Exercise 5. Rigid Transform Blending and Variational Methods

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MATLAB R2016b version was used for coding and testing:

MathWorks, MATLAB R2016b (9.1.0.441655)
64-bit (maci64)

The ***code*** directory contains the followings:

part2 script .m file for exercise part 2.

lotr.jpg original image file.

results directory contains result images of part 2.

For running ***part2.m*** script, adjust parameters first. TODO
Note that **these script was only tested in MATLAB R2016b environment**.

1 EXERCISE PART 1: UNDERSTANDING AND UTILIZING DUAL QUATERNION

1.1 TASK 1: THINKING ABOUT FUNDAMENTAL PROPERTIES

- How do dual quaternions represent rotations and translations?

  Unit dual quaternions naturally represent 3D rotation, when the dual part $\mathbf{q}_\epsilon = 0$ (thus $\hat{\mathbf{q}} = \mathbf{q_0} + \epsilon\mathbf{q}_\epsilon = \mathbf{q_0}$). Dual quaternion multiplication with unit dual quaternion $\hat{\mathbf{t}} = 1 + \frac{\epsilon}{2}(t_0 i + t_1 j + t_2 k)$ corresponds to translation by vector (t_0, t_1, t_2) represent 3D translation.

- What is the advantage of representing rigid transformations with dual quaternions for blending?

  The linear combination of dual quaternions does not make artifacts or skin-collapsing effect, thus blending using dual quaternions is fast and more robust than using homogeneous matrix. Moreover, since dual quaternions require only 8 floats per transformation, instead of the 12 required by matrices, they are more memory efficient.

- Briefly explain one fundamental disadvantage of using quaternion based shortest path blending for rotations as compared to linear blend skinning?

  TODO Dual quaternions causes "flipping artifacts" which occurs with joint rotations of more than 180 degrees, This is a corollary of the shortest path property: when the other path becomes shorter, the skin changes its shape discontinuously.

1.2 TASK 2: DERIVATIONS AND DEEPER UNDERSTANDING

- For a dual quaternion $\hat{\mathbf{q}} = \cos(\hat{\theta}/2) + \hat{\mathbf{s}}\sin(\hat{\theta}/2)$, prove that $\hat{\mathbf{q}}^t = \cos(t\hat{\theta}/2) + \hat{\mathbf{s}}\sin(t\hat{\theta}/2)$

  Starting with $\hat{\mathbf{q}}$:

  $$\hat{\mathbf{q}} = \cos(\hat{\theta}/2) + \hat{\mathbf{s}}\sin(\hat{\theta}/2) \tag{1.1}$$

  As $\hat{\mathbf{q}}^t = \exp\big(t\log(\hat{\mathbf{q}})\big)$,

  $$\begin{aligned} \hat{\mathbf{q}}^t &= \exp\big(t\log(\hat{\mathbf{q}})\big) && (1.2)\\ &= \exp\Big(t\log\big(\cos(\hat{\theta}/2) + \hat{\mathbf{s}}\sin(\hat{\theta}/2)\big)\Big) && (1.3) \end{aligned}$$

Plug in $\log\big(\cos(\hat{\theta}/2) + \hat{\mathbf{s}}\sin(\hat{\theta}/2)\big) = \hat{\mathbf{s}}\frac{\hat{\theta}}{2}$ to equation (1.3):

$$\hat{\mathbf{q}}^t = \exp\Big(\frac{t\hat{\theta}}{2}\hat{\mathbf{s}}\Big) \tag{1.4}$$

Let $\hat{\mathbf{a}} = \frac{t\hat{\theta}}{2}\hat{\mathbf{s}}$. Since $\hat{\theta} = \theta_0 + \epsilon\theta_\epsilon$ and $\hat{\mathbf{s}} = \mathbf{s}_0 + \epsilon\mathbf{s}_\epsilon$.

$$\begin{aligned}
\hat{\mathbf{a}} &= \frac{t\hat{\theta}}{2}\hat{\mathbf{s}} &(1.5)\\
&= \frac{t}{2}(\theta_0 + \epsilon\theta_\epsilon)(\mathbf{s}_0 + \epsilon\mathbf{s}_\epsilon) &(1.6)\\
&= \frac{t}{2}(\theta_0\mathbf{s}_0 + \epsilon\theta_0\mathbf{s}_\epsilon + \epsilon\theta_\epsilon\mathbf{s}_0 + \cancelto{0}{\epsilon^2\theta_\epsilon\mathbf{s}_\epsilon}) &(1.7)\\
&= \underbrace{\Big(\frac{t}{2}\theta_0\mathbf{s}_0\Big)}_{:=\mathbf{a}_0} + \epsilon\underbrace{\Big(\frac{t}{2}\theta_0\mathbf{s}_\epsilon + \frac{t}{2}\theta_\epsilon\mathbf{s}_0\Big)}_{:=\mathbf{a}_\epsilon} &(1.8)
\end{aligned}$$

