Q1

(a) $f(x,y) = 2x+2y+z$, $g(x,y) = x^2+y^2+z^2-25=0$

$L = f+\lambda g = 2x+2y+z+\lambda(x^2+y^2+z^2-25)$

$$\because \begin{cases} \frac{\partial L}{\partial x} = 2+2x\lambda=0 \Rightarrow x = -\frac{1}{\lambda} \\ \frac{\partial L}{\partial y} = 2+2y\lambda = 0 \Rightarrow y = -\frac{1}{\lambda} \\ \frac{\partial L}{\partial y} = 1+2z\lambda = 0 \Rightarrow z = -\frac{1}{2\lambda} \\ \frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-25=0 \Rightarrow \frac{1}{\lambda^2}+\frac{1}{\lambda^2}+\frac{1}{4\lambda^2}-25=0 \Rightarrow \lambda = \pm\frac{3}{10} \end{cases}$$

when $\lambda = \frac{3}{10}$, $\left(-\frac{10}{3}, -\frac{10}{3}, -\frac{5}{3}\right)$

$\lambda = -\frac{3}{10}$, $\left(\frac{10}{3}, \frac{10}{3}, \frac{5}{3}\right)$

$f\left(-\frac{10}{3}, -\frac{10}{3}, -\frac{5}{3}\right) = -\frac{20}{3} - \frac{20}{3} - \frac{5}{3} = -\frac{45}{3} = -15$

$f\left(\frac{10}{3}, \frac{10}{3}, \frac{5}{3}\right) = \frac{20}{3} + \frac{20}{3} + \frac{5}{3} = 15$

$\therefore$ The maximum value is 15.

minimum value is -15.

(b) $f = x^2+y^2+z^2$, $g = x^4+y^4+z^4-1=0$

$L = f+\lambda g = x^2+y^2+z^2+\lambda(x^4+y^4+z^4-1)$

$$\begin{cases} \frac{\partial L}{\partial x} = 2x+4x^3\lambda = 0 \Rightarrow x=0 \text{ or } x = \pm\sqrt{\frac{-1}{2\lambda}} \\ \frac{\partial L}{\partial y} = 2y+4y^3\lambda=0 \Rightarrow y=0 \text{ or } y = \pm\sqrt{\frac{-1}{2\lambda}} \\ \frac{\partial L}{\partial z} = 2z+4z^3\lambda=0 \Rightarrow z=0, \text{ or } z = \pm\sqrt{\frac{-1}{2\lambda}} \\ \frac{\partial L}{\partial \lambda} = x^4+y^4+z^4-1=0 \end{cases}$$

$$\left.\begin{array}{l} \text{when } x=y=0 \;.\; (0,0,\pm 1)\,; \\ \qquad x=z=0 \;,\; (0,\pm 1,0)\,; \\ \qquad y=z=0 \;,\; (\pm 1,0,0)\,; \end{array}\right\} \begin{array}{l} \text{2 point each situation} \\ f=1 \end{array}$$

$$\left.\begin{array}{l} x=0 \;\; \left(0,\pm\frac{1}{\sqrt[4]{2}},\pm\frac{1}{\sqrt[4]{2}}\right); \\ y=0 \;\; \left(\pm\frac{1}{\sqrt[4]{2}},0,\pm\frac{1}{\sqrt[4]{2}}\right); \\ z=0 \;\; \left(\pm\frac{1}{\sqrt[4]{2}},0,\pm\frac{1}{\sqrt[4]{2}}\right); \end{array}\right\} \begin{array}{l} \text{4 points each situation} \\ f=\sqrt{2} \end{array}$$

$$x=y=z\neq 0 \;\; \left(\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}}\right) \qquad f=\sqrt{3}$$

$$\left(-\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}}\right)$$

$\therefore$ maximum : $f=\sqrt{3}$

minimum : $f=1$

(c) $f(x,y,z)=yz+xy \quad g(x,y,z)=xy-1=0$

$h(x,y,z)=y^2+z^2-1=0$

$\therefore L=f+\lambda_1 g+\lambda_2 h = yz+xy+\lambda_1(xy-1)+\lambda_2(y^2+z^2-1)$

