Section 1.4

12. **a)** Since $0 + 1 > 2 \cdot 0$, we know that $Q(0)$ is true.
b) Since $(-1) + 1 > 2 \cdot (-1)$, we know that $Q(-1)$ is true.
c) Since $1 + 1 \not> 2 \cdot 1$, we know that $Q(1)$ is false.
d) From part **(a)** we know that there is at least one x that makes $Q(x)$ true, so $\exists x\, Q(x)$ is true.
e) From part **(c)** we know that there is at least one x that makes $Q(x)$ false, so $\forall x\, Q(x)$ is false.
f) From part **(c)** we know that there is at least one x that makes $Q(x)$ false, so $\exists x\, \neg Q(x)$ is true.
g) From part **(a)** we know that there is at least one x that makes $Q(x)$ true, so $\forall x\, \neg Q(x)$ is false.

31. In each case we just have to list all the possibilities, joining them with $\vee$ if the quantifier is $\exists$, and joining them with $\wedge$ if the quantifier is $\forall$.
a) $Q(0,0,0) \wedge Q(0,1,0)$ **b)** $Q(0,1,1) \vee Q(1,1,1) \vee Q(2,1,1)$
c) $\neg Q(0,0,0) \vee \neg Q(0,0,1)$ **d)** $\neg Q(0,0,1) \vee \neg Q(1,0,1) \vee \neg Q(2,0,1)$

35. **a)** As we saw in Example 13, this is true, so there is no counterexample.
b) Since 0 is neither greater than nor less than 0, this is a counterexample.
c) This proposition says that 1 is the only integer—that every integer equals 1. If is obviously false, and any other integer, such as -111749, provides a counterexample.

43. A conditional statement is true if the hypothesis is false. Thus it is very easy for the second of these propositions to be true—just have $P(x)$ be something that is not always true, such as "The integer x is a multiple of 2." On the other hand, it is certainly not always true that if a number is a multiple of 2, then it is also a multiple of 4, so if we let $Q(x)$ be "The integer x is a multiple of 4," then $\forall x(P(x) \to Q(x))$ will be false. Thus these two propositions can have different truth values. Of course, for some choices of P and Q, they will have the same truth values, such as when P and Q are true all the time.

45. Both are true precisely when at least one of $P(x)$ and $Q(x)$ is true for at least one value of x in the domain (universe of discourse).

Section 1.5

28. **a)** true (let $y = x^2$) **b)** false (no such y exists if x is negative) **c)** true (let $x = 0$)
d) false (the commutative law for addition always holds) **e)** true (let $y = 1/x$)
f) false (the reciprocal of y depends on y—there is not one x that works for all y) **g)** true (let $y = 1 - x$)
h) false (this system of equations is inconsistent)
i) false (this system has only one solution; if $x = 0$, for example, then no y satisfies $y = 2 \wedge -y = 1$)
j) true (let $z = (x + y)/2$)

31. As we push the negation symbol toward the inside, each quantifier it passes must change its type. For logical connectives we either use De Morgan's laws or recall that $\neg(p \to q) \equiv p \wedge \neg q$.

a)
$$\begin{aligned}\neg\forall x\exists y\forall z\, T(x,y,z) &\equiv \exists x\neg\exists y\forall z\, T(x,y,z)\\ &\equiv \exists x\forall y\neg\forall z\, T(x,y,z)\\ &\equiv \exists x\forall y\exists z\, \neg T(x,y,z)\end{aligned}$$

b)
$$\begin{aligned}\neg(\forall x\exists y\, P(x,y) \vee \forall x\exists y\, Q(x,y)) &\equiv \neg\forall x\exists y\, P(x,y) \wedge \neg\forall x\exists y\, Q(x,y)\\ &\equiv \exists x\neg\exists y\, P(x,y) \wedge \exists x\neg\exists y\, Q(x,y)\\ &\equiv \exists x\forall y\, \neg P(x,y) \wedge \exists x\forall y\, \neg Q(x,y)\end{aligned}$$

c)

$$\neg\forall x\exists y(P(x,y) \wedge \exists z\, R(x,y,z)) \equiv \exists x\neg\exists y(P(x,y) \wedge \exists z\, R(x,y,z))$$
$$\equiv \exists x\forall y\, \neg(P(x,y) \wedge \exists z\, R(x,y,z))$$
$$\equiv \exists x\forall y(\neg P(x,y) \vee \neg\exists z\, R(x,y,z))$$
$$\equiv \exists x\forall y(\neg P(x,y) \vee \forall z\, \neg R(x,y,z))$$

d)

$$\neg\forall x\exists y(P(x,y) \to Q(x,y)) \equiv \exists x\, \neg\exists y(P(x,y) \to Q(x,y))$$
$$\equiv \exists x\forall y\, \neg(P(x,y) \to Q(x,y))$$
$$\equiv \exists x\forall y(P(x,y) \wedge \neg Q(x,y))$$

39. a) Since the square of a number and its additive inverse are the same, we have many counterexamples, such as $x = 2$ and $y = -2$.
b) This statement is saying that every number has a square root. If x is negative (like $x = -4$), or, since we are working in the domain of the integers, x is not a perfect square (like $x = 6$), then the equation $y^2 = x$ has no solution.
c) Since negative numbers are not larger than positive numbers, we can take something like $x = 17$ and $y = -1$ for our counterexample.

