Q1

(a) $\iint_D \frac{y}{x^5+1}\,dA$, $D=\{(x,y)\mid x\in[0,1],\ 0\le y\le x^2\}$

$$\iint_D \frac{y}{x^5+1}\,dA = \int_0^1\int_0^{x^2}\frac{y}{x^5+1}\,dy\,dx = \int_0^1 \frac{1}{x^5+1}\int_0^{x^2} y\,dy\,dx$$

$$= \int_0^1 \frac{x^4}{2(x^5+1)}\,dx$$

let $u=x^5+1$, $du=5x^4dx$

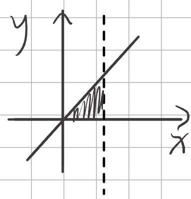
$$\int_0^1 \frac{x^4}{2(x^5+1)}\,dx = \int_1^2 \frac{1}{10u}\,du = \frac{1}{10}(\ln 2-\ln 1) = \frac{\ln 2}{10}$$

(b) $\iint_D x\,dA = \int_0^{\pi}\int_0^{\sin x} x\,dy\,dx = \int_0^{\pi} x\cdot\sin x\cdot dx$

let $u=x$, $dv=\sin x\,dx$, then $du=dx$, $v=-\cos x$

$$\int_0^{\pi} x\sin x\,dx = \Big[-x\cos x\Big]_0^{\pi} + \int_0^{\pi}\cos x\,dx = \pi-0+0=\pi$$

Q2



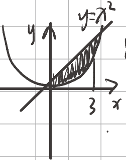
(a) Type ① $D=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le x\}$

$$\iint_D x\,dA = \int_0^1\int_0^x x\,dy\,dx = \int_0^1 x\int_0^x dy\,dx = \int_0^1 x^2dx = \frac{1}{3}$$

Type ② $D=\{(x,y)\mid 0\le y\le 1,\ y\le x\le 1\}$

$$\iint_D x\,dA = \int_0^1\int_y^1 x\,dx\,dy = \int_0^1 \frac{1}{2}(1-y^2)\,dy = \frac{1}{2}\left(1-\frac{1}{3}-0\right) = \frac{1}{3}$$

(b) Type ① $D=\{(x,y)\mid 0\le x\le 3,\ x^2\le y\le 3x\}$

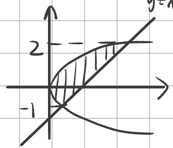


$$\iint_D xy\,dA = \int_0^3\int_{x^2}^{3x} xy\,dy\,dx = \int_0^3 \frac{x}{2}(9x^2-x^4)\,dx = \int_0^3\left(\frac{9}{2}x^3-\frac{1}{2}x^5\right)dx = \frac{729}{8}-\frac{486}{8}=\frac{243}{8}$$

Type ② $D=\{(x,y)\mid 0\le y\le 9,\ \frac{y}{3}\le x\le \sqrt{y}\}$

$$\iint_D xy\,dA = \int_0^9\int_{\frac{y}{3}}^{\sqrt{y}} xy\,dx\,dy = \int_0^9 \frac{y}{2}\left(y-\frac{y^2}{9}\right)dy = \int_0^9\left(\frac{y^2}{2}-\frac{y^3}{18}\right) = \frac{9^3}{6}-\frac{9^4}{72} = \frac{243}{8}$$

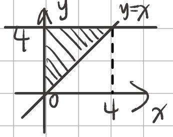
Q3



(a)

$$\therefore \iint_D y\,dA = \int_{-1}^2\int_{y^2}^{y+2} y\,dx\,dy = \int_{-1}^2 y(y+2-y^2)\,dy = \int_{-1}^2(2y+y^2-y^3)\,dy$$

$$= 4+\frac{8}{3}-4-1+\frac{1}{3}+\frac{1}{4} = \frac{9}{4}$$

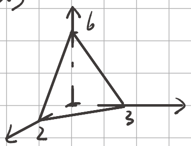


(b) $\iint_D y^2 e^{xy}\,dA = \int_0^4\int_0^y y^2 e^{xy}\,dx\,dy = \int_0^4 (ye^{y^2}-y)\,dy$

$= \int_0^4 ye^{y^2}\,dy - \int_0^4 y\,dy = \frac{1}{2}(e^{16}-1) - \frac{1}{2}\times 16$

$= \frac{1}{2}e^{16} - \frac{17}{2}$

Q4 (a)



$D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le -\frac{3}{2}x+3\}$

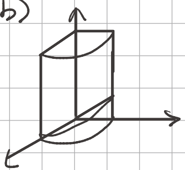
$V = \iint_D (6-3x-2y)\,dA = \int_0^2\int_0^{-\frac{3}{2}x+3} (6-3x-2y)\,dy\,dx$

$= \int_0^2 \left[6y-3xy-y^2\right]_{y=0}^{y=-\frac{3}{2}x+3} dx$

$= \int_0^2 (\frac{9}{4}x^2-9x+9)\,dx = \left[\frac{3}{4}x^3-\frac{9}{2}x^2+9x\right]_{x=0}^{x=2}$

$= 6-18+18-0 = 6$

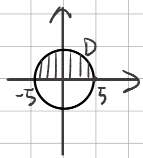
(b)