Since exponential of dual quaternion is given by $e^{\hat{\mathbf{q}}} = \cos\big(\|\hat{\mathbf{q}}\|\big) + \frac{\hat{\mathbf{q}}}{\|\hat{\mathbf{q}}\|}\sin\big(\|\hat{\mathbf{q}}\|\big)$, then $e^{\hat{\mathbf{a}}} = \cos\big(\|\hat{\mathbf{a}}\|\big) + \frac{\hat{\mathbf{a}}}{\|\hat{\mathbf{a}}\|}\sin\big(\|\hat{\mathbf{a}}\|\big)$. Here, the norm of dual quaternion $\hat{\mathbf{a}}$ is $\|\hat{\mathbf{a}}\| = \|\mathbf{a}_0\| + \epsilon\frac{<\mathbf{a}_0,\mathbf{a}_\epsilon>}{\|\mathbf{a}_0\|}$. Looking into $<\mathbf{a}_0, \mathbf{a}_\epsilon>$ and $\|\mathbf{a}_0\|$,

$$\begin{aligned}
<\mathbf{a}_0, \mathbf{a}_\epsilon> &=< \frac{t}{2}\theta_0\mathbf{s}_0, \frac{t}{2}\theta_0\mathbf{s}_\epsilon + \frac{t}{2}\theta_\epsilon\mathbf{s}_0 > &(1.9)\\
&=< \frac{t}{2}\theta_0\mathbf{s}_0, \frac{t}{2}\theta_0\mathbf{s}_\epsilon > + < \frac{t}{2}\theta_0\mathbf{s}_0, \frac{t}{2}\theta_\epsilon\mathbf{s}_0 > &(1.10)
\end{aligned}$$

$$\begin{aligned}
\|\mathbf{a}_0\| &= \sqrt{<\mathbf{a}_0, \mathbf{a}_0>} &(1.11)\\
&= \sqrt{< \frac{t}{2}\theta_0\mathbf{s}_0, \frac{t}{2}\theta_0\mathbf{s}_0 >} &(1.12)
\end{aligned}$$

Note that $<\mathbf{s}_0, \mathbf{s}_0>= 1$ and $<\mathbf{s}_0, \mathbf{s}_\epsilon>= 0$. Thus equation (1.10) and (1.12) can be expressed as follows:

$$<\mathbf{a}_0, \mathbf{a}_\epsilon> = \Big(\frac{t}{2}\Big)^2\theta_0\theta_\epsilon \tag{1.13}$$

$$\|\mathbf{a}_0\| = \sqrt{\Big(\frac{t}{2}\theta_0\Big)^2} = \frac{t}{2}\theta_0 \tag{1.14}$$

Plug (1.13) and (1.14) into $\|\hat{\mathbf{a}}\| = \|\mathbf{a}_0\| + \epsilon\frac{<\mathbf{a}_0,\mathbf{a}_\epsilon>}{\|\mathbf{a}_0\|}$:

$$\begin{aligned}
\|\hat{\mathbf{a}}\| &= \frac{t}{2}\theta_0 + \epsilon\frac{\big(\frac{t}{2}\big)^2\theta_0\theta_\epsilon}{\frac{t}{2}\theta_0} &(1.15)\\
&= \frac{t}{2}\theta_0 + \epsilon\frac{t}{2}\theta_\epsilon &(1.16)\\
&= \frac{t}{2}(\theta_0 + \epsilon\theta_\epsilon) = \frac{t}{2}\hat{\theta} &(1.17)
\end{aligned}$$

