Prove the maximum flow is equal to the minimum cut capacity.

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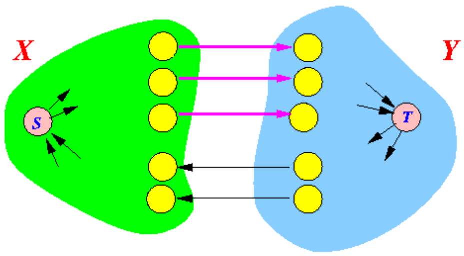
Let G be a directed graph with S and T being the source and target of G respectively. We have to show that:

There exists a cut C such that:

Capacity of cut C >= maximum flow

Assume that we have found the maximum flow that is computed by Ford-Fulkerson algorithm. We can divide/cut the graph into two subsets of vertices as depicted in the picture below. Note that this shows the residual graph (G_f).



The two subsets are:

1. X: the set of vertices reachable from s in G_f
2. Y: the set of remaining vertices

The above construction yields a cut. This means that the source S and the target T are in different sets. Since the construct yields a cut, we have:

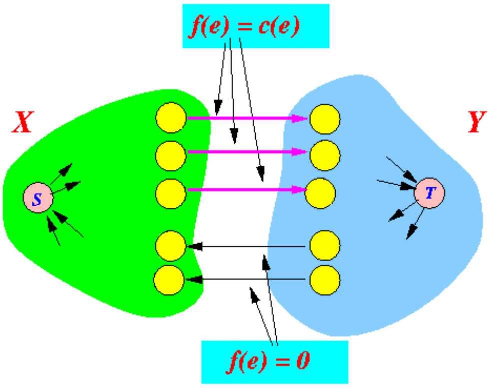
S ∈ X

T ∈ Y

Suppose that T ∈ X. From the construction of the cut C, X = all vertices that can be reached from source S by an augmenting path, but because T ∈ X, there exists an augmenting path from S → T which means that the current flow is not maximum. That **contradicts** the given fact that the current flow is maximum.

We know that the max flow is equal to flow of outgoing edges from X less incoming edges from Y in Gf. Therefore for max flow = Capacity of cut(X, Y) we need:

- All *outgoing edges* from the cut must be **fully saturated**.
- All *incoming edges* to the cut must have **zero flow**.



To prove the above claim we consider two cases:

1- f(e) = c(e) // outgoing edges are fully saturated

Assume there is an arbitrary forward edge A→B of the cut

A ∈ X, and

B ∈ Y (i.e., there is no augmenting path from S to B)

Because A ∈ X, there is an augmenting path from S → A: Suppose **f(e) < c(e),** then there would be an augmenting path from S → B (by increasing the flow through A → B) which would mean that B ∈ Y which contradicts the fact that: **B ∈ Y**.

Therefore, all outgoing edges are **fully saturated i.e f(e) = c(e)** and all edges in the cut are **parts of augmenting paths.**

To prove that all incoming edges have 0 flow **(f(e)=0)** we show the following.

Assume there is an arbitrary backward edge B→A of the cut

A ∈ X, and

B ∈ Y (i.e., there is no augmenting path from S to B)

Because A ∈ X, there is a flow augmenting path from S → A

Suppose f(e) > 0, then there would be an augmenting path from S → B (by increasing the flow through A → B) If we decrease some flow on A→B, which means B ∈ X. This **contradicts** the fact that: B ∈ Y. Therefore all incoming edges from Y to X carry **no flow.**

Both of the above statements complete the proof that the capacity of cut obtained above is equal to the flow obtained in the network. Also, since the flow is obtained by Ford-Fulkerson algorithm, it is the max.flow which means **the cut has to be min-cut** because **any flow in the network is always less than or equal** to the capacity of every cut in a network.