Solution V2.1

LST

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1 problem definition

$$\min \sum_{i=1}^{n} \frac{\nabla {f_i}^2}{1 - F_i(c_i)}$$

$$s.t. \quad \sum_{i=1}^{n} \int_{c_i}^{M} x dF_i(x) \leq B$$

where $\forall c_i, 0 \leq c_i \leq M$,and$F(0) = 0, F(M) = 1$

2 solution

First consider the unconstrained problem. If $\overline{y}$ is the local minimum of the functional $J(y)$ if y is the local niminum of $J(y)$, then it holds that $\forall \hat{y}$ in a function space V

$$\delta J|_y(\hat{y} - \overline{y}) \geq 0$$

where ,$\delta J|_y(\hat{y} - \overline{y})$is the Gateaux deravative of J in the direction of $\hat{y} - \overline{y}$.

We then consider the constraint problem. For the constraint optimization problem, we have that if y is the extremal of the constraint problem, then it is also the extremal of the augmented cost functional(Lagarangian) $J(y) + \lambda C(y)$,where the λ is the Lagarange Multiplier, and $C(y)$ is the constraint.

We come back to our problem. We first give our function space $V = \{y|y(0) = 0, y(M) = 1\}$. And we denote our cost function as

$$M(F_1, , , F_n) = \sum_{i=1}^{n} \frac{\nabla f_i^2}{1 - F_i(c_i)}.$$

Then the augmented cost function is derived as

$$J(F_1, , , F_n, \lambda) = M(F_1, , , F_n) + \lambda(\sum_{i=1}^{n} \int_{c_i}^{M} x dF_i(x) - B)$$

According to the calculation, we obtain that for $\forall \hat{F} \in V$

$$\delta J|_{F_t}(\hat{F}_t - F_t) = \int_{c_t}^{M} (-\frac{\alpha_t}{(1-F_t(c_t))^2} + \lambda x)(\hat{f}(x) - f(x))dx$$

if $\overline{F}$ is the local minimum, then we have

$$\delta J(\hat{F}_t - \overline{F_t}) \geq 0$$

holds for every $\hat{F} \in V$.Noticing that

$$\int_0^M f_t(x) - f(x)dx = 0$$

We must have

$$-\frac{\nabla f_t{}^2}{(1-F_t(c_t))^2} + \lambda x \geq 0$$

hold on every where on $[c_t, M]$ We assume that

$$-\frac{\nabla f_t{}^2}{(1-F_t(c_t))^2} + \lambda c_t = \beta$$

where $\beta \geq 0$. thus we obtain that

$$F_t(c) = \begin{cases} 1 - \frac{\nabla f_t}{\sqrt{\lambda c - \beta}} & c \in (\frac{\nabla f_t^2 + \beta}{\lambda}, M] \\ 0 & else \end{cases} \tag{1}$$

Noticing that $F(x)$ is not continuous, according to Stieltjes Integral, we rewrite the constraint as following

$$\begin{aligned}
&\sum_{t=1}^{T}(\int_{c_t}^{M} x dF_t(x)) \\
=&\sum_{t=1}^{T}(\int_{c_t}^{M} x f_i(x)dx + (1 - F_i(M)M) \\
\leq&\sum_{t=1}^{T} \nabla f_t(\frac{2}{\lambda}\sqrt{\lambda M - \beta} + \frac{c_t}{\sqrt{\lambda c_t - \beta}} - \frac{2}{\lambda}\sqrt{\lambda c_t - \beta}) \\
\leq&B
\end{aligned}$$

The Stieltjes Integral here has its practical significance. Because we assume that the cost lies between $[0, M]$, in other word, the mechanism do not accept any price higher than M, thus for all posted price c that are higher than M, the mechanism will only pay M instead of c.

In order to obtain the best f_t we can get, we need to find the K-T point of the following optmization problem

$$\min \sum_{i=1}^{n} \nabla f_t \sqrt{\lambda c_t - \beta}$$

$$s.t. \sum_{t=1}^{T} \nabla f_t (\frac{2}{\lambda}\sqrt{\lambda M - \beta} + \frac{c_t}{\sqrt{\lambda c_t - \beta}} - \frac{2}{\lambda}\sqrt{\lambda c_t - \beta}) \leq B$$

$$\beta \geq 0$$

$$\mu \geq 0$$

. The Lagrangian is thus given as follows

$$L(\mu, \beta, \lambda) = \sum_t \left(\nabla f_t \Big(\sqrt{\lambda c_t - \beta} + \mu (\frac{2}{\lambda}\sqrt{\lambda M - \beta} + \frac{c_t}{\sqrt{\lambda c_t - \beta}} - \frac{2}{\lambda}\sqrt{\lambda c_t - \beta}) \Big) \right) - \mu B \tag{2}$$

and get its gradient

$$\frac{\partial L}{\partial \lambda} = \sum_t \nabla f_t \big((\frac{c_t}{2} - \frac{\mu}{\lambda}) \frac{1}{\sqrt{\lambda c_t - \beta}} + \frac{\mu}{\lambda} \frac{1}{\sqrt{\lambda M - \beta}} - \frac{1}{2} \frac{c_t^2 \mu}{\sqrt{(\lambda c_t - \beta)^3}})$$

$$\frac{\partial L}{\partial \beta} = \sum_t \nabla f_t \big((\frac{\mu}{\lambda} - \frac{1}{2}) \frac{1}{\sqrt{\lambda c_t - \beta}} - \frac{\mu}{\lambda} \frac{1}{\sqrt{\lambda M - \beta}} + \frac{\mu}{2} \frac{c_t}{\sqrt{(\lambda c_t - \beta)^3}})$$

Plus, we can easily show that μ can not equal to 0, thus we have the equation

$$B - \sum_{t=1}^{T} \nabla f_t (\frac{2}{\lambda}\sqrt{\lambda M - \beta} + \frac{c_t}{\sqrt{\lambda c_t - \beta}} - \frac{2}{\lambda}\sqrt{\lambda c_t - \beta}) = 0$$

To solve the analytic solution of λ and β is infeasible. There are two solution here, first is to use a fixed parameter of λ and a fixed β. The second one is that we first set a fixed β, and then update λ in each time t

$$\theta_0^{(t)} = \sum_{i=1}^{t-1} \frac{\nabla f_t(h_t)}{t-1} \sqrt{M} \tag{3}$$

$$\lambda^{(t)} = \frac{T^2}{B^2 M} {\theta_0^{(t)}}^2 + \frac{\beta}{M} \tag{4}$$

3 Note

The definition of Gateaux Deravative is

$$\delta J|_y(\eta) = \lim_{\epsilon \to 0} \frac{J(y + \epsilon \eta) - J(y)}{\epsilon}$$