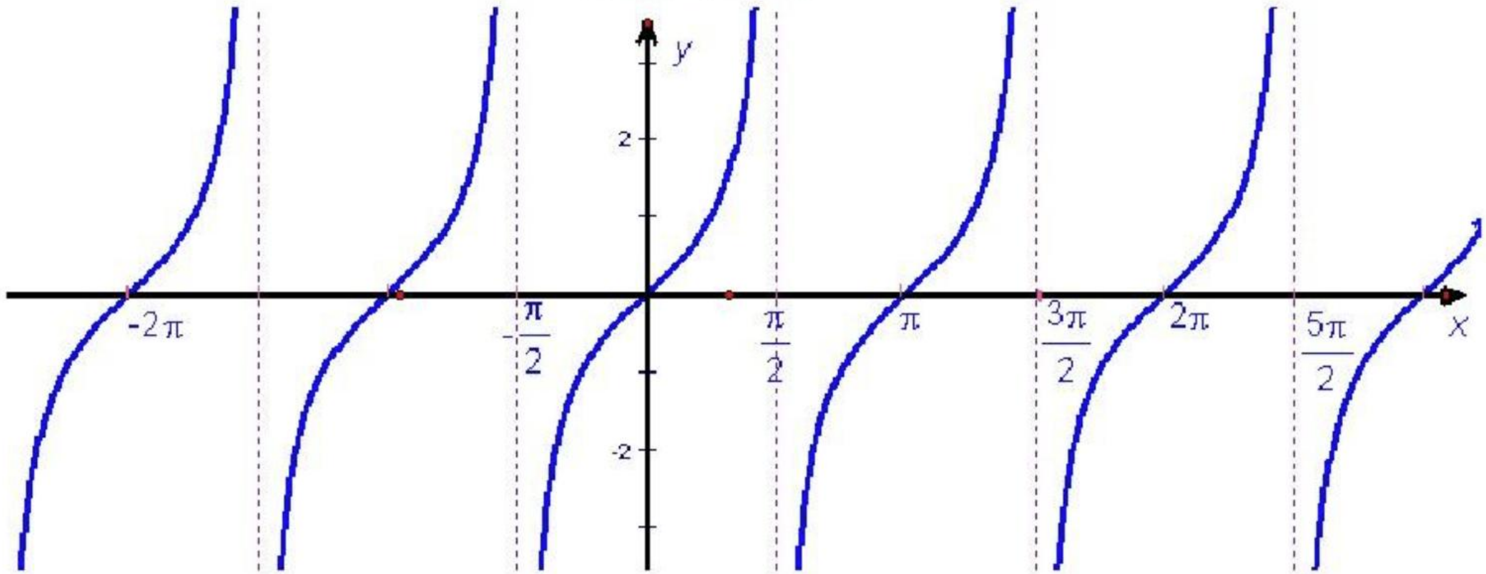
$$A = \sqrt{x_0{}^2 + \frac{v_0^2}{\omega^2}} \qquad \tan\varphi_0 = -\frac{v_0}{\omega x_0}$$

φ_0 is only determined by the direction v_0

φ_0 要由 v_0 的方向唯一确定

$$\tan\varphi_0 = -\frac{v_0}{\omega x_0}$$

正切曲线



Discuss

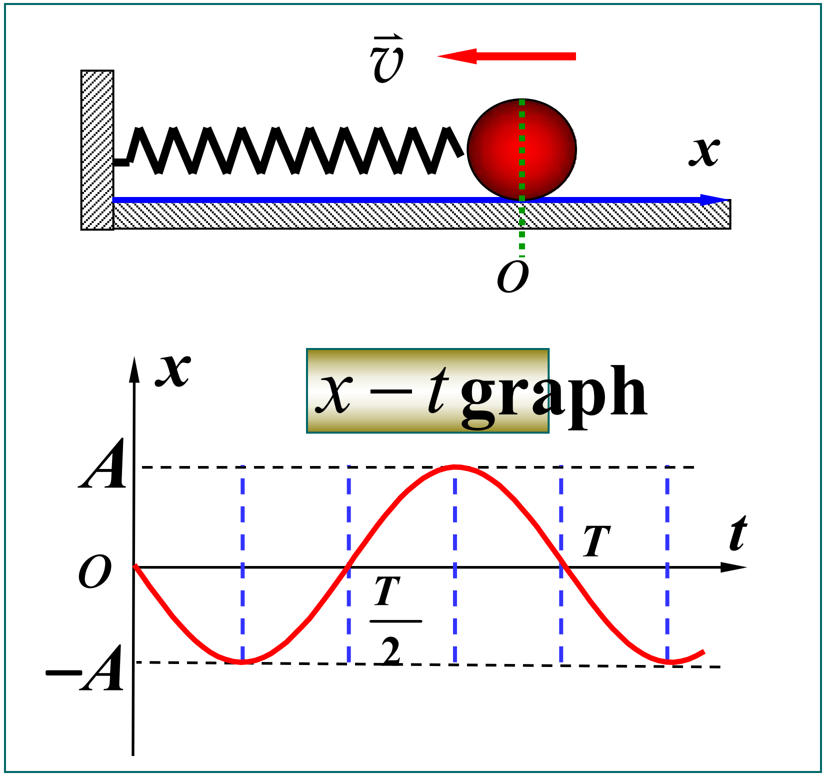
known: $t = 0, x = 0, v_0 < 0$ **ask** φ

$$0 = A\cos\varphi \Rightarrow \varphi = \pm\frac{\pi}{2}$$

$$\because v_0 = -A\omega\sin\varphi < 0$$

$$\therefore \sin\varphi > 0 \qquad \varphi = \frac{\pi}{2}$$

$$x = A\cos(\omega t + \frac{\pi}{2})$$



2. Rotational vector representation of simple harmonic vibration

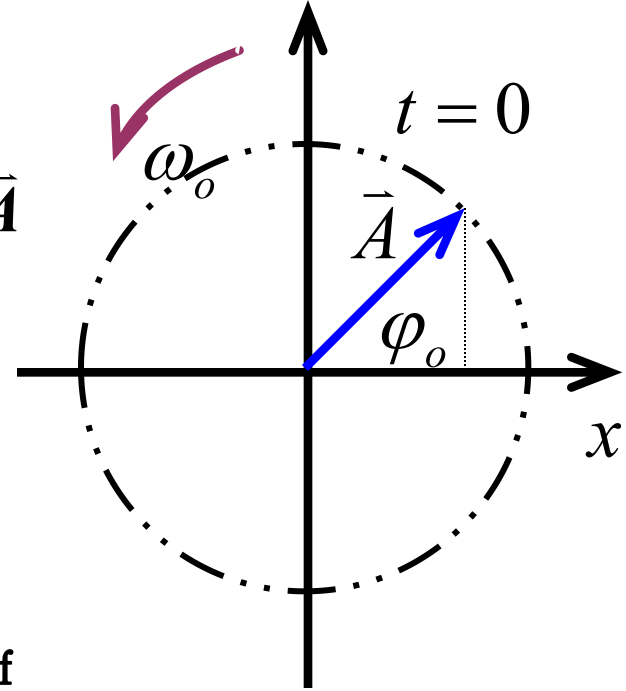
O point as a starting point to obtain a vector $\vec{A}$

Rotational vector, or an amplitude vector

Length is equal to the amplitude A of the simple harmonic vibration

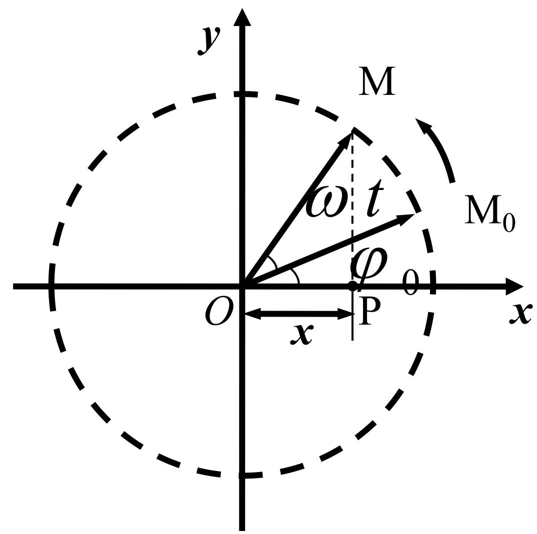
The vector rotates counterclockwise around the O point in the Oxy plane

Its angular velocity and the angular frequency ω of the simple harmonic vibration



At time t, the projection of the rotation vector on the x-axis is

$$x = A\cos(\omega t + \varphi_0)$$



Corresponding: the projection of the rotation vector endpoint M on the x-axis

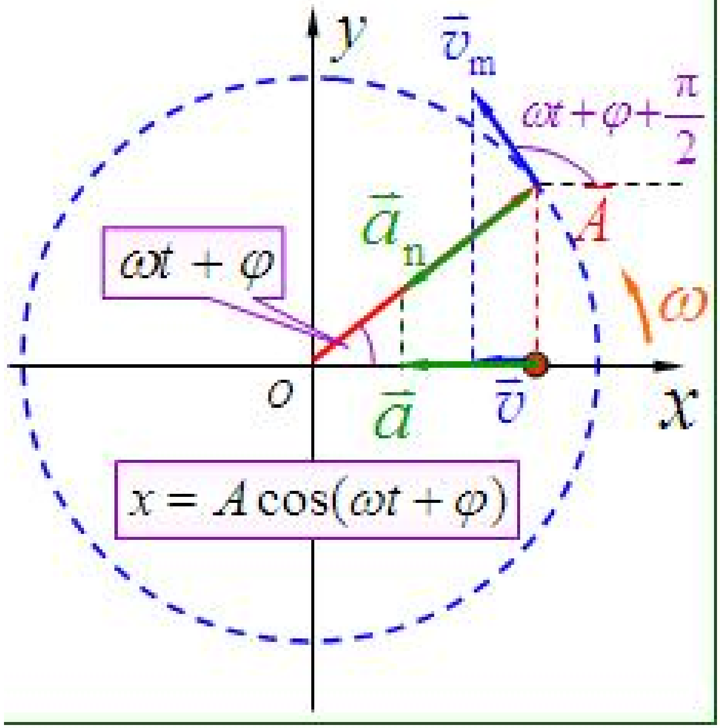
P is the origin simple harmonic vibration with O on the x axis

The velocity of the M point is

$$v_{\mathrm{M}} = A\omega$$

The velocity of point P is

$$v_{\mathrm{P}} = -\omega A \cos(\omega t + \varphi_0)$$



The acceleration at point M is the centripetal acceleration

$$a_{\mathrm{M}} = A\omega^2$$

The acceleration at point P is

$$a_{\mathrm{P}} = -A\omega^2 \cos(\omega t + \varphi_0)$$

Discuss

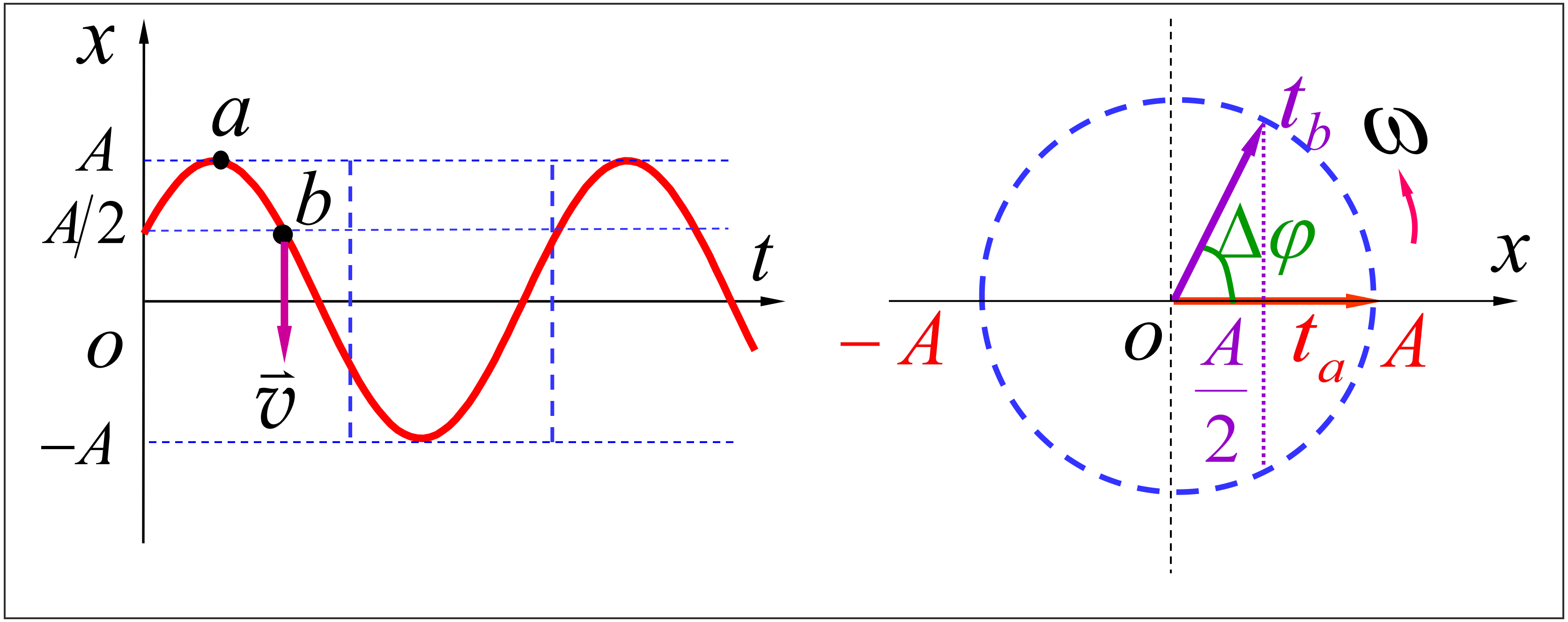
➤ **Phase difference: indicates the difference between the two phases**

