STAT 1151
Introduction to Probability

Lecture 3 Probability; Random Variables and Probability Distributions

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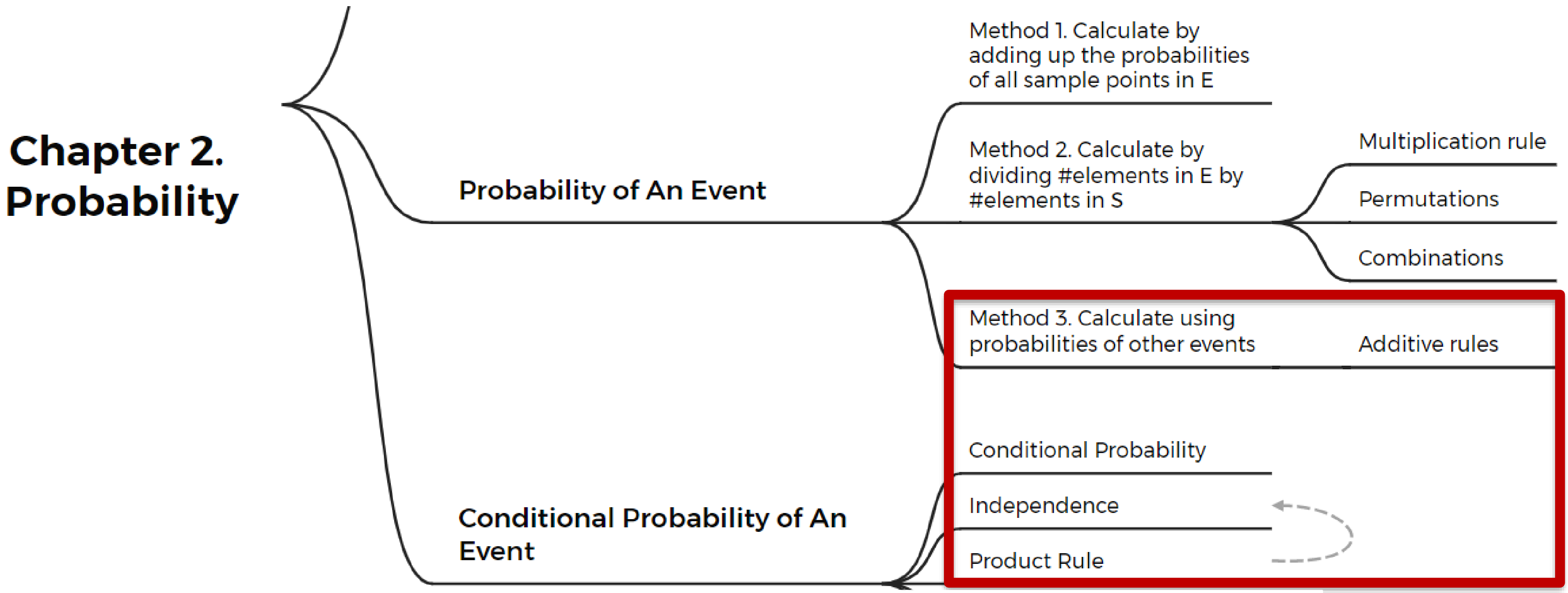
Last Lecture



Outline

- Chapter 2 Probability
  - Total Probability
  - Bayes' Rule
- Chapter 3 Random Variables and Probability Distributions
  - Discrete/Continuous Random Variables
  - Discrete/Continuous Probability Distributions

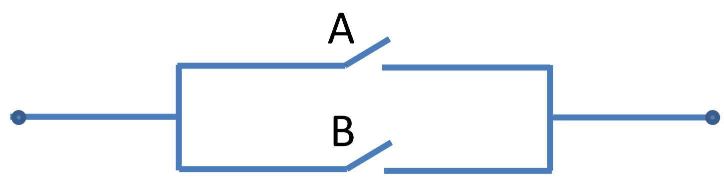
Product Rule & Independence

Example:
Switches in electrical circuits are often assumed to work (or fail) independently of each other. These switches may be set up as follows:

In series:

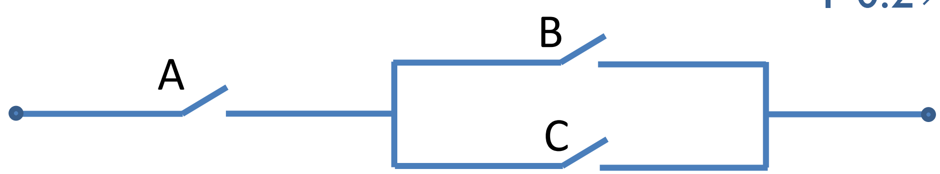
$P(AB)=P(A)P(B)$
$=0.8\times 0.8 = 0.64$

In parallel:



$P(A\cup B)=1-P(A')P(B')$
$=1-0.2\times 0.2 = 0.96$

In combination:

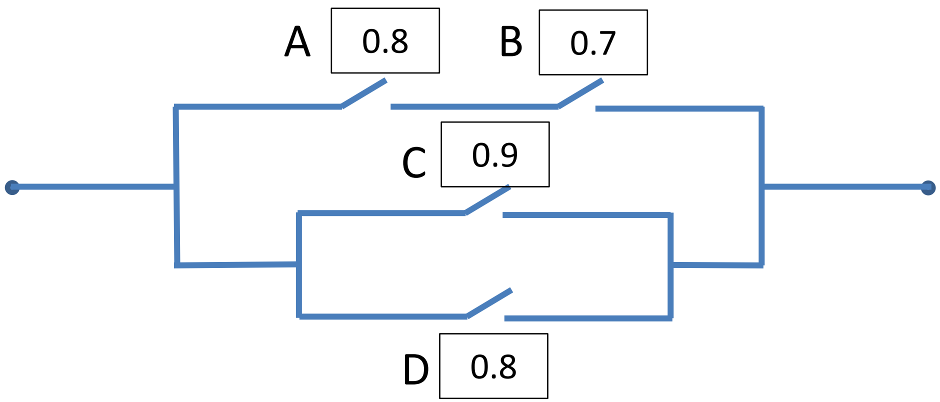


$P[A\cap(B\cup C)]=P(A)[1-P(B')P(C')]=0.8\times(1-0.2\times 0.2) = 0.768$

When a switch is flipped, it will close with a probability 0.8. Suppose that all the switches are in open status, find the probability that the current will flow through when all the switches are flipped.

