Contents

1 Question 1

We have an system of linear equations for which:

$$x + y + z = 2 \tag{1}$$
$$x + 4y - z = k \tag{2}$$
$$2x - y + 4z = k^2 \tag{3}$$

We can throw it into an augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 4 & -1 & k \\ 2 & -1 & 4 & k^2 \end{array}\right)$$

The row operations (which reduces it into row echelon form are as follows:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & -3 & 2 & k^2-4 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \longmapsto R_2 - R_1 \\ R_3 \longmapsto R_3 - 2R_1 \end{array} \tag{4}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & 0 & 0 & k^2+k-6 \end{array}\right) \tag{5}$$

The left side of the last row of the augmented matrix contains a row of zeros so we can solve for k:

$$k^2 + k - 6 = 0 \tag{6}$$

If we solve equation (6) for k we will get $\boldsymbol{k = -3}$ and $\boldsymbol{k = 2}$.

1.1 Cases Where There Are Infinite Amount of Solutions

If we substitute $k = -3$ and $k = 2$ into the last row of (5) will be $0x+0y+0z = 0$. Hence for $k = -3$ and $k = 2$, there will be **infinite** amount of solutions to this system. The planes will meet in a infinite long line if $k = -3$ and $k = -2$.

To solve for the line we can make:

$$z = t \tag{7}$$

and solve R_2 in (5) for y which is:

$$3y - 2t = k - 2$$

Because k has two possible values due to R_2 in (5), we will end up with 2 lines depending on the value of k

1.1.1 If $k = 2$

Let $k = 2$:

$$3y - 2t = 0 \tag{8}$$

$$y = \frac{2t}{3} \tag{9}$$

If we look back at R_1 of (5) we get a equation of:

$$x + y + z = 2 \tag{10}$$

Substitute (7) and (9) into (10) gives:

$$x + \frac{5t}{3} = 2 \tag{11}$$

$$x = 2 - \frac{5t}{3} \tag{12}$$

Therefore in the end our line when $k = 2$ will be:

$$x = 2 - \frac{5t}{3}$$
$$y = \frac{2t}{3}$$
$$z = t$$

1.1.2 If $\boldsymbol{k = -3}$

Let $k = -3$:

$$3y - 2t = -5 \tag{13}$$
$$y = \frac{2t-5}{3} \tag{14}$$

If we look back at R_1 of (5) again we get a equation of:

$$x + y + z = 2$$

Substitute (7) and (14) into (10) gives:

$$x + \frac{2t-5}{3} + \frac{3t}{3} = 2 \tag{15}$$
$$x = 2 - \frac{5t-5}{3} \tag{16}$$

Therefore in the end our line when $k = -3$ will be:

$$\begin{aligned} x &= 2 - \frac{5t-5}{3} \\ y &= \frac{2t-5}{3} \\ z &= t \end{aligned}$$

1.2 Cases Where There Are No Solutions

If $\boldsymbol{k \neq 2}$ or $\boldsymbol{k \neq -3}$ there will be **no** solutions for this system as $0x+0y+0z \neq k^2 + k - 6$ when $k \neq 2$ or $k \neq -3$.

1.3 Cases Where The Planes Meet At A Point

It is not possible for this system to meet at one point because the left side of last row in (5) is full of zeros.

2 Question 2

We want to prove that $\boldsymbol{A(B+C) = AB + AC}$.
Let's $\boldsymbol{A}$ to be a $\boldsymbol{m \times n}$ matrix and $\boldsymbol{B, C}$ to be a $\boldsymbol{n \times p}$ matrix. So AB and AC are both $\boldsymbol{m \times p}$ matrix.
Let $1 \leq i \leq m$ and $i \leq j \leq p$.
Let's consider the entry in i-th row and j-th column of $A(B+C)$.

$$[A(B+C)]_{ij} = \sum_{k=1}^{n} (A)_{ik}(B+C)_{kj}$$

$$= \sum_{k=1}^{n} (A)_{ik} \left[(B)_{kj} + (C)_{kj}\right]$$

Because We are manipulating real numbers here the following is allowed:

$$= \sum_{k=1}^{n} \left[(A)_{ik}(B)_{kj} + (A)_{ik}(C)_{kj}\right]$$

$$= \sum_{k=1}^{n} (A)_{ik}(B)_{kj} + \sum_{k=1}^{n} (A)_{ik}(C)_{kj}$$

Using the definition of matrix multiplication we get:

$$= (AB)_{ij} + (AC)_{ij}$$

$$= (AB + AC)_{ij}$$

The results above implies that:

$$A(B+C) = AB + AC$$

Q.E.D

3 Question 3

3.1 a)

