Maximizing the Range of A Projectile (non-zero initial height)

This problem can be done in a straightforward manner with calculus, though it's long. A much better method was found some years ago by P. Palffy-Muhoray and D. Balzarini (*Amer. Jour. Phys.* **50** (1982) 181). Here are both versions.

1. Calculus

Call the initial velocity $\boldsymbol{v_o} = (v_o, \angle\theta) = (v_o\cos\theta, v_o\sin\theta)$, the initial position $\boldsymbol{r_o} = (0, h)$ and the final position $\boldsymbol{r} = (R, 0)$. The coordinate equations are as usual

$$\begin{aligned} x &= v_o \cos\theta\, t \\ y &= h + v_o \sin\theta\, t - \tfrac{1}{2} g t^2 \end{aligned}$$

To find the distance travelled, set $y = 0$ and solve for the time with the quadratic formula, discarding the negative root:

$$t = \frac{v_o \sin\theta + \sqrt{v_o^2 \sin^2\theta + 2gh}}{g}$$

Plug this into the x equation to obtain

$$R = \frac{v_o^2 \cos\theta \sin\theta + v_o \cos\theta \sqrt{v_o^2 \sin^2\theta + 2gh}}{g}$$

This can be cleaned up a little and written as

$$R = \frac{v_o^2}{g} f(\theta)$$

where

$$f(\theta) = \cos\theta \sin\theta + \cos\theta \sqrt{\sin^2\theta + p}$$

and $p = 2gh/v_o^2$. The range R will be maximized by maximizing $f(\theta)$. Then

$$\frac{df}{d\theta} = \cos^2\theta - \sin^2\theta - \sin\theta\sqrt{\cdots} + \tfrac{1}{2}\cos\theta \cdot 2\sin\theta\cos\theta \cdot \frac{1}{\sqrt{\cdots}} = 0$$

or, cleaning up and rearranging,

$$\cos^2\theta - \sin^2\theta = \sin\theta\sqrt{\cdots} - \cos^2\theta \sin\theta \cdot \frac{1}{\sqrt{\cdots}}$$

Square both sides to clear up the radicals;

$$(\cos^2\theta - \sin^2\theta)^2 = \sin^2\theta\,(\sin^2\theta + p) - 2\sin^2\theta\cos^2\theta + \frac{\cos^4\theta \sin^2\theta}{\sin^2\theta + p}$$

Expand and cancel:

$$\cos^4\theta - \cancel{2\cos^2\theta \sin^2\theta} + \cancel{\sin^4\theta} = \cancel{\sin^4\theta} + p\sin^2\theta - \cancel{2\cos^2\theta\sin^2\theta} + \frac{\cos^4\theta \sin^2\theta}{\sin^2\theta + p}$$

which becomes

$$\cos^4\theta = p\sin^2\theta + \frac{\cos^4\theta \sin^2\theta}{\sin^2\theta + p}$$

or

$$\cancel{\cos^4\theta \sin^2\theta} + p\cos^4\theta = p\sin^4\theta + p^2 \sin^2\theta + \cancel{\cos^4\theta \sin^2\theta}$$

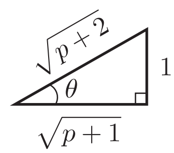
Divide by p and rewrite $\cos^2\theta$ as $(1 - \sin^2\theta)$:

$$(1 - \sin^2\theta)^2 = \sin^4\theta + p\sin^2\theta$$

or, finally,

$$(p + 2)\sin^2\theta = 1 \quad \Rightarrow \quad \sin\theta = \frac{1}{\sqrt{p+2}}$$

Find the tangent of this angle in the usual way; draw a diagram:



Then

$$\tan\theta = \frac{1}{\sqrt{p+1}} = \frac{1}{\sqrt{(2gh/v_o^2) + 1}} = \frac{v_o}{\sqrt{v_o^2 + 2gh}}$$

Observe that this reduces, as it must, to $\theta = 45°$ when $h = 0$; for then $\tan\theta = 1$.

2. Vector algebra (Palffy-Muhoray & Balzarini)

The kinematic equations of Galileo remain true even when the quantities are vectors, if the acceleration is constant. Letting $\boldsymbol{v}$ be the final velocity, and $\Delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r_o} = (R, -h)$, Galileo's kinematics may be written (t is the time of flight)

$$\boldsymbol{v} - \boldsymbol{v_o} = \boldsymbol{g}t$$

$$\boldsymbol{v_o} + \boldsymbol{v} = \frac{2\Delta\boldsymbol{r}}{t}$$

Take the dot product between these two equations and obtain

$$v^2 - v_o^2 = 2\boldsymbol{g} \cdot \Delta\boldsymbol{r} = 2gh$$

which is simply the conservation of energy. Take the cross product between them to obtain

$$2\boldsymbol{v} \times \boldsymbol{v_o} = 2\boldsymbol{g} \times \Delta\boldsymbol{r} = 2gR$$

or, letting α be the angle between the velocities,

$$vv_o \sin\alpha = gR$$

Because $v = \sqrt{v_o^2 + 2gh}$, and is not dependent on its direction, it is easy to see how to maximize R: the angle $\alpha = 90°$. That is, if the range is maximal, the initial and final velocities must be perpendicular to each other (as is true for a zero initial height; the initial velocity has a launch angle of 45° and an impact angle of 135°.) Return to the first equation, and take the dot product of it with $\boldsymbol{v_o}$, recalling that the initial and final velocities are perpendicular:

$$\boldsymbol{v_o} \cdot (\boldsymbol{v} - \boldsymbol{v_o}) = \boldsymbol{v_o} \cdot \boldsymbol{g}t \quad \Rightarrow \quad -v_o^2 = v_o gt\cos(90 + \theta)$$

or what comes to the same thing, $v_o = gt\sin\theta$. Take the cross product of $\boldsymbol{v_o}$ with the first equation, and find

$$\boldsymbol{v_o} \times (\boldsymbol{v} - \boldsymbol{v_o}) = \boldsymbol{v_o} \times \boldsymbol{g}t \quad \Rightarrow \quad v_o v = v_o gt\sin(90 + \theta)$$

so that $v = gt\cos\theta$. Taking the ratio of these two equations gives, finally,

$$\frac{v_o}{v} = \frac{gt\sin\theta}{gt\cos\theta} = \tan\theta = \frac{v_o}{\sqrt{v_o^2 + 2gh}}$$

exactly as before.

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