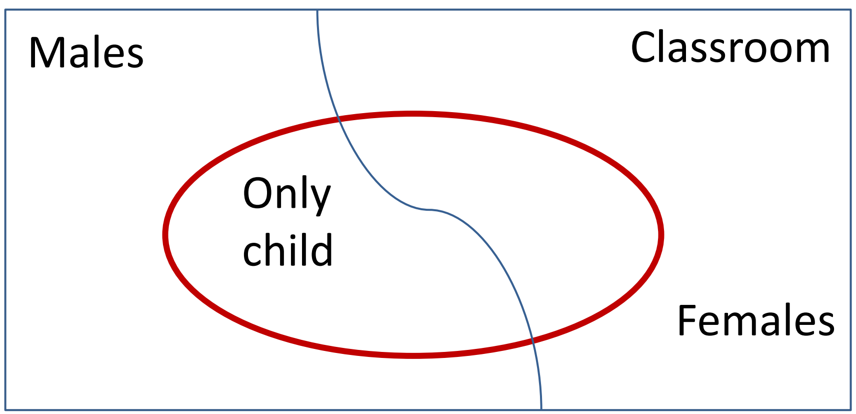
Product Rule & Independence

In-class exercise:
An electrical circuit is displayed below. The switches operate independently of each other, and the probability that each switch closes when it is flipped is displayed in the figure. What is the probability that current will flow through when the switches are flipped?



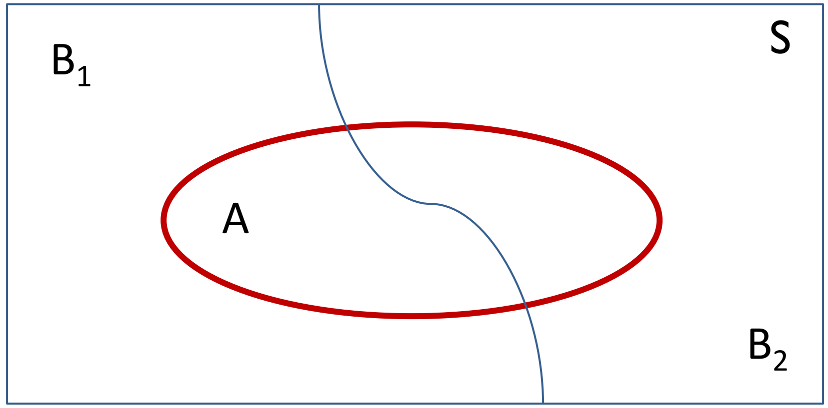
Total Probability

Example:
In our classroom, there are XX female and YY male students. Z1 females and Z2 males have no siblings. Now a student is being selected at random for giving a self-introduction. Find the probability of the event that the student selected is the only child.



Total Probability

Definition:



$B_1 + B_2 = S$

$$A = (AB_1) \cup (AB_2)$$

where AB_1 and AB_2 are mutually exclusive; therefore we can find

$$P(A) = P(AB_1) + P(AB_2)$$

If conditional probabilities $P(A|B_1)$ and $P(A|B_2)$ are known, then

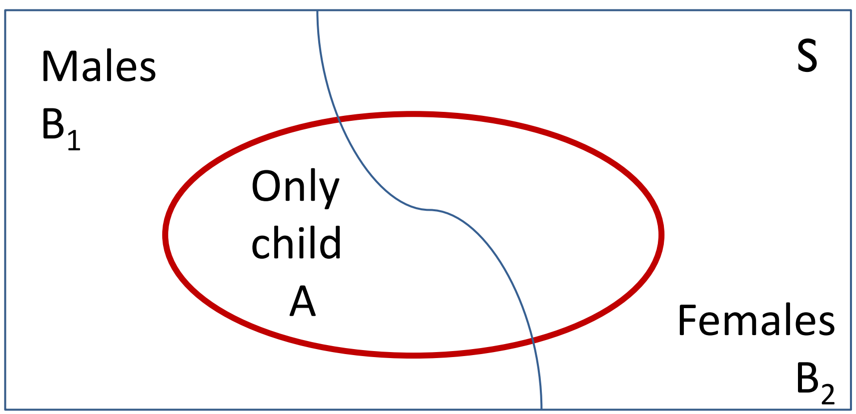
$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2)$$

This is known as the theorem of **total probability**.

Total Probability

Example:

In our classroom, there are males and females who are the only child, and males and females who are not the only child. Now a student is being selected at random for giving a self-introduction. Find the probability of the event A that the student selected is the only child.