(1) For the same simple harmonic motion, the phase difference can give the time required for the change between the two motion states.

$$x_1 = A\cos(\omega t_1 + \varphi) \qquad x_2 = A\cos(\omega t_2 + \varphi)$$

$$\Delta\varphi = (\omega t_2 + \varphi) - (\omega t_1 + \varphi)$$

$$\Delta t = t_2 - t_1 = \frac{\Delta\varphi}{\omega}$$



$$\Delta\varphi = \frac{\pi}{3} \qquad \Delta t = \frac{\pi/3}{2\pi}T = \frac{1}{6}T$$

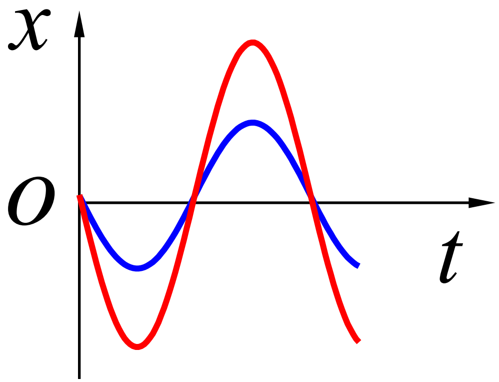
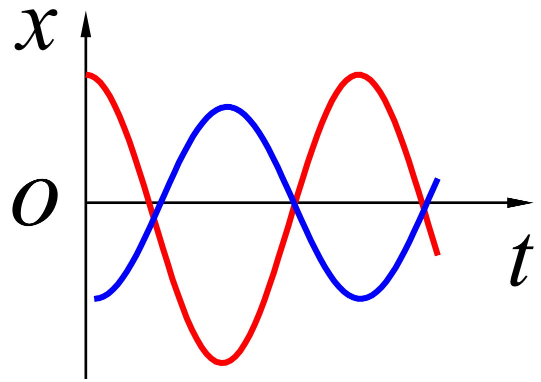
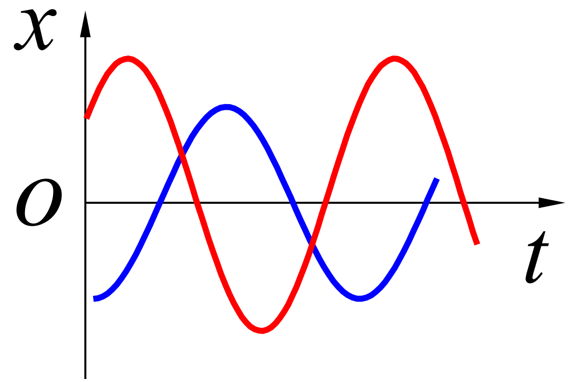
(2) For two simple harmonic motions of the same frequency, the phase difference indicates the difference in their steps (solving the problem of vibration synthesis).

$$x_1 = A_1 \cos(\omega t + \varphi_1) \quad x_2 = A_2 \cos(\omega t + \varphi_2)$$

$$\Delta\varphi = (\omega t + \varphi_2) - (\omega t + \varphi_1)$$

$$\boxed{\Delta\varphi = \varphi_2 - \varphi_1}$$

$$\Delta\varphi = \varphi_2 - \varphi_1$$

synchronization	inverted	
$\Delta\varphi = 0$	$\Delta\varphi = \pm\pi$	$\Delta\varphi$ For the $\left\{\begin{array}{l}\text{before}\\ \text{after}\end{array}\right.$ other
		

Phase difference of two simple harmonic vibrations of the same frequency

$$(\omega_0 t+\varphi_{20})-(\omega_0 t+\varphi_{10})=\varphi_{20}-\varphi_{10}$$

$\varphi_{20}-\varphi_{10}$

>0 φ_{20} **before** φ_{10}

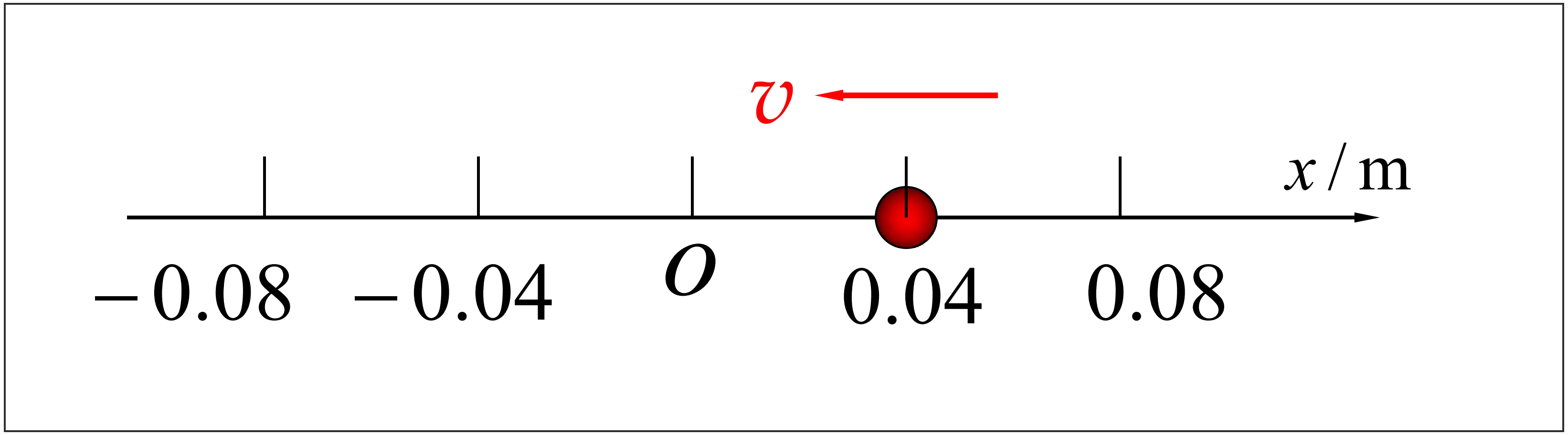
<0 φ_{20} **after** φ_{10}

(2n±1)π for inverted phase

2nπ for the same phase

An object with a mass of 0.01 kg makes a simple harmonic motion, its amplitude is 0.08 m, and the period is 4s. At the beginning, the object moves negatively at x =0.04 m (as shown in the figure). Try to calculate:

(1) At t =1.0s, the position and force of the object;



known

$$m = 0.01\,\mathrm{kg}, A = 0.08\,\mathrm{m}, T = 4\,\mathrm{s}$$

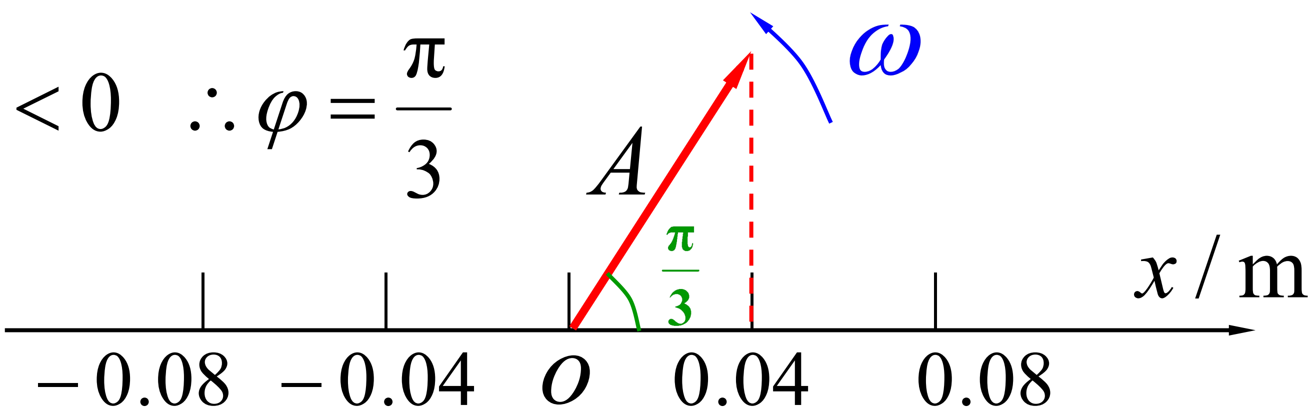
$t = 0, x = 0.04\,\mathrm{m}, v_0 < 0$ **For (1)** $t = 1.0\,\mathrm{s}, x, F$

Solution $A = 0.08\,\mathrm{m} \quad \omega = \dfrac{2\pi}{T} = \dfrac{\pi}{2}\,\mathrm{s}^{-1}$

$$t = 0, \quad x = 0.04\ \mathrm{m}$$

Substitute $x = A\cos(\omega t + \varphi) \Rightarrow \varphi = \pm\dfrac{\pi}{3}$

$$\because v_0 < 0 \quad \therefore \varphi = \frac{\pi}{3}$$



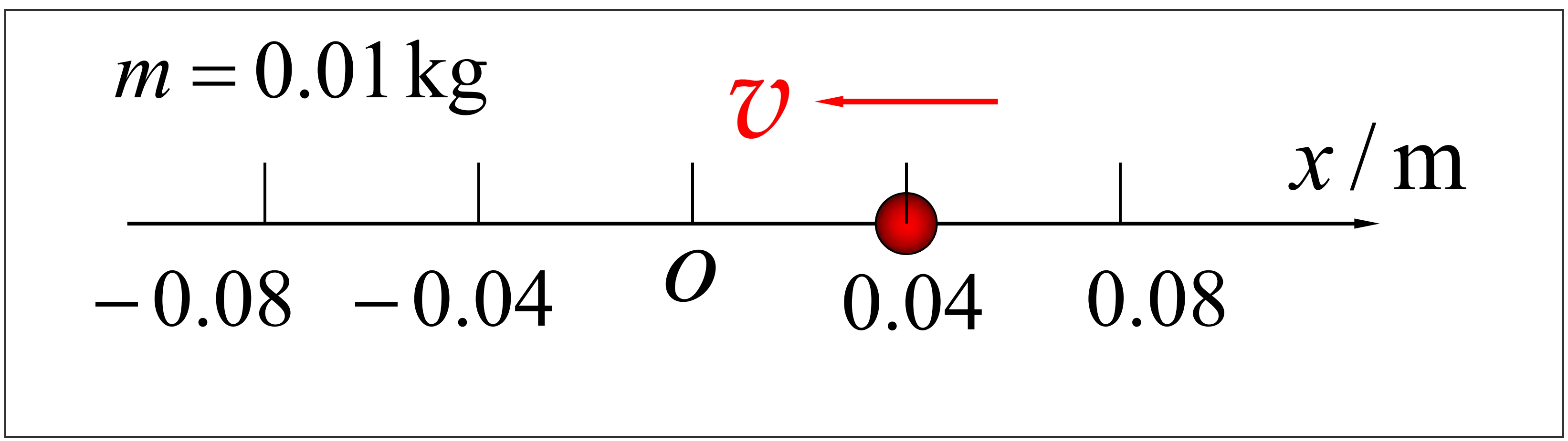
$$\because \varphi = \frac{\pi}{3}$$

$$\therefore x = 0.08\cos(\frac{\pi}{2}t + \frac{\pi}{3})$$

(1) $t = 1.0\ \text{s}, x, F$

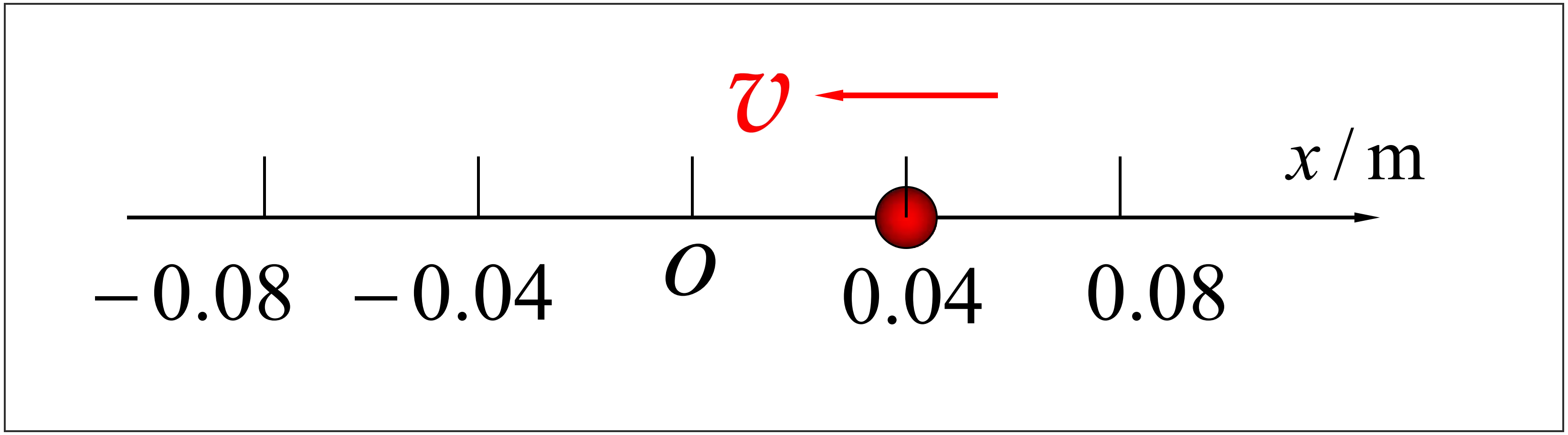
$t = 1.0\ \text{s}$ **Substitute into the above equation to obtain** $x = -0.069\ \text{m}$

