Problem 1 (Posted by dreammath)**.** Find all functions $f:\mathbb{R}^{>0}\to\mathbb{R}^{>0}$ such that

$$f(x)f(yf(x))=f(x+y)$$

holds for all reals x and y.

(Link to AoPS)

Solution 1 (by N.T.TUAN)**.**
I know a solution, but what is your solution?

Solution 2 (by andyciup)**.**
This is from IMC, 2000 : `http://www.mathlinks.ro/Forum/viewtopic.php?search_id=1113833585&t=58521`

Solution 3 (by aviateurpilot)**.**
it's easy, $f(x)=\frac{|2-x|+2-x}{(2-x)^2}\ if\ x\neq 2\ and\ f(2)=0$
or $f(x)=\frac{Max(2-x,0)}{(2-x)+[10^{-|x-2|}]}$

Solution 4 (by N.T.TUAN)**.**
aviateurpilot, can you post concrete?

Solution 5 (by pco)**.**

> Find all functions: R* to R* (R* means the set of positive reals.) $f(x)f(yf(x))=f(x+y)$. It 's very interesting for anybody who wants to solve it.

Claim 1 : $f(x)\leq 1\forall x>0$. Let $x_0>0$ such that $f(x_0)>1$. Then let $y_0=\frac{x_0}{f(x_0)-1}>0$. $f(x_0)f(y_0f(x_0))=f(x_0+y_0)\implies f(x_0)f(\frac{x_0f(x_0)}{f(x_0)-1})=f(\frac{x_0f(x_0)}{f(x_0)-1})$ and, since $f(x)>0\ \forall x:\ f(x_0)=1$, which is a contradiction. So Claim 1 is true.

Claim 2: $f(x)$ is a non-increasing function Obvious since $y>x\implies f(x)f((y-x)f(x))=f(y)$ and so $f(y)\leq f(x)$ since $f((y-x)f(x))\leq 1$ (with Claim 1)

Claim 3 : if it exists $x_0>0$ such that $f(x_0)=1$, then $f(x)=1$ for all $x>0$ Just put $x=x_0$ in the original equation and you get $f(y+x_0)=f(y)\ \forall\ y>0$. and so $f(x)$ is a constant (since it is a non increasing function (claim 2)) and so $f(x)=f(x_0)=1\ \forall\ x>$ and Claim 3 is true.

Claim 4: if $f(x)$ is non constant, then $f(x)$ is an injective function (and so is strictly decreasing) Let $f(u)=f(v)$ with $u>v$. Then $f(v)f((u-v)f(v))=f(u)$ and so $f((u-v)f(v))=1$ and so $f(x)=1$ (with claim 3). Q.E.D.

Solutions : $f(x)=1$ is a solution. Consider now $f(x)$ non is constant, that's to say $f(x)$ injective (claim 4) : Let $y=\frac{z}{f(x)}$. Then the initial equation becomes $f(x)f(z) = f(x + \frac{z}{f(x)})$. But we also have $f(z)f(x) = f(z + \frac{x}{f(z)})$ So $f(x + \frac{z}{f(x)}) = f(z + \frac{x}{f(z)})$ and, since $f(x)$ is injective : $x + \frac{z}{f(x)} = z + \frac{x}{f(z)}$ and so (dividing by xz) $\frac{1}{z} + \frac{1}{xf(x)} = \frac{1}{x} + \frac{1}{zf(z)}$.

So $\frac{1}{z} - \frac{1}{zf(z)} = \frac{1}{x} - \frac{1}{xf(x)} = a$ and $f(x) = \frac{1}{1-ax}$

And it is easy to verify that this expression is a solution of initial equation as soon as $a \leq 0$ (since we want $f(x) > 0$)

So the general solution is $f(x) = \frac{1}{1+ax}$ for any $a \geq 0$

Solution 6 (by pco).

> it's easy, $f(x) = \frac{|2-x|+2-x}{(2-x)^2}\ if\ x \neq 2\ and\ f(2) = 0$
> or $f(x) = \frac{Max(2-x,0)}{(2-x)+[10^{-|x-2|}]}$

That's wrong since $f(x) > 0\ \forall x > 0$ (f : $R^* \to R^*$)

Solution 7 (by aviateurpilot).

> it's easy, $f(x) = \frac{|2-x|+2-x}{(2-x)^2}\ if\ x \neq 2\ and\ f(2) = 0$
> or $f(x) = \frac{Max(2-x,0)}{(2-x)+[10^{-|x-2|}]}$

sorry **N.T.TUAN**, here it's solution for another problem (easy) where $f(2) = 0$ and $f(x + y) = f(x)f(yf(x)), \forall (x, y) \in (R^+)^2$ and $f(x) = 0\ in [0, 2[$:rotfl: .

Solution 8 (by N.T.TUAN).

joke :P is it a problem from IMO? pco are right!

****

Problem 2 (Posted by ehsan2004). Find all the function $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that for all $x, y \in \mathbb{R}^{\geq 0}$,

$$\sqrt[2]{f(\frac{x^2 + y^2}{2})} = \frac{f(x) + f(y)}{2}.$$

(Link to AoPS)

Solution 9 (by pco).

> Find all the function $f : R^+ \cup \{0\} \longrightarrow R^+ \cup \{0\}$ such that: $\forall x, y \geq 0$: $\sqrt[2]{f(\frac{x^2+y^2}{2})} = \frac{f(x)+f(y)}{2}$.

Let $g(x) = \sqrt{f(x)}$

We have $g(\frac{x^2+y^2}{2})=\frac{g^2(x)+g^2(y)}{2}$

And so, taking $x=y=0$: $g(0)=g^2(0)$ and either $g(0)=0$, or $g(0)=1$

1) If $g(0)=0$, taking $y=0$, we have $g(\frac{x^2}{2})=\frac{g^2(x)}{2}$ and so $g(\frac{x^2+y^2}{2})=g(\frac{x^2}{2})+g(\frac{y^2}{2})$ And so $g(x+y)=g(x)+g(y)$ (Cauchy equation) and since $g(x)\geq 0$, $g(x)=ax$ (since any non continuous solution of Cauchy's equation is unbounded on any non empty open interval). Putting back this expression in the original equation, we find $a=0$ or $a=1$

And two solutions : $f(x)=0$ $f(x)=x^2$

2) If $g(0)=1$, taking $y=0$, we have $g(\frac{x^2}{2})=\frac{g^2(x)}{2}+\frac{1}{2}$ and so $g(\frac{x^2+y^2}{2})=g(\frac{x^2}{2})+g(\frac{y^2}{2})-1$ And so $g(x+y)-1=(g(x)-1)+(g(y)-1)$ (Cauchy equation) and, since $g(x)\geq 0$, $g(x)=ax+1$ (since any non continuous solution of Cauchy's equation is unbounded on any non empty open interval). Putting back this expression in the original equation, we find $a=0$

And a third solution : $f(x)=1$

****

—————————————————————————-

Problem 3 (Posted by harazi)**.** Suppose f is a real function such that for all x,y we have $|f(x)+f(y)|=|f(x+y)|$. Then f is additive.

(Link to AoPS)

Solution 10 (by pco)**.**

> Suppose f is a real function such that for all x,y we have $|f(x)+f(y)|=|f(x+y)|$. Then f is additive.

For any pair (x,y), either $f(x+y)=f(x)+f(y)$, or $f(x+y)=-f(x)-f(y)$

It's rather immediate to see that $f(0)=0$ and that $f(-x)=-f(x)\ \forall x$.

Let then x,y such that $f(x+y)=-f(x)-f(y)$ We have either $f(x+(-2x))=f(x)+f(-2x)$, or $f(x+(-2x))=-f(x)-f(-2x)$ 1) If $f(x+(-2x))=-f(x)-f(-2x)$, it means $f(-x)=-f(x)-f(-2x)$ and since $f(-x)=-f(x)$: $f(2x)=0$ and so $f(x)=0$ 2) If $f(x+(-2x))=f(x)+f(-2x)$, then We have $f(x+y+(-2x))=\epsilon_1 f(x+y)+\epsilon_1 f(-2x)=-\epsilon_1 f(x)-\epsilon_1 f(y)+\epsilon_1 f(-2x)$ where $\epsilon_1\in\{-1,+1\}$ We also have $f(x+y+(-2x))=\epsilon_2 f(x+(-2x))+\epsilon_2 f(y)=\epsilon_2 f(x)+\epsilon_2 f(y)+\epsilon_2 f(-2x)$ where $\epsilon_2\in\{-1,+1\}$

And so, by subtracting the two equalities, we have : $(\epsilon_1+\epsilon_2)(f(x)+f(y))+(\epsilon_2-\epsilon_1)f(-2x)=0$ Then : 2.1) either $\epsilon_1=\epsilon_2$ and we have $f(x)+f(y)=0$ 2.2) or $\epsilon_1=-\epsilon_2$ and we have $f(-2x)=0$ and so $f(x)=0$

So, $f(x+y)=-f(x)-f(y)$ implies either $f(x)=0$, or $f(x)+f(y)=0$ With the same demonstration, using $f(y+(-2y))$ instead of $f(x+(-2x))$, we have : So, $f(x+y)=-f(x)-f(y)$ implies either $f(y)=0$, or $f(x)+f(y)=0$

Then $f(x+y)=-f(x)-f(y)$ implies always $f(x)+f(y)=0$ (since $f(x)=0$ and $f(y)=0$ imply too $f(x)+f(y)=0$).

So $f(x+y) = -f(x) - f(y)$ implies that always $f(x+y) = 0$ and so $f(x+y) = -f(x+y) = f(x) + f(y)$

So we always have $f(x+y) = f(x) + f(y)$ and $f(x)$ is additive.

****

Problem 4 (Posted by Cezar Lupu)**.** Prove that there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = -x$, for any $x \in \mathbb{R}$;)

(Link to AoPS)

Solution 11 (by harazi)**.**

Very well, too bad that it's already posted.

Solution 12 (by Cezar Lupu)**.**

I apologise, I didn't know that. :blush:

Solution 13 (by enescu)**.**

There are many such fuctions. However, there's an interesting geometric observation: rotating the graph of such f with $90°$ around the origin we obtain the same graph (I think this is an old russian problem). Here's an example:

****

Problem 5 (Posted by Mathx)**.** find all function $f : \mathbb{N} \to \mathbb{R}$ such that 1. $f(x+22) = f(x)$ for all $x \in \mathbb{N}$, and 2. $f(x^2y) = f(f(x))^2 \cdot f(y)$ for all $x, y \in \mathbb{N}$.

(Link to AoPS)

Solution 14 (by pco)**.**

> find all function $f : \mathbb{N} \to \mathbb{R}$ such that:
> 1.$f(x+22) = f(x)$ 2.$\forall x, y \in \mathbb{N}$ $f(x^2y) = f(f(x))^2.f(y)$

First notice that although $f(x)$ is stated $f : \mathbb{N} \to \mathbb{R}$, we need $f : \mathbb{N} \to \mathbb{N}$ in order to have $f(f(x))$ defined in the equation in point 2.

Then, notice that $f(x+22) = f(x)$ implies that $f(n)$ take at most 22 different values when $n \in \mathbb{N}$

Now, since $f(x) \in \mathbb{N}$, $f(f(x))^2 \in \mathbb{N}$. Assume that $\exists x_0 \in \mathbb{N}$ such that $f(f(x_0)) > 1$. If $x_0 = 1$, take $x_0 = 23$ (since $f(23) = f(1)$). So we can consider $x_0 > 1$. Then $f(x_0^2 y) > f(y)$, but also $f(x_0^4 y) > f(x_0^2 y) > f(y)$ and we can easily find more than 22 different values for $f(x)$, which is impossible.

So $f(f(x)) = 1$ $\forall x \in \mathbb{N}$ and $f(x^2y) = f(y)$ $\forall x, y \in \mathbb{N}$

Then let $x > y \in \mathbb{N}$: $f(22^2x) = f(x)$ $f(22^2y) = f(y)$ $f(22^2x) = f(22^2y + 22k)$ with $k = 22(x-y)$ and so $f(22^x) = f(22^2y)$ and so $f(x) = f(y)$ and so $f(x) = c$ and so $f(x) = 1$ (since $f(f(x)) = 1$)

And the only solution to these equations is $f(x) = 1$ $\forall x \in \mathbb{N}$

Solution 15 (by N.T.TUAN)**.**

This isn't a problem from APMO 2002, in APMO 2002 equational problem is http://www.mathlinks.ro/Forum/viewtopic.php?p=474887 .

Solution 16 (by noname69)**.**

Actually it is similar to APMO 2003 - Problem 7 The conditions are the same but $f(x^2y) = (f(x))^2 f(y)$

****

Problem 6 (Posted by Arne)**.** Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f^{(f(n))}(n) = n + 1, \ \forall n \in \mathbb{N},$$

where $f^{(k)}(n) = f\left(f^{(k-1)}(n)\right), \ \forall k \in \mathbb{N}$?

(Link to AoPS)

Solution 17 (by pco)**.**

> Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that
>
> $$f^{(f(n))}(n) = n + 1, \ \forall n \in \mathbb{N},$$
>
> where $f^{(k)}(n) = f\left(f^{(k-1)}(n)\right), \ \forall k \in \mathbb{N}$?

I suppose that, as generally in this forum, $0 \notin \mathbb{N}$

As a consequence, there does not exist any $n_0 \in \mathbb{N}$ such that $f(n_0) = 1$. Else we would have $n_0 + 1 = f^{f(n_0)}(n_0) = f(n_0) = 1$ and so $n_0 = 0$

1) We know that $f(1) \neq 1$. Assume then $f(1) = 2$. Then $f^{f(1)}(1) = 2$ and so $f^2(1) = 2$ and so $f(f(1)) = 2$ and $f(2) = 2$. But this is impossible since we would have then $f^k(2) = 2 \ \forall k$ and $f^{f(2)}(2) = 3$ would be impossible. So $f(1) > 2$.

2) $f(x)$ is injective : $f(a) = f(b) > 1$ implies $a + 1 = f^{f(a)}(a) = f^{f(b)}(a)$ $= f^{f(b)-1}(f(a)) = f^{f(b)-1}(f(b)) = f^{f(b)}(b) = b + 1$ and so $a = b$

3) $f^{f(1)}(1) = 2$, then $f^{f(1)+f(2)}(1) = f^{f(2)}(2) = 3$ and so on : So $\boxed{f^{\sum_{k=1}^{n} f(k)}(1) = n + 1 \ \forall n > 0}$

4) But, since $f(1) > 2$ and so $f^{\sum_{k=1}^{f(1)-1} f(k)}(1) = f(1)$ and so $f(f^{(\sum_{k=1}^{f(1)-1} f(k))-1}(1)) =$ $f(1)$ and, since $f(x)$ is injective :

$f(f^{(\sum_{k=1}^{f(1)-1} f(k))-2}(1)) = 1$ which is impossible (consider that $f(1) > 2$ implies that $(\sum_{k=1}^{f(1)-1} f(k)) - 2 > 0$)

So, such function $f(x)$ does not exist.

****

Problem 7 (Posted by Rushil)**.** For real numbers a, b, c, d not all equal to 0 , define a real function $f(x) = a + b\cos 2x + c\sin 5x + d\cos 8x$. Suppose $f(t) = 4a$ for some real t. prove that there exists a real number s s.t. $f(s) < 0$

(Link to AoPS)

Solution 18 (by pco)**.**

Hello,

Here is a not-so-simple solution

> For real numbers a, b, c, d not all equal to 0 , define a real function $f(x) = a + b\cos 2x + c\sin 5x + d\cos 8x$. Suppose $f(t) = 4a$ for some real t. prove that there exist a real number s s.t. $f(s) < 0$

I'll study $g(x) = b\cos 2x + c\sin 5x + d\cos 8x$ and show that $(P1)$:$\max_{x\in[0,2\pi]} g(x) + 3\min_{x\in[0,2\pi]} g(x) < 0\ \forall(b, c, d)$

This will solve the problem since : $f(t) = 4a$ means $g(t) = 3a \Rightarrow \max_{x\in[0,2\pi]} g(x) \geq 3a$ and, using $P1$, we then have $\min_{x\in[0,2\pi]} g(x) < -a \Rightarrow f(x_{min}) = a + \min_{x\in[0,2\pi]} g(x) < 0$

Let us now demonstrate $(P1)$. For the following of the demo, I'll say $\max_{x\in[0,2\pi]} g(x) \leq |b| + |c| + |d|$ and shall find : Either value $g(x_i)$ such that $S1 = |b|+|c|+|d|+3g(x_i) < 0$ and that will be enough (using $\min_{x\in[0,2\pi]} g(x) \leq g(x_i)$). or values $g(x_i)$ and $g(x_j)$ such that $S2 = |b| + |c| + |d| + 3\frac{g(x_i)+g(x_j)}{2} < 0$ and that will be enough too ($\min_{x\in[0,2\pi]} g(x) \leq \frac{g(x_i)+g(x_j)}{2}$).

0) First, I just want to establish some inequalities : I1) $3\sin\frac{pi}{4} - 1 > 0$ (obvious) I2) $3\cos\frac{pi}{4} - 1 > 0$ (obvious) I3) $3\sin\frac{pi}{8} - 1 > 0$ (demo : $\frac{1}{2} < \frac{49}{81} \Rightarrow \cos\frac{\pi}{4} < \frac{7}{9} \Rightarrow (\sin\frac{\pi}{8})^2 > \frac{1}{9} \Rightarrow 3\sin\frac{pi}{8} - 1 > 0$) I4) $3\cos\frac{pi}{8} - 1 > 0$ (obvious since $\cos\frac{pi}{8} > \sin\frac{pi}{8}$) I5) $\frac{3}{2}\sin\frac{\pi}{4} - 1 > 0$ (easy to check)

Notice for the following that b, c and d are not all-zero since this would imply $f(x) = a \neq 0$, and it would be impossible to find t such that $f(t) = 4a$.

1) Case : $b \geq 0$, $c \geq 0$, $d \geq 0$: With $x = \frac{3\pi}{8}$, we have $g(x) = -b\sin\frac{pi}{4} - c\sin\frac{\pi}{8} - d \Rightarrow S1 = -b(3\sin\frac{pi}{4} - 1) - c(3\sin\frac{\pi}{8} - 1) - 2d < 0$. We have $S1 < 0$ and not $S1 \leq 0$ since b, c, and d are not all-zero.

2) Case : $b \geq 0$, $c \geq 0$, $d \leq 0$: With $x = \frac{3\pi}{2}$, we have $g(x) = -b - c + d \Rightarrow S1 = -2b - 2c + 2d < 0$.

3) Case : $b \geq 0$, $c \leq 0$, $d \geq 0$: With $x = \frac{11\pi}{8}$, we have $g(x) = -b\cos\frac{pi}{4} + c\sin\frac{\pi}{8} - d \Rightarrow S1 = -b(3\sin\frac{pi}{4} - 1) + c(3\sin\frac{\pi}{8} - 1) - 2d < 0$.

4) Case : $b \geq 0$, $c \leq 0$, $d \leq 0$: With $x = \frac{\pi}{2}$, we have $g(x) = -b + c + d \Rightarrow S1 = -2b + 2c + 2d < 0$.

5) Case : $b \leq 0$, $c \geq 0$, $d \geq 0$: With $x = \frac{9\pi}{8}$, we have $g(x) = b\cos\frac{pi}{4} - c\cos\frac{\pi}{8} - d \Rightarrow S1 = b(3\cos\frac{pi}{4} - 1) - c(3\cos\frac{\pi}{8} - 1) - 2d < 0$.

6) Case : $b \leq 0$, $c \geq 0$, $d \leq 0$: With $x = 0$ and $y = \frac{\pi}{4}$, we have $\frac{g(x)+g(y)}{2} = b\frac{1}{2} - c\frac{1}{2}\sin\frac{\pi}{4} + d \Rightarrow S2 = b\frac{1}{2} - c(\frac{3}{2}\sin\frac{\pi}{4} - 1) + 2d < 0$.

7) Case : $b \leq 0$, $c \leq 0$, $d \geq 0$: With $x = \frac{\pi}{8}$, we have $g(x) = b\cos\frac{pi}{4} + c\cos\frac{\pi}{8} - d \Rightarrow S1 = b(3\cos\frac{pi}{4} - 1) + c(3\cos\frac{\pi}{8} - 1) - 2d < 0$.

8) Case : $b \leq 0$, $c \leq 0$, $d \leq 0$: With $x = 0$ and $y = \frac{5\pi}{4}$, we have $\frac{g(x)+g(y)}{2} = b\frac{1}{2} + c\frac{1}{2}\sin\frac{\pi}{4} + d \Rightarrow S2 = b\frac{1}{2} + c(\frac{3}{2}\sin\frac{\pi}{4} - 1) + 2d < 0$.

And this closes the demo (except in case of errors ... :( ) I know this demo is rather long but since this topic have been posted more than 1 year ago, I think noone have found a simpler solution ... for the moment.

– Patrick

Solution 19 (by Agr˙94˙Math)**.**

Can someone post a simpler solution for this problem? I am sure there exists some elegant and nice solution.

Great effort pco.

Solution 20 (by Ashutoshmaths)**.**

thanks pco. Any simpler solution?

Solution 21 (by joybangla)**.**

Well this one is almost trivial.I don't understand its usage in IMOTC.Anyway let $g(x) = be^{2ix} - ice^{5ix} + de^{8ix}$ see that $g(x) + g\left(x + \frac{2\pi}{3}\right) + g\left(x + \frac{4\pi}{3}\right) = g(x)\left(1 + e^{\frac{2\pi i}{3}} + e^{\frac{4\pi i}{3}}\right) = 0$ now since $f(x) = a + \Re(g(x))$ we have $f(x) + f\left(x + \frac{2\pi}{3}\right) + f\left(x + \frac{4\pi}{3}\right) = 3a....(*)$ now if $a < 0$ then $s = t$ will be enough.But if $a > 0$ then see that putting $x = t$ in $(*)$ we have $f\left(t + \frac{2\pi}{3}\right) + f\left(t + \frac{4\pi}{3}\right) = -a$ so one of $t + \frac{2\pi}{3}, t + \frac{4\pi}{3}$ works.Finally let $a = 0$ then in $(*)$ at least one term should be negative for some x otherwise we get $f \equiv 0$ but that implies $a = b = c = d = 0$ which is a contradiction.

****

Problem 8 (Posted by Arne)**.** Find all functions $f : \mathbb{N} \to \mathbb{Z}$ (where $\mathbb{N}$ is the set of positive integers) such that

$$f(ab) + f\left(a^2 + b^2\right) = f(a) + f(b),\ \forall a, b \in \mathbb{N}$$

and such that $f(a) \geq f(b)$ if $a|b$ $(\forall a, b \in \mathbb{N}.)$

(Link to AoPS)

Solution 22 (by pco)**.**

> Find all functions $f : \mathbb{N} \to \mathbb{Z}$ (where $\mathbb{N}$ is the set of positive integers) such that
>
> $$f(ab) + f\left(a^2 + b^2\right) = f(a) + f(b),\ \forall a, b \in \mathbb{N}$$
>
> and such that $f(a) \geq f(b)$ if $a|b$ $(\forall a, b \in \mathbb{N}.)$

I have some difficulties to solve this one.

Some intermediate results : Let $M = f(1)$ the greatest value of $f(x)$ and $A_0 = \{n \text{ such that } f(n) = M\}$

1) $1 \in A_0$ (since 1 divides any positive natural number)

2) $2 \in A_0$ (since $f(1 \times 1) + f(1+1) = f(1) + f(1)$ and so $f(2) = f(1)$)

3) $p \in A_0$ for any prime $p = 1 \pmod 4$: Consider that -1 is then a quadratic residue mod p and so that there exists a such that $a^2 = -1+kp$ Then $f(a)+f(a^2+1) = f(a)+f(1)$ and so $f(a^2+1) = f(kp) = f(1)$ but $f(p) \geq f(kp)$ since $p|kp$ and so $f(p) \geq f(1)$ and so $f(p) = f(1)$

4) $a, b \in A_0$ implies $ab \in A_0$ since $f(ab) < f(a) = f(b) = M$ would imply $f(a^2+b^2) > f(a) = f(b) = M$, which is impossible.

5) $a \in A_0$ implies $d \in A_0$ $\forall d|a$

6) $\gcd(a, b) = 1$ implies $a^2 + b^2 \in A_0$. It's easy to see : If prime $p|a^2 + b^2$, then $a^2 = -b^2 \neq 0 \pmod p$ and so -1 is a quadratic residu mod p. So $p = 1 \pmod 4$ and $a^2 + b^2 \in A_0$ (using points 1 and 4 above).

7) Let $P_0 = \{\text{primes } p = 1 \pmod 4\} \cup \{2\}$. Then we have $\{\prod_k p_k^{n_k}, p_k \in P_0\} \subseteq A_0$.

Some examples of $f(x)$:

$E1$) $f(x) = c$

$E2$) Let $a < b$ and let prime $p = 3 \pmod 4$. $f(x) = a$ if $p|x$ and $f(x) = b$ if not.

But I think these are not the complete set of solutions.

Solution 23 (by pco)**.**

> Find all functions $f : \mathbb{N} \to \mathbb{Z}$ (where $\mathbb{N}$ is the set of positive integers) such that
>
> $$f(ab) + f\left(a^2 + b^2\right) = f(a) + f(b),\ \forall a, b \in \mathbb{N}$$
>
> and such that $f(a) \geq f(b)$ if $a|b$ ($\forall a, b \in \mathbb{N}$.)

Here is a general solution :

If $f(x)$ is a solution, $f(x) + c$ is also a solution. So I'll study only solutions where $f(1) = 0$. Notice that since $1|n$, $f(n) \leq 0$ $\forall n$.

1) Since $f(1 \times 1) + f(1+1) = f(1) + f(1)$, $f(2) = f(1) = 0$ and $\boxed{f(2) = 0}$

2) Let n an integer such that -1 is a quadratic residue mod n. Then it exists a such that $a^2 = -1 + kn$. Then $f(a) + f(a^2+1) = f(a) + f(1)$ and so $f(a^2+1) = f(kn) = f(1) = 0$ but $f(n) \geq f(kn)$ since $n|kn$ and so $f(n) \geq f(1)$ and so $f(n) = f(1) = 0$ So $\boxed{u^2 = -1 \pmod n \implies f(n) = 0}$

3) A consequence of point 2 above is that $f(p) = 0$ for any prime $p = 1 \pmod 4$ since -1 is always quadratic residue modulus such primes.

4) If $f(a) = f(b) = 0$, then $f(ab) < f(a) = f(b) = 0$ would imply $f(a^2+b^2) > 0$, which is impossible. So $\boxed{f(a) = f(b) = 0 \implies f(ab) = 0}$

5) Let a, b such that $\gcd(a, b) = 1$, then, let p a prime divisor of $a^2 + b^2$. We have $a^2 + b^2 = 0 \pmod p$ So $a^2 = -b^2 \pmod p$. We also have $a^2 \neq 0 \pmod p$ else p would divide a and b but $\gcd(a, b) = 1$. So b have an inverse (mod p) and so $(a/b)^2 = -1 \pmod p$ and so, according to point 2 above, $f(p) = 0$. So $a^2 + b^2$ is a product of primes p_i such that $f(p_i) = 0$. And so, according to

point 4 above, $f(a^2+b^2)=0$. So, since $f(ab)+f(a^2+b^2)=f(a)+f(b)$, we can conclude that : $\boxed{\gcd(a,b)=1 \implies f(ab)=f(a)+f(b)}$

6) using $a=bc$ in the original equation, we have $f(b^2c)+f(b^2(c^2+1))=f(bc)+f(b)$. But $f(b)\geq f(b^2(c^2+1))$ and $f(bc)\geq f(b^2c)$. Hence $f(b^2c)=f(bc)$. So (taking $c=1$), $f(b^2)=f(b)$. Then (taking $c=b$), $f(b^3)=f(b^2)$. So, with an easy induction : $\boxed{f(b^k)=f(b)\ \forall k\geq 1}$

7) Using points 5 and 6 above, we have $f(\prod p_i^{n_i})=\sum f(p_i)$ where p_i are primes.

As a conclusion, we have : $f(1)=0$ $f(2)=0$ $f(p)=0$ for any prime p such that $p=1 \pmod 4$ $f(\prod p_i^{n_i})=\sum f(p_i)$ where p_i are primes.

We can now show that these necessary conditions are sufficient :

Let $f(x)$ defined as : $f(1)=0$ $f(2)=0$ $f(p)=0$ for any prime p such that $p=1 \pmod 4$ $f(p)=a_p\leq 0$ for any other prime p (where a_p is a nonpositive integer) $f(\prod p_i^{n_i})=\sum f(p_i)$ where p_i are primes.

We can show that $f(x)$ has the required properties :

Obviously $a|b$ implies $f(a)\geq f(b)$ $f(1\times 1)+f(1^2+1^2)=f(1)+f(1)=0$ $f(a\times 1)+f(a^2+1)=f(a)+f(1)$ since all prime divisors p of a^2+1 are such that $p=1 \pmod 4$. Let then $a,b>1$ Let p_i prime divisors of a not dividing b Let q_i prime divisors of b not dividing a Let r_i prime divisors of a and b $f(a)=\sum f(p_i)+\sum f(r_i)$ $f(b)=\sum f(q_i)+\sum f(r_i)$ $f(ab)=\sum f(p_i)+\sum f(q_i)+\sum f(r_i)$ $f(a^2+b^2)=\sum f(r_i)+\sum f(s_i)$ where s_i are prime divisor of $A=(\frac{a}{\gcd(a,b)})^2+(\frac{b}{\gcd(a,b)})^2$. But, as we demonstrate in point 5 above, all prime divisors of A are such that $s_i=1 \pmod 4$ (since -1 is quadratic residue mod s_i) and so $f(A)=0$ So $f(a^2+b^2)=\sum f(r_i)$

And so $f(ab)+f(a^2+b^2)=f(a)+f(b)$

And so we have the general solution : Let M be any integer. $f(1)=M$ $f(2)=M$ $f(p)=M$ for any prime p such that $p=1 \pmod 4$ $f(p)=M+a_p$ for any other prime p, where a_p is any <u>nonpositive integer</u>$f(\prod p_i^{n_i})=M+\sum(f(p_i)-M)$ where p_i are primes.

Solution 24 (by Yustas)**.**

Very nice solution, long but beautiful.

****

Problem 9 (Posted by Rushil)**.** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+y)=f(x)f(y)f(xy)$ for all $x,y\in\mathbb{R}$.

(Link to AoPS)

Solution 25 (by t̆00b5t̆00b5)**.**

First let's note that if $f(x)=0$ for some x then $f(x)=0$ for all x by translation. So let's suppose $f(x)\neq 0\forall x$ If f is a solution so is $-f$

$x=y=0: f(0)^2=1$ If f is a solution so is $-f$ so we take for the moment $f(0)=1$

Taking successively values for x, y: $1, -1; -2, 1; -1, -1$ we get that $f(1) = f(-1) = f(-2) = f(2) = 2$ Then taking $x, -x$, $x-1, 1$, we get $f(-x) = f(-x^2)$ or $f(x) = f(-x)$

Looking again $x, -x$ gives $f(x)^3 = 1$ i.e $f(x) = 1$

Only solutions are $-1, 0, 1$

Mod edit: see pco's solution http://artofproblemsolving.com/community/c6h146995p832434

Solution 26 (by rem)**.**

1. Take $x = y = 2$: $f(4) = f^2(2)f(4)$ $f^2(2) = 1$ $f(2) = -1 or 1$. 2. Take $x = y = 1$: $f(2) = f^3(1)$ $f(1) = -1 or 1$, same sign as $f(2)$. 3. Take $x = 0, y = 1$: $f(1) = f(1)f^2(0)$ $f^2(0) = 1$, so $f(0) = -1 or 1$ 4. Take $x = 1, y = -1$: $f(0) = f(1)f^2(-1)$. $f^2(-1) = \frac{f(0)}{f(1)}$. (*) 5. Take $x = y = -1$: $f(-2) = f^2(-1)f(1)$, so $f(2)$ has same sign as $f(1)$. Subst (*): $f(-2) = f(0)$. So, $f(-2) = f(0) = -1 or 1$, have same sign as $f(1)$. 6. Take $x = -2, y = 1$: $f(-1) = f(-2)^2 f(1)$ $f(-1) = f(1) = -1 or 1$. 7. Take $x = k, y = -1$ and $x = k-1, y = 1$: $f(k-1) = f^2(k)f(1)$, so $f(k-1)$ has same sign as $f(1)$ $f(k) = f^2(k-1)f(-1)$ so $f(k)$ has same sign as $f(1)$ Dividing, get: $f^3(k-1) = f^3(k)$ so $f(k) = f(k-1)$ Multiplying, get: $f(k-1)f(k) = 1$. Hence $f(k) = -1 or 1$ and $f(k)$ has same sign as $f(1)$. 8. Note that if $f(0) = 0$,then take $x = 0, y = k$ to get $f(a) = 0$. So the solutions are: $f(x) = 0 = const$ $f(x) = 1 = const$ $f(x) = -1 = const$.

Solution 27 (by sayantanchakraborty)**.**

$$f(x+y) = f(x)f(y)f(xy)$$

.....(1) We note that if f(r)=0 for some real r then f(x)=0 for all reals x, We thus work now fully on the assumption that f(r) not equals zero for any r. Taking x=a+b and y=c in (1) we get

$$f(a+b+c) = f(a+b)f(c)f(ac+bc) = f(a)f(b)f(ab)f(c)f(ac)f(bc)f(abc^2)$$

Thus we get

$$f(a+b+c) = f(a)f(b)f(c)f(ab)f(bc)f(ca)f(abc^2)$$

.....(2) Again taking x=a and y=b+c in (1) we get

$$f(a+b+c) = f(a)f(b+c)f(ab+ac) = f(a)f(b)f(c)f(bc)f(ab)f(ac)f(a^2bc)$$

Thus we get

$$f(a+b+c) = f(a)f(b)f(c)f(ab)f(bc)f(ca)f(a^2bc)$$

......(3) Comparing (2) and (3) we get

$$f(abc^2) = f(a^2bc)$$

...(4) If a,b,c are not equal to 0, then we put b=1/ac in (4) to get

$$f(c) = f(a)$$

Since a and c are arbitrary reals, it follows that f is a constant function, say

$$f(x) = c$$

for all reals x. Then substitution in (1) leads to

$$c = c^3$$

giving values c=0,1 and -1 Thus we get three solutions:

$$f(x) = 0$$

$$f(x) = -1$$

$$f(x) = 1$$

each extending over all reals x.

****

Problem 10 (Posted by silouan)**.** Prove that there is NOT existing function f:$\in R$ to R such that $f(f(x)) = -x$ for each $x \in R$

(Link to AoPS)

Solution 28 (by Christian Hirsch)**.**

Perhaps I'm missing something, but are you sure that we can't find such f :? Just divide $\mathbb{R}\backslash 0$ in disjoint pairs; for each pair $(a|b)$ define: $f(a) = b, f(b) = -a, f(-a) = -b, f(-b) = a$

Solution 29 (by silouan)**.**

I think the problem is correct maybe yu thought that any orbit was infinite but this is not true...

Solution 30 (by Tellah)**.**

take :$f(x) = x + 1$ if $x > 0$ and $[x]$ is even $-x + 1$ if $x > 0$ and $[x]$ is odd $x - 1$ if $x < 0$ and $[x]$ is even $-x - 1$ if $x < 0$ and $[x]$ is odd i think $f(f(x)) = -x$ $\forall x$;)

Solution 31 (by enescu)**.**

see [url=http://www.mathlinks.ro/Forum/viewtopic.php?t=49191]this[/url], you'll have a graph as well. You probably missed something like "continuous" or "monotonic"... Anyway, searching a bit before posting does not hurt....

****

Problem 11 (Posted by orl)**.** A function f defined on the positive integers (and taking positive integers values) is given by:

$$f(1) = 1, f(3) = 3$$
$$f(2 \cdot n) = f(n)$$
$$f(4 \cdot n + 1) = 2 \cdot f(2 \cdot n + 1) - f(n)$$
$$f(4 \cdot n + 3) = 3 \cdot f(2 \cdot n + 1) - 2 \cdot f(n),$$

for all positive integers n. Determine with proof the number of positive integers ≤ 1988 for which $f(n) = n$.

(Link to AoPS)

Solution 32 (by pco)**.**

A function f defined on the positive integers (and taking positive integers values) is given by:

$$f(1) = 1, f(3) = 3$$
$$f(2 \cdot n) = f(n)$$
$$f(4 \cdot n + 1) = 2 \cdot f(2 \cdot n + 1) - f(n)$$
$$f(4 \cdot n + 3) = 3 \cdot f(2 \cdot n + 1) - 2 \cdot f(n),$$

for all positive integers n. Determine with proof the number of positive integers ≤ 1988 for which $f(n) = n$.

Considering that $f(n) = f(2n)$, the two last equations give : $f(4n + 1) - f(4n) = 2(f(2n + 1) - f(2n))$ $f(4n + 3) - f(4n + 2) = 2(f(2n + 1) - f(2n))$

And so, if n is even and $2^{p+1} > n \geq 2^p > 1$, we have $f(n + 1) - f(n) = 2^p$

So if we have an even $n = \sum_{i=1}^{k} 2^{a_i}$, where $\{a_i\}$ is a strictly increasing sequence with $a_1 > 0$ (n even) : $f(n+1) = 2^{a_k} + f(n)$ Then $f(n) = f(\sum_{i=1}^{k} 2^{a_i})$ $= f(\sum_{i=1}^{k} 2^{a_i - a_1}) = 2^{a_k - a_1} + f(\sum_{i=2}^{k} 2^{a_i - a_1})$ And so $f((\sum_{i=1}^{k} 2^{a_i}) + 1) = 2^{a_k} + 2^{a_k - a_1} + f(\sum_{i=2}^{k} 2^{a_i - a_1})$

And it is easy to conclude that

$$f\left(\sum_{i=1}^{k} 2^{a_i}\right) = \sum_{i=1}^{k} 2^{a_k - a_i}$$

And that applying $f(n)$ means the reverse order of binary representation of n (and this could be also easily shown with induction).

So $f(n) = n$ occurs if and only if the binary representation of n is symmetrical.

It remains to count these "symetric" numbers. We have exactly $2^{\lceil \frac{m-1}{2} \rceil}$ such numbers in $[2^m, 2^{m+1})$. So : We have exactly 1 such numbers in $[1, 2)$ We have exactly 1 such numbers in $[2, 4)$ We have exactly 2 such numbers in $[4, 8)$ We have exactly 2 such numbers in $[8, 16)$ We have exactly 4 such numbers in $[16, 32)$ We have exactly 4 such numbers in $[32, 64)$ We have exactly 8 such numbers in $[64, 128)$ We have exactly 8 such numbers in $[128, 256)$ We have exactly 16 such numbers in $[256, 512)$ We have exactly 16 such numbers in $[512, 1024)$

Since $1988 = B11111000100$, positions 2 to 6 may be any between 00000 and 11101, and so : We have exactly 30 such numbers in $[1024, 1988]$

And so the requested number is $1+1+2+2+4+4+8+8+16+16+30 = 92$

Solution 33 (by muhammad-alhafi1)**.**
my solution:

****

Problem 12 (Posted by maky)**.** Find all functions $f : (0, \infty) \to (0, \infty)$ with the following properties: $f(x+1) = f(x) + 1$ and $f\left(\frac{1}{f(x)}\right) = \frac{1}{x}$.

Proposed by P. Volkmann

(Link to AoPS)

Solution 34 (by solyaris)**.**

It can be shown that $f(x) = x$ is the only function from $(0, \infty)$ to $(0, \infty)$ with the following properties : (1) $f(x+1) = f(x) + 1$ and (2) $f\left(\frac{1}{f(x)}\right) = \frac{1}{x}$.

Proof: It is clear that $f(x) = x$ works.

Now suppose that f is any function with the desired properties.

By (1) f is periodic and by (2) it follows that f is injective and surjective, i.e. bijective. We conclude that $f((n-1, n]) = (n-1, n]$ for all natural n. Again by (2) we conclude $f(1) = 1$, so $f(n) = n$ for natural n and $f(n-1, n) = (n-1, n)$.

Now we proceed by induction: Using (2) we can show that for every natural k we have the following: $f((n-1)/k, n/k) = ((n-1)/k, n/k)$ and $f(n/k) = n/k$ for all natural n.

Now it is easy to show that $f(x) = x$ for all positive real x. :)

I admit I have left out some details, but I suppose they are easy to fill in. :D

Solution 35 (by Rust)**.**

Let $g(x) = f(x) - x$. From (1), g(x) is periodic (but f is not periodic). From (2) f(x) is bijective. It gives f((0,1))=(0,1) and bijective in (0,1). f(1)=1 (f(1)¿1 and f(1)¡1 gives contradiction. From (1) $f(n) = n, f((n, n+1)) = (n, n+1)$. From (2) $f((\frac{1}{n+1}, \frac{1}{n})) = (\frac{1}{n+1}, \frac{1}{n})$, and $f((n + \frac{1}{k+1}, n + \frac{1}{k})) = (n + \frac{1}{k+1}, n + \frac{1}{k}), f(n + \frac{1}{k}) = n + \frac{1}{k}$ and

$$f((n_1 + \frac{1}{n_2 + \frac{1}{n_3}}, n_1 + \frac{1}{n_2 + \frac{1}{n_3+1}})) = (n_1 + \frac{1}{n_2 + \frac{1}{n_3}}, n_1 + \frac{1}{n_2 + \frac{1}{n_3+1}})$$

e.t.c. It gives (by continiosly rations) $f(x) = x$.

Solution 36 (by AYMANE)**.**

But f is not contenious :!:

Solution 37 (by Rust)**.**

> But f is not contenious :!:

I don't say that f is continuous, this property proved that. Let

$$x(n_0,n_1,...,n_k)=n_0+\frac{1}{n_1+\frac{1}{n_2+...+\frac{1}{n_k}}}.$$

We have $f(x(n_0,n_1,...,n_k))=x(n_0,n_1,n_2,...,n_k), n_0\geq 0, n_i\geq 1, i\geq 1$ Let $I(n_0,n_1,...,n_k)=(x(n_0,n_1,...,n_k,x(n_0,n_1,...,n_k+1))$ if k is even and $I(n_0,n_1,...,n_k)=(x(n_0,n_1,...,n_k+1),x(n_0,n_1,...,n_k))$ if k is odd. We have for any k $if\ x\in I(n_0,n_1,n_2,...,n_k)$ then $f(x)\in I(n_0,n_1,n_2,...n_k)$. It give f(x)=x.

Solution 38 (by barasawala)**.**

> Now we proceed by induction: Using (2) we can show that for every natural k we have the following: $f((n-1)/k,n/k)=((n-1)/k,n/k)$ and $f(n/k)=n/k$ for all natural n.

How to do it? I can't see.

Solution 39 (by barasawala)**.**

Can anyone show me how to do the induction?

Solution 40 (by venatrix)**.**

Hope this is correct, let's try

By induction on the first condition, it's easy to show that, for all $n\in\mathbb{N}$, $f(x+n)=f(x)+n$. The induction base is the first condition, now let's suppose that $f(x+n)=f(x)+n$ holds, consequently $f(x+n+1)=f((x+n)+1)=f(x+n)+1=f(x)+n+1$ as desired.

Now we will prove that $f(x)$ is bijective. Let $h(x)=\frac{1}{x}$. Then $\frac{1}{x}=f(h(f(x)))$, and then the conslusion follows.

Lets now define the following equality $\frac{f(x+1)}{f(x)}=\frac{1}{f(\frac{x}{x+1})}$ (*) From the first condition we've got, defining $y=\frac{1}{f(x)}$, $f(y+1)=f(y)+1\Rightarrow f(\frac{1+f(x)}{f(x)})=f(\frac{f(x+1)}{f(x)})=f(\frac{1}{f(x)})+1=\frac{x+1}{x}$. From the second we've got $f(\frac{1}{f(\frac{x}{x+1})})=\frac{x+1}{x}$. Comparing the two equalities, and recalling that $f(x)$ is injective, we have the result.

Now, let's define $g(x)=f(x)-x$, so $g(x)=g(x+1)$, so $g(x)$ is $1-$periodic, as well as $n\in\mathbb{N}-$periodic

Let's suppose that $g(x)$ is non constant. Now the following holds $g(\frac{1}{x})=g(\frac{1}{x}+1)=g(\frac{x+1}{x})\Rightarrow g(f(\frac{1}{f(x)}))=g(f(\frac{1}{f(\frac{x}{x+1})}))$ (**)

Since we supposed that $g(x)$ is non constant, this implies that $f(\frac{1}{f(x)})=f(\frac{1}{f(\frac{x}{x+1})})+n$ Using (*) we get that $f(\frac{1}{f(x)})=f(\frac{f(x+1)}{f(x)})+n=f(\frac{f(x+1)+nf(x)}{f(x)})$, where the last equality of the chain is authorized by the induction on n shown at the beginning.

Now, since $f(x)$ is bijective, we've got $\frac{1}{f(x)}=\frac{f(x+1)+nf(x)}{f(x)}\Rightarrow 1=f(x+1)+nf(x)$ (***)

Comparing (***) with the first condition, we've got $f(x+1)+nf(x) = f(x+1)-f(x)$, and so $f(x) \equiv 0$, absurd.

So $g(x)$ is constant, let's say $g(x) = k$, so $f(x) = x + k$. But even if this solutions fits the first condition, when plugged into the second, it leads to $f(\frac{1}{f(x)}) = \frac{1}{x+k} + k = \frac{1}{x}$, so $k = 0$ and finally we've got our solution, since $\boxed{f(x) = x}$

Solution 41 (by solyaris).

> Can anyone show me how to do the induction?

OK, I will give some more details. We will show that $f(x) = x$ is the only function from $(0, \infty)$ to $(0, \infty)$ with (1) $f(x+1) = f(x)+1$ and (2) $f\left(\frac{1}{f(x)}\right) = \frac{1}{x}$.

It is clear that $f(x) = x$ works. On the other hand let f have the above properties. Then by (2) it follows that f is injective and surjective, i.e. bijective. By induction over k we will show that the following holds: $f((n-1)/k, n/k) = ((n-1)/k, n/k)$ and $f(n/k) = n/k$ for all natural n.

Then we are done as this implies $f(x) = x$ for all rational x, and for irrational x this implies $f(x) \in (a, b)$ for all rationals a, b such that $x \in (a, b)$ and thus also $f(x) = x$.

For the induction we first do the case $k = 1$. By the bijectivity of f the sets $A_n := f(n, n+1] = f(0, 1] + n$ (using (1)) have to be disjoint, and the union has to be $(0, \infty)$. Thus we have $A_0 = (0, 1]$. (Indeed for $x \notin A_0$ we have $x \in A_{n+1}$ for some $n \geq 0$, so $x - 1 \in A_n$ by (1), so $x > 1$. And for $x > 1$ we have $x - 1 \in A_n$ for some $n \geq 0$, so by (1) we have $x \in A_{n+1}$, so $x \notin A_0$.) Now it suffices to show $f(1) = 1$. Then the rest of the case $k = 1$ follows from (1). $f(1) \leq 1$ follows from the above and $f(1) \geq 1$ follows from the following: Let $a = f(1)$, then by (2) we have $f(1/a) = 1$, and thus by the above $1/a \leq 1$, so $a = f(1) \geq 1$.

For the inductive step we assume the assertion is true for $1, \ldots, k-1$. By (1) it suffices to show that $f((n-1)/k, n/k) = ((n-1)/k, n/k)$ for all $1 \leq n \leq k$ and $f(n/k) = n/k$ for all natural $1 \leq n < k$. The second assertion follows from $f(n/k) = f(1/(k/n)) = f(1/f(k/n)) = 1/(k/n) = n/k$, where the second step holds by the inductive hypothesis as $n < k$, and the third step holds by (2). Likewise the first assertion follows from $f((n-1)/k, n/k) = \{f(x) : (n-1)/k < x < n/k\} = \{f(1/y) : k/n < y < k/(n-1)\} = \{f(1/f(y)) : k/n < y < k/(n-1)\} = \{1/y : k/n < y < k/(n-1)\} = ((n-1)/k, n/k)$, where in the first step we substituted $y = 1/x$, in the second step we have used the inductive hypothesis as $n, n-1 < k$, so $f(k/n, (k+1)/n) = (k/n, (k+1)/n)$ and $f((k-1)/(n-1), k/(n-1)) = ((k-1)/(n-1), k/(n-1))$, which implies $f(k/n, k/(n-1)) = (k/n, k/(n-1))$, and in the third step we have used (2).

I hope this answers you question. (And I am sorry for the delay.)

Solution 42 (by edriv).

> Since we supposed that $g(x)$ is non constant, this implies that $f(\frac{1}{f(x)}) = f(\frac{1}{f(\frac{x}{x+1})}) + n$

I think this passage is wrong.. :maybe:

Solution 43 (by reveryu)**.**

> By (1) f is periodic

> From (1) g(x) is periodic (but f is not periodic).