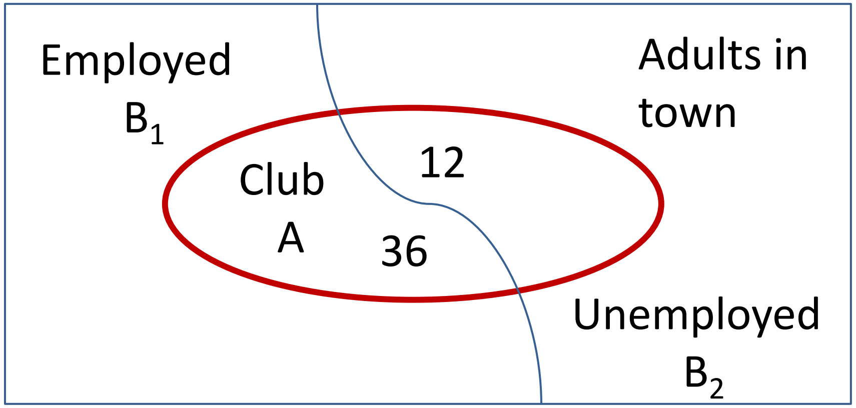


Respond at PollEv.com/xtan166

Total Probability

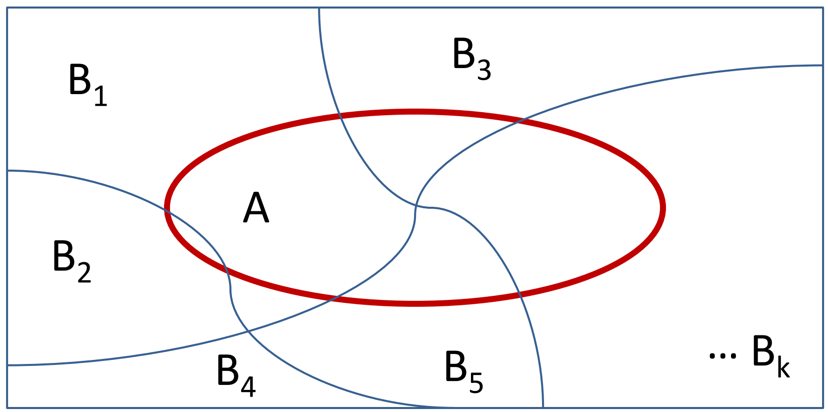
Example:
A resident is being selected at random from the adults of a small town for a tour throughout the country to publicize the advantages of establishing new industries in the town. Suppose that we are now given the additional information that 36 of those employed and 12 of those unemployed are members of the Rotary Club. Find the probability of the event A that the individual selected is a member of the Rotary Club.

	Employed	Unemployed	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900



Total Probability

Generalization:



$B_1 + B_2 + \ldots + B_k = S$

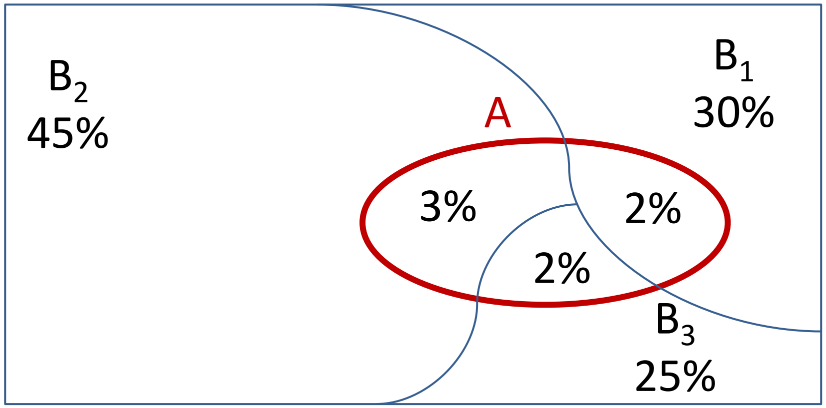
If events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for i = 1, 2,…, k, then

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i)$$

Total Probability

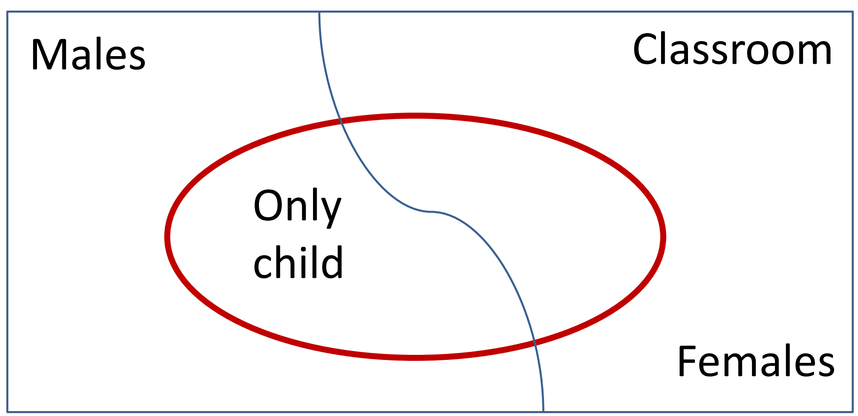
Example:
In a certain assembly plant, three machines, B1, B2, and B3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Let A be the event that the selected product is defective.



Bayes' Rule

Example:
In our classroom, there are XX female and YY male students. Z1 females and Z2 males have no siblings. Now a student is being selected at random for giving a self-introduction. Suppose the student selected is the only child. What is the probability that the student is a male?



