Section 2.1

5. **a)** Yes; order and repetition do not matter.
b) No; the first set has one element, and the second has two elements.
c) No; the first set has no elements, and the second has one element (namely the empty set).

10. **a)** true **b)** true **c)** false—see part **(a)** **d)** true
e) true—the one element in the set on the left is an element of the set on the right, and the sets are not equal
f) true—similar to part **(e)** **g)** false—the two sets are equal

11. **a)** T (in fact x is the only element) **b)** T (every set is a subset of itself)
c) F (the only element of $\{x\}$ is a letter, not a set) **d)** T (in fact, $\{x\}$ is the only element)
e) T (the empty set is a subset of every set) **f)** F (the only element of $\{x\}$ is a letter, not a set)

23. **a)** Since the set we are working with has 3 elements, the power set has $2^3 = 8$ elements.
b) Since the set we are working with has 4 elements, the power set has $2^4 = 16$ elements.
c) The power set of the empty set has $2^0 = 1$ element. The power set of this set therefore has $2^1 = 2$ elements. In particular, it is $\{\emptyset, \{\emptyset\}\}$. (See Example 14.)

25. We need to prove two things, because this is an "if and only if" statement. First, let us prove the "if" part. We are given that $A \subseteq B$. We want to prove that the power set of A is a subset of the power set of B, which means that if $C \subseteq A$ then $C \subseteq B$. But this follows directly from Exercise 17. For the "only if" part, we are given that the power set of A is a subset of the power set of B. We must show that every element of A is also an element of B. So suppose a is an arbitrary element of A. Then $\{a\}$ is a subset of A, so it is an element of the power set of A. Since the power set of A is a subset of the power set of B, it follows that $\{a\}$ is an element of the power set of B, which means that $\{a\}$ is a subset of B. But this means that the element of $\{a\}$, namely a, is an element of B, as desired.

26. We need to show that every element of $A \times B$ is also an element of $C \times D$. By definition, a typical element of $A \times B$ is a pair (a, b) where $a \in A$ and $b \in B$. Because $A \subseteq C$, we know that $a \in C$; similarly, $b \in D$. Therefore $(a, b) \in C \times D$.

30. We can conclude that $A = \emptyset$ or $B = \emptyset$. To prove this, suppose that neither A nor B were empty. Then there would be elements $a \in A$ and $b \in B$. This would give at last one element, namely (a, b), in $A \times B$, so $A \times B$ would not be the empty set. This contradiction shows that either A or B (or both, it goes without saying) is empty.

Section 2.2

18. **a)** Suppose that $x \in A \cup B$. Then either $x \in A$ or $x \in B$. In either case, certainly $x \in A \cup B \cup C$. This establishes the desired inclusion.
b) Suppose that $x \in A \cap B \cap C$. Then x is in all three of these sets. In particular, it is in both A and B and therefore in $A \cap B$, as desired.
c) Suppose that $x \in (A - B) - C$. Then x is in $A - B$ but not in C. Since $x \in A - B$, we know that $x \in A$ (we also know that $x \notin B$, but that won't be used here). Since we have established that $x \in A$ but $x \notin C$, we have proved that $x \in A - C$.
d) To show that the set given on the left-hand side is empty, it suffices to assume that x is some element in that set and derive a contradiction, thereby showing that no such x exists. So suppose that $x \in (A-C) \cap (C-B)$. Then $x \in A - C$ and $x \in C - B$. The first of these statements implies by definition that $x \notin C$, while the second implies that $x \in C$. This is impossible, so our proof by contradiction is complete.
e) To establish the equality, we need to prove inclusion in both directions. To prove that $(B-A) \cup (C-A) \subseteq (B \cup C) - A$, suppose that $x \in (B - A) \cup (C - A)$. Then either $x \in (B - A)$ or $x \in (C - A)$. Without loss of

generality, assume the former (the proof in the latter case is exactly parallel.) Then $x \in B$ and $x \notin A$. From the first of these assertions, it follows that $x \in B \cup C$. Thus we can conclude that $x \in (B \cup C) - A$, as desired. For the converse, that is, to show that $(B \cup C) - A \subseteq (B - A) \cup (C - A)$, suppose that $x \in (B \cup C) - A$. This means that $x \in (B \cup C)$ and $x \notin A$. The first of these assertions tells us that either $x \in B$ or $x \in C$. Thus either $x \in B - A$ or $x \in C - A$. In either case, $x \in (B - A) \cup (C - A)$. (An alternative proof could be given by using Venn diagrams, showing that both sides represent the same region.)