$$\begin{cases} \frac{\partial L}{\partial x}=y+y\lambda_1=y(1+\lambda_1) \Rightarrow \lambda_1=-1 \\ \frac{\partial L}{\partial x}=x+z+x\lambda_1+2y\lambda_2=0 \Rightarrow z=2y\lambda_2 \\ \frac{\partial L}{\partial z}=y+2z\lambda_2=0 \Rightarrow \lambda_2=\pm\frac{1}{2} \\ \frac{\partial L}{\partial \lambda_1}=xy-1=0 \qquad \frac{\partial L}{\partial \lambda_2}=y^2+z^2-1=0 \end{cases}$$

$\therefore$ when $\lambda_2=\frac{1}{2}$, $z=y$, $\Rightarrow (\sqrt{2},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, $(-\sqrt{2},-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$

when $\lambda_2=-\frac{1}{2}$, $z=-y \Rightarrow (\sqrt{2},\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$, $(-\sqrt{2},-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$

$f(\sqrt{2},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})=f(-\sqrt{2},-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})=\frac{3}{2}$

$f(\sqrt{2},\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})=f(-\sqrt{2},-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})=\frac{1}{2}$

$\therefore$ Maximum value of f is $\frac{3}{2}$.

minimum is $\frac{1}{2}$

(d) $f(x,y,z)=x^2+y^2+z^2$, $\quad g(x,y,z)=x-y-1=0$

$h(x,y,z)=y^2-z^2-1=0$

$\therefore L=f+\lambda_1 g+\lambda_2 h=x^2+y^2+z+\lambda_1(x-y-1)+\lambda_2(y^2-z^2-1)$

$$\begin{cases} \frac{\partial L}{\partial x}=2x+\lambda_1=0 \\ \frac{\partial L}{\partial y}=2y-\lambda_1+2y\lambda_2=0 \\ \frac{\partial L}{\partial z}=2z-2z\lambda_2=0 \quad \Rightarrow z=0 \text{ or } \lambda_2=1 \\ \frac{\partial L}{\partial \lambda_1}=x-y-1=0 \end{cases}$$

$\frac{\partial L}{\partial \lambda_2}=y^2-z^2-1=0$

when $z=0$, $y^2=1$, $y=\pm1 \Rightarrow (2,1,0)\ (0,-1,0)$

$\lambda_2=1$, $y=\pm\lambda$, $x=-\frac{\lambda_1}{2} \Rightarrow$ not exist

$f(2\,1\,0)=5$, $\quad f(0,-1,0)=1$

maximum value of f is 5

minimum is 1.

Q2

(a) $f(x,y) = x^2+y^2+4x-4y$. $D=\{(x,y) \mid x^2+y^2 \le 9\}$

$$\nabla f = \vec{0} \Rightarrow \begin{cases} 2x+4=0 \\ 2y-4=0 \end{cases} \Rightarrow (-2,2) \in D$$

$$g(x,y) = x^2+y^2-9=0$$

$$\therefore L = x^2+y^2+4x-4y+\lambda(x^2+y^2-9)$$

$$\begin{cases} \frac{dL}{dx} = 2x+4+2x\lambda=0 \Rightarrow x=\frac{-2}{1+\lambda} \\ \frac{dL}{dx} = 2y-4+2y\lambda=0 \Rightarrow y=\frac{2}{1+\lambda} \\ \frac{dL}{d\lambda} = x^2+y^2-9=0 \Rightarrow \frac{4}{(1+\lambda)^2}+\frac{4}{(1+\lambda)^2}-9=0 \Rightarrow \lambda=\pm\frac{2\sqrt{3}}{3}-1 \end{cases}$$

when $\lambda = \frac{2}{3}\sqrt{2}-1 \Rightarrow \left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$

$\lambda = -\frac{2\sqrt{2}}{3}-1 \Rightarrow \left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$

$f(-2,2) = 4+4-8-8=-8$ (Minimum)