Finally, by (1.4), (1.17) and $e^{\hat{\mathbf{a}}} = \cos\left(\|\hat{\mathbf{a}}\|\right) + \frac{\hat{\mathbf{a}}}{\|\hat{\mathbf{a}}\|}\sin\left(\|\hat{\mathbf{a}}\|\right)$,

$$\begin{aligned}
\hat{\mathbf{q}}^t &= e^{\hat{\mathbf{a}}} && (1.18)\\
&= \cos\left(\frac{t}{2}\hat{\theta}\right) + \frac{\hat{\mathbf{a}}}{\frac{t}{2}\hat{\theta}}\sin\left(\frac{t}{2}\hat{\theta}\right) && (1.19)\\
&= \cos\left(\frac{t}{2}\hat{\theta}\right) + \frac{\cancel{\frac{t}{2}\hat{\theta}}\,\hat{\mathbf{s}}}{\cancel{\frac{t}{2}\hat{\theta}}}\sin\left(\frac{t}{2}\hat{\theta}\right) && (1.20)\\
&= \cos\left(\frac{t}{2}\hat{\theta}\right) + \hat{\mathbf{s}}\sin\left(\frac{t}{2}\hat{\theta}\right) && (1.21)
\end{aligned}$$

The proof has been done.

- Consider rigid transformations in the 2D xy-plane. For these transformations, the rotation is always around the z (or $-z$)-axis, i.e. $\mathbf{s}_0$ is fixed to the z-axis. On the other hand, a dual quaternion encodes translations only along $\mathbf{s}_0$, which are in this case always zero, since we can only translate in the xy-plane. Then, how can a dual quaternion represent a rotation and translation in the xy-plane, such as the one depicted in Figure 1.1 (a)?

Figure 1.1: Dual quaternion (or screw) representation of 2D translation

(a) An object (left) is first rotated around its center of mass (middle) and then translated (right)

(b) Shifting the screw axis

Dual quaternion can be intuitively represented by a screw which of unit vector $\mathbf{s}_0$ as a screw axis, θ_0 the angle of rotation and θ_ϵ is the amount of translation. For the 2D case of Figure 1.1, if the axis of screw is shifted to point $\mathbf{r}$, as shown in Figure 1.1 (b) such transformation can be represented as a screw axis i.e. dual quaternion. In this case, $\mathbf{s}_\epsilon = \mathbf{r} \times \mathbf{s}_0$

TODO additional explanation + r draw

2 Exercise part 2: Variational Methods - Denoising problems

In this section, three different denoising methods were implemented and compared:

- Filtering
- Heat diffusion
- Variational approach

2.1 Task 1: Filtering

2.2 Task 2: Heat diffusion

2.3 Task 3: Variational approach

2.3.1 Description

Finally, variational approach was implemented. This method considers the image as function of the space of all images. TODO

Derivation of Euler-Lagrange equation The energy function for denoising problem can be defined as follows:

$$E(I) = \int_\Omega \left[\left(I(\mathbf{x}) - I_0(\mathbf{x}) \right)^2 + \lambda \left\| \nabla_\mathbf{x} I(\mathbf{x}) \right\|^2 \right] d\mathbf{x} \tag{2.1}$$

where, $\Omega = \mathbb{R}^2$ is the domain of the 2D image, I_0 the noisy image, and λ a regularization parameter. $I, I_0 \in \mathcal{V} = \mathcal{L}^2(\Omega)$.

Let's define the function $L(I, \nabla I, \mathbf{x})$ as follows:

$$L(I, \nabla I, \mathbf{x}) = \left[\left(I(\mathbf{x}) - I_0(\mathbf{x}) \right)^2 + \lambda \left\| \nabla_\mathbf{x} I(\mathbf{x}) \right\|^2 \right] \tag{2.2}$$

Then the Gâteaux derivative is given by

$$\delta E(I; h) = \lim_{\alpha \to 0} \frac{1}{\alpha} \Big(E(I + \alpha h) - E(I) \Big) \tag{2.3}$$

$$= \lim_{\alpha \to 0} \frac{1}{\alpha} \int_\Omega \Big(L(I + \alpha h, \nabla I + \alpha \nabla h, \mathbf{x}) - L(I, \nabla I, \mathbf{x}) \Big) d\mathbf{x} \tag{2.4}$$

apply Taylor expansion:

$$= \lim_{\alpha \to 0} \frac{1}{\alpha} \int_\Omega \Big(\cancel{L(I, \nabla I, \mathbf{x})} + \alpha \frac{\partial L}{\partial I} h + \alpha \frac{\partial L}{\partial (\nabla I)} \nabla h + o(\alpha^2) - \cancel{L(I, \nabla I, \mathbf{x})} \Big) d\mathbf{x} \tag{2.5}$$