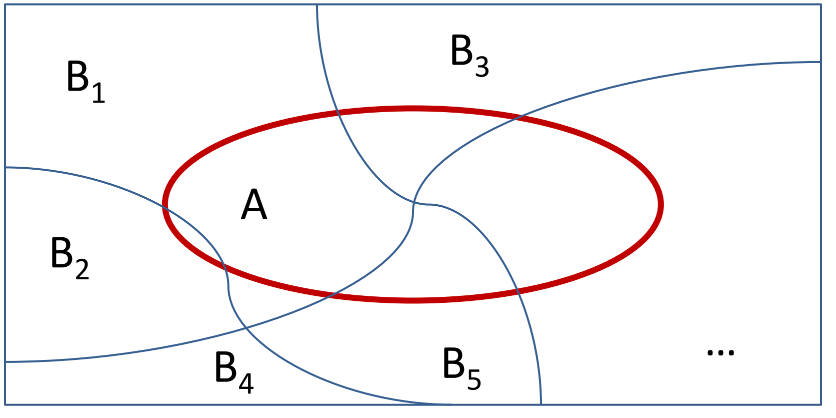
Bayes' Rule

Definition of **Bayes' Rule**:

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A in S such that $P(A) \neq 0$,

$$P(B_r|A) = \frac{P(B_r \cap A)}{P(A)} = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(A \cap B_i)} = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}$$

for $r = 1, 2, \ldots, k$,

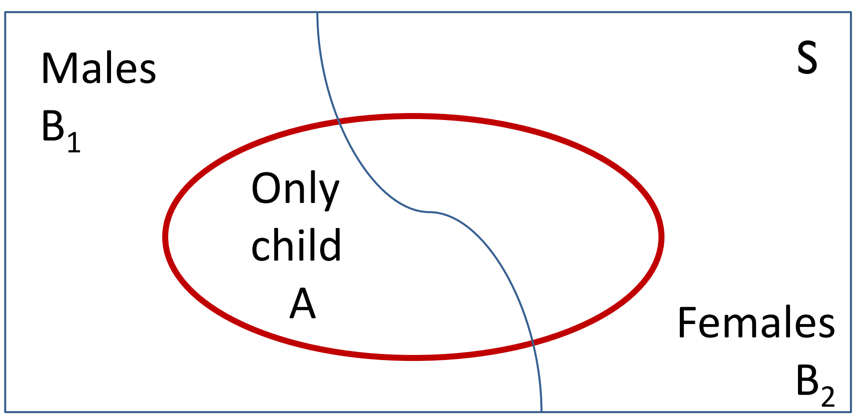


$B_1 + B_2 + \ldots + B_k = S$

Bayes' Rule

Example:

In our classroom, there are males and females who are the only child, and males and females who are not the only child. Now a student is being selected at random for giving a self-introduction. Suppose the student selected is the only child. What is the probability that the student is a male?

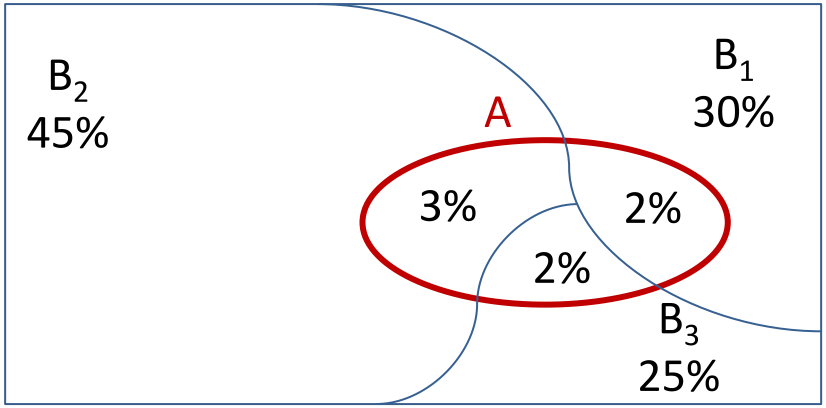


Bayes' Rule

Revisit example:

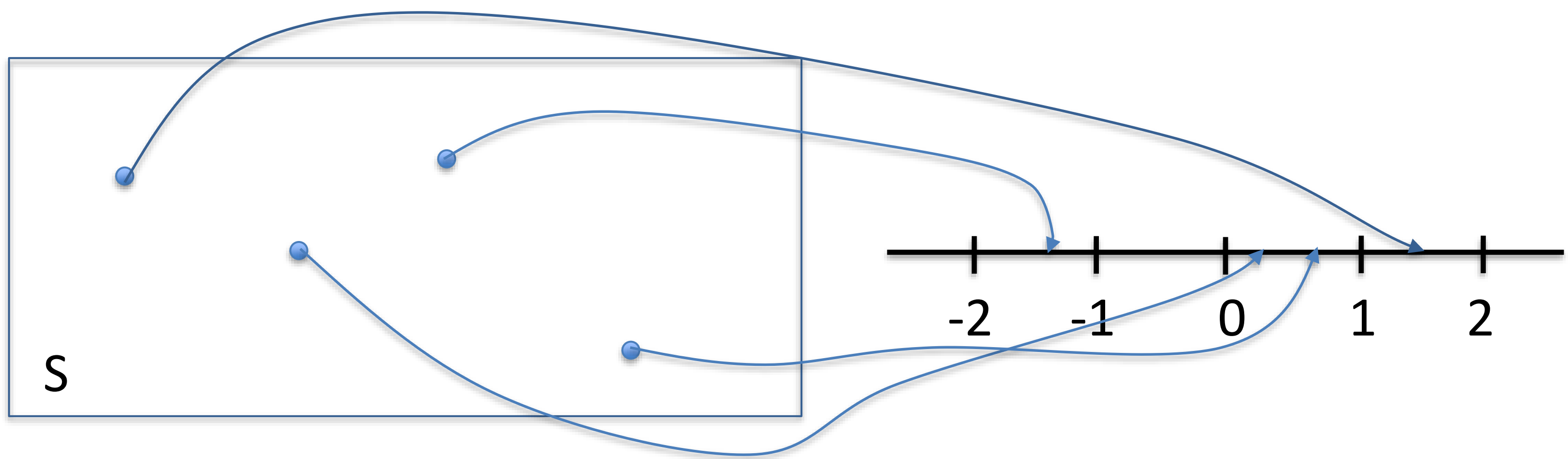
In a certain assembly plant, three machines, B1, B2, and B3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected and it is defective. What is the probability that this product was made by machine B_1?