$$F = -kx = -m\omega^2 x = 1.70\times10^{-3}\ \text{N}$$



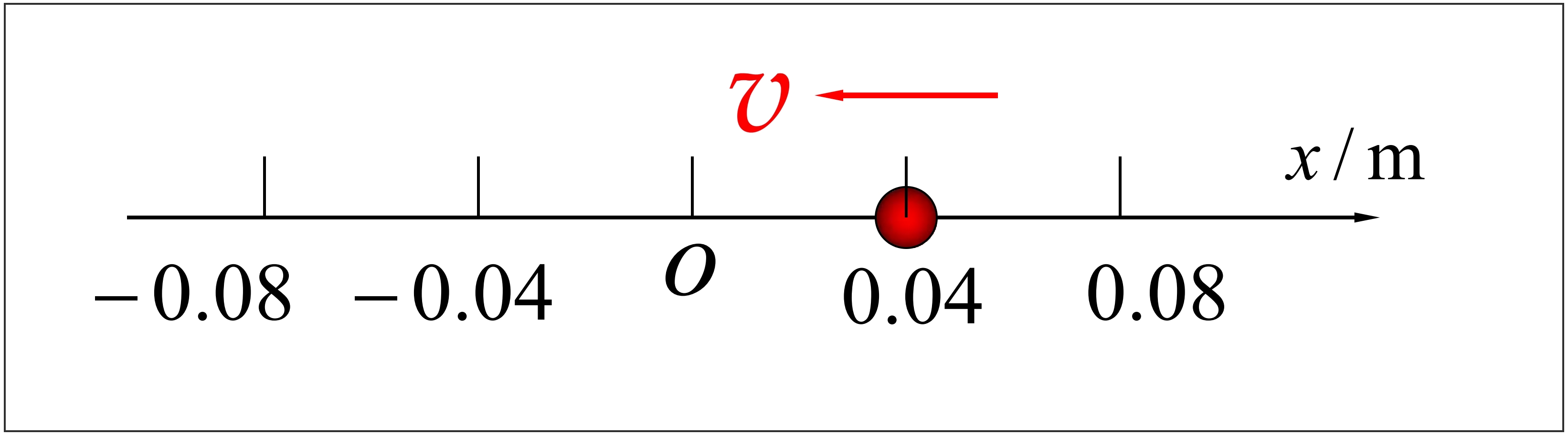
(2) The shortest time required to move from the starting position to x = -0.04 m.

The shortest time required to move from the starting position to x= -0.04 m is *t*

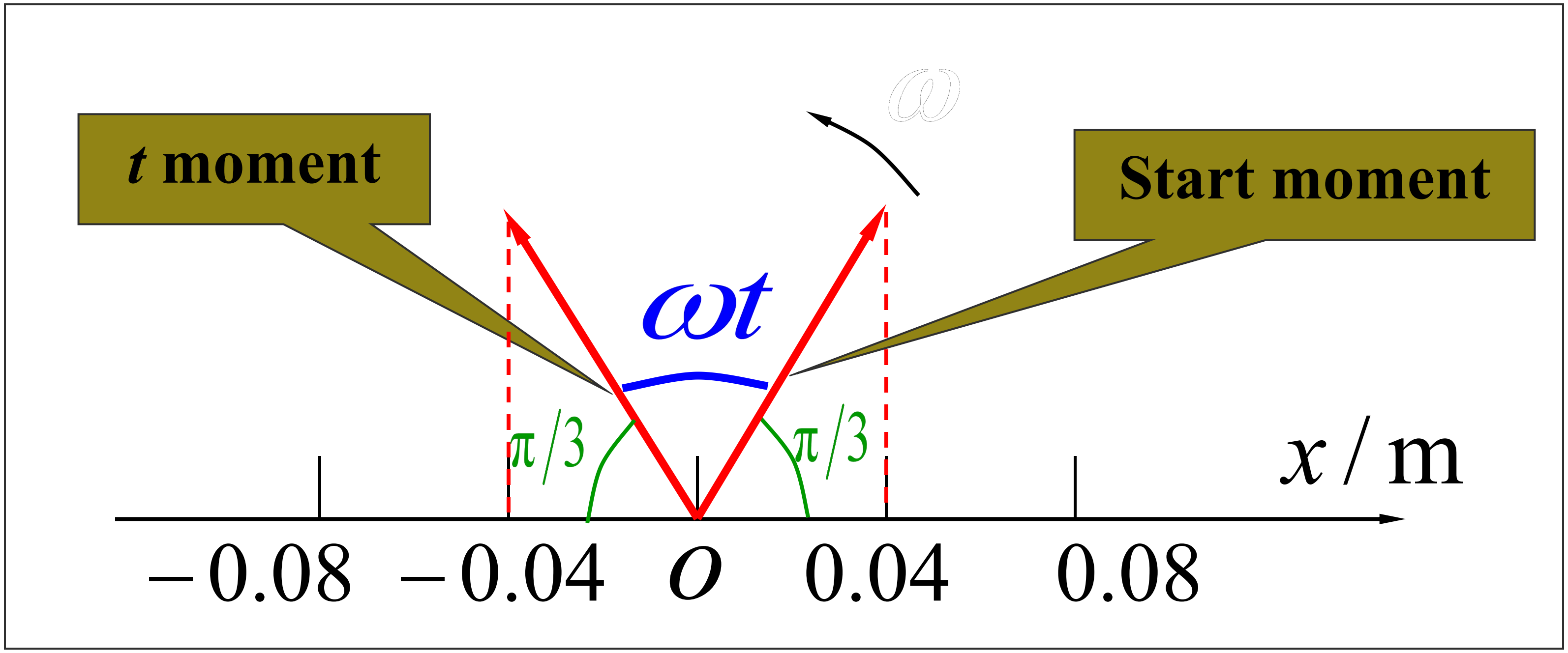


$$x = 0.08\cos\left(\frac{\pi}{2}t + \frac{\pi}{3}\right) \Rightarrow -0.04 = 0.08\cos\left(\frac{\pi}{2}t + \frac{\pi}{3}\right)$$

$$t = \frac{\arccos\left(-\frac{1}{2}\right) - \frac{\pi}{3}}{\pi/2} = \frac{2}{3} = 0.667\ \mathrm{s}$$



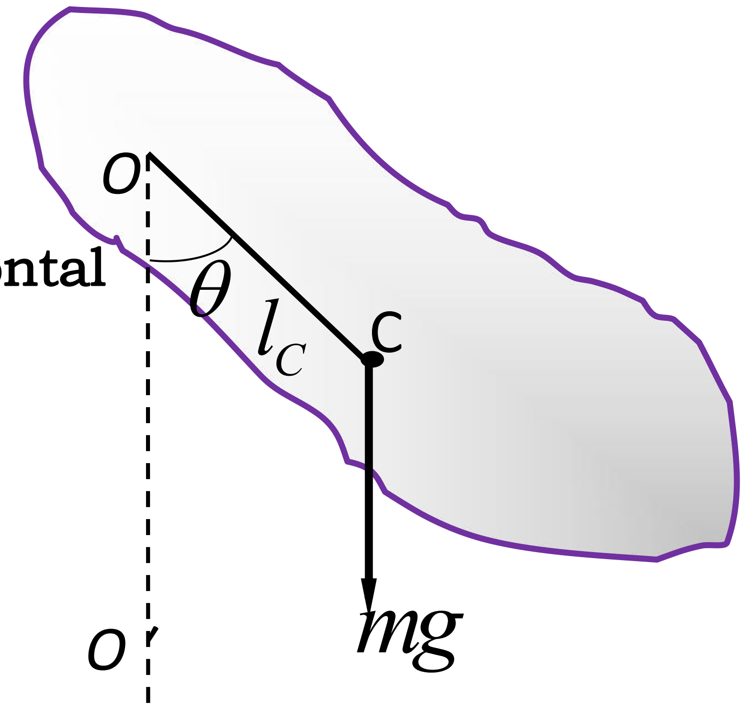
Method 2



$$\omega t=\frac{\pi}{3} \qquad \omega=\frac{\pi}{2}\,\mathrm{rad}\cdot\mathrm{s}^{-1} \qquad t=\frac{2}{3}=0.667\ \mathrm{s}$$

3. Typical problem of simple harmonic vibration

- Compound pendulum



The rigid body swings around the horizontal axis of O

Law of rigid body in axial rotation

inertia

$$J\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -mgl_C\sin\theta$$

The negative sign indicates that the torque always brings the rotation back to the equilibrium position

The Angle is very small

$$\sin\theta \sim \theta$$

$$J\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mgl_C\theta = 0$$

$$J\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+mgl_C\theta=0$$

Suppose

$$\omega^2=\frac{mgl_C}{J}$$

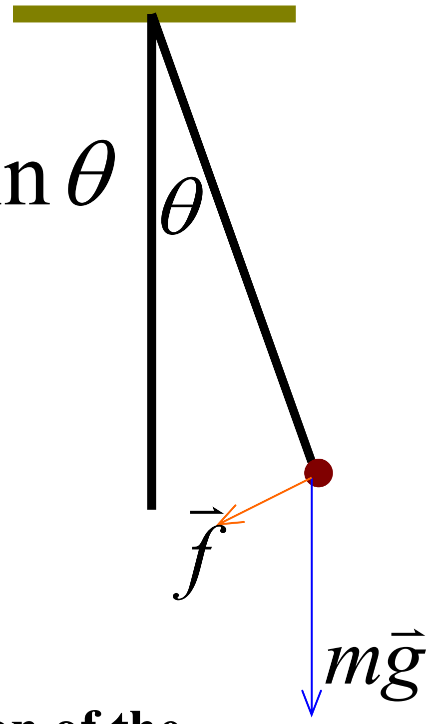
$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+\omega^2\theta=0$$

To solve $\theta=\theta_0\cos(\omega t+\varphi_0)$

It can be seen that the small Angle rotation of the fixed axis is simple harmonic vibration

Single pendulum

The law of rotation $ml\dfrac{d^2\theta}{dt^2} = -mg\sin\theta$



when $\sin\theta \approx \theta$ **then**

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

When the angular displacement is very small, the vibration of the single pendulum is the simple harmonic vibration angular frequency, and the period of the vibration is respectively:

$$\omega_0 = \sqrt{\frac{g}{l}} \qquad T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}$$

The angular frequency and period of vibration are completely composed of vibration system itself.