$f\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right) = 9-12\sqrt{2}$

$f\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right) = 9+12\sqrt{2}$ (Maximum)

(b) $f(x,y) = e^{-xy}$, $D=\{(x,y) \mid x^2+y^2 \le 1\}$

$$\because \nabla f = \vec{0} \Rightarrow \begin{cases} -ye^{xy}=0 \Rightarrow (0,0) \in D \\ -xe^{xy}=0 \end{cases}$$

$$g(x,y) = x^2+y^2-1=0$$

$$L = e^{-xy}+\lambda(x^2+y^2-1)$$

$$\begin{cases} \frac{dL}{dx} = -ye^{xy}+2x\lambda=0 \Rightarrow -xye^{xy}+2x^2\lambda=0 \quad (1) \\ \frac{dL}{dy} = -xe^{xy}+2y\lambda=0 \Rightarrow -xye^{xy}+2y^2\lambda=0 \quad (2) \end{cases}$$

(1) − (2): $(x^2-y^2)\lambda=0 \Rightarrow \lambda=0 / x=y / x=-y$

$\frac{\partial L}{\partial \lambda}=x^2+y^2-1=0$

when $\lambda=0$, $x=0$, $y=0$, $x^2+y^2\neq 1$. ∴ not exist

when $x=y$, $2x^2=1 \Rightarrow x=\pm\frac{1}{\sqrt{2}} \Rightarrow (\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})\ (-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$

$f(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})=f(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})=e^{-\frac{1}{2}}$ (min)

when $x=-y$ $2x^2=1 \Rightarrow x=\pm\frac{1}{\sqrt{2}} \Rightarrow (\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$

$f(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})=f(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})=e^{\frac{1}{2}}$ (max)

$f(0,0)=e^0=1$

Q3.

$f(x,y,z)=x^2+y^2+z^2$, $g(x,y,z)=x+y+2z-2=0$

$h(x,y,z)=z-x^2-y^2=0$

$\nabla f=\langle 2x,2y,2z\rangle$, $\nabla g=\langle 1,1,2\rangle$ $\nabla h=\langle -2x,-2y,1\rangle$

$$\begin{cases} 2x=\lambda_1-2x\lambda_2 & (1)\\ 2y=\lambda_1-2y\lambda_2 & (2)\\ 2z=2\lambda_1+\lambda_2 \\ x+y+2z=2 \\ z-x^2-y^2=0 \end{cases}$$

(2) − (1): $2(y-x)=-2(y-x)\lambda_2$

∴ $x=y$ or $\lambda_2=-1$

when $\lambda_2=-1$, $2x=\lambda_1+2x \Rightarrow \lambda_1=0$, $z=-\frac{1}{2}$ $\begin{cases} x+y=3 \\ x^2+y^2=-\frac{1}{2}\end{cases}$ not exist

when $x=y$, $\begin{cases} x+z=1 \\ z=2x^2 \end{cases} \Rightarrow 2x^2+x-1=0 \Rightarrow x=\frac{1}{2}$ or $x=-1$

∴ $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ $(-1,-1,2)$

$f(\frac{1}{2},\frac{1}{2},\frac{1}{2})=\frac{3}{4}$, $f(-1,-1,2)=6$

$\therefore (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is nearest, $(-1, -1, 2)$ is farthest

Q4

(a) $f(x,y) = x$ $\quad g(x,y) = x^4 - x^3 + y^2 = 0$

$\nabla f = \langle 1, 0 \rangle \quad \nabla g = \langle 4x^3 - 3x^2, 2y \rangle$

$$\begin{cases} 1 = \lambda(4x^3 - 3x^2) \\ 0 = \lambda(2y) \Rightarrow y = 0 / \lambda = 0 \text{ (not exist)} \\ y^2 + x^4 - x^3 = 0 \end{cases}$$

when $y = 0$, $x^3(x-1) = 0 \Rightarrow x = 0$ (not exist) $/ x = 1$.

$\therefore f(1,0) = 1$.

