Exercise 5. Rigid Transform Blending and Variational Methods

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MATLAB R2016b version was used for coding and testing:

MathWorks, MATLAB R2016b (9.1.0.441655)
64-bit (maci64)

The ***code*** directory contains the followings:

part1_1.m script .m file for exercise part 1.1.

part1_2.m script .m file for exercise part 1.2.

part2_1.m script .m file for exercise part 2.1.

part2_2.m script .m file for exercise part 2.2.

PART I provided directory for part 1.

PART II directory which contains implementation of part 2 and provided files including skeleton code etc.

img directory which contains images for testing part 2.2

result result image of part 1 and part 2.

For running each .m script, check dependencies (especially for ***part2_1.m*** and ***part2_2.m***) and adjust parameters first. Note that **these scripts only work properly in MATLAB R2016b environment** and **have done in Mac OS 10.11.6.** More details are stated in the *Running* section of each parts.

1 Exercise part 1: Understanding and Utilizing Dual Quaternion

1.1 Task 1: Thinking about fundamental properties

- How do dual quaternions represent rotations and translations?

  Unit dual quaternions naturally represent 3D rotation, when the dual part $\mathbf{q}_\epsilon = 0$ (thus $\hat{\mathbf{q}} = \mathbf{q_0} + \epsilon\mathbf{q}_\epsilon = \mathbf{q_0}$). Dual quaternion multiplication with unit dual quaternion $\hat{\mathbf{t}} = 1 + \frac{\epsilon}{2}(t_0 i + t_1 j + t_2 k)$ corresponds to translation by vector (t_0, t_1, t_2) represent 3D translation.

- What is the advantage of representing rigid transformations with dual quaternions for blending?

  The linear combination of dual quaternions does not make artifacts or skin-collapsing effect, thus blending using dual quaternions is fast and more robust than using homogeneous matrix. Moreover, since dual quaternions require only 8 floats per transformation, instead of the 12 required by matrices, they are more memory efficient.

- Briefly explain one fundamental disadvantage of using quaternion based shortest path blending for rotations as compared to linear blend skinning?

  TODO Dual quaternions causes "flipping artifacts" which occurs with joint rotations of more than 180 degrees, This is a corollary of the shortest path property: when the other path becomes shorter, the skin changes its shape discontinuously.

1.2 Task 2: Derivations and deeper understanding

- For a dual quaternion $\hat{\mathbf{q}} = \cos(\hat{\theta}/2) + \hat{\mathbf{s}}\sin(\hat{\theta}/2)$, prove that $\hat{\mathbf{q}}^t = \cos(t\hat{\theta}/2) + \hat{\mathbf{s}}\sin(t\hat{\theta}/2)$

  Starting with $\hat{\mathbf{q}}$:

$$\hat{\mathbf{q}} = \cos(\hat{\theta}/2) + \hat{\mathbf{s}}\sin(\hat{\theta}/2) \tag{1.1}$$

  As $\hat{\mathbf{q}}^t = \exp\big(t\log(\hat{\mathbf{q}})\big)$,

$$\begin{aligned}\hat{\mathbf{q}}^t &= \exp\big(t\log(\hat{\mathbf{q}})\big) && (1.2)\\ &= \exp\Big(t\log\big(\cos(\hat{\theta}/2) + \hat{\mathbf{s}}\sin(\hat{\theta}/2)\big)\Big) && (1.3)\end{aligned}$$

Plug in $\log\big(\cos(\hat{\theta}/2)+\hat{\mathbf{s}}\sin(\hat{\theta}/2)\big)=\hat{\mathbf{s}}\frac{\hat{\theta}}{2}$ to equation (1.3):

$$\hat{\mathbf{q}}^t=\exp\Big(\frac{t\hat{\theta}}{2}\hat{\mathbf{s}}\Big) \tag{1.4}$$

Let $\hat{\mathbf{a}}=\frac{t\hat{\theta}}{2}\hat{\mathbf{s}}$. Since $\hat{\theta}=\theta_0+\epsilon\theta_\epsilon$ and $\hat{\mathbf{s}}=\mathbf{s}_0+\epsilon\mathbf{s}_\epsilon$.

$$\begin{aligned}
\hat{\mathbf{a}}&=\frac{t\hat{\theta}}{2}\hat{\mathbf{s}} && (1.5)\\
&=\frac{t}{2}(\theta_0+\epsilon\theta_\epsilon)(\mathbf{s}_0+\epsilon\mathbf{s}_\epsilon) && (1.6)\\
&=\frac{t}{2}(\theta_0\mathbf{s}_0+\epsilon\theta_0\mathbf{s}_\epsilon+\epsilon\theta_\epsilon\mathbf{s}_0+\cancelto{0}{\epsilon^2\theta_\epsilon\mathbf{s}_\epsilon}) && (1.7)\\
&=\underbrace{\Big(\frac{t}{2}\theta_0\mathbf{s}_0\Big)}_{:=\mathbf{a}_0}+\epsilon\underbrace{\Big(\frac{t}{2}\theta_0\mathbf{s}_\epsilon+\frac{t}{2}\theta_\epsilon\mathbf{s}_0\Big)}_{:=\mathbf{a}_\epsilon} && (1.8)
\end{aligned}$$