:what?:

Solution 44 (by polyethylene)**.**

I still do not understand why $f((0,1)) = (0,1)$. Can anyone gives an explanation please

Solution 45 (by ThE-dArK-lOrD)**.**

For each subset $S \subseteq \mathbb{R}^+$, denote $f(S) = \{f(x) \mid x \in S\}$. We say that a set S is *good*if $f(S) = S$. It's obvious that f is bijective. The proof is divided into three main steps, I'll show only the detailed proof of the first one:

1. $(0,1]$ is good. Proof: From the first condition, we can easily deduce that $f(x) \geq \lfloor x \rfloor$ for all $x \in \mathbb{R}^+$. For each $r \in (0,1]$, there exists $s = \frac{1}{f(\frac{1}{r})}$ such that $f(s) = r$. We've $\frac{1}{r} > 1 \implies f(\frac{1}{r}) \geq 1 \implies s \leq 1$. So, for each $r \in (0,1]$, there exists $s \in (0,1]$ that $f(s) = r$. Suppose there exists $s' \in (0,1]$ that $f(s') > 1$. We've proved that there must exist $r' \in (0,1]$ that $f(r') = f(s') - \lceil f(s') \rceil + 1$. By the first condition, we get $f(r' + \lceil f(s') \rceil - 1) = f(s')$. Injectivity implies $r' + \lceil f(s') \rceil - 1 = s' \in (0,1]$, clearly impossible. So, for each $s \in (0,1]$, we get $f(s) \in (0,1]$, this completes the first part.

From now on, we'll use the (infinite) continued fraction representation of positive real numbers. 2. For any positive integers n, the interval set

$$\Big([a_0, a_1, a_2, ..., a_{n-1}, a_n], [a_0, a_1, a_2, ..., a_{n-1}, a_n + 1]\Big]$$

is good for all non-negative integers a_0 and positive integers $a_1, a_2, ..., a_n$. Proof: Induction on n.

3. For all positive real numbers r and ϵ, there exists $a, b \in \mathbb{R}^+$ that $a < r < b$ and $|a - r|, |b - r| < \epsilon$ and the set (a, b) is good.

After this, if there exists $r \in \mathbb{R}^+$ that $f(r) \neq r$, there exists $\epsilon \in \mathbb{R}^+$ such that for all $a, b \in \mathbb{R}^+$ that $a < r < b$ and $|a - r|, |b - r| < \epsilon$,

$$r \in (a, b], f(r) \notin (a, b].$$

The third result gives us the contradiction.

****

—————————————————————————-

Problem 13 (Posted by N.T.TUAN)**.** Find all pairs of positive real numbers (a, b) such that for every $n \in \mathbb{N}$ and every real root x_n of the equation $4n^2x = \log_2(2n^2x + 1)$ we have $a^{x_n} + b^{x_n} \geq 2 + 3x_n$.

(Link to AoPS)

Solution 46 (by kyoshiro˙hp)**.**
In the first equation we obtain $x_n = 0$ or $x_n = \frac{-1}{4n^2}$ so we can get $ab \leq e^3$

Solution 47 (by N.T.TUAN)**.**
Can you post your solution? Concrete!

Solution 48 (by pco)**.**
Hello N.T.TUAN

> Find all pairs of positive real numbers (a, b) such that for every $n \in \mathbb{N}$ and every real root x_n of the equation $4n^2x = \log_2(2n^2x + 1)$ we have $a^{x_n} + b^{x_n} \geq 2 + 3x_n$.

For $n = 0$, every real x is root of the equation, and so, no matter what are the x_i for $i > 0$, the requirement is

What are the pairs of positive real numbers (a, b) such that for every $x \in \mathbb{R}$ $a^x + b^x \geq 2 + 3x$.

Let$f(x) = a^x + b^x$ and $g(x) = 3x + 2$ $f(x)$ is convex and $f(0) = g(0)$ So the requirement is $f'(0) = g'(0) \Rightarrow ln(a) + ln(b) = 3$

Hence $ab = e^3$

– Patrick

Solution 49 (by N.T.TUAN)**.**
But here $\mathbb{N}$ is set of all positive integer numbers.

Solution 50 (by pco)**.**

> But here $\mathbb{N}$ is set of all positive integer numbers.

You're right. I always make the same error. Here is the correction.

We have $f(x) = a^x + b^x$ and $g(x) = 3x + 2$ and we want $f(x_n) \geq g(x_n)$ For $x_n = 0$, the requirement is fulfilled.

So we want only $f(-\frac{1}{4n^2}) \geq g(-\frac{1}{4n^2})$ $\forall n \in \mathbb{N}$

Since $f(0) = g(0)$ and $f(x)$ is convex, it's immediate to see that : If $f(-\frac{1}{4n1^2}) \geq g(-\frac{1}{4n1^2})$ and $n2 < n1$ then $f(-\frac{1}{4n2^2}) \geq g(-\frac{1}{4n2^2})$

So the requirement is for the limit (when $n \to +\infty$) and so is $f'(0) \leq g'(0)$, so $ln(a) + ln(b) \leq 3$, so $ab \leq e^3$ as kyoshiro˙hp said (shortly :)).

– Patrick

Solution 51 (by N.T.TUAN).

Sorry, I don't understand here.

> Since $f(0) = g(0)$ and $f(x)$ is convex, it's immediate to see that : If $f(-\frac{1}{4n1^2}) \geq g(-\frac{1}{4n1^2})$ and $n2 < n1$ then $f(-\frac{1}{4n2^2}) \geq g(-\frac{1}{4n2^2})$

But I can have $ab \leq e^3$, by $f'(0) \leq g'(0)$. (Not using above text) . In fact, It is from

$$\frac{f(\frac{-1}{4n^2}) - f(0)}{\frac{-1}{4n^2} - 0} \leq \frac{g(\frac{-1}{4n^2}) - g(0)}{\frac{-1}{4n^2} - 0},$$

now let $n \to \infty$. Maybe I know little on convex functions. Finally, if (a, b) is answer then $ab \leq e^3$, but if $ab \leq e^3$ then is it answer? :maybe:

Solution 52 (by pco).

> Final, if (a, b) is answer then $ab \leq e^3$, but if $ab \leq e^3$ then is it answer? :maybe:

Ok, I'll try to be more precise :

$f(x) = a^x + b^x$ with $a > 0$ and $b > 0$ $g(x) = 3x + 2$

Let $x_n = \frac{-1}{4n^2}$

We have $f(x_n) = f(0) + f'(0)x_n + f''(h_n)\frac{x_n^2}{2}$ for some $h_n \in [x_n, 0]$ and $g(x_n) = g(0) + x_n g'(0)$ $f(x_n) \geq g(x_n) \Leftrightarrow f(0) + f'(0)x_n + f''(h_n)\frac{x_n^2}{2} \geq g(0) + x_n g'(0)$ and, since $f(0) = g(0)$ and $x_n < 0$: $f(x_n) \geq g(x_n) \Leftrightarrow f'(0) + f''(h_n)\frac{x_n}{2} \leq g'(0)$ and : $f(x_n) \geq g(x_n) \Leftrightarrow f'(0) - g'(0) \leq f''(h_n)\frac{-x_n}{2}$

Then, and since $f''(h_n) > 0$ (f is convex) : 1) $f(x_n) \geq g(x_n) \Rightarrow f'(0) - g'(0) \leq f''(h_n)\frac{-x_n}{2}$ $\forall n \in \mathbb{N} \Rightarrow f'(0) - g'(0) \leq 0$ and $f'(0) - g'(0) \leq 0$ is a necessary condition 2) $f'(0) - g'(0) \leq 0 \Rightarrow f'(0) - g'(0) \leq 0 \leq f''(h_n)\frac{-x_n}{2}$ $\forall n \in \mathbb{N} \Rightarrow f(x_n) \geq g(x_n)$ and $f'(0) - g'(0) \leq 0$ is a sufficient condition

So $ab \leq e^3$ is a necessary condition AND a sufficient condition. I hope this would be clear enough (even with my poor english language) :)

– Patrick

Solution 53 (by N.T.TUAN).

Thank you very much! But this

> $f'(0) - g'(0) \leq f''(h_n)\frac{-x_n}{2}$ $\forall n \in \mathbb{N} \Rightarrow f'(0) - g'(0) \leq 0$

Why?

Solution 54 (by pco).

> Thank you very much! But this
>
> > $f'(0) - g'(0) \le f''(h_n)\frac{-x_n}{2}\ \forall n \in \mathbb{N} \Rightarrow f'(0) - g'(0) \le 0$
>
> Why?

You just have to do $n \to \infty$ in the inequality. Then $x_n \to 0$, $h_n \to 0$ and $f''(h_n)\frac{-x_n}{2} \to 0$.

– Patrick

Solution 55 (by N.T.TUAN)**.**

Ok!

> Find all pairs of positive real numbers (a, b) such that for every $n \in \mathbb{N}$ and every real root x_n of the equation $4n^2x = \log_2(2n^2x + 1)$ we have $a^{x_n} + b^{x_n} \ge 2 + 3x_n$.

I will post solution i know. First, we need solve the equation $4n^2x = \log_2(2n^2x + 1)$. Condition $x > \frac{-1}{2n^2}$. This equation is equivalent to $2^{4n^2x} = 2n^2x + 1$. Put $t = 4n^2x + 1$ then $2^t = t + 1$, this equation is equivalent to $t \in \{0, 1\}$ because $f(0) = f(1) = 0$ and $f''(t) > 0 \forall t$, here $f(t) = 2^t - t - 1$. Finally $4n^2x = \log_2(2n^2x + 1)$ has roots 0 and $\frac{-1}{4n^2}$.

If (a, b) satisfy then $a^{\frac{-1}{4n^2}} + b^{\frac{-1}{4n^2}} \ge 2 + 3(\frac{-1}{4n^2})\forall n$ or $\frac{1}{2}(a^{\frac{-1}{4n^2}} + b^{\frac{-1}{4n^2}}) \ge 1 + \frac{3}{2}(\frac{-1}{4n^2})\forall n$ or $(\frac{a^{\frac{-1}{4n^2}} + b^{\frac{-1}{4n^2}}}{2})^{\frac{-1}{4n^2}} \ge (2 + 3(\frac{-1}{4n^2}))^{\frac{-1}{4n^2}}\forall n$, here let $n \to \infty$ we have $\sqrt{ab} \le e^{\frac{3}{2}}$.

If (a, b) satisfies $\sqrt{ab} \le e^{\frac{3}{2}}$ then $\frac{1}{2}(a^{\frac{-1}{4n^2}} + b^{\frac{-1}{4n^2}}) \ge (\sqrt{ab})^{\frac{-1}{4n^2}} \ge e^{\frac{3}{2} \cdot \frac{-1}{4n^2}} \ge 1 + \frac{3}{2}(\frac{-1}{4n^2})\forall n$ and $a^0 + b^0 = 2 = 2 + 3.0$.

Answer $\sqrt{ab} \le e^{\frac{3}{2}}$.

Solution 56 (by pco)**.**

Hello ! Nice demo. Just a little error :

> If (a, b) satisfy then $a^{\frac{-1}{4n^2}} + b^{\frac{-1}{4n^2}} \ge 2 + 3(\frac{-1}{4n^2})\forall n$ or $\frac{1}{2}(a^{\frac{-1}{4n^2}} + b^{\frac{-1}{4n^2}}) \ge 1 + \frac{3}{2}(\frac{-1}{4n^2})\forall n$ or $(\frac{a^{\frac{-1}{4n^2}} + b^{\frac{-1}{4n^2}}}{2})^{\frac{-1}{4n^2}} \ge (2 + 3(\frac{-1}{4n^2}))^{\frac{-1}{4n^2}}\forall n$, here let $n \to \infty$ we have $\sqrt{ab} \le e^{\frac{3}{2}}$

I think $\frac{1}{2}(a^{\frac{-1}{4n^2}} + b^{\frac{-1}{4n^2}}) \ge 1 + \frac{3}{2}(\frac{-1}{4n^2})\forall n$ implies $(\frac{a^{\frac{-1}{4n^2}} + b^{\frac{-1}{4n^2}}}{2})^{\frac{-1}{4n^2}} \le (1 + \frac{3}{2}(\frac{-1}{4n^2}))^{\frac{-1}{4n^2}}\forall n$ (and not "$\ge$")

Then, when $n \to +\infty$, LHS and RHS $\to 1$ and I don't see how you conclude $\sqrt{ab} \le e^{\frac{3}{2}}$ The mistake is in the exponent : You need to use $(-4n^2)$ and not $\frac{-1}{4n^2}$. Then :

If (a,b) satisfy then $a^{\frac{-1}{4n^2}}+b^{\frac{-1}{4n^2}}\geq 2+3(\frac{-1}{4n^2})\forall n$ or $\frac{1}{2}(a^{\frac{-1}{4n^2}}+b^{\frac{-1}{4n^2}})\geq 1+\frac{3}{2}(\frac{-1}{4n^2})\forall n$ or $(\frac{a^{\frac{-1}{4n^2}}+b^{\frac{-1}{4n^2}}}{2})^{-4n^2}\leq(1+\frac{3}{2}(\frac{-1}{4n^2}))^{-4n^2}\forall n$, here let $n\to\infty$ we have $\sqrt{ab}\leq e^{\frac{3}{2}}$

I think your demo is quicker than mine.

– Patrick

Solution 57 (by N.T.TUAN)**.**

Sorry :blush: Thanks! Now this topic has got the two solutions

****

Problem 14 (Posted by anuj kumar)**.** Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that

$$f(m^2+f(n))=f(m)^2+n,$$

for all $m,n\in\mathbb{N}$.

(Link to AoPS)

Solution 58 (by pco)**.**

Hello anuj kumar!

> find all f:N-¿N such that
>
> $$f(m^2+f(n))=f(m)^2+n.\ m, nbelongtoN$$

Let the proposal P(m,n) be : $f(m^2+f(n))=f(m)^2+n$

1) $f(x)$ is injective $f(a)=f(b)$ ==¿ $f(m^2+f(a))=f(m^2+f(b))\Rightarrow f(m)^2+a=f(m)^2+b\Rightarrow a=b$ Q.E.D.

2) f(0) = 0 Let $f(0)=a>0$ Then $P(0,n)\Rightarrow f(f(n))=n+a^2\Rightarrow f(n+a^2)=f(n)+a^2\Rightarrow f(n)=n+f(mod(n,a^2))-mod(n,a^2))=n+d(n)$ with $d(n)$ bounded Then $f(m^2+f(n))=f(m^2+n+d(n))=m^2+n+d(n)+d(m^2+n+d(n))$ and $f(m)^2+n=(m+d(m))^2+n=m^2+n+2md(m)+d(m)^2$ And $d(n)+d(m^2+n+d(n))=2md(m)+d(m)^2$ But L.H.S is bounded and R.H.S is not (except if $d(m)=0\ \forall m$, which implies $f(n)=n$ which is in contradiction with $f(0)=a>0$ so $f(0)=0$ Q.E.D.

3)f(n) is bijective $P(0,n)\Rightarrow f(f(n))=n\Rightarrow$ f is surjective $\Rightarrow$ f is bijective (with point 1)) Q.E.D.

4) $f(n)=n$ is the only solution $P(n,0)\Rightarrow f(n^2)=(f(n))^2$. Since f is bijective, for a given n I can find an integer p ¿ 0 such that $f(p)=2n+1$ Then $P(n,p)\Rightarrow f(n^2+2n+1)=(f(n))^2+p\Rightarrow f((n+1)^2)=(f(n))^2+p$ $\Rightarrow(f(n+1))^2=(f(n))^2+p\Rightarrow f(n+1)>f(n)$ So f is a strictly increasing bijection from N in N ==¿ f(n)=n Q.E.D.

I'm afraid my demonstration is quite long and I think there exists some simpler one.

– Patrick

Solution 59 (by me@home).

Sorry, I'm not sure what you mean by $d(n)$ is bounded? Do you mean linear?

Solution 60 (by pco).

> Sorry, I'm not sure what you mean by $d(n)$ is bounded? Do you mean linear?

No I mean it exist reals A and B such that $A < d(n) < B \ \forall n$

And this is easy to see since $d(n) = f(mod(n, a^2)) - mod(n, a^2)$: d(n) can only take at most a^2 different values ==¿ a finite set of values is always "bounded" (I dont know if it is the good English word : French one is "born")

– Patrick

Solution 61 (by pardesi).

> find all f:N-¿N such that
> $$f(m^2 + f(n)) = f(m)^2 + n.\ m, nbelongtoN$$

> f(0) = 0

Since the function is only defined for natural no.s how can you prove that $f(0) = 0$

OK pco i read your other post so it's the problem with your country but you should seriously like to revisit the proof.

Solution 62 (by lovejrz).

what's QED mean?

Solution 63 (by pardesi).

> what's QED mean?

http://en.wikipedia.org/wiki/Q.E.D.

Solution 64 (by pco).

> ok pco i read your other post so it's the problem with your country but you should seriously like to revisit the proof.

OK, let's go :

1) f is injective : $f(a) = f(b) \Rightarrow f(m + f(a)) = f(m + f(b)) \Rightarrow f(m)^2 + a = f(m)^2 + b \Rightarrow a = b$ Q.E.D.

2) f is strictly increasing : If $f(a) > f(b)$, then $f(a)^2 + n = f(b)^2 + (n + f(a)^2 - f(b)^2)$ and then $f(a^2 + f(n)) = f(b^2 + f(n + f(a)^2 - f(b)^2))$ and, since f is injective

: $a^2 + f(n) = b^2 + f(n + f(a)^2 - f(b)^2)$ which implies $f(n + f(a)^2 - f(b)^2) = f(n) + (a^2 - b^2) \Rightarrow f(n + k(f(a)^2 - f(b)^2)) = f(n) + k(a^2 - b^2) \Rightarrow a > b$. $f(a) > f(b) \Rightarrow a > b$ shows that f is monotonously increasing. Hence, since f is also injective, f is strictly increasing. QED

3) $f(n) = n$ Since f is strictly increasing, $f(a) = f(b) + 1 \Rightarrow a = b + 1$ But we have $f(m^2 + f(n+1)) = f(m)^2 + n + 1 = f(m^2 + f(n)) + 1$ and so $m^2+f(n+1) = m^2+f(n)+1$ and $f(n+1) = f(n)+1$ So $f(n) = n+f(1)-1$ When entering this expression in the functional equation, we conclude that $f(1) = 1$ and $f(n) = n$ Q.E.D.

– Patrick

Solution 65 (by pardesi)**.**

:coolspeak: though i haven't gone through the proof (in no mood to do so :sleep2:)

****
——————————————————————————-

Problem 15 (Posted by spix)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\left|\sum_{k=1}^{n} 2^k (f(x+ky) - f(x-ky))\right| \leq 1,$$

for all integers $n \geq 0$ and all $x, y \in \mathbb{R}$.

(Link to AoPS)

Solution 66 (by pco)**.**

Hello spix !

> Find all $f : R \to R$ such that: $\left|\sum_{k=1}^{n} 2^k (f(x+ky) - f(x-ky))\right| \leq 1, \forall n \in N^\star, \forall x, y \in R$

Let $A = \sum_{k=1}^{n} 2^k (f(x+ky) - f(x-ky))$

We have $|A| \leq 1$

Let $B = \sum_{k=1}^{n+1} 2^k (f(x+ky) - f(x-ky)) = A + 2^{n+1}(f(x+(n+1)y) - f(x-(n+1)y))$

We have $|B| \leq 1$, so $\left|A + 2^{n+1}(f(x+(n+1)y) - f(x-(n+1)y))\right| \leq 1$

But : $\left|A + 2^{n+1}(f(x+(n+1)y) - f(x-(n+1)y))\right| \geq \left|2^{n+1}(f(x+(n+1)y) - f(x-(n+1)y))\right| - |A|$

So : $1 \geq \left|2^{n+1}(f(x+(n+1)y) - f(x-(n+1)y))\right| - |A|$

So : $1 + |A| \geq \left|2^{n+1}(f(x+(n+1)y) - f(x-(n+1)y))\right|$

So : $\left|2^{n+1}(f(x+(n+1)y) - f(x-(n+1)y))\right| \leq 2$

And : $|f(x+(n+1)y) - f(x-(n+1)y)| \leq 2^{-n}, \forall n \in N^\star, \forall x, y \in R$

Let a and b two reals.

Let $x = \frac{a+b}{2}$ and $y = \frac{b-a}{2(n+1)}$

Then $|f(x+(n+1)y) - f(x-(n+1)y)| \le 2^{-n}$ becomes $|f(b) - f(a)| \le 2^{-n}, \forall n \in N^\star, \forall a, b \in R$

And f is any constant function.

– Patrick

****
———————————————————————————————————-

Problem 16 (Posted by pohoatza)**.** Let a be a real number and let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(0) = \frac{1}{2}$ and

$$f(x+y) = f(x)f(a-y) + f(y)f(a-x), \quad \forall x, y \in \mathbb{R}.$$

Prove that f is constant.

(Link to AoPS)

Solution 67 (by scorpius119)**.**

First substitute $x = y = 0$ to get $f(a) = \frac{1}{2}$. Then substituting $y = 0$, we get

$$f(x) = \frac{f(x)}{2} + \frac{f(a-x)}{2} \Rightarrow f(x) = f(a-x)$$

This turns the original functional equation into $f(x+y) = 2f(x)f(y)$. In particular, $x = y$ gives

$$f(x) = 2f(x)^2 \Rightarrow f(x)(2f(x) - 1) = 0$$

We must show $f(x) = \frac{1}{2}$ for all x, so suppose otherwise: that there is some b such that $f(b) = 0$. If this were to happen, substituting $x = -b, y = b$ into $f(x+y) = 2f(x)f(y)$ gives $\frac{1}{2} = 0$ which is BAD! Therefore f is the constant 1/2.

Solution 68 (by Jan)**.**

I could be wrong, but I think you made a mistake scorpius:

> This turns the original functional equation into $f(x+y) = 2f(x)f(y)$. In particular, $x = y$ gives
>
> $$f(x) = 2f(x)^2 \Rightarrow f(x)(2f(x) - 1) = 0$$

You only know that $f(2x) = 2f(x)^2$

Solution 69 (by pohoatza)**.**

> First substitute $x = y = 0$ to get $f(a) = \frac{1}{2}$. Then substituting $y = 0$, we get
>
> $$f(x) = \frac{f(x)}{2} + \frac{f(a-x)}{2} \Rightarrow f(x) = f(a-x)$$

Continuing from here, just take $y = a - x$, therefore $f(0) = f^2(x) + f^2(a-x)$, so $f(x) = \frac{1}{2}$ or $-\frac{1}{2}$. But now we have $f(\frac{x}{2}) = \frac{1}{2}$ or $-\frac{1}{2}$ and $f(\frac{a-x}{2}) = f(\frac{x}{2})$. Hence $f(x) = f(\frac{x}{2} + \frac{x}{2}) = 2f(x)f(a - \frac{x}{2}) = \frac{1}{2}$.

Solution 70 (by me@home)**.**

Okay, first I see an easy way why $f(a) = f^2(x) + f^2(a-x)$ follows, which then shows that $f(0) = \frac{a}{2} = f(a) = ...$ so why did you write $f(0)$?... I guess that doesn't really matter Next, what does $f\left(\frac{a-x}{2}\right) = f\left(\frac{x}{2}\right)$ follows from? Because

$$\frac{1}{2} = f\left(2 \cdot \frac{a}{2}\right) = 2f\left(\frac{a}{2}\right)^2$$

Also $f(x) = f(y) \implies 2f(x)^2 = 2f(y)^2 \implies f(2x) = f(2y)$ but from the other direction, I think it can only show that $f(x) = f(y) \implies f\left(\frac{x}{2}\right) = \pm f\left(\frac{y}{2}\right)$... can you please explain your logic behind the steps because it seems like there are a couple mistakes or just sloppiness

Solution 71 (by pco)**.**

Hello you at home !

> ... can you please explain your logic behind the steps because it seems like there are a couple mistakes or just sloppiness

Let $P(x, y)$ be the proposal $f(x + y) = f(x)f(a - y) + f(y)f(a - x)$
$P(0, a)$ gives $f(a) = \frac{1}{2}$
$P(x, 0)$ gives then $f(a - x) = f(x)$
Replacing then $f(a-x)$ by $f(x)$ and $f(a-y)$ by $f(y)$ in $P(x, y)$, we have $Q(x, y)$: $f(x + y) = 2f(x)f(y)$ $Q(\frac{x}{2}, \frac{x}{2})$ gives $f(x) \geq 0$ $\forall$ x and $Q(x, -x)$ shows that $f(x) \neq 0$ and $f(-x) = \frac{1}{4f(x)}$ $Q(a, -x)$ gives $f(a - x) = f(-x) = \frac{1}{4f(x)}$ but $f(a - x) = f(x)$ and so $\frac{1}{4f(x)} = f(x) \Rightarrow f(x)^2 = \frac{1}{4}$ Since we have shown that $f(x) \geq 0$, we can conclude that $f(x) = \frac{1}{2}$ $\forall$ $x \in \mathbb{R}$

Is it rather clear ?

– Patrick

Solution 72 (by efang)**.**

crastybow and I did this last night which was the night right after the all-nighter of AMSP i.e. in 40 hours we each had a total of 2 hours of sleep so we weren't able to finish the problem but this morning after 12 hours of sleep I solved it.

Anyways

crastybow part:

Substitute $x = y = 0$ to see that $f(0) = f(a) \to f(a) = \frac{1}{2}$

Now substitute $x = y = a$ (which we thought to be the crux move in our tired state) Notice that then $f(2a) = f(a)f(0) + f(a)f(0) = f(a) = \frac{1}{2}$

Yesterday night we made the mistake of assuming that from here we could assume that $f(a) = f(2a) = f(4a)$.... which, although is true, we didn't actually prove it and when you sub $x = y = 2a$ you don't obtain $f(4a) = f(2a)$ but also that it doesn't cover when $a = 0$ (we noticed that it didn't cover the $a = 0$ part last night which is why crastybow didn't post the solution)

And then I finished the rest this morning (no more random talk)

I first noticed that

$f(4a) = f(a + 3a) = f(a)f(0) + f(3a)f(-2a) = f(a - 2a) = f(a)f(0) + f(-2a)f(3a)$

Perhaps there's a generalization...

$f(a + ba) = f(a)f(0) + f(ba)f(a - ba) = f(a)f(0) + f(a - ba)f(ba) = f(2a - ba) \to f((b+1)a) = f((2-b)a)$

From here we conclude that

$f(a) = f(2a) = f(0) = f(3a) = f(-a) = f(4a) = f(-2a) = f(5a)\dots$

So this means that the function $f(x, y) - \frac{1}{2} = 0$ has infinite roots $\implies f(x, y)$ is a constant polynomial.

That is of course if $a \neq 0$.

If a=0 then claiming that $f(a) = f(2a) = f(0)\dots$ would be claiming that $f(0) = f(0) = f(0)\dots$

So now we solve for when $a = 0$. We notice that if $a = 0$

$f(x + y) = f(x)f(-y) + f(-x)f(y) = f(-(x + y))$

Now plug in $x = -y$

$f(0) = f(x)f(y)+f(x)f(y) = 2f(x)f(y) = 2f(x)f(-x) = 2f(x)f(x) = \frac{1}{4} \to f(x) = \pm\frac{1}{2}$

We already knew that $f(0) = \frac{1}{2}$ so this means $f(x) = \frac{1}{2}$ for all x.

Therefore the function of f must always be constant.

EDIT:

you know what screw it. I don't think it's ok just to say $f(0) = \frac{1}{2}$ so $f(x) = \frac{1}{2}$

Let me try to prove that $f(x) > 0$ for all x which would be a more satisfactory finish

$f(\frac{x}{2} + \frac{x}{2}) = 2f(\frac{x}{2})f(-\frac{x}{2}) = 2f(\frac{x}{2})^2$ which is always positive.

So $f(x) > 0 \to f(x) = \frac{1}{2}$ so f is constant.

$\square$.

****

Problem 17 (Posted by pohoatza)**.** Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + f(y)) = f(x) + \sin y$, for all reals x, y ?

(Link to AoPS)

Solution 73 (by pco)**.**

Hello pohoatza,

> Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+f(y)) = f(x)+\sin y$, for all reals x, y ?

1) $f(x)$ is surjective :

1.1) : $f(x+f(y)) = f(x)+sin(y) \to f(f(x)) = f(0)+sin(x) ==> f(R) contains [f(0)-$

$1, f(0)+1]$
$1.2): \ f(x+f(y)) = f(x)+sin(y) \rightarrow f(x+nf(y)) = f(x)+n*sin(y) \rightarrow f(x+nf(pi/2)) = f(x)+nandf(x+nf(-pi/2)) = f(x)-n$

So $f(R)$ contains one interval of width 2 (see 1.1) and any translation by $\pm$ n of such an interval (see 1.2) $\rightarrow f(R)=R$ Q.E.D.

2) f is bounded
f is surjective $\rightarrow$ it exists $x0$ such that $f(x0)=0$ f is surjective $\rightarrow$ for any x, it exists z such that $f(z)=x-x0$ Then :
f(x) = f(x0 + f(z)) = f(x0) + sin(z) = sin(z) ==¿ f(x) is in [-1,+1] for any x Q.E.D.

3) f doesn't exist
It's obvious since 1) and 2) are in contradiction.

– Patrick

Solution 74 (by Erken)**.**

substitute 0 instead of x into the main equation:

$f(f(x)) = f(0)+\sin x$

so $f(f(x)) \in [f(0)-1, f(0)+1]$

It means that a boundary of all possible values of $f(f(x))$ is limited, consequently, the same claim should hold for $f(x)$.

so if there is some x such that $f(x)=y$, then:

$f(x+f(z)) = y+\sin z$, so for every $y_0 \in [y-1, y+1]$ there should exist x_0 such that $f(x_0)=y_0$, it is enough to take $x_0 = x+f(z_0)$, where $\sin z_0 = y_0 - y$.

So the image of f can be infinitely extended, thus the boundary of its values is unlimited. Causing a Contradiction.

Solution 75 (by sansae)**.**

> It means that a boundary of all possible values of $f(f(x))$ is limited, consequently, the same claim should hold for $f(x)$.

I think this is not right... is it? If this statement is true, why?

Solution 76 (by pco)**.**

Clearly wrong :

Let $f(x)=|x|-x$ so that $f(f(x)) \equiv 0$

We have an example of unbounded continuous function $f(x)$ such that $f(f(x))$ is bounded.

****

Problem 18 (Posted by stergiu)**.** Find all real functions f defined on $\mathbb{R}$, such that

$$f(f(x)+y) = f(f(x)-y)+4f(x)y,$$

for all real numbers x, y.

(Link to AoPS)

Solution 77 (by Rust).

$y = f(x)$ gives $f(2y) = f(0) + (2y)^2$.
$y = -f(x)$ give $f(-2y) = f(0) + (2y)^2$.
It is easy to check, that $f(x) = x^2 + c$ is the solution.
As proof, if that equation had no other solution sufficient enough to show, for any y exists x, such that $|y| = |f(x)|$.

Solution 78 (by maky).

i don't think that the surjectivity of f is very simple. in fact, i think it's harder than the "offical" solution.

Solution 79 (by pco).

Hello,

$IM(f)$ is a serious problem.

For example, $f(x) = 0$ is a solution (different from $x^2 + c$)

– Patrick

Solution 80 (by Anto).

The solution that I like is as follows.

Let $g(x) = f(x) - x^2$, then the functional relation becomes $g(g(x) + x^2 + y) = g(g(x) + x^2 - y)$ for all real x, y. Then $g(g(x) + x^2 - g(y) - y^2 + z) = g(g(x) + x^2 + g(y) + y^2 - z) = g(g(y) + y^2 - g(x) - x^2 + z)$, thus $g(z) = g(2 * (g(x) + x^2 - g(y) - y^2) + z)$ for all real x, y, z. If $g(x) + x^2$ is constant, then we get the solution $f(x) = 0$ for all real x.
If $g(x) + x^2$ is not constant, then g is periodic, say with period T. Then from the last equation we put $x = y + T$ to get $g(z) = g(2 * (2 * y * T + T^2) + z)$ for all real y and z. But we can choose y at our will to get $g(z) = g(0)$ for all real z and hence $f(x) = x^2 + g(0)$ for all real x.

Solution 81 (by maky).

why is g periodic ?

Solution 82 (by Persu Madalina).

If $g(x) + x^2$ is not constant then there exists a different from b such that $a = g(x) + x^2$ and $b = g(z) + z^2$ for some x and z.

$g(a + y) = g(a - y)$ (*) and $g(b + y) = g(b - y)$ for all y.
Substitute y by $a - y - b$ in the second relation and you obtain $g(b + (a - y - b)) = g(b - (a - y - b)) \iff g(a - y) = g(y + 2 * b - a)$, but from (*), $g(a - y) = g(y + a) \implies g(y + a) = g(y + 2 * b - a)$. If we take $z = y + a$ we will get $g(z) = g(z + 2 * (b - a))$ and because z undergoes $\mathbb{R} \implies g$ is periodic of period $2 * (b - a)$.

Solution 83 (by pco).

Hello Maky!

> why is g periodic ?

Because he has $g(z) = g(2(g(x) + x^2 - g(y) - y^2) + z)$ for all real x,y,z So, if $g(x) + x^2$ is not a constant, it exists x and y such that $g(x) + x^2 \neq g(y) + y^2$ Then, if I call $T = 2(g(x) + x^2 - g(y) - y^2)$, $T \neq 0$ and for any real z, $g(z) = g(2(g(x) + x^2 - g(y) - y^2) + z)$ ==¿ $g(z) = g(z + T)$

And g is periodic.

– Patrick

Solution 84 (by behemont)**.**

> The solution that i like is as follows :
> Let $g(x) = f(x) - x^2$, then the functional relation becomes $g(g(x) + x^2 + y) = g(g(x) + x^2 - y)$ for all real x, y. Then
>
> $$g(g(x)+x^2-g(y)-y^2+z) = g(g(x)+x^2+g(y)+y^2-z) = g(g(y)+y^2-g(x)-x^2+z)$$
>
> and thus $g(z) = g(2 * (g(x) + x^2 - g(y) - y^2) + z)$ for all real x, y, z. If $g(x) + x^2$ is constant then we get the solution $f(x) = 0$ for all real x. If $g(x)+x^2$ is not constant then g is periodic say with period T. Then from the last equation we put $x = y+T$ to get $g(z) = g(2*(2*y*T+T^2)+z)$ for all real y and z. But we can choose y at our will to get $g(z) = g(0)$ for all real z and hence $f(x) = x^2 + g(0)$ for all real x.

can someone turn this into $LaTeX$ and explain a bit?

Solution 85 (by maky)**.**

all the solutions i know (including mine, in the contest) use the idea that $\mathrm{Im}\, f - \mathrm{Im}\, f = \mathbb{R}$. after that, it's pure algebraic manipulations.

Solution 86 (by behemont)**.**

> all the solutions i know (including mine, in the contest) use the idea that $\mathrm{Im}\, f - \mathrm{Im}\, f = \mathbb{R}$. after that, it's pure algebraic manipulations.

can you show that?

Solution 87 (by idioteque)**.**

What is IR? :blush:

Solution 88 (by me@home)**.**

The solution that i like is as follows :
Let $g(x) = f(x) - x^2$, then the functional relation becomes

$$g\left(g(x) + x^2 + y\right) = g\left(g(x) + x^2 - y\right) \qquad \forall \text{ real } x, y$$

Then

$$g\left(g(x) + x^2 - g(y) - y^2 + z\right) = g\left(g(x) + x^2 + g(y) + y^2 - z\right) = g\left(g(y) + y^2 - g(x) - x^2 + z\right)$$

thus

$$g(z) = g\left(2(g(x) + x^2 - g(y) - y^2) + z\right) \qquad \forall \text{ real } x, y, z$$

If $g(x) + x^2$ is constant then we get the solution $f(x) = 0$ for all real x.
If $g(x) + x^2$ is not constant then g is periodic say with period T. Then from the last equation we put $x = y + T$ to get

$$g(z) = g\left(2 \cdot (2yT + T^2) + z\right) \qquad \forall \text{ real } y, z$$

But we can choose y at our will to get

$$g(z) = g(0) \qquad \forall \text{ real } z$$

and hence

$$\boxed{f(x) = x^2 + g(0) \qquad \forall \text{ real } x}$$

IR is pure imaginaries.

Solution 89 (by N.T.TUAN)**.**
In problem IR is $\mathbb{R}$ = set of all real numbers.

Solution 90 (by maky)**.**
i will post here my solution. i gave it in the contest as well
the solutions are $f \equiv 0$ and $f(x) = x^2 + a$, with a real. obviously $f \equiv 0$ satisfies the equation, so i will choose an x_0 now such that $f(x_0) \neq 0$. i first claim that any real number can be written as $f(u) - f(v)$, with u, v reals. denote $f(x_0)$ with $t \neq 0$. then, putting in the equation, it follows that $f(t+y) - f(t-y) = 4ty$. since $t \neq 0$ and y is an arbitrary real number, it follows that any real d can be written as $f(u) - f(v)$, with u, v reals (take $y = d/4t$ above). this proves my claim above. now, let‘s see that $f\big(f(x) + f(y) + z\big) = f\big(f(x) - f(y) - z\big) + 4f(x)\big(f(y) + z\big)$ and $f\big(f(y) + f(x) + z\big) = f\big(f(y) - f(x) - z\big) + 4f(y)\big(f(x) + z\big)$ (the relations are deduced from the hypothesis) for all reals x, y, z. i will denote by d the difference $f(x) - f(y)$. substracting the above relations will give me $f(d - z) - f(-d - z) + 4zd = 0$. now, let‘s see that this is true for all reals d, z, because z was chosen arbitrary and $d = f(x) - f(y)$ can be choosen arbitrary because of the first claim. now it‘s easy. just take $d = z = -x/2$ and it gets

that $f(0) - f(x) + x^2 = 0$, or $f(x) = x^2 + f(0)$. this obviously satisfies the relation, and this ends the proof.

Solution 91 (by RaleD).

First $f(x) = 0$ is solution. Now suppose $f(x) \neq 0$ for some x. We can see that any $y \in R$ can be represented by $f(a) - f(b)$ and also $2(f(a) - f(b))$ for some reals a, b (1). Now placing $f(x) = y$ gives $f(2f(x)) = f(0) + 4f(x)^2$. Put $y = f(x) - 2f(y)$ and we get

$f(2(f(x)-f(y))) = f(2f(y))+4f(x)^2-8f(x)f(y) = f(0)+4f(y)^2+4f(x)^2-8f(x)f(y) = f(0) + 4(f(x) - f(y))^2$.

Because of (1) we see that all solutions other than $f(x) = 0$ are $f(x) = x^2 + f(0)$

Solution 92 (by ShahinBJK).

From which source did you find this question. It is International Zhautykov olympiad 2011.But you wrote it at 2007 How?

Solution 93 (by crazyfehmy).

> From which source did you find this question. It is International Zhautykov olympiad 2011.But you wrote it at 2007 How?

It is Balkan MO 2007 second problem.

Solution 94 (by mavropnevma).

Not quite. Swapping x and y, the Zhautykov writes as $f(f(x)+y) = f(y - f(x)) + 4f(x)y$ (the difference is, at the Balkan MO, the start of RHS was $f(f(x)-y)$). But, of course, the method is identical, so the problem is a spoof. See also my comment posted at the Zhautykov link here

Solution 95 (by ShahinBJK).

> Not quite. Swapping x and y, the Zhautykov writes as $f(f(x) + y) = f(y - f(x)) + 4f(x)y$ (the difference is, at the Balkan MO, the start of RHS was $f(f(x) - y)$). But, of course, the method is identical, so the problem is a spoof. See also my comment posted at the Zhautykov link here

yes i didn't see the difference but $f : even$ can be easily found.

Solution 96 (by Blitzkrieg97).

IR means irrational?

Solution 97 (by john111111).

No,it is the set of real numbers.

Solution 98 (by MathPanda1).

Does this work? $f \equiv 0$ is trivial. Assume there exists x_0 such that $f(x_0) \neq 0$. The equation is equivalent to $f(x+y) = f(x-y) + 4xy$ for all $x \in \operatorname{Im} f$, $y \in \mathbb{R}$.

(1) Since y can range over all real numbers, sub $x = x_0$ gives $\operatorname{Im} f - \operatorname{Im} f = \mathbb{R}$. Let $f(x) - x^2 = g(x)$. Then, (1) is equivalent to $g(x+y) = g(x-y)$ or $g((x-z)+y+z) = g((x-z)-y+z)$ for all $x, z \in \operatorname{Im} f$, $y \in \mathbb{R}$. Since $\operatorname{Im} f - \operatorname{Im} f = \mathbb{R}$, we get $g(w+y+z) = g(w-y+z)$ for all $z \in \operatorname{Im} f$, $y, w \in \mathbb{R}$. Let $y = \frac{u-v}{2}$ and $w = \frac{u+v-2z}{2}$ for arbitrary real numbers u, v. Then, $g(u) = g(v)$ for all $u, v \in \mathbb{R}$ i.e. g is a constant i.e. $f(x) = x^2 + c$ for some constant c, as required.

Solution 99 (by navi˙09220114)**.**

Hi MathPanda1,

I am not sure why if you fix a real z in Im(f), then x-z can represent all reals? If not, then w is only the set Im(f)-z, which is not R. (since Im(f) is not necessarily R)

Solution 100 (by Vrangr)**.**

Redacted...

Solution 101 (by navi˙09220114)**.**

Im(f) here i mean image of f, not the imaginary part of f. I am sorry for the confusion

****

————————————————————————————-

Problem 19 (Posted by Jan)**.** Determine all maps $f : \mathbb{N} \to [1, +\infty)$ that satisfy the following conditions: [list] [*]$f(2) = 2$, [*]$f(mn) = f(m)f(n)$, for all $m, n \in \mathbb{N}$, and [*]$f(m) < f(n)$ if $m < n$. [/list]

(Link to AoPS)

Solution 102 (by Rust)**.**

If $n = \prod p_i^{k_i}$, then $f(n) = \prod_i (f(p_i))^{k_i}$. Let $g(p) = \frac{ln(f(p))}{ln(p)}$ function $P \to R$. Let p and q primes and $\frac{P_k}{Q_k}$ is goot rational approach $\frac{ln(p)}{ln(q))}$. Then $(-1)^k p^{Q_k} > (-1)^k q^{Q_k}$, therefore $(-1)^k f(p^{Q_k})) > (-1)^k f(q^{P_k})$. It give $(-1)^k \frac{ln(f(p))}{ln(f(q))} > (-1)^k \frac{P_k}{Q_k}$ or $|\frac{lnf(p)}{ln(f(q)} - \frac{ln(p)}{ln(q}| < \epsilon_k \to 0$. Therefore $g(p)$ is constant and $f(n) = n^a, a \geq 0$.

Solution 103 (by pco)**.**

Hello Jan

> Determine all maps $f : \mathbb{N}_0 \to [1, +\infty[$ that satisfy: [list] [*]$f(2) = 2$ [*]$f(mn) = f(m)f(n),\ \forall\ m, n \in \mathbb{N}_0$ [*]$f(m) < f(n)$ if $m < n$[/list]

First, we can ignore the requirement $f(2) = 2$ Obviously, $f(m^r) = f(m)^r$ Let a,b two integers ¿ 1 and n,p two integers ¿ 0 such that $\frac{n}{p} \geq \frac{ln(a)}{ln(b)}$ Then, $b^n \geq a^p$ hence $f(b^n) \geq f(a^p)$ then $f(b)^n \geq f(a)^p$ and $\frac{n}{p} \geq \frac{ln(f(a))}{ln(f(b))}$

So : Let a,b two integers ¿ 1 and n,p two integers ¿ 0 : $\frac{n}{p} \geq \frac{ln(a)}{ln(b)}$ ==¿ $\frac{n}{p} \geq \frac{ln(f(a))}{ln(f(b))}$

But, it is always possible to find two integers p and q such that $\frac{n}{p}$ be as near as possible and above $\frac{ln(a)}{ln(b)}$ ==¿ $\frac{ln(a)}{ln(b)} \geq \frac{ln(f(a))}{ln(f(b))}, \forall a, b$ integers ¿ 1 Since we also have then $\frac{ln(b)}{ln(a)} \geq \frac{ln(f(b))}{ln(f(a))}$

The conclusion is $\frac{ln(a)}{ln(b)} = \frac{ln(f(a))}{ln(f(b))}$, hence $\frac{ln(f(a))}{ln(a)} = \frac{ln(f(b))}{ln(b)}$

Hence $f(x) = x^c$

But f(2)=2

Hence f(x) = x

– Patrick

Solution 104 (by Diogene)**.**

$f(1) = (f(1))^2 \implies f(1) = 1$. Let a be an integer $a > 2$. It's easy to see that $f(a^n) = (f(a))^n$. Let u_n be an integer such that $2^{u_n - 1} \leq a^n \leq 2^{u_n}$ so $2^{\frac{u_n - 1}{n}} \leq a \leq 2^{\frac{u_n}{n}}$ so $\frac{u_n}{n}$ is convergent to $\frac{\ln(a)}{\ln(2)}$ We can write : $2^{u_n - 1} \leq a^n \leq 2^{u_n} \implies (f(2))^{u_n - 1} \leq (f(a))^n \leq (f(2))2^{u_n} \implies 2^{\frac{u_n - 1}{n}} \leq f(a) \leq 2^{\frac{u_n}{n}} \implies f(a) = 2^{\frac{\ln(a)}{\ln(2)}} = a$ Of course this proof is verry known .. :cool:

****

———————————————————————————————-

Problem 20 (Posted by Lee Sang Hoon)**.** Find all functions $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ such that $f(1) = 1$ and

$$f(m^2 + n^2) = f(m)^2 + f(n)^2,$$

for all $m, n \in \mathbb{N} \cup \{0\}$.

(Link to AoPS)

Solution 105 (by N.T.TUAN)**.**

[hide="hint"]Choose $m = n = 0$ we have $f(0) = 2f^2(0)$, so $f(0) = 0$ because $f(0) \in \mathbb{Z}$. Choose $m = 0$ we have $f(m^2) = f^2(m) \forall m \in S$, so $f(m^2 + n^2) = f(m^2) + f(n^2) \forall m, n \in S$.

Choose $m = n = 1$ we have $f(2) = 2$.

Easy see that $f(4) = 4, f(5) = 5, f(8) = 8, f(26) = 26$. Therefore $25 = f^2(5) = f(25) = f(3^2 + 4^2) = f^2(3) + f^2(4)$, so $f(3) = 3$, by this method we have $f(k) = k \forall 0 \leq k \leq 26$.

Assume that $f(i) = i \forall i \in \{0, 1, ..., k\} (k \geq 26)$, we need prove $f(k+1) = k+1$. Use $(12n)^2 + n^2 = (9n)^2 + (8n)^2$ we can know $f(12n)$, use $(12n+1)^2 + (n-12)^2 = (9n+8)^2 + (8n-9)^2$ we can know $f(12n+1)$, etc.

Solution 106 (by pco)**.**

Hello N.T.TUAN

> ...by this method we have $f(k) = k \forall 0 \leq k \leq 26$.

I'm sorry but this seems not so simple. Finding $f(7)$, for example, need to use $7^2+24^2=25^2$ and hence to know $f(24)$. this is possible (not immediate) thru $24^2+10^2=26^2$ and $10=3^2+1^2$ and $26=5^2+1^2$

But, for computing $f(11)$, the only equation available is $11^2=61^2-60^2$ so we need to know $f(61)$ and $f(60)$. For $f(61)$, the only equation is $61^2=1861^2-1860^2$!

So, in order to compute $f(11)$, we need to compute $f(60)$, $f(1861)$ and $f(1860)$!

So, how could you show that $f(11)=11$?

– Patrick

Solution 107 (by N.T.TUAN)**.**

Yes, But we have $11^2+2^2=125=10^2+5^2$, so $f^2(11)+f^2(2)=f^2(10)+f^2(5)$.

Solution 108 (by pco)**.**

> Yes, But we have $11^2+2^2=125=10^2+5^2$, so $f^2(11)+f^2(2)=f^2(10)+f^2(5)$.

