CS0441

Discrete Structures Section 2

KP. Wang **Assignment #3** **Solutions**

1. Section 1.7 # 14, 18, 19, 20, 25, 28
2. Section 1.8 # 2, 7, 17
3. Section 2.1 # 10, 20, 21, 25, 32, 44
4. Section 2.2 # 14, 15
5. Consider sets A, B, and C, show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$ by mutual subsets.

1. *Solution:* If x is rational and not zero, then by definition we can write $x = p/q$, where p and q are nonzero integers. Since $1/x$ is then q/p and $p \neq 0$, we can conclude that $1/x$ is rational.

2. *Solution:*

   (a) We must prove the contrapositive: If n is odd, then $3n + 2$ is odd. Assume that n is odd. Then we can write $n = 2k+1$ for some integer k. Then $3n+2 = 3(2k+1)+2 = 6k + 5 = 2(3k + 2) + 1$. Thus $3n + 2$ is two times some integer plus 1 , so it is odd.

   (b) Suppose that $3n+2$ is even and that n is odd. Since $3n+2$ is even, so is $3n$. If we add subtract an odd number from an even number, we get an odd number, so $3n - n = 2n$ is odd. But this is obviously not true. Therefore our supposition was wrong, and the proof by contradiction is complete.

3. *Solution:* The proposition we are trying to prove is "If 0 is a positive integer greater than 1 , then $0^2 > 0$." Our proof is a vacuous one. Since the hypothesis is false, the conditional statement is automatically true.

4. *Solution:* We need to prove the proposition "If 1 is a positive integer, then $1^2 \geqslant 1$." The conclusion is the true statement $1 \geqslant 1$. Therefore the conditional statement is true. This is an example of a trivial proof, since we merely showed that the conclusion was true.

5. *Solution:* Suppose by way of contradiction that a/b is a rational root, where a and b are integers and this fraction is in lowest terms (that is, a and b have no common divisor greater than 1). Plug this proposed root into the equation to obtain $a^3/b^3 + a/b + 1 = 0$. Multiply through by b^3 to obtain $a^3 + ab^2 + b^3 = 0$. If a and b are both odd, then the left-hand side is the sum of three odd numbers and therefore must be odd. If a is odd and b is even, then the left-hand side is odd + even + even, which is again odd. Similarly, if a is even and b is odd, then the left-hand side is even + even +odd, which is again odd. Because the fraction a/b is in simplest terms, it cannot happen that both a and b are even. Thus in all cases, the left-hand side is odd, and therefore cannot equal 0 . This contradiction shows that no such root exists.

6. *Solution:* There are two things to prove. For the "if" part, there are two cases. If $m = n$, then of course $m^2 = n^2$; if $m = -n$, then $m^2 = (-n)^2 = (-1)^2 n^2 = n^2$. For the "only if" part, we suppose that $m^2 = n^2$. Putting everything on the left and factoring, we have $(m + n)(m - n) = 0$. Now the only way that a product of two numbers can be zero is if

one of them is zero. Therefore we conclude that either $m+n=0$ (in which case $m=-n$), or else $m-n=0$ (in which case $m=n$), and our proof is complete.

7. *Solution:* The cubes that might go into the sum are $1, 8, 27, 64, 125, 216, 343, 512$, and 729 . We must show that no two of these sum to a number on this list. If we try the 45 combinations $(1+1, 1+8, \ldots, 1+729, 8+8,\ 8+27, \ldots 8+729, \ldots, 729+729)$, we see that none of them works. Having exhausted the possibilities, we conclude that no cube less than 1000 is the sum of two cubes.

8. *Solution:* There are several cases to consider. If x and y are both nonnegative, then $|x|+|y|=x+y=|x+y|$. Similarly, if both are negative, then $|x|+|y|=(-x)+(-y)=-(x+y)=|x+y|$, since $x+y$ is negative in this case. The complication (and strict inequality) comes if one of the variables is nonnegative and the other is negative. By the symmetry of the roles of x and y here (strictly speaking, by the commutativity of addition), we can assume without loss of generality that it is x that is nonnegative and y that is negative. So we have $x \geq 0$ and $y<0$.

9. *Solution:* The equation $|a-c|=|b-c|$ is equivalent to the disjunction of two equations: $a-c=b-c$ or $a-c=-b+c$. The first of these is equivalent to $a=b$, which contradicts the assumptions made in this problem, so the original equation is equivalent to $a-c=-b+c$. By adding $b+c$ to both sides and dividing by 2 , we see that this equation is equivalent to $c=(a+b)/2$. Thus there is a unique solution. Furthermore, this c is an integer, because the sum of the odd integers a and b is even.

10. *Solution:*

    a) true,

    b) true,

    c) false,

    d) true,

    e) true - the one element in the set on the left is an element of the set on the right, and the sets are not equal,

    f) true,

    g) false - the two sets are equal.

11. *Solution:*

    a) The empty set has no elements, so its cardinality is 0 .

    b) This set has one element (the empty set), so its cardinality is 1 .

    c) This set has two elements, so its cardinality is 2 .

    d) This set has three elements, so its cardinality is 3 .