Since exponential of dual quaternion is given by $e^{\hat{\mathbf{q}}}=\cos\big(\|\hat{\mathbf{q}}\|\big)+\frac{\hat{\mathbf{q}}}{\|\hat{\mathbf{q}}\|}\sin\big(\|\hat{\mathbf{q}}\|\big)$, then $e^{\hat{\mathbf{a}}}=\cos\big(\|\hat{\mathbf{a}}\|\big)+\frac{\hat{\mathbf{a}}}{\|\hat{\mathbf{a}}\|}\sin\big(\|\hat{\mathbf{a}}\|\big)$. Here, the norm of dual quaternion $\hat{\mathbf{a}}$ is $\|\hat{\mathbf{a}}\|=\|\mathbf{a}_0\|+\epsilon\frac{<\mathbf{a}_0,\mathbf{a}_\epsilon>}{\|\mathbf{a}_0\|}$. Looking into $<\mathbf{a}_0,\mathbf{a}_\epsilon>$ and $\|\mathbf{a}_0\|$,

$$\begin{aligned}
<\mathbf{a}_0,\mathbf{a}_\epsilon>&=<\frac{t}{2}\theta_0\mathbf{s}_0,\frac{t}{2}\theta_0\mathbf{s}_\epsilon+\frac{t}{2}\theta_\epsilon\mathbf{s}_0> && (1.9)\\
&=<\frac{t}{2}\theta_0\mathbf{s}_0,\frac{t}{2}\theta_0\mathbf{s}_\epsilon>+<\frac{t}{2}\theta_0\mathbf{s}_0,\frac{t}{2}\theta_\epsilon\mathbf{s}_0> && (1.10)
\end{aligned}$$

$$\begin{aligned}
\|\mathbf{a}_0\|&=\sqrt{<\mathbf{a}_0,\mathbf{a}_0>} && (1.11)\\
&=\sqrt{<\frac{t}{2}\theta_0\mathbf{s}_0,\frac{t}{2}\theta_0\mathbf{s}_0>} && (1.12)
\end{aligned}$$

Note that $<\mathbf{s}_0,\mathbf{s}_0>=1$ and $<\mathbf{s}_0,\mathbf{s}_\epsilon>=0$. Thus equation (1.10) and (1.12) can be expressed as follows:

$$<\mathbf{a}_0,\mathbf{a}_\epsilon>=\Big(\frac{t}{2}\Big)^2\theta_0\theta_\epsilon \tag{1.13}$$

$$\|\mathbf{a}_0\|=\sqrt{\Big(\frac{t}{2}\theta_0\Big)^2}=\frac{t}{2}\theta_0 \tag{1.14}$$

Plug (1.13) and (1.14) into $\|\hat{\mathbf{a}}\|=\|\mathbf{a}_0\|+\epsilon\frac{<\mathbf{a}_0,\mathbf{a}_\epsilon>}{\|\mathbf{a}_0\|}$:

$$\begin{aligned}
\|\hat{\mathbf{a}}\|&=\frac{t}{2}\theta_0+\epsilon\frac{\big(\frac{t}{2}\big)^2\theta_0\theta_\epsilon}{\frac{t}{2}\theta_0} && (1.15)\\
&=\frac{t}{2}\theta_0+\epsilon\frac{t}{2}\theta_\epsilon && (1.16)\\
&=\frac{t}{2}(\theta_0+\epsilon\theta_\epsilon)=\frac{t}{2}\hat{\theta} && (1.17)
\end{aligned}$$

Finally, by (1.4), (1.17) and $e^{\hat{\mathbf{a}}} = \cos\left(\|\hat{\mathbf{a}}\|\right) + \frac{\hat{\mathbf{a}}}{\|\hat{\mathbf{a}}\|}\sin\left(\|\hat{\mathbf{a}}\|\right)$,

$$
\begin{aligned}
\hat{\mathbf{q}}^t &= e^{\hat{\mathbf{a}}} && (1.18)\\
&= \cos\left(\frac{t}{2}\hat{\theta}\right) + \frac{\hat{\mathbf{a}}}{\frac{t}{2}\hat{\theta}}\sin\left(\frac{t}{2}\hat{\theta}\right) && (1.19)\\
&= \cos\left(\frac{t}{2}\hat{\theta}\right) + \frac{\cancel{\frac{t}{2}\hat{\theta}}\hat{\mathbf{s}}}{\cancel{\frac{t}{2}\hat{\theta}}}\sin\left(\frac{t}{2}\hat{\theta}\right) && (1.20)\\
&= \cos\left(\frac{t}{2}\hat{\theta}\right) + \hat{\mathbf{s}}\sin\left(\frac{t}{2}\hat{\theta}\right) && (1.21)
\end{aligned}
$$

The proof has been done.

