Linear Algebra II
Equivalent Quadratic Forms

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Equivalent Quadratic Forms

Given two quadratic forms $Q_1(\vec{x}^t)$ and $Q_2(\vec{x}^t)$ on F^n we say $Q_1(\vec{x}^t)$ and $Q_2(\vec{x}^t)$ are equivalent if there exists an invertible matrix P such that $Q_2(\vec{x}^t) = Q_1(\vec{x}^t P)$. Let A_1 and A_2 be matrix representation of Q_1 and Q_2 respectively. When A_1 is diagonal, we say Q_1 is a diagonal form. Then we have

$$A_2 = PA_1P^t.$$

When two quadratic forms are equivalent, they can be obtained from each other by changing of variables.

In this course, we always use $\vec{x}$ to denote the column vector. On the other hands, it is common to write variables of quadratic forms in row vectors.

Diagonalization of Quadratic Forms

Suppose $Q(\vec{x}^t) = \vec{x}^t A\vec{x}$ is real. Let $\alpha = \{\vec{v}_1, \cdots, \vec{v}_n\}$ be an orthonormal eigenbasis of A such that $A\vec{v}_i = \lambda_i \vec{v}_i$ for all i. Let $P = (\vec{v}_1 \ \cdots \ \vec{v}_n)$ and $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$. Recall that we have $A = PDP^t$. Let $\vec{y}^t = \vec{x}^t P$. Then

$$Q(\vec{x}^t) = \vec{y}^t D\vec{y}.$$

Theorem

Every real quadratic form is equivalent to a diagonal quadratic form.

Instead of using orthonormal basis α, let us consider an orthogonal eigenspace basis $\beta = \{r_1\vec{v}_1, \cdots, r_n\vec{v}_n\}$. Then the corresponding change-of-basis matrix is

$$\tilde{P} = (r_1\vec{v}_1 \ \cdots \ r_n\vec{v}_n) = (\vec{v}_1 \ \cdots \ \vec{v}_n)\begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix} = P\begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix}.$$

Then

$$\begin{aligned} A = PDP^t &= \tilde{P}\begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^{-1}\begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{pmatrix}^{-1}\tilde{P}^t \\ &= \tilde{P}\begin{pmatrix} \frac{\lambda_1}{r_1^2} & & \\ & \ddots & \\ & & \frac{\lambda_n}{r_n^2} \end{pmatrix}\tilde{P}^t. \end{aligned}$$

Now let $\vec{y}^t = \vec{x}^t\tilde{P}$, then $Q(\vec{x}) = \sum \frac{\lambda_i}{r_i^2}y_i^2$.

Diagonalization of Quadratic Forms

We summarize the above discussion as the following theorem.

Theorem

Suppose $Q(\vec{x}^t) = \vec{x}^t A\vec{x}$ *is real. Let* $\alpha = \{\vec{v}_1, \cdots, \vec{v}_n\}$ *be an orthogonal eigenbasis of* A *such that* $A\vec{v}_i = \lambda_i\vec{v}_i$ *for all* i*. Let* $P = (\vec{v}_1 \ \cdots \ \vec{v}_n)$ *and* $\vec{y}^t = \vec{x}^t P$*. Then*

$$Q(\vec{x}) = \sum_i \frac{\lambda_i}{\|\vec{v}_i\|^2} y_i^2.$$

Note that we can rescale $\vec{v}_i$ and assume that $\lambda_i/\|\vec{v}_i\|^2 = \pm 1$ for all i with $\lambda_i \neq 0$. In other words, every real quadratic form is equivalent to the following standard form

$$y_1^2 + \cdots + y_{n_1}^2 - y_{n_1+1}^2 - \cdots - y_{n_1+n_{-1}}^2.$$

Here n_1 is the number of positive eigenvalues and n_{-1} is the number of negative eigenvalues.

Definition

The number $n_1 - n_{-1}$ is called the signature of the real quadratic form Q. The number $n_1 + n_{-1}$ is called the rank of Q.

Theorem

Two quadratic forms on F^n are equivalent if and only if they have the same signature and the same rank.

Example 1

Let

$$Q_1(x_1, x_2, x_3) = (x_1\, x_2\, x_3) A_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \text{where } A_1 = \begin{pmatrix} -1 & 0 & 2 \\ 0 & -1 & 1 \\ 2 & 1 & 3 \end{pmatrix}.$$

Let $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ be the eigenvectors of A corresponding to the eigenvalues $4, -2, -1$. Let $P_1 = (\vec{v}_1\, \vec{v}_2\, \vec{v}_3)$. Then

$$A_1 = P_1 \begin{pmatrix} \frac{4}{\|\vec{v}_1\|^2} & & \\ & \frac{-2}{\|\vec{v}_2\|^2} & \\ & & \frac{-1}{\|\vec{v}_3\|^2} \end{pmatrix} P_1^t = P_1 \begin{pmatrix} \frac{2}{15} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{5} \end{pmatrix} P_1^t.$$

If we let $\tilde{P}_1 = (\sqrt{\frac{2}{15}}\vec{v}_1\ \frac{1}{\sqrt{3}}\vec{v}_2\ \frac{1}{\sqrt{5}}\vec{v}_3)$, then

$$A_1 = \tilde{P}_1 \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \tilde{P}_1^t.$$

We conclude that the signature of Q_1 is -1, the rank of Q_1 is 