You're right. I forgot the equation $n^2=a^2+b^2-c^2$

Here is another solution

1) First we can show that $f(n)$ $(\forall n>1)$ may always be computed just using others $f(i)$ with $i<n$: Let $n=(2k+1)2^p$

1.1) if $k>1$ $n^2=a^2+b^2-c^2$ with $a=(2k-1)2^p<n$, $b=(k+2)2^p<n$, and $c=(k-2)2^p<n$ Then $f(n)^2+f(c)^2=f(n^2+c^2)=f(a^2+b^2)=f(a)^2+f(b)^2$ $\Rightarrow f(n)^2=f(a)^2+f(b)^2-f(c)^2$

1.2) if $k=1$ and p is odd $(n=3*2^{2q+1})$: $n^2=a^2+b^2-c^2$ with $a=3*2^{2q}<n$, $b=2^{2q}<n$, and $c=(2^{2q+1})<n$ Then $f(n)^2=f(a)^2+f(b)^2-f(c)^2$

1.3) if $k=1$ and p is even $(n=3*2^{2q})$: $f(5*2^{2q})=f((2^{q+1})^2+(2^q)^2)=(f(2^{q+1}))^2+(f(2^q))^2)$ with $2^{q+1}<n$ and $2^q<n$ $f(4*2^{2q})=(f(2^{q+1}))^2$ $f(n)^2=f((3*2^{2q})^2)=(f(5*2^{2q}))^2-(f(4*2^{2q}))^2$

1.4) if $k=0$ and p is odd $(n=2^{2q+1})$: $f(n)=f((2^q)^2+(2^q)^2)=2(f(2^q))^2$ with $2^q<n$

1.5) if $k=0$ and p is even $(n=2^{2q})$: $f(n)=(f(2^q))^2$ with $2^q<n$

2) It's immediate then, with induction starting with $f(0)=0$ and $f(1)=1$ and using the above formulas, to show that $f(n)=n$ $\forall n\geq 0$ is a necessary condition.

3) Last easy step is to show that necessary condition $f(n)=n$ is also sufficient.

– Patrick

Solution 109 (by N.T.TUAN)**.**

You are wrong at 1.2 !

Solution 110 (by pco)**.**

You are wrong at 1.2 !

Yes! Sorry ! :blush: :blush:

1.2.new) If $k=1$ and p is odd $(n=3*2^{2q+1})$ $f(10*2^{2q})=f(3*2^q)^2+f(2^q)^2$ $f(8*2^{2q})=f(2^{q+1})^2+f(2^{q+1})^2$ $f(3*2^{2q+1})^2=f(10*2^{2q})^2-f(8*2^{2q})^2$

– Patrick

Solution 111 (by N.T.TUAN).

Now, you are true, i think so. Congratulate for new solution !

****

Problem 21 (Posted by perfect˙radio). Find all pairs of functions $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ such that

$$f\left(x+g(y)\right)=xf(y)-yf(x)+g(x)\quad \text{for all } x,y\in\mathbb{R}.$$

(Link to AoPS)

Solution 112 (by TomciO).

I have a solution to this one (not mine) and I can post it if you want (it's not short but not very long). If you are still interested in trying it yourself here are 2 hints: 1)Prove that g is surjective (that's not very easy). 2)Therefore, it takes value 0, using it solve the rest of a problem (that's the easier part).

Solution 113 (by pco).

Hello !

My way to prove that g(x) may be not surjective is very short :

$(f(x)=0, g(x)=0)$ is a solution in which g is not surjective.

– Patrick

Solution 114 (by perfect˙radio).

I think that you're totally misunderstanding e.lopes and TomciO! When $f(0)=0$ (that's the first thing I tried) it's easy to prove that g takes the value 0. In the other case, I think it may be that g is surjective.

Solution 115 (by TomciO).

Exactly, in fact, I wasn't clear in my previous post. We need only that g takes value 0, if $f(0)=0$ then we can do it by hand, in other case we prove that g is surjective.

Solution 116 (by e.lopes).

I found one more nice and short way to solve our equation, my friends!

We will prove first that $g(i)=0$, for any i.

If f(0) = 0, we find that $f(x+g(0))=g(x)$. Take $x=-g(0).g(-g(0))=f(0)=0$.

If f(0) isn't, is more difficult. See: Take x = 0 in the original equation. We see that f is surjective and g is injective!, because $f(g(y))=g(0)-y.f(0)$

In original equation, put $x = g(x)$. We find $f(g(x) + g(y)) = g(x)f(y) - yf(g(x) + g(g(x)) = g(x)f(y) - y(g(0) - xf(0)) + g(g(x))$

This is symetric, so, $g(x)f(y) - y(g(0) - xf(0)) + g(g(x)) = g(y)f(x) - x(g(0) - yf(0)) + g(g(y))$ (*).

Cause f is surjetive, we can take $y = m$, where $f(m) = 0$.

Do this in (*)!

We find $-f(0)m + g(g(x)) = g(m)f(x) - f(0)x + g(g(m))$. Putting $g(m) = p$ and $g(g(m)) + f(0)m = q$,

this becomes $g(g(x)) = pf(x) - f(0)x + q$.

Substituting back into (*) we find $g(x)f(y) + pf(x) = g(y)f(x) + pf(y)$ (**)

Let y = 0 in (**). We find g(x) = $\frac{(f(0)-p)f(x)}{g(0)} + k$. So, g is surjective! and we can take g(i) = 0! for one real i.

Now, we will prove that f and g are linears! So, in the original equation, let y = i!

We find $g(x) = (i + 1)f(x) - f(i)x$. Substituing back into original equality, we find:

$f(x + g(y)) = (i + 1 - y)f(x) + x(f(y) - f(i))$ (***)

Let $y = i + 1$ in (***).

We find that $f(x + r) = x(f(i + 1) - f(i))$, where $f = g(i + 1)$! So, f is linear! :)

In the equality g(x) = (i+1)f(x)-f(i)x, we see the that g is also linear!

Now, is easy, let f(x) = ax+b and g(x) = cx+d, and good luck with the computations!

The solutions are f(x) = g(x) = 0 and $f(x) = \frac{k(x-k)}{k+1}$, $g(x) = k(x - k)$! for a real k!

Solution 117 (by Pedram-Safaei)**.**

suppose that $P(x, y)$ be the following assertion:$f(x+g(y)) = xf(y)-yf(x)+g(x)$ $P(x, 0) : f(x + g(0)) = xf(0) + g(x)$. (put $g(0) = c$) (1) $P(x + t + c, y) : f(x + c + t + g(y)) = (x + t + c)f(y) - yf(x + c + t) + g(x + c + t)$ $P(x+c, y) : f(x+c+g(y)) = (x+c)f(y)-yf(x+c)+g(x+c)$.now subtracting two equations and use of (1) we have:$tf(0) + g(x + t + g(y)) - g(x + g(y)) = tf(y) - y(tf(0) + g(x + c + t) - g(x + c)) + g(x + t) - g(x)$. so if the previous assertion be $Q(x, y)$,then $Q(x, 0) : g(x + t + c) - g(x + c) = g(x + t) - g(x)$ (2) then with use of (2) we have $Q(x, 1) : g(x + t + g(1)) - g(x + g(1)) = t(f(1) - f(0))$ so for all x we have:$g(x + t) - g(x) = ta$ (put $a = f(1) - f(0)$) put $x = 0$ we have:$g(t) = at + c$ so:$(putb = f(0))$ $f(x + c) = bx + ax + c$ so $f(x) = (a + b)x + c(1 - a - b)$ now by putting it into main equation we conclude that:$a(a + b) = -c(1 - a - b) = -b$ and $f(x) = (a + b)x + b, g(x) = ax + c$.or for some real $a \neq -1.f(x) = \frac{a(x-a)}{a+1}, g(x) = a(x - a)$.

Solution 118 (by CanVQ)**.**

> Find all pairs of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x + g(y)) = x \cdot f(y) - y \cdot f(x) + g(x)$ (1) for all real x, y.

Let $f(0) = a$ and $g(0) = b$. Taking $x = 0$ in (1), we get

$$f\big(g(y)\big) = -ay + b, \quad \forall y \in \mathbb{R}. \quad (2)$$

There are two cases to consider:

Case 1: $\mathbf{a = 0}$. Taking $y = 0$ in (1), we get

$$f(x + b) = g(x), \quad \forall x \in \mathbb{R}. \quad (3)$$

From this, we deduce that $g(-b) = 0$. Now, replacing $y = -b$ in (1), we have

$$f(x) = x \cdot f(-b) + b \cdot f(x) + g(x), \quad \forall x \in \mathbb{R}. \quad (4)$$

Taking $x = -b$ in (4), we get $f(-b) = g(-b) = 0$ and hence, the equality (4) can be rewritten as

$$(1 - b) \cdot f(x) = g(x), \quad \forall x \in \mathbb{R}. \quad (5)$$

Taking $x = 0$ in (5), we get $b = g(0) = 0$. Thus, the identities (2) and (5) can be written as

$$f\big(g(x)\big) = 0, \quad \forall x \in \mathbb{R} \quad (6)$$

and

$$f(x) = g(x), \quad \forall x \in \mathbb{R}. \quad (7)$$

From (6) and (7), we get $g\big(g(x)\big) = 0$, $\forall x \in \mathbb{R}$. Now, replacing y by $g(x)$ in (1), we can easily deduce that $f(x) = g(x) = 0$, $\forall x \in \mathbb{R}$. These functions satisfy our condition.

Case 2: $\mathbf{a \neq 0}$. From (2), we can easily see that f is surjective and g is injective. Replacing x by $g(x)$ in (1) and using (2), we get

$$f\big(g(x) + g(y)\big) = g(x) \cdot f(y) + axy - by + g\big(g(x)\big), \quad \forall x, y \in \mathbb{R}. \quad (8)$$

Changing the position of x and y in (8), we get

$$g(x) \cdot f(y) - by + g\big(g(x)\big) = g(y) \cdot f(x) - bx + g\big(g(y)\big), \quad \forall x, y \in \mathbb{R}. \quad (9)$$

Replacing $y = 0$ in (1), we have

$$f(x + b) = ax + g(x), \quad \forall x \in \mathbb{R}. \quad (10)$$

Replacing x by $g(x)$ in (10), we get

$$g\big(g(x)\big) + a \cdot g(x) = f\big(b + g(x)\big) = b \cdot f(x) - x \cdot f(b) + g(b),$$

or

$$g\big(g(x)\big) = b \cdot f(x) - a \cdot g(x) - x \cdot f(b) + g(b), \quad \forall x \in \mathbb{R}. \quad (11)$$

Plugging this result into (9), we get

$$\begin{aligned} g(x) \cdot f(y) - by + b \cdot f(x) - a \cdot g(x) - x \cdot f(b) &= \\ = g(y) \cdot f(x) - bx + b \cdot f(y) - a \cdot g(y) - y \cdot f(b). \quad (12) \end{aligned}$$

Since $f(b) = b$ (we can easily obtain this by setting $y = 0$ in (2)), the above identity can be written as

$$g(x) \cdot \big[f(y) - a\big] + b \cdot f(x) = g(y) \cdot \big[f(x) - a\big] + b \cdot f(y),$$

or

$$\big[g(x) - b\big]\big[f(y) - a\big] = \big[g(y) - b\big]\big[f(x) - a\big], \quad \forall x, y \in \mathbb{R}. \quad (13)$$

Since f is surjective, there exists y_0 such that $f(y_0) \neq a$. Taking $y = y_0$ in (13), we get

$$g(x) - b = \frac{g(y_0) - b}{f(y_0) - a}\big[f(x) - a\big], \quad \forall x \in \mathbb{R}. \quad (14)$$

If $g(y_0) = b$, then we have $g(x) = b$, $\forall x \in \mathbb{R}$ which is a contradiction since g is injective. So we must have $g(y_0) \neq b$. From this and the previous results, we can easily prove that f and g are bijectives. Thus, the exists a unique number c such that $g(c) = 0$. Taking $x = y = c$ in (1), we have $f(c) = g(c) = 0$. Continuously, replacing $y = c$ in (1), we get

$$(1 + c) \cdot f(x) = g(x), \quad \forall x \in \mathbb{R}. \quad (15)$$

Since g is bijective, it is clearly that $c \neq -1$, from which it follows that

$$f(x) = \frac{1}{1 + c} \cdot g(x), \quad \forall x \in \mathbb{R}. \quad (16)$$

Now, replacing $y = 1 + c$ in (1) and using (15), we get

$$f\big(x + g(1 + c)\big) = x \cdot f(1 + c) = \frac{x \cdot g(1 + c)}{1 + c}, \quad \forall x \in \mathbb{R}. \quad (17)$$

Replacing $x = c - g(1 + c)$ in (17), we get

$$g(1 + c) \cdot \big[c - g(1 + c)\big] = 0.$$

Since g is bijective, we have $g(1 + c) \neq 0$ and hence, it follows that $g(1 + c) = c$. From this, we have

$$f(x + c) = \frac{cx}{1 + c}, \quad \forall x \in \mathbb{R}. \quad (18)$$

Replacing x by $x - c$ in (18), we get

$$f(x) = \frac{c}{1 + c}(x - c), \quad \forall x \in \mathbb{R}.$$

It follows that

$$g(x) = c(x - c), \quad \forall x \in \mathbb{R}.$$

These functions satisfy our condition.

Solution 119 (by JuanOrtiz)**.**

Notice f(g(x))=g(0)-f(0)x. (1) If f(0)=0 we get g(-g(0))=0. Otherwise f is surjective and g is injective. taking x=g(x) and considering y as a consant, we get an equation. if we change the value of y in this equation, we find that g(x)=0 for some x. So g(a)=0 for some a always. Then in our original equation f(x)=(f(x+g(a))=xf(a)-af(x)+g(x) so f(x)=(g(x)+xf(a))/(a+1), unless a=-1 but in this case we finish easily. And we get f(a)=af(a)/(a+1). so if a=0 we finish easily, otherwise f(a)=0 and we get g=cf for a constant c. also f is bijective and from here we finish easily.

answer: linear f and g=cf for a constant c. additional equations must be satisfied by the coefficients of f, we obtain these relations by just plugging in.

Solution 120 (by efang)**.**

I proved the $f(0) = 0$ the same way as CanVQ. I have a slightly different approach in proving $f(0) \neq 0$, specifically in proving bijectivity of f and g

[size=150]f(0) is not equal to 0[/size]

Then $f(g(y)) = -y * f(0) + g(0)$ (1).

Lemma: f and g are both bijective

Proof:

Clearly $f \circ g$ is a bijective function as it's linear. Thus we can immediately conclude from it's injectivity that g is injective and from it's surjectivity that f is surjective.

For any $c \in \mathbb{R}$ note that if $f(c) = a$ we have $f(c) = a = \frac{a-g(0)}{f(0)} * f(0) + g(0) = f(g(\frac{g(0)-a}{f(0)}))$ and since $f(0)$ is non-zero we have that there necessarily exists some real number a satisfying $g(a) = c$ for all real c proving surjectivity

Now, we had $a, b \in \mathbb{R}$ such that $f(a) = f(b) = c$. Then, we know there exist real m and n such that $g(m) = a$ and $g(n) = b$

Then we would have $f(g(m)) = f(a) = -m * f(0) + g(0)$ and $f(g(n)) = f(b) = -n * f(0) + g(0)$ Thus we have $m = n$ and $a = b$ so f is injective. This completes the lemma.

$\square$

Now this means there exists a unique value a such that $g(a) = 0$. Plug in $x = y = a$ into the original equation to get $f(a) = g(a) = 0$. (2)

Now, plug in just $y = a$ and use the result from (2) to obtain $f(x) = -a * f(x) + g(x)$ or $(a+1)f(x) = g(x)$ (3)

Plugging (3) into (1) reveals $f((a+1)f(x)) = -x * f(0) + (a+1)f(0)$. Now notice that when $x = a+1$ we have that $f((a+1)f(a+1)) = 0$. Thus $f(a+1) = \frac{a}{a+1}$ and thus also $g(a+1) = a$ (4)

Now, plug (3) back into the original equation and we get

$f(x + (a+1)f(y)) = xf(y) - yf(x) + (a+1)f(x)$. Plug in $y = a+1$ and use (4) and we get:

$f(x+a) = \frac{xa}{a+1}$ or $f(x) = \frac{(x-a)a}{a+1}$ and thus from (3) $g(x) = (x-a)a$ for some real constant a which completes the problem.

$\square$

(note that since $(a+1)*g(0) = f(0)$ and we have that $g(0) \neq 0$ and $f(0) \neq 0$ this means that $a+1 \neq 0$ and $a \neq -1$.)

****

Problem 22 (Posted by perfect˙radio)**.** Prove that there is a unique function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$f\left(m + f(n)\right) = n + f\left(m + 95\right), \forall m, n \in \mathbb{Z}_{>0}.$$

Compute $\sum_{k=1}^{19} f(k)$.

(Link to AoPS)

Solution 121 (by pco)**.**
Hello perfect˙radio,
I don't know $\mathbb{Z}_{>0}$. I assume it is $1, 2, 3, \ldots$..

> Prove that there is a unique function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that
>
> $$f\left(m + f(n)\right) = n + f\left(m + 95\right), \forall m, n \in \mathbb{Z}_{>0}.$$

$f(a) = f(b) \Rightarrow f(m+f(a)) = f(m+f(b)) \Rightarrow a+f(m+95) = b+f(m+95)$ $\Rightarrow a = b \Rightarrow f(x)$ is injective

$f(1 + f(1)) = 1 + f(96) \Rightarrow f(1 + f(1) + f(n+1)) = n+1+f(1+f(1)) = n+2+f(96)$ $f(1+f(2)) = 2+f(96) \Rightarrow f(1+f(2)+f(n)) = n+f(1+f(2)) = n+2+f(96)$ So $f(1+f(1)+f(n+1)) = f(1+f(2)+f(n))$ and, since f is injective, $1+f(1)+f(n+1) = 1+f(2)+f(n)$ and $f(n+1) = f(n)+f(2)-f(1)$

Hence $f(n) = kn + b$ and so $f(m + f(n)) = km + k^2n + kb + b$ and $n + f(m + 95) = n + km + 95k + b$ So $km + k^2n + kb + b = n + km + 95k + b$ and $(k^2 - 1)n + k(b - 95) = 0 \Rightarrow k = 1$ $(f > 0)$ and $b = 95$

So, $f(n) = n + 95$

> Compute $\sum_{k=1}^{19} f(k)$.

$\sum_{k=1}^{19} f(k) = 1995$
– Patrick

Solution 122 (by N.T.TUAN)**.**

> Prove that there is a unique function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that
>
> $$f\left(m + f(n)\right) = n + f\left(m + 95\right), \forall m, n \in \mathbb{Z}_{>0}.$$
>
> Compute $\sum_{k=1}^{19} f(k)$.

1) f is injective. 2) $f(f(m)+f(n)) = n+f(95+f(m)) = n+m+f(2\cdot 95)$ therefore $f(f(n)+f(n+2)) = f(2f(n+1))$, by injective we have $f(n+2)+f(n) = 2f(n+1)$.

And that is all!

****

Problem 23 (Posted by N.T.TUAN)**.** Given $n \in \mathbb{N}$, nd all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$\sum_{k=0}^{n} \binom{n}{k} f(x^{2^k}) = 0.$$

(Link to AoPS)

Solution 123 (by Rust)**.**

Your condition equavalent to $f(x) + f(x^2) = 0$ (consider $x \to x^{2^m}, m = 0, 1, ..., n$). If $g(y) = f(e^y)$, then $g(y) + g(2y) = 0$, if $g(\pm 2^t) = s_{\pm}(t)$, then $s_{\pm}(t+1) = -s_{\pm}(t)$ ($s_{\pm}(t)$ periodic with period 2). For all $s_{\pm}(t)$, $t \in [0, 1)$ we get solution $f(x)$.

Solution 124 (by pco)**.**

> Given $n \in \mathbb{N}$, nd all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$,
> $$\sum_{k=0}^{n} \binom{n}{k} f(x^{2^k}) = 0.$$

$\sum_{k=0}^{n} \binom{n}{k} f(x^{2^k}) = 0 \Leftrightarrow \sum_{k=0}^{n-1} \binom{n-1}{k} f(x^{2^k}) + \sum_{k=0}^{n-1} \binom{n-1}{k} f(x^{2^{k+1}}) = 0$. If I call $g(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} f(x^{2^k})$, I have $g(x^2) = -g(x)$ and so $g(x) = g(x^{4^{-p}})$ for $x \geq 0$ and so, with continuiity (when p$\to +\infty$) $g(x) = c$ for any $x \geq 0$ and, since $g(x) = -g(x^2)$ for any x, we have $g(x) = c$ for any $x \geq 0$x, and $g(x) = -c$ for any $x \leq 0$ and so $g(x) = 0$ for any x

So $\sum_{k=0}^{n-1} \binom{n-1}{k} f(x^{2^k}) = 0$

So any solution for rank n is also solution for rank $n-1$ The only solution at rank 0 is $f(x) = 0$

So the only solution at rank n is $f(x) = 0$.

– Patrick

Solution 125 (by Rust)**.**

Why $f(x) + f(x^2)$ give $f(x) \equiv 0$?

Solution 126 (by pco)**.**

> Why $f(x) + f(x^2)$ give $f(x) \equiv 0$?

As I said : $g(x)+g(x^2)=0 \Rightarrow g(x^2)=-g(x) \Rightarrow g(x)=-g(x^{\frac{1}{2}})$ for $x \geq 0 \Rightarrow g(x)=g(x^{\frac{1}{4}})$ for any $x \geq 0$ Hence $g(x)=g(x^{4^{-p}})$ for any $x \geq 0$ When $p \to +\infty$ and because g is continuous, $g(x)=g(1)$ for any $x>0$ Since $g(x)=-g(x^2)$ for any x, we have then $g(x)=-g(1)$ for any $x<0$ With the continuity at 0, the result is $g(x)=0$

– Patrick

Solution 127 (by Rust)**.**

Oh! I forgot about continiosly.

****

————————————————————————————————-

Problem 24 (Posted by N.T.TUAN)**.** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(1)=1$ and

$$f\left(f(x)y+\frac{x}{y}\right)=xyf(x^2+y^2),$$

for all real numbers x and y with $y \neq 0$.

(Link to AoPS)

Solution 128 (by nayel)**.**

i have the following:

let $f(0)=c$. $x=0$ gives $f(cy)=0, \forall y \neq 0$. $x=c$ gives $f(\frac{c}{y})=cyf(c^2+y^2)=0, \forall y \neq 0$. so $c=0$ or $f(c^2+y^2)=0, \forall y \neq 0$.

assume that $c \neq 0$. then $f(cy)=0$ for $y \neq 0$ gives $f(c^2)=0$. combining this with $f(c^2+y^2)=0$ for $y \neq 0$ we get f is constant and $f \equiv 0$. so this is a solution. now we turn our attention to $c=f(0)=0$.

$x=1$ implies $f(y+\frac{1}{y})=yf(y^2+1)$. on the other hand, $y=1$ gives $f(x+f(x))=xf(x^2+1)$. from these two equations we get $f(x+\frac{1}{x})=f(x+f(x))$ for $x \neq 0$. from this i inspect $f(x)=0$ for $x=0$, otherwise $f(x)=\frac{1}{x}$. just need to show that f is injective...

probably that didn't help.. :blush:

Solution 129 (by scorpius119)**.**

[hide="hmm..."] We can get $f(0)=0$ because if $f(0)\neq 0$, then we can plug in $x=0,\ y=\frac{1}{f(0)}$ to get $f(1)=0$: a contradiction.

For any nonzero x, there exists a nonzero y such that

$$f(x)y+\frac{x}{y}=x^2+y^2$$

as a cubic equation has at least one real root. Let c be this common value; then we have

$$f(c)=xyf(c) \Rightarrow f(c)(xy-1)=0$$

Now to show that if $f(c)=0$, then $c=0$... :maybe: If we can show this, then we'll be done, since $xy=1$ and $f(x)y+\frac{x}{y}=x^2+y^2$ implies $f(x)=\frac{1}{x}$.

****

Problem 25 (Posted by Jutaro)**.** Given the positive real numbers $a_1 < a_2 < \cdots < a_n$, consider the function

$$f(x) = \frac{a_1}{x + a_1} + \frac{a_2}{x + a_2} + \cdots + \frac{a_n}{x + a_n}$$

Determine the sum of the lengths of the disjoint intervals formed by all the values of x such that $f(x) > 1$.

(Link to AoPS)

Solution 130 (by pco)**.**

Hello Jutaro,

> Given the positive real numbers $a_1 < a_2 < \cdots < a_n$, consider the function
> $$f(x) = \frac{a_1}{x + a_1} + \frac{a_2}{x + a_2} + \cdots + \frac{a_n}{x + a_n}$$
> Determine the sum of the lengths of the disjoint intervals formed by all the values of x such that $f(x) > 1$.

Very nice question.

Let $f(x) = \frac{P(x)}{Q(x)}$ with $P(x) = \sum_i a_i \prod_{j \neq i}(x + a_j)$ and $Q(x) = \prod_i (x + a_i)$

$f(x) = 1 \Leftrightarrow Q(x) - P(x) = 0$ and this polynomial have exactly n roots $-a_n < r_n < -a_{n-1} < r_{n-1} < \cdots < -a_1 < r_1$

The number requested is $\sum_i (r_i + a_i) = \sum_i r_i + \sum_i a_i$ $\sum_i r_i$ is the opposite of the coefficient of x^{n-1} in $Q(x) - P(x)$ Coefficient of x^{n-1} in $Q(x)$ is $\sum_i a_i$ Coefficient of x^{n-1} in $P(x)$ is $\sum_i a_i$ Hence $\sum_i r_i = 0$

The the number requested is $\sum_i a_i$

– Patrick

Solution 131 (by Rust)**.**

$f(x) - 1 = 0 \to \frac{Q(x)}{P(x)} = 0$, were $P(x) = \prod_i (x + a_i)$, $Q(x) = -P(x) + \sum_i a_i \frac{P(x)}{x + a_i}$. Because $f(x) \to -\infty$, when $x \to a_i - 0$, and $f(x) \to +\infty$, when $x \to a_i + 0$ we have n roots Q(x)$-a_n < y_n < -a_{n-1} < ... < -a_1 < y_1 < \infty$. We calculate $Q(x) = -x^n - \sum_i a_i x^{n-1} - ... + x^{n-1}(a_1 + ... + a_n + ...$ It give $\sum_i y_i = 0$. Therefore $\sum_i L_i = \sum_i (y_i - (-a_i)) = \sum_i a_i$.

Solution 132 (by modularmarc101)**.**

> ... The number requested is $\sum_i (r_i + a_i) = \sum_i r_i + \sum_i a_i$ ...

Sorry, but I'm having trouble understanding why it is this sum. I know I'm wrong, but I thought it would be something like $(r_1 - r_2) + (r_3 - r_4) + ...$ (that or $(r_2 - r_3) + (r_4 - r_5) + ...$).

EDIT: Thanks pco!

Solution 133 (by pco).

No, you need to go back to the original expression of $f(x)$:

$f((-\infty, -a_n)) = (-\infty, 0)$

$f((-a_n, r_n)) = (1, +\infty)$ and so we get a "good" interval $(-a_n, r_n)$ whose length is $r_n + a_n$ $f((r_n, -a_{n-1})) = (-\infty, 1)$

$f((-a_{n-1}, r_{n-1})) = (1, +\infty)$ and so we get another "good" interval $(-a_{n-1}, r_{n-1})$ whose length is $r_{n-1}+a_{n-1}$ $f((r_{n-1}, -a_{n-2})) = (-\infty, 1)$ ... $f((-a_{n-k}, r_{n-k})) = (1, +\infty)$ and so we get a "good" interval $(-a_{n-k}, r_{n-k})$ whose length is $r_{n-k} + a_{n-k}$ $f((r_{n-k}, -a_{n-k-1})) = (-\infty, 1)$

Hence the result $\sum_i (r_i + a_i)$

****

Problem 26 (Posted by silouan). Let n be a positive integer. Find all monotone functions f from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f(x + f(y)) = f(x) + y^n,$$

for all $x, y \in \mathbb{R}$.

(Link to AoPS)

Solution 134 (by N.T.TUAN).

From hypothesis we have f is injective. Choose $y = 0$ then $f(x + f(0)) = f(x)$, so $f(0) = 0$. Choose $x = 0$ we have $f(f(y)) = y^n$, so n is odd. Therefore f is surjective and f is additive, so $f(x) = ax$, check! We will find a.

Solution 135 (by mathisfun1).

> From hypothesis we have f is injective. Choose $y = 0$ then $f(x+f(0)) = f(x)$, so $f(0) = 0$. Choose $x = 0$ we have $f(f(y)) = y^n$, so n is odd. Therefore f is surjective and f is additive, so $f(x) = ax$, check! We will find a.

I'm sorry, could you explain how $f(f(y)) = y^n$ implies f is surjective and f is additive?

Solution 136 (by N.T.TUAN).

1)If n is odd then $\{y^n | y \in \mathbb{R}\} = \mathbb{R}$, therefore f is surjective. 2)From above we have $f(x + f(y)) = f(x) + f(f(y))$ and so f is additive.

Solution 137 (by Yosh...).

why n must be odd?

and why if we have $f(x)$ bijective and additive,so we have $f(x) = ax$? thx..

Solution 138 (by N.T.TUAN).

> why n must be odd?

Because $f(f(y)) = y^n$ and $f_o f$ is increasing!

Solution 139 (by pco).

> Let $n \in N$. Find all monotone functions f from R to R such that $f(x + f(y)) = f(x) + y^n$

I don't agree with N.T.TUAN when he said that hypothesis imply $f(x)$ is an injective function. It's true but not immediate (a monotonous function may be non-injective). BTW, we don't need to show that $f(x)$ is injective.

I suggest a little modification to N.T.TUAn demo (which is rather good indeed) :

$E1(x, y)$: $f(x + f(y)) = f(x) + y^n$ $E1(x, 0)$: $f(x + f(0)) = f(x)$ and if $f(0) \neq 0$, we have a monotonous periodic function, so a constant and no constant satisfy the problem, so $f(0) = 0$ Then $E1(0, x)$ gives $f(f(x)) = x^n$ and n is odd since $f(x)$ monotonous implies $f(f(x))$ increasing and x^n increasing implies n odd. Then n odd and $f(f(x)) = x^n$ implies $f(x)$ is a surjective function. $E1(0, y)$: $f(f(y)) = y^n$ and so $E1(x, y)$ becomes $f(x + f(y)) = f(x) + f(f(y))$ and since $f(x)$ is surjective $f(x + z) = f(x) + f(z)$ and $f(x)$ is additive.

If $f(x)$ is additive, it is well known that $f(x) = f(1)x$ for any $x \in \mathbb{Q}$ and then $f(x) = f(1)x$ for any $x \in \mathbb{R}$ since $f(x)$ is monotonous.

So, if we write $f(1) = a$, $ax + a^2y = ax + y^n$ and $n = 1$ and $a \in \{-1, +1\}$

Solution 140 (by N.T.TUAN).

I mean that "monotone functions"= increasing or decreasing. Therefore f is injective!

Solution 141 (by pco).

> I mean that "monotone functions"= increasing or decreasing. Therefore f is injective!

OK, but I think that "monotonic" function is a "non-decreasing" or "non increasing" one.

Solution 142 (by N.T.TUAN).

Thanks for your link! Wiki is GOOD! :P

****

Problem 27 (Posted by Zamfirmihai). Build a bijective function $f : [0; 1] \to \mathbb{R}$.

(Link to AoPS)

Solution 143 (by perfect˙radio).

It's really well-known.

Note that $g(x) = \frac{1}{x - \frac{1}{2}} - 2$ is a bijection from $\left(\frac{1}{2}, 1\right]$ to $[0, \infty)$.

Also, $h(x) = \frac{1}{x - \frac{1}{2}} + 2$ is a bijection from $\left[0, \frac{1}{2}\right)$ to $(-\infty, 0]$. Define $a_n = h^{-1}\left(-\frac{1}{n}\right)$, for all integers $n \geq 1$.

We define f as follows:

- $f(x) = g(x)$ if $x > \frac{1}{2}$; - $f(x) = h(x)$ if $x \in \left[0, \frac{1}{2}\right) \setminus \{0, a_1, a_2, a_3, \ldots\}$; - $f(x) = -1$ if $x = \frac{1}{2}$; - $f(x) = -\frac{1}{2}$ if $x = 0$; - $f(a_n) = -\frac{1}{n+2}$ for all $n \geq 1$.

Solution 144 (by pco)**.**

> Build a bijective function $f, f : [0; 1] \to \mathbb{R}$

I have a rather similar solution. Since it is obvious to have bijection between $(0,1) \leftrightarrow \mathbb{R}$, the problem is to find a bijective function between $[0,1]$ and $(0,1)$. I suggest the following one :

Let $A = \{2^{-n} \forall n \in \mathbb{N} \cup \{0\}\}$ $f(0) = \frac{1}{2}$ $f(x) = \frac{x}{4}$ $\forall x \in A$ $f(x) = x$ everywhere else (in $(0,1] - A$)

– Patrick

****

——————————————————————————-

Problem 28 (Posted by N.T.TUAN)**.** Find all pairs functions $f, g : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f(m) - f(n) = (m - n)(g(m) + g(n)) \quad \forall m, n \in \mathbb{N}_0,$$

where $\mathbb{N}_0 = \{0, 1, 2, ...\}$.

(Link to AoPS)

Solution 145 (by pco)**.**

Hello N.T.TUAN

> Find all pairs functions $f, g : \mathbb{N}_0 \to \mathbb{N}_0$ such that
>
> $$f(m) - f(n) = (m - n)(g(m) + g(n)) \, \forall m, n \in \mathbb{N}_0,$$
>
> where $\mathbb{N}_0 = \{0, 1, 2, ...\}$.

$f(m) - f(n) = (m - n)(g(m) + g(n)) \Rightarrow f(m) = m(g(m) + g(0)) + f(0) \Rightarrow f(m) - f(n) = m(g(m) + g(0)) - n(g(n) + g(0))$ And so : $m(g(m) + g(0)) - n(g(n) + g(0)) = (m - n)(g(m) + g(n))$, which implies $n(g(m) - g(0)) = m(g(n) - g(0))$ and $g(n) = a * n + b$

Then $f(n) = n(g(n) + g(0)) + f(0) = an^2 + 2bn + c$

And it is easy to verify that these necessary conditions work.

So the solution is : $g(n) = a * n + b$ $f(n) = an^2 + 2bn + c$ $a, b, c \in \mathbb{N}_0$

– Patrick

****

——————————————————————————-

Problem 29 (Posted by Rust)**.** Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(2x) = 2f^2(x) - a$ for all $x \in \mathbb{R}$ and a fixed real a (example: $a = 1$ and $f(x) = \cos x$).

(Link to AoPS)

Solution 146 (by pco)**.**

> Find continiosly function $R \to R$, satisfyed $f(2x) = 2f^2(x) - a$, a fixed real parametr (for example a=1 f(x)=cosx).

Hello Rust! The answer to this problem is strongly linked to the value of a (in fact to the existence of fixed points for $g(x) = 2x^2 - a$).

1) First it is obvious that we can choose $f(x)$ on $[1, 2]$, for example, with the only constraints of continuity and $f(2) = 2f^2(1) - a$ and then we can define $f(x)$ on $[1, +\infty)$ without difficulty. Same thing for an arbitrary definition on $[-2, -1]$, with continuity and $f(-2) = 2f^2(-1) - a$, and then complete definition on $(-\infty, -1]$.

2) the problem is to be able to build $f(x)$ on $[0, 1)$ and to have the continuity in 0.

2.1) if $a < -\frac{1}{8}$, the function $g(x) = 2x^2 - a$ has no fixed point and it is rather easy to see that it is impossible to define $f(x)$ about $0 \Rightarrow$ no solution.

2.2) if $-\frac{1}{8} \le a \le 0$, it is generally possible to build $f(x)$ on $(0, 1]$ by using the formula $f(x) = \sqrt{\frac{f(2x)+a}{2}}$. We must take the positive value of the square root and not the negative one. One condition for this to work is that the reference value ($f(x)$ in $[1, 2]$ or $[-2, -1]$) must be either the lowest fixed point of $g(x)$ (and the function is constant), either $>$ to this lowest fixed point and then $f(0) =$ the highest fixed point of $g(x)$.

2.3) if $0 < a < \frac{3}{8}$. it is again possible to build $f(x)$ on $(0, 1]$ by using the formula $f(x) = \sqrt{\frac{f(2x)+a}{2}}$. One condition for this to work is that the reference value ($f(x)$ in $[1, 2]$ or $[-2, -1]$) must be $> -a$ and then $f(0) =$ the highest fixed point of $g(x)$. But it is then possible to use the negative value of the square ($f(x) = -\sqrt{\frac{f(2x)+a}{2}}$ in some conditions). Due to the fact that $g'(x) > -1$ on the lowest fixed point, it is possible to have this lowest fixed point as a value for $f(0)$.

2.4) if $\frac{3}{8} \le a$, it is impossible (except $f(x)$ is constant) to have $f(0) =$ lowest fixed point. So $f(0) =$ highest fixed point. The reference value ($f(x)$ in $[1, 2]$ or $[-2, -1]$) must be $> -a$. It is possible during some steps to use $f(x) = -\sqrt{\frac{f(2x)+a}{2}}$ but, at the end, we must use $f(x) = \sqrt{\frac{f(2x)+a}{2}}$

As a conclusion : if $a < -\frac{1}{8}$: no solution if $-\frac{1}{8} \le a \le 0$: infinite many solutions (starting wih values $>$ lowest fixed point), all non constant solutions having $f(0) =$ highest fixed point. if $0 < a < \frac{3}{8}$: infinite many solutions and two possible values for $f(0)$. if $\frac{3}{8} \le a$: infinite many solutions and all non constant solutions having $f(0) =$ highest fixed point.

– Patrick

Solution 147 (by Rust)**.**

Ok! Had these equation nonconstant continiosly periodic solution for $a \geq \frac{3}{8}$? Is these periodic solution with period 2π unique (a=1, f(x)=cos x)? Obviosly, if f(x) is solution, then f(bx) is solution. Therefore I interested about minimal period 2π.

Solution 148 (by pco)**.**

> Ok! Had these equation nonconstant continiosly periodic solution for $a \geq \frac{3}{8}$? Is these periodic solution with period 2π unique (a=1, f(x)=cos x)? Obviosly, if f(x) is solution, then f(bx) is solution. Therefore I interested about minimal period 2π.

For the moment, I can show that if $f(x)$ is continuous, non constant and periodic, then $a = 1$:

We have $f(2x) = g(f(x))$ with $g(x) = 2x^2 - a$ and $f(x)$ periodic (period T).

Let $p = \frac{1-\sqrt{1+8a}}{4}$ and $q = \frac{1+\sqrt{1+8a}}{4}$ the two fixed points of $g(x)$ (which exist since $a \geq \frac{3}{8} > -\frac{1}{8}$).

Since $f(x)$ is continuous and periodic, it exists lower and upper bounds for $f(x)$: $A \leq f(x) \leq B$ $\forall x \in \mathbb{R}$. But, if $|f(x_0)| > q$ for some x_0 it is easy to see that $\lim_{n\to+\infty} f(2^n x_0) = +\infty$ and so we can conclude $|f(x)| \leq q$ $\forall x \in \mathbb{R}$

I have shown in the previous post that $f(0) = q$.

If $f(\frac{kT}{2^n}) = q$ $\forall k, n \in \mathbb{N}$, it is immediate (since f is continuous), that $f(x) = q$ $\forall x$). So, since $f(x)$ is supposed periodic non constant, it exist k and n such that $f(\frac{kT}{2^n}) \neq q$. But $f(kT) = f(0) = q$. So it exist in $[1, n]$ an integer i such that $f(\frac{kT}{2^i}) = q$. and $f(\frac{kT}{2^{i+1}}) \neq q$. But, if $f(x) = q$, and since $q = 2q^2 - a$, $f(\frac{x}{2}) = q$ or $f(\frac{x}{2}) = -q$. So $f(\frac{kT}{2^{i+1}}) = -q$.

So (remember that $|f(x)| \leq q$ $\forall x \in \mathbb{R}$) lower and upper bounds of $f(x)$ are $-q$ and $+q$ and these values may be reached. Since $f(x)$ is continuous with values in $[-q, +q]$, it exist x_1 such that $f(x_1) = 0$. Then $f(2x_1) = -a$ and so $-a \in [-q, +q]$, so $a \leq q$ But, since $f(x) = 2f^2(\frac{x}{2}) - a$, we have $f(x) \geq -a$ $\forall x$, and so $-q \geq -a$ and $q \leq a$

So $q = a$, and, since $q = \frac{1+\sqrt{1+8a}}{4}$, we can conclude $a = 1$

So $a = 1$

– Patrick

****

Problem 30 (Posted by N.T.TUAN)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x^3 + y^3) = x^2 f(x) + y f(y^2)$$

for all $x, y \in \mathbb{R}$.

(Link to AoPS)

Solution 149 (by pco).

> Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
> $$f(x^3 + y^3) = x^2 f(x) + y f(y^2)$$
> for all $x, y \in \mathbb{R}$.

$x = 0$ and $y = 0 \Rightarrow f(0) = 0$

$f(x^3 + 0^3) = x^2 f(x)$ $f(0^3 + x^3) = x f(x^2)$

So $f(x^3) = x^2 f(x) = x f(x^2)$ and $f(x^3 + y^3) = f(x^3) + f(y^3)$

So $f(x + y) = f(x) + f(y)$ $y = -x \Rightarrow f(-x) = -f(x)$

Then $x^2 f(x) = x f(x^2) \Rightarrow f(x^2) = x f(x)$ $\forall x \neq 0$ So $f((x + 1)^2) = (x + 1)f(x + 1) = (x + 1)(f(x) + f(1)) = x f(x) + f(x) + x f(1) + f(1)$ $\forall x \neq -1$ But $f((x+1)^2) = f(x^2+x+x+1) = f(x^2)+f(x)+f(x)+f(1) = x f(x)+2f(x)+f(1)$ $\forall x \neq 0$ So $x f(x)+f(x)+x f(1)+f(1) = x f(x)+2f(x)+f(1)$ and $f(x) = x f(1)$ $\forall x \neq 0$ and $x \neq -1$

The two cases $x = 0$ and $x = -1$ are very easy to check ($f(0) = 0$ and $f(-1) = -f(1)$)

So $f(x) = ax$ and it is easy to check that this necessary condition works.

Solution 150 (by quangpbc).

The same functional equation Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$f(x^3 - y^3) = x^2 f(x) - y f(y^2)$ for all $x, y \in \mathbb{R}$.

Solution 151 (by quangpbc).

Nobody want to solve it :D

Solution 152 (by lasha).

Here are the solution of both functional equations: $x = 0$ follows $f(y^3) = y f(y^2)$ (1), If $y = 0$, $f(x^3) = x^2 f(x)$ (2). So, $f(x^3 + y^3) = x^2 f(x) + y f(y^2) = f(x^3) + f(y^3)$. So, for any real pair (a, b) we have: $f(a + b) = f(a) + f(b)$ (3). $x = y$ follows: $x^2 f(x) = x f(x^2)$, $f(x^2) = x f(x)$ (4). (3) and (4) gives: $f(x^2 + 2x + 1) = f(x^2) + f(2x) + f(1) = x f(x) + 2f(x) + f(1)$ (5). (4) gives: $f(x^2 + 2x + 1) = f((x+1)^2) = (x+1)f(x+1) = (x+1)(f(x)+f(1)) = x f(x) + x + f(x) + f(1)$ (6). (5) and (6) follows: $f(x) = x f(1) = kx$, where k is any real number. So, $f(x) = kx$. Here is the second one: $y = 0$ follows: $f(x^3) = x^2 f(x)$ (1). $x = y = 0$ follows $f(0) = 0$ (2). $x = 0$ follows:$f(-y^3) = -y f(y^2)$ (3). $x = -y$ and (3) follows:$f(x^3) = x f(x^2)$ (4). (1) and (4) follows:$f(x^2) = x f(x)$ (5). (1) and (3) follows that for any pair of real numbers (z, t), $f(z + t) = f(z) + f(t)$. (6). $f(x^2 + 2x + 1) = f(x^2) + 2f(x) + f(1) = x f(x) + 2f(x) + f(1)$. $f(x^2 + 2x + 1) = f((x + 1)^2) = (x + 1)f(x + 1) = (x + 1)(f(x) + f(1)) = x f(x) + x f(1) + f(x) + f(1)$. Last two equality gives:$f(x) = x f(1) = kx$, where k is any real number. So, $f(x) = kx$. :blush:

Solution 153 (by quangpbc).

Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that

$f(x^5 - y^5) = x^2 f(x^3) - y^2 f(y^3)$
And the same problem :
1,Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that
$f(x^5 - y^5) = x^4 f(x) - y^4 f(y)$
2,Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that
$f(x^5 - y^5) = x^3 f(x^2) - y^3 f(y^2)$

Solution 154 (by hsbhatt)**.**
Cannot these be solved using Jensen's Inequality

Solution 155 (by quangpbc)**.**

> Cannot these be solved using Jensen's Inequality

[color=green]Hnm, what do you mean, **hsbhatt**? What makes you think that we can use Jensen's Inequality to prove them? :roll: :o [/color]

Solution 156 (by sargeist)**.**
Hi,
Can we just do the following?
We are given:
$f(x^3 + y^3) = x^2 f(x) + y f(y^2)$
so substitute $x = y = 0$ to get $f(0) = 0$.
Then, put $x = 0$ and $y = y$ to get: $f(y^3) = y f(y^2)$
and $x = x$ and $y = 0$ to get: $f(x^3) = x^2 f(x)$.
Add these together and compare with the original equation, to get:
$f(x^3 + y^3) = f(x^3) + f(y^3)$
which is just $f(u + v) = f(u) + f(v)$, which is Cauchy's functional equation. This has solution $f(x) = kx$ for some k.
And if we substitute this into the original equation, we find that this works with any k.
Is this ok? Am I allowed to just say: "Ah, the solution of cauchy's functional equation is known to be 'blah'?"

Solution 157 (by TaiPan SP!)**.**
Well the flaw is that we are working with $f : \mathbb{R} \to \mathbb{R}$ and there are more than one function that satisfies $f(x + y) = f(x) + f(y)$ for $x, y, \in \mathbb{R}$.

Solution 158 (by Math pro)**.**

> Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that
> $f(x^5 - y^5) = x^2 f(x^3) - y^2 f(y^3)$
> And the same problem :
> 1,Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that
> $f(x^5 - y^5) = x^4 f(x) - y^4 f(y)$
> 2,Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that
> $f(x^5 - y^5) = x^3 f(x^2) - y^3 f(y^2)$

From NGUYEN TRONG TUAN 's book,anh QUANG! :oops:

Solution 159 (by harpeng)**.**

> Hi,
> Can we just do the following?
> We are given:
> $f(x^3 + y^3) = x^2 f(x) + yf(y^2)$
> so substitute $x = y = 0$ to get $f(0) = 0$.
> Then, put $x = 0$ and $y = y$ to get: $f(y^3) = yf(y^2)$
> and $x = x$ and $y = 0$ to get: $f(x^3) = x^2 f(x)$.
> Add these together and compare with the original equation, to get:
> $f(x^3 + y^3) = f(x^3) + f(y^3)$
> which is just $f(u+v) = f(u) + f(v)$, which is Cauchy's functional equation. This has solution $f(x) = kx$ for some k.
> And if we substitute this into the original equation, we find that this works with any k.
> Is this ok? Am I allowed to just say: "Ah, the solution of cauchy's functional equation is known to be 'blah'?"

As I know, it is allowed to say "the solution of cauchy's functional equation is known to be" when the function is about relative

numbers. You need some more conditions(I can't and won't remember what was them) to say when it is about real numbers.

Solution 160 (by mlequi)**.**

> Hi,
> Can we just do the following?
> We are given:
> $f(x^3 + y^3) = x^2 f(x) + yf(y^2)$
> so substitute $x = y = 0$ to get $f(0) = 0$.
> Then, put $x = 0$ and $y = y$ to get: $f(y^3) = yf(y^2)$
> and $x = x$ and $y = 0$ to get: $f(x^3) = x^2 f(x)$.
> Add these together and compare with the original equation, to get:
> $f(x^3 + y^3) = f(x^3) + f(y^3)$
> which is just $f(u+v) = f(u) + f(v)$, which is Cauchy's functional equation. This has solution $f(x) = kx$ for some k.
> And if we substitute this into the original equation, we find that this works with any k.
> Is this ok? Am I allowed to just say: "Ah, the solution of cauchy's functional equation is known to be 'blah'?"

Continuing from this solution, we have $f(x^3) = xf(x^2) = x^2 f(x)$. Thus, we have $f(x^2) = xf(x)$. Substituting to the original functional equation, we get $f(x^3 + y^3) = xf(x)^2 + yf(y)^2$....(1) Now, suppose a and b are two real numbers such that $a \geq b$. Then, substitute $x = \sqrt[3]{a-b}$ and $y = \sqrt[3]{b}$ to (1), we have $f(a) = \sqrt[3]{a-b}f(\sqrt[3]{a-b})^2 + \sqrt[3]{b}f(\sqrt[3]{b})^2$ Since $f(x^3) = xf(x^2)$, then we have $f(a) = \sqrt[3]{a-b}f(\sqrt[3]{a-b})^2 + f(b)$. We have $f(a) \geq f(b)$. Thus, the function is increasing. Now, it means we can use the Cauchy equation... right? Sorry for my bad English

Solution 161 (by littletush)**.**

by letting $x = 0$ or $y = 0$ etc. we can easily get $f(x^2) = xf(x)$ and $f(x^3 + y^3) = f(x^3) + f(y^3)$ hence f is a Cauchy's function then $f(x^2 + 2x + 1) = (x+1)f(x+1)$ hence $(x+2)f(x) + 1 = (x+1)(f(x)+1)$ yielding $f(x) = x$. QED

Solution 162 (by pco)**.**

> yielding $f(x) = x$.

