CS323, Quiz #10 – Tamer Avcı

Prove that a matching *M* is maximum iff there is no augmenting path.

Since this is an if-and-only-if statement we have to prove the following two statements:

- If a matching *M* is maximum then there is no augmenting path.
- If there is no augmenting path then matching *M* is maximum.

We can prove the second statement by proving its contrapositive.

Let us assume that there is an augmenting path in *G(V,E).* This augmenting path *P* contains *v0, v1, v2, …, v2m+1*. Note that the augmenting path must be of odd length. This is because we need to have alternating edges in *M* and $E \setminus M$. Therefore we can generally define *P* as *E(P)* given that *E(P)* is an M-alternating path meaning its edges are alternatively in $E \setminus M$ and M. Then a new maximum can be defined as

$$M' = (M \setminus (E(P) \cap M) \cup (E(P) \setminus M)$$

Which is a matching because its size $|M'| = |M| - 1 + x$ where *x* denotes free vertices, which is a necessary condition assuming there is an augmenting path. Because $|M'| > |M|$, *M* cannot be a maximum.

Similarly assume that *M* is not a maximum matching, then we have to show that there is an augmenting path. Let *M'* represent the maximum matching in *G(E,V).* Then $|M'| > |M|$.

Let $H = (M' \setminus M) \cup (M \setminus M')$ Since |M'| > |M|, H contains more edges of *M'* than of *M*, and *H* must contain a component which is a path, let's call it P, that starts and ends with edges in *M'*. Since the start vertex and end vertex of *P* are incident or saturated in *M'* in *H* they must be non-incident or *M*-unsaturated in *G*. Thus, *P* is an *M*-augmenting path in *G*. Hence we proved both ways.