12. *Solution:*

    a) $\{\emptyset,\{a\}\}$

    b) $\{\emptyset,\{a\},\{b\},\{a, b\}\}$

    c) $\{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\}$

13. *Solution:* We need to prove two things, because this is an "if and only if" statement. First, let us prove the "if" part. We are given that $A \subseteq B$. We want to prove that the power set of A is a subset of the power set of B, which means that if $C \subseteq A$ then $C \subseteq B$. But this follows directly from Exercise 17. For the "only if" part, we are given that the power set of A is a subset of the power set of B. We must show that every element of A is also an element of B. So suppose a is an arbitrary element of A. Then $\{a\}$ is a subset of A, so it is an element of the power set of A. Since the power set of A is a subset of the power set of B, it follows that $\{a\}$ is an element of the power set of B, which means that $\{a\}$ is a subset of B. But this means that the element of $\{a\}$, namely a, is an element of B, as desired.

14. *Solution:* In each case the answer is a set of 3 -tuples.

    a) $\{(a, x, 0),(a, x, 1),(a, y, 0),(a, y, 1),(b, x, 0),(b, x, 1),$
    $(b, y, 0),(b, y, 1),(c, x, 0),(c, x, 1),(c, y, 0),(c, y, 1)\}$

    b) $\{(0, x, a),(0, x, b),(0, x, c),(0, y, a),(0, y, b),(0, y, c),$
    $(1, x, a),(1, x, b),(1, x, c),(1, y, a),(1, y, b),(1, y, c)\}$

    c) $\{(0, a, x),(0, a, y),(0, b, x),(0, b, y),(0, c, x),(0, c, y),$
    $(1, a, x),(1, a, y),(1, b, x),(1, b, y),(1, c, x),(1, c, y)\}$

    d) $\{(x, x, x),(x, x, y),(x, y, x),(x, y, y),(y, x, x),(y, x, y),(y, y, x),(y, y, y)\}$

15. *Solution:* In each case we want the set of all values of x in the domain (the set of integers) that satisfy the given equation or inequality.

    a) It is exactly the positive integers that satisfy this inequality. Therefore the truth set is $\left\{x \in \mathbb{Z} \mid x^{3} \geqslant 1\right\}=\{x \in \mathbb{Z} \mid x \geqslant 1\}=\{1,2,3, \ldots\}$.

    b) The square roots of 2 are not integers, so the truth set is the empty set, $\varnothing$.

    c) Negative integers certainly satisfy this inequality, as do all positive integers greater than 1 . However, $0 \nless 0^{2}$ and $1 \nless 1^{2}$. Thus the truth set is $\left\{x \in \mathbb{Z} \mid x<x^{2}\right\}=\{x \in \mathbb{Z} \mid x \neq 0 \wedge x \neq 1\}=\{\ldots,-3,-2,-1,2,3, \ldots\}$.

16. *Solution:* Since $A=(A-B) \cup(A \cap B)$, we conclude that $A=\{1,5,7,8\} \cup\{3,6,9\}=\{1,3,5,6,7,8,9\}$. Similarly $B=(B-A) \cup(A \cap B)=\{2,10\} \cup\{3,6,9\}=\{2,3,6,9,10\}$.

17. *Solution:* a) This proof is similar to the proof of the dual property. Suppose $x \in \overline{A \cup B}$. Then $x \notin A \cup B$, which means that x is in neither A nor B. In other words, $x \notin A$ and $x \notin B$. This is equivalent to saying that $x \in \bar{A}$ and $x \in \bar{B}$. Therefore $x \in \bar{A} \cap \bar{B}$, as desired. Conversely, if $x \in \bar{A} \cap \bar{B}$, then $x \in \bar{A}$ and $x \in \bar{B}$. This means $x \notin A$ and $x \notin B$, so x cannot be in the union of A and B. Since $x \notin A \cup B$, we conclude that $x \in \overline{A \cup B}$, as desired.

    b) The following membership table gives the desired equality, since columns four and seven are identical.

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

18. *Solution:* First we will show that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$. Suppose $(a, b) \in A \times (B \cap C)$. By definition of the Cartesian product, this means $a \in A$ and $b \in B \cap C$. By definition of intersection, it follows that $b \in B$ and $b \in C$. Thus, since $a \in A$ and $b \in B$, it follows that $(a, b) \in A \times B$. Also, since $a \in A$ and $b \in C$, it follows that $(a, b) \in A \times C$. Now we have $(a, b) \in A \times B$ and $(a, b) \in A \times C$, so $(a, b) \in (A \times B) \cap (A \times C)$. We've shown that $(a, b) \in A \times (B \cap C)$ implies $(a, b) \in (A \times B) \cap (A \times C)$ so we have $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

    Next we will show that $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. Suppose $(a, b) \in (A \times B) \cap (A \times C)$.

    By definition of intersection, this means $(a, b) \in A \times B$ and $(a, b) \in A \times C$. By definition of the Cartesian product, $(a, b) \in A \times B$ means $a \in A$ and $b \in B$. By definition of the Cartesian product, $(a, b) \in A \times C$ means $a \in A$ and $b \in C$. We now have $b \in B$ and $b \in C$, so $b \in B \cap C$, by definition of intersection. Thus we've deduced that $a \in A$ and $b \in B \cap C$, so $(a, b) \in A \times (B \cap C)$. In summary, we've shown that $(a, b) \in (A \times B) \cap (A \times C)$ implies $(a, b) \in A \times (B \cap C)$ so we have $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

    The previous two paragraphs show that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ and $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$, so it follows that $(A \times B) \cap (A \times C) = A \times (B \cap C)$.