Section 1.6

12. Applying Exercise 11, we want to show that the conclusion r follows from the five premises $(p \wedge t) \to (r \vee s)$, $q \to (u \wedge t)$, $u \to p$, $\neg s$, and q. From q and $q \to (u \wedge t)$ we get $u \wedge t$ by modus ponens. From there we get both u and t by simplification (and the commutative law). From u and $u \to p$ we get p by modus ponens. From p and t we get $p \wedge t$ by conjunction. From that and $(p \wedge t) \to (r \vee s)$ we get $r \vee s$ by modus ponens. From that and $\neg s$ we finally get r by disjunctive syllogism.

19. a) This is the fallacy of affirming the conclusion, since it has the form "$p \to q$ and q implies p."
b) This reasoning is valid; it is modus tollens.
c) This is the fallacy of denying the hypothesis, since it has the form "$p \to q$ and $\neg p$ implies $\neg q$."

20. a) This is invalid. It is the fallacy of affirming the conclusion. Letting $a = -2$ provides a counterexample.
b) This is valid; it is modus ponens.

29. We can set this up in two-column format. The proof is rather long but straightforward if we go one step at a time.

Step	**Reason**
1. $\exists x \neg P(x)$	Premise
2. $\neg P(c)$	Existential instantiation using (1)
3. $\forall x(P(x) \vee Q(x))$	Premise
4. $P(c) \vee Q(c)$	Universal instantiation using (3)
5. $Q(c)$	Disjunctive syllogism using (4) and (2)
6. $\forall x(\neg Q(x) \vee S(x))$	Premise
7. $\neg Q(c) \vee S(c)$	Universal instantiation using (6)
8. $S(c)$	Disjunctive syllogism using (5) and (7), since $\neg\neg Q(c) \equiv Q(c)$
9. $\forall x(R(x) \to \neg S(x))$	Premise
10. $R(c) \to \neg S(c)$	Universal instantiation using (9)
11. $\neg R(c)$	Modus tollens using (8) and (10), since $\neg\neg S(c) \equiv S(c)$
12. $\exists x \neg R(x)$	Existential generalization using (11)

Section 1.7

11. To disprove this proposition it is enough to find a counterexample, since the proposition has an implied universal quantification. We know from Example 10 that $\sqrt{2}$ is irrational. If we take the product of the irrational number $\sqrt{2}$ and the irrational number $\sqrt{2}$, then we obtain the rational number 2. This counterexample refutes the proposition.

27. We must prove two conditional statements. First, we assume that n is odd and show that $5n+6$ is odd (this is a direct proof). By assumption, $n = 2k+1$ for some integer k. Then $5n+6 = 5(2k+1)+6 = 10k+11 = 2(5k+5)+1$. Since we have written $5n+6$ as 2 times an integer plus 1, we have showed that $5n+6$ is odd, as desired. Now we give an proof by contraposition of the converse. Suppose that n is not odd—in other words, that n is even. Then $n = 2k$ for some integer k. Then $5n+6 = 10k+6 = 2(5k+3)$. Since we have written $5n+6$ as 2 times an integer, we have showed that $5n+6$ is even. This completes the proof by contraposition of this conditional statement.

42. We show that each of these is equivalent to the statement (v) n is odd, say $n = 2k+1$. Example 1 showed that (v) implies (i), and Example 8 showed that (i) implies (v). For $(v) \to (ii)$ we see that $1-n = 1-(2k+1) = 2(-k)$ is even. Conversely, if n were even, say $n = 2m$, then we would have $1-n = 1-2m = 2(-m)+1$, so $1-n$ would be odd, and this completes the proof by contraposition that $(ii) \to (v)$. For $(v) \to (iii)$, we see that $n^3 = (2k+1)^3 = 8k^3+12k^2+6k+1 = 2(4k^3+6k^2+3k)+1$ is odd. Conversely, if n were even, say $n = 2m$, then we would have $n^3 = 2(4m^3)$, so n^3 would be even, and this completes the proof by contraposition that $(iii) \to (v)$. Finally, for $(v) \to (iv)$, we see that $n^2+1 = (2k+1)^2+1 = 4k^2+4k+2 = 2(2k^2+2k+1)$ is even. Conversely, if n were even, say $n = 2m$, then we would have $n^2+1 = 2(2m^2)+1$, so n^2+1 would be odd, and this completes the proof by contraposition that $(iv) \to (v)$.