

# Solution V1.1

LST

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## 1 problem definition

$$\begin{aligned} \min \sum_{i=1}^n \frac{\alpha_i}{1 - F_i(c_i)} \\ s.t. \quad \sum_{i=1}^n \int_{c_i}^M x dF_i(x) \leq B \end{aligned}$$

where  $\forall c_i, 0 \leq c_i \leq M$ , and  $F(0) = 0, F(M) = 1$

## 2 solution

First consider the unconstrained problem. If  $\bar{y}$  is the local minimum of the functional  $J(y)$  if  $y$  is the local minimum of  $J(y)$ , then it holds that  $\forall \hat{y}$  in a function space  $V$

$$\delta J|_y(\hat{y} - \bar{y}) \geq 0$$

where  $\delta J|_y(\hat{y} - \bar{y})$  is the Gateaux derivative of  $J$  in the direction of  $\hat{y} - \bar{y}$ .

We then consider the constraint problem. For the constraint optimization problem, we have that if  $y$  is the extremal of the constraint problem, then it is also the extremal of the augmented cost functional (Lagrangian)  $J(y) + \lambda C(y)$ , where the  $\lambda$  is the Lagrange Multiplier, and  $C(y)$  is the constraint.

We come back to our problem. We first give our function space  $V = \{y | y(0) = 0, y(M) = 1\}$ . And we denote our cost function as

$$M(F_1, \dots, F_n) = \sum_{i=1}^n \frac{\alpha_i}{1 - F_i(c_i)}.$$

Then the augmented cost function is derived as

$$J(F_1, \dots, F_n, \lambda) = M(F_1, \dots, F_n) + \lambda \left( \sum_{i=1}^n \int_{c_i}^M x dF_i(x) - B \right)$$

According to the calculation, we obtain that for  $\forall \hat{F} \in V$

$$\delta J|_{F_t}(\hat{F}_t - F_t) = \int_{c_t}^M \left( -\frac{\alpha_t}{(1 - F_t(c_t))^2} + \lambda x \right) (\hat{f}(x) - f(x)) dx$$

if  $\bar{F}$  is the local minimum, then we have

$$\delta J(\hat{F}_t - \bar{F}_t) \geq 0$$

holds for every  $\hat{F} \in V$ . Noticing that

$$\int_0^M f_t(x) - f(x) dx = 0$$

We must have

$$-\frac{\alpha_t}{(1 - F_t(c_t))^2} + \lambda x \geq 0$$

hold on every where on  $[c_t, M]$  thus we obtain that

$$F_t(c) = 1 - \sqrt{\frac{\alpha_t}{\lambda c}} \quad c \in (0, M)$$

Noticing that  $F(x)$  is not continuous, according to Stieltjes Integral, we rewrite the constraint as following

$$\sum_{i=1}^n \left( \int_{c_i}^M x f_i(x) dx + (1 - F_i(M))M \right) \leq B$$

According to the property of Lagrangian, we have

$$\frac{\partial J}{\partial \lambda} = \sum_{i=1}^n \left( \int_{c_i}^M x f_i(x) dx + (1 - F_i(M))M \right) - B = 0$$

we can solve the  $\lambda$  when we have the form of  $F_t$

### 3 Note

The definition of Gateaux Derivative is

$$\delta J|_y(\eta) = \lim_{\epsilon \rightarrow 0} \frac{J(y + \epsilon \eta) - J(y)}{\epsilon}$$