And what about $f(x) = 0\ \forall x$? And what about $f(x) = 2x\ \forall x$?
You should pay more attention when copying previous posts of the thread.

Solution 163 (by andreass)**.**

You can't be serious! :(Me, after proving that $f(x+y) = f(x)+f(y)$, which is true regardless of the number of summands, I said that it immediately follows that $f(nx) = nf(x)\ \forall n \in N$ and $x \in R$ but since $f(-x) = -f(x)$ and $f(0) = 0$ then $f(nx) = nf(x)\ \forall n \in Z$ Then, I tried to generalise this step by step to n becoming real, by first becoming rational. $f(nx) = nf(x) \Rightarrow f(x) = \frac{1}{n}f(nx)$ $\forall n \in Z - 0 \Rightarrow f(x) = \frac{1}{n} \times mf(\frac{n}{m}x)$ where $m \in Z$. Hence if we substitute $q = \frac{n}{m}$ we get that $f(x) = \frac{1}{q}f(qx)\ \forall q \in Q$ i.e. $f(qx) = qf(x) \Rightarrow f(q) = qf(1)$ $\forall q \in Q$. Now consider the real number r. Denote by $[r]$ the floor function and a_k the k-th decimal digit of r after the point. Note the obviously every $\frac{a_k}{10^k}$ term will be rational. Hence $r = [r] + \sum_{k=1}^{\infty} \frac{a_k}{10^k} \Rightarrow f(r) = f([r] + \sum_{k=1}^{\infty} \frac{a_k}{10^k})$ $= [r]f(1) + \sum_{k=1}^{\infty} \frac{a_k}{10^k}f(1) = ([r] + \sum_{k=1}^{\infty} \frac{a_k}{10^k})f(1) = rf(1)\forall r \in R$ and finally it is generalized to reals!!!! Can anyone comment or tell me if I am correct or not?

Solution 164 (by panos˙lo)**.**

Andreass you are wrong. I 'll give you two reasons to make you realise it: 1) Firstly, it's a well known fact that Cauchy's functional equation has also noncontinuous solutions, as we can see using Hamel basis. 2) Yes, additivity implies that the additive property holds regardless of the number of summands, but it doesn't mean that it holds for the case that they are infinite.

Solution 165 (by Takeya.O)**.**

> which is just $f(u+v) = f(u) + f(v)$, which is Cauchy's functional equation. This has solution $f(x) = kx$ for some k.

You are wrong. :o

Cauchy Equationimplies that $c \in \mathbb{R}$ s.t. $\forall x \in \mathbb{Q} : f(x) = cx$ not $\forall x \in \mathbb{R} : f(x) = cx$.

Thanks, Takeya.O :P

Solution 166 (by Takeya.O)**.**

> Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
> $$f(x^3 + y^3) = x^2 f(x) + y f(y^2)$$
> for all $x, y \in \mathbb{R}$.

Let $P(x, y)$ be $f(x^3 + y^3) = x^2 f(x) + y f(y^2)$.

$P(x, 0) \to f(x^3) = x^2 f(x), P(0, x) \to f(x^3) = x f(x^2) \to f(0) = 0$ and $\forall x \neq 0, x^2 f(x) = x f(x^2) \Leftrightarrow f(x^2) = x f(x)$ which also holds at $x = 0$.

Thus $f(x^3 + y^3) = f(x^3) + f(y^3) \to \forall a, \forall b \in \mathbb{R} : f(a + b) = f(a) + f(b)$. Then $\forall x \in \mathbb{R}, \forall k \in \mathbb{Q} : f(kx) = kf(x)$.

On the other hand,for $\forall x \in \mathbb{R}, \forall k \in \mathbb{Q}$, $f((x+k)^3) = (x+k)^2 f(x+k).LHS = f(x^3 + 3x^2k + 3xk^2 + k^3) = f(x^3) + 3kf(x^2) + 3k^2 f(x) + k^3 f(1).RHS = (x^2 + 2xk + k^2)(f(x) + kf(1)) = x^2 f(x) + x^2 f(1)k + 2xf(x)k + 2xf(1)k^2 + f(x)k^2 + f(1)k^3$.Since $LHS = RHS$, $2(f(x) - f(1)x)k^2 + (xf(x) - f(1)x^2)k = 0$.For fixed $x \in \mathbb{R}$,this holds at infinitely many $k \in \mathbb{Q}$.Hence $f(x) - f(1)x = 0 \Leftrightarrow f(x) = f(1)x$.Conversely $\forall a \in \mathbb{R}, f(x) = ax(\forall x \in \mathbb{R})$ which satisfies the condition.

Therefore the answer is $\boxed{\forall a \in \mathbb{R} : f(x) = ax(\forall x \in \mathbb{R})}$ ■ :coool:

****

Problem 31 (Posted by N.T.TUAN)**.** Let n be a natural number divisible by 4. Determine the number of bijections f on the set $\{1, 2, ..., n\}$ such that $f(j) + f^{-1}(j) = n + 1$ for $j = 1, ..., n$.

(Link to AoPS)

Solution 167 (by pco)**.**

$j = f(x) \Rightarrow f(f(x)) = n + 1 - x$ (notice here that $f(x) \neq x$ since n is even).

So, if I call $y = f(x)$, we have, applying $f()$ many times, 4-numbers cycles $x, y, n + 1 - x, n + 1 - y, x, y, n + 1 - x, n + 1 - y, ...)$

In each cycle, the four numbers are different and exactly two of them are in $\{1, 2, ..., \frac{n}{2}\}$. There are $(\frac{n}{2} - 1)(\frac{n}{2} - 3)...3 * 1$ ways for grouping (x, y)s and each of the $\frac{n}{4}$ couples may lead to 2 cycles $(x, y, n + 1 - x, n + 1 - y)$ or $(x, n + 1 - y, n + 1 - x, y)$

So the requested number is $((\frac{n}{2} - 1)(\frac{n}{2} - 3)...3 * 1)2^{\frac{n}{4}}$ And, since $(\frac{n}{2} - 1)(\frac{n}{2} - 3)...3 * 1 = \frac{(\frac{n}{2})!}{2^{\frac{n}{4}}(\frac{n}{4})!}$, the result is $\frac{(\frac{n}{2})!}{(\frac{n}{4})!}$ – Patrick

****

Problem 32 (Posted by N.T.TUAN)**.** Let $\mathbb{R}^*$ denote the set of nonzero real numbers. Find all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ such that

$$f(x^2+y) = f^2(x) + \frac{f(xy)}{f(x)}\ \forall x, y \in \mathbb{R}^*, x^2+y \in \mathbb{R}^*.$$

(Link to AoPS)

Solution 168 (by pco)**.**

Let $\mathbb{R}^*$ denote the set of nonzero real numbers. Find all functions f : $\mathbb{R}^* \to \mathbb{R}^*$ such that

$$f(x^2+y) = f^2(x) + \frac{f(xy)}{f(x)}\ \forall x, y \in \mathbb{R}^*, x^2+y \in \mathbb{R}^*.$$

P1 : $f(x^2+y) = f^2(x) + \frac{f(xy)}{f(x)}\ \forall x, y \in \mathbb{R}^*, x^2+y \in \mathbb{R}^*$

1) P2 : $f(x^2+1) = f^2(x)+1\ \forall x, \in \mathbb{R}^*$ Easy with P1 and $y=1$

2) $f(-x) = -f(x)\ \forall x \neq 0$ $-x$ in P2 $\Rightarrow f((-x)^2+1) = f^2(-x)+1 = f(x^2+1) = f^2(x)+1 \Rightarrow f(-x) = \pm f(x)$ Let then $\phi = \frac{1+\sqrt{5}}{2}$, we have $\phi^2 = \phi+1$ If $f(-\phi) = f(\phi)$, then take $x=\phi$ and $y=-1$ in P1 and $f(\phi^2-1) = f^2(\phi)+1$,$\Rightarrow f(\phi) = f^2(\phi)+1$ which have no real solution $\Rightarrow f(-\phi) = -f(\phi)$ Then P1 gives : $f(\phi^2+\frac{x}{\phi}) = f^2(\phi) + \frac{f(x)}{f(\phi)}$ but also $f((-\phi)^2+\frac{x}{\phi}) = f^2(-\phi) + \frac{f(-x)}{f(-\phi)}$ and so $\frac{f(x)}{f(\phi)} = \frac{f(-x)}{f(-\phi)}$ and so $f(-x) = -f(x)$. Q.E.D.

3) $f(1) = 1$ Let $a = f(1)$, $b = f(2)$ and $c = f(3)$ Then $x=1$ and $y=1$ in P1 gives $b = a^2+1$ Then $x=1$ and $y=2$ in P1 gives $c = a^2+\frac{b}{a}$ Then $x=2$ and $y=-1$ in P1 gives $c = b^2-1$ This system is easy to solve ang gives $f(1)=1$

4) $f(x+1) = f(x)+1\ \forall\ x \neq 0$ and $x \neq -1$ Just use P1 with 1 and x

5) $f(x^2) = f^2(x)\ \forall\ x \neq 0$ (and, as a consequence, $f(x) \geq 0\ \forall\ x > 0$) Immediate from points 1) and 4) above.

6) $f(a+b) = f(a)+f(b)$ for $a \neq 0$, $b \neq 0$, $a+b \neq 0$ P1 gives $f(x^2+y) = f^2(x) + \frac{f(xy)}{f(x)}$ Using $y+1$ instead of y, P1 gives $f(x^2+y+1) = f^2(x) + \frac{f(xy+x)}{f(x)}$ And, since (from 4) above), $f(x^2+y+1) = f(x^2+y)+1$ for $x^2+y \neq 0, -1$: $f^2(x) + \frac{f(xy+x)}{f(x)} = f^2(x) + \frac{f(xy)}{f(x)} + 1$ and $f(xy+x) = f(xy)+f(x)$ Taking $x=b$ and $y=\frac{a}{b}$ gives the result. The problem of "$x^2+y \neq 0, -1$", if $b^2+\frac{a}{b} = 0, -1$ may be solved (I've not checked) by using $y=b$ and $x=\frac{a}{b}$ or $y-1$ and not $y+1$ in P1.

7) $f(x) = x$ $f(a+b) = f(a)+f(b)$ gives easily $f(x) = x\ \forall x \in \mathbb{Q}^*$ And since $f(b) \geq 0$ for $b > 0$ (see 5)), then $f(x)$ is monotonous increasing and so $f(x) = x$ $\forall x \in \mathbb{R}^*$

And it is immediate to verify that this necessary condition works.

****

—————————————————————————-

Problem 33 (Posted by sinajackson)**.** Find all real numbers α for which there exists a non constant function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, we have

$$f(\alpha(x + y)) = f(x) + f(y).$$

(Link to AoPS)

Solution 169 (by Hawk Tiger)**.**

> Find all numbers like α such that there exist a non constant function $f : \mathbb{R} \to \mathbb{R}$ where for $x, y \in \mathbb{R}$ we have: $f(\alpha(x + y)) = f(x) + f(y)$

The only possible α is 1. Let $x = y = 0$ We get that $f(0) = 0$ Let $x = 0$ we get that $f(\alpha y) = f(y)$ i.e.$f(y) = f(\frac{1}{\alpha}y)(\alpha \neq 0$ is obvious)......................(1) Then let $x = \frac{1}{\alpha}u, y = \frac{1}{\alpha}v$ We get that $f(x+y) = f(\frac{1}{\alpha}x)+f(\frac{1}{\alpha}y) = f(x)+f(y)$ (from (1))................................(2) We assume that $\alpha \neq 1$ In sake of contradition,we consider $f(x) - f(y)$

$$f(x)-f(y) = f(\frac{(\alpha-1)x-y}{\alpha-1}+\frac{\alpha y}{\alpha-1})-f(y) = f(\frac{(\alpha-1)x-y}{\alpha-1})+f(\frac{\alpha y}{\alpha-1})-f(y)$$

(becuase of (2))

$$= f(\frac{(\alpha-1)x-y}{\alpha-1}) + f(\frac{y}{\alpha-1}) - f(y)$$

(because of (1))

$$= f(\frac{(\alpha-1)x-y}{\alpha-1} + \frac{y}{\alpha-1}) - f(y) = f(y) - f(y) = 0$$

(becuase of (2)) Then $f(x) = f(y)$, contradition!

Solution 170 (by pco)**.**

> Find all numbers like α such that there exist a non constant function $f : \mathbb{R} \to \mathbb{R}$ where for $x, y \in \mathbb{R}$ we have: $f(\alpha(x + y)) = f(x) + f(y)$

If $\alpha \neq 1$, Let $y = \frac{-\alpha x}{\alpha-1}$ then $\alpha(x + y) = y$ and $f(y) = f(x) + f(y)$ and $f(x) = 0\ \forall x$

So the only value is $\alpha = 1$ (which allows for example $f(x) = x$)

****

Problem 34 (Posted by sinajackson)**.** Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have $f(x) = f(x^2 + c)$, where $c \in \mathbb{R}$ is a constant number!

(Link to AoPS)

Solution 171 (by Rust)**.**

For $x \in R$ define set $C(x) = \{x_0 = x, x_{n+1} = x_n^2 + c\}$ For any x,y and ϵ exist $z, t, |z| \leq |c| + 1, |t| \leq |c| + 1$, suth that $x \in C_z, y \in C_t, |z - t| < \epsilon$. Therefore only constant functions work.

Solution 172 (by pco)**.**

> Therefore only constant functions work.

Surely not. If $c > \frac{1}{4}$, for example, there are infinety many nonconstant solutions :

Let g(x) be any continuous function defined on $[0, c]$ and such that $g(c) = g(0)$ Then let a_i be : $a_0 = 0$ $a_{n+1} = a_n^2 + c$ $\forall n \geq 0$

Since $c > \frac{1}{4}$, a_n is strictly increasing and $\lim_{n \to +\infty} a_n = +\infty$

Then we can define $f(x)$: For $x \in [a_0, a_1)$ $f(x) = g(x)$ For $x \in [a_n, a_{n+1})$ $f(x) = f(\sqrt{x - c})$ $\forall n > 0$ For $x < 0$, $f(x) = f(-x)$

****

———————————————————————————-

Problem 35 (Posted by Chessy)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) + f\left(1 - \frac{1}{x}\right) = x,$$

holds for all real x.

(Link to AoPS)

Solution 173 (by N.T.TUAN)**.**

First, this is a topic has bad title : http://www.mathlinks.ro/Forum/viewtopic.php?t=144403 . But I'll not lock it because you are a new member

2-nd, I will help you :D . If put $g(x) = 1 - \frac{1}{x}$ then $g(g(g(x))) = x$ and that's all!

Solution 174 (by pco)**.**

> What is $f(x)$ in this equation, can anyone solve it in a "step by step" way for me? Thank you !
> $f(x) + f(1 - \frac{1}{x}) = x$

To go further in the remark of N.T.TUAN, you have :

E1 : $f(x) + f(g(x)) = x$ with $g(x) = 1 - \frac{1}{x}$. So you have : E2 : $f(g(x)) + f(g(g(x))) = g(x)$ and you look at $g(g(x)) = \frac{1}{1-x}$ (nothing special) E3 : $f(g(g(x))) + f(g(g(g(x)))) = g(g(x))$ and then you discover that $g(g(g(x))) = x$ So you have : E3 : $f(g(g(x))) + f(x) = g(g(x))$

Then E1-E2+E3 gives : $2f(x) = x - g(x) + g(g(x))$ and then $f(x) = \frac{1}{2}(x - 1 + \frac{1}{x} + \frac{1}{1-x})$ And it is easy to check that this necessary condition works.

Solution 175 (by Chessy)**.**

Thank you very much sir, you have been a great help.

Take care

****

Problem 36 (Posted by Zaratustra)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(0) = 2f^2(x) - f(2x) \quad \forall x \in \mathbb{R}.$$

(Link to AoPS)

Solution 176 (by pco)**.**

> Find all functions $f : \mathbb{R} \to \mathbb{R}$ who satisfy:
> $f(0) = 2f^2(x) - f(2x)$

This is a rather frequent problem whose general form is $f(2x) = h(f(x))$ (here $h(x) = 2x^2 - a$)

In this case, you write : 1) $f(0) = h(f(0))$, so $f(0)$ is any fixed point of h. In this special case, $f(0) = 0$ or $f(0) = 1$ 2) for $x > 0$ $f(x) = g(\frac{\ln(x)}{\ln(2)})$ with $g(x)$ such that $g(x+1) = h(g(x))$, so $g(x+p) = h^{\circ p}(g(x))$ (where $f^{\circ p}(x)$ means $f \circ f \circ f \circ \ldots \circ f(x)$ (p times)).

So you just have to choose any function $a(x)$ defined on $(0, 1]$ and then $g(x) = h^{\circ \lfloor x \rfloor}(a(x - \lfloor x \rfloor)$

But CAUTION : in order $h^{\circ p}(x)$ exists for any $p < 0$, the best case is $h(x)$ bijective function. When this is not the case, special attention must be made on these "negative" compositions.

3) For $x < 0$ $f(x) = g(\frac{\ln(-x)}{\ln(2)})$ with $g(x)$ such that $g(x + 1) = h(g(x))$ ...

So the general answer is : Let $a(x)$ and $b(x)$ be any functions defined on $(0, 1]$. Then the solutions to the equation $f(2x) = h(f(x))$ are :

1) $f(0)$ is any fixed point of $h(x)$ 2) For $x > 0$, $f(x) = h^{\circ \lfloor \frac{\ln(x)}{\ln(2)} \rfloor}(a(\frac{\ln(x)}{\ln(2)} - \lfloor \frac{\ln(x)}{\ln(2)} \rfloor)$ 3) For $x < 0$, $f(x) = h^{\circ \lfloor \frac{\ln(-x)}{\ln(2)} \rfloor}(b(\frac{\ln(-x)}{\ln(2)} - \lfloor \frac{\ln(-x)}{\ln(2)} \rfloor)$

In our special case, there are two subtle modification : 1) Let $f(0) = a$ with either $a = 0$, either $a = 1$ and let $h(x) = 2x^2 - a$

We need to define $h^{\circ -1}(x)$ and I suggest $h^{\circ -1}(x) = \sqrt{\frac{x+a}{2}}$ and so we need $a(x) \geq 0$ and $b(x) \geq 0$ (for $a = 1$, this choice may lead to forget some solutions, I think).

2) For $x > 0$, $f(x) = h^{\circ \lfloor \frac{\ln(x)}{\ln(2)} \rfloor}(a(\frac{\ln(x)}{\ln(2)} - \lfloor \frac{\ln(x)}{\ln(2)} \rfloor)$ 3) For $x < 0$, $f(x) = h^{\circ \lfloor \frac{\ln(-x)}{\ln(2)} \rfloor}(b(\frac{\ln(-x)}{\ln(2)} - \lfloor \frac{\ln(-x)}{\ln(2)} \rfloor)$

If the continuity is demanded, some conditions more on $a(x)$ and $b(x)$ exist.

****

Problem 37 (Posted by A3K08)**.** Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) + f(n+1) = n+2, \quad \forall n \in \mathbb{N}.$$

(Link to AoPS)

Solution 177 (by pco)**.**

> Find all functions $f : N \to N$ such that: $f(f(n))+f(n+1) = n+2, \forall n \in N$

Nice question. We have $P(n) : f(f(n)) + f(n+1) = n+2$

We have obviously $f(n+1) < n+2 \ \forall n \in N$, so $f(n) \leq n \ \forall n > 1$ Let $a = f(1)$

$P(1) \implies f(a) + f(2) = 3$ and then $f(2)$ may be 1 or 2

1) $f(2) = 1$ We have then : (i)$f(1) = a$, (ii)$f(2) = 1$ and (iii)$f(a) = 2$ Then $P(2) \implies f(f(2)) + f(3) = 4 \implies$ (iv)$f(3) = 4 - a$. Then a can only be $1, 2$ or 3. $a = 1 \implies$ (i) and (iii) are in contradiction. $a = 2 \implies$ (ii) and (iii) are in contradiction. $a = 3 \implies$ (iii) and (iv) are in contradiction. So $f(2) \neq 1$

2) then $f(2) = 2$ We have then : $f(1) = a$, $f(2) = 2$ and $f(a) = 1$ Then $P(2) \implies f(f(2))+f(3) = 4 \implies f(3) = 2$. Then $P(3) \implies f(f(3))+f(4) = 5 \implies f(4) = 3$. Then $P(4) \implies f(f(4)) + f(5) = 6 \implies f(5) = 4$.

By induction, it is easy to show that $n > f(n) \geq 2 \ \forall n \geq 3$: a) : It's OK for $n = 3 : 3 > f(3) = 2 \geq 2$ b) : If it's OK $\forall k \in [3, n]$, then $P(n) \implies f(n+1) = n+2-f(f(n))$ Then, if $f(n) = 2$, $f(n+1) = n$ and $n+1 > f(n+1) \geq 2$ Else, if $f(n) > 2$, $f(n) \in [3, n)$ and so $f(n) > f(f(n)) \geq 2$ and so $n \geq n+2-f(f(n)) > n+2-f(n) > 2$ and so $n+1 > f(n+1) > 2$ and the induction is verified.

So, $f(n) \geq 2 \forall n \geq 2$ and since $f(a) = 1$, we must have $a = 1$

Then, since we have $f(n) \leq n \ \forall n > 1$ and $f(1) = 1 \leq 1$, we have $f(n) \leq n$ $\forall n > 0$

And so, we have a unique solution completely defined by : $f(1) = 1$ $f(n) = n+1-f(f(n-1)) \ \forall n > 1$ Which gives values $1, 2, 2, 3, 4, 4, 5, 5, 6, 7, 7, ...$

I don't know if there is a closed form for this unique solution but it's near of $\frac{-1+\sqrt{5}}{2}n$

I'll continue to search for such a closed form.

Solution 178 (by pco)**.**

> And so, we have a unique solution completely defined by : $f(1) = 1$ $f(n) = n+1-f(f(n-1)) \ \forall n > 1$ Which gives values $1, 2, 2, 3, 4, 4, 5, 5, 6, 7, 7, ...$
> I don't know if there is a closed form for this unique solution but it's near of $\frac{-1+\sqrt{5}}{2}n$
> I'll continue to search for such a closed form.

Ok, the closed form for the unique solution is $f(n) = 1 + \lfloor \frac{-1+\sqrt{5}}{2}n \rfloor$

Demo by induction : Let $\alpha = \frac{-1+\sqrt{5}}{2} = 0.618\ldots$. Notice that we have $\alpha^2 + \alpha - 1 = 0$ a) It's true for $n = 1$: $f(1) = 1 = 1 + \lfloor\alpha\rfloor$

b) assume it's true for any $0 < k \leq n$ Then we have $f(n+1) = n + 2 - f(f(n)) = n + 2 - 1 - \lfloor\alpha f(n)\rfloor = n + 1 - \lfloor\alpha + \alpha\lfloor\alpha * n\rfloor\rfloor$ We must show that $f(n+1) = 1 + \lfloor\alpha * n + \alpha\rfloor$ So we must show that $\lfloor\alpha * n + \alpha\rfloor = n - \lfloor\alpha + \alpha\lfloor\alpha * n\rfloor\rfloor$ Or : $n = \lfloor\alpha * n + \alpha\rfloor + \lfloor\alpha + \alpha\lfloor\alpha * n\rfloor\rfloor$

This equality is rather easy to verify: Let $\alpha * n = k + r$ with $r \in (0, 1)$ Then $S = \lfloor\alpha * n + \alpha\rfloor + \lfloor\alpha + \alpha\lfloor\alpha * n\rfloor\rfloor = \lfloor k + r + \alpha\rfloor + \lfloor\alpha + \alpha * k\rfloor$ So $S = \lfloor k + r + \alpha\rfloor + \lfloor\alpha + \alpha^2 * n - \alpha * r\rfloor$ So $S = \lfloor k + r + \alpha\rfloor + \lfloor\alpha + n - \alpha * n - \alpha * r\rfloor$ So $S = \lfloor k + r + \alpha\rfloor + \lfloor\alpha + n - k - r - \alpha * r\rfloor$

Then : 1) $r = 1 - \alpha$ is impossible since we would have $\alpha * n = k + 1 - \alpha$ and α would be rational. 2) If $0 < r < 1 - \alpha$, then $r(1 + \alpha) < 1 - \alpha^2 = \alpha$ then $0 < \alpha - r - \alpha * r < \alpha < 1$ and we have $\lfloor k + r + \alpha\rfloor = k$ $\lfloor\alpha + n - k - r - \alpha * r\rfloor = n - k$ And $S = n$ 3) If $2 > r + \alpha > 1$, then $r(1 + \alpha) > 1 - \alpha^2 = \alpha$ then $-1 < -r < \alpha - r - \alpha * r < 0$ and we have $\lfloor k + r + \alpha\rfloor = k + 1$ $\lfloor\alpha + n - k - r - \alpha * r\rfloor = n - k - 1$ And $S = n$ Q.E.D.

Solution 179 (by A3K08)**.**

I have tried for two hours to prove this problem but I not sure is correct. I find that $f(n) = [\frac{1+\sqrt{5}}{2}n] - n + 1$ and I used the following results: i) For each $n \in N$, $[\alpha([n\alpha] - n + 1)] = n$ or $n + 1$ ii)For each $n \in N$, $[(n + 1)\alpha] = [n\alpha] + 2$ if $[\alpha([n\alpha] - n + 1)] = n$ where $\alpha = \frac{1+\sqrt{5}}{2}$

****

Problem 38 (Posted by A3K08)**.** Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers n, we have

$$f(n) + f(n + 1) = f(n + 2)f(n + 3) - k,$$

where $k = p - 1$ for some prime p.

(Link to AoPS)

Solution 180 (by pco)**.**

> Find all $f : N \to N$ such that: $f(n) + f(n + 1) = f(n + 2)f(n + 3) - k$, where $k = p - 1$ for some prime p

Quite nice !

$E1(n)$: $f(n) + f(n+1) = f(n+2)f(n+3) - p + 1$ with p prime $E1(n+1)$: $f(n+1) + f(n+2) = f(n+3)f(n+4) - p + 1$

$E1(n+1) - E1(n)$: $E2(n)$: $f(n+2) - f(n) = f(n+3)(f(n+4) - f(n+2)$

1) $f(1) = f(3) = a$ and $f(2) = f(4) = b$ Then property E2 shows that $f(2k) = b$ and $f(2k - 1) = a$ for any $k > 0$ Then property E1 shows that $a + b = ab + 1 - p$, so $(a - 1)(b - 1) = p$, so $(a, b) = (2, p + 1)$ or $(p + 1, 2)$

2) $f(1)=f(3)=a$ and $f(4)=f(2)+c$ with $c\in\mathbb{Z}^*$ Then property E2 shows that $f(2k-1)=a$ for any $k>0$ And property E2 shows that $f(4)-f(2)=a(f(6)-f(4))=a^2(f(8)-f(6))=a^3(f(10)-f(8))=...$ And obviously $a=1$ and $f(2k)=f(2)+(k-1)c$ and $c>0$ Then $E1(2k) \implies f(2)+(k-1)c+1=f(2)+kc-p+1$ and so $c=p$ and $E2(2k+1)$ is also verified

3) $f(3)=f(1)+c$ with $c\in\mathbb{Z}^*$ Then property E2 shows that $f(3)-f(1)=c=f(4)(f(5)-f(3))=f(4)f(6)(f(7)-f(5))=f(4)f(6)f(8)(f(9)-f(7))=...$ And obviously $f(2k)=1$ for any $k>1$ and $f(2k-1)=f(1)+(k-1)c$ for any $k>0$ and c must be >0 With $E2(2)$ we also have $f(2)=1$ Then $E1(2k) \implies 1+f(1)+kc=f(1)+(k+1)c-p+1$ and so $c=p$ and $E2(2k+1)$ is also verified

So, we have 4 families of solutions :

Family F1 : $f(1,2,3,...)=2,p{+}1,2,p{+}1,2,p{+}1,...$ Family F2 : $f(1,2,3,...)=p{+}1,2,p{+}1,2,p{+}1,2,...$ Family F3 : $f(1,2,3,...)=1,a,1,a{+}p,1,a{+}2p,1,a+3p,1,a+4p,...$ Family F4 : $f(1,2,3,...)=a,1,a+p,1,a+2p,1,a+3p,1,a+4p,1,...$

Solution 181 (by aviateurpilot)**.**

> So, we have 4 families of solutions :
> Family F1 : $f(1,2,3,...)=2,p+1,2,p+1,2,p+1,...$ Family F2 : $f(1,2,3,...)=p+1,2,p+1,2,p+1,2,...$ Family F3 : $f(1,2,3,...)=1,a,1,a+p,1,a+2p,1,a+3p,1,a+4p,...$ Family F4 : $f(1,2,3,...)=a,1,a+p,1,a+2p,1,a+3p,1,a+4p,1,...$

good
F1: $2f(n)=(-1)^n(p-1)+p+3$
F2: $2f(n)=(-1)^{n+1}(p-1)+p+3$
F3: $4f(n)=(-1)^n(p(n-2)+2a-2)+2a+2+p(n-2)$
F4: $4f(n)=(-1)^{n+1}(p(n-1)+2a-2)+2a+2+p(n-1)$

****

Problem 39 (Posted by Modul)**.** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(f(x))=5f(x)-4x,\quad \forall x\in\mathbb{R}.$$

(Link to AoPS)

Solution 182 (by pco)**.**

> Find all functions $f:R\to R$ such that $f(f(x))=5f(x)-4x, \forall x\in R$.

Besides the two immediate trivial solutions $f(x)=x$ and $f(x)=4x+b$, this problem has infinitely many solutions.

From $f(f(x))=5f(x)-4x$, it is easy to see that :

1) $f(x)$ is a bijective function 2) $f^{\circ n}(x) = \frac{4^n-1}{3}f(x) - \frac{4^n-4}{3}x$ (where $f^{\circ n}(x) = f \circ f \circ \ldots \circ f(x)$ n times) 3) $f(f(x)) - f(x) = 4(f(x) - x)$ and so $f(x) > x \Leftrightarrow f(f(x)) > f(x)$ 4) $g^{\circ n}(x) = \frac{4}{3}(1 - \frac{1}{4^n})g(x) - \frac{1}{3}(1 - \frac{1}{4^{n-1}})x$ where $g(x) = f^{\circ(-1)}(x)$ is the reciprocical function of $f(x)$ 5) $g(g(x)) - g(x) = \frac{1}{4}(g(x) - x)$ and so $g(x) > x \Leftrightarrow g(g(x)) > g(x)$

With these 5 statements, it is easy to build a lot of solutions :

a) Take any couple $(a, b > a)$ and any bijective function $h(x)$ $[a, b] \to [b, 5b - 4a]$ b) Define $f(x) = h(x)$ for any $x \in [a, b]$ c) Using points 2) and 3) above, it's easy to build $f(x)$ on $[b, +\infty)$ d) Using points 4) and 5) above, it's easy to build $f(x)$ on $(\frac{4a-b}{3}, a)$ e) Take $f(\frac{4a-b}{3}) = \frac{4a-b}{3}$ f) Take any couple $(c < \frac{4a-b}{3}, d < c)$ and any bijective function $k(x)$ $[d, c] \to [5d - 4c, d]$ g) Define $f(x) = k(x)$ for any $x \in [d, c]$ h) Using points 2) and 3) above, it's easy to build $f(x)$ on $(-\infty, d]$ i) Using points 4) and 5) above, it's easy to build $f(x)$ on $[c, \frac{4a-b}{3})$

Such construction gives infinitely many continuous increasing solutions.

If you add the constraint that $f(x)$ must be differentiable, this constraint for $x = \frac{4a-b}{3}$ would imply $f(x) = x$ or $f(x) = 4x + c$

****

Problem 40 (Posted by The soul of rock)**.** Let $a > 0$ be a constant not equal to 1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(ax) = f(x) + 2f(-x),$$

for all real x.

(Link to AoPS)

Solution 183 (by pco)**.**

> Let $a = const > 0$ and $a \neq 1$. Find all functions $f(x)$ such that
>
> $$f(ax) = f(x) + 2f(-x)$$

We have $f(0) = 0$. Consider now that $f(x)$ is defined for $x > 0$, then it will be defined, for $x < 0$ by $f(x) = \frac{f(-ax) - f(-x)}{2}$ and the property is verified for any $x \geq 0$.

In order the property be verified for any $x < 0$, we must have, for any $x > 0$, $f(-ax) = f(-x) + 2f(x)$, and so $\frac{f(a^2x) - f(ax)}{2} = \frac{f(ax) - f(x)}{2} + 2f(x)$, and so $f(a^2x) = 2f(ax) + 3f(x)$

So the problem is equivalent to find a function $f(x)$ such that $f(a^2x) = 2f(ax) + 3f(x)$ $\forall x > 0$, and to expand $f(0) = 0$ and $f(x) = \frac{f(-ax) - f(-x)}{2}$ $\forall x < 0$

For $x > 0$, let $f(x) = g(\frac{\ln(x)}{\ln(a)})$. We have then $g(\frac{\ln(x)}{\ln(a)} + 2) = 2g(\frac{\ln(x)}{\ln(a)} + 1) + 3g(\frac{\ln(x)}{\ln(a)})$ $\forall x > 0$, and hence $g(x+2) = 2g(x+1) + 3g(x)$ $\forall x$ This last equation is rather well known and we have $g(x+n) = \frac{3^n - (-1)^n}{4}g(x+1) + \frac{3^n + 3(-1)^n}{4}g(x)$

So the solution for $g(x)$ is : $g(x)=\frac{3^{\lfloor x\rfloor}-(-1)^{\lfloor x\rfloor}}{4}h(x-\lfloor x\rfloor+1)+\frac{3^{\lfloor x\rfloor}+3(-1)^{\lfloor x\rfloor}}{4}h(x-\lfloor x\rfloor)$, with $h(x)$ any function defined on $[0,2)$

And so the solution for $f(x)$ is :

For $x>0$, $f(x)=\frac{3^{\lfloor\frac{\ln(x)}{\ln(a)}\rfloor}-(-1)^{\lfloor\frac{\ln(x)}{\ln(a)}\rfloor}}{4}h(\frac{\ln(x)}{\ln(a)}-\lfloor\frac{\ln(x)}{\ln(a)}\rfloor+1)+\frac{3^{\lfloor\frac{\ln(x)}{\ln(a)}\rfloor}+3(-1)^{\lfloor\frac{\ln(x)}{\ln(a)}\rfloor}}{4}h(\frac{\ln(x)}{\ln(a)}-\lfloor\frac{\ln(x)}{\ln(a)}\rfloor)$, with $h(x)$ any function defined on $[0,2)$

For $x=0$, $f(0)=0$;

For $x<0$, $f(x)=\frac{f(-ax)-f(-x)}{2}$

Examples : 1) $h(x)=0\implies f(x)=0$

2) $h(x)=2\implies$ For $x>0$, $f(x)=3^{\lfloor\frac{\ln(x)}{\ln(a)}\rfloor}+(-1)^{\lfloor\frac{\ln(x)}{\ln(a)}\rfloor}$ For $x=0$, $f(0)=0$; For $x<0$, $f(x)=3^{\lfloor\frac{\ln(x)}{\ln(a)}\rfloor}-(-1)^{\lfloor\frac{\ln(x)}{\ln(a)}\rfloor}$

Solution 184 (by The soul of rock)**.**

I need a solution base on cyclic function! Could you help me! Thanks!

Solution 185 (by pco)**.**

> I need a solution base on cyclic function! Could you help me! Thanks!

I don't understand. Give some precisions.

My solution is, imho, the general one (all solutions has this form for some h(x)).

Solution 186 (by Just)**.**

pco,

Could you please give some details about the solutions to the "rather well known equation". Thanks!

Solution 187 (by pco)**.**

> pco,
> Could you please give some details about the solutions to the "rather well known equation". Thanks!

I'm sorry if I was not clear enough. I say that functional equation $f(x+2)=af(x+1)+bf(x)$ ($a\neq 0$ and $b\neq 0$) is rather simple and well known (according to me :blush:) : If you define $f(x)$ as any function on $[0,2)$, then you can define it (thru $f(x+2)=af(x+1)+bf(x)$ with x in $[0,1)$) on $[2,3)$ and then on $[3,4)$ and so on for any $x\geq 0$. And $f(x+2)=af(x+1)+bf(x)$ ($a\neq 0$ and $b\neq 0$) $\implies f(x)=-\frac{a}{b}f(x+1)-\frac{1}{b}f(x+2)$ and so $f(x)$ can be defined on $(-\infty,0)$ in the same way.

In order to have a concrete expression of the solution, we can write $f(x+n)=u_nf(x+1)+v_nf(x)$ with u_n and v_n defined as : $u_{n+1}=au_n+v_n$ $v_{n+1}=bu_n$ $u_0=0$, $u_1=1$, $v_0=1$, $v_1=0$,.

Or else : $u_{n+1}=au_n+bu_{n-1}$, $u_0=0$, $u_1=1$ and $v_n=bu_{n-1}$

Once you have u_n and v_n, you say : $f(x) = h(x)\ \forall x \in [0,2)$ with $h(x)$ being any function defined in $[0,2)$ and : $f(x) = f(\lfloor x\rfloor + \{x\}) = u_{\lfloor x\rfloor} f(\{x\}+1) + v_{\lfloor x\rfloor} f(\{x\})$ (with $\{x\}$ being the fractional part of x) and then :

$f(x) = u_{\lfloor x\rfloor} h(\{x\}+1) + v_{\lfloor x\rfloor} h(\{x\})$

Solution 188 (by Just)**.**

Ok, got you. Thanks.

I was thinking of another method, what do you think?

$f(ax) = f(x)+2f(-x)$ and $f(-ax) = f(-x)+2f(x)$ hence $f(ax)+f(-ax) = 3(f(x)+f(-x))$.

Letting $g(x) = f(x)+f(-x)$ we get $g(ax) = 3g(x)$ whose solutions are (considering that $g(0) = 0$) $kx^{\frac{\ln 3}{\ln a}}$ (not sure they are the only ones).

Then, we try solving $f(x)+f(-x) = kx^{\frac{\ln 3}{\ln a}}$.

Solution 189 (by pco)**.**

> Ok, got you. Thanks.
> I was thinking of another method, what do you think?
> $f(ax) = f(x)+2f(-x)$ and $f(-ax) = f(-x)+2f(x)$ hence $f(ax)+f(-ax) = 3(f(x)+f(-x))$.
> Letting $g(x) = f(x)+f(-x)$ we get $g(ax) = 3g(x)$ whose solutions are (considering that $g(0) = 0$) $kx^{\frac{\ln 3}{\ln a}}$ (not sure they are the only ones).
> Then, we try solving $f(x)+f(-x) = kx^{\frac{\ln 3}{\ln a}}$.

1) Yes, it could be a method and I think you'll obtain similar results.

2) Caution : $kx^{\frac{\ln 3}{\ln a}}$ is absolutly not the only solution of $g(ax) = 3g(x)$

****

Problem 41 (Posted by Kondr)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(xf(x)+f(y)\right) = \left(f(f(x))\right)^2 + y \quad \forall x,y \in \mathbb{R}.$$

(Link to AoPS)

Solution 190 (by pco)**.**

> Find all functions $f : \mathbb{R} \to \mathbb{R}$:
> $f\left(xf(x)+f(y)\right) = \left(f(f(x))\right)^2 + y$

$E1(x,y) :\ f\left(xf(x)+f(y)\right) = \left(f(f(x))\right)^2 + y$ Let $f(0) = a$ and $f(a) = b$

$E1(0,x) :\ f(f(x)) = x + b^2 \implies E2(x) :\ f(x+b^2) = f(x) + b^2$

$E1(0,0) :\ b = b^2 \implies b = 0$ or $b = 1$

1) If $b = 1$, we have $f(f(x)) = x+1$ and $f(x+1) = f(x)+1$ and so $E1(x,y)$ becomes $f(xf(x)+f(y)) = (x+1)^2 + y$. Then $f(0) = a$ implies $f(-1) = a-1$ Then $E1(-1,0)$ gives $f(a - f(-1)) = 0$ so $f(1) = 0$ and $f(f(1)) = a$ but

$f(f(1)) = 2$ and so $a = 2$ and, since $f(0) = a = 2$, we have $f(n) = n+2$ for any integer n. But then $E1(1,0)$ is wrong

2) So $b = 0$, and then $E1(0,x)$ gives $f(f(x)) = x$ and then $E1(x,0)$ gives $f(xf(x)) = x^2$ and $E1(f(x),0)$ gives $f(f(x)x) = f(x)^2$. So $f(x)^2 = x^2$

So either $f(x) = x$, either $f(x) = -x$. The question then is to check if it is possible to have $f(x) = x$ for some x and $f(x) = -x$ for some other x : Let x such that $f(x) = x$ and y such that $f(y) = -y$ Then $E1(x,y)$ gives $f(x^2 - y) = x^2 + y$ and so either $x^2 - y = x^2 + y$, either $-x^2 + y = x^2 + y$, and so either $x = 0$, either $y = 0$

So either $f(x) = x$ for any x, either $f(x) = -x$ for any x

It is easy to verify that these two solutions are available.

So the two solutions are $f(x) = x$ and $f(x) = -x$

****

Problem 42 (Posted by yrhc@eht)**.** Find all real functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f\left(x^2 + y\right) = 2f(x) + f(y^2),$$

for all x and y in the domain.

(Link to AoPS)

Solution 191 (by arkhammedos)**.**

did u mean this $f(x^2 + y) = 2f(x) + f(y^2)$:huh:

Solution 192 (by pco)**.**

> did u mean this $f(x^2 + y) = 2f(x) + f(y^2)$:huh:

If it is $P(x,y)$: $f(x^2+y) = 2f(x)+f(y^2)$, then : $P(0,0) \implies f(0) = 3f(0) \implies f(0) = 0$ $P(0,x) \implies f(x) = f(x^2)$ $P(x,0) \implies f(x^2) = 2f(x)$ So $2f(x) = f(x)$ and the unique solution is $f(x) = 0$

Solution 193 (by yrhc@eht)**.**

sorry, is = anyway, thank for try helping.

****

Problem 43 (Posted by shyong)**.** Suppose that two given functions f and g are defined on the interval $[0, 2k]$ for some $k > 0$. Can we always find a pair (x, y) of real numbers such that $|xy + f(x) + g(y)| \geq k^2$?

(Link to AoPS)

Solution 194 (by pco).

> Let the domain of two given functions f and g be defined on $[0, 2k]$,$k > 0$. Can we always find a pair of $(x, y) \in \mathbb{R}$ such that $|xy + f(x) + g(y)| \geq k^2$?

Yes, we can.

1) Case 1:$g(0) - g(2k) \leq 2k^2 \implies k^2 - g(2k) - 2k * 2k \leq -k^2 - g(0) - 2k * 0$. Then: Either $f(2k) \geq k^2 - g(2k) - 2k * 2k$, either $f(2k) < -k^2 - g(0) - 2k * 0$. So : Either $2k * 2k + f(2k) + g(2k) \geq k^2$, either $2k * 0 + f(2k) + g(0) < -k^2$. So : Either $|2k * 2k + f(2k) + g(2k)| \geq k^2$, either $|2k * 0 + f(2k) + g(0)| > k^2$

2) Case 2 : $g(0) - g(2k) > 2k^2 \implies k^2 - g(0) - 0 * 0 < -k^2 - g(2k) - 0 * 2k$. Then : Either $f(0) \geq k^2 - g(0) - 0 * 0$, either $f(0) < -k^2 - g(2k) - 0 * 2k$. So : Either $0 * 0 + f(0) + g(0) \geq k^2$, either $0 * 2k + f(0) + g(2k) < -k^2$. So : Either $|0 * 0 + f(0) + g(0)| \geq k^2$, either $|0 * 2k + f(0) + g(2k)| > k^2$

And, as a consequence : Either $|2k * 2k + f(2k) + g(2k)| \geq k^2$, Either $|2k * 0 + f(2k) + g(0)| > k^2$, Either $|0 * 0 + f(0) + g(0)| \geq k^2$, Either $|0 * 2k + f(0) + g(2k)| > k^2$

And the requested property is verified.

****

Problem 44 (Posted by djuro)**.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all real x, we have

$$f(x)^2 = f(x^2) + 2.$$

(Link to AoPS)

Solution 195 (by goc).

haven't seen it, but i'll try to explain an idea for a solution.

its' easy to see that $f(1) = 2$ just by pluging 1into the equation and that $f(x) > \sqrt{2}$. $f(x) = \sqrt{2 + f(x^2)} \geq \sqrt{2 + \sqrt{2}}$ since the limit of $\sqrt{2 + \sqrt{2 + ...}} = 2$ we get that $f(x) \geq 2$ for all x now you need an infinite partition of R where each subset is closed under the operation of squaring and square-rooting or equivalently if x is in A then x^{2^n} is in A for all integers n for any such subset A pick one number a_0 and let $f(a_0)$ be an arbitrary value greater than 2. from that you can calculate every value for a from A by solving the recurence relation $a_{n+1} + 2 = a_n^2$ which holds for every integer n and all a_n are positive... djuro gave me an idea for this one. set $f(a_0) = x + \frac{1}{x}$ for $x > 1$ then $f(a_n) = x^{2^n} + \frac{1}{x^{2^n}}$ for any integer n since $x = \frac{f(a_0) + \sqrt{f(a_0)^2 - 4}}{2}$ we now have a direct formula for a_n

Solution 196 (by mornik).

Hmm..., goc, just a little thing, function is $f : R^+ \to R^+$ and therefore $f(0)$ does not exist, as I can see it.

Solution 197 (by goc).

thanks. doesn't change much, totaly forgot bout it... for mornik nadam se da vjezbas(cini mi se bar da vjezbas posto si konstantno na forumu ;))...ak trebas kaj,javi se

Solution 198 (by djuro).

I see the solution $f(x) = x + \frac{1}{x}$.

ja sam inace smislio taj zadatak al ga ne znam rijesit.

satja

Solution 199 (by mornik).

Well, there may be other solutions, but anyhow you should have posted it in Algebra: Proposed & Own if you came up with the problem yourself.

@goc: A neto malo radim. Nemam kaj doma radit pa sjedim na forumu.

Solution 200 (by Ravi B).

More generally, the function $f(x) = x^c + x^{-c}$ works, where c is a real constant.

djuro, did you want to assume that the function f satisfies some continuity property? Otherwise, you could get crazy examples by choosing different c above for different x.

Solution 201 (by goc).

he doesn't know anything about continuity yet, he's a student of mine... :) try to solve it with assumption of continuity...

Solution 202 (by djuro).

> he doesn't know anything about continuity yet, he's a student of mine... :) try to solve it with assumption of continuity...

Let's make a good problem then :) I want not less than 50

p.s. jel continuous znaci neprekidna?

Solution 203 (by mornik).

Da. Pretty much all the nice functions are continuous, even though I just know goc will find one which is discontinuous and try to convince us it's really nicer than, say, $f(x) = x^2$. :D

Solution 204 (by pco).

> Find all functions $f : R^+ \to R^+$ such that
>
> $$f(x)^2 = f(x^2) + 2.$$
>
> I'd like to know has anyone seen this before?

Using the idea of Goc, it is possible to show that a general solution of this functional equation is :

Let $u(x)$ and $v(x)$ any two funtions defined in $[0,1)$ and whose values are strictly positive. Then $f(x)$ is defined as :

For $x>1$, $f(x)=u(\{\frac{\ln(\ln(x))}{\ln(2)}\})^{2^{\lfloor\frac{\ln(\ln(x))}{\ln(2)}\rfloor}}+u(\{\frac{\ln(\ln(x))}{\ln(2)}\})^{-2^{\lfloor\frac{\ln(\ln(x))}{\ln(2)}\rfloor}}$ For $x=1$, $f(x)=2$ For $0<x<1$, $f(x)=v(\{\frac{\ln(-\ln(x))}{\ln(2)}\})^{2^{\lfloor\frac{\ln(-\ln(x))}{\ln(2)}\rfloor}}+v(\{\frac{\ln(-\ln(x))}{\ln(2)}\})^{-2^{\lfloor\frac{\ln(-\ln(x))}{\ln(2)}\rfloor}}$

Where $\{y\}$ means fractional part of y

Solution 205 (by behemont)**.**

pco, how do you come up with something like that?

Solution 206 (by delegat)**.**

> p.s. jel continuous znaci neprekidna?

Jeste.

Solution 207 (by djuro)**.**

Would the problem be better if one adds more conditions? Is this problem worth something?

I ask because it's mine :) I put Copyright on this prob.

Solution 208 (by pco)**.**

> pco, how do you come up with something like that?