$$= \int_\Omega \Big(\frac{\partial L}{\partial I} h + \frac{\partial L}{\partial (\nabla I)} \nabla h \Big) d\mathbf{x} \tag{2.6}$$

apply integration by parts and $h = 0$ on boundary :

$$= \int_\Omega \frac{\partial L}{\partial I} h \, d\mathbf{x} + \cancelto{0}{h \frac{\partial L}{\partial (\nabla I)} \Big|_\Omega} - \int_\Omega \nabla \frac{\partial L}{\partial (\nabla I)} h \, d\mathbf{x} \tag{2.7}$$

$$= \int_\Omega \Big(\frac{\partial L}{\partial I} - \nabla \frac{\partial L}{\partial (\nabla I)} \Big) h(\mathbf{x}) \, d\mathbf{x} \tag{2.8}$$

Remark following theorem:

Theorem 1. *If $\hat{u}$ is an extremum of a functional $E : \mathcal{V} \to \mathbb{R}$, then*

$$\delta E(\hat{u}, h) = 0 \quad \forall h \in \mathcal{V}.$$

By (2.8) and **Theorem 1**,

$$\frac{\partial L}{\partial I} - \nabla \frac{\partial L}{\partial(\nabla I)} = 0 \tag{2.9}$$

Plug (2.2) into (2.9):

$$\frac{\partial L}{\partial I} = 2\big(I - I_0\big) \tag{2.10}$$

$$\nabla \frac{\partial L}{\partial(\nabla I)} = 2\lambda \nabla \cdot (\nabla I) \tag{2.11}$$

$$= 2\lambda \, \mathbf{div}(\nabla I) \tag{2.12}$$

TODO CHECK!

VECTORIZATION AND LINEAR OPERATION

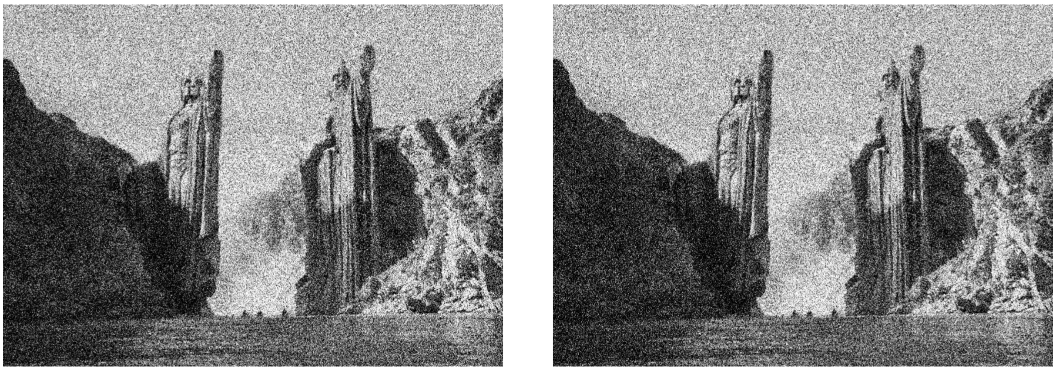
2.3.2 RESULTS

2.4 TASK 4: COMPARISON

- How can you describe the results? Does any of these methods give better results than the others?
- What are the benefits and drawbacks of each methods?
- Can you explain the motivations behind each of the methods?