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2 EXERCISE PART 2: INTERACTIVE SEGMENTATION WITH GRAPH CUT

In this section, three different denoising methods were implemented and compared:

- Filtering
- Heat diffusion
- Variational approach

2.1 TASK 1: FILTERING

2.2 TASK 2: HEAT DIFFUSION

2.3 TASK 3: VARIATIONAL APPROACH

2.3.1 DESCRIPTION

Finally, variational approach was implemented. This method considers the image as function of the space of all images. TODO

DERIVATION OF EULER-LAGRANGE EQUATION The energy function for denoising problem can be defined as follows:

$$E(I) = \int_\Omega \left[\left(I(\mathbf{x}) - I_0(\mathbf{x})\right)^2 + \lambda \left\|\nabla_\mathbf{x} I(\mathbf{x})\right\|^2\right] d\mathbf{x} \tag{2.1}$$

where, $\Omega = \mathbb{R}^2$ is the domain of the 2D image, I_0 the noisy image, and λ a regularization parameter. $I, I_0 \in \mathcal{V} = \mathcal{L}^2(\Omega)$.

Let's define the function $L(I, \nabla I, \mathbf{x})$ as follows:

$$L(I, \nabla I, \mathbf{x}) = \left[\left(I(\mathbf{x}) - I_0(\mathbf{x})\right)^2 + \lambda \left\|\nabla_\mathbf{x} I(\mathbf{x})\right\|^2\right] \tag{2.2}$$

Then the Gâteaux derivative is given by

$$
\begin{aligned}
\delta E(I;h) &= \lim_{\alpha\to 0}\frac{1}{\alpha}\Big(E(I+\alpha h)-E(I)\Big) && (2.3)\\
&= \lim_{\alpha\to 0}\frac{1}{\alpha}\int_\Omega \Big(L(I+\alpha h,\nabla I+\alpha\nabla h,\mathbf{x})-L(I,\nabla I,\mathbf{x})\Big)d\mathbf{x} && (2.4)
\end{aligned}
$$

apply Taylor expansion:

$$
\begin{aligned}
&= \lim_{\alpha\to 0}\frac{1}{\alpha}\int_\Omega \Big(\cancel{L(I,\nabla I,\mathbf{x})}+\alpha\frac{\partial L}{\partial I}h+\alpha\frac{\partial L}{\partial(\nabla I)}\nabla h+o(\alpha^2)-\cancel{L(I,\nabla I,\mathbf{x})}\Big)d\mathbf{x} && (2.5)\\
&= \int_\Omega \Big(\frac{\partial L}{\partial I}h+\frac{\partial L}{\partial(\nabla I)}\nabla h\Big)d\mathbf{x} && (2.6)
\end{aligned}
$$

apply integration by parts and $h=0$ on boundary :

$$
\begin{aligned}
&= \int_\Omega \frac{\partial L}{\partial I}h\,d\mathbf{x}+\cancelto{0}{h\frac{\partial L}{\partial(\nabla I)}\Big|_\Omega}-\int_\Omega \nabla\frac{\partial L}{\partial(\nabla I)}h\,d\mathbf{x} && (2.7)\\
&= \int_\Omega \Big(\frac{\partial L}{\partial I}-\nabla\frac{\partial L}{\partial(\nabla I)}\Big)h(\mathbf{x})\,d\mathbf{x} && (2.8)
\end{aligned}
$$

Remark following theorem:

Theorem 1. *If $\hat{u}$ is an extremum of a functional $E:\mathcal{V}\to\mathbb{R}$, then*

$$\delta E(\hat{u},h)=0 \quad \forall h\in\mathcal{V}.$$

By (2.8) and **Theorem 1**,

$$\frac{\partial L}{\partial I}-\nabla\frac{\partial L}{\partial(\nabla I)}=0 \tag{2.9}$$

Plug (2.2) into (2.9):

$$
\begin{aligned}
\frac{\partial L}{\partial I} &= 2\big(I-I_0\big) && (2.10)\\
\nabla\frac{\partial L}{\partial(\nabla I)} &= 2\lambda\nabla\cdot(\nabla I) && (2.11)\\
&= 2\lambda\,\mathbf{div}(\nabla I) && (2.12)
\end{aligned}
$$

TODO CHECK!

VECTORIZATION AND LINEAR OPERATION

2.3.2 Results

2.4 Task 4: Comparison

- How can you describe the results? Does any of these methods give better results than the others?
- What are the benefits and drawbacks of each methods?
- Can you explain the motivations behind each of the methods?

References