1) $f(x)\geq 2\ \forall x>0$ We have $f(x)>0$. Then, if it exists $a>0$such that $0<f(a)<2$, then we have $0<f(a^2)<f(a)<2$ and the sequence $a_n=f(a^{2^n})$ is a strictly decreasing sequence whose each element is in $(0,2)$. So this sequence has a limit which verifies $L=L^2-2$ and so $L=-1$ or $L=2$, which is impossible if $0<f(a)<2$. So $f(x)\geq 2\ \forall x>0$

2) $f(x)=a(x)+\frac{1}{a(x)}$ with $a(x)\geq 1$ and $a(x)^2=a(x^2)$. For any $y\geq 2$, it exists a unique real $z\geq 1$ such that $y=z+\frac{1}{z}$. So it exists a unique function $a(x)\geq 1$ such that $f(x)=a(x)+\frac{1}{a(x)}$. Then $f(x)^2=f(x^2)+2$ becomes $a(x)^2+\frac{1}{a(x)^2}=a(x^2)+\frac{1}{a(x^2)}$. Then, since there is a unique way of splitting a number ≥ 2 in a sum of a number ≥ 1 and its inverse, we have $a(x)^2=a(x^2)$

3) Solutions of $a(x)\geq 1$ and $a(x)^2=a(x^2)\ \forall\ x>0$ Since $x>0$, we can write $a(x)=b(\ln(x))$ and we have $b(\ln(x))^2=b(2\ln(x))\ \forall\ x>0$ and so $b(x)^2=b(2x)$ $\forall\ x$ Then, we can distinguish 3 situations :

3.1) $x>0$. Let then $b(x)=c(\frac{\ln(x)}{\ln(2)})$ and we have $c(\frac{\ln(x)}{\ln(2)})^2=c(1+\frac{\ln(x)}{\ln(2)}$ and so $c(x+1)=c(x)^2$ So $c(x+n)=c(x)^{2^n}$ and so we can define $c(x)=u(x)$ for $x\in[0,1)$ ($u(x)$ is any function defined on $[0,1)$ and ≥ 1) and then : $c(x)=c(\lfloor x\rfloor+\{x\})=c(\{x\})^{2^{\lfloor x\rfloor}}=u(\{x\})^{2^{\lfloor x\rfloor}}$

3.2) $x=0\implies b(0)^2=b(0)$ and, since $a(x)\geq 1$ and $b(x)\geq 1$: $b(0)=1$.

3.3) $x<0$. Let then $b(x)=c(\frac{\ln(-x)}{\ln(2)})$ and we have $c(\frac{\ln(-x)}{\ln(2)})^2=c(1+\frac{\ln(-x)}{\ln(2)}$ and so $c(x+1)=c(x)^2$ So $c(x+n)=c(x)^{2^n}$ and so we can define $c(x)=v(x)$

for $x\in[0,1)$ ($v(x)$ is any function defined on $[0,1)$ and ≥ 1) and then : $c(x)=c(\lfloor x\rfloor+\{x\})=c(\{x\})^{2^{\lfloor x\rfloor}}=v(\{x\})^{2^{\lfloor x\rfloor}}$

4) General solution of the requested functional equation : Let $u(x)$ and $v(x)$ any two functions ≥ 1 and defined in $[0,1)$: $x>1$, $f(x)=u(\{\frac{\ln(\ln(x))}{\ln(2)}\})^{2^{\lfloor\frac{\ln(\ln(x))}{\ln(2)}\rfloor}}+u(\{\frac{\ln(\ln(x))}{\ln(2)}\})^{-2^{\lfloor\frac{\ln(\ln(x))}{\ln(2)}\rfloor}}$ $x=1$, $f(x)=2$ $x<1$, $f(x)=v(\{\frac{\ln(-\ln(x))}{\ln(2)}\})^{2^{\lfloor\frac{\ln(-\ln(x))}{\ln(2)}\rfloor}}+v(\{\frac{\ln(-\ln(x))}{\ln(2)}\})^{-2^{\lfloor\frac{\ln(-\ln(x))}{\ln(2)}\rfloor}}$

5) We can retrieve the solution $f(x)=x^a+\frac{1}{x^a}$: Just take $u(x)=v(x)=e^{a2^x}$

Solution 209 (by pco)**.**

> Find all functions $f:R^+\to R^+$ such that
>
> $$f(x)^2=f(x^2)+2.$$
>
> I'd like to know has anyone seen this before?

About your second question : this equation is in the very frequent class of equations like : $f\circ u=v\circ f$ where $u(x)$ and $v(x)$ are bijective functions.

You must first check that if $u(x)$ has fixed points, $v(x)$ has too (since $u(x)=x \implies v(f(x))=f(x)$) For such an equation, you have $f\circ u^{\circ n}=v^{\circ n}\circ f$ (where $h^{\circ n}=h\circ h\circ h\cdots\circ h$ n times). Then the key point is to find (if possible) a subset S of the domain of $f(x)$ such that any real y may be written in a unique way as $y=u^{\circ n}(x)$ with $x\in S$ and $n\in\mathbb{Z}$

If this is possible, an if we call $n(x)$ (values in $\mathbb{Z}$) and $seed(x)$ (values in S) the two numbers such that $x=u^{\circ n(x)}(seed(x))$, and if we have $h(x)$ any function defined in S, then :

Obviously $n(u(x))=n(x)+1$ and $seed(u(x))=seed(x)$ and then :

$f(x)=v^{\circ n(x)}(h(seed(x)))$

Trivial examples :

1) $f(x+1)=f(x)+7$ We have $u(x)=x+1$ and $v(x)=x+7$ We can take $S=[0,1)$, $n(x)=\lfloor x\rfloor$ and $seed(x)=\{x\}$ and we have the general solution $f(x)=h(\{x\})+7\lfloor x\rfloor$ with $h(x)$ beeing any function defined in $[0,1)$ Obviously, in this case, we can write $k(x)=h(x)-7x$ and then $f(x)=k(\{x\})+7x$

2) $f(x+1)=(f(x))^2$ We have $u(x)=x+1$ and $v(x)=x^2$ We can take $S=[0,1)$, $n(x)=\lfloor x\rfloor$ and $seed(x)=\{x\}$ and we have the general solution $f(x)=h(\{x\})^{2^{\lfloor x\rfloor}}$ with $h(x)$ beeing any function defined in $[0,1)$ and with values ≥ 0 For example, with $h(x)=a^{2^x}$, we have the simple solution $f(x)=a^{2^x}$

3) $f(x^3)=f(x)+9$ We have $u(x)=x^3$ and $v(x)=x+9$ Impossible since $u(x)$ has fixed points -1 and $+1$ but $v(x)$ has no fixed points.

.... In your case, we have $f(x^2)=f(x)^2-2$ We have $u(x)=x^2$ and $v(x)=x^2-2$ and $u(x)$ is a bijective function $R^+\to R^+$ ...

****

Problem 45 (Posted by delegat)**.** Find all funcions $f : \mathbb{R} \to \mathbb{R}$ such that for all reals x and y the following equality holds:

$$f(x)f(yf(x)-1) = x^2f(y) - f(x).$$

(Link to AoPS)

Solution 210 (by behemont)**.**

Let $*$ denotes the initial equation. Plugging $x = 0$ into $*$, we get $f(0) = 0$, then plugging $y = 0$ we get $f(-1) = -1$. Plugging $x = -1$ into $*$ we get $f(-y-1) = -f(y) - 1$. Using that, the inital equation becomes $-f(x)f(-yf(x)) = x^2f(y)$, denote it by $**$, and plugging here $y = -1$ we get $f(x)f(f(x)) = x^2$. (1)

Let's suppose $f(x_1) = f(x_2)$. Then also $f(yf(x_1)) = f(yf(x_2))$, so $f(x_1) = f(x_2) \to x_1^2 = x_2^2$. (2)

Look at (1). It's obvious that $f(x) < 0 \to f(f(x)) < 0$ and $f(x) > 0 \to f(f(x)) > 0$, meaning that t and $f(t)$ have the same sign $\forall t \in Im(f)$. Then also f is injective $\forall t \in Im(f)$ because of (1). (3)

Now take $t \in Im(f)$, $t \neq 0$. We know that $f(t)f(f(t)) = t^2$. Because of (3) we know that f is nonincreasing or nondecreasing for such t's. So suppose $f(t) > t \to f(f(t)) \geq f(t) > t$. By multiplying it, we get f(t)$f(f(t)) > t^2$. Contradiction.. We'll get the same for $f(t) < t$. So $f(t) = t \longleftrightarrow f(f(x)) = f(x) \forall x \in \mathbb{R}$. Using that in (1), we get $[f(x)]^2 = x^2$.

Should be easy to prove that there isn't a such that $f(a) = -a$, but i am in a hurry now so i'll finish it later..

Solution 211 (by goc)**.**

> Because of (3) we know that f is nonincreasing or nondecreasing for such t

why should it be monotone? :huh: there are plenty of nonmonotone bijections(and therefore injections) in the world... monotone-¿bijective and not the other way arround edit: one more thing

> we get $f(-1) = -1$

that's not quite true either. you let a solution slip by here. from that condition you can get that $f(x) = 0$ for all x, or $f(-1) = -1$...

Solution 212 (by behemont)**.**

ok i skipped that trivial solution, but why do you say it's not monotone? :(

Solution 213 (by pco)**.**

> ..., but why do you say it's not monotone? :(

It exist non monotone function which are injective. For example $f(x) = 2\lfloor x\rfloor - x + 1$. In this example, we have : $f(t)$ and t has the same sign $f(x)$ is a bijective function. $f(x)$ is not monotone.

Solution 214 (by pco)**.**

> Find all funcions $f:\mathbb{R}\to\mathbb{R}$ such that for all reals x and y the following equality holds:
> $f(x)f(yf(x)-1) = x^2f(y) - f(x)$

1) $x = 0$ implies $f(0) = 0$

2) $y = 0$ implies $f(x)f(-1) = -f(x)$ and either $f(x) = 0\ \forall x$, either $f(-1) = -1$

Assume now $f(x)$ is not the null function. 3) $x = -1$ implies $f(-y-1) = -f(y) - 1$ and $f(-\frac{1}{2}) = -\frac{1}{2}$

4) $x = -\frac{1}{2}$ and $y = -2$ implies $f(-2) = -2$

5) $x = -1$ and $y = -2$ implies then $f(1) = 1$

6) $x = 1$ implies $f(y-1) = f(y) - 1$ and so $f(x+1) = f(x) + 1\ \forall x$

7) $y = -1$ implies $f(x)f(-f(x)-1) = -x^2 - f(x)$ and, with point 3, $f(x)f(f(x)) = x^2$

8) Replace x by $x+1$ in 7 above, and, using 6 : $f(x+1)f(f(x+1)) = (x+1)^2$ implies $(f(x)+1)(f(f(x))+1) = x^2+2x+1$, so $f(x)f(f(x))+f(x)+f(f(x))+1 = x^2+2x+1$ and so $f(f(x)) + f(x) = 2x$

9) $f(f(x))f(x) = x^2$ and $f(f(x)) + f(x) = 2x$ implies $f(f(x))$ and $f(x)$ are the two roots of $Y^2 - 2xY + x^2 = 0$ and then $f(x) = x$

So the only two solutions could be $f(x) = 0$ and $f(x) = x$ and it is easy to check that these two solutions are OK.

****

Problem 46 (Posted by radio)**.** Find all functions $f:\mathbb{R}^+\to\mathbb{R}$ such that

$$f(x)f(y) = f(xy) + 2005\left(\frac{1}{x} + \frac{1}{y} + 2004\right),$$

for all $x, y > 0$.

(Link to AoPS)

Solution 215 (by pco)**.**

> Find all $f : R^+ \to R$ such that
> $f(x)f(y) = f(xy) + 2005(\frac{1}{x} + \frac{1}{y} + 2004)$ for $x, y > 0$

1) $x = y = 1$ implies $f(1)^2 - f(1) - 2005 * 2006 = 0$ and $f(1) = 2006$ or $f(1) = -2005$

2) $y = 1$ gives $f(x)(f(1)-1) = 2005 * (2005 + \frac{1}{x})$ and, since $f(1) \neq 1$, $f(x) = 2005\frac{2005x+1}{(f(1)-1)x}$

So $f(x)$ could only be $f(x) = 2005 + \frac{1}{x}$ or $f(x) = 2005\frac{2005x+1}{-2006x}$

And, by checking back in the initial equation, the only solution is $f(x) = 2005 + \frac{1}{x}$

****

Problem 47 (Posted by N.T.TUAN). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that a)$f(x) \geq e^{2004x}$ for all $x \in \mathbb{R}$, and b)$f(x+y) \geq f(x)f(y)$ for all $x, y \in \mathbb{R}$.

(Link to AoPS)

Solution 216 (by pco).

> Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that a)$f(x) \geq e^{2004x} \forall x \in \mathbb{R}$ and b)$f(x+y) \geq f(x) + f(y) \forall x, y \in \mathbb{R}$.

From a), we get $f(0) \geq 1$ From b) we get $f(x+0) \geq f(x) + f(0)$ and so $f(0) \leq 0$

So we have a contradiction and no such function exists.

Solution 217 (by N.T.TUAN).

I am wrong! I edited it , please continuou!

Solution 218 (by Kunihiko˙Chikaya).

> Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that a)$f(x) \geq e^{2004x} \forall x \in \mathbb{R}$ and .

I think $f(x) \geq 1 + 2004x$. :maybe:

Solution 219 (by N.T.TUAN).

No, problem is true now.

Solution 220 (by mszew).

[hide="Is it correct?"]let $f(x) = e^{2004x}g(x)$ from $a)g(x) \geq 1\ \forall x$

$f(x+y) = e^{2004x}e^{2004y}g(x+y)$ $f(x)f(y) = e^{2004x}e^{2004y}g(x)g(y)$ from $b)g(x+y) \geq g(x)g(y)\ \forall x, y$ $y = 0$ $g(x) \geq g(x)g(0)$ then $g(0) = 1$ $y = -x$ $g(0) \geq g(x)g(-x)$ $1 \geq g(x)g(-x)$ then $g(x) = 1$

Solution 221 (by N.T.TUAN).

> let $f(x) = e^{2004x}g(x)$ from $a)g(x) \geq 1\ \forall x$
> $f(x+y) = e^{2004x}e^{2004y}g(x+y)$ $f(x)f(y) = e^{2004x}e^{2004y}g(x)g(y)$ from $b)g(x+y) \geq g(x)g(y)\ \forall x, y$ $y = 0$ $g(x) \geq g(x)g(0)$ then $g(0) = 1$ $y = -x$ $g(0) \geq g(x)g(-x)$ $1 \geq g(x)g(-x)$ then $g(x) = 1$

You are true!

****

Problem 48 (Posted by friendlist). Find all funtions $f : \mathbb{R} \to \mathbb{R}$ such that the following equality holds for all $x, y \in \mathbb{R}$:

$$f(x^3 - y) + 2y(3f^2(x) + y^2) = f(y + f(x)).$$

(Link to AoPS)

Solution 222 (by sinajackson).

[hide="some useful moves!"] by $x = y = 0$ we have: $f(0) = f(f(0))$ by putting $-y$ instead of y we have: $f(x^3 + y) - 2y\left(3f^2(x) + y^2\right) = f(f(x) - y)$ and by sum of this and the main one: we have: $f(x^3 + y) + f(x^3 - y) = f(f(x) - y) + f(f(x) + y)$ by putting $y = f(x)$ we have: $f(x^3 + f(x)) - 8f^3(x) = f(0)$ now this will help us to prve that $f(0) = 0$, by putting $x = 0$ we have: $f(f(0)) - 8f^3(x) = f(0) \to -8f^3(x) = 0 \to f(0) = 0$. as a consequence we must have: $f(x^3 + f(x)) = 8f^3(x)$:)

Solution 223 (by pco).

> Find all funtions such that the following equality holds for all $x; y \in R$
> $f(x^3 - y) + 2y(3f^2(x) + y^2) = f(y + f(x))$

Choose y such that $x^3 - y = y + f(x)$ which implies equality of the leftmost and rightmost summands in the equation :

$y = \frac{x^3 - f(x)}{2} \implies f(x^3 - y) = f(y + f(x))$ and so $y(3f^2(x) + y^2) = 0$ and so $y = 0$ (since $y \in \mathbb{R}$ and $f(x) \in \mathbb{R}$) and so $f(x) = x^3$

And it is rather easy to check that this necessary condition is sufficient.

So the only solution is $f(x) = x^3$

****

Problem 49 (Posted by santosguzella). Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

for all rational x and y.

(Link to AoPS)

Solution 224 (by pco).

> Find all functions f from the rational to the rational numbers satisfying
> $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ for all rational x and y.

$x = y = 0 \implies f(0) = 0$ $x = 0 \implies f(-y) = f(y)$ $\forall$ y rational.

Now, we can show with induction that $f(nx) = n^2 f(x)$ for every nonnegative integer n : a) It's true for $n = 0$ and $n = 1$ b) If it is true up to $n \geq 1$, then :$f(nx+x)+f(nx-x) = 2f(nx)+2f(x)$ and so : $f((n+1)x) = 2f(nx)+2f(x)-f((n-1)x) = (2n^2+2-(n-1)^2)f(x) = (n^2+2n+1)f(x) = (n+1)^2 f(x)$ and it's true for $n+1$

Then $f(x) = f(q(\frac{x}{q})) = q^2 f(\frac{x}{q})$ and so $f(\frac{x}{q}) = \frac{f(x)}{q^2}$

So $f(\frac{p}{q}) = f(p(\frac{1}{q})) = p^2 f(\frac{1}{q}) = p^2 \frac{f(1)}{q^2}$

And, since $f(x)$ is an even function : $f(x) = ax^2\ \forall x \in \mathbb{Q}$

****

Problem 50 (Posted by N.T.TUAN)**.** For which integers $n > 1$ and real numbers r does the curve $y = x^n + rx$ contain the vertices of a rectangle?

(Link to AoPS)

Solution 225 (by pco)**.**

> For which integers $n > 1$ and real numbers r does the curve $y = x^n + rx$ contain the vertices of a rectangle?

Let $f(x) = x^n + rx$

If the four vertices of the rectangle are $(x_A, y_A), (x_B, y_B), (x_C, y_C), (x_D, y_D)$ with $x_A \leq x_B \leq x_C \leq x_D$, then we have obviously $x_A < x_B < x_C < x_D$ and $f'(x)$ have at least two zeroes in $[x_A, x_D]$. Then, since $f'(x) = nx^{n-1} + r$, we must have n odd and $r < 0$.

Since the two values where $f'(x) = 0$ are in $[x_A, x_D]$, we have $x_A < 0$ and $x_D > 0$

Now, we can consider that none of the slopes of the sides of the rectangle are zero (either the other sides would be "vertical", which is impossible). Consider then the slope "a" which is positive (the slope of AB and CD).

It is rather easy to see that if $x_C < 0$, then the length of CD is greater than the length of AB (don't forget they are parallel). It is easy too in the same way to see that if if $x_B > 0$, then the length of AB is greater than the length of CD.

So $x_A < x_B < 0 < x_C < x_D$

Then if we consider the intersections C and D of $f(x) = x^n + r$ (n odd and $r < 0$) with $y = ax + b$ ($a > 0$) with $x > 0$, we have at most one couple of points C and D for a given length CD.

Hence, since $f(x)$ is an odd function, since AB is parallel to CD and since $AB = CD$, we need to have A symetric to D and B symetric to C and the four points are $A = (-b, -b^n - rb)$, $B = (-a, -a^n - ra)$, $C = (a, a^n + ra)$ and $D = (b, b^n + rb)$ for some $0 < a < b$.

It remains to have BD perpendiculat to CD :

Slope of BD is $\frac{a^n+b^n+ra+rb}{a+b}$

Slope of CD is $\frac{a^n-b^n+ra-rb}{a-b}$

BD perpendicular to CD means product of slopes equal to -1, and so $(a^n + ra)^2 - (b^n + rb)^2 = b^2 - a^2$ And so $(a^n + ra)^2 + a^2 = (b^n + rb)^2 + b^2$ with $0 < a < b$

This is possible iff $g(x) = (x^n + rx)^2 + x^2$ has at least one postive root (wth sign changing) for $g'(x) = 0$ $g'(x) = 2(x^n + rx)(nx^{n-1} + r) + 2x = 2x[(Z + r)(nZ + r) + 1]$ with $Z = x^{n-1}$ The equation $(Z + r)(nZ + r) + 1 = 0$ has a positive root with sign changing iff $r^2 < \frac{4n}{(n-1)^2}$, that's to say $r < -\frac{2\sqrt{n}}{n-1}$ (since we know $r < 0$)

So the curve $f(x) = x^n + rx$ ($n > 1$) contains the four vertices of a rectangle if and only if $n$is odd and $r < -\frac{2\sqrt{n}}{n-1}$.

****

Problem 51 (Posted by April)**.** Suppose that f is a real-valued function for which

$$f(xy) + f(y - x) \geq f(y + x)$$

for all real numbers x and y.

a) Give a non-constant polynomial that satisfies the condition. b) Prove that $f(x) \geq 0$ for all real x.

(Link to AoPS)

Solution 226 (by pco)**.**

> Suppose that f is a real-valued function for which
>
> $$f(xy) + f(y - x) \geq f(y + x)$$
>
> for all real numbers x and y.
> a. Give a nonconstant polynomial that satisfies the condition. b. Prove that $f(x) \geq 0$ for all real x.

a. $f(x) = x^2 + 4$ Proof : $f(xy) + f(y - x) - f(x + y) = (xy)^2 + 4 + (y - x)^2 - (x + y)^2$ So $f(xy) + f(y - x) - f(x + y) = (xy)^2 - 4xy + 4$ So $f(xy) + f(y - x) - f(x + y) = (xy - 2)^2 \geq 0$ So $f(xy) + f(y - x) \geq f(x + y)$

b. Investigate then the case where $xy = y + x$, that's to say $y = \frac{x}{x-1}$. Then the inequation becomes $f(y - x) \geq 0$. So we always have $f(\frac{x}{x-1} - x) \geq 0$

So $f(\frac{x(2-x)}{x-1}) \geq 0$ $\forall x \in R - \{1\}$

But the equation $\frac{x(2-x)}{x-1} = a$ is equivalent to $x^2 + (a - 2)x - a = 0$ and this equation always has real roots$\neq 1$ (discriminant is $a^2 + 4 > 0$). So, for any real a, exists $x \neq 1$ such that $\frac{x(2-x)}{x-1} = a$, and so $f(a) \geq 0$

So $f(a) \geq 0$ $\forall$ $a \in R$

Solution 227 (by AwesomeToad)**.**

How would one go about finding such a polynomial for a)? (And I'm also curious what other polynomials suffice)

Solution 228 (by quantumbyte).

I solved it by looking at part b). I knew that the easiest nonconstant polynomial such that $f(x) \geq 0$ for all real x would have to be something of the form $x^2 + c$. Now finding c is easy by just completing the square.

Solution 229 (by vsathiam).

Not sure if this works:

From the given inequality we have: f(0) $\geq$ 0

Suppose there exists a k such that f(k)¡0. Substituting x=x, y= $\frac{k}{x}$ into the original equation gives:

f(k)+f(0) $\geq$ f(x+$\frac{k}{x}$).

The function g(x)= $\frac{k}{x}$ for some constant k has a range of all real values (besides k) for all x (besides x=0).

So f(x) $\leq$ f(k)+f(0) ¡ f(0).

But f(0)=f(0). Since f(k) is a finite negative value, the function has a discontinuity at f(0). This is a contradiction since all polynomials are continuous.

So f(x) $\geq$ 0 for all x.

****

Problem 52 (Posted by N.T.TUAN). Find all functions $f : (0, \infty) \to (0, \infty)$ such that $f(x)f(y) = f(x + yf(x))$ for all $x, y > 0$.

(Link to AoPS)

Solution 230 (by Yuriy Solovyov).

> Find all functions $f : (0, \infty) \to (0, \infty)$ such that $f(x)f(y) = f(x + yf(x)) \forall x, y > 0$.

Is the function of monotonous? (if $f(x) = f(y)$ then $x = y$). If yes, then: $i)x = y = 1, f(1) = c; f(c + 1) = c^2$; $ii)x = c + 1; c^2 f(y) = f(c + 1 + yf(c + 1))$; $y = c+1; c^2 f(x) = f(x+(c+1)f(x))$; So: $f(x+(c+1)f(x)) = f(c+1+xf(c+1))$ and $c^2x + c + 1 = x + (c + 1)f(x)$; $f(x) = (c - 1)x + 1$.

Solution 231 (by aviateurpilot).

> Find all functions $f : (0, \infty) \to (0, \infty)$ such that $f(x)f(y) = f(x + yf(x)) \forall x, y > 0$.

we suppose that $\exists x, y > 0 :\ f(x) < f(y)$ then $\exists a > 0 :\ x+af(x) = y+af(y)$ so $f(x)f(a) = f(x+af(x)) = f(y+af(y)) = f(y)f(a)$ gives $f(x) = f(y)$ (impossible) then f is *increasing*. and we have $\forall x, y > 0 :\ f(x+yf(x)) = f(x)f(y) = f(y+xf(y))$ then $\forall x, y > 0 : \exists c_{x,y} > 0$ *such that* $\forall h$ *between* $x + yf(x)$ *and* $y + xf(y)]$ $f(x) = c_{x,y}$

Solution 232 (by N.T.TUAN)**.**

> we suppose that $\exists x, y > 0: \ f(x) < f(y)$ then $\exists a > 0: \ x + af(x) = y + af(y)$

Why you have $a > 0$? :maybe:

Solution 233 (by pco)**.**

Very nice demo. We can add two parts : 1) f is constant or strictly increasing (which allows the end of your demo) 2) $c \geq 1$

Demo : 1) f is constant or strictly increasing 1.1) $f(x) \geq 1 \ \forall x > 0$ Assume we have $x_0 > 0$ such that $f(x_0) < 1$. Take then $y = \frac{x_0}{1-f(x_0}$. We have $y > 0$ and the equation $f(x)f(y) = f(x+yf(x))$ becomes : $f(x_0)f(\frac{x_0}{1-f(x_0}) = f(x_0 + \frac{x_0 f(x_0)}{1-f(x_0}) = f(\frac{x_0}{1-f(x_0})$ But this is impossible since $f(x_0) \neq 1$ and $f(y) \neq 0$. So $f(x) \geq 1$ $\forall x > 0$; Q.E.D.

1.2) $f(x)$ is a non decreasing function. Let $a > b > 0$. Then let $x = b$ and $y = \frac{a-b}{f(b)}$. The equation $f(x)f(y) = f(x + yf(x))$ becomes : $f(b)f(y) = f(a)$. And since $f(y) \geq 1$, we can conclude $f(a) \geq f(b)$ So $a > b > 0$implies $f(a) \geq f(b)$ and $f(x)$ is non decreasing. Q.E.D.

1.3) $f(x) \equiv 1$ or $f(x)$ is strictly increasing. If it exists $u > 0$ such that $f(u) = 1$, then $x = u$ in the original equation gives $f(y) = f(y+u) \ \forall y > 0$ and since $f(x)$ is a non decreasing function, $f(x)$ is a constant and so $f(x) = 1$ $\forall x > 0$ If $f(x) > 1 \ \forall x > 0$, then the equation $f(b)f(y) = f(a)$ in 1.2 above implies $f(a) > f(b)$ and $f(x)$ is strictly increasing. Q.E.D.

So now, we know that either $f(x) = 1 \ \forall x > 0$, either $f(x)$ is strictly increasing and so, injective, which allows the implication $f(x + (c+1)f(x)) = f(c+1+xf(c+1)) \implies c^2x + c + 1 = x + (c+1)f(x)$ in your demonstration.

2) $c \geq 1$ This is immediatly implied by 1.1 above.

Hope this will complement your demo.

****

Problem 53 (Posted by Euler-ls)**.** Find all function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying (a) $f(f(x)f(y)) = xy$ for all $x, y \in \mathbb{R}^+$, and (b) $f(x) \neq x$ for all $x > 1$.

(Link to AoPS)

Solution 234 (by pco)**.**

> Find all function $f : R^+ \to R^+$ satisfying (a) $f(f(x)f(y)) = xy$ for all $x, y \in R^+$ (b) $f(x) \neq x$ for all $x > 1$
> I think $f(x) = \frac{1}{x}$... but I don't know how to prove it....

$P(x, y) : \ f(f(x)f(y)) = xy \ f(a) = f(b) \implies a = f(f(a)f(1)) = f(f(b)f(1)) = b \implies f(x)$ is injective. $P(1,1) : \ f(f(1)f(1)) = 1$ and so $f(a) = 1$ with

$a = f(1)^2$ $P(x, a)$: $f(f(x)) = ax$ and, for example, $f(f(a)) = a^2$ $P(f(a), f(a))$: $f(f(f(a))f(f(a))) = f(a)f(a)$ and so $f(a^4) = 1 = f(1)$ and so $a^4 = a$ qince $f(x)$ is injective. So $a = 1$

$P(f(x)f(y))$: $f(f(f(x))f(f(y))) = f(x)f(y) \implies f(xy) = f(x)f(y)$

So the equation (a) is equivalent to involutive solutions of Cauchy's equation $f(xy) = f(x)f(y)$.

The general involutive solutions of Cauchy's equation $f(xy) = f(x)f(y)$ may be written $f(x) = e^{a(\ln(x)) - b(\ln(x))}$ where $a(x)$ and $b(x)$ are the projections of x on A or B where A and B are supplementary $\mathbb{Q} - vectorspace$ in R

With $A = \mathbb{R}$ and $B = \{0\}$, we have the solution $f(x) = x$ With $A = \{0\}$ and $B = \mathbb{R}$, we have the solution $f(x) = \frac{1}{x}$ With axiom of Choice, a lot of other pairs (A, B) exist and a lot of solutions of equation (a) exist.

But, if $A \neq \{0\}$, it exists $a > 0$, $a \in A$ and then $f(e^a) = e^a$ with $e^a > 1$ and the second condition is not verified.

So the only solution is continuous and is $f(x) = \frac{1}{x}$

****

Problem 54 (Posted by cckek)**.** Find a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(1) = 2$ and for every positive integer n,

$$f(f(n)) = f(n) + n \quad \text{and} \quad f(n) < f(n+1).$$

(Link to AoPS)

Solution 235 (by Rust)**.**

Obviosly $f(n) = [\phi n + 1]$, were $\phi = \frac{1+\sqrt{5}}{2}$ is solution. I think it posted before.

Solution 236 (by pco)**.**

> Obviosly $f(n) = [\phi n + 1]$, were $\phi = \frac{1+\sqrt{5}}{2}$ is solution. I think it posted before.

Obviosly $f(n) = [\phi n + 1]$, were $\phi = \frac{1+\sqrt{5}}{2}$ is not a solution : $f(1) = 2$, $f(2) = 4$ and $f(f(1)) \neq f(1) + 1$

In fact, an answer could be $f(n) = [\phi n + \frac{1}{2}]$, were $\phi = \frac{1+\sqrt{5}}{2}$

Solution 237 (by Rust)**.**

> In fact, an answer could be $f(n) = [\phi n + \frac{1}{2}]$, were $\phi = \frac{1+\sqrt{5}}{2}$

If $f(n) = [\phi n + c]$, then $-(\phi - 1) - 1 + n + \phi c < f(f(n)) - f(n) = [(\phi - 1)[\phi n + c] + c] < n + \phi c$, therefore work c, suth that $c\phi < 1$ and $(\phi - 1)(c - 1) + c \geq 0$. It give $2 - \phi \leq c < \phi - 1$. (c=1/2 work). But these functional equation had infinetely many solutions. For example $f(4)$ may be 6 or 7.

Solution 238 (by pco)**.**

> Find a function $f:\mathbb{N}\to\mathbb{N},\ f(1)=2, f(f(n))=f(n)+n,\quad f(n)<f(n+1)$.

In fact $f(n)=\lfloor \phi n+a\rfloor$, with $\phi=\frac{1+\sqrt{5}}{2}$ and $a\in[2-\phi,\phi-1)$ is a possible solution :

1) $f(1)=2:\ f(1)=\lfloor\phi+a\rfloor=2$ since $2-\phi\le a<\phi-1<3-\phi$ and so $2\le\phi+a<3$ Q.E.D.

2) $f(n)<f(n+1)$ We have $f(n)=\lfloor\phi n+a\rfloor<\lfloor\phi n+1+a\rfloor\le\lfloor\phi n+\phi+a\rfloor=f(n+1)$ Q.E.D.

3) $f(f(n))=f(n)+n$ Since $f(n)=\lfloor\phi n+a\rfloor$, we have $\phi n+a=f(n)+\{f(n)\}$ and so $f(n)=\phi n+a-\{f(n)\}$ So $f(f(n))=\lfloor\phi f(n)+a\rfloor=\lfloor\phi^2 n+\phi a-\phi\{f(n)\}+a\rfloor$ So $f(f(n))=\lfloor\phi n+n+\phi a-\phi\{f(n)\}+a\rfloor$ (remember $\phi^2=\phi+1$) So $f(f(n))=n+\lfloor(\phi n+a)+\phi a-\phi\{f(n)\}\rfloor$ So $f(f(n))=n+\lfloor f(n)+\{f(n)\}+\phi a-\phi\{f(n)\}\rfloor$ So $f(f(n))=f(n)+n+\lfloor\phi a-(\phi-1)\{f(n)\}\rfloor$

It remains to show that $\lfloor\phi a-(\phi-1)\{f(n)\}\rfloor=0$, which is easy : $\{f(n)\}\ge 0\implies\ \phi a-(\phi-1)\{f(n)\}\le\phi a<\phi(\phi-1)=1\ \{f(n)\}<1\implies\ \phi a-(\phi-1)\{f(n)\}>\phi a-(\phi-1)\ge\phi(2-\phi)-(\phi-1)=0$

So $0<\phi a-(\phi-1)\{f(n)\}<1$ and so $\lfloor\phi a-(\phi-1)\{f(n)\}\rfloor=0$

And so $f(f(n))=f(n)+n$ Q.E.D.

****

—————————————————————————-

Problem 55 (Posted by pohoatza)**.** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x^2+y+f(y))=2y+f^2(x)$ for all real numbers x and y.

(Link to AoPS)

Solution 239 (by pco)**.**

> Find every function $f:\mathbb{R}\to\mathbb{R}$ such that $f(x^2+y+f(y))=2y+f^2(x)$ for all real numbers x and y.

Here is a rather complex solution (I think some simplier one must exist) :

We have the property $P(x,y):\ f(x^2+y+f(y))=2y+f^2(x)$

1) $f(x)$ is surjective : $P(x,\frac{a-f^2(x)}{2}):\ f(\ldots)=a$. Q.E.D.

2) $f(-x)=f(x)$ or $f(-x)=-f(x)$ $P(x,y):\ f(x^2+y+f(y))=2y+f^2(x)$ $P(-x,y):\ f(x^2+y+f(y))=2y+f^2(-x)$ So $f^2(-x)=f^2(x)$. Q.E.D.

3) $f(x)=0\Leftrightarrow x=0$ Since $f(x)$ is surjective (see 1). above), it exist a such that $f(a)=0$ Then $P(0,a)$ gives $0=2a+f^2(0)$ and so $a\le 0$ But $f(a)=0$ implies $f(-a)=0$ (from 2). above) and so, with the same method, $-a\le 0$ And so $a=0$. Q.E.D.

4) $f(x^2)=f^2(x)$ and so $f(x)>0\ \forall x>0$ Immediate with $P(x,0)$.

5) $f(x+f(x))=2x\ \forall x$ Immediate with $P(0,x)$

6) $f(x) \le -x \ \forall x \le 0$ Since $f(x) \ge 0 \ \forall x \ge 0$ (see 4.), and since $f(x+f(x)) = 2x$ (see 5.), $f(x+f(x)) \le 0 \ \forall x \le 0$ and so $x + f(x) \le 0 \ \forall x \le 0$ Q.E.D.

7) $f(-x) = -f(x) \ \forall x$ We know (see 2.) that $f(-x) = f(x)$ or $f(-x) = -f(x) \ \forall x$ Assume then there exists $a > 0$ such that $f(a) = f(-a)$ $P(\sqrt{a}, -a)$: $f(f(a)) = -2a + f(a)$ And since $a > 0$, $f(a) > 0$ and so $f(f(a)) > 0$. So $-2a + f(a) > 0$ and $f(a) > 2a$ But $f(a) = f(-a)$ and $f(-a) \le a$ (see 6.). So $0 < 2a < f(a) \le a$ which is impossible and no such a exist. So $f(-x) = -f(x)$ $\forall x$ Q.E.D.

8) $f(x) + x$ is a surjective function. Let $x \ge 0$. $P(\sqrt{\frac{x}{2}}, -\frac{x}{2})$: $-f(f(\frac{x}{2})) = -x + f(\frac{x}{2})$ and so $f(f(\frac{x}{2})) + f(\frac{x}{2}) = x$ So $f(x) + x$ may have any nonnegative value, and since $f(x) + x$ is an odd function, $f(x) + x$ may have any real value. Q.E.D

9) $f(x+y) = f(x) + f(y) \ \forall x, y$ From 4. and 5. above, the functional equation may be written : $f(x^2 + y + f(y)) = f(x^2) + f(y + f(y))$ And so, since $x + f(x)$ is surjective : $f(x+y) = f(x) + f(y) \ \forall x \ge 0$, $\forall y$ And since $f(x)$ is an odd function : $f(x+y) = f(x) + f(y) \ \forall x, y$ Q.E.D.

10) $f(x) = x \ \forall x$ From 9., we know that $f(x)$ is a solution of Cauchy's equation. From 4., we know that $f(x) > 0 \ \forall x > 0$ and so $f(x)$ is continuous (all non continuous solutions of Cauchy's equation have neither upper, nor lower bound on any non empty interval). So $f(x) = ax$ And, putting back this solution in the original equation, we get $a = 1$

And so $f(x) = x$ Q.E.D.

Solution 240 (by Rust)**.**
very nice.

Solution 241 (by Albanian Eagle)**.**
I also posted it some time ago without knowing that it had already been posted three times before here :)

Solution 242 (by pohoatza)**.**
Well the good part is that we would had not seen pco's beautiful solution if not reposting it :)

Solution 243 (by Albanian Eagle)**.**
:maybe: Isn't it like Darij's solution at http://www.mathlinks.ro/Forum/viewtopic.php?t=68972 ??

****

Problem 56 (Posted by Erken)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$(x+y)(f(x) - f(y)) = (x-y)(f(x+y))$$

for all reals x and y.

(Link to AoPS)

Solution 244 (by pco).

> Find all functions $f : R \to R$,that satisfy to the following condition: For all $x, y \in R$: $(x+y)(f(x)-f(y)) = (x-y)(f(x)+f(y))$.

Developping the products, we have $xf(y) = yf(x)$, or also $\frac{f(x)}{x} = \frac{f(y)}{y} = c$ $\forall x, y \neq 0$

And so $f(x) = cx$ (even for $x = 0$ since $f(0) = 0$ and putting back in the original equation, any value c fit.

So the general solution is $f(x) = cx$.

Solution 245 (by Erken).

Sorry for my mistake; Here is the right version: Find all $f : R \to R$,that satisfy the following condition: For all $x, y \in R$ we have that $(x+y)(f(x)-f(y)) = (x-y)(f(x+y))$

Solution 246 (by pco).

> Sorry for my mistake; Here is the right version: Find all $f : R \to R$,that satisfy the following condition: For all $x, y \in R$ we have that $(x+y)(f(x)-f(y)) = (x-y)(f(x+y))$

With $y = 2$, we have $(i) : f(x+2) = \frac{x+2}{x-2}(f(x)-f(2))$

With $y = 1$, we have $f(x+1) = \frac{x+1}{x-1}(f(x)-f(1))$

So $f(x+2) = \frac{x+2}{x}(f(x+1)-f(1))$

So $f(x+2) = \frac{x+2}{x}(\frac{x+1}{x-1}(f(x)-f(1))-f(1))$

So $(ii) : f(x+2) = \frac{x+2}{x}(\frac{x+1}{x-1}f(x)-2f(1)\frac{x}{x-1})$

Comparing (i) and (ii), we get :

$\frac{x+2}{x-2}(f(x)-f(2)) = \frac{x+2}{x}(\frac{x+1}{x-1}f(x)-2f(1)\frac{x}{x-1})$

$\frac{1}{x-2}(f(x)-f(2)) = \frac{x+1}{x(x-1)}f(x)-2f(1)\frac{1}{x-1}$

$2f(1)\frac{1}{x-1} - \frac{1}{x-2}f(2) = (\frac{x+1}{x(x-1)} - \frac{1}{x-2})f(x)$

$f(x) = \frac{f(2)}{2}x(x-1) - f(1)x(x-2)$

$f(x) = ax^2 + bx$

Putting back this expression in original equation, we can verify that this solution works for any values of (a, b)

Solution 247 (by Erken).

Nice solution,i've solved it the same way .

****

Problem 57 (Posted by santosguzella)**.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such for all $x, y \in \mathbb{R}^+$, we have

$$f(x)f(yf(x)) = f(x+y).$$

(Link to AoPS)

Solution 248 (by pco)**.**

> Let R^+ be the set of positive real numbers. Find all functions $f : R^+ \to R^+$ such for all $x, y \in R^+$,
>
> $$f(x)f(yf(x)) = f(x+y)$$

We have property $P(x,y) : f(x)f(yf(x)) = f(x+y)\ \forall x, y > 0$

1). $f(x) \le 1\ \forall x$ If it exists $a > 0$ such that $f(a) > 1$. Then $P(a, \frac{a}{f(a)-1})$ gives $f(a)f(\frac{af(a)}{f(a)-1}) = f(\frac{af(a)}{f(a)-1})$ which is impossible since $f(a) \neq 1$ and $f(\frac{af(a)}{f(a)-1}) \neq 0$. So $f(x) \le 1\ \forall x > 0$ Q.E.D.

2). Either $f(x) = 1\ \forall x > 0$, either $f(x) < 1\ \forall x > 0$ and $f(x)$ is strictly decreasing, and so is an injective function. $f(x)f(yf(x)) = f(x+y)$ and $f(yf(x)) \le 1$ imply $f(x+y) \le f(x)$ and $f(x)$ is a non increasing function. If it exists $a > 0$ such that $f(a) = 1$, then $P(a,x)$ gives $f(x) = f(x+a)\ \forall x$ and so, since $f(x)$ is non increasing, $f(x) = 1\ \forall x > 0$ If it does not exist any $a > 0$ such that $f(a) = 1$, then $f(x) < 1\ \forall x > 0$ and then $f(x)f(yf(x)) = f(x+y)$ and $f(yf(x)) < 1$ imply $f(x+y) < f(x)$ and $f(x)$ is a strictly decreasing function, and so $f(x)$ is injective. Q.E.D.

3). If $f(x) \neq 1$, then $f(x) = \frac{1}{ax+1}$ for some $a > 0$ Assume $f(x) \neq 1\ \forall x$ and let $x > y > 0$ and compare $P(x, \frac{y}{f(x)})$ and $P(y, x - y + \frac{y}{f(x)})$:

$P(x, \frac{y}{f(x)})$ gives $f(x)f(y) = f(x + \frac{y}{f(x)})$

$P(y, x - y + \frac{y}{f(x)})$ gives $f(y)f((x - y + \frac{y}{f(x)})f(y)) = f(x + \frac{y}{f(x)})$

So $f(x)f(y) = f(y)f((x - y + \frac{y}{f(x)})f(y))$ and, since $f(x)$ is injective : $x = (x - y + \frac{y}{f(x)})f(y)$

So $\frac{x}{f(y)} - x = \frac{y}{f(x)} - y$

So $\frac{1}{yf(y)} - \frac{1}{y} = \frac{1}{xf(x)} - \frac{1}{x}$

So $\frac{1}{xf(x)} - \frac{1}{x}$ is a constant a and we have :

$f(x) = \frac{1}{ax+1}$ Putting back this expression in original equation, we find that any value a fit. But, we know that $f(x) > 0\ \forall x > 0$ and so we need $a > 0$. (inequality is strict since we assumed $f(x) \neq 1$) Q.E.D.

So the general solution of requested equation is $f(x) = \frac{1}{ax+1}$ for any $a \ge 0$ (the case $a = 0$ gives the solution $f(x) \equiv 1$)

****

Problem 58 (Posted by nayel)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant and continuous function such that

$$f(x+y) + f(x-y) = 2f(x)f(y),$$

for all x, y in $\mathbb{R}$. Assume that there exists $a \in \mathbb{R}$ such that $f(a) < 1$ Show that there exists infinitely many real numbers r such that $f(r) = 0$.

(Link to AoPS)

Solution 249 (by pco).

> Let $f : \mathbf{R} \to \mathbf{R}$ be a non-constant and continuous function such that
>
> $$f(x+y)+f(x-y)=2f(x)f(y)$$
>
> for all x, y in $\mathbf{R}$. Show that there exists infinitely many real numbers r such that $f(r)=0$.

I'm afraid it's wrong : take $f(x)=\cosh(x)$

Solution 250 (by nayel).

I think this one is true(thanks pco):

Let $f : \mathbf{R} \to \mathbf{R}$ be continuous such that

$$f(x+y)+f(x-y)=2f(x)f(y) \qquad (1)$$

for all x, y in $\mathbf{R}$. Assume that there exists $a \in \mathbf{R}$ such that $f(a) < 1$. Prove that there exists infinitely many real numbers r such that $f(r)=0$.

Solution 251 (by TTsphn).

In this case $f(x)=bcox$ and $bcosa<1$

Solution 252 (by nayel).

I think It would be helpful if you post your solution. Actually I didn't pretty much understand what you mean. :maybe:

Solution 253 (by MellowMelon).

[hide="Solution with small amount of ugliness"] With $y=0$, we find that either $f(x)=0$, solving the problem immediately, or $f(0)=1$. So assume the latter.

Lemma: There exists a $c \neq 0$ such that $f(c)=0$. Proof: Consider the value of a such that $f(a)<1$. If $f(x)<0$ for any x, then by $f(0)=1$ and f continuous, f must have a nonzero root, proving the lemma. So assume $0<f(a)<1$. $x=y=a$ gives $f(2a)+1=2f(a)^2$. If $f(a)<\sqrt{2}/2$, we can get that $f(2a)$ is negative, finishing the problem. So assume $f(a)>\sqrt{2}/2$. Let $f(a)=1-\alpha$ for an α that gives $\sqrt{2}/2<f(a)<1$. Then $f(2a)+1=2-4\alpha+2\alpha^2$. Since $\alpha^2<\alpha$, $f(2a)=1-4\alpha+2\alpha^2<1-2\alpha=f(a)-\alpha$. This shows that $f(2a)<f(a)-\alpha$. So then if we consider $f(2a)=1-\beta$, we get that $\beta<2\alpha$. This means that $f(2^n a)<f(a)-2^n\alpha$, and since α is positive, we must eventually get an $f(2^n a)<\sqrt{2}/2$. Then $f(2^{n+1}a)<0$, and we have proved the lemma. [hide="Remark"] There has got to be a better way to prove this lemma. The method here seemed cleaner than doing calculus on $2x^2-1$ or $2x^2-x-1$, but perhaps I'm not thinking along the best lines in general.

Now that we have our c, plug $x=y=c$ in. $f(2c)+1=0$ so $f(2c)=-1$. Now plug $x=2c, y=c$ in. $f(3c)=0$. So then $f(3^n c)=0$, finishing the proof.

Solution 254 (by nayel).

Hey that is like *exactly* my solution!!! How did you guess? :D

Solution 255 (by nayel).

My solution: First assume that $\exists r : f(r) = 0$. Now we easily get $f(2r) = -1, f(3r) = 0$, inductively implying $f(3^n r) = 0$. Thus there is an infinite number of rs that satisfy $f(r) = 0$.

Now we will prove that $\exists r : f(r) = 0$. Put $y = 0$ in (1) to get $f(0) = 1$ and $x = y$ to get $f(2x) + 1 = (f(x))^2$. If $f(x) < 0$ for some real x, we are done by continuity as $f(0) = 1$. So assume that $f(x) > 0 \ \forall x \in \mathbf{R}$.

Let $0 < f(a) < 1$. Then $f(2a) = 2(f(a))^2 - 1 < 2f(a) - 1$. An inductive argument shows that $f(2^n a) < 2^n f(a) - 2^n + 1 \implies f(2^n a) - f(a) < (2^n - 1)(f(a) - 1)$. Now as $f(a) - 1 < 0$, we can choose n large enough so that $f(2^n a) - f(a) < -1 \implies f(2^n a) < 0$. So we are done, because $f(2^n a) < 0, f(0) = 1$ and by continuity there exists r such that $f(r) = 0$.

****

Problem 59 (Posted by delegat). Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the identity:

$$f(x)f(y) = f(x + yf(x))$$

for all positive reals x and y

(Link to AoPS)

Solution 256 (by MellowMelon).

[hide="Incomplete solution - one large hole"] Observe first that $f(x) = ax+1$ for all nonnegative a works.