REFERENCES

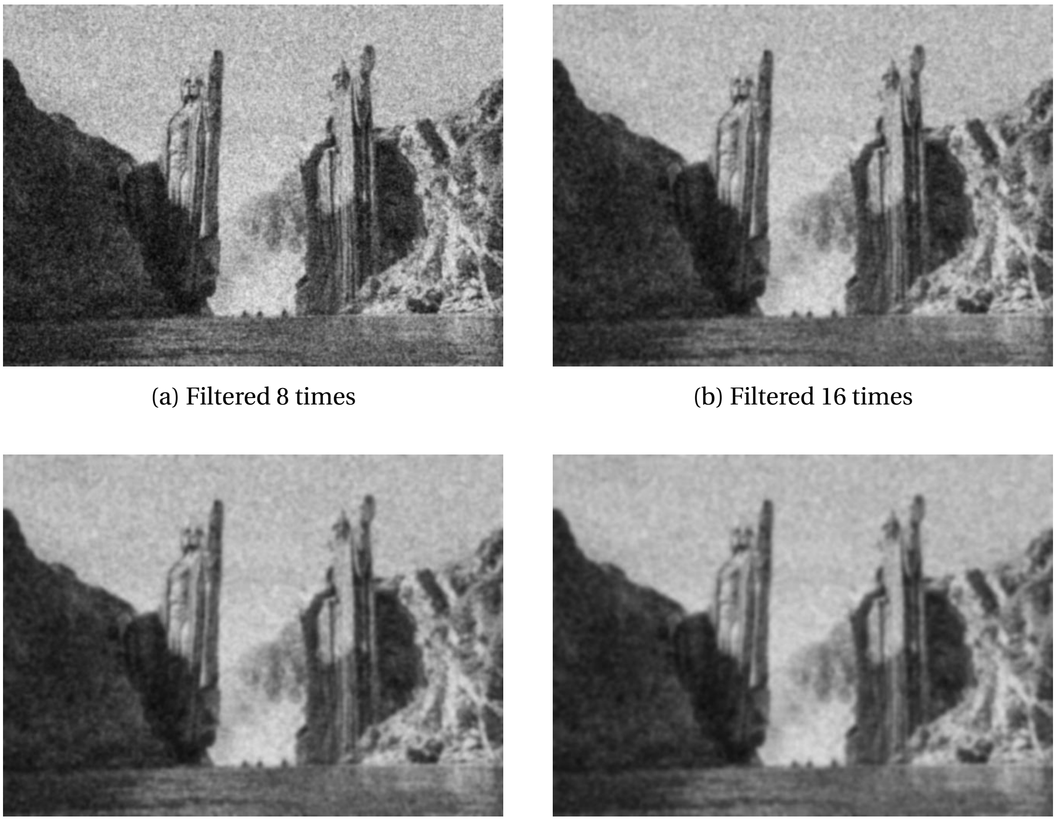
Figure 2.1: Denoised image with variational method



(a) Noisy image

(b) Variational denoising

Figure 2.3: Denoised image after filtering multiple time



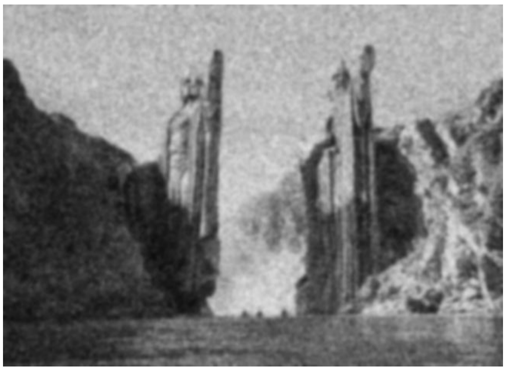
(a) Filtered 8 times

(b) Filtered 16 times

(c) Filtered 24 times

(d) Filtered 32 times

Figure 2.5: Denoised image after diffusion



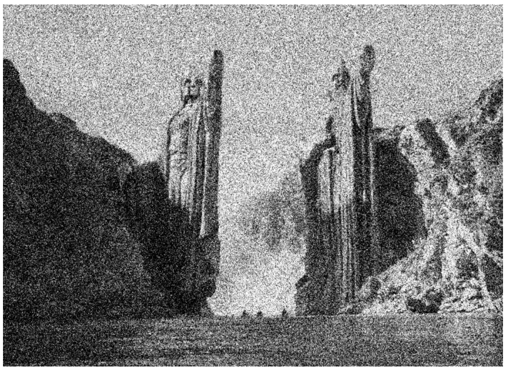
(a) Diffusion at time 25

(b) Diffusion at time 50

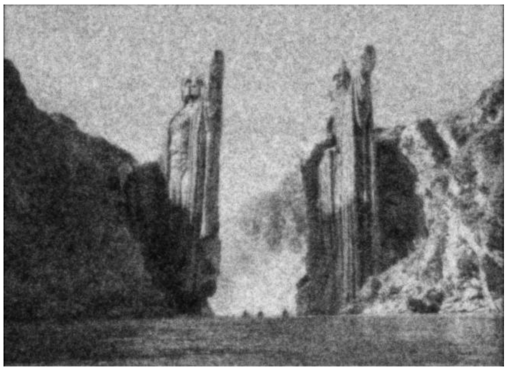
(c) Diffusion at time 75

(d) Diffusion at time 100

Figure 2.7: Denoised image with variational method



(a) Noisy image



(b) Variational denoising