We have a matrix A that:

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

We can put it into an augmented matrix such that $(A|I)$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

The row operations which turns $(A|I)$ into $(I|A^{-1})$ are as follows:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \longmapsto R_2 - R_1 \\ R_3 \end{array} \tag{17}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \longmapsto R_3 - R_2 \end{array} \tag{18}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & -1 \\ 0 & 2 & 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right) \begin{array}{l} R_1 \longmapsto 2R_1 - R_3 \\ R_2 \longmapsto 2R_2 + R_1 \\ R_3 \end{array} \tag{19}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \begin{array}{l} R_1 \longmapsto \frac{1}{2}R_1 \\ R_2 \longmapsto \frac{1}{2}R_2 \\ R_3 \longmapsto \frac{1}{2}R_3 \end{array} \tag{20}$$

Therefore A^{-1}

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{21}$$

3.2 b)

We can express row operations done from (17) to (20) as elementary matrices E_1 to E_4 For operation (17)

$$(17) = E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(18) = E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$(19) = E_3 = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(20) = E_4 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

We can now state that:

$$E_4 E_3 E_2 E_1 A = I \tag{22}$$

Where:

$$E_4 E_3 E_2 E_1 = A^{-1} \tag{23}$$

Therefore:

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3.3 c)

If we multiply the both sides of (22) by $(E_4)^{-1}$ to the left then we will get:

$$E_3E_2E_1A = (E_4)^{-1} \tag{24}$$

We can keep doing the same for E_3, E_2, E_1 then we will get:

$$A = I(E_1)^{-1}(E_2)^{-1}(E_3)^{-1}(E_4)^{-1} \tag{25}$$

The Inverse of E_1, E_2, E_3, E_4 are as follows:

$$(E_1)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$(E_2)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$
$$(E_3)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$
$$(E_4)^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Therefore A can be expressed as:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

4 Question 4

We have a matrix A for which:

$$A = \begin{pmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{pmatrix}$$

4.1 a)

The LU-Decompsition can be started by first Reducing A to upper triangular form to find U:

$$\begin{pmatrix} 2 & -4 & 0 \\ 0 & 10 & 8 \\ 0 & 0 & 4 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \longmapsto 2R_2 - 3R_1 \\ R_3 \longmapsto 2R_3 + R_1 \end{array} \tag{26}$$

Let the equivalent matrix for the row operations in (26) to be L^{-1}:

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \tag{27}$$

Because $L^{-1}A = U \Rightarrow A = LU$ if we multiply both sides by $(L^{-1})^{-1} = L$ to the left. Hence L is the inverse of (27):

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \tag{28}$$

Therefore:

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & -4 & 0 \\ 0 & 10 & 8 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{pmatrix} \tag{29}$$

$$LU = A \tag{30}$$

4.2 b)

We are asked to solve the linear system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$. We can rewrite $A\mathbf{x} = \mathbf{b}$ because of (30) where $LU = A$:

$$LU\mathbf{x} = \mathbf{b} \tag{31}$$

Let $\mathbf{y} = U\mathbf{x}$ from (31) so $L\mathbf{y} = \mathbf{b}$: We can solve for $\mathbf{y}$ first, so let $\mathbf{y} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} \tag{32}$$

Then we get 3 equations from (32):

$$\begin{aligned} a &= 2 \\ \frac{3}{2}a + \frac{1}{2}b &= 0 \\ -\frac{1}{2}a + \frac{1}{2}c &= -5 \end{aligned}$$

Solving them for a, b, c gives:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ -8 \end{pmatrix} \tag{33}$$

Throwing the result from (33) into $U\mathbf{x} = \mathbf{y}$:

$$\begin{pmatrix} 2 & -4 & 0 \\ 0 & 10 & 8 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ -8 \end{pmatrix} \tag{34}$$

From (34) We get yet another 3 equations:

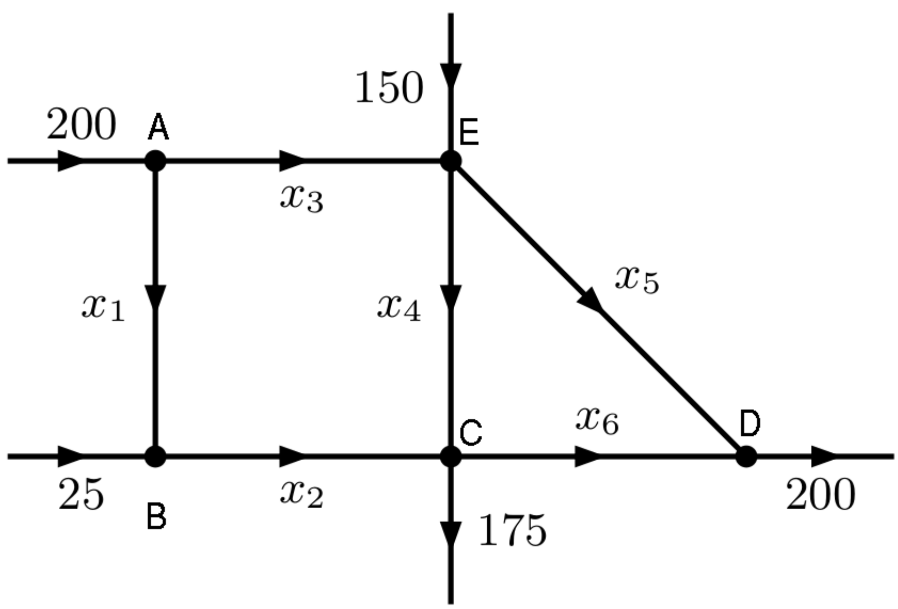
$$\begin{aligned} 2x - 4y &= 2 \\ 10y + 8z &= -6 \\ 4z &= -8 \end{aligned}$$

Solving the equations above for x, y, z gives:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

5 Question 5

The diagram shows the known flow rates of hydrocarbon in a network of pipes. Its nodes are labelled as A to E respectively.



5.1 a)

The network equation and node equations are as follows:

$$\begin{aligned}
\text{Network: } 200 + 25 + 150 - 200 - 175 &= 0 \\
\text{Node A: } 200 - x_1 - x_3 &= 0 \\
\text{Node B: } 25 + x_1 - x_2 &= 0 \\
\text{Node C: } x_4 + x_2 - x_6 - 175 &= 0 \\
\text{Node D: } x_6 + x_5 - 200 &= 0 \\
\text{Node E: } 150 + x_3 - x_4 - x_5 &= 0
\end{aligned}$$

After a bit of rearranging we will get:

$$\text{Network: } 375 = 375 \tag{35}$$
$$\text{Node A: } x_1 + x_3 = 200 \tag{36}$$
$$\text{Node B: } x_1 - x_2 = -25 \tag{37}$$
$$\text{Node C: } x_2 + x_4 - x_6 = 175 \tag{38}$$
$$\text{Node D: } x_5 + x_6 = 200 \tag{39}$$
$$\text{Node E: } x_3 - x_4 - x_5 = -150 \tag{40}$$

Equation (35) tells us that the amount of hydrocarbons flowing in equals to the amount flowing out.

5.2 b)

Putting equations from (36) to (40) into an augmented matrix we get:

$$\left(\begin{array}{cccccc|c}
1 & 0 & 1 & 0 & 0 & 0 & 200 \\
1 & -1 & 0 & 0 & 0 & 0 & -25 \\
0 & 1 & 0 & 1 & 0 & -1 & 175 \\
0 & 0 & 0 & 0 & 1 & 1 & 200 \\
0 & 0 & 1 & -1 & -1 & 0 & -150
\end{array}\right) \tag{41}$$

Using GNU octave, the reduced echelon form of (41) is:

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & -1 & 150 \\ 0 & 1 & 0 & 1 & 0 & -1 & 175 \\ 0 & 0 & 1 & -1 & 0 & 1 & 50 \\ 0 & 0 & 0 & 0 & 1 & 1 & 200 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{42}$$

x_1, x_2, x_3, x_5 are leading variables and x_4, x_6 are free variables. We let that $x_4 = s$ and $x_6 = t$.

$$x_1 + x_4 - x_6 = 150 \tag{43}$$
$$x_2 + x_4 - x_6 = 175 \tag{44}$$
$$x_3 - x_4 + x_6 = 50 \tag{45}$$
$$x_5 + x_6 = 200 \tag{46}$$
$$x_4 = s \tag{47}$$
$$x_6 = t \tag{48}$$

Solving each equation in turn gives:

$$x_1 = 150 - s + t \tag{49}$$
$$x_2 = 175 - s + t \tag{50}$$
$$x_3 = 50 + s - t \tag{51}$$
$$x_5 = 200 - t \tag{52}$$
$$x_4 = s \tag{53}$$
$$x_6 = t \tag{54}$$

5.3 c)

If $x_4 = 50$ and $x_6 = 0$, We just throw the numbers back into Equation (49) to (54)

$$x_1 = 100$$
$$x_2 = 125$$
$$x_3 = 100$$
$$x_5 = 200$$
$$x_4 = 50$$
$$x_6 = 0$$