Let A be the event that the selected product is defective.



Chapter 3
Random Variables and Probability Distributions

Rule of association is called a **random variable.**



Random Variables

Definition:

- Formal terms (in textbook): A random variable is a function that associates a real number with each element in the sample space.
- Simple terms: A random variable is the outcome of a statistical experiment, that you can measure or count.

Examples:

Toss a coin twice: Sample space = {HH, HT, TH, TT}

- Let a random variable X be the number of head:
  X = {0, 1, 2}
- Let a random variable Y be the experimental outcomes:
  Y = {y=1 for HH; y=2 for HT; y=3 for TH; y=4 for TT}

A capital letter for a random variable

Its corresponding small letter for its values

Bernoulli Random Variables

Definition:

- Any random variable whose only possible values are 0 and 1 is called a **Bernoulli random variable**.

Examples:

1. Toss a coin:
   X = {0 for head; 1 for tail}
2. The gender of selecting a student in our classroom:
   Y = {0 for female; 1 for male}
3. Make a phone call:
   Z = {0 for fail; 1 for success}

Two Types of Random Variables

Examples:
Randomly select 10 students in our classroom and record their height.

- Let a random variable X be the number of students who are between 160-170cm.
  X = {0, 1, 2, …, 10}

  Discrete Random Variable

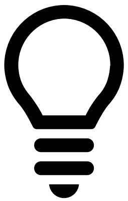
- Let a random variable Y be the height of students.
  Y takes on all values y for which y > 0 and y is a real number

  Continuous Random Variable

Two Types of Random Variables

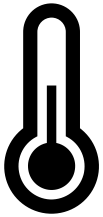
Discrete Random Variable: can only take on a countable number of values (either a finite set or else can be listed in an infinite sequence)

- Single dice roll
- Number of dice rolls until 6 appears
- Number of defective light bulbs in a box of 100



Continuous Random Variable: can take on continuous values in a single interval on the number line or in a disjoint union of such intervals

- Heights of humans
- Temperature in Chengdu
- Time to have lunch

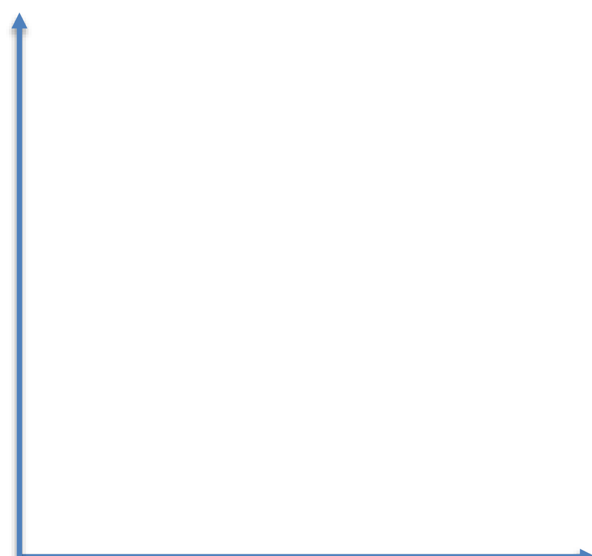


Probability Distributions

Probability Distribution: is the representation of random variable values and their associated probabilities.

Random variable X

Probability
P(X = x)