First, suppose that $f(a) = f(b) = c$. Then $cf(y) = f(a + yc) = f(b + yc)$.

Suppose $a \neq b$, then WLOG $a > b$. Let $d = \frac{a-b}{c}$. Then we get that $cf(y) = f(a+yc) = f(b+(y+d))c = cf(y+d)$. Thus f is periodic with period d.

You can use the same argument above to show that $f(x + d/f(x)) = f(x)$ for all x. Assume there is an x with $f(x) > 1$ and apply this identity infinitely many times to get a contradiction. So now $f(x) \leq 1$. Here's the part of the solution I haven't finished: I am pretty sure there is a way to proceed from here to get $f(x) = 1$ for all x, but I haven't figured out exactly how. (though it's entirely possible I missed an answer for when $f(x) < 1$ for some x)

Now we look for solutions other than those above, so we must remove our assumption that $a \neq b$ above. Therefore $a = b$ and f is injective. Observe $f(x+yf(x)) = f(x)f(y) = f(y)f(x) = f(y+xf(y))$. By f injective, $x+yf(x) = y + xf(y)$. Then plug in $y = 1$ to get $f(x) = 1 + xf(1) - x$. Let $f(1)$ be any value greater than 1 and we get the remaining solutions: $f(x) = ax + 1$ for a positive.

Solution 257 (by pco).

Find all functions $f : \mathbb{R}+ \to \mathbb{R}+$ which satisfy the following identity: $f(x)f(y) = f(x + yf(x))$ for all positive reals x and y

We have the property $P(x, y) : f(x)f(y) = f(x + yf(x)) \ \forall x, y > 0$

1). $f(x) \geq 1 \ \forall x > 0$ Let $a > 0$ such that $0 < f(a) < 1$. Then $\frac{a}{1-f(a)} > 0$ and $P(a, \frac{a}{1-f(a)})$ gives $f(a)f(\frac{a}{1-f(a)}) = f(\frac{a}{1-f(a)})$ which is impossible since $f(a) \neq 1$ and $f(\frac{a}{1-f(a)}) \neq 0$. So, such a don't exist and $f(x) \geq 1 \ \forall x > 0$. Q.E.D.

2). $f(x)$ is a non decreasing function. Since $f(x) \geq 1 \ \forall x > 0$, $P(x, \frac{h}{f(x)})$ gives $f(x) \leq f(x)f(\frac{h}{f(x)}) = f(x + h)$. Q.E.D.

3). Either $f(x) = 1 \ \forall x > 0$, either $f(x)$ is a strictly increasing function, and so is injective. If it exists $a > 0$ such that $f(a) = 1$, then $P(a, x)$ gives $f(x) = f(x + a)$ and, since $f(x)$ is non decreasing, $f(x) = 1 \ \forall x > 0$ If $f(x) > 1$ $\forall x > 0$, $P(x, \frac{h}{f(x)})$ gives $f(x) < f(x)f(\frac{h}{f(x)}) = f(x + h)$. Q.E.D.

4). If $f(x) \neq 1$, then $f(x) = ax + 1$ for some $a > 0$ If $f(x) \neq 1$, $f(x)$ is injective. Compare then $P(x, y)$ and $P(y, x)$: $P(x, y) : f(x)f(y) = f(x+yf(x))$ $P(y, x) : f(y)f(x) = f(y+xf(y))$ So $f(x+yf(x)) = f(y+xf(y))$ and so ($f(x)$ is injective) $x + yf(x) = y + xf(y)$ So $\frac{f(x)}{x} - \frac{1}{x} = \frac{f(y)}{y} - \frac{1}{y}$ So $\frac{f(x)}{x} - \frac{1}{x}$ is the constant function and $\frac{f(x)}{x} - \frac{1}{x} = a$ and so $f(x) = ax + 1$ Putting back this expression in $P(x, y)$, we get that any value a fit but we need to have $a > 0$ in order to have $f(x) > 0$ and $f(x) \neq 1$ Q.E.D.

The general solution of the equation is $f(x) = ax + 1$ for any value $a \geq 0$ (the value 0 gives the constant solution $f(x) \equiv 1$)

Solution 258 (by quangpbc)**.**

Solution : Suppose there exists x_0 so that $f(x_0) < 1$ .Let $y = \frac{x_0}{1-f(x_0)}$,we have $f(x_0).f(\frac{x_0}{1-f(x_0)}) = f(\frac{x_0}{1-f(x_0)}) \to f(x_0) = 1$ contracdition. Then $f(x) \geq 1$ for all $x > 0$. Now we have two cases :

Case 1 . There exists x_1 so that $f(x_1) = 1$. For any $u (0 < u < x_1)$ there exists $v (0 < v < x_1)$ so that $u + vf(u) = x_1$ Then $f(u).f(v) = f(x_1) = 1$ but $f(u), f(v) \geq 1$ so $f(u) = f(v) = 1$. Hence $f(x) = 1$ for all $0 < x \leq x_1$. In the equation , replace x by x_0 we have $f(x_1 + y) = f(y)$ so $f(x)$ is a circulation fucntion with cycle less than x_1 . Then we have $f(x) = 1$ for all $x \in \mathbb{R}^+$.

Case 2 : $f(x) > 1$ for all $x \in \mathbb{R}^+$. For any pairs (x, y) with $x > y$ there exists t so that $x = y + tf(y)$ .Then $f(x) = f(y).f(t) \to f(x) > f(y)$ so $f(x)$ is increasing fucntion In the equation replace x by y we have :$f(x).f(y) = f(x + yf(x)) = f(y + xf(y))$. Then $x + yf(x) = y + xf(y)$. Hence $\frac{f(x)-1}{x} = constant$ .We get $f(x) = ax + 1$ for any value $a \geq 0$.

****

Problem 60 (Posted by Leonhard Euler)**.** Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f^{19}(n) + 97f(n) = 98n + 232$. Notice in this problem that $f^{19}(n)$ means the composition of f with itself 19 times.

(Link to AoPS)

Solution 259 (by pco).

> Find all functions f:N-¿N such that $f^{19}(n)+97f(n)=98n+232$

I consider that $f^{19}(n)$ is the composition of $f(n)$ 19 times (and not the 19^{th} power; in such a case, there are no solution).

First, we have $97f(n)<98n+232$ and so $f(n)<\frac{98}{97}n+\frac{232}{97}$ $\forall n$ and so $f^{19}(n)<(\frac{98}{97})^{19}n+232((\frac{98}{97})^{19}-1)$

But $f^{19}(n)<(\frac{98}{97})^{19}n+232((\frac{98}{97})^{19}-1)$ implies $97f(n)>98n+232-(\frac{98}{97})^{19}n-232((\frac{98}{97})^{19}-1)$

So $(\frac{98}{97}-\frac{1}{97}(\frac{98}{97})^{19})n+\frac{232}{97}(2-(\frac{98}{97})^{19})<f(n)<\frac{98}{97}n+\frac{232}{97}$ $\forall n$ which implies :

$0.997n+1.877<f(n)<1.011n+2.392$ $\forall n$ or $n+2-(0.003n+0.123)<f(n)<n+2+(0.011n+0.392)$

This implies $f(n)=n+2$ $\forall n<40$ and now it is easy to extend with induction :

If $f(p)=p+2$ $\forall p<n$ for $n>38$, then : $f(n-38)=n-36$ $f^{19}(n-38)+97f(n-38)=98(n-38)+232$ and so $f^{19}(n-38)+97(n-36)=98(n-38)+232$ and so $f^{19}(n-38)=n$ $f^{19}(n-36)+97f(n-36)=98(n-36)+232$ and so $f^{19}(n-36)+97(n-34)=98(n-36)+232$ and so $f^{19}(n-36)=n+2$

So $n+2=f^{19}(n-36)=f^{19}(f(n-38))=f(f^{19}(n-38))=f(n)$ and we have $f(n)=n+2$ and so the induction is OK.

So the unique solution could be $f(n)=n+2$ (and an immediate verification in original equation shows that this solution fit).

****

Problem 61 (Posted by Leonhard Euler). Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+f(x)+y)=f(y)+2x$$

holds for all $x,y\in\mathbb{R}$.

(Link to AoPS)

Solution 260 (by pco).

> Find all functions f:R-¿R such that $f(x+f(x)+y)=f(y)+2x$

We have the property $P(x,y):\ f(x+f(x)+y)=f(y)+2x$ $\forall x,y$

1). $f(x)$ is bijective $P(\frac{a-f(0)}{2},0):\ f(\ldots)=a$ and $f(x)$ is surjective $P(x,-f(x))$: $f(x)=f(-f(x))+2x$ and so $f(x_1)=f(x_2)$ implies $x_1=x_2$ and $f(x)$ is injective Q.E.D.

2.) $f(0)=0$ $P(0,x):\ f(x+f(0))=f(x)$ and, since $f(x)$ is injective, $f(0)=0$ Q.E.D.

3.) $x+f(x)$ is bijective $P(x,0)$: $f(x+f(x))=2x$ and so $x+f(x)=f^{-1}(2x)$ and $x+f(x)$ is bijective. Q.E.D.

4). $f(x+y)=f(x)+f(y)$ Since $f(x+f(x))=2x$, $P(x,y)$ may be written $f(x+f(x)+y)=f(y)+f(x+f(x))$ and since $x+f(x)$ is surjective, for any z it exists x such that $x+f(x)=z$ and so $f(z+y)=f(y)+f(z)$. Q.E.D.

5). the only continuous solutions are $f(x)=x$ and $f(x)=-2x$ The only continuous solutions of this Cauchy's equation are $f(x)=ax$ and so we have $ax+a^2x+ay=ay+2x$ and so $a^2+a-2=0$ and so $a=1$ or $a=-2$ Q.E.D.

6). With axiom of Choice, it exists non continuous solutions. Using (classical way) a base $\{x_i\}$ for the $\mathbb{Q}$-vector space $\mathbb{R}$, it is immediate to see that $f(x_i)=x_i$ or $f(x_i)=-2x_i$. So we can build two $\mathbb{Q}$-vector space A and B such that $A\cap B=\{0\}$ and $A+B=\mathbb{R}$ (A using the part of the base such that $f(x_i)=x_i$ and B using the part of the base such that $f(x_i)=-2x_i$). Then, if $a(x)$ and $b(x)$ are the projections of a real x on A and B (such that there is a unique decomposition $x=a(x)+b(x)$ with $a(x)\in A$ and $b(x)\in B$, then $f(x)=a(x)-2b(x)$

So the general solution of the original equation is the following one :

Let A and B be two $\mathbb{Q}$-vector space A and B such that $A\cap B=\{0\}$ and $A+B=\mathbb{R}$ Let $a(x)$ and $b(x)$ be the projections of a real x on A and B (such that there is a unique decomposition $x=a(x)+b(x)$ with $a(x)\in A$ and $b(x)\in B$

Then $f(x)=a(x)-2b(x)$ is a solution (and all solutions are of this form).

If $A=\mathbb{R}$ and $B=\{0\}$, then we have the continuous solution $f(x)=x$ If $A=\{0\}$ and $B=\mathbb{R}$, then we have the continuous solution $f(x)=-2x$ With axiom of Choice, a lot of other pairs (A,B) exist and so a lot of non continuous solutions.

****

Problem 62 (Posted by mathemagics)**.** Find all functions $f:\mathbb{R}-[0,1]\to\mathbb{R}$ such that

$$f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)},$$

for all $x\in\mathbb{R}$.

(Link to AoPS)

Solution 261 (by Rust)**.**

> Find all functions $f:\mathbb{R}-[0,1]\to\mathbb{R}$ such that
> $f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)}$
> India Competition

1. $f(x)+f(\frac{1}{1-x})=\frac{2}{x}+\frac{2}{x-1}$. $x=\frac{1}{1-y},y\to x$ give 2. $f(\frac{1}{1-x})+f(\frac{x-1}{x})=\frac{2}{x}-2x$ $x=\frac{y-1}{y},y\to x$ give 3. $f(\frac{x-1}{x})+f(x)=\frac{2x}{x-1}-2x$ (1.+3.-2.)/2 give $f(x)=\frac{x+1}{x-1}$.

Solution 262 (by asdrojas)**.**

Rust see that, in (*) x must be greater than 1, so by (**) $0 < y < 1$ but this implies that $\frac{1}{2} < \frac{1}{1-x} < 1$ in (***) but it's out of the domaind of f

Solution 263 (by asdrojas)**.**

> Find all functions $f : \mathbb{R} - [0, 1] \to \mathbb{R}$ such that
> $f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}$ (*)
> India Competition

The domain of f is $\mathbb{R} - [0, 1]$ so in the condition (*) we must have $x \in (-\infty, 0) \cup (1, \infty)$ and $\frac{1}{1-x} \in (-\infty, 0) \cup (1, \infty)$ that implies $x > 1$ and $\frac{1}{1-x} < 0$. (*) is equivalent to $f(x) + f\left(\frac{1}{1-x}\right) = \frac{2}{x-1} + \frac{2}{x} = \left(1 + \frac{2}{x-1}\right) + \left(\frac{2}{x} - 1\right) = \left(1 + \frac{2}{x-1}\right) + \left(1 + \frac{2}{\frac{1}{1-x} - 1}\right)$ then $f(x) - \left(1 + \frac{2}{x-1}\right) + f\left(\frac{1}{1-x}\right) - \left(1 + \frac{2}{\frac{1}{1-x} - 1}\right) = 0$ Let $g(x) = f(x) - \left(1 + \frac{2}{x-1}\right)$ then $g(x) + g\left(\frac{1}{1-x}\right) = 0$ $g(x) = -g\left(\frac{1}{1-x}\right)$ Now if x goes from 1 to ∞, $\frac{1}{1-x}$ goes from -∞ to 0, so we can set $g(x)$ an arbitrary function for $x > 1$, and construct g for negatives values as $g(x) = -g\left(\frac{1}{1-x}\right)$, so $f(x) = g(x) + 1 + \frac{2}{x-1}$

Solution 264 (by Virgil Nicula)**.**

Proof. Denote the function $g : A \to \mathbb{R}$, where $g(x) = \frac{2(1-2x)}{x(1-x)}$ and the function $\phi : A \to \mathbb{R}$, where $\phi(x) = \frac{1}{1-x}$. Observe that the function ϕ is injective and $\mathrm{Im}\phi \equiv \phi(A) = A$, i.e. the function $\phi : A \to A$ is bijective (it is a **dynamic function**). Prove easily that $\phi \circ \phi \circ \phi = 1_A$ (the identical function) and

$$\left\{ \begin{array}{c} f + f \circ \phi = g \\ f \circ \phi + f \circ \phi \circ \phi = g \circ \phi \\ f \circ \phi \circ \phi + f = g \circ \phi \circ \phi \end{array} \right\| \implies f = \frac{1}{2} \cdot (g - g \circ \phi + g \circ \phi \circ \phi) \text{ a.s.o.}$$

Solution 265 (by asdrojas)**.**

Virgil ϕ is not bijective, supose $x = -1$ then $x \in A$ but $\phi(x) = \frac{1}{2} \notin A$

Solution 266 (by pco)**.**

> Find all functions $f : \mathbb{R} - [0, 1] \to \mathbb{R}$ such that
> $f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}$
> India Competition

The problem is not well defined, as the scope of values for which the equation is verified is not given.

By the way, I think there is a typo in the problem and the domain is $f : \mathbb{R} - \{0, 1\} \to \mathbb{R}$ instead of $f : \mathbb{R} - [0, 1] \to \mathbb{R}$.

Then Rust's solution would be the good one.

Solution 267 (by mathemagics).

This is India 1992 and the problem was proposed with the domain $\mathbb{R} - [0, 1]$. Do you have any book that contains Indian Olympiads to check it ?

****

Problem 63 (Posted by Leonhard Euler). Show that there exist function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = n^2$.

(Link to AoPS)

Solution 268 (by pco).

> Show that there exist function $f : n \to n$ such that $f(f(n)) = n^2$

Let A be the set of positive integers which are not perfect squares. Let B and C two disjoint infinite subsets of A such that $A = B \cup C$ and $h(n)$ a bijective function from B to C. Let $k(n)$ the function which gives, for any integer $n > 1$, the littlest integer m such that it exists an integer $p \geq 0$ such that $n = m^{2^p}$. Let $e(n)$ the integer $p \geq 0$ such that $n = k(n)^{2^p}$ Obviously, $k(n)$ is not a perfect square and either $k(n) \in B$, either $k(n) \in C$

Then we can define $f(n)$: $f(1) = 1$ $\forall n > 1$: if $k(n) \in B$, $f(n) = (h(k(n)))^{2^{e(n)}}$ if $k(n) \in C$, $f(n) = (h^{-1}(k(n)))^{2^{e(n)+1}}$

Solution 269 (by BaBaK Ghalebi).

`https://artofproblemsolving.com/community/c6h128748` `https://artofproblemsolving.com/community/c6h24087`

****

Problem 64 (Posted by sylow˙theory). Let $f(x) = x^n + \cdots + x + 1$ and let $g(x) = f(x^{n+1})$. Find the remainder when $g(x)$ is divided by $f(x)$.

(Link to AoPS)

Solution 270 (by minsoens).

I have a solution, but I must admit that I have seen a similar problem before.. $f(x) = \sum_{i=0}^{n} x^i \implies f(x^{n+1}) = \sum_{i=0}^{n} x^{(n+1)i} = n + 1 + \sum_{i=0}^{n} \left(x^{(n+1)i} - 1\right)$.

$$\begin{aligned} x^{(n+1)i} - 1 &= \left(x^{n+1} - 1\right)\left(x^{(n+1)(i-1)} + x^{(n+1)(i-2)} + \cdots + x^{n+1} + 1\right) \\ &= (x-1)\left(x^n + x^{n-1} + \cdots + x + 1\right)\left(x^{(n+1)(i-1)} + x^{(n+1)(i-2)} + \cdots + x^{n+1} + 1\right) \\ &= (x-1)\, f(x)\left(x^{(n+1)(i-1)} + x^{(n+1)(i-2)} + \cdots + x^{n+1} + 1\right) \end{aligned}$$

So all of $x^{(n+1)i} - 1$ in $n+1+\sum_{i=0}^{n} \left(x^{(n+1)i} - 1\right)$ are divisible by $f(x)$. Therefore the remainder is $\boxed{n+1}$. Have I made any mistake? :maybe:

Solution 271 (by sylow˙theory).

> Therefore the remainder is $\boxed{n+1}$. Have I made any mistake? :maybe:

I forgot what the answer is :P , but it looks right (to lazy to check). But, I have a really short solution. :ninja:

Solution 272 (by K81o7).

> But, I have a really short solution. :ninja:

Would this be it, by any chance? :D

$f(x)|x^{n+1}-1|f(x^{n+1})-f(1)=g(x)-f(1) \implies g(x)\equiv f(1)\equiv n+1 \mod f(x)$

Solution 273 (by pco).

> I was doing a problem, and I generalized it. Its not so hard, but its a nice result.
> Let $f(x)=x^n+...+x+1$ and let $g(x)=f(x^{n+1})$. Find the remainder when $g(x)$ is divided by $f(x)$.

[hide="Short solution using complex numbers"] Let $z_k=e^{\frac{2ki\pi}{n+1}}$ for $k\in\{1,2,\dots,n\}$. We have $z_k^{n+1}=1$ and $f(z_k)=0\ \forall k\in\{1,2,\dots,n\}$

We have $g(x)=f(x)q(x)+r(x)$ with degree of $r(x)<n$. So $g(z_k)=f(z_k^{n+1})=f(1)=n+1=r(z_k)$

So $r(x)$ is a polynomial of degree $\le n-1$ and taking the same value for n different values. So $r(x)$ is a constant.

And $r(x)=n+1$

Solution 274 (by sylow˙theory).

Good solutions guys :)

But I did it like Patrick. Write it like $g(x)=q(x)f(x)+r(x)$ and work with $(n+1)$-th roots of unity. Substitute those in and claim $r(x)$ is a constant polynomial.

****

—————————————————————————-

Problem 65 (Posted by N.T.TUAN). Let $0<a<1$ and $I=(0,a)$. Find all functions $f:I\to\mathbb{R}$ satisfying at least one of the conditions below: a)f is continuous on I and $f(xy)=xf(y)+yf(x)$ for all $x,y\in I$. b)$f(xy)=xf(x)+yf(y)$ for all $x,y\in I$.

(Link to AoPS)

Solution 275 (by pco).

> Let $0 < a < 1$ and $I = (0, a)$. Find all functions $f : I \to \mathbb{R}$ satisfying at least one of the conditions below: a)f is continuous on I and $f(xy) = xf(y) + yf(x)\forall x, y \in I$. b)$f(xy) = xf(x) + yf(y)\forall x, y \in I$.

a) : Let $g(x) = a^{-x}f(a^x)$. We have g continuous, defined on $(1, +\infty)$ and $g(x+y) = g(x)+g(y)\ \forall x \in (1, +\infty)$ It's then easy, for $b > 1$ and $p \geq q \geq 1 \in \mathbb{N}$ to write : $g(q\frac{pb}{q}) = qg(\frac{pb}{q})$ since $\frac{pb}{q} \in (1, +\infty)$ and $g(pb) = pg(b)$ since $b \in (1, +\infty)$ And so $g(\frac{p}{q}b) = \frac{p}{q}g(b)\ \forall b > 1 \in \mathbb{R}$ and $\forall p \geq q \geq 1 \in \mathbb{N}$ And so, since g is continuous, $g(x) = cx\ \forall x > 1$ So $f(x) = \alpha x\ln(x)\ \forall x \in I$ and it is easy to check that this necessary condition is sufficient.

b) : $f(xy) = xf(x) + yf(y) \implies f(x^2) = 2xf(x)$ and then also $2f(xy) = f(x^2) + f(y^2)$. So we can write $f(x^2y^2)$ in two ways : $f(x^2y^2) = f(XY) = Xf(X)+Yf(Y) = x^2f(x^2)+y^2f(y^2)$ $f(x^2y^2) = f(Z^2) = 2Zf(Z) = 2xyf(xy) = xy(f(x^2)+f(y^2))$ Comparing these two expressions,we get, for $x \neq y$, $xf(x^2) = yf(y^2)$ and so $f(x) = \frac{c}{\sqrt{x}}$ Putting back this expression in $f(x^2) = 2xf(x)$, we get $\frac{c}{x} = 2c\sqrt{x}$ and so $c = 0$ And so $f(x) = 0$

So the requested functions are $f(x) = \alpha x\ln(x)$ for any $\alpha \in \mathbb{R}$

****

—————————————————————————————————-

Problem 66 (Posted by Nbach). Find all continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that :

$$f\left(\frac{1}{f(xy)}\right) = f(x)f(y)$$

for all $x, y \in \mathbb{R}^+$.

(Link to AoPS)

Solution 276 (by pco).

> Find all continous function $f : R_+ \mapsto R_+$ such that : $f(\frac{1}{f(xy)}) = f(x)f(y)$ for all $x, y \in R_+$

Let $g(x) = f(\frac{1}{x})$ We have $g(g(xy)) = g(x)g(y)$ and so $g(g(x)) = g(1)g(x)$ and so $g(x) = g(1)x\ \forall x \in g(\mathbb{R}^+)$ Then, since $g(xy) \in g(\mathbb{R}^+)$, $g(g(xy)) = g(1)g(xy)$ and so $g(1)g(xy) = g(x)g(y)\ \forall x \in \mathbb{R}^+$

Let then $g(x) = g(1)h(x)$: we have $h(xy) = h(x)h(y)$ and $h(x)$ continuous. So $h(x) = x^c$ for some $c \in \mathbb{R}$

Putting back this expression in the original equation, we find $f(x) = 1$ or $f(x) = \frac{a}{x}$, with $a > 0$

Solution 277 (by quangpbc).

Let $f(1) = a$.

Let $y = 1$, $f(\frac{1}{x}) = af(x)$.(1) On (1) move x by xy , we get :
$f(\frac{1}{xy}) = af(xy) = f(x).f(y)$.(2)

Let $g(x) = \frac{f(x)}{a}$ so $g(x)$ is a continous function . From (2), we have $g(xy) = g(x)g(y)$. Then $g(x) = x^t$ or $f(x) = ax^t$ And easily to show that $t = 0$ or $t = -1$.

-$t = 0$, we have $a = a^2$, then $a = 1 \to f(x) = 1$ -$t = -1$. we have $f(x) = \frac{a}{x}$ with $a \in \mathbb{R}^+$

****

Problem 67 (Posted by Amir.S)**.** Find all $f : \mathbb{N} \to \mathbb{R}$ that for a given $n \in \mathbb{N}$,

$$f(m + k) = f(mk - n)$$

holds for all positive integers k and m with $mk > n$.

(Link to AoPS)

Solution 278 (by pco)**.**

> find all $f : \mathbb{N} \to \mathbb{R}$ that for a given $n \in \mathbb{N}$ we have: $f(m + k) = f(mk - n)$, $m, k \in \mathbb{N}$, $mk > n$

$m = x - 1$ and $n = 1 \implies f(x) = f(x - (n + 1))$ $\forall x > n + 1$ and so $f(x) = f(x+n+1)$ $\forall x > 0$ $k = n+1$ implies then $f(m+n+1) = f((m-1)(n+1)+1)$ $\forall m > 0, n > 0$ But, since $f(x) = f(x + n + 1)$ $\forall x > 0$, this last equation may be written $f(m) = f(m + n + 1) = f((m - 1)(n + 1) + 1) = f(1)$

And so the unique solution $f(m) = c$ $c \in \mathbb{R}$

****

Problem 68 (Posted by quangpbc)**.** Find all increasing functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(xf(y)) = yf(2x), \quad \forall x, y \in \mathbb{R}.$$

(Link to AoPS)

Solution 279 (by pco)**.**

> Find all $f : \mathbb{R} \to \mathbb{R}$ so that :
> 1, f is an increasing function.
> 2, $f(xf(y)) = yf(2x)$
> :)

Let $P(x, y)$ be the property $f(xf(y)) = yf(2x)$

$P(1, x) \implies f(f(x)) = f(2)x$. $f(2) \neq 0$ since $f(x)$ is strictly increasing, so $f(x)$ is a bijective function. $P(x, 0) \implies f(xf(0)) = 0$ and so $f(0) = 0$

since $f(x)$ is an injective function. $P(f(x), y) \implies f(f(x)f(y)) = yf(2f(x))$ $P(f(y), x) \implies f(f(x)f(y)) = xf(2f(y))$ So $yf(2f(x)) = xf(2f(y))$ and so $\frac{f(2f(x))}{x} = \frac{f(2f(y))}{y} = a \ \forall x, y \neq 0$ So $f(2f(x)) = ax \ \forall x$ with $a \neq 0$ So we have a new property $Q(x, y) : f(f(x)f(y)) = axy$ Let then $b \neq 0$ such that $f(b) = 1$ (b exists since $f(x)$ is a surjective function) $Q(x, b) \implies f(f(x)) = abx$ and so $f(abx) = abf(x)$ $Q(f(\frac{x}{ab}), f(\frac{y}{ab})) \implies f(xy) = \frac{1}{ab^2}f(x)f(y)$

And this is a well known Cauchy equation which gives $f(x) = ux$ for some u and $\forall x \in \mathbb{Q}$ and then $f(x) = ux \ \forall x$ since $f(x)$ is an increasing function.

Putting back this solution in the original equation, we find $u = 2$

And the only solution is $f(x) = 2x$

****

——————————————————————————-

Problem 69 (Posted by hoangclub)**.** Determine all function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(m+n)f(m-n) = f(m^2),$$

for all $m, n \in \mathbb{N}$.

(Link to AoPS)

Solution 280 (by pco)**.**

> Denote Z^+ is set of all positive integer numbers. determine all function f:Z^+ -¿Z^+ such that $f(m+n)f(m-n)$=$f(m^2)$ for all m,n $\in Z^+$.

$f(n)f(n+4) = f((n+2)^2) = f(n+1)f(n+3)$ and so $\frac{f(n+4)}{f(n+3)} = \frac{f(n+1)}{f(n)}$ and so It exists u, a, b, c such that : $f(3p+1) = ua^pb^pc^p$ $f(3p+2) = ua^{p+1}b^pc^p$ $f(3p+3) = ua^{p+1}b^{p+1}c^p$

Then : $f(1)f(3) = f(4) \implies uuab = uabc$ and $u = c$ $f(1)f(5) = f(9) \implies uua^2bc = ua^3b^3c^2$ and $u = ab^2c$ $f(1)f(7) = f(16) \implies uua^2b^2c^2 = ua^5b^5c^5$ and $u = a^3b^3c^3$ $f(1)f(9) = f(25) \implies uua^3b^3c^2 = ua^8b^8c^8$ and $u = a^5b^5c^6$

And so $a = b = c = u = 1$

And $f(n) = 1 \ \forall n$

****

——————————————————————————-

Problem 70 (Posted by greatestmaths)**.** Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all real x, we have

$$f(f(f(x))) - 3f(f(x)) + 6f(x) = 4x + 3.$$

(Link to AoPS)

Solution 281 (by fermat3)**.**

no. i thing it's easy your problem.its classical: f is increasing.and use the classical method of iteration .and you will find solution:$f(x) = x + 1$:P

Solution 282 (by pco).

> find all function f: R—–¿R such that for all x in R.
> f(f(f(x)))-3f(f(x))+6f(x)= 4x+3

The difficulty is to study the non-trivial solutions (the trivial one is $f(x) = x+1$).

I think I have shown there is no other bijective solution with an Abel form $f(x) = g^{-1}(1 + g(x))$ But it remains job to do.

I'm glad you posted this meassage in this forum, greatestmaths, since it means you have the solution.

Thanks to post it or just a hint.

Solution 283 (by fermat3).

here you are a hint:let be $x_n = f^n - n$ then its easy to see that $x_{n+3} - 3x_{n+2} + 6x_{n+1} - 4x_n = 0$ then study this sequence via the fact that f is increasing

Solution 284 (by pco).

> here you are a hint:let be $x_n = f^n - n$ then its easy to see that $x_{n+3} - 3x_{n+2} + 6x_{n+1} - 4x_n = 0$ then study this sequence via the fact that f is increasing

Why is $f(x)$ increasing ?

Solution 285 (by fermat3).

supose that $f is decreasing$ then $f0f0f - 3f0f + 6f$ is deacreasing wich absurd because $f^3 - 3f^2 + 6f = 4x + 3$

Solution 286 (by pco).

> supose that $f is decreasing$ then $f0f0f - 3f0f + 6f$ is deacreasing wich absurd because $f^3 - 3f^2 + 6f = 4x + 3$

But f can be neither decreasing, neither increasing ! (increasing on some parts, decreasing on others), so your demo seems wrong.

Solution 287 (by daniel73).

Yep, I agree with pco; we know that f is injective (it is enough to assume that x, y exist such that $f(x) = f(y)$ and see that this leads to $4x+3 = 4y+3$), but I do not know how you can go from there to saying that f needs to be monotonous. For all that we know (unless proved otherwise), it could even be not continuous at any x!

My own efforts go more in the direction of defining $g(x) = f(f(x)) - 2f(x) + 4x$ so that obviously $g(f(x)) - g(x) = 3$ for all x, and $h(x) = f(x) - x$ so that $h(f(f(x))) - 2h(f(x)) + 4h(x) = 3$, but I have not been able to crack it yet (the idea is showing that $h(x)$ is constant and/or $g(x) = 3x$).

Solution 288 (by pco)**.**

> find all function f: R—–¿R such that for all x in R.
> f(f(f(x)))-3f(f(x))+6f(x)= 4x+3

Here is a partial result : $f(x) = x + 1$ is the unique continuous solution.

Demo : Let $x_0 \in \mathbb{R}$ and the sequence $a_1 = x_0$, $a_{n+1} = f(a_n)$. We have $a_{n+3} = 3a_{n+2} - 6a_{n+1} + 4a_n + 3$ Let then $b_n = a_{n+1} - a_n$. We have $b_{n+2} = 2b_{n+1} - 4b_n + 3$ and so $b_{n+3} - 1 = -8(b_n - 1)$

So, if $b_n \neq 1$ for some n, we are sure that the sequence b_n will have some $b_i > 0$ and some $b_j < 0$. Or $a_{i+1} > a_i$ and $a_{j+1} < a_j$ or again $f(a_i) < a_i$ and $f(a_j) > a_j$. Alternatively, $b_1 = 1$ implies $a_2 = a_1 + 1$ and so $f(x_0) = x_0 + 1$

So : either $f(x_0) = x_0 + 1$, either it exists reals u and v such that $f(u) < u$ and $f(v) > v$. Then, if $f(x)$ is continous, it exists w such that $f(w) = w$. But then $f(f(f(w))) = w$ and $3f(f(w)) - 6f(w) + 4w + 3 = 3w - 6w + 4w + 3 = w + 3$ and so $f(f(f(w))) \neq 3f(f(w)) - 6f(w) + 4w + 3$

So, either $f(x) = x + 1$ $\forall x$, either $f(x)$ is not continuous.

And, since $x + 1$ is a solution, the only continuous soluition is $f(x) = x + 1$ Q.E.D.

It remains to study non continuous solutions.

And, once again, greatestmaths, I would be happy to have some hint (or the complete solution).

****
—————————————————————————–

Problem 71 (Posted by chessfreak)**.** Do there exist four real polynomials such that the sum of any three of them has a real zero while the sum of no two polynomials has a real zero?

(Link to AoPS)

Solution 289 (by pco)**.**

> Do there exist four real polynomials such that the sum of any three of them has a real zero while the sum of no two polynomials has a real zero?

No, it does not :

$\forall i, j \in \{1, 2, 3, 4\}$, we have either $P_i + P_j > 0$ $\forall x$, either $P_i + P_j < 0$ $\forall x$

But $P_1 + P_2$, $P_1 + P_3$ and $P_2 + P_3$ can't have same sign, else their sum would have constant sign and $P_1 + P_2 + P_3$ would never be 0.

So, WLOG (choice or order + replacement P_i by $-P_i$) say : $P_1 + P_2 > 0$ $P_2 + P_3 > 0$ $P_1 + P_3 < 0$

Since $P_1 + P_3 < 0$, and since $P_1 + P_3$, $P_1 + P_4$ and $P_3 + P_4$ can't have same sign, $P_1 + P_4$ or $P_3 + P_4$ is > 0. WLOG say : $P_1 + P_4 > 0$

Since $P_1 + P_2 > 0$ and $P_1 + P_4 > 0$, and since $P_1 + P_2$, $P_1 + P_4$ and $P_2 + P_4$ can't have same sign : $P_2 + P_4 < 0$

Then since $P_2 + P_3 > 0$ and $P_1 + P_4 > 0$, then $P_1 + P_2 + P_3 + P_4 > 0$ But since $P_2 + P_4 < 0$ and $P_1 + P_3 < 0$, then $P_1 + P_2 + P_3 + P_4 < 0$

And so contradiction. (and this property is available for any set of continuous functions)

Solution 290 (by venkata)**.**

i dunno how many times u need to be told that IMOTC postals are not to be posted at Mathlinks. trust me, ppl who have easy access to the net have a gr8 advantage over those who dont. in due recognition of the fact that u r not from IMOTC, i request u to kindly not post ANY OF THE IMOTC postal probs.

Solution 291 (by borislav˙mirchev)**.**

Internet is great place for people to share their knowledge. You should post all problems that you like! I dislike monopolism in math!!!

Solution 292 (by BaBaK Ghalebi)**.**

> Internet is great place for people to share their knowledge. You should post all problems that you like! I dislike monopolism in math!!!

sure but I think that venkata meant that these problems are in a contest which is still open,so the students are supposed to solve the questions by them selves untill the given time,so it is not a good idea to post it here,its kind of cheating...

Solution 293 (by borislav˙mirchev)**.**

Yes, you are maybe right. But I think the problems are took from somewhere maybe some other competition or some textbook. What if the student use same textbook? I think, at this time when the world is a "globbal village" some competition without presence of the students if it is not time limited or aren't there some other limitations may be a warranty only that students have the desire and they want to solve proposed problems. They also may ask for help their friends parents, or other web sites not so good but, different from mathlinks... I think after I-st round there should be some communication with the students...maybe interview or round where they solve problems. Other idea I have is electronic competition with fixed time limit, but you again cannot be sure who is in front of the PC...

Solution 294 (by pardesi)**.**

chessfreaki suppose u r relly intesrested in seeing the Postal problems solved :P How is it that u r not a IMOTC member(as u had claimed) and yet u r the first each time to post the postal problems. well we have given u enough warnings i think it's time u better stp posting them otherwise we have to egt each of ur posts scrutinized and deleted if they are have postal problems .and if u still continue this u know what lies next. now no more requests just STOP POSTING THEM BECAUSE OF U EVERYONE ELSE IS DOING SO

Solution 295 (by venkata)**.**

Dear borislav˙mirchev As Bhabak Ghalebi mentions, this is an ongoing competition , and the rules clrly state: NO EXTERNAL HELP TO BE TAKEN. as far as "global villaging" goes, no 1 minds the posting of problems ONCE THE DEADLINE IS OVER. also, in all these competitions, a lot hinges on TRUST and ETHICS. so things like asking frnds, parents etc usually dont happen at this level(so i bliv) :maybe: btw, ill surely suggest ur examination methods to our profs

****

Problem 72 (Posted by pco)**.** For those who think that $f(f(x))$ is always an increasing function :

Find all functions $\mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = -x\ \forall x \in \mathbb{R}$

(Link to AoPS)

Solution 296 (by lasha)**.**

Suppose there exists such interval P, that for any reals (x, y) from P, where $x > y$, $f(x) > f(y)$. Than, $f(f(x)) > f(f(y))$, but as $f(f(x)) = -x$,$f(f(y)) = -y$, we have $-x > -y$, but $x > y$-contradiction. So, f is increasing function. Again take reals x and y with $x > y$. $f(x) < f(y)$. So, $f(f(x)) > f(f(y))$, or $-x > -y$-again contradiction because $x > y$. So, there is no solution.

Solution 297 (by pco)**.**

> Suppose there exists such interval P, that for any reals (x, y) from P, where $x > y$, $f(x) > f(y)$. Than, $f(f(x)) > f(f(y))$, but as $f(f(x)) = -x$,$f(f(y)) = -y$, we have $-x > -y$, but $x > y$-contradiction. So, f is increasing function. Again take reals x and y with $x > y$. $f(x) < f(y)$. So, $f(f(x)) > f(f(y))$, or $-x > -y$-again contradiction because $x > y$. So, there is no solution.

You're wrong. Solutions exist

When you write "Suppose there exists such interval P, that for any reals (x, y) from P, where $x > y$, $f(x) > f(y)$. Then, $f(f(x)) > f(f(y))$", the conclusion "Then, $f(f(x)) > f(f(y))$" is wrong because maybe $f(x) \notin P$ when $x \in P$

Solution 298 (by Rust)**.**

$f(0) = 0$. For all $x \neq 0$ we can consider pairs $(x, y), x \neq 0 \neq y,\ |x| \neq |y|$ and define $f(x) = y, f(y) = -x, f(-x) = -y, f(-y) = x$.

Solution 299 (by pco)**.**

> $f(0)=0$. For all $x \neq 0$ we can consider pairs $(x,y), x\neq 0 \neq y,\ |x| \neq |y|$ and define $f(x)=y, f(y)=-x, f(-x)=-y, f(-y)=x$.

Perfecty right, Rust.

In other words : Let $\mathbb{A}$ and $\mathbb{B}$ to subsets of $\mathbb{R}^+$ such that : $\mathbb{A}\cup\mathbb{B}=\mathbb{R}^+$ $\mathbb{A}\cap\mathbb{B}=\emptyset$ $\exists$ $h(x)$ bijective function $\mathbb{A}\to\mathbb{B}$

$f(x)$ may be defined as : $f(0)=0$ $f(x)=h(x)$ $\forall x\in\mathbb{A}$ $f(x)=-h^{-1}(x)$ $\forall x\in\mathbb{B}$ $f(x)=-f(-x)$ $\forall x<0$

Example : $f(0)=0$ $f(x)=\frac{|x|}{x}-(-1)^{[-|x|]}x$ $\forall x\neq 0$

Solution 300 (by Rust)**.**

But there are not continiosly function $f(x):R^n\to R^n$ suth that $f(f(x))=-x$ if n is odd.

Solution 301 (by lasha)**.**

You are right pco! Sorry, I made mistake. :(

Solution 302 (by perfect˙radio)**.**

For pco: http://www.mathlinks.ro/viewtopic.php?search_id=1499275189&t=113408.

[In the link it is proved that every such function has infinitely many discontinuities.]

Solution 303 (by silouan)**.**

PLease see this and the link of enescu http://www.mathlinks.ro/viewtopic.php?search_id=831547553&t=56637

****

Problem 73 (Posted by Shishkin)**.** Find all functions $f:\mathbb{Q}\to\mathbb{Q}$ such that $f(x^2+y+f(xy))=3+(x+f(y)-2)f(x)$ for all $x,y\in\mathbb{Q}$.

(Link to AoPS)

Solution 304 (by pco)**.**

> Find all functions f $Q\to Q$ such that $f(x^2+y+f(xy))=3+(x+f(y)-2)f(x)$.

I have a rather complex solution to prove $f(x)=x+1$:

Let $P(x,y)$ be the property : $f(x^2+y+f(xy))=3+(x+f(y)-2)f(x)$ Let $f(0)=a$

1) $P(0,x)$ implies $f(x+a)=af(x)+3-2a$ and so : $\boxed{f(x+na)=a^nf(x)+(3-2a)(1+a+a^2+\ldots+a^{n-1})}$

2) $P(x,0)$ and $P(-x,0)$ imply $f(x^2+a)=(x+a-2)f(x)+3$ and $f(x^2+a)=(-x+a-2)f(-x)+3$ and so : $\boxed{(x+a-2)f(x)=(-x+a-2)f(-x)}$

3) $P(1,0)$ implies $f(a+1) = (a-1)f(1)+3$ but 1) implies $f(a+1) = af(1)+3-2a$ and so $af(1)+3-2a = (a-1)f(1)+3$ and : $\boxed{f(1)=2a}$

4) If $a=2$, then 1) gives $f(2n) = 2^n+1$ and 1)+3) gives $f(2n+1) = 3\times 2^n+1$ But then $P(2,2)$ gives $f(6+f(4)) = 12$, so $f(11) = 12$ but $f(11) = f(2\times 5+1) = 3\times 2^5+1 = 97$ and so $a \neq 2$ If $a=0$, then $P(0,0)$ gives $f(0) = a = 3$ and so $a \neq 0$ If $a=3$, then 2) above with $x=1$ gives $2f(1) = 0$ but 3) said $f(1) = 2a = 6$ and so : $\boxed{a \notin \{0,2,3\}}$

5) Using $x = a-2$ in 2) above, we get $2(a-2)f(a-2) = 0$ and so, since, from 4), $a \neq 2$: $\boxed{f(a-2)=0}$

6) Using $x=1$ in 2) above gives $f(-1) = 2a\frac{a-1}{a-3}$ (remember with 4) above that $a \neq 3$). Then $P(-1,-1)$ gives $f(2a) = 3+(2a\frac{a-1}{a-3}-3)2a\frac{a-1}{a-3}$ But 1) gives $f(2a) = a^3-2a^2+a+3$ and so : $a^3-2a^2+a+3 = 3+(2a\frac{a-1}{a-3}-3)2a\frac{a-1}{a-3}$ which gives : $a(a-1)(a^3-11a^2+25a-27) = 0$

It's rather easy to see that $a^3-11a^2+25a-27$ has no rational root and so : $\boxed{f(0)=a=1}$

7) Then 1) gives $f(x+n) = f(x)+n$ and $f(n) = n+1$ Then, Let p and q coprimes positive integers. $P(q,\frac{p}{q})$ gives $f(q^2+\frac{p}{q}+p+1) = 3+(q+f(\frac{p}{q})-2)(q+1)$ But $f(q^2+\frac{p}{q}+p+1) = q^2+p+1+f(\frac{p}{q})$ (since $f(x+n) = f(x)+n$) and so :

$q^2+p+1+f(\frac{p}{q}) = 3+(q+f(\frac{p}{q})-2)(q+1)$

$p+q = qf(\frac{p}{q}$ and so $f(\frac{p}{q}) = \frac{p}{q}+1$

So $f(x) = x+1\ \forall x \in \mathbb{Q}^+$ Using 2), we have then $f(-x) = -x+1\ \forall x \in \mathbb{Q}^+$

So $f(x) = x+1\ \forall x \in \mathbb{Q}$

And putting back this expression in original equation, we find that this necessary condition is sufficient.

And the only solution is $\boxed{f(x)=x+1}$

Solution 305 (by nguyenvuthanhha)**.**

A very good solution , Pco . You really have talent in Algebra .

Solution 306 (by Dumel)**.**

> It's rather easy to see that $a^3-11a^2+25a-27$ has no rational root and so : $\boxed{f(0)=a=1}$

you're wrong

Solution 307 (by pco)**.**

Please, could you kindly show me where I am wrong ?

Is there a rational root to $a^3-11a^2+25a-27$? (please give us)

Or is my conclusion $a=1$ wrong ?, and why ?

Solution 308 (by Dumel)**.**

let $g(x) = x^3-11x^2+25x-27$ $-27 = g(0) < 0$ and $g(11) > 0$ g is a continuous funcion so there exist x_0 in $(0,11)$ such that $g(x_0) = 0$

edit: oh oh I forgot that f is $Q \to Q$ and all roots of g are surd so your solution is (almost) correct :-)

Solution 309 (by pco)**.**

> let $g(x) = x^3 - 11x^2 + 25x - 27$ $-27 = g(0) < 0$ and $g(11) > 0$ g is a continuous funcion so there exist x_0 in $(0, 11)$ such that $g(x_0) = 0$
> edit: oh oh I forgot that f is $Q \to Q$ and all roots of g are surd so your solution is (almost) correct :-)

Yes, I said that $x^3 - 11x^2 + 25x - 27$ had no rational root.

And, why the "almost" word in your post ? What else is not correct in my solution ?

Solution 310 (by Dumel)**.**

"almost" was just due to the word "rational"

Solution 311 (by mathuz)**.**

:roll: Oh...ho, I found very nice solution of the problem! Let $g(x) = f(x) - 1$ and $g : Q \to Q$. Then from original equation we have

$$g(x^2 + y + g(xy) + 1) = 2 + (x + g(y) - 1)(g(x) + 1).$$

(1) $g(0) = 0$; really it's true! (2) from (1), $g(1) = 1$ and

$$g(x + 1) = g(x) + 1$$

and

$$g(x^2) = x + g(x)(x - 1)(*)$$

for any ratsional x. Hence, $g(n) = n$ any integer n. Let $x = \frac{m}{n}$ some integers m and n, then at $P(x, n), \to g(x) = x$ for any ratsional numbers x. Therefore, $f(x) = x + 1$, any ratsional x.

Solution 312 (by mathuz)**.**

Sorry, I have latex mistake. Original version: :roll: Oh...ho, I found very nice solution of the problem! Let $g(x) = f(x) - 1$ and $g : Q \to Q$. Then from original equation we have

$$g(x^2 + y + g(xy) + 1) = 2 + (x + g(y) - 1)(g(x) + 1).$$

(1) $g(0) = 0$; really it's true! (2) from (1), $g(1) = 1$ and

$$g(x + 1) = g(x) + 1$$

and

$$g(x^2) = x + g(x)(x - 1)(*)$$

for any ratsional x. Hence, $g(n) = n$ any integer n. Let $x = \frac{m}{n}$ some integers m and n, then from $(*)$, at $P(x, n)$, $\to g(x) = x$ for any ratsional numbers x. Therefore, $f(x) = x + 1$, any ratsional x.

Solution 313 (by tenplusten)**.**

> Sorry, I have latex mistake. Original version: :roll: Oh...ho, I found very nice solution of the problem! Let $g(x) = f(x) - 1$ and $g : Q \to Q$. Then from original equation we have
>
> $$g(x^2 + y + g(xy) + 1) = 2 + (x + g(y) - 1)(g(x) + 1).$$
>
> (1) $g(0) = 0$; really it's true! (2) from (1), $g(1) = 1$ and
>
> $$g(x + 1) = g(x) + 1$$
>
> and
>
> $$g(x^2) = x + g(x)(x - 1)(*)$$
>
> for any ratsional x. Hence, $g(n) = n$ any integer n. Let $x = \frac{m}{n}$ some integers m and n, then from $(*)$, at $P(x, n)$, $\to g(x) = x$ for any ratsional numbers x. Therefore, $f(x) = x + 1$, any ratsional x.

I would like to see your proof for both claims????

****

Problem 74 (Posted by delegat)**.** Find all functions $\mathbb{R} \to \mathbb{R}$ that satisfy

$$f(xf(y) + f(x)) = 2f(x) + xy,$$

for all reals x and y.

(Link to AoPS)

Solution 314 (by pco)**.**

> Find all functions $\mathbb{R} \to \mathbb{R}$ that satisfy:
> $f(xf(y) + f(x)) = 2f(x) + xy$ for all reals x and y.