Summary: Description and characteristics of simple harmonic motion

（1） **Object subjected to linear restoring force** $\boxed{F=-kx}$

equilibrium position $\boxed{x=0}$

（2） **Kinetic description of simple harmonic motion**

$$\boxed{\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}=-\omega^2 x}$$

（3） **Kinematic description of simple harmonic motion**

$$\boxed{x = A\cos(\omega t+\varphi)} \qquad \boxed{v=-A\omega\sin(\omega t+\varphi)}$$

（4） **Acceleration is proportional to displacement and in the opposite direction**

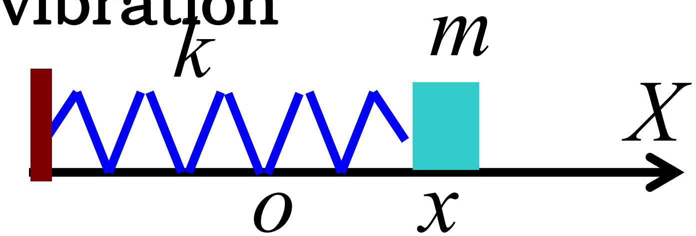
$$\boxed{a=-\omega^2 x}$$

spring oscillator $\omega = \sqrt{k/m}$

simple pendulum $\omega = \sqrt{g/l}$

compound pendulum $\omega = \sqrt{mgl/J}$

Energy of a simple harmonic vibration



The kinetic energy of the simple harmonic vibration:

Take the horizontal spring oscillator, for example

$$\omega_0 = \sqrt{k/m}$$

$$x(t) = A\cos(\omega_0 t + \varphi_0)$$

$$E_k = \frac{1}{2}mV^2 = \frac{1}{2}mA^2\omega_0^2 \sin^2(\omega_0 t + \varphi_0)$$

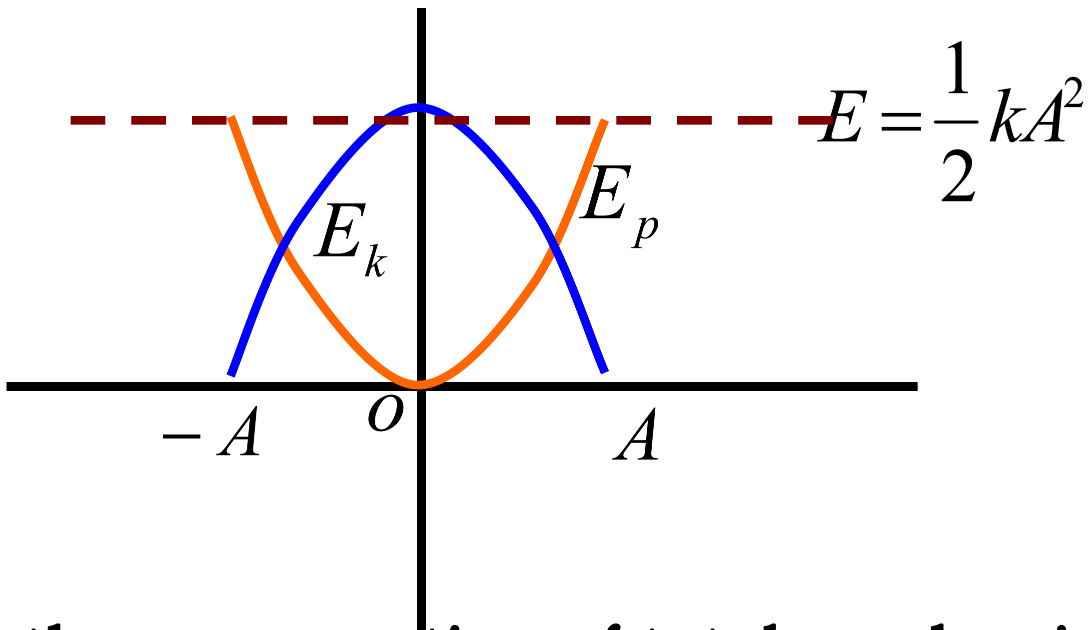
$$= \frac{1}{2}kA^2 \sin^2(\omega_0 t + \varphi_0)$$

Potential energy of simple harmonic vibration: $f = -kx = \frac{\mathrm{d}E_p}{\mathrm{d}x}$

$$E_p = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t + \varphi_0);$$

Total energy of the simple harmonic vibration

$$E = E_k + E_p$$
$$= \frac{1}{2}kA^2[\sin^2(\omega_0 t + \varphi_0) + \cos^2(\omega_0 t + \varphi_0)] = \frac{1}{2}kA^2$$



Elastic force is the conservation of total mechanical energy,
That is, the total energy does not change with time

Time average of the kinetic energy:

$$\overline{E_k} = \frac{1}{T}\int_0^T \frac{1}{2}kA^2 \sin^2(\omega_0 t + \varphi_0)dt$$

$$= \frac{kA^2}{2T\omega_0}\int_{\varphi_0}^{2\pi+\varphi_0} \sin^2 x \cdot dx = \frac{1}{4}kA^2$$

Time average of the potential energy:

$$\overline{E_P} = \frac{1}{T}\int_0^T \frac{1}{2}kA^2 \cos^2(\omega_0 t + \varphi_0)dt$$

$$= \frac{kA^2}{2T\omega_0}\int_{\varphi_0}^{2\pi+\varphi_0} \cos^2 x \cdot dx = \frac{1}{4}kA^2$$

conclusion:

* The average value of the kinetic energy and the potential energy of the spring oscillator are equal, both are half of the total mechanical energy

* The total energy of any simple harmonic vibration is proportional to the square of the amplitude

* Amplitude not only indicates the range of simple harmonic vibration motion, but also reflects the size of the total energy of the vibration system and the intensity of the vibration.

These conclusions apply equally to any simple harmonic vibration

deduction

conservation of energy $\Longrightarrow$ **Simple harmonic equations of motion**

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = 0$$

$$m\cancel{v}\frac{\mathrm{d}v}{\mathrm{d}t} + kx\cancel{\frac{\mathrm{d}x}{\mathrm{d}t}} = 0$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{k}{m}x = 0$$

For an object with mass $0.10\ \mathrm{kg}$, making simple harmonic motion with amplitude $1.0\times10^{-2}\ \mathrm{m}$, its maximum acceleration is: $4.0\ \mathrm{m\cdot s^{-2}}$

(1) Period of vibration;

(2) The kinetic energy passing through the equilibrium position;

(3) Total energy;

(4) Where does the object have equal kinetic energy and potential energy?

known

$$m = 0.10\ \text{kg},\quad A = 1.0\times 10^{-2}\ \text{m},$$

$$a_{\max} = 4.0\ \text{m}\cdot\text{s}^{-2}$$ **For: (1)** T; (2) $E_{\text{k,max}}$

Solution (1) $a_{\max} = A\omega^2 \quad \omega = \sqrt{\dfrac{a_{\max}}{A}} = 20\ \text{s}^{-1}$

$$T = \frac{2\pi}{\omega} = 0.314\ \text{s}$$

(2) $E_{\text{k,max}} = \dfrac{1}{2} m v_{\max}^2 = \dfrac{1}{2} m\omega^2 A^2$

$$= 2.0\times 10^{-3}\ \text{J}$$

known

$$m = 0.10\,\text{kg}，\ A = 1.0\times 10^{-2}\ \text{m}，$$

$a_{max} = 4.0\,\text{m}\cdot\text{s}^{-2}$ **For: (3)** E_{sum} ；

(4) The position when $E_K = E_P$

solution （3） $E_{sum} = E_{k,max} = 2.0\times 10^{-3}\ \text{J}$

（4） $E_k = E_p$ $\qquad E_p = 1.0\times 10^{-3}\ \text{J}$

$$E_p = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$$

$$x^2 = \frac{2E_p}{m\omega^2} = 0.5\times 10^{-4}\ \text{m}^2 \qquad x = \pm 0.707\ \text{cm}$$

Synthesis of simple harmonic vibrations

1. Synthesis of simple harmonic vibrations in the same direction and at the same frequency

conclusion:

Algebraic method: let two vibrations have the same frequency, move on the same straight line with different amplitudes and initial phases

$$x_1(t) = A_1 \cos(\omega t + \varphi_1) \qquad x_2(t) = A_2 \cos(\omega t + \varphi_2)$$

$$\begin{aligned} x(t) = x_1(t) + x_2(t) &= (A_1 \cos\varphi_1 + A_2 \cos\varphi_2)\cos\omega t \\ &\quad - (A_1 \sin\varphi_1 + A_2 \sin\varphi_2)\sin\omega t \end{aligned}$$

$$= A\cos\varphi \cdot \cos\omega t - A\sin\varphi \cdot \sin\omega t$$

$$= A\cos(\omega t + \varphi)$$

Amplitudes

Still a simple harmonic vibration of the same frequency

In formula:

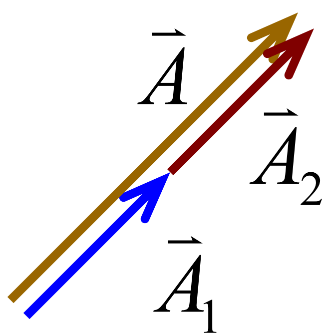
$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_2 - \varphi_1)}$$

$$\varphi = arctg\frac{A_1\sin\varphi_1 + A_2\sin\varphi_2}{A_1\cos\varphi_1 + A_2\cos\varphi_2}$$

Therefore, when

$$\varphi_2 - \varphi_1 = 2k\pi \qquad k = 0, \pm 1, \pm 2, \cdots$$

$$A = A_1 + A_2$$



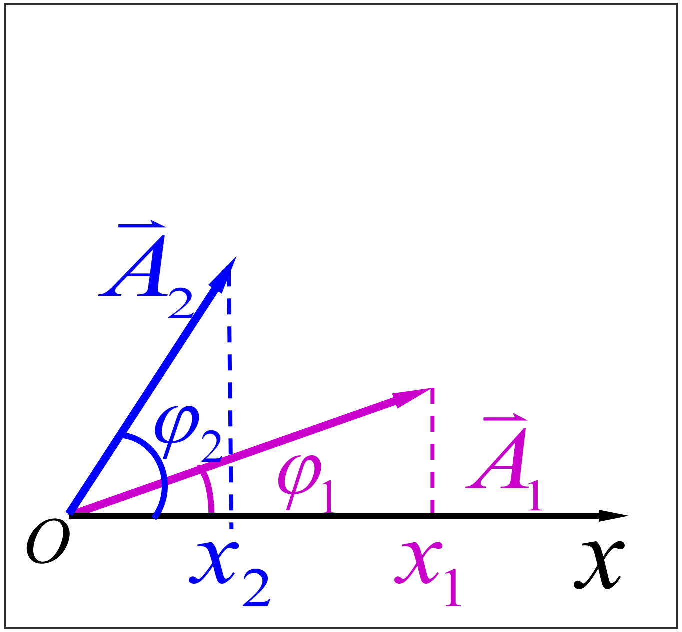
The maximum amplitude is obtained

Geometric method

Let a particle participate in two independent simple harmonic vibration of the same direction and the same frequency:

$$x_1 = A_1 \cos(\omega t + \varphi_1)$$

$$x_2 = A_2 \cos(\omega t + \varphi_2)$$

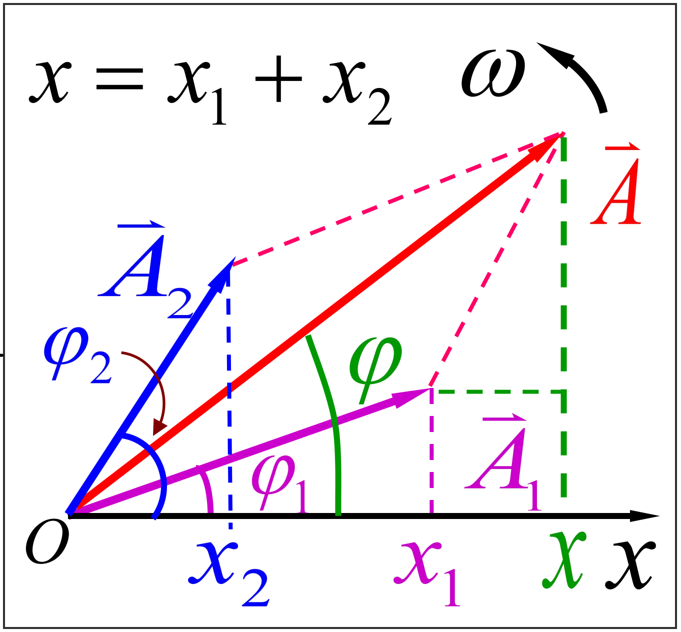


Phase difference between the two vibrations

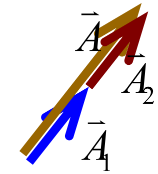
$$\Delta\varphi = \varphi_2 - \varphi_1 = \textbf{constant}$$

$$\boxed{x = A\cos(\omega t + \varphi)}$$

$$\begin{cases} A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_2 - \varphi_1)} \\ \tan\varphi = \dfrac{A_1\sin\varphi_1 + A_2\sin\varphi_2}{A_1\cos\varphi_1 + A_2\cos\varphi_2} \end{cases}$$