Using Lagrange multipliers, the minimum Value of f is 1

(b) $y^2 = (x^3 - x^4) = x^3(1-x) \geq 0$

$0 \leq x \leq 1$

$\therefore$ the minimum is $f(0,0) = 0$

$\nabla f(0,0) = \langle 1, 0 \rangle \quad \nabla g(0,0) = \langle 0, 0 \rangle$

$\therefore \nabla f(0,0) = \lambda \nabla g(0,0)$ can't be established for any value of λ.

(c) Lagrange's method can only be applied when $\nabla g \neq \vec{0}$

Q5

(a) $\int_1^3 \int_1^5 \frac{\ln y}{xy}\, dy\, dx$

$= \int_1^3 \frac{1}{x} \int_1^5 \frac{\ln y}{y}\, dy\, dx$ let $u = \ln y$ $du = \frac{1}{y} dy$

$= \int_1^3 \frac{1}{x} \int_0^{\ln 5} u\, du\, dx = \int_1^3 \frac{1}{x} \frac{1}{2} [u^2]_0^{\ln 5} dx = \int_1^3 \frac{1}{2x} (\ln 5)^2 dx$

$= (\frac{\ln 5}{2})^2 \cdot (\ln x)_1^3$

$= \ln 3 \cdot \frac{\ln^2 5}{2}$

(b) $\int_0^1 \int_0^1 \sqrt{s+t}\, ds\, dt$

$= \int_0^1 \frac{2}{3} (t+s)^{\frac{3}{2}} \Big|_{s=0}^{s=1} = \int_0^1 \frac{2}{3} [(1+t)^{\frac{3}{2}} - t^{\frac{3}{2}}] dt$

$= \frac{2}{3} \left[\frac{2}{5}(1+t)^{\frac{5}{2}} - \frac{2}{5} t^{\frac{5}{2}} \right]_{t=0}^{t=1}$

$= \frac{2}{3} (\frac{8}{5}\sqrt{2} - \frac{2}{5} - \frac{2}{5})$

$= \frac{16\sqrt{2} - 8}{15}$

(c) $\int_0^1 \int_0^1 xy \sqrt{x^2+y^2}\, dy\, dx$, Let $u = x^2 + y^2$, $du = 2y\, dy$

$= \int_0^1 \frac{x}{2} \int_{x^2}^{x^2+1} u^{\frac{1}{2}} du\, dx = \int_0^1 \frac{x}{2} [\frac{2}{3} u^{\frac{3}{2}}]_{x^2}^{x^2+1} dx$

$= \int_0^1 \frac{x}{2} [\frac{2}{3} (x^2+1)^{\frac{3}{2}} - \frac{2}{3} x^3] dx$

$= \int_0^1 \frac{1}{3} x (x^2+1)^{\frac{3}{2}} dx - \int_0^1 \frac{x^4}{3} dx = \frac{1}{15} (x^2+1)^{\frac{5}{2}} \Big|_0^1 - \frac{1}{15} (x^5) \Big|_0^1$

$= \frac{1}{15} (4\sqrt{2} - 1) - \frac{1}{15} = \frac{4\sqrt{2} - 2}{15}$

(d) $\int_0^1 \int_0^1 \frac{x}{1+xy}\, dy\, dx$

$= \int_0^1 x \int_0^1 \frac{1}{1+xy}\, dy\, dx = \int_0^1 x \cdot \frac{1}{x} \left[\ln(1+xy) \right]_0^1 dx$

$= \int_0^1 \ln(1+x) dx$

Let $u = \ln(1+x)$, $dv = dx$

$du = \frac{1}{(1+x)} dx$, $v = x$

$\therefore \int \ln(1+x)\, dx = x \ln(1+x) - \int \frac{x}{1+x}\, dx$

$= \ln(1+x) \cdot x - \int (1 - \frac{1}{1+x})\, dx$

$= \ln(1+x) \cdot x - x + \ln(1+x)$

$\therefore \int_0^1 \ln(1+x)\, dx = \left[x \cdot \ln(x+1) + \ln(1+x) - x \right]_{x=0}^{x=1}$

$= \ln 2 + \ln 2 - 1 = 2\ln 2 - 1$