Here is a rather long method to show that $f(x) = x + 1$. I think someone could could find a shorter one.

Let $P1(x, y)$ be the property $P1(x, y) : f(xf(y) + f(x)) = 2f(x) + xy$

1) $f(x)$ is a bijective function : $f(y_1) = f(y_2)$ implies $2f(1) + y_1 = f(f(y_1) + f(1)) = f(f(y_2) + f(1)) = 2f(1) + y_2$ implies $y_1 = y_2$ and $f(x)$ is injective. $P1(1, z - 2f(1)) : f(f(z - 2f(1)) + f(1)) = z$ and $f(x)$ is surjective.

2) $f(-1) = 0$, $f(0) = 1$, $f(1) = 2$ and $f(2) = 3$ Let $f(0) = a$, $f(b) = 0$ and $f(c) = 1$ $P1(b, c)$ gives $f(b) = bc = 0$ and so either $b = 0$, either $c = 0$ If $b = 0$, $f(0) = 0$ and then $P1(f^{-1}(x), 0)$ gives $f(x) = 2x$ and it is easy to see that this solution is wrong. So $c = 0$ and $f(0) = 1$ and $a = 1$. Then $P1(0, y)$ gives $f(1) = 2$ Then $P1(b, b)$ gives $1 = b^2$ and so $b = -1$ since $f(1) = 2 \neq 0$ and $f(-1) = 0$ Finally, $P(1, -1)$ gives $f(2) = 3$

3) Four other interesting properties : $P1(x,-1)$ gives $P2(x)$: $f(f(x)) = 2f(x) - x$ $P1(x,0)$ gives $P3(x)$: $f(x+f(x)) = 2f(x)$ Applying $f(x)$ on both sides of P3, we have $f(2f(x)) = f(f(x+f(x)))$. Using then P2 on RHS : $f(2f(x)) = 2f(x+f(x)) - x - f(x)$. Using then P3 in RHS : $f(f(x)) = 3f(x)-x = (2f(x)-x)+f(x) = f(f(x))+f(x)$. And so, since $f(x)$ is surjective : $P4(x)$: $f(2x) = f(x)+x$ Applying then $f(x)$ on both sides of $P1(x,y)$: $f(2f(x)+xy) = f(f(xf(y)+f(x)))$. Using P2 on RHS : $f(2f(x)+xy) = 2f(xf(y)+f(x)) - xf(y) - f(x)$. Using P1 on RHS : $P5(x,y)$: $f(2f(x)+xy) = 3f(x)+2xy-xf(y)$

As a synthesis now, we have : $P1(x,y)$: $f(xf(y)+f(x)) = 2f(x)+xy$ $P2(x)$: $f(f(x)) = 2f(x)-x$ $P3(x)$: $f(x+f(x)) = 2f(x)$ $P4(x)$: $f(2x) = f(x)+x$ $P5(x,y)$: $f(2f(x)+xy) = 3f(x)+2xy-xf(y)$ $f(-1)=0$ $f(0)=1$ $f(1)=2$ $f(2)=3$

4) $f(x)=x+1$ $P5(2,x-3)$ gives $f(2x) = 4x-3-2f(x-3)$. Using then P4 on LHS : $f(x) = 3x-3-2f(x-3)$ $P5(1,x-3)$ gives $f(x+1) = 2x-f(x-3)$ Eliminating $f(x-3)$ in these two equations, we obtain $f(x) = -x-3+2f(x+1)$ From $f(x) = -x-3+2f(x+1)$, we have $f(x+1) = -x-4+2f(x+2)$ and so $f(x) = -3x-11+4f(x+2)$ and so $f(2x) = -6x-11+4f(2x+2)$ Using P4 on both sides, we have $f(x) = -3x-7+4f(x+1)$ Eliminating $f(x+1)$ between $f(x) = -x-3+2f(x+1)$ and $f(x) = -3x-7+4f(x+1)$, we have $f(x) = x+1$

And we just have to check that this value fit in the original equation.

Solution 315 (by kucheto)**.**

Let $P(x,y)$ be assertion of $f(xf(y)+f(x)) = 2f(x)+xy$. $P(1,y-2f(1)) \Rightarrow f(f(y-2f(1))+f(1)) = y \Rightarrow f(x)$ is surjective. Let $f(a)=0$, $f(b)=1$; $P(a,b) \Rightarrow ab=0$. If $a=0$, then $P(x,0) \Rightarrow f(f(x)) = 2f(x) \Rightarrow f(z)=2z$ which is not a solution. Therefore, $b=0 \Rightarrow f(0)=1$; $P(0,0) \Rightarrow f(1)=2$. $P(a,a) \Rightarrow a^2=1 \Rightarrow a=-1$. $f(c)=-1$, $P(c,-1) \Rightarrow 0=-2-c \Rightarrow c=-2$. $P(x,-2) \Rightarrow f(f(x)-x) = 2(f(x)-x) \Rightarrow f(u)=2u$ for $u \in f(x)-x$. $P(-1,y) \Rightarrow f(-f(y)) = -y$. (1) $P(y,-1) \Rightarrow f(f(y)) = 2f(y)-y$. (2) (2)−(1) $\Rightarrow$ $2f(y) = f(f(y))-f(-f(y)) \Rightarrow 2z = f(z)-f(-z)$. $z=u \Rightarrow f(-u)=0 \Rightarrow -u = a = -1 \Rightarrow u=1 \Rightarrow f(x)=x+1$ which indeed is a solution.

Hence, the only solution to the equation is $f(x)=x+1$.

****

Problem 75 (Posted by ekaragoz)**.** Prove that there are no continuous functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the functional equation

$$f(1+f(x)) = 2-3x \quad \forall x \in \mathbb{R}.$$

(Link to AoPS)

Solution 316 (by Rust)**.**

There are no function (without continiosly). Because 2-3x is bijective f(x) must be bijective (for any y exist $x=(2-y)/3, z=1+f(x)$ suth that $f(z)=y$, and $f(x)=f(y) \to f(1+f(x))=3-2x=f(1+f(y))=3-2y \to x=y$). From $f(3-3x)=f(1+2-3x)=f(1+f(1+f(x)))=2-3(1+f(x))=-3f(x)-1$ we get $f(3/4)=-1/4$.

Solution 317 (by pco)**.**

> Prove that there are no continuous function satisfying the functional equation below. (f:R-R)
> f(1+f(x))=2-3x

$f(x)$ is clearly bijective, so, if $f(x)$ is continuous, $f(x)$ must be stricty monotonous (since bijective and continuous), so $f(1+f(x))$ would be stricty increasing, which is wrong since $f(1+f(x))=2-3x$, strictly decreasing.

So $f(x)$ can't be continuous.

Solution 318 (by MellowMelon)**.**

pco, can you elaborate more on how you can conclude that $f(x)$ is strictly increasing? I got the rest of the solution.

Solution 319 (by Hamster1800)**.**

Suppose that $f(x)$ is strictly increasing, then if $y>x$,

$f(1+f(y))=2-3y<f(1+f(x))$

But since f is strictly increasing,

$1+f(y)>1+f(x)$ $f(1+f(y))>f(1+f(x))$, but $f(1+f(y))<f(1+f(x))$. Contradiction.

The other way follows similarly.

Solution 320 (by pco)**.**

> pco, can you elaborate more on how you can conclude that $f(x)$ is strictly increasing? I got the rest of the solution.

$g(x)=1+f(x)$ is strictly monotonous. So $g(g(x))$ is stricty increasing. So $g(g(x))-1=f(1+f(x))$ is strictly increasing.

Solution 321 (by Rust)**.**

I found mistake in my solution. Let $g(x)=\frac{1}{4}+f(\frac{3}{4}+x)$, then equation equavalent to $g(g(x))=-3x$ and it had infinetely many (non continiosly) solutions.

Solution 322 (by ekaragoz)**.**

Thank you all

****

Problem 76 (Posted by apollo)**.** Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that $f(m+n)f(m-n)=f(m^2)$ holds for all integers m and n with $m>n>0$.

(Link to AoPS)

Solution 323 (by Rust)**.**

$m = n$ give $f(m^2) = 0$. If $0 \notin N$ conradition, else $m = 1, n = 2, 3, ...$ give $f(n) = 9\ \forall n \in N$.

Solution 324 (by pco)**.**

> Let denote the set of positive integers.Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(m+n)f(m-n) = f(m^2)$

I think that the condition $f(m+n)f(m-n) = f(m^2)$ is supposed to be true $\forall m > n > 0$

Let then $a > 2$: $f(2a-1)f(1) = f(a^2) = f(2a-2)f(2)$ and so $E1$: $\frac{f(2a-1)}{f(2a-2)} = \frac{f(2)}{f(1)} = q$. With $a = 3$, we have $\frac{f(5)}{f(4)} = \frac{f(2)}{f(1)}$.

Let then $a > 3$: $f(2a)f(4) = f((a+2)^2) = f(2a-1)f(5)$ and so $E2$: $\frac{f(2a)}{f(2a-1)} = \frac{f(5)}{f(4)} = \frac{f(2)}{f(1)} = q$.

And so $\frac{f(x+1)}{f(x)} = q\ \forall x > 6$ and $f(x) = \alpha q^x\ \forall x > 6$

Then, for $m > n+6$, $f(m+n)f(m-n) = f(m^2)$ becomes $\alpha^2 q^{2m} = \alpha q^{m^2}$ and $\alpha = q = 1$ and $f(x) = 1\ \forall x > 6$

Let then $a > 0$ and $b > 3$: $f(a+2b)f(a) = f((a+b)^2)$. We have $a+2b > 6$ and $(a+b)^2 > 6$ and so $f(a+2b) = f((a+b)^2) = 1$ and $f(a) = 1\ \forall a > 0$

And the only solution is $f(n) = 1\ \forall n \in \mathbb{N}$

****

Problem 77 (Posted by hien)**.** Suppose that the function $f : \mathbb{N} \to \mathbb{N}$ satisfies the following conditions: a) $f(1) = 1$, b) $3f(n) \cdot f(2n+1) = f(2n) \cdot [1 + 3f(n)]$ for all $n \in \mathbb{N}$, and c) $f(2n) < 6f(n)$ for all $n \in \mathbb{N}$.

Find positive integers k and m satisfying $f(k) + f(m) = 293$.

(Link to AoPS)

Solution 325 (by hjbrasch)**.**

b) yields $f(2n)$ is divisible by $3f(n)$ and then together with c) we get the recursions:

$f(2n) = 3f(n), f(2n+1) = 1 + 3f(n)$

Claim: $f(\sum_{i=0}^{m} \alpha_i 2^i) = \sum_{i=0}^{m} \alpha_i 3^i$ with $\alpha_i \in \{0, 1\}$

Proof: Apply $f(\sum_{i=0}^{m} \alpha_i 2^i) = \alpha_0 + 3\sum_{i=0}^{m-1} \alpha_{i+1} 2^i$ plus induction on m.

as for the solution, e.g., set $k = 47, m = 5$

Solution 326 (by Rust)**.**

> Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satify: a) $f(1) = 1$ b) $3f(n).f(2n+1) = f(2n).[1 + 3f(n)]$ c) $f(2n) < 6f(n)$ Find numbers k, m satifying $f(k) + f(m) = 293$

Because $(3f(n), 1+3f(n)) = 1$ we get $f(2n+1) = k(1+3f(n)), f(2n) = 3kf(n)$, from c) we get, that k=1. It give $f(2n) = 3f(n), f(2n+1) = 3f(n)+1$. Because f(1)=1, for $n = a_0 + a_1 * 2 + ... + a_k * 2^k, a_i = 0, 1$ 2-adic digits we get $f(n) = a_0 + 3 * a_1 + ... + a_k * 3^k$. I think it posted before.

Solution 327 (by pco).

> b) yields $f(2n)$ is divisible by $3f(n)$ and then together with c) we get the recursions:
> $f(2n) = 3f(n), f(2n+1) = 1 + 3f(n)$
> Claim: $f(\sum_{i=0}^{m} \alpha_i 2^i) = \sum_{i=0}^{m} \alpha_i 3^i$ with $\alpha_i \in \{0, 1\}$
> Proof: Apply $f(\sum_{i=0}^{m} \alpha_i 2^i) = \alpha_0 + 3\sum_{i=0}^{m-1} \alpha_{i+1} 2^i$ plus induction on m.
> as for the solution, e.g., set $k = 47, m = 5$

Quite OK.
And all couples (k, m) are $(5, 47), (7, 45), (13, 39), (15, 37), (37, 15), (39, 13), (45, 7), (47, 5)$

Solution 328 (by hjbrasch).
actually, one can easily show that all possible solutions are given by
$k = 32\alpha + 8\beta + 2\gamma + 5$ $m = 32(1-\alpha) + 8(1-\beta) + 2(1-\gamma) + 5 = 52 - k$
with $\alpha, \beta, \gamma \in \{0, 1\}$ (i.e. 8 different ordered solutions)

Solution 329 (by hien).

> actually, one can easily show that all possible solutions are given by
> $k = 32\alpha + 8\beta + 2\gamma + 5$ $m = 32(1-\alpha) + 8(1-\beta) + 2(1-\gamma) + 5 = 52 - k$
> with $\alpha, \beta, \gamma \in \{0, 1\}$ (i.e. 8 different ordered solutions)

How can you find exactly it?

Solution 330 (by hjbrasch).
$293 = 1 \cdot 3^5 + 1 \cdot 3^3 + 2 \cdot 3^2 + 1 \cdot 3^1 + 2 \cdot 3^0$ and now you look for decompositions
$293 = f(k) + f(m) = \sum_{i=0}^{5} (a_i + b_i) 3^i$ with $0 \le a_i, b_i \le 1$ from which it follows that $0 \le a_i + b_i \le 2$ and all sums $a_i + b_i$ are uniquely determined.

Solution 331 (by hien).
Okay, i see, Thanks for your response

Solution 332 (by quangpbc).

> Okay, i see, thanks so much for reply

Hi teacher Hien :D . I saw it is a problem from book " Bi ton hm s qua cc k thi Omlympic " of Nguyn Trng Tun , it is so nice

Solution 333 (by hien).
Yes, the idea of solving is that a number is presented based on base 2 and the function values in base 3

****

Problem 78 (Posted by N.T.TUAN)**.** Find a function $f(x)$ defined for all real values of x such that for all x, $f(x+2)-f(x)=x^2+2x+4$, and if $x \in [0,2)$, then $f(x)=x^2$.

(Link to AoPS)

Solution 334 (by TTsphn)**.**

We consider $f(x)$ on the $[0,2),[2,4),..,[2n,2n+2)$ and $[-(2n+2),-2n)$ Consider $f(x)$ when $x \in [2,4)$ From condition we has $f(x)-f(x-2)=x^2-2x+4$ Because $x \in [2,4)$ so $x-2 \in [0,2)$ so $f(x-2)=(x-2)^2$ So $f(x)=x^2-2x+4+(x-2)^2=2x^2-6x+8=2(x^2-3x+4)$ We prove that exist $f(x)$ by induction.

Solution 335 (by pco)**.**

> Find a function $f(x)$ defined for all real values of x such that for all x, $f(x+2)-f(x)=x^2+2x+4$, and if $x \in [0,2)$, then $f(x)=x^2$.

Although I think that TTsphn's answer is correct, here is a closed form of the unique solution :
$f(x)=\frac{x^3+8x}{6}-\frac{4}{3}\{\frac{x}{2}\}^3+4\{\frac{x}{2}\}^2-\frac{8}{3}\{\frac{x}{2}\}$
Where $\{a\}$ is the fractional part of a

****

Problem 79 (Posted by Dtrung)**.** Let a,b,c, and d be given real numbers. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(ax+b)=c\cdot f(x)+d,$$

for all $x \in \mathbb{R}$.

(Link to AoPS)

Solution 336 (by pco)**.**

> Let a,b,c,d be real numbers. Find all functions $f:R\to R$ such that : $f(ax+b)=c.f(x)+d$ for all $x\in R$.

It's a rather easy problem but you must be clever about all the different cases. Here is the answer (demo is below in the second part of this post). Let's go

$(a,b,c,d)=(0,b,1,0)$. Solution is $f(x)=u$ for any real u. $(a,b,c,d)=(0,b,1,d\neq 0)$. No solution $(a,b,c,d)=(0,b,c\neq 1,d)$. Solution is $f(x)=\frac{d}{1-c}$ constant. $(a,b,c,d)=(1,0,1,0)$. Solution is any function $f(x)$. $(a,b,c,d)=(1,0,1,d\neq 0)$. No solution $(a,b,c,d)=(1,b\neq 0,1,d)$. Solution is $f(x)=$

$h(\{\frac{x}{b}\}) + \frac{dx}{b}$ for any function $h(x)$ defined on $[0, 1)$. $(a, b, c, d) = (1, b \neq 0, c \neq 1, d)$. Solution is $f(x) = c^{\lfloor \frac{x}{b} \rfloor} h(\{\frac{x}{b}\}) - \frac{d}{c-1}$ for any function $h(x)$ defined on $[0, 1)$. $(a, b, c, d) = (-1, b, c = 1, d \neq 0)$. No solution $(a, b, c, d) = (-1, b, c = 1, d = 0)$. Solution is $f(x) = h(x - \frac{b}{2}) + h(\frac{b}{2} - x)$ for any function $h(x)$ $(a, b, c, d) = (-1, b, c = -1, d)$. Solution is $f(x) = h(\frac{b}{2} - x) - h(x - \frac{b}{2}) + \frac{d}{2}$ for any function $h(x)$ $(a, b, c, d) = (-1, b, c \notin \{-1, +1\}, d)$. Solution is $f(x) = \frac{d}{1-c}$ constant. $(a, b, c, d) = (a \notin \{-1, 0, +1\}, b, 1, d \neq 0)$. No solution $(a, b, c, d) = (a \notin \{-1, 0, +1\}, b, 1, 0)$. Solution is $f(x) = h(Sign(x + \frac{b}{a-1})Sign(a)^{\lfloor \frac{\ln(|x+\frac{b}{a-1}|)}{\ln(|a|)} \rfloor}\{\frac{\ln(|x+\frac{b}{a-1}|)}{\ln(|a|)}\})$ for any function $h(x)$ defined on $(-1, 1)$ $(a, b, c, d) = (a \notin \{-1, 0, +1\}, b, c \neq 1, d)$; Solution is $f(x) = c^{\lfloor \frac{\ln(|x-\frac{b}{1-a}|)}{\ln(|a|)} \rfloor} h(Sign(x - \frac{b}{1-a})Sign(a)^{\lfloor \frac{\ln(|x-\frac{b}{1-a}|)}{\ln(|a|)} \rfloor}\{\frac{\ln(|x-\frac{b}{1-a}|)}{\ln(|a|)}\}) + \frac{d}{1-c}$ for any function $h(x)$ defined on $(-1, 1)$

Demos :

Case 1:$a = 0$, $c = 1$ and $d = 0$. The equation is $f(b) = f(x)$ and the solutions are the constant functions $f(x) = u$.

Case 2: $a = 0$, $c = 1$ and $d \neq 0$. The equation is $f(b) = f(x) + d$ and have no solution (since, for $x = b$, we have $0 = d$)

Case 3: $a = 0$ and $c \neq 1$. The equation is $f(b) = cf(x) + d$ and the solution is $f(x) = \frac{d}{1-c}$

Case 4: $a = 1$, $b = 0$, $c = 1$ and $d = 0$. The equation is $f(x) = f(x)$ and any function $f(x)$ is solution.

Case 5: $a = 1$, $b = 0$, $c = 1$ and $d \neq 0$. The equation is $0 = d$ and no solution exist.

Case 6: $a = 1, b \neq 0, c = 1$. The equation is $f(x + b) = f(x) + d$, so $f(x + nb) = f(x) + nd$, so

$$f(x) = f(b\{\frac{x}{b}\} + \lfloor \frac{x}{b} \rfloor b) = f(b\{\frac{x}{b}\}) + \lfloor \frac{x}{b} \rfloor d$$

So $f(x)$ is completely defined on $\mathbb{R}$ as soon as $f(x)$ is defined on $[0, b)$ (or $(b, 0]$). The general solution is then $f(x) = g(b\{\frac{x}{b}\}) + \lfloor \frac{x}{b} \rfloor d$ for any function $g(x)$ defined on $[0, b)$ (or $(b, 0]$). Considering $h(x) = g(bx) - dx$, we have a prettier form for the solution : $f(x) = h(\{\frac{x}{b}\}) + \frac{dx}{b}$ for any function $h(x)$ defined on $[0, 1)$. And it is easy to check that this necessary condition is sufficient.

Case 7: $a = 1$ and $c \neq 1$. The equation is $f(x + b) = cf(x) + d$ and so $f(x + nb) = c^n f(x) + d\frac{c^n-1}{c-1}$ and so $f(x) = f(b\{\frac{x}{b}\} + \lfloor \frac{x}{b} \rfloor b) = c^{\lfloor \frac{x}{b} \rfloor} f(b\{\frac{x}{b}\}) + d\frac{c^{\lfloor \frac{x}{b} \rfloor}-1}{c-1}$. So $f(x)$ is completely defined on $\mathbb{R}$ as soon as $f(x)$ is defined on $[0, b)$ (or $(b, 0]$). The general solution is then $f(x) = c^{\lfloor \frac{x}{b} \rfloor} g(b\{\frac{x}{b}\}) + d\frac{c^{\lfloor \frac{x}{b} \rfloor}-1}{c-1}$ for any function $g(x)$ defined on $[0, b)$ (or $(b, 0]$). Considering $h(x) = g(bx) + \frac{d}{c-1}$, we have a prettier form for the solution : $f(x) = c^{\lfloor \frac{x}{b} \rfloor} h(\{\frac{x}{b}\}) - \frac{d}{c-1}$ for any function $h(x)$ defined on $[0, 1)$. And it is easy to check that this necessary condition is sufficient.

Case 8: $a = -1$,$c = 1$ and $d \neq 0$; The equation is $f(b - x) = f(x) + d$ and so, with $x = \frac{b}{2}$: $0 = d$ and no solution exist.

Case 9: $a=-1$,$c=1$ and $d=0$; The equation is $f(b-x)=f(x)$. Let then $g(x)=f(x+\frac{b/2}{)}$. we have $f(x)=g(x-\frac{b}{2})$ and the equation becomes $g(\frac{b}{2}-x)=g(x-\frac{b}{2})$ and so $g(-x)=g(x)$. The solution is then $f(x)=g(x-\frac{b}{2})$ for any even function $g(x)$. Considering that any even function can be written $g(x)=h(x)+h(-x)$, we have another form for the solution : $f(x)=h(x-\frac{b}{2})+h(\frac{b}{2}-x)$ for any function $h(x)$ And it is easy to check that this necessary condition is sufficient.

Case 10: $a=-1$ and $c=-1$; The equation is $f(b-x)=d-f(x)$. Let then $g(x)=f(\frac{b}{2}-x)-\frac{d}{2}$. We have $f(x)=g(\frac{b}{2}-x)+\frac{d}{2}$ and the equation becomes $g(x-\frac{b}{2})=-g(\frac{b}{2}-x)$ and so $g(-x)=-g(x)$. The solution is then $f(x)=g(\frac{b}{2}-x)+\frac{d}{2}$ for any odd function $g(x)$. Considering that any odd function can be written $g(x)=h(x)-h(-x)$, we have another form for the solution : $f(x)=h(\frac{b}{2}-x)-h(x-\frac{b}{2})+\frac{d}{2}$ for any function $h(x)$ And it is easy to check that this necessary condition is sufficient.

Case 11: $a=-1$ and $|c|\neq 1$ The equation is $f(b-x)=cf(x)+d$ and so $f(x)=f(b-(b-x))=cf(b-x)+d=c^2f(x)+cd+d$. Then : $f(x)=\frac{d}{1-c}$ and the original equation becomes $\frac{d}{1-c}=\frac{cd}{1-c}+d$ which is true. So the only solution is $f(x)=\frac{d}{1-c}$. And it is easy to check that this necessary condition is sufficient.

Case 12: $a\neq 0$, $|a|\neq 1$, $c=1$ and $d\neq 0$. The equation is $f(ax+b)=f(x)+d$. let then $x=\frac{b}{1-a}$. We have $0=d$ and no solution exist.

Case 13: $a\neq 0$, $|a|\neq 1$, $c=1$ and $d=0$. The equation is $f(ax+b)=f(x)$. let then $k(x)=f(x-\frac{b}{a-1})$. We have $f(x)=k(x+\frac{b}{a-1})$ and the equation is $k(a(x+\frac{b}{a-1}))=k(x+\frac{b}{a-1})$ and so $k(ax)=k(x)$. This equation is well known and a general solution is : $k(0)=u$ and, for $x\neq 0$, $k(x)=h(Sign(x)Sign(a)^{\lfloor\frac{\ln(|x|)}{\ln(|a|)}\rfloor}\{\frac{\ln(|x|)}{\ln(|a|)\}}$ for any function $h(x)$ defined on $(-1,1)$ And so a general solution : $f(\frac{b}{1-a})=u$ (any real constant) For $x\neq\frac{b}{1-a}$: $f(x)=h(Sign(x+\frac{b}{a-1})Sign(a)^{\lfloor\frac{\ln(|x+\frac{b}{a-1}|)}{\ln(|a|)}\rfloor}\{\frac{\ln(|x+\frac{b}{a-1}|)}{\ln(|a|)}\})$ for any function $h(x)$ defined on $(-1,1)$ And it is easy to check that this necessary condition is sufficient.

Case 14: $a\neq 0$, $|a|\neq 1$ and $c\neq 1$. The equation is $f(ax+b)=cf(x)+d$. Let then $k(x)=f(x+\frac{b}{1-a})-\frac{d}{1-c}$. We have $f(x)=k(x-\frac{b}{1-a})+\frac{d}{1-c}$ and the equation becomes : $k(ax)=ck(x)$ This equation is well known and a general solution is : $k(0)=0$ and, for $x\neq 0$, $k(x)=c^{\lfloor\frac{\ln(|x|)}{\ln(|a|)}\rfloor}h(Sign(x)Sign(a)^{\lfloor\frac{\ln(|x|)}{\ln(|a|)}\rfloor}\{\frac{\ln(|x|)}{\ln(|a|)\}}$ for any function $h(x)$ defined on $(-1,1)$ And so a general solution : $f(\frac{b}{1-a})=0$ For $x\neq\frac{b}{1-a}$: $f(x)=c^{\lfloor\frac{\ln(|x-\frac{b}{1-a}|)}{\ln(|a|)}\rfloor}h(Sign(x-\frac{b}{1-a})Sign(a)^{\lfloor\frac{\ln(|x-\frac{b}{1-a}|)}{\ln(|a|)}\rfloor}\{\frac{\ln(|x-\frac{b}{1-a}|)}{\ln(|a|)}\})+\frac{d}{1-c}$ for any function $h(x)$ defined on $(-1,1)$ And it is easy to check that this necessary condition is sufficient.

****

Problem 80 (Posted by Yuriy Solovyov). Let $n=111\cdots 11$, where the 1's have been repeated 2007 times. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that for all real x not equal to 0 or 1, we have

$$f^n(x) = \left(1 - \frac{1}{\sqrt[n]{x}}\right)^n?$$

Note that $f^n(x)$ denotes composition of f with itself n times.

(Link to AoPS)

Solution 337 (by pco)**.**

> Let $n = 111...11$, where we have 2007 number 1. Is there exist such function $f(x)$, that for all x, except 1 and 0, we have: $f(f(f(...(f(x))...))) = (1 - \frac{1}{\sqrt[n]{x}})^n$? In the left side $f(x)$ takes n times.

The answer is yes.

Let $E = \mathbb{R} - \{0, 1\}$

Let $g(x) = (1 - \frac{1}{\sqrt[n]{x}})^n$ from $E \to E$. It's easy to see that $g(x)$ is bijective and that $g(g(g(x))) = x$. It's also clear that $g(x) \neq x\ \forall x \in E$ and (since $g(g(g(x))) = x$) that $g(g(x)) \neq x\ \forall x \in E$

Let then : $\mathbb{A} = \{x \in E$ such that $x < g(x)$ and $x < g(g(x))\}$ $\mathbb{B} = \{g(x), x \in \mathbb{A}\}$ $\mathbb{C} = \{g(x), x \in \mathbb{B}\}$

Clearly, $\mathbb{A} \cap \mathbb{B} = \emptyset$, $\mathbb{A} \cap \mathbb{C} = \emptyset$, $\mathbb{B} \cap \mathbb{C} = \emptyset$ and $\mathbb{A} \cup \mathbb{B} \cup \mathbb{C} = E$ and these three sets are infinite.

Let then $A_1, A_2, ..., A_n$ a partition of $\mathbb{A}$ in n bijective subsets and $h_i(x)$, $i \in \{1, 2, ..., n-1\}$ the bijections from A_i in A_{i+1}.

Let then $f(x)$ defined as : $f(0) = 0$ $f(1) = 1$

$\forall x \in E$, either $x \in \mathbb{A}$, either $x \in \mathbb{B}$, either $x \in \mathbb{C}$ and :

1) If $x \in \mathbb{A}$, $\exists k \in \{1, 2, ..., n\}$ such that $x \in A_k$. If $k < n$, $f(x) = h_k(x)$ If $k = n$, $f(x) = g(h_{n-1}^{-1}(h_{n-2}^{-1}(...(h_2^{-1}(h_1^{-1}(x)))...)))$

2) If $x \in \mathbb{B}$, $g^{-1}(x) = g(g(x)) \in \mathbb{A}$ and let $f(x) = g(f(g(g(x)))) = g(f(g^{-1}(x)))$

3) If $x \in \mathbb{C}$, $g(x) = g^{-1}(g^1(x)) \in \mathbb{A}$ and let $f(x) = g(g(f(g(x)))) = g^{-1}(f(g(x)))$

We have $f^n(x) = g(x)\ \forall x \in E$ (I could give a full precise demo of this point if anybody wants).

And this is avaible for any odd n.

****

Problem 81 (Posted by Gib Z)**.** Find a function $f(x)$ such that $f(f(x)) = \exp(x)$.

(Link to AoPS)

Solution 338 (by pco).

> Find A function f(x) such that f(f(x)) = exp(x)

Rather classical problem and solution :

Let $u < 0$ Let $h(x)$ be any strictly increasing continuous bijection from $(-\infty, u] \to (u, 0]$ such that : $\lim_{x\to-\infty} h(x) = u$ and $h(u) = 0$ Let $\{a_n, n > 0\}$ the strictly increasing sequence defined by $a_1 = u$, $a_2 = 0$ and $a_{n+2} = e^{a_n}$ $\forall n > 0$

Let then the sequence of functions $\{g_k(x), k > 0\}$ defined as : $g_1(x)$ is the bijection from $(-\infty, a_1) \to (a_1, a_2)$ such that $g_1(x) = h(x)$ $\forall k > 1$: $g_k(x)$ is the bijection from $[a_{k-1}, a_k) \to [a_k, a_{k+1})$ such that $g_k(x) = e^{g_{k-1}^{-1}(x)}$

Let then $f(x)$ the strictly increasing continuous function defined as :

$\forall x \in (-\infty, a_1)$ $f(x) = g_1(x)$ $\forall x \in [a_{k-1}, a_k)$, $f(x) = g_k(x)$ $\forall k > 1$

Then, if $x < a_1$, $f(x) = h(x) \in (a_1, a_2)$, so $f(f(x)) = e^{g_1^{-1}(f(x))} = e^{h^{-1}(h(x))} = e^x$ And, $\forall k > 1$: $\forall x \in [a_{k-1}, a_k)$, $f(x) = g_k(x) \in [a_k, a_{k+1})$. So $f(f(x)) = e^{g_k^{-1}(f(x))} = e^{g_k^{-1}(g_k(x))} = e^x$

And so $f(f(x)) = e^x$ $\forall x \in \mathbb{R}$

****

Problem 82 (Posted by nedo477). Given $a, s \in \mathbb{N}$, prove that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) = a \cdot n^s,$$

holds for all $n \in \mathbb{N}$.

(Link to AoPS)

Solution 339 (by pco).

> Given $a, s \in$N, prove that exist function $f : N \to N$ such that $f(f(n)) = a.n^s$ with all $n \in N$

1) If $a = s = 1$, then $f(n) = n$ is a solution

2) Consider now that either $a > 1$, either $s > 1$. $\forall x \in \mathbb{N}$, let $g(x)$ the fewest positive integer such that it exists a non negative integer k such that $x = a^{\sum_{i=1}^{k} s^{i-1}}(g(x))^{s^k}$ (with the convention that $\sum_{i=1}^{0} s^{i-1} = 0$). Let then $h(x)$ this nonnegative integer k.

Let $E = g(\mathbb{N}$. E is a countable infinite subset of $\mathbb{N}$ (since it contains at least all the primes different from a). Let U any infinite subset of E such that $V = E - U$ is also an infinite subset. U and V are bijective subsets (each is a countable infinite set) and let $b(x)$ a bijection from U in V.

Let then $f(x)$ defined as :

If $g(x) \in U$, $f(x) = a^{\sum_{i=1}^{h(x)} s^{i-1}}(b(g(x)))^{s^{h(x)}}$

If $g(x) \in V$, $f(x) = a^{\sum_{i=1}^{h(x)+1} s^{i-1}}(b^{-1}(g(x)))^{s^{h(x)+1}}$

It's rather easy to check that $f(f(x)) = ax^s$:

If $g(x) \in U$, $f(x) = a^{\sum_{i=1}^{h(x)} s^{i-1}}(b(g(x)))^{s^{h(x)}}$. Then $g(f(x)) = b(g(x)) \in V$ and $h(f(x)) = h(x)$. And so :

$f(f(x)) = a^{\sum_{i=1}^{h(f(x))+1} s^{i-1}}(b^{-1}(g(f(x))))^{s^{h(f(x))+1}} = a^{\sum_{i=1}^{h(x)+1} s^{i-1}}(b^{-1}(b(g(x))))^{s^{h(x)+1}}$
$= a(a^{\sum_{i=1}^{h(x)} s^{i-1}}(g(x))^{s^{h(x)}})^s = a \cdot x^s$

If $g(x) \in V$, $f(x) = a^{\sum_{i=1}^{h(x)+1} s^{i-1}}(b^{-1}(g(x)))^{s^{h(x)+1}}$. Then $g(f(x)) = b^{-1}(g(x)) \in U$ and $h(f(x)) = h(x) + 1$. And so :

$f(f(x)) = a^{\sum_{i=1}^{h(f(x))} s^{i-1}}(b(g(f(x))))^{s^{h(f(x))}} = a^{\sum_{i=1}^{h(x)+1} s^{i-1}}(b(b^{-1}(g(x))))^{s^{h(x)+1}}$
$= a(a^{\sum_{i=1}^{h(x)} s^{i-1}}(g(x))^{s^{h(x)}})^s = a \cdot x^s$

Solution 340 (by nedo477)**.**

I've read your solution. It's great! And here is my way *Let $g(n) = a.n^s; g_m(n) = g(..(g(n))$ (m times) and $g_0(n) = n$ $B = (b \ \in N | b \neq g(n) \forall n)$

*First we'll prove that $\forall n$, $\exists! b \in B, m \in N$ such that $n = g_m(b)(1)$ $+n \in B \to m = 0$, (1) right $n \notin B \to \exists n' : n = g(n') \to n' < n \to n' = g_{m'}(b') \to n = g_{m'+1}(b')$, (1) right +Assume $\exists (m', b') \neq (m, b)$ so that $g_{m'}(b') = n = g_m(b)$ If $m' = m$, then $b' = b$,wrong If $m' > m$, then $b = g_{m'-m}(b') \notin B$, wrong If $m' < m$, then $b' = g_{m-m'}(b) \notin B$, wrong So (m, b) is unique. (1) is proved

*Now, we'll finish this problem $B = (b_1, b_2, ...)$ $(b_1 < b_2 < ...)$ $n = g_m(b_i)$ If $i \vdots 2$, $f(n) = g_m(b_{i-1})$ else $f(n) = g_{m+1}(b_{i+1})$

Then $f(n)$ is the function we need (easy to check).

Solution 341 (by pco)**.**

> I've read your solution. It's great! And here is my way *Let $g(n) = a.n^s; g_m(n) = g(..(g(n))$ (m times) and $g_0(n) = n$ $B = (b \ \in N | b \neq g(n) \forall n)$
> *First we'll prove that $\forall n$, $\exists! b \in B, m \in N$ such that $n = g_m(b)(1)$ $+n \in B \to m = 0$, (1) right $n \notin B \to \exists n' : n = g(n') \to n' < n \to n' = g_{m'}(b') \to n = g_{m'+1}(b')$, (1) right +Assume $\exists (m', b') \neq (m, b)$ so that $g_{m'}(b') = n = g_m(b)$ If $m' = m$, then $b' = b$,wrong If $m' > m$, then $b = g_{m'-m}(b') \notin B$, wrong If $m' < m$, then $b' = g_{m-m'}(b) \notin B$, wrong So (m, b) is unique. (1) is proved
> *Now, we'll finish this problem $B = (b_1, b_2, ...)$ $(b_1 < b_2 < ...)$ $n = g_m(b_i)$
> If $i \vdots 2$, $f(n) = g_m(b_{i-1})$ else $f(n) = g_{m+1}(b_{i+1})$
> Then $f(n)$ is the function we need (easy to check).

Quite OK, and we have nearly the same demos :

My E is your B My $g(n)$ is your function mapping any n on the b_i such that $n = g_m(b_i)$ And you choose as splitting U, V of E : $E = B = \{b_1, b_2, b_3, \ldots\}$ $= U \cup V = \{b_1, b_3, b_5, \ldots\} \cup \{b_2, b_4, b_6, \ldots\}$

****

Problem 83 (Posted by Erken)**.** Find all functions $f : \mathbb{N} \to \mathbb{R}$ such that for all $x, y, z \in \mathbb{N}$,

$$f(xy) + f(xz) \geq f(x)f(yz) + 1.$$

(Link to AoPS)

Solution 342 (by ashwinrk˙jain)**.**

this is quite interesting question $f(xy)+f(xz) >= f(x)f(yz)+1$ hence we get $f(0)+\frac{1}{f(0)} <= 2$ but we know $f(0)+\frac{1}{f(0)} >= 2$ hence we can say $f(0) = 1$ now, put $x = y = z = 1$ we get $f(1)[2 - f(1)] >= 1$ which again gives $(f(1)-1)^2 = 0$ that is $f(1) = 1$ now, put y=0 $f(0)+f(xz) >= f(x)f(0)+1$ solving this we get $f(xz) >= f(x)$..........................(1) put x=1 $f(z) >= f(1)$............................(2) also put z=1 in (1) we $f(0) >= f(x)$......................(3) combining (2) and (3) we get $f(1) <= f(x) <= f(0)$ that is $1 <= f(x) <= 1$ hence we get $f(x) = 1$

Solution 343 (by Erken)**.**

In this question $\mathbb{N}$ is a set of natural numbers,i.e $\mathbb{N} = \{1, 2, 3 \dots\}$.So you can't consider $f(0)$.

Solution 344 (by pco)**.**

> Find all functions $f : \mathbb{N} \to \mathbb{R}$,such that for all $x, y, z \in \mathbb{N}$: $f(xy)+f(xz) \geq f(x)f(yz) + 1$.

Let $P(x, y, z)$ the property $f(xy) + f(xz) \geq f(x)f(yz) + 1$

1) $P(1, 1, 1)$ gives $2f(1) \geq f(1)^2+1$, and so $0 \geq f(1)^2-2f(1)+1 = (f(1)-1)^2$ and so $\boxed{f(1) = 1}$

2) $P(x, 1, 1)$ gives $2f(x) \geq f(x)f(1)+1 = f(x)+1$ and so $\boxed{f(x) \geq 1\, \forall x \in \mathbb{N}}$

3) $P(x, x, x)$ gives $2f(x^2) \geq f(x)f(x^2)+1$ and so $f(x^2)(f(x)-2) \leq -1$ and so $\boxed{f(x) < 2\, \forall x \in \mathbb{N}}$

4) Let then $a = \sup(\{f(n), n \in \mathbb{N}\})$ the least upper bound of $f(\mathbb{N})$. a exists since $f(\mathbb{N}) \subseteq [1, 2)$ and $2 \geq a \geq 1$ Let then $a > \epsilon > 0$ and p such that $0 \leq a - f(p) < \epsilon$ $P(p, p, 1)$ gives $f(p^2) + f(p) \geq (f(p))^2 + 1$. And, since LHS is less tha $a + a$: $(f(p))^2 + 1 \leq 2a$ and so $(a - \epsilon)^2 + 1 \leq 2a$ And so $(a-1)^2 < 2\epsilon a - \epsilon^2 < 4\epsilon\ \forall \epsilon \in (0, 1)$ And so $a = 1$ And so $\boxed{f(x) = 1\ \forall x \in \mathbb{N}}$

****

Problem 84 (Posted by Svejk)**.** Let $f : (0, \infty) \to (0, \infty)$ be a non-constant function such that

$$f(x) \cdot f(yf(x)) \cdot f(zf(x+y)) = f(x+y+z) \quad \forall x, y, z \in (0, \infty).$$

a) Prove that f is injective. b) Find f.

(Link to AoPS)

Solution 345 (by pco).

> Let $f:(0,\infty)\to(0,\infty)$, such that :
> $f(x)\cdot f(yf(x))\cdot f(zf(x+y))=f(x+y+z)\ \forall x,y,z\in(0,\infty)$
> a)prove that f is injective b) find f

Obviously, it is impossible to prove that $f(x)$ is injective since $f(x)=1$, non injective, is a solution of the equation.

Solution 346 (by Svejk).

yes you are right..I forgot to mention that f must not be constant.I edited my initial post now.I m sorry.

Solution 347 (by pco).

> Let $f:(0,\infty)\to(0,\infty)$ a non constant function such that :
> $f(x)\cdot f(yf(x))\cdot f(zf(x+y))=f(x+y+z)\ \forall x,y,z\in(0,\infty)$
> a)prove that f is injective b) find f

Let $P_0(x,y,z)$ the property $f(x)f(yf(x))f(zf(x+y))=f(x+y+z)$

1) Assume it exists $x,y>0$ such that $f(x+y)>1$. Then $\frac{x+y}{f(x+y)-1}>0$ and : $P_0(x,y,\frac{x+y}{f(x+y)-1})$ gives $f(x)f(yf(x))f(f(x+y)\frac{x+y}{f(x+y)-1})=f(x+y+\frac{x+y}{f(x+y)-1})$ And since $f(x+y)\frac{x+y}{f(x+y)-1}=x+y+\frac{x+y}{f(x+y)-1}$, we have $f(x)f(yf(x))=1$ Then $P_0(x,y,\frac{x+y}{f(x+y})$ gives $f(x)f(yf(x))f(x+y)=f(x+y+\frac{x+y}{f(x+y)})$ and, since $f(x)f(yf(x))=1$ and $f(x+y)>1$: $f(x+y+\frac{x+y)}{f(x+y})>1$

So, $f(x+y)>1$ implies $\boxed{f(yf(x))=\dfrac{1}{f(x)}}$ and $f(x+y+\frac{x+y}{f(x+y})>1$ Then,

$f(x+y+\frac{x+y}{f(x+y)})>1$ implies $\boxed{f(\dfrac{x+y}{f(x+y)}f(x+y))=\dfrac{1}{f(x+y)}}$

So, $P_0(x,y,\frac{x+y}{f(x+y)})$ gives $f(x)f(yf(x))f(\frac{x+y}{f(x+y)}f(x+y))=f(x+y+\frac{x+y}{f(x+y)})$ and so : $f(x)\frac{1}{f(x)}\frac{1}{f(x+y)}=f(x+y+\frac{x+y}{f(x+y)})$ and so : $f(x+y+\frac{x+y}{f(x+y)})f(x+y)=1$ which is impossible since both factors are >1

So $\boxed{f(x)\le 1\ \forall x>0}$

2) Since $f(x)\le 1\ \forall x$, then $f(x+y+z)=f(x)f(yf(x))f(zf(x+y))\le f(x)$ and $f(x)$ is a non increasing function. Assume then it exists $a>0$ such that $f(a)=1$. Since $f(x)$ is a non increasing function, $f(x)=1\ \forall x\le a$. Then $P_0(\frac{a}{2},\frac{a}{2},\frac{a}{2})$ gives $f(\frac{a}{2})f(\frac{a}{2}f(\frac{a}{2}))f(\frac{a}{2}f(a))=f(\frac{3a}{2})$ But each factor in LHS is 1 and so $f(\frac{3a}{2})=1$ and so $f((\frac{3}{2})^n a)=1\ \forall n$ and so $f(x)=1\ \forall x\le(\frac{3}{2})^n a\ \forall n$ And so $f(x)=1\ \forall x$. So, since $f(x)$ is a non-constant function, such $a>0$ respecting $f(a)=1$ does not exist and $\boxed{0<f(x)<1\ \forall x>0}$

3) Since $f(u)<1\ \forall u>0$: $f(x+y+z)=f(x)f(yf(x))f(zf(x+y))<f(x)$ and $f(x)$ is a stricty decreasing function. As a consequence, $f(x)$ is an injective function.

4) Let then $x, y, z > 0$. Let $u = \frac{zf(x+y)}{f(x)}$ and $v = \frac{yf(x)}{f(x+u)}$; We have : $uf(x) = zf(x+y)$ and $vf(x+u) = yf(x)$ and so $f(x)f(yf(x))f(zf(x+y)) = f(x)f(uf(x))f(vf(x+u))$ and so $f(x+y+z) = f(x+u+v)$ and, since $f(x)$ is injective, $y+z = u+v$:

So $y + \frac{uf(x)}{f(x+y)} = u + \frac{yf(x)}{f(x+u)}$

So $\frac{f(x)}{yf(x+y)} - \frac{1}{y} = \frac{f(x)}{uf(x+u)} - \frac{1}{u} = c$

Sp $f(x+y) = \frac{f(x)}{cy+1}$ $\forall x, y > 0$

So $f(y) = \frac{f(x)}{cy-cx+1}$ $\forall y > x > 0$

And so $f(x) = \frac{1}{ax+b}$ for $x > d$ and some a and b

Putting back this expression in the orinal equation, we find $\boxed{f(x) = \frac{1}{cx+1}}$

****

Problem 85 (Posted by N.T.TUAN)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(xy)) = f(x) + xf(y) \quad \forall x, y \in \mathbb{R}.$$

(Link to AoPS)

Solution 348 (by pco)**.**

> Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
>
> $$f(x + f(xy)) = f(x) + xf(y)\, \forall x, y \in \mathbb{R}.$$

Let $P(x, y)$ be the property $f(x + f(xy)) = f(x) + xf(y)$

1) Assume it exists $u \neq 0$ such that $f(u) = 0$. Let then $v \neq 0$. $P(\frac{u}{v}, v)$ gives $f(\frac{u}{v} + f(u)) = f(\frac{u}{v}) + \frac{u}{v}f(v)$ and so $0 = \frac{u}{v}f(v)$ and so $f(v) = 0$ and $f(x) = 0$ $\forall x \neq 0$. Then, if $f(0) = a \neq 0$, $P(a, 0)$ gives $f(2a) = f(a) + af(0)$ and so $f(0) = 0$ and contradiction. So, if it exists $u \neq 0$ such that $f(u) = 0$, then $f(x) = 0$ $\forall x$ and this is a first solution to the problem.