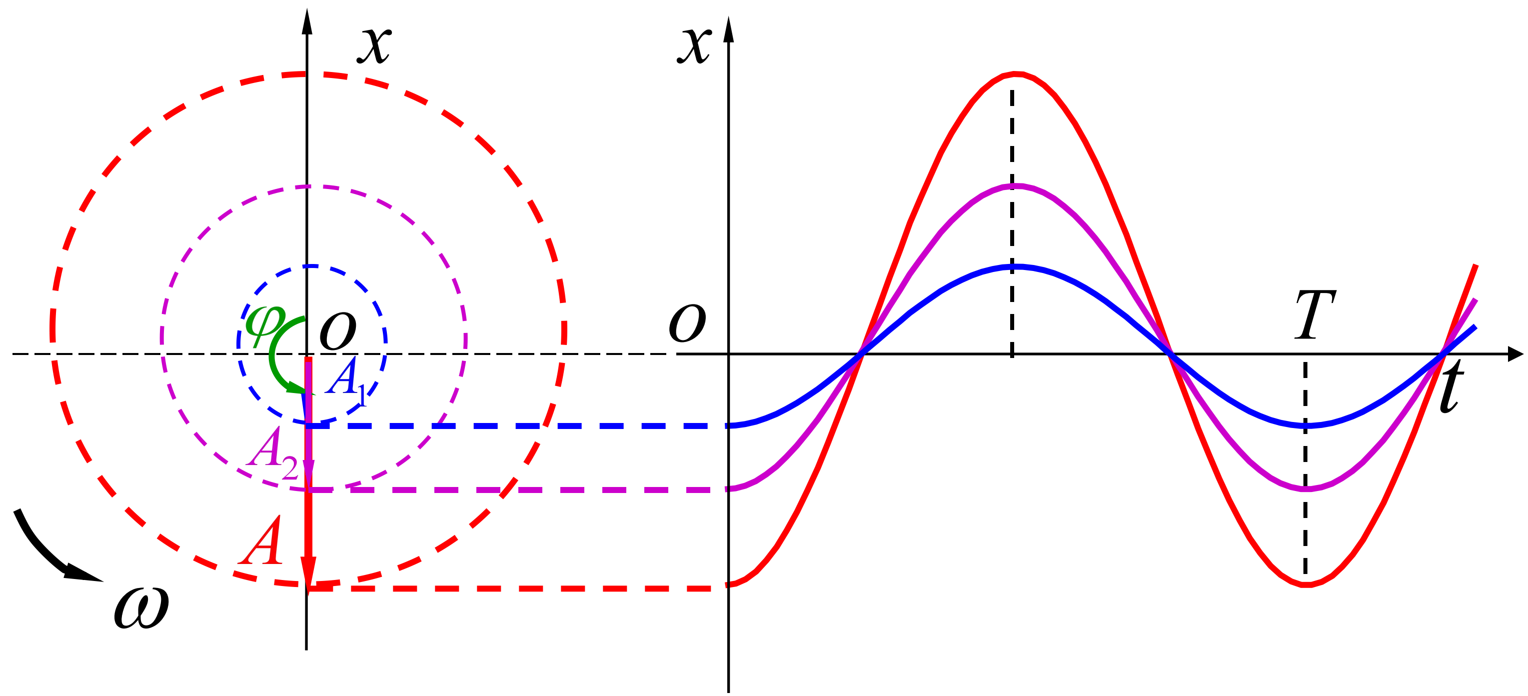


Two simple harmonic motions in the same direction and frequency are still simple harmonic motions of the same frequency after being synthesized.



(1) Phase difference

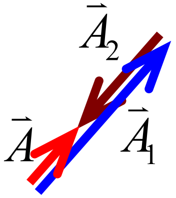
$$\Delta\varphi = \varphi_2 - \varphi_1 = 2k\pi \quad (k = 0, \pm 1, \pm 2, \cdots)$$



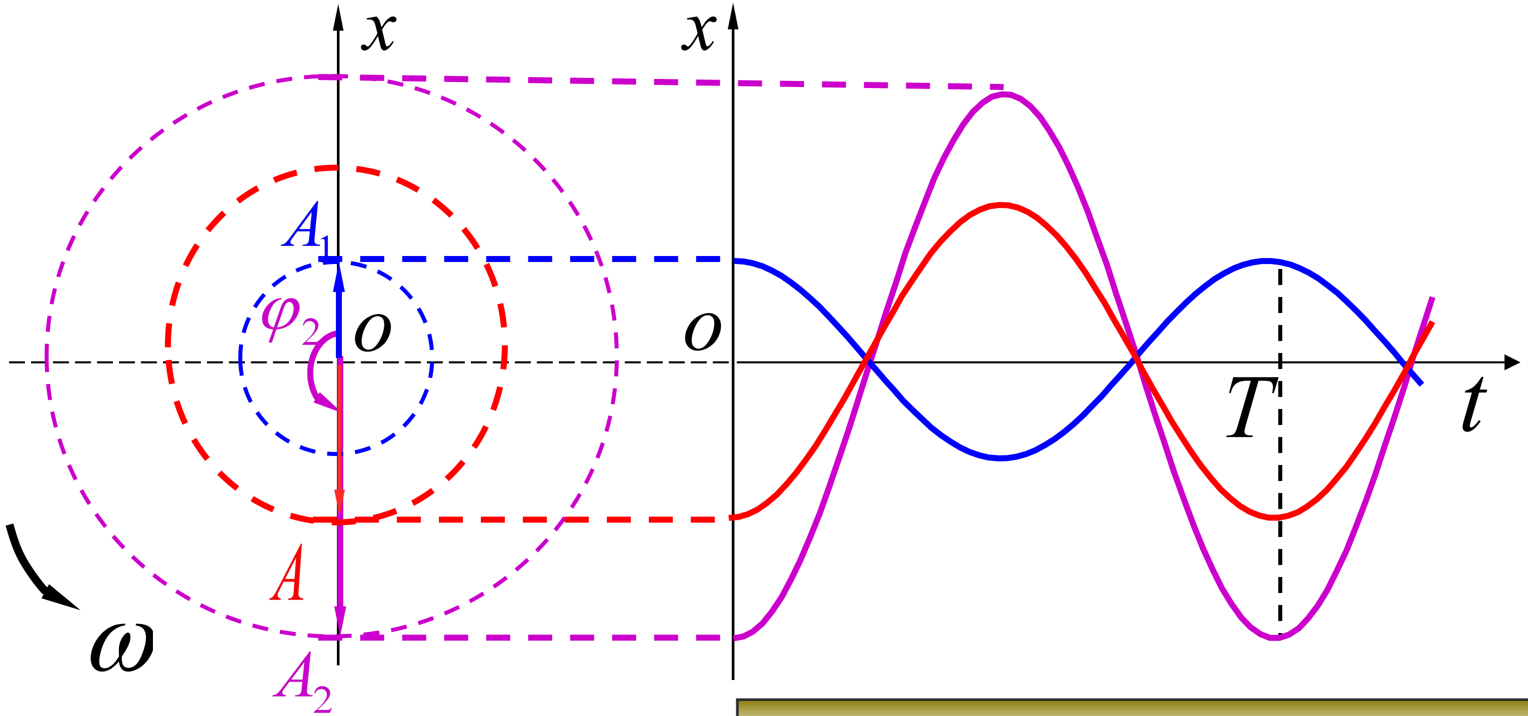
$$\boxed{x = (A_1 + A_2)\cos(\omega t + \varphi)}$$

$$\begin{cases} A = A_1 + A_2 \\ \varphi = \varphi_2 = \varphi_1 + 2k\pi \end{cases}$$

(2) Phase difference



$$\Delta\varphi = \varphi_2 - \varphi_1 = (2k+1)\pi\,(k = 0\,, \pm 1, \cdots)$$



$$\begin{cases} A = \left| A_1 - A_2 \right| \\ \varphi = \varphi_2 = \varphi_1 + (2k+1)\pi \end{cases}$$

$$\boxed{x = (A_2 - A_1)\cos(\omega t + \varphi)}$$

brief summary

(1) Phase difference $\varphi_2 - \varphi_1 = 2k\pi \quad (k = 0, \pm 1, \cdots)$

$$A = A_1 + A_2$$

strengthen

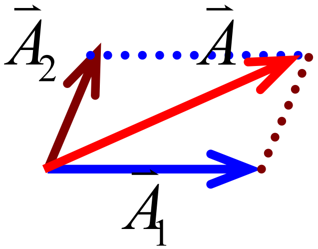
(2) Phase difference $\varphi_2 - \varphi_1 = (2k+1)\pi \quad (k = 0, \pm 1, \cdots)$

$$A = |A_1 - A_2|$$

attenuate

(3) General situation

$$A_1 + A_2 > A > |A_1 - A_2|$$



[Appendix] The synthesis of N simple harmonic vibrations of the same frequency in the same direction
(Vector synthesis method)

Let their amplitudes be equal, and the initial phase difference is one constant quantity
Its expression is:

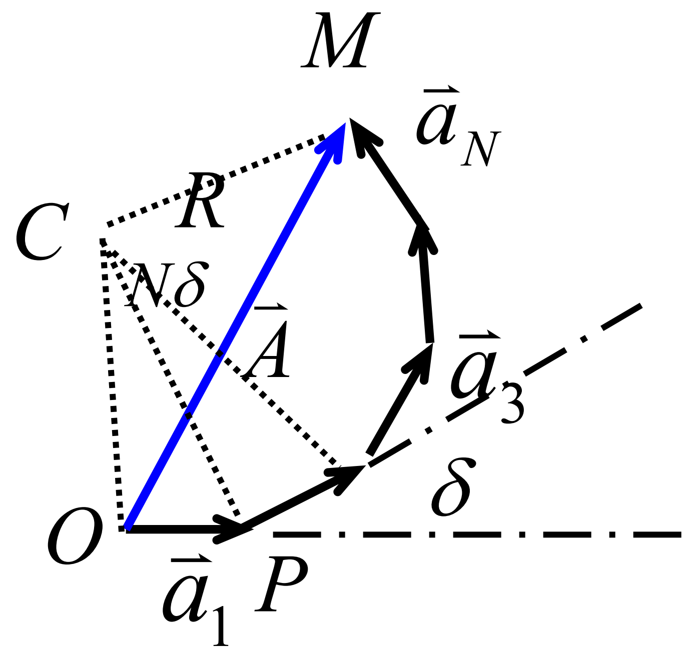
$$x_1(t) = a\cos\omega t$$

$$x_2(t) = a\cos(\omega t + \delta)$$

$$x_3(t) = a\cos(\omega t + 2\delta)$$

$$\vdots$$

$$x_N(t) = a\cos(\omega t + N\delta)$$



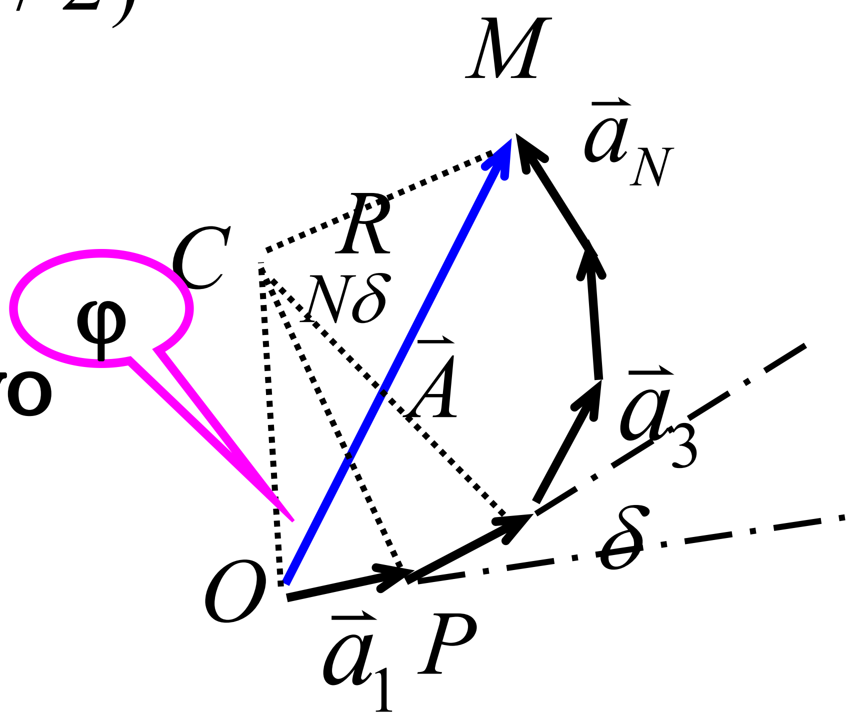
$$A = 2R\sin(N\delta/2)$$

In the ΔOCP:

$$a = 2R\sin(\delta/2)$$

The division of top two formula

$$A = a\frac{\sin(N\delta/2)}{\sin\delta/2}$$



$$\because \angle COM = (\pi - N\delta)/2$$

$$\because \angle COP = (\pi - \delta)/2$$

$$\varphi = \angle COP - \angle COM = \frac{N-1}{2}\delta$$

The expression for the combined vibration

$$x(t) = A\cos(\omega t + \varphi)$$

$$= a\frac{\sin(N\delta/2)}{\sin(\delta/2)}\cos(\omega t + \frac{N-1}{2}\delta)$$

Discussion 1:

equal

$$\delta = 2k\pi \qquad k = 0, \pm 1, \pm 2, \cdots$$

$$A = \lim a\frac{\sin(N\delta/2)}{\sin(\delta/2)} = Na$$

That is, when the vibrations are same phase, the amplitude of the vibration is the largest

Discussion 2:

$$x(t)=a\frac{\sin(N\delta/2)}{\sin(\delta/2)}\cos(\omega t+\frac{N-1}{2}\delta)$$

When $\delta=2k'\pi/N$ and $k'\neq kN$

$$A=a\frac{\sin(k'\pi)}{\sin(k'\pi/N)}=0$$

That is : $N\delta=2k\pi \qquad k=0,\pm1,\pm2,\cdots$

At this time, each vibration vector in turn, forming a closed positive regular polygon, the amplitude of the combined vibration is zero

The synthesis of the multiple partition vibrations discussed above illustrates the interference of the light and diffraction rules have important applications

2. Synthesis of simple harmonic vibrations in the same direction and at different frequencies

For simplicity, firstly discussing the condition: two identical amplitudes and initial phase are same, and the frequencies are different in the same direction

Synthesis of the vibrations.

