Solution V2.0

LST

July 14, 2016

1 problem definition

$$\min \sum_{i=1}^{n} \frac{\nabla {f_i}^2}{1 - F_i(c_i)}$$

$$s.t. \quad \sum_{i=1}^{n} \int_{c_i}^{M} x dF_i(x) \leq B$$

where $\forall c_i, 0 \leq c_i \leq M$,and$F(0) = 0, F(M) = 1$

2 solution

First consider the unconstrained problem. If $\overline{y}$ is the local minimum of the functional $J(y)$ if y is the local niminum of $J(y)$, then it holds that $\forall \hat{y}$ in a function space V

$$\delta J|_y(\hat{y} - \overline{y}) \geq 0$$

where ,$\delta J|_y(\hat{y} - \overline{y})$is the Gateaux deravative of J in the direction of $\hat{y} - \overline{y}$.

We then consider the constraint problem. For the constraint optimization problem, we have that if y is the extremal of the constraint problem, then it is also the extremal of the augmented cost functional(Lagarangian) $J(y) + \lambda C(y)$,where the λ is the Lagarange Multiplier, and $C(y)$ is the constraint.

We come back to our problem. We first give our function space $V = \{y|y(0) = 0, y(M) = 1\}$. And we denote our cost function as

$$M(F_1, , , F_n) = \sum_{i=1}^{n} \frac{\nabla f_i^2}{1 - F_i(c_i)}.$$

Then the augmented cost function is derived as

$$J(F_1, , , F_n, \lambda) = M(F_1, , , F_n) + \lambda(\sum_{i=1}^{n} \int_{c_i}^{M} x dF_i(x) - B)$$

According to the calculation, we obtain that for $\forall \hat{F} \in V$

$$\delta J|_{F_t}(\hat{F}_t - F_t) = \int_{c_t}^{M} (-\frac{\alpha_t}{(1-F_t(c_t))^2} + \lambda x)(\hat{f}(x) - f(x))dx$$

if $\overline{F}$ is the local minimum, then we have

$$\delta J(\hat{F}_t - \overline{F_t}) \geq 0$$

holds for every $\hat{F} \in V$.Noticing that

$$\int_0^M f_t(x) - f(x)dx = 0$$

We must have

$$-\frac{\nabla f_t{}^2}{(1-F_t(c_t))^2} + \lambda x \geq 0$$

hold on every where on $[c_t, M]$ We assume that

$$-\frac{\nabla f_t{}^2}{(1-F_t(c_t))^2} + \lambda c_t = \beta$$

where $\beta \geq 0$. thus we obtain that

$$F_t(c) = 1 - \frac{\nabla f_t}{\sqrt{\lambda c - \beta}} \quad c \in (0, M)$$

In order to obtain the best f_t we can get, we need to find the K-T point of the following optmization problem

$$\min \sum_{i=1}^{n} \nabla f_t \sqrt{\lambda c_t - \beta}$$

$$s.t. \nabla f_t \left[\frac{M}{\sqrt{\lambda M - \beta} + \frac{2}{3\lambda^2}\left[(\lambda M + 2\beta)\sqrt{\lambda M - \beta} - (\lambda c_t + 2\beta)\sqrt{\lambda c_t - \beta}\right]} \right] \leq B$$

$$\beta \geq 0$$

$$\mu \geq 0$$

We then write the Lagaranian of the problem

$$L = \sum_t \nabla f_t \sqrt{\lambda c_t - \beta} - \mu \sum_t \left[B - \nabla f_t \left[\frac{M}{\sqrt{\lambda M - \beta} + \frac{2}{3\lambda^2}\left[(\lambda M + 2\beta)\sqrt{\lambda M - \beta} - (\lambda c_t + 2\beta)\sqrt{\lambda c_t - \beta}\right]} \right] \right]$$

and get its gradient

$$\frac{\partial L}{\partial \lambda} =$$

$$\frac{\partial L}{\partial \beta} =$$

Noticing that $F(x)$ is not continuous, according to Stieltjes Integral, we rewrite the constraint as following

$$\sum_{i=1}^{n}(\int_{c_i}^{M} xf_i(x)dx + (1 - F_i(M)M) \leq B$$

According to the property of Lagarangian, we have

$$\frac{\partial J}{\partial \lambda} = \sum_{i=1}^{n}(\int_{c_i}^{M} xf_i(x)dx + (1 - F_i(M)M) - B = 0$$

we can solve the λ when we have the form of F_t

3 Note

The definition of Gateaux Deravative is

$$\delta J|_y(\eta) = \lim_{\epsilon \to 0} \frac{J(y + \epsilon\eta) - J(y)}{\epsilon}$$