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19. **a)** This is clear, since both of these sets are precisely $\{x \mid x \in A \wedge x \notin B\}$.
b) One approach here is to use the distributive law; see the answer section for that approach. Alternatively, we can argue directly as follows. Suppose $x \in (A \cap B) \cup (A \cap \overline{B})$. Then we know that either $x \in A \cap B$ or $x \in A \cap \overline{B}$ (or both). If either case, this forces $x \in A$. Thus we have shown that the left-hand side is a subset of the right-hand side. For the opposite direction, suppose $x \in A$. There are two cases: $x \in B$ and $x \notin B$. In the former case, x is then an element of $A \cap B$ and therefore also an element of $(A \cap B) \cup (A \cap \overline{B})$. In the latter cases, $x \in \overline{B}$ and therefore x is an element of $A \cap \overline{B}$ and therefore also an element of $(A \cap B) \cup (A \cap \overline{B})$.

36. There are precisely two ways that an item can be in either A or B but not both. It can be in A but not B (which is equivalent to saying that it is in $A - B$), or it can be in B but not A (which is equivalent to saying that it is in $B - A$). Thus an element is in $A \oplus B$ if and only if it is in $(A - B) \cup (B - A)$.

51. **a)** As i increases, the sets get larger: $A_1 \subset A_2 \subset A_3 \cdots$. All the sets are subsets of the set of integers, and every integer is included eventually, so $\bigcup_{i=1}^{\infty} A_i = \mathbf{Z}$. Because A_1 is a subset of each of the others, $\bigcap_{i=1}^{\infty} A_i = A_1 = \{-1, 0, 1\}$.
b) All the sets are subsets of the set of integers, and every nonzero integer is in exactly one of the sets, so $\bigcup_{i=1}^{\infty} A_i = \mathbf{Z} - \{0\}$. Each pair of these sets are disjoint, so no element is common to all of the sets. Therefore $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

c) This is similar to part **(a)**, the only difference being that here we are working with real numbers. Therefore $\bigcup_{i=1}^{\infty} A_i = \mathbf{R}$ (the set of all real numbers), and $\bigcap_{i=1}^{\infty} A_i = A_1 = [-1, 1]$ (the interval of all real numbers between -1 and 1, inclusive).
d) This time the sets are getting smaller as i increases: $\cdots \subset A_3 \subset A_2 \subset A_1$. Because A_1 includes all the others, $\bigcup_{i=1}^{\infty} A_i = A_1 = [1, \infty)$ (all real numbers greater than or equal to 1). Every number eventually gets excluded as i increases, so $\bigcap_{i=1}^{\infty} A_i = \emptyset$. Notice that ∞ is not a real number, so we cannot write $\bigcap_{i=1}^{\infty} A_i = \{\infty\}$.

61. **a)** The multiplicity of a in the union is the maximum of 3 and 2, the multiplicities of a in A and B. Since the maximum is 3, we find that a occurs with multiplicity 3 in the union. Working similarly with b, c (which appears with multiplicity 0 in B), and d (which appears with multiplicity 0 in A), we find that $A \cup B = \{3 \cdot a, 3 \cdot b, 1 \cdot c, 4 \cdot d\}$.
b) This is similar to part **(a)**, with "maximum" replaced by "minimum." Thus $A \cap B = \{2 \cdot a, 2 \cdot b\}$. (In particular, c and d appear with multiplicity 0—i.e., do not appear—in the intersection.)
c) In this case we subtract multiplicities, but never go below 0. Thus the answer is $\{1 \cdot a, 1 \cdot c\}$.
d) Similar to part **(c)** (subtraction in the opposite order); the answer is $\{1 \cdot b, 4 \cdot d\}$.
e) We add multiplicities here, to get $\{5 \cdot a, 5 \cdot b, 1 \cdot c, 4 \cdot d\}$.