The vibration expressions are respectively:

$$x_1(t) = A\cos(\omega_1 t + \varphi) \qquad x_2(t) = A\cos(\omega_2 t + \varphi)$$

Using the trigonometric function relation:

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}$$

Synthetic vibration expression:

$$x(t) = A\cos(\omega_1 t + \varphi) + A\cos(\omega_2 t + \varphi)$$

Appendix: Proof of the triangular function relation

$$\frac{4}{2}\cos\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}$$

$$=\frac{4}{2}(\cos\frac{\alpha}{2}\cdot\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\cdot\sin\frac{\beta}{2})(\cos\frac{\alpha}{2}\cdot\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\cdot\sin\frac{\beta}{2})$$

$$=\frac{4}{2}(\cos^2\frac{\alpha}{2}\cdot\cos^2\frac{\beta}{2}-\sin^2\frac{\alpha}{2}\cdot\sin^2\frac{\beta}{2})$$

$$=\frac{4}{2}[\frac{1}{2}(1+\cos\alpha)\cdot\frac{1}{2}(1+\cos\beta)-\frac{1}{2}(1-\cos\alpha)\cdot\frac{1}{2}(1-\cos\beta)]$$

$$=\cos\alpha+\cos\beta$$

Synthetic vibration expression:

$$x(t) = A\cos(\omega_1 t + \varphi) + A\cos(\omega_2 t + \varphi)$$

$$= 2A\cos\frac{(\omega_2 - \omega_1)t}{2} \cdot \cos[\frac{(\omega_2 + \omega_1)t}{2} + \varphi]$$

When ω_1 and ω_2 are very large and have very little difference,

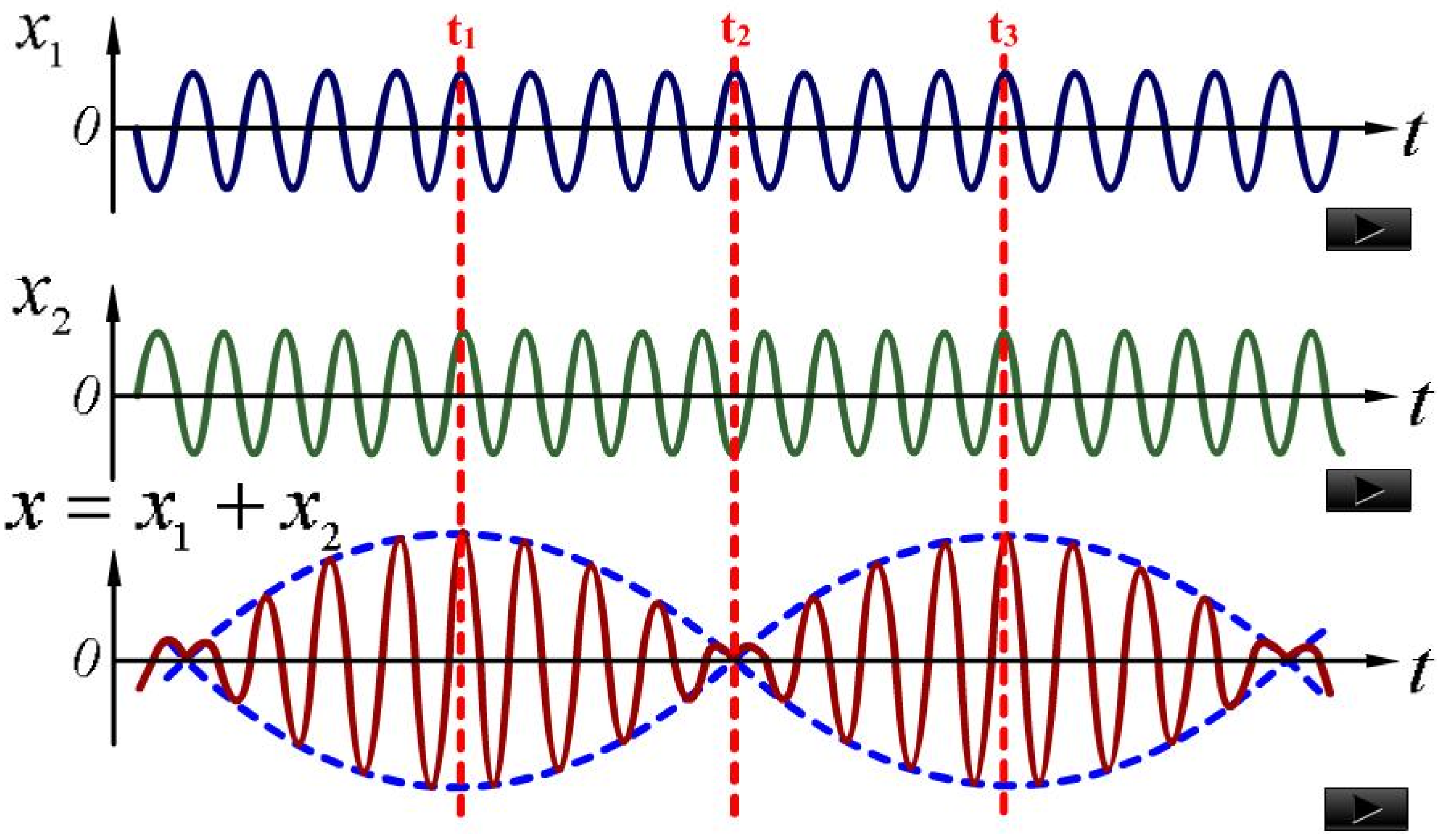
$|2A\cos(\omega_2 - \omega_1)t/2|$ **is considered as the amplitude change part,**

Synthetic vibration is a harmonic vibration of the angular frequency

$$(\omega_2 + \omega_1)/2$$

The period of its amplitude change is determined by the absolute value of the amplitude change, that is, the vibration is sometimes too strong and sometimes too weak, so it is an approximate harmonic vibration

The phenomenon of synthetic vibration fluctuating between strength and weakness is called beat.



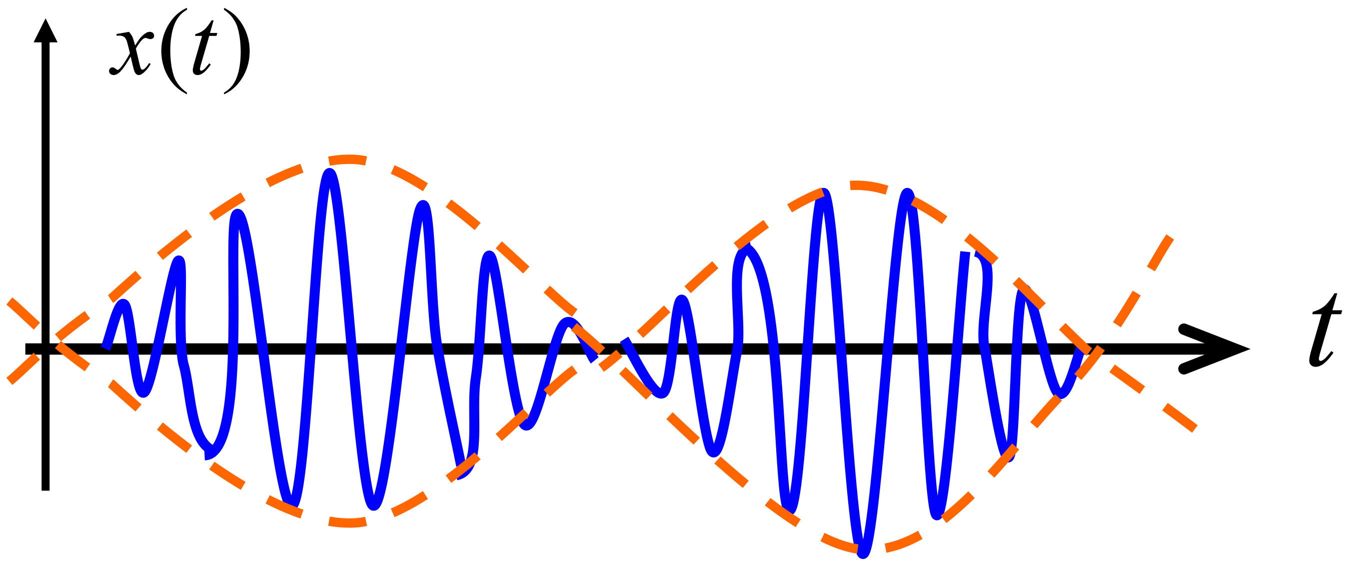
x_1
t_1
t_2
t_3
0
t
x_2
0
t
$x = x_1 + x_2$
0
t

The number of times the vibration strengthens or weakens per unit time is called the beat frequency

Obviously, the beat frequency is twice the frequency of the vibration

That is, the beat frequency is: $\cos(\frac{\omega_2-\omega_1}{2}t)$

$$\nu = 2\frac{1}{2\pi}(\frac{\omega_2-\omega_1}{2}) = \nu_2-\nu_1$$



3. Synthesis of simple harmonic vibration direction

Let a particle participate in the two vibration directions with each other vertical simple harmonic vibration of the same frequency, i. e

$$x = A_1 \cos(\omega t + \varphi_{10}); \qquad y = A_2 \cos(\omega t + \varphi_{20})$$

$$\frac{x}{A_1} = \cos \omega t \cdot \cos \varphi_{10} - \sin \omega t \cdot \sin \varphi_{10}$$

$$\frac{y}{A_2} = \cos \omega t \cdot \cos \varphi_{20} - \sin \omega t \cdot \sin \varphi_{20}$$

$$\frac{x}{A_1} \cos \varphi_{20} - \frac{y}{A_2} \cos \varphi_{10} = \sin \omega t \cdot \sin(\varphi_{20} - \varphi_{10})$$

$$\frac{x}{A_1}\cos\varphi_{20} - \frac{y}{A_2}\cos\varphi_{10} = \sin\omega t \cdot \sin(\varphi_{20} - \varphi_{10})$$

$$\frac{x}{A_1}\sin\varphi_{20} - \frac{y}{A_2}\sin\varphi_{10} = \cos\omega t \cdot \sin(\varphi_{20} - \varphi_{10})$$

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2}\cos\Delta\varphi = \sin^2\Delta\varphi$$

Elliptic equation

Specific shape is determined by the phase difference

$$\Delta\varphi = (\varphi_{20} - \varphi_{10})$$

The direction of the movement of the particle is related to $\Delta\varphi$

when $0 < \Delta\varphi < \pi$, particle moves clockwise; at that time,

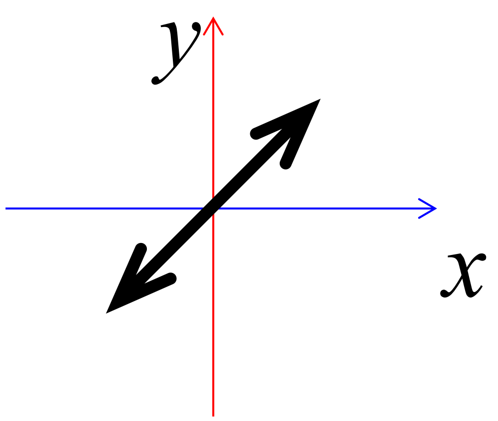
$\pi < \Delta\varphi < 2\pi$, particle moves in a counterclockwise direction

At $A_1 = A_2$, positive ellipses degenerated into circles

$$\frac{x^2}{A_1^2}+\frac{y^2}{A_2^2}-\frac{2xy}{A_1 A_2}\cos\Delta\varphi=\sin^2\Delta\varphi$$

Discussion 1 $(\varphi_{20}-\varphi_{10})=0$

$$\frac{x^2}{A_1^2}+\frac{y^2}{A_2^2}-\frac{2xy}{A_1 A_2}=0$$



In $y=\frac{A_2}{A_1}x$

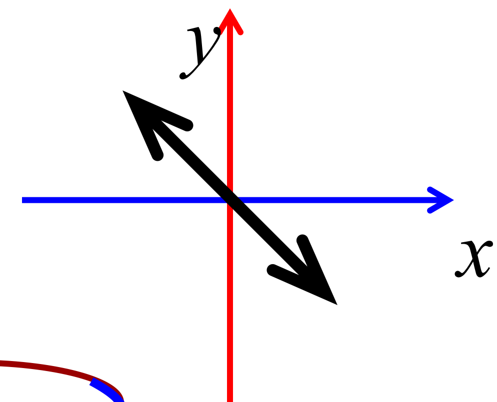
Movement in a straight line

Discussion 2

$$(\varphi_{20}-\varphi_{10})=\pi \qquad \frac{x^2}{A_1^2}+\frac{y^2}{A_2^2}+\frac{2xy}{A_1A_2}=0$$

So it's a vibration in a straight line.