$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\}$

$V = \iint_D y\,dA = \int_0^1 \frac{1}{2}(1-x^2-0)\,dx = \left[\frac{1}{2}x - \frac{x^3}{6}\right]_0^1 = \frac{1}{2}-\frac{1}{6}-0 = \frac{1}{3}$

Q5 (a) $D = \{(r,\theta) \mid 0 \le r \le 5,\ 0 \le \theta \le \pi\}$



$\iint_D x^2 y\,dA = \int_0^\pi\int_0^5 r^2\cos^2\theta \cdot r\sin\theta\, r\,dr\,d\theta$

$= \int_0^\pi \cos^2\theta\sin\theta \left[\frac{r^5}{5}\right]_{r=0}^{r=5} d\theta$

$= 625\int_0^\pi \cos^2\theta\,\sin\theta\,d\theta$

Let $u=\cos\theta$, then $du = -\sin\theta\,d\theta$

$625\int_0^\pi \cos^2\theta\,\sin\theta\,d\theta = 625\int_1^{-1} -u^2\,du = 625(\frac{1}{3}+\frac{1}{3}) = \frac{1250}{3}$

(b) $D = \{(r,\theta) \mid 1 \le r \le 3,\ 0 \le \theta \le \frac{\pi}{2}\}$

$\iint_D \sin(x^2+y^2)\,dA = \int_0^{\frac{\pi}{2}}\int_1^3 \sin(r^2)\,r\,dr\,d\theta$ let $u=r^2$, then $du=2r\,dr$

$= \int_0^{\frac{\pi}{2}}\int_1^9 \sin u \cdot \frac{1}{2}\,du\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}(-\cos u)_{u=1}^{u=9}$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(-\cos 9+\cos 1)\,d\theta = \frac{\pi}{4}(\cos 1 - \cos 9)$

Q6

$$\sqrt{x^2+y^2}=\sqrt{1-x^2-y^2} \quad \therefore x^2+y^2=\frac{1}{2}$$

$$D=\left\{(r,\theta)\,\middle|\, 0\le r\le \frac{\sqrt{2}}{2},\ 0\le\theta\le 2\pi\right\}$$

$$V=\iint_D \sqrt{1-x^2-y^2}-\sqrt{x^2+y^2}\,dA=\int_0^{2\pi}\int_0^{\frac{\sqrt{2}}{2}}(\sqrt{1-r^2}-r)\,r\,dr\cdot d\theta$$

$$=\int_0^{2\pi}\left(\int_0^{\frac{\sqrt{2}}{2}}(\sqrt{1-r^2})r\,dr-\int_0^{\frac{\sqrt{2}}{2}}r^2dr\right)d\theta \quad \text{let } u=1-r^2,\ du=2rdr$$

$$=\int_0^{2\pi}\left(\int_1^{\frac{1}{2}}(-2u^{\frac{1}{2}})du-\left[\frac{r^3}{3}\right]_{r=0}^{r=\frac{\sqrt{2}}{2}}\right)d\theta$$

$$=\int_0^{2\pi}\left(-\frac{\sqrt{2}}{12}+\frac{1}{3}-\frac{\sqrt{2}}{12}\right)d\theta=\left(\frac{1}{3}-\frac{\sqrt{2}}{6}\right)\cdot 2\pi=\frac{(2-\sqrt{2})}{3}\pi$$

Q7 (a) $\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\sin(x^2+y^2)\,dy\,dx$

$$\because D=\{(r,\theta)\,|\,0\le r\le 3,\ 0\le\theta\le\pi\}$$

$$\therefore \int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\sin(x^2+y^2)\,dy\,dx=\iint_D\sin(x^2+y^2)\,dA$$

$$=\int_0^{\pi}\int_0^{3}\sin(r^2)\,r\,dr\,d\theta,\quad \text{let } u=r^2,\ du=2rdr$$

$$=\int_0^{\pi}\int_0^{9}\sin u\cdot\frac{1}{2}du\,d\theta=\int_0^{\pi}-\frac{1}{2}(\cos 9-1)\,d\theta$$

$$=-\frac{\pi}{2}(\cos 9-1)$$

(b) $\int_0^1\int_y^{\sqrt{2-y^2}}(x+y)\,dx\,dy$

$$\because D=\left\{(r,\theta)\,\middle|\,0\le r\le\sqrt{2},\ 0\le\theta\le\frac{\pi}{4}\right\}$$

$$\therefore \int_0^1\int_y^{\sqrt{2-y^2}}(x+y)\,dx\,dy=\iint_D(x+y)\,dA$$

$$=\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}}(r\cos\theta+r\sin\theta)\,r\,dr\,d\theta=\int_0^{\frac{\pi}{4}}(\cos\theta+\sin\theta)\left[\frac{r^3}{3}\right]_{r=0}^{r=\sqrt{2}}d\theta=\frac{2\sqrt{2}}{3}\left[\sin\theta-\cos\theta\right]_0^{\frac{\pi}{4}}$$

$$=\frac{2\sqrt{2}}{3}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\ 0+1\right)=\frac{2}{3}\sqrt{2}$$