Random variable value X = x

$$(x, P(X = x))$$

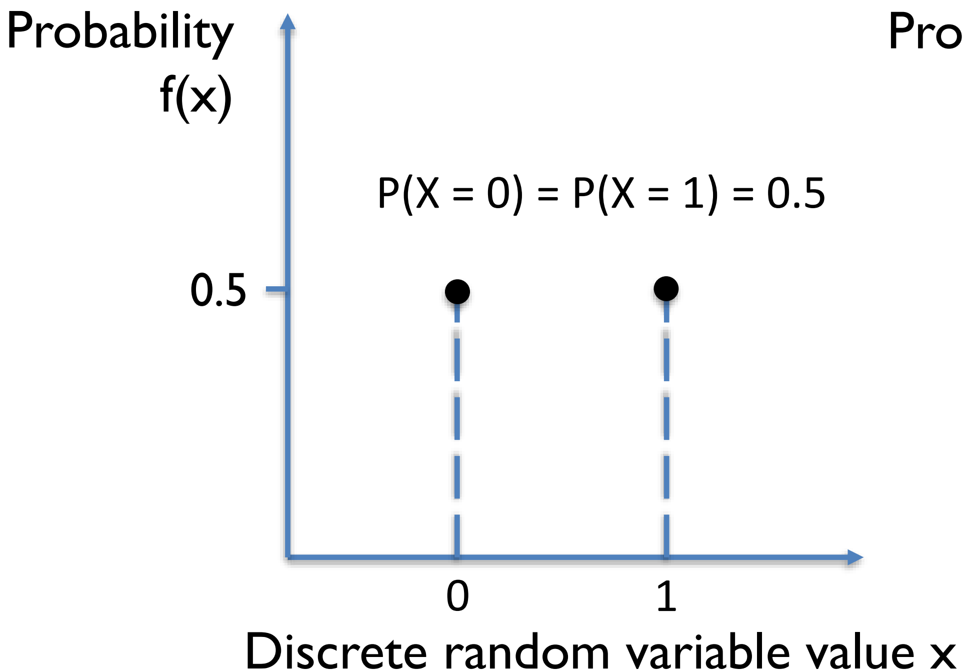
$$f(x) = P(X = x)$$

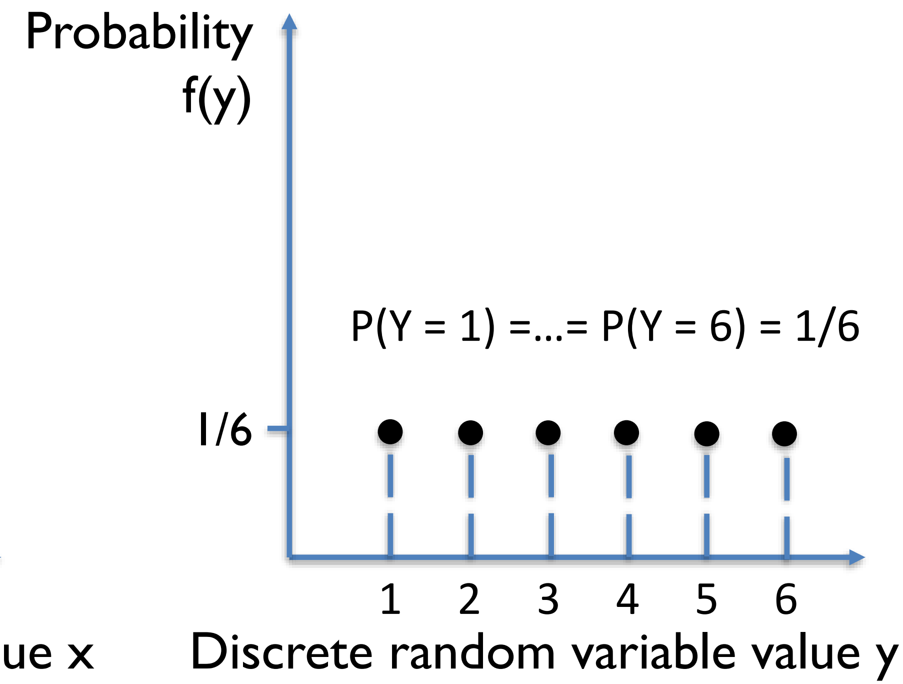
The <u>probability distribution</u> or <u>probability function</u> of random variable X.

Discrete Probability Distributions

Examples:

- Let X = single coin toss
- Let Y = single dice toss

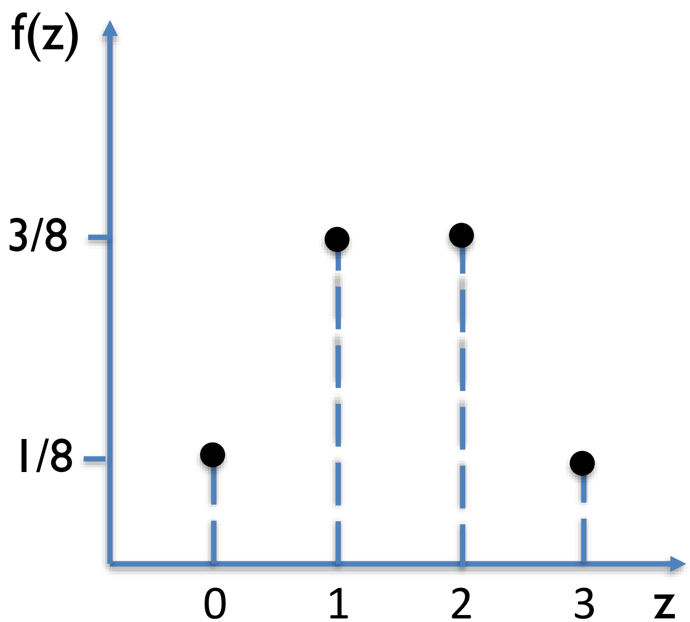


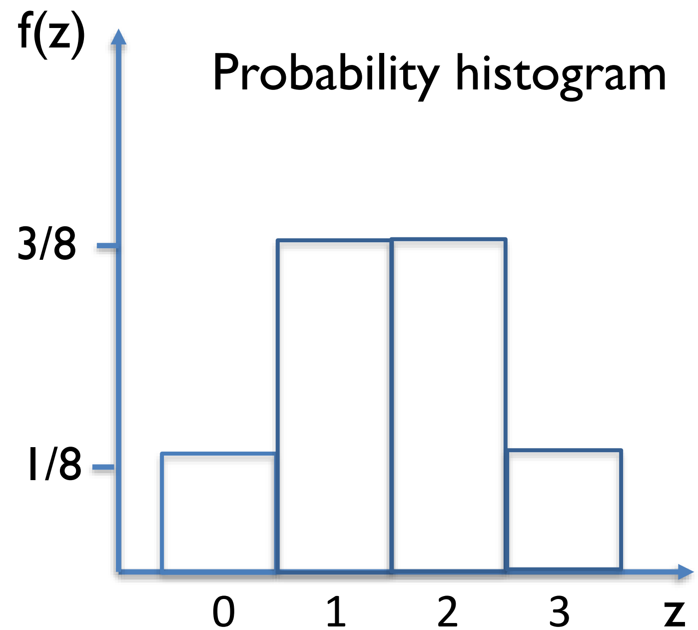


Discrete Probability Distributions

Examples: Let Z = the number of heads tossing a coin three times

z	0	1	2	3
P(Z=z)	1/8	3/8	3/8	1/8





Discrete Probability Distributions

Definition:

The set of ordered pairs $(x, f(x))$ is a **probability function** or **probability distribution** of the discrete random variable X if, for each possible outcome x,

1. $f(x) \geq 0$,
2. $\sum f(x) = 1$, ⬅ The values of X exhaust all possible outcomes
3. $P(X = x) = f(x)$.