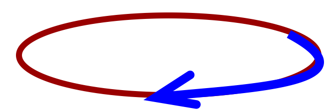
$$y=-\frac{A_2}{A_1}x$$



Discussion 3

$$(\varphi_{20}-\varphi_{10})=\frac{\pi}{2}$$

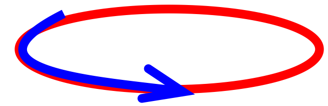
$$\frac{x^2}{A_1^2}+\frac{y^2}{A_2^2}=1$$



So it is an elliptic equation with semi-axis A1 on the X-axis and semi-axis A2 on the Y-axis, and rotated clockwise.

Discussion 4 $(\varphi_{20}-\varphi_{10})=\frac{3\pi}{2}$

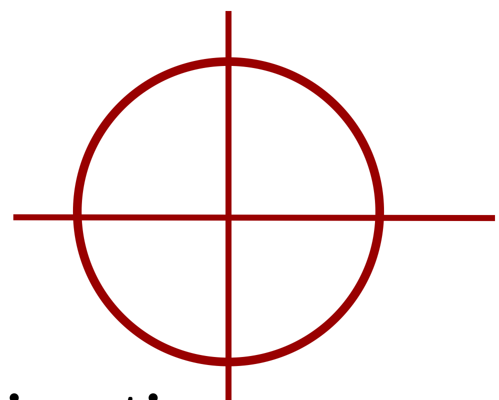
$$\frac{x^2}{A_1^2}+\frac{y^2}{A_2^2}=-1$$



So it is an elliptic equation with the length of A1 on the X-axis and A2 on the Y-axis, and rotated counterclockwise.

Discussion 5

$$A_1=A_2$$



The orbit of the particle is circular.
The phase difference in the X and Y directions determines the direction of rotation.

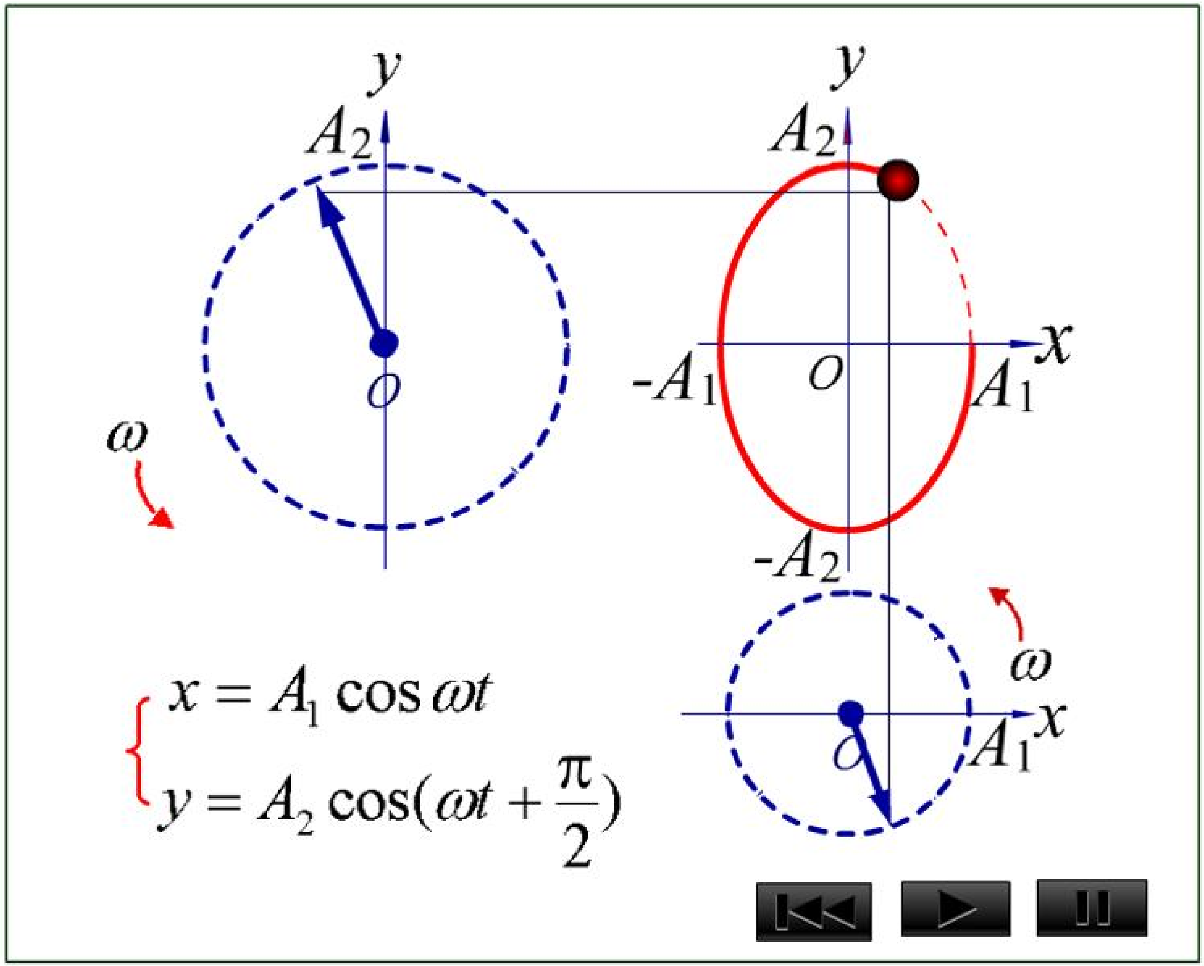
Discussion 6

$$\varphi_{20}-\varphi_{10}\neq\frac{2k+1}{2}\pi \qquad k=0,1,2,3\cdots\cdots$$

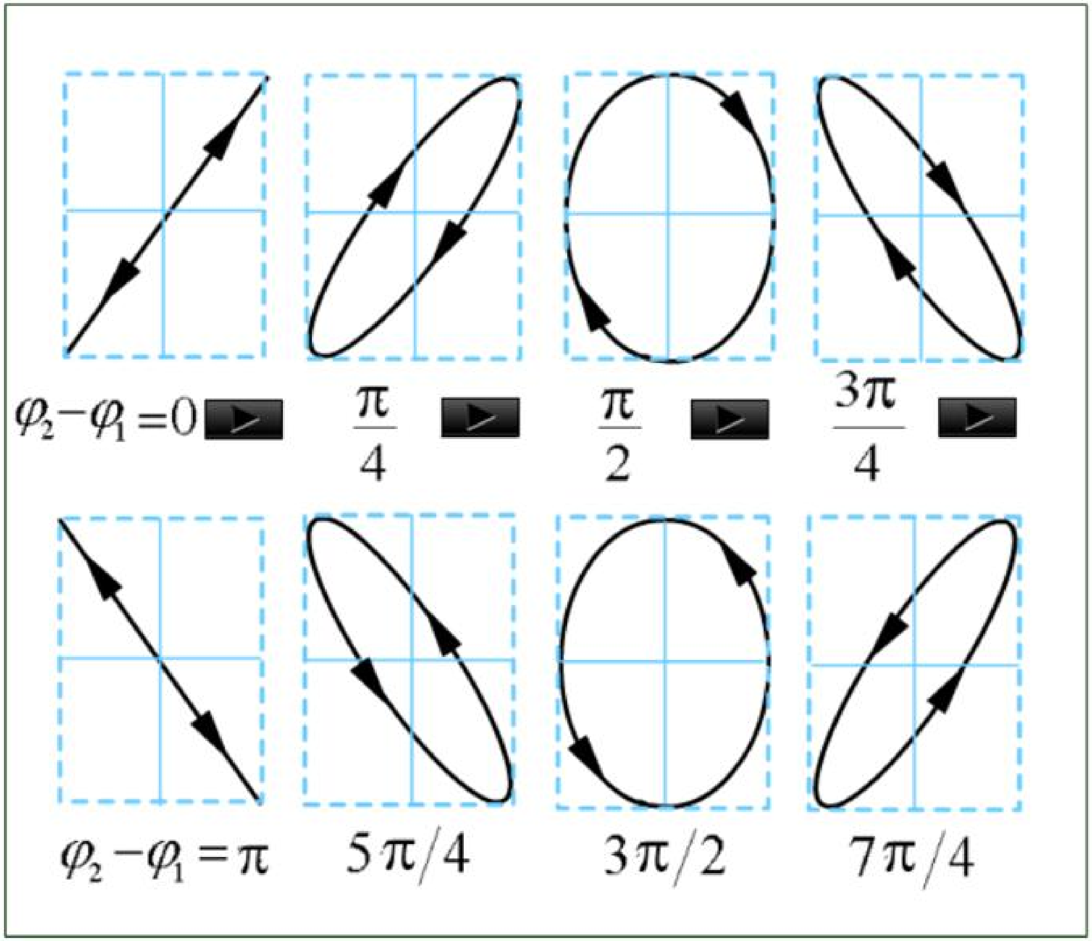
$$\varphi_{20}-\varphi_{10}\neq 2k\pi$$ Any arbitrary elliptic equation

In summary, after the synthesis of two simple harmonic vibration of the same frequency, the combined vibration is carried out in a straight line or on an ellipse (a straight line is a degraded ellipse). When the amplitude of two partial vibrations is equal, the elliptical orbit becomes a circle

A vibration synthesis diagram is depicted with a rotation vector



Synthetic diagram of simple harmonic motion of two mutual perpendicular same frequencies with different phase differences



4. Synthesis of simple harmonic vibrations with vertical vibration directions and different frequencies

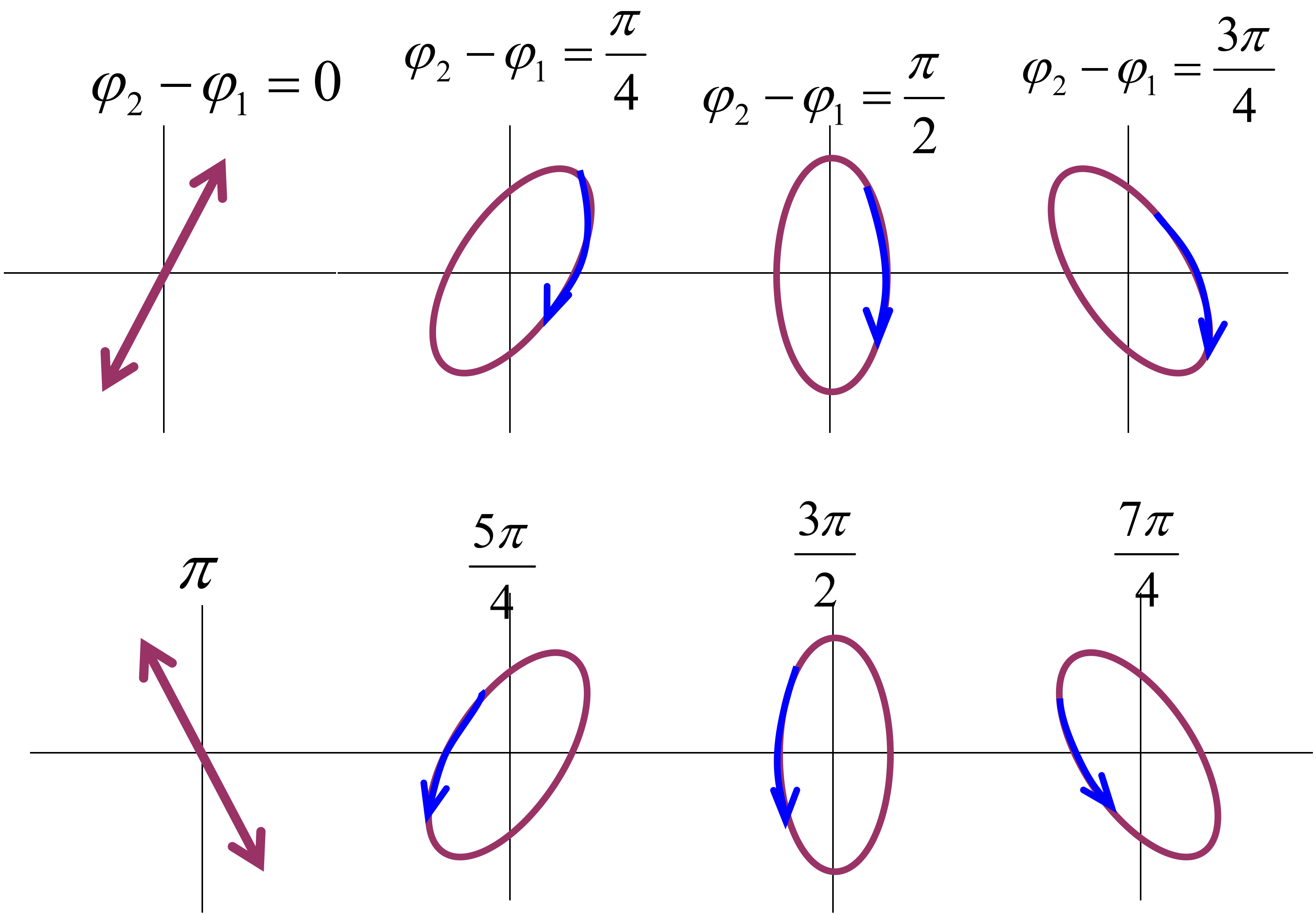
Generally, complex motion tracks are not closed curves, i. e., the synthetic movement is not periodic movement