Now, we'll consider $f(x)$ not always 0. And so $\boxed{f(u) = 0 \implies u = 0}$

2) $P(-1, -1)$ gives $f(-1 + f(1)) = f(-1) - f(-1) = 0$ and so $\boxed{f(1) = 1}$ and $\boxed{f(0) = 0}$

3) We have $f(0) = 0$ and $f(1) = 1$. Assume $f(n) = n$. Then $P(1, n)$ gives $f(n+1) = n+1$ and so, with induction, $\boxed{f(n) = n \;\forall n \in \mathbb{N} \cup \{0\}}$

4) Let $n \geq 1$ $P(-1, -n)$ gives $f(n-1) = f(-1) - f(-n)$ and so $f(-n) = 1 + f(-1) - n$ Then $P(-n, -1)$ gives $f(0) = 0 = f(-n) - nf(-1) = 1 + f(-1) - n - nf(-1)$= $(1 + f(-1))(1-n)$ and so $f(-1) = -1$ and so $f(-n) = -n$ So $\boxed{f(n) = n \;\forall n \in \mathbb{Z}}$

5) $P(-1,x)$ gives $f(-1+f(-x)) = -1-f(x)$ and so $1+f(-1+f(-x)) = -f(x)$; Then $P(1,-1+f(-x))$ gives $\boxed{f(-f(x)) = -f(x)}$ Then $-f(-f(x)) = f(x)$ and so $f(-f(-f(x))) = -f(-f(x))$ and so $\boxed{f(f(x)) = f(x)}$

6) $P(1,x)$ gives $f(f(x)+1) = f(x)+1$ and, replacing x with $f(x)+1$, we obtain $f(f(x)+2) = f(x)+2$ and, with immediate induction, $f(f(x)+n) = f(x)+n$ $\forall n \in \mathbb{N}\cup\{0\}$ $P(-1,x)$ gives $f(-1+f(-x)) = -1-f(x)$ and so $2+f(-1+f(-x)) = 1-f(x)$ and so (since $f(f(x)+n) = f(x)+n$) $f(1-f(x)) = 1-f(x)$ So $-f(1-f(x)) = f(x)-1$ and, since $f(-f(x)) = -f(x)$: $f(f(x)-1) = f(x)-1$, and with induction $f(f(x)-n) = f(x)-n$ So $\boxed{f(f(x)+n) = f(x)+n \ \forall n \in \mathbb{Z}}$

7) Now, replacing x by nx in point 6 above : $f(f(nx)+n) = f(nx)+n$. But $P(n,x)$ gives $f(n+f(nx)) = f(n)+nf(x) = n+nf(x)$ and so, combining these two lines : $f(nx)+n = nf(x)+n$ and so $\boxed{f(nx) = nf(x) \ \forall n \in \mathbb{Z}}$

8) Assume, for a given u, $f(u) = -u$, then $f(-u) = u$ (since, according to point 7 above, $f(nu) = nf(u)$; take $n=-1$). Then $P(-1,-u)$ gives $f(-1+f(u)) = -1-f(-u)$ and so $f(-1-u) = -1-u$ Then $P(1,-1-u)$ gives $f(1+f(-1-u)) = 1+f(-1-u)$ and so $f(-u) = -u$ and, since $f(-u) = u$, $u = -u = 0$ So, $\boxed{f(u) = -u \implies u = 0}$

9) $P(x,-1)$ gives $f(x+f(-x)) = f(x)-x$ and so $f(x-f(x)) = f(x)-x$, and so (according to point 8 above) $x - f(x) = 0$ And so $\boxed{f(x) = x \ \forall x}$

So the two only solutions are : $f(x) = 0 \ \forall x$ $f(x) = x \ \forall x$

****

Problem 86 (Posted by cyshine)**.** Let $f(x) = x^2 + 2007x + 1$. Prove that for every positive integer n, the equation $\underbrace{f(f(\ldots(f}_{n \text{ times}}(x))\ldots)) = 0$ has at least one real solution.

(Link to AoPS)

Solution 349 (by TTsphn)**.**

Solution 1 : Result Consider the equation : $f(x) = a$ where $a \geq \frac{-2007^2+4}{4}$ has solution x_1, x_2 then $x_1 = \max\{x_1, x_2 \geq \frac{-2007}{2}$ Proof : From $x_1+x_2 = -2007$ We prove by induction $f_n x = a$ has at least real solution if $a > \frac{-2007^2+4}{4}$ Suppose it is true for n We prove it is true for $n+1$ $f_{n+1}(x) = f_n(x)^2 + 2007 f_n(x) + 1$ If $f_n(x_n) = x_1$ then x_n is a root of $f_{n+1}(x) = 0$ But we have $x_1 > \frac{-2007^2+4}{4}$ so the equation has solution .Call it is $x_{11} \geq \frac{-2007}{4}$ Contine consider equation $f_{n-1} = x_{11}$ .. It has solution. Our induction claim. Apply this result our problem was be claim. I know that have other solution so please contine discuss it .

Solution 350 (by pco)**.**

Let $f(x) = x^2 + 2007x + 1$. Prove that for every positive integer n, the equation $\underbrace{f(f(\dots(f}_{n\text{ times}}(x))\dots)) = 0$ has at least one real solution.

The equation $f(x) = x$ has two solutions $a < b < 0$.

Then, if $g(x) = \underbrace{f(f(\dots(f}_{n\text{ times}}(x))\dots))$, we have :

1) $g(b) = b < 0$ 2) $g(x)$ is a polynomial whose top term is x^{2^n} and so $\lim_{x\to+\infty} g(x) = \infty > 0$

And so $g(x)$ has at least one real root in $(b, +\infty)$

Solution 351 (by hjbrasch)**.**

Define $a = -\frac{2007}{2}$, $b = -a^2 + 1$ and the intervals $D = [a, \infty[$, $W = [b, \infty[$, then we have $0 \in D \subset W$ and $f(x) = (x-a)^2 + b$, i.e. $f(\mathbb{R}) = f(D) = W$ and therefore $f(\dots f(\dots(\mathbb{R})\dots) = W$

Solution 352 (by Mithril)**.**

$f(x) = q$ has solution if $q > 1 - \frac{2007^2}{4}$. One solution for $f(x) = q$ is $x_1 = \frac{\sqrt{s}-2007}{2} > -2007 > 1 - \frac{2007^2}{4}$. So $f(x) = x_1$ has a root x_2 such that $f(x) = x_2$ has a root x_3..., and if we assume $q = 0$, we can do this process indefinitely, which is what we had to prove.

Solution 353 (by TTsphn)**.**

Here is general problem (can solve with my method) Find then number of solution of the equation : $f_n(x) = 0$

Solution 354 (by SnowEverywhere)**.**

I think that this works.

Solution

Let $Q(x) = \underbrace{f(f(\dots(f}_{n\text{ times}}(x))\dots))$.

We have that

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$

Therefore we also have that

$$x \to \infty \quad \Rightarrow \quad Q(x) \to \infty$$

Hence there exists arbitrarily large value α such that $Q(\alpha) > 0$.

Let β denote a root of the equation $f(x) = x$. The quadratic formula yields that $\beta < 0$. This yields that $Q(\beta) = \beta < 0$.

By the intermediate value theorem, there exists x between α and β such that $Q(x) = 0$.

****

—

Problem 87 (Posted by primoz2)**.** Prove that there are no functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(f(n)) = n + 1$ for all positive integers n.

(Link to AoPS)

Solution 355 (by pco)**.**

> Prove that there are no funkcions satisfying f(f(n))=n+1, where n is a natural number

1) Wrong : $f(n) = n + \frac{1}{2}$ is a solution.

2) If you add to your problem that $f(x)$ is a function from $\mathbb{N} \to \mathbb{N}$, then :

First solution: If $a = 1$, $f(f(a) = f(f(1)) = f(a) = f(1) = a = 1 \neq 1 + 1$. hence $f(1) = a > 1$.

Now, the notation $f^p(x)$ means $f(f(..(x))..)$ p times and not the power.

Then $f^2(1) = 2$, $f^4(1) = 3$, ... and $f^{2a-2}(1) = a$ and $f^{2a-1}(1) = f(a) = f(f(1)) = 2$ But $f^{2a-1}(1) = f^{2a-2}(f(1)) = f(1) + a - 1 = 2a - 1$

So $2a - 1 = 2$ and $a = \frac{3}{2}$ which is impossible. hence the result.

Second solution : another quicker method: $f(f(n)) = n + 1$ implies $f(n + 1) = f(n) + 1$ and so $f(n) = n + f(1) - 1$ Then $f(f(n)) = n + 1$ implies $n + 2(f(1) - 1)) = n + 1$ and so $f(1) = \frac{3}{2}$ which is impossible.

Solution 356 (by primoz2)**.**

I ment it like in 2, but i didnt't know how to write that what you aded.

****

Problem 88 (Posted by tanpham90)**.** Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(f(x)) = 3f(x) - 2x, \quad \forall x \in \mathbb{R}.$$

(Link to AoPS)

Solution 357 (by pco)**.**

> Find all continuous functions $f : R \to R$ which is satisfy the requirements : $f(f(x)) = 3f(x) - 2x \ \forall x \in R$

1) $f(x) = x$ is a solution. So we'll now consider functions for which it exists an x_0 such that $f(x_0) \neq x_0$

2) $f(x)$ is obviously injective, and so monotonous (since continuous). If $f(x_0) > x_0$, then $f(f(x_0)) = 3f(x_0) - 2x_0 > f(x_0)$ and so $f(x)$ is strictly increasing. If $f(x_0) < x_0$, then $f(f(x_0)) = 3f(x_0) - 2x_0 < f(x_0)$ and so $f(x)$ is strictly increasing. If $\lim_{x\to+\infty} f(x) = L$, then, in $f(f(x)) = 3f(x) - 2x$, when $x \to +\infty$, LHS$\to$ $f(f(L))$ and RHS$\to$ $-\infty$. So $\lim_{x\to+\infty} f(x) = +\infty$ If $\lim_{x\to-\infty} f(x) = l$, then, in $f(f(x)) = 3f(x) - 2x$, when $x \to -\infty$, LHS$\to$

$f(f(l))$ and RHS$\to +\infty$. So $\lim_{x\to -\infty} f(x) = -\infty$ So $f(x)$ is a strictly increasing bijection and $f(\mathbb{R}) = \mathbb{R}$

3) Using $f(f(x)) = 3f(x) - 2x$, it is easy to establish that $f(u_n) = u_{n+1}$ with $u_0 = x_0$, $u_1 = f(x_0)$ and $u_{n+2} = 3u_{n+1} - 2u_n$ for any $n \geq 0$. Then, since $u_n = (2x_0 - f(x_0)) + (f(x_0) - x_0)2^n$, we have $f(u_n) = 2u_n - (2x_0 - f(x_0))$ Using that $f(x)$ is a bijection and that $f(f(x)) = 3f(x) - 2x$ implies $2f^{-1}(x) = 3x - f(x)$, it is possible to show that the previous equalities are true even for $n < 0$ and so : $f(x) = 2x - L_0$ with $L_0 = 2x_0 - f(x_0)$ for any $x \in \{(2x_0 - f(x_0)) + (f(x_0) - x_0)2^n \ \forall n \in \mathbb{Z}\}$. With continuity, we have, when $n \to -\infty$, $f(L_0) = L_0$

4) Now, consider $f(x_0) > x_0$. If it exists x_1(Wlog $x_1 > x_0$) such that $f(x_1) > x_1$ and $f(x_1) \neq 2x_1 - L_0$, then we have $f(L_0) = L_0$ and $f(L_1) = L_1$. For any $L \in (L_0, L_1)$, with continuity, it exist a value a such that $f(a) = 2a - L$. Applying then the demo of point 3 above starting with a and $f(a)$, we conclude $f(L) = L$, and so $f(x) = x \ \forall x \in (L_0, L_1)$ which is in contradiction with $f(x) = 2x - L_0$ for any $x \in \{(2x_0 - f(x_0)) + (f(x_0) - x_0)2^n \ \forall n \in \mathbb{Z}\}$. So $f(x_0) > x_0$ implies : $f(x) \leq x \ \forall x \leq L_0$ $f(x) = 2x - L_0 \ \forall x \geq L_0$

5) Now, consider $f(x_0) < x_0$. The same demo as above gives : $f(x) \geq x$ $\forall x \geq L_0$ $f(x) = 2x - L_0 \ \forall x \leq L_0$

And so the solutions :

a) $f(x) = x$ *b*) Let $L \in \mathbb{R}$. $f(x) = x \ \forall x \leq L$ and $f(x) = 2x - L \ \forall x \geq L$ *c*) Let $L \in \mathbb{R}$. $f(x) = x \ \forall x \geq L$ and $f(x) = 2x - L \ \forall x \leq L$ *d*) Let $L_0 \leq L_1 \in \mathbb{R}$. $f(x) = 2x - L_0 \ \forall x \leq L_0$, $f(x) = x \ \forall x \in [L_0, L_1]$ and $f(x) = 2x - L_1 \ \forall x \geq L_1$

Solution 358 (by tanpham90)**.**

Thank a million , Patrick :) !

> 4) Now, consider $f(x_0) > x_0$. If it exists x_1(Wlog $x_1 > x_0$) such that $f(x_1) > x_1$ and $f(x_1) \neq 2x_1 - L_0$, then we have $f(L_0) = L_0$ and $f(L_1) = L_1$. For any $L \in (L_0, L_1)$, with continuity, it exist a value a such that $f(a) = 2a - L$. Applying then the demo of point 3 above starting with a and $f(a)$, we conclude $f(L) = L$, and so $f(x) = x$ $\forall x \in (L_0, L_1)$ which is in contradiction with $f(x) = 2x - L_0$ for any $x \in \{(2x_0 - f(x_0)) + (f(x_0) - x_0)2^n \ \forall n \in \mathbb{Z}\}$. So $f(x_0) > x_0$ implies : $f(x) \leq x \ \forall x \leq L_0$ $f(x) = 2x - L_0 \ \forall x \geq L_0$

Can you explain more detail , please ! What is L_1 and why do we have $f(L_1) = L_1$

Solution 359 (by pco)**.**

> Thank a million , Patrick :) !
>
> > 4) Now, consider $f(x_0) > x_0$. If it exists x_1(Wlog $x_1 > x_0$) such that $f(x_1) > x_1$ and $f(x_1) \neq 2x_1 - L_0$, then we have $f(L_0) = L_0$ and $f(L_1) = L_1$. For any $L \in (L_0, L_1)$, with continuity, it exist a value a such that $f(a) = 2a - L$. Applying then the demo of point 3 above starting with a and $f(a)$, we conclude $f(L) = L$, and so $f(x) = x\ \forall x \in (L_0, L_1)$ which is in contradiction with $f(x) = 2x - L_0$ for any $x \in \{(2x_0 - f(x_0)) + (f(x_0) - x_0)2^n \forall n \in \mathbb{Z}\}$. So $f(x_0) > x_0$ implies : $f(x) \le x\ \forall x \le L_0\ f(x) = 2x - L_0\ \forall x \ge L_0$
>
> Can you explain more detail , please ! What is L_1 and why do we have $f(L_1) = L_1$

Sure, it's not very clear. i'm sorry. let's try in a different manner :

Let x_0 and x_1 such that $f(x_0) > x_0$ and $f(x_1) > x_1$

Then we have shown in the point 3 above that : $f(x) = 2x - L_0$ with $L_0 = 2x_0 - f(x_0)$ for any $x \in \{(2x_0 - f(x_0)) + (f(x_0) - x_0)2^n\ \ \forall n \in \mathbb{Z}\}$. $f(L_0) = L_0$ And similarly : $f(x) = 2x - L_1$ with $L_1 = 2x_1 - f(x_1)$ for any $x \in \{(2x_1 - f(x_1)) + (f(x_1) - x_1)2^n\ \ \forall n \in \mathbb{Z}\}$. $f(L_1) = L_1$

Then, if $L_0 \neq L_1$, WLOG say $L_0 < L_1$, Let $L \in (L_0, L_1)$ The line $g(x) = 2x - L$ is below the curve $f(x)$ at point $x = L_0$ and is above the curve $f(x)$ at point $x = L_1$. So, using continuity, it exist a value $a \in (L_0, L_1)$ such that $f(x) = g(x)$ and so $f(a) = 2a - L$. Applying then the demo of point 3 above starting with a and $f(a)$, we conclude $f(L) = L$, and so $f(x) = x\ \forall x \in (L_0, L_1)$ which is in contradiction with $f(x) = 2x - L_0$ for any $x \in \{(2x_0 - f(x_0)) + (f(x_0) - x_0)2^n\ \ \forall n \in \mathbb{Z}\}$. So $L_0 = L_1$ and :

$f(x_0) > x_0$ implies : $f(x) \le x\ \forall x \le L_0\ f(x) = 2x - L_0\ \forall x \ge L_0$

And the reminder of the demo.

Is it a bit more clear ?.

Solution 360 (by tanpham90)**.**

Now I understand ! Thank you Patrick

****

Problem 89 (Posted by tdl)**.** Find all function $f : (0,1) \to \mathbb{R}$ so that

$$f(xy) = xf(x) + yf(y), \quad \forall x, y \in (0,1).$$

(Link to AoPS)

Solution 361 (by TTsphn)**.**

Good problem . $f(x) \equiv 0$

Solution 362 (by pco).

> Find all function $f:(0;1)\to R$ so that: $f(xy)=xf(x)+yf(y)\forall x,y\in(0;1)$

$f(xy^2)=f(x(y^2))=xf(x)+y^2f(y^2)=xf(x)+2y^3f(y)$ $f(xy^2)=f((xy)y)=xyf(xy)+yf(y)=x^2yf(x)+xy^2f(y)+yf(y)=x^2yf(x)+y(xy+1)f(y)$

And so $xf(x)+2y^3f(y)=x^2yf(x)+y(xy+1)f(y)$ And so $y(2y^2-xy-1)f(y)=x(xy-1)f(x)$

Taking then $y=\frac{x+\sqrt{x^2+8}}{4}$, we have $y\in(0,1)$ and $2y^2-xy-1=0$ and $xy-1\neq 0$ (since $x\in(0,1)$ and $y\in(0,1)$) and so $f(x)=0\ \forall x$

Solution 363 (by tdl).

Another solution: $f(x^2)=2xf(x), f(x^4)=2x^2f(x^2)=4x^3f(x)$ $f(x^4)=xf(x)+x^3f(x^3)=xf(x)+x^3(xf(x)+x^2f(x^2))=(2x^6+x^4+x)f(x)$ Then $f(x)g(x)=0\forall x\in(0;1)$ with $g(x)=2x^6+x^4-4x^3+x$ It mean $f(x)=0$ with all x isn't root of $g(x)$ If exit $f(x_0)\neq 0$, it mean $g(x_0)=0$ then $f(x_0^2)=2x_0f(x_0)\neq 0$, hence $g(x_0^2)=0$ Similarly we have $g(x)$ have infinity number of root, contradiction!

****

Problem 90 (Posted by FOURRIER). Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that

$$f(x^2+x+3)+2f(x^2-3x+5)=6x^2-10x+17$$

holds for all real x. Find $f(85)$.

(Link to AoPS)

Solution 364 (by Rust).

$f(x)=2x-3$.

Solution 365 (by TTsphn).

> $f(x)=2x-3$.

Are you sure?Note that $f(x)$ is not a polynomial. Let $g(x)=f(x)-2x+\frac{9}{2}$ Then $g(x^2+x+3)+2g(x^2-3x+5)=0$ From this equation we can file some value of $g(x)$ but can not find $g(x)$ on R because $h(x)=x^2+x+3$ is not inject in R.

Solution 366 (by Rust).

Let $t(x)=f(x^2-3x+5)-2(x^2-3x+5)+3$, then $f(x^2+x+3)-2(x^2+x+3)+3=t(x+2)$ and $t(x+2)+2t(x)=0, t(3-x)=t(x)$. Therefore $f(n+0.5)\equiv 0\ \forall n\in Z$. For prove $t(x)\equiv 0$ we need continiosly f.

Solution 367 (by pco).

> Let $t(x) = f(x^2-3x+5) - 2(x^2-3x+5)+3$, then $f(x^2+x+3) - 2(x^2+x+3)+3 = t(x+2)$ and $t(x+2)+2t(x) = 0, t(3-x) = t(x)$. Therefore $f(n+0.5) \equiv 0$ $\forall n \in Z$. For prove $t(x) \equiv 0$ we need continiosly f.

The job is nearly finished :

You have $t(3-x) = t(x)$ and so $t(x+2) = t(1-x)$ So $t(x+2)+2t(x) = 0$ becomes $t(1-x) = -2t(x)$ and so $t(x) = t(1-(1-x)) = -2t(1-x) = 4t(x)$ and so $t(x) = 0$ $\forall x$

And so $f(x^2-3x+5) = 2(x^2-3x+5) - 3$ And so $f(85) = 167$

Solution 368 (by TTsphn).

$f(x)$ is not equalities $2x-3$ for all x. Rust have a mistake with his solution. x^2-3x+5 not take all value on R. We can file out an solution of this equation as follow : $f(x) = c, \forall x \in (-\infty, 0)$ and $f(x) = 0$ on other interval.

Solution 369 (by pco).

> $f(x)$ is not equalities $2x-3$ for all x. Rust have a mistake with his solution. x^2-3x+5 not take all value on R. We can file out an solution of this equation as follow : $f(x) = c, \forall x \in (-\infty, 0)$ and $f(x) = 0$ on other interval.

Rust gave a second more precise (and, according to me, correct) demo :

Let $t(x) = f(x^2-3x+5) - 2(x^2-3x+5)+3$ Then $t(x+2) = f(x^2+x+3) - 2(x^2+x+3)+3$

So we have $f(x^2-3x+5) = t(x) + 2(x^2-3x+5) - 3$ and $f(x^2+x+3) = t(x+2) + 2(x^2+x+3) - 3$

And so $6x^2-10x+17 = f(x^2+x+3)+2f(x^2-3x+5) = t(x+2)+2(x^2+x+3) - 3 + 2(t(x) + 2(x^2-3x+5) - 3) = t(x+2) + 2t(x) + 6x^2 - 10x + 17$

And so $t(x+2) = -2t(x)$ We also have $t(3-x) = f((3-x)^2 - 3(3-x) + 5) - 2((3-x)^2 - 3(3-x)+5)+3 = f(x^2-3x+5) - 2(x^2-3x+5)+3 = t(x)$

So $t(3-x) = t(x)$ and so $t(x+2) = t(1-x)$ So $t(x+2) = -2t(x)$ becomes $t(1-x) = -2t(x)$ and so $t(x) = t(1-(1-x)) = -2t(1-x) = 4t(x)$ and so $t(x) = 0$ $\forall x$

And so $f(x^2-3x+5) = 2(x^2-3x+5)-3$ And so $f(x) = 2x-3$ $\forall x \in [\frac{11}{4}, +\infty)$

And so $f(85) = 167$

Solution 370 (by FOURRIER).

Why don't we do it like this :

Lets look for a simple function f wich satisfies the conditions,

Let $f(x) = ax + b$

We can find easily $f(x) = 2x - 3$

and we deduce

Is this method true?

I think this is Rust's Method

Solution 371 (by tchebytchev).

you can find find $f(85)$ using this method , but in general cases it isn't true , for example . let$f$be function defined from R to R such that $(f(x))^2 = f(2x)$. find $f(0)$. here there is two value 0 and 1 if $f(x) = 0$ for every x then $f(0) = 0$ but if $f(x) = e^x$ then $f(0) = 1$.

Solution 372 (by pco).

> Why don't we do it like this :
> Lets look for a simple function f wich satisfies the conditions,
> Let $f(x) = ax + b$
> We can find easily $f(x) = 2x - 3$
> and we deduce
> Is this method true?
> I think this is Rust's Method

In fact, just by saying $f(x) = 2x - 3$ is a solution, you can find **<u>a</u>**value for $f(85)$. The problem is that maybe different solutions exist leading to different values of $f(85)$

So : Either you consider that the problem says in an implicit way that there is a unique solution for $f(85)$ and then the solution is trivial and you are right. Either you consider that the problem does not say such a thing and you must show that there is a unique value 167

And RUST used the first methoid in its first post and nearly the second in its second post.

Solution 373 (by FOURRIER).

> Let $t(x) = f(x^2 - 3x + 5) - 2(x^2 - 3x + 5) + 3$

> Let $t(x) = f(x^2 - 3x + 5) - 2(x^2 - 3x + 5) + 3$

How do you think of this?

Thanks :)

Solution 374 (by Rust).

Let $g(x) = x^2 - 3x + 5 = g(3 - x)$, then $x^2 + x + 3 = g(x + 2)$, therefore $t(x) = f(g(x)) - 2g(x) + 3$ satisfyed $t(x + 2) + 2t(x) = 0, t(3 - x) = t(x)$.

Solution 375 (by pco).

What do you mean ?

The phrase before, in my post (and that you carefully omitted) was : "Rust gave a second more precise (and, according to me, correct) demo : "

I was just explaining completely Rust's demo (and my own end of Rust's demo) So I used Rusts's phrases.

Is there any problem with this ?

And have you read all the posts ?

Solution 376 (by FOURRIER).

> What do you mean ?

I mean where the Idea of considering $t(x) = f(x^2-3x+5)-2(x^2-3x+5)+3$ came from , and Rust answered me but if you have somthn to add thank you

****

Problem 91 (Posted by stergiu). Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the property that

$$f(f(x) - f(y)) = f(f(x)) - y$$

holds true for all reals x and y. Prove that f is an odd function.

(Link to AoPS)

Solution 377 (by TTsphn).
Easy to check that $f(x)$ is inject. Suppose exist a, b satisfy $f(a) = f(b)$ Let $y = a, y = b$ then $f(f(x) - f(a)) = f(x) - a$ $f(f(x) - f(b)) = f(x) - b$ s but $f(a) = f(b)$ then $a = b$ Let $y = 0$ then $f(f(x) - f(0)) = f(f(x))$ so $f(0) = 0$ Let $x = y$ then $f(f(x)) = x$ Let $x = 0$ then $f(-f(y)) = -y$ Let $y = f(y)$ then $f(-f(f(y)) = -f(y)$ But from $f(f(y)) = y$ we have $f(-y) = -f(y)$ It mean that $f(x)$ is a odd function.

Solution 378 (by pco).

> If function f has the propeprty
> $f(f(x) - f(y)) = f(f(x)) - y$
> for every reals x, y , prove that f is an odd function.

Let $P(x, y)$ be the property $f(f(x) - f(y)) = f(f(x)) - y$

$P(x, x)$ implies $f(f(x)) = x + f(0)$ and then $f(x)$ is bijective. $P(0, 0)$ implies $f(f(0)) = f(0)$ and, since $f(x)$ is bijective, $f(0) = 0$

So we have $f(f(x)) = x$ and $P(0, f(x))$ gives $f(-x) = -f(x)$

Q.E.D.

Solution 379 (by silouan).
Could we find all the functions f with propeprty
$f(f(x) - f(y)) = f(f(x)) - y$:) :?:

Solution 380 (by TTsphn).
Let $x \to f(x), y \to f(y)$ then $f(x-y) = f(x)-f(y)$ so $f(x+y) = f(x)+f(y)$ and more : $f(f(x)) = x$ But with conditon we can find out it.

Solution 381 (by pco).

> Could we find all the functions f with propeprty
> $f(f(x)-f(y))=f(f(x))-y$:) :?:

We have $f(f(x))=x$ and $f(-x)=-f(x)$ so $f(f(x)+f(y))=x+y$ and so $f(x+y)=f(x)+f(y)$

So the solutions are :

Let $\mathbb{A}$ and $\mathbb{B}$ two supplementary $\mathbb{Q}$-vectorspaces of $\mathbb{R}$ ($\mathbb{A}\cap\mathbb{B}=\{0\}$ and $\mathbb{A}+\mathbb{B}=\mathbb{R}$

Let $a(x)$ and $b(x)$ the projections of x on $\mathbb{A}$ and $\mathbb{B}$ (We have in a unique manner $x=a(x)+b(x)$)

Then $f(x)=a(x)-b(x)$

Note: Without Axiom of Choice, the two only possibilities for $\mathbb{A}$ and $\mathbb{B}$ are ($\mathbb{A}=\mathbb{R}$ and$\mathbb{A}=\{0\}$) or ($\mathbb{A}=\{0\}$ and $\mathbb{B}=\mathbb{R}$)

which give the two solutions $f(x)=x$ and $f(x)=-x$

With Axiom of Choice, we have infinitly many others.

****

Problem 92 (Posted by N.T.TUAN)**.** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(2xy+y)=2f(xy)+f(y)\ \ \forall x,y\in\mathbb{R}$.

(Link to AoPS)

Solution 382 (by TTsphn)**.**

It follow that $f(x+y)=f(x)+f(y)$ it is the Cauchy's function.

Solution 383 (by pco)**.**

> Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(2xy+y)=2f(xy)+f(y)\ \ \forall x,y\in\mathbb{R}$.

Let $P(x,y)$ be the property $f(2xy+y)=2f(xy)+f(y)$

$P(0,0)$ implies $f(0)=0$

$P(-1,x)$ implies $f(-x)=-f(x)$

$P(-\frac{1}{2},x)$ implies $2f(\frac{x}{2})=f(x)$

$P(\frac{u}{2v},v)$ implies $f(u+v)=2f(\frac{u}{2})+f(v)=f(u)+f(v)\ \forall v\neq 0$. But $f(u+0)=f(u)+f(0)$

So $P(x,y)\ \ \forall x,y$ implies $f(x+y)=f(x)+f(y)\ \ \forall x,y$ But obviously $f(x+y)=f(x)+f(y)\ \ \forall x,y$ implies $P(x,y)\ \ \forall x,y$

So the set of solution of $P(x,y)$ is the set of solutions of Cauchy's equation $f(x+y)=f(x)+f(y)$: Continuous solutions : $f(x)=ax$ + Infinitely many non continuous solutions with AC.

****

Problem 93 (Posted by duytungct)**.** Given a prime p, find all functions $f : \mathbb{N} \to \mathbb{Z}$ such that $f(n+p) = f(n)$ and $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

(Link to AoPS)

Solution 384 (by TTsphn)**.**

Because value of $f(n) \in \{f(1), f(2), ..., f(p+1)\}$ Call t is the value of $f(n)$ such that : $|f(t)| = \max\{|f(1)|, .., |f(p+1)|$ Let $m = t$ then $f(nt) = f(n)f(t)$ so $|f(nt)| = |f(n) \| f(t)|$ But from $|f(t)|$ is maximum of $|f(n)|$ so $f(n) \in \{0, 1\}$ Case 1Exist $n_0 \in N$ such that $f(n_0) = 0$ then $f(mn_0) = 0, \forall m \in N$ But $\{mn_0\}_{m=1}^{p}$ is a complete reside mod p . So $f(n) = 0, \forall n \in N$. Case 2$f(n) \equiv 1$ So has two function satisfy condition $f(n) = 1, f(n) = 0$

Solution 385 (by pco)**.**

> So has two function satisfy condition $f(n) = 1, f(n) = 0$

I've not checked your demo, but your conclusion seems not completely right.

Here is at least a third solution : If $n = 0 \pmod{p}$, $f(n) = 0$ Else $f(n) = 1$

For $p = 3$, here is a fourth solution : If $n = 0 \pmod{3}$, $f(n) = 0$ If $n = 1 \pmod{3}$, $f(n) = 1$ If $n = 2 \pmod{3}$, $f(n) = -1$

Solution 386 (by TTsphn)**.**

Oh sorry, i have a mistake : Must be $|f(n)| = 1$ and contine as above.

Solution 387 (by pco)**.**

> Oh sorry, i have a mistake : Must be $|f(n)| = 1$ and contine as above.

Continue up to what conclusion ?

The fact that $f(n) \in \{-1, 0, 1\}$ is immediate : $f(n+p) = f(n)$ implies $f(n)$ takes at most p values $f(n^k) = (f(n))^k$ implies then $f(n) \in \{-1, 0, 1\}$, else $f(n)$ would take infinitely many values.

The question is to find the general solution :

For $p = 2$, we have exactly three solutions : $(0, 0)$, $(0, 1)$ and $(1, 1)$

For $p = 3$, we have exactly four solutions : $(0, 0, 0)$, $(0, 1, -1)$, $(0, 1, 1)$ and $(1, 1, 1)$

For $p = 5$, we have exactly four solutions : $(0, 0, 0, 0, 0)$, $(0, 1, -1, -1, 1)$, $(0, 1, 1, 1, 1)$ and $(1, 1, 1, 1, 1)$

Solution 388 (by pco)**.**

> Given $p \in P$. Find $f : N \to Z$ satisfy $f(n+p) = f(n)$ and $f(mn) = f(m)f(n)\ \forall\ m, n \in N$

I think that we generally have four solutions :

1) $f(n) = 0$ 2) $f(n) = 1$ 3) $f(n) =$Legendre Symbol(n, p) 4) $f(n) =$ (Legendre Symbol$(n, p))^2 = 0$ for any $n = 0 \pmod{p}$ and 1 elsewhere.

These four solutions obviously respect the problem. It remains to prove there are no other solution.

Solution 389 (by duytungct)**.**
What does Legendre Symbol(n,p) mean?

Solution 390 (by TTsphn)**.**
The Lengendre symbol is the symbol of quadric reside mod p .

Solution 391 (by TTsphn)**.**
Good idea pco. Let $n = kp$ then $f(p(k+1)) = f(pk) \Longleftrightarrow f(p)(f(k+1) - f(k)) = 0$ Case 1$f(p)$ is different from 0 . So $f(k+1) = f(k), \forall k \in N$ It mean that $f(k) = 1$ Case 2$f(p) = 0$ If $f(1) = 0$ then $f(n) = 0, \forall n \in N$ If $f(1) = -1$ then let $n = 1$ then we have $f(n) = 0$,tradition. So $f(1) = 1$ we will prove that : $f(x^2) = 1, \forall \gcd(x,p) = 1$ from $f(x^2) = (f(x))^2 \in \{1,0\}$ If $f(x^2) = 0$ for some $x \in N$ then $f(x) = 0$ So $f(mx) = 0$ But from $\{mx\}$ is a complete reside mod p and $f(1) = 1$ it is tradition. It mean that $f(x^2) = 1, \forall x \in N$ so $f(a) = 0$ when $(\frac{a}{p}) = 1$ We have two case. **1**$f(x) \equiv 1$ **2**Exist $n_0 \in N$ such that $f(n_0) = -1$ so n_0 is non quadric reside mod p Call $a_1, , .., a_{\frac{p-1}{2}}$ is the quadric reside mod p. Then $f(a_i.n_0) = -1$ But $\{a_i n_0\}$ take all nonquadric reside mod p . So $f(b) = -1$ when $(\frac{b}{p}) = -1$ Solution complete.

Solution 392 (by duytungct)**.**
Here is my way, thought it is also similar with TTsphn's Let $m = n = 1$ then $f(1) = (f(1))^2$, so $1)f(1) = 0$ then $f(n) = 0 \forall n$ $2)f(1) = 1$. Let $m = n = 0$ then $+)f(0) = 1 \to f(n) = 1 \forall n$ $+)f(0) = 0 \to f(n) = 0 \ \forall \ p|n$ this case is solved like TTsphn's

Solution 393 (by TTsphn)**.**
I think so ,but careful .This probblem take from a contest .(I don't remember year) but N is not contain 0. Although your solution is still true as I find $f(p)$

****

Problem 94 (Posted by toanIneq)**.** Find all the functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(f(f(n))) + 6f(n) = 3f(f(n)) + 4n + 2007$$

for all $n \in \mathbb{N}$.

(Link to AoPS)

Solution 394 (by pco)**.**

> find all the functions $f : N \to N$ satisfying $f(f(f(n)) + 6f(n) = 3f(f(n)) + 4n + 2007$

Parenthesis missing.
Is it : $f(f(f(n))) + 6f(n) = 3f(f(n)) + 4n + 2007$?
Or : $f(f(f(n)) + 6f(n)) = 3f(f(n)) + 4n + 2007$?

Solution 395 (by toanIneq)**.**
oh,I don't think so.my problem is true and try $f(n) = n + 669$

Solution 396 (by pco)**.**

> oh,I don't think so.my problem is true and try $f(n) = n + 669$

You don't think so what ?
In $f(f(f(n)) + 6f(n)$, you have four left parenthesis and only three right parenthesis. So, parenthesis missing.

Solution 397 (by toanIneq)**.**
yes, i see and I'm sorry that for the parenthesis missing in my problem. I edited it $f(f(f(n))) + 6f(n) = 3f(f(n)) + 4n + 2007$

Solution 398 (by pco)**.**

> yes, i see and I'm sorry that for the parenthesis missing in my problem. I edited it $f(f(f(n))) + 6f(n) = 3f(f(n)) + 4n + 2007$

Let $f(n) = g(n) + 669$ We have $g(g(g(n))) + 6g(n) = 3g(g(n)) + 4n$
Let the the sequence $a_k(n)$ defined as : $a_0(n) = n$ $a_{k+1}(n) = g(a_k(n))$
We have $a_{k+3}(n) = 3a_{k+2}(n) - 6a_{k+1}(n) + 4a_k(n)$
And so $a_k(n) = (\frac{2n}{3} + \frac{g(g(n))}{3} - 2\frac{g(n)-n}{3\sqrt{3}}) + (\frac{n}{3} - \frac{g(g(n))}{3} + 2\frac{g(n)-n}{3\sqrt{3}})2^k\cos(k\frac{\pi}{3}) + \frac{g(n)-n}{\sqrt{3}}2^k\sin(k\frac{\pi}{3})$ which may be written :
$a_k(n) = u(n) + 2^k(v(n)\cos(k\frac{\pi}{3}) + w(n)\sin(k\frac{\pi}{3}))$ If $v(n)$ and $w(n)$ are not zero, $|a_k(n)|$ grows up to $+\infty$ and sign changes between $a_k(n)$ and $a_{k+3}(n)$
So, since $a_k(n)$ must $\in \mathbb{N}$, we must have $v(n) = w(n) = 0$
And so $g(n) = n$
And so $f(n) = n + 669$, unique solution.

****

Problem 95 (Posted by phuong)**.** Finf all funtions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) + f(x)f(y) = f(xy) + f(x) + f(y), \quad \forall x, y \in \mathbb{R}.$$

(Link to AoPS)

Solution 399 (by shyamkumar)**.**
By observation, f(x) = 0 is one such function. Now,plug in x=0,y=0this yields f(0)= 0 or 2. Then plug in y=0 to give f(x)=2 (if f(0) = 2). Hence another possible function is f(x)=2 (constant function)..
Another possibility is f(x+y) = f(x) + f(y) and f(xy)=f(x)*f(y) This gives (on plugging x=y=0 and later y=0,x=1,2,3..) f(x) =x
This is all Ive been able to dothough I feel (intuitively) that there are no more functions, I havent been able to prove/disprove that.:oops: Shyam

Solution 400 (by pco).

> Finf all funtions $f : \mathbb{R} \to \mathbb{R}$ such that for arbitrary real numbers x and y: $f(x+y) + f(x)f(y) = f(xy) + f(x) + f(y), \forall x, y \in \mathbb{R}$

1) If $f(x) = c$ constant, we have $c + c^2 = 3c$ and we find immediatly two solutions $f(x) = 0$ and $f(x) = 2$

2) If $f(x)$ is not constant : Let $P(x, y)$ the property $f(x+y) + f(x)f(y) = f(xy) + f(x) + f(y)$ $P(x, 0)$ gives $f(x) + f(x)f(0) = f(0) + f(x) + f(0)$ and so $f(0) = 0$ (since $f(x)$ is not the constant 2). $P(x, 1)$ gives $f(x+1) = (2 - f(1))f(x) + f(1)$ and so $P(x+1, 1)$ gives $f(x+2) = (2 - f(1))f(x+1) + f(1)$ $= (2 - f(1))^2 f(x) + f(1)(3 - f(1))$ Since $f(0) = 0$, we can also write this last equation $f(x+2) = (2 - f(1))^2 f(x) + f(2)$, with $f(2) = f(1)(3 - f(1))$ But $P(x, 2)$ gives $f(x+2) = f(2x) + f(x)(1 - f(2)) + f(2)$

And so $(2 - f(1))^2 f(x) + f(2) = f(2x) + f(x)(1 - f(2)) + f(2)$ implies $(2 - f(1))^2 f(x) = f(2x) + f(x)(1 - 3f(1) + f(1)^2)$ implies $f(2x) = (3 - f(1))f(x)$

So we have $f(2x) = af(x)$ and $f(4x) = a^2 f(x)$ with $a = 3 - f(1)$.

Then $P(2x, 2y)$ gives $af(x+y) + a^2 f(x)f(y) = a^2 f(xy) + af(x) + af(y)$. We also have $P(x, y)$: $f(x+y) + f(x)f(y) = f(xy) + f(x) + f(y)$ which implies $a^2 f(x+y) + a^2 f(x)f(y) = a^2 f(xy) + a^2 f(x) + a^2 f(y)$

Subtracting these two equations, we have : $a(a-1)f(x+y) = a(a-1)(f(x) + f(y))$

But, we have : $a \neq 0$, else $f(2x) = af(x)$ would imply $f(x) = 0$, constant. $a \neq 1$, else $f(1) = 3 - a = 2$ and, since $f(x+1) = (2 - f(1))f(x) + f(1)$, $f(x+1) = 2$ and $f(x) = 2$, constant.

So we have $f(x+y) = f(x) + f(y)$ and, as a consequence $f(xy) = f(x)f(y)$

And this well known system has a unique solution $f(x) = x$

So we just have three solutions, as Shyamkumar thought : $f(x) = 0$ $f(x) = 2$ $f(x) = x$

Solution 401 (by greentreeroad).

> So we have $f(x+y) = f(x) + f(y)$ and, as a consequence $f(xy) = f(x)f(y)$
> And this well known system has a unique solution $f(x) = x$

how do you show this, thanks :roll:

Solution 402 (by nguyenvuthanhha).

Pco's solution is correct and I have just found an other one

Solution 403 (by pco).

$f(x+y = f(x) + f(y)$ is Cauchy's equation and gives $f(x) = f(1)x$ $\forall x \in \mathbb{Q}$ $f(xy) = f(x)f(y)$ gives $f(x^2) = f(x)^2$ and so $f(x) \geq 0$ $\forall x \geq 0$

So $f(x+y) \geq f(x)$ $\forall x, \forall y \geq 0$ and so $f(x)$ is monotonous (non decreasing).

And $f(x) = f(1)x$ $\forall x \in \mathbb{Q}$ PLUS $f(x)$ monotonous implies $f(x) = f(1)x$ $\forall x \in \mathbb{R}$

Now $f(xy)=f(x)f(y) \implies f(1)^2=f(1)$ and so two solutions : $f(x)=x$ $f(x)=0$

And, since in paragraph 2 we supposed that $f(x)$ was not a constant function : $f(x)=x$

****

Problem 96 (Posted by massnet)**.** Find all pairs of functions $f,g:\mathbb{R}\to\mathbb{R}$ such that

(a) $f(xg(y+1))+y=xf(y)+f(x+g(y))$ for any $x,y\in\mathbb{R}$, and

(b) $f(0)+g(0)=0$.

(Link to AoPS)

Solution 404 (by pco)**.**

> Find all pairs of functions $f,g:R\to R$ such that
> (a) $f(xg(y+1))+y=xf(y)+f(x+g(y))$ for any $x,y\in R$
> (b) $f(0)+g(0)=0$.

1) $x=0$ implies $f(g(y))=y+f(0)$. And so $f(x)$ is surjective and $g(x)$ is injective.

2) If $g(1)\neq 1$, then, using $x=\frac{g(0)}{g(1)-1}$ and $y=0$ in a), we get: $f\left(\frac{g(0)g(1)}{g(1)-1}\right)=\frac{f(0)g(0)}{g(1)-1}+f\left(\frac{g(0)g(1)}{g(1)-1}\right)$ and so $f(0)g(0)=0$ and so $f(0)=g(0)=0$

Using then $y=-1$ in a), we have $f(x+g(-1))=-f(-1)x-1$ $\forall x$ and so $f(x)=ax+b$ for some a and b

We know that $a\neq 0$, else $f(x)$ would not be surjective. And so, since $f(x)=ax+b$ and $f(g(x))=x$ (see point 1 above) : $g(x)=\frac{x-b}{a}$. Putting these two values in equation a), we find $f(x)=g(x)=x$, which is a contradiction in this paragraph since we supposed $g(1)\neq 1$ So $\boxed{g(1)=1}$

3) Since $f(x)$ is surjective, exists u such that $f(u)=0$ If $g(u+1)\neq 1$, Putting $x=\frac{g(u)}{g(u+1)-1}$ and $y=u$ in a), we get :

$f\left(\frac{g(u)g(u+1)}{g(u+1)-1}\right)+u=f\left(\frac{g(u)g(u+1)}{g(u+1)-1}\right)$ and so $u=0$, which is a contradiction since it implies $1\neq g(u+1)=g(1)=1$

So $g(u+1)=1$, then $g(u+1)=g(1)$ and so $u=0$ since $g(x)$ is injective (see point 1 above).

So $f(0)=0$ and then $g(0)=0$ and the same demo as above (in point 2) works :

Using then $y=-1$ in a), we have $f(x+g(-1))=-f(-1)x-1$ $\forall x$ and so $f(x)=ax+b$ for some a and b $a\neq 0$, else $f(x)$ would not be surjective. And so, since $f(x)=ax+b$ and $f(g(x))=x$ (see 1) above) : $g(x)=\frac{x-b}{a}$. Putting these two values in a), we find $\boxed{f(x)=g(x)=x}$, unique solution.

****

Problem 97 (Posted by Jure the frEEEk)**.** Let $f:\mathbb{R}\to\mathbb{R}$. prove that if $f(xy+y+x)=f(xy)+f(x)+f(y)$, then $f(x+y)=f(x)+f(y)$.

(Link to AoPS)

Solution 405 (by pco)**.**

> Let $f:\mathbb{R}\to\mathbb{R}$. prove that if $f(xy+y+x)=f(xy)+f(x)+f(y)$ than $f(x+y)=f(x)+f(y)$

A not-so-simple solution :

Let $P(x,y)$ be the property $f(xy+y+x)=f(xy)+f(x)+f(y)$.

$P(0,0)$ implies $\boxed{f(0)=0}$

$P(x,-x)$ implies $\boxed{f(-x)=-f(x)}$ $\forall x$

Let $x\neq -1$. Then : $P(x,\frac{x}{x+1})$ gives $f(2x)=f\left(\frac{x^2}{x+1}\right)+f(x)+f\left(\frac{x}{x+1}\right)$ $P(x,-\frac{x}{x+1})$ gives $f(0)=-f\left(\frac{x^2}{x+1}\right)+f(x)-f\left(\frac{x}{x+1}\right)$ Adding these two equalities, we have $f(2x)=2f(x)$ $\forall x\neq -1$. So, for $x=1$, we have $f(2)=2f(1)$ and so $-f(2)=-2f(1)$ and so $f(-2)=2f(-1)$ and so $\boxed{f(2x)=2f(x)}$ $\forall x$

$P(-x,-y)$ gives then $f(xy-y-x)=f(xy)-f(x)-f(y)$. Adding with $P(x,y)$, we get $f(xy+x+y)+f(xy-x-y)=2f(xy)$ and, since $2f(xy)=f(2xy)$: $\boxed{f(xy+x+y)=f(2xy)+f(x+y-xy)}$ $\forall x,y$

Let then $u\leq 0$ and $v\in\mathbb{R}$. It is always possible to find x and y such that $xy=\frac{u}{2}$ and $x+y=v+\frac{u}{2}$ Hence, since $f(xy+x+y)=f(2xy)+f(x+y-xy)$, we have $f(u+v)=f(u)+f(v)$. Now, if $u>0$, we have $-u<0$ and also $f(-u-v)=f(-u)+f(-v)$ and so $f(u+v)=f(u)+f(v)$

And so $\boxed{f(u+v)=f(u)+f(v)}$ $\forall u,v$

Solution 406 (by Jure the frEEEk)**.**

thanks very much. i had problems with proving it for irationals

Solution 407 (by behemont)**.**

I've found a nice solution..

Plugging $y=1$ to the equation we get $f(2x+1)=2f(x)+c$, where $c=f(1)$. Call this property $(*)$.

Now substituting y with $2y+1$ in the initial equation we get $f(2xy+2x+2y+1)=f(2xy+x)+f(x)+f(2y+1)$, and now using $(*)$ after some simplification we get $2f(xy)+f(x)=f(2xy+x)$, which obviously means $f(2a+b)=2f(a)+f(b),\forall a,b$. Call this property $(**)$.

Now substituting x with $2x+1$ and y with $2y+1$ in the initial equation, we get $f(4xy+4x+4y+3)=f(4xy+2x+2y+1)+f(2x+1)+f(2y+1)$. Now using $(*)$ and $(**)$ this equality leads to $f(x+y)=f(x)+f(y)$, and that is what we wanted...

****

Problem 98 (Posted by perfect˙radio)**.** What's the minimal number n of discontinuities that a real function f which satisfies $f(f(x)) = -x$ can have?

(Link to AoPS)

Solution 408 (by PTynan89)**.**

Over all of the reals, or just some interval?

And also, it's pretty obvious that it has to be a bijection.

One more thing, the function f has order 4, meaning that $f^4(x) = e(x) = x$, where e is the identity function.

That's all I have so far, so it's probably not very helpful.

Solution 409 (by jmerry)**.**

This only really makes sense as a question about all of $\mathbb{R}$.

It's not just that f has order 4; every orbit except $\{0\}$ has exactly four elements.

Infinitely many discontinuities are required. Proof: First, we note that $f(0) = 0$. Suppose f has only n discontinuities $x_1, \ldots, x_n$ (including zero), and let $S = \{x : x = f^i(x_j) \text{ for some } i, j\} \cup \{0\}$. S is still finite, and contains $4k+1$ elements for some integer $k \leq n$. Also, $f(S) = S$ and $f^{-1}(S) = S$. $\mathbb{R} \setminus S$ is the union of $4k+2$ open intervals, and f is continuous on each of these intervals. Since f maps $\mathbb{R} \setminus S$ to itself bijectively, these intervals must be mapped to each other by f. Let A be the set of these intervals; we define f on A in the natural way. Since each element of A is either entirely positive or entirely negative, $f^2(U) \neq U$ for each $U \in A$. On the other hand, f^4 is the identity on U, so each orbit in U has exactly four elements. The number of elements in U is not divisible by 4, and we have a contradiction.

****