Discrete Probability Distributions

Example:
A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Let X = the number of defective computers purchased by the school

X = {0, 1, 2}

$$f(0) = P(X = 0) =$$

$$f(1) = P(X = 1) =$$

$$f(2) = P(X = 2) =$$

Formula for probability distribution:

$$f(x) = P(X = x) =$$

Cumulative Distribution Function

For many problems, we may be interested in computing the probability that the observed value of a random variable *X* will be less than or equal to some real number *x*.

Ex. What is the probability of purchasing no more than 1 defective computer? $P(X=0)+P(X=1) \ = P(X \leq 1) \ = F(x)$

Definition:

The **cumulative distribution function F(x)** of a discrete random variable X with probability distribution f(x) is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \qquad for\ a\ real\ number\ x$$

Cumulative Distribution Function

Revisit example:

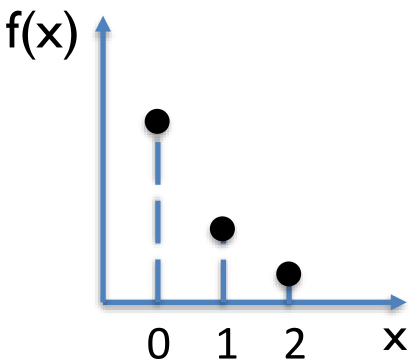
A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the cumulative distribution function for the number of defectives.

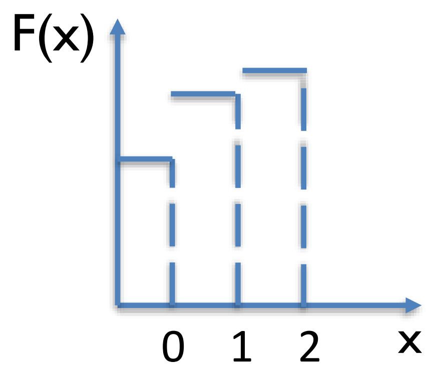
$$f(0) = P(X = 0) = \frac{136}{190}$$

$$f(1) = P(X = 1) = \frac{51}{190}$$

$$f(2) = P(X = 2) = \frac{3}{190}$$

$$F(x) = P(X \leq x) = \begin{cases} \frac{136}{190}, & \text{for } x \leq 0 \\ \frac{187}{190}, & \text{for } 0 < x \leq 1 \\ 1, & \text{for } 1 < x \leq 2 \end{cases}$$





Monotone
Nondecreasing

Continuous Probability Distributions

Example:

Let X be a continuous random variable whose values are the heights of sophomores.

- What is the probability of selecting a person at random who is exactly 165 cm?

  Why are you sure the person is exactly 165 cm, not 165.000001 or 164.999999 cm?

A continuous random variable has a probability of 0 of assuming *exactly* any of its values.

Continuous Probability Distributions

- What is the probability of selecting a person at random who is at least 164 cm but not more than 166 cm?

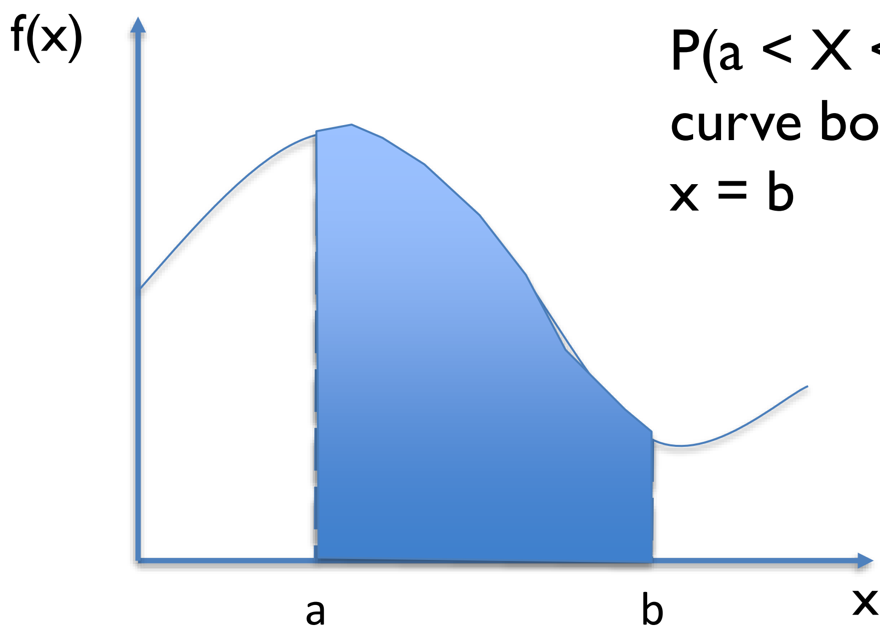
Now we are dealing with an interval rather than a point value of the continuous random variable.

$$P(164 \leq X \leq 166) = P(164 < X < 166) + P(X = 164) + P(X = 166)$$
$$= P(164 < X < 166)$$

It dose not matter whether we include an endpoint of the interval or not.

Probability Density Function

The probability distribution of a continuous random variable X cannot be presented in tabular form, but can be stated as a formula $f(x)$, which is called the **probability density function** of X.



P(a < X < b) = The area under f(x) curve bounded by x axis, x = a, and x = b

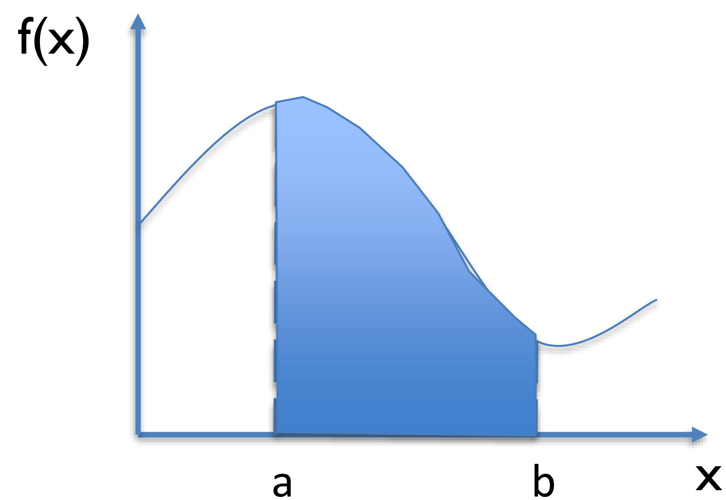
Probability Density Function

Definition:
The function $f(x)$ is a **probability density function** (pdf) for the continuous random variable X, defined over the set of real numbers, if

1. $f(x) \geq 0, for\ all\ x \in R.$

2. $\int_{-\infty}^{\infty} f(x)dx = 1.$

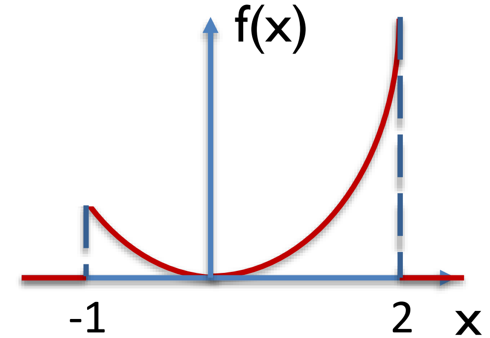
3. $P(a < X < b) = \int_{a}^{b} f(x)dx.$



Probability Density Function

Example:
Suppose that the error in the reaction temperature (◦C) for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & elsewhere \end{cases}$$



(a) Verify that f(x) is a density function.

1. $f(x) \geq 0, for\ all\ x \in R$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $P(a < X < b) = \int_a^b f(x)dx$

(b) Find $P(0 < X \leq 1)$.

(c) Find $P(X \leq 1)$. ? Cumulative distribution function

Cumulative Distribution Function

Definition:

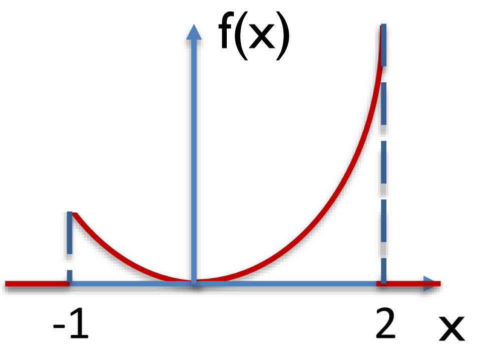
The **cumulative distribution function F(x)** of a continuous random variable X with density function f(x) is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt, \qquad for\ -\infty < x < \infty$$

where $f(x) = \frac{dF(x)}{dx}$, if the derivative exists.

Example in last page:

- For -1 ≤ x < 2, $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-1}^{x} \frac{t^2}{3} dt = \frac{x^3 + 1}{9}$
- For x < -1, $F(x) = 0$
- For x ≥ 2, $F(x) = 1$



Cumulative Distribution Function

Example:
The Department of Energy (DOE) puts projects out on bid and generally estimates what a reasonable bid should be. Call the estimate b. The DOE has determined that the density function of the winning (low) bid is

$$f(x) = \begin{cases} \frac{5}{8b}, & \frac{2b}{5} \le x \le 2b \\ 0, & elsewhere \end{cases}$$

Find F(x).

- For $2b/5 \le x \le 2b$,

- For $x < 2b/5$,

- For $x > 2b$,

Using F(x) to Compute Probabilities

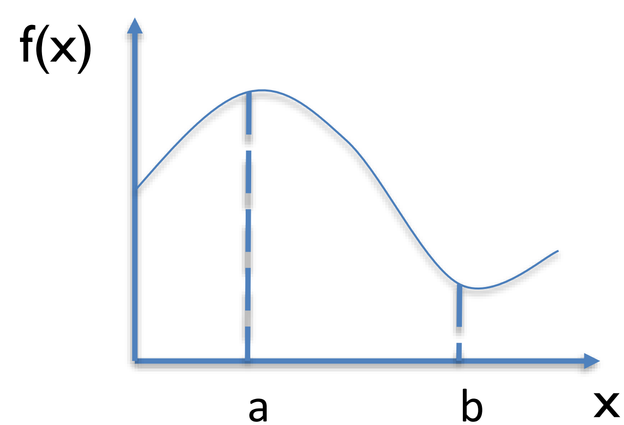
Proposition:
Let X be a continuous random variable with probability density function f(x) and cumulative distribution function F(x).

- For any number a,

$$P(X > a) = 1 - F(a)$$

- For any two numbers a and b with a < b,

$$P(a \leq X \leq b) = F(b) - F(a)$$



Using F(x) to Compute Probabilities

Example:

Suppose the pdf of the magnitude X of a dynamic load on a bridge (in newtons) is given by

$$f(x) = \begin{cases} \dfrac{1}{8} + \dfrac{3}{8}x, & 0 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$$

- For $0 \leq x \leq 2$,
- For $x < 0$,
- For $x > 2$,

Using F(x) to Compute Probabilities

Example:
Suppose the pdf of the magnitude X of a dynamic load on a bridge (in newtons) is given by

$$f(x) = \begin{cases} \frac{1}{8} + \frac{3}{8}x, & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{8} + \frac{3}{16}x^2, & 0 \le x \le 2 \\ 1, & x > 2 \end{cases}$$

$$P(1 \le X \le 1.5) = F(1.5) - F(1)$$

$$P(X > 1) = 1 - F(1)$$