Here are two cases discussed

$\nu_2 - \nu_1 \approx 0$ is regarded as the synthesis of the same frequency, but the phase difference between the two vibrations is changing slowly, so the orbit of the particle movement will constantly change from the sequence shown in the figure below

$0 < \varphi_2 - \varphi_1 < \pi$ clockwise

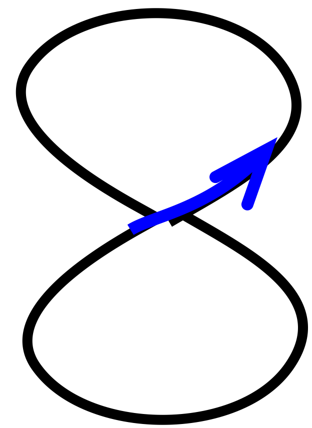
$\pi < \varphi_2 - \varphi_1 < 2\pi$ counterclockwise



$\varphi_2 - \varphi_1 = 0$
$\varphi_2 - \varphi_1 = \frac{\pi}{4}$
$\varphi_2 - \varphi_1 = \frac{\pi}{2}$
$\varphi_2 - \varphi_1 = \frac{3\pi}{4}$
π
$\frac{5\pi}{4}$
$\frac{3\pi}{2}$
$\frac{7\pi}{4}$

2. If the two vibration perpendicular to each other and the frequencies are in an integer ratio, the trajectory of the synthetic motion is a closed curve, and the motion also has the pattern of the period——motion trajectory is called the Lisa pattern

Frequency can be measured in RF technology:



$$T_x : T_y = 1 : 2$$

On the oscilloscope, two vibrations are input in the vertical direction and the horizontal direction. If one of the frequencies is known, the other unknown frequency can be obtained according to the known standard Lisa figure

2 Damped vibration

Free vibration without damping

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{K}}$$

Damped vibration of the harmonic oscillator

The vibration system is subject to the viscous resistance of the medium, which is proportional to the velocity size and is in the opposite direction

$$f_r = -\gamma v = -\gamma \frac{dx}{dt}$$

Elastic or quasi-elastic forces and the above

$$m\ddot{x} = -kx - \gamma\dot{x}$$

$$\omega_0^2 = \frac{k}{m};$$

$$m\ddot{x} = -kx - \gamma\dot{x}$$

$$\beta = \frac{\gamma}{2m}$$

ω_0 :the intrinsic angular frequency of the vibration system,

β : the damping coefficient

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = 0$$

(1) With small damping, $\beta^2 < \omega_0^2$ the solution of this equation:

$$x(t) = Ae^{-\beta\cdot t}\cos(\omega t + \varphi_0) \qquad \omega = \sqrt{\omega_0^2 - \beta^2}$$

The resistance increases the cycle

This situation is called underdamped

A and the initial phase φ_0 are determined by the initial conditions

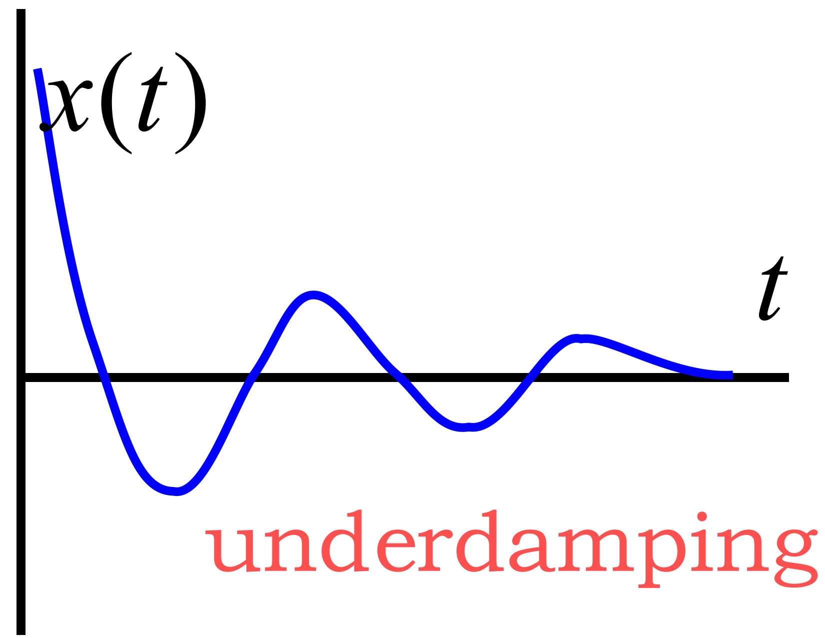
$$t = 0, \quad x(0) = x_0, \quad \left.\frac{dx}{dt}\right|_{t=0} = V_0$$

That is: $x_0 = A\cos\varphi_0$

$$V_0 = -A\omega\sin\varphi_0 - A\beta\cos\varphi_0$$

$$A = \sqrt{x_0^2 + \frac{(V_0 + \beta x_0)^2}{\omega^2}},$$

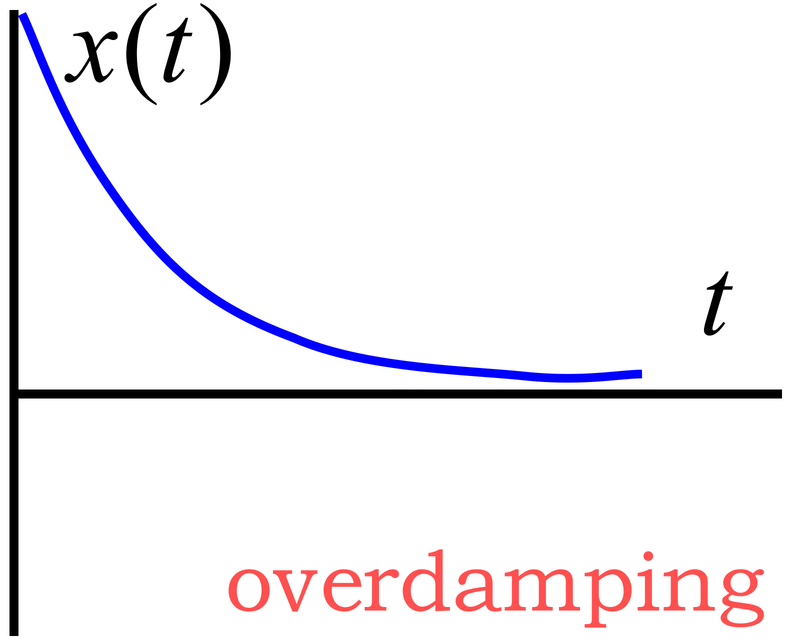
$$tg\varphi_0 = -\frac{V_0 - \beta x_0}{\omega x_0}$$



(2) Solution of the equation when the damping is large $\beta^2 > \omega_0^2$:

$$x(t) = C_1 e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t} + C_2 e^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t}$$

C_1, C_2 are the integral Constant, as given by the initial condition To determine, in this case Known as overdamping

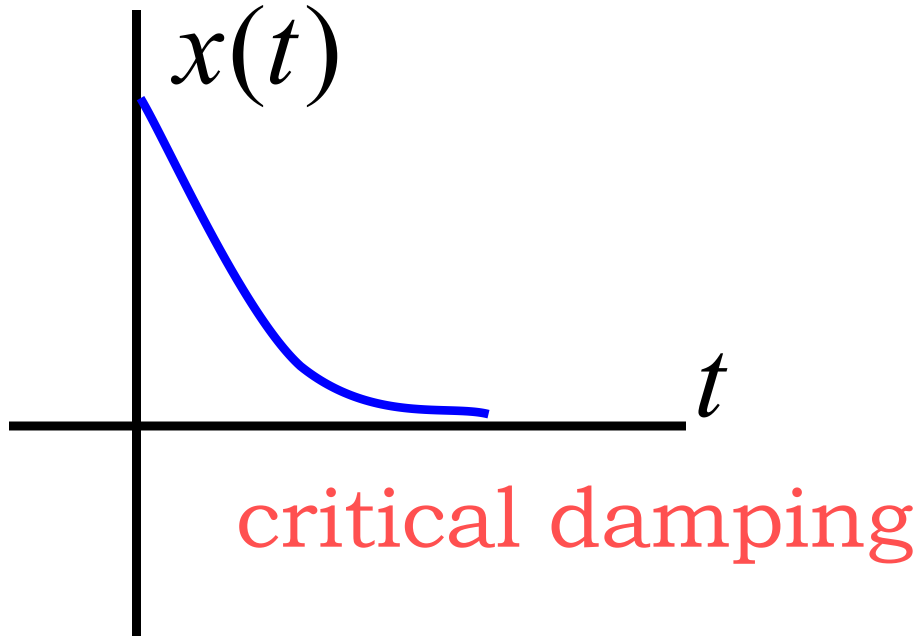


No vibration occurs

overdamping

(3) If the solution of the equation $\beta^2 = \omega_0^2$ is:

$$x(t) = (C_1 + C_2 t)e^{-\beta t}$$



C_1, C_2 are made by the initial condition
The integral constants that are determined
$\beta^2 = \omega_0^2$

Call it the critical damping case. It is the vibration system just cannot make the periodic vibration, and soon back to the equilibrium position, applied in the balance adjustment

Is a critical point from a periodic factor to no periodicity $\omega = \sqrt{\omega_0^2 - \beta^2}$

3 Forced vibration and resonance

The forced vibration of the harmonic oscillator

Set force $$f = H\cos pt$$

damping force: $$f_r = -\gamma v = -\gamma \cdot \dot{x}$$

$$令\ \omega_0^2 = \frac{k}{m};\ \beta = \frac{\gamma}{2m};\quad h = \frac{H}{m}$$

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = h\cos pt$$

Is the typical constant coefficient, second order, linear, inhomogeneous differential equations

By the differential equation theory:

General solutions of inhomogeneous differential equations = Solution of homogeneous differential equations + a special solution of inhomogeneous

$\beta^2 < \omega_0^2$ **Its solution is:**

$$x(t) = Ae^{-\beta \cdot t}\cos(\sqrt{\omega_0^2 - \beta^2}\,t + \varphi_0) + A_p\cos(pt + \phi_0)$$

After a long enough time, called the stationary solution:

$$x(t) = A_p \cos(pt + \phi_0)$$

The angular frequency of the equal vibration is the frequency of the forcing force;

The amplitude and the phase difference between the steady state and the forced force are respectively:

$$A_p = \frac{h}{\sqrt{(\omega_0^2 - p^2)^2 + 4\beta^2 p^2}}$$

$$\phi_0 = arctg\,\frac{-2\beta p}{\omega_0^2 - p^2}$$

Discuss:

$$p \gg \omega_0, \qquad A_p = \frac{H/m}{p^2} = \frac{h}{p^2} \quad \text{less}$$

$$p \ll \omega_0, \qquad A_p = \frac{H/m}{\omega_0^2} = \frac{H}{k}$$

$$p = \omega_0, \qquad A_p = \frac{H/m}{2\beta\omega_0}$$

If β very small

A_p, very large.

- resonate

Solving for the extreme value of amplitude $A_p = \dfrac{h}{\sqrt{(\omega_0^2 - p^2)^2 + 4\beta^2 p^2}}$ versus frequency

Amplitude has a maximum value:

$$A_r = \frac{h}{2\beta\sqrt{\omega_0^2 - \beta^2}}$$

The amplitude of the resonance.

$$p_r = \sqrt{\omega_0^2 - 2\beta^2}$$

The angular frequency of the resonance.

$$p_r = \sqrt{\omega_0^2 - 2\beta^2}$$ The angular frequency of the resonance.

substitution $$\phi_0 = arctg \frac{-2\beta p}{\omega_0^2 - p^2}$$

The initial phase at the resonance $$\phi_{0r} = arctg \frac{-\sqrt{\omega_0^2 - 2\beta}}{\beta}$$

When the frequency of the forced force is a certain value
The phenomenon where the displacement amplitude appears maximum, called the shift resonance,
Resonance for short ***(resonance).***

When $\beta \rightarrow 0$ **weakly damped**

$$\therefore p_r = \omega_0, \qquad A_r \longrightarrow \infty, \qquad \phi_{0r} = -\pi/2$$

Resonances occur at the natural frequency, called sharp resonances.

The cause of the sharp increase in the vibration amplitude

The forced vibration phase $\pi/2$ lags behind the forced force phase, namely, the vibration velocity

In the same phase with the forced force, namely, the external force always does positive work to the system, right

The velocity increases has the maximum efficiency vibration amplitude

As the amplitude increases, the power of the resistance also increases, and finally it is offset with the power of the forced force, thus keeping the amplitude constant. From the energy point of view, in the resonance, this energy changes into the energy of the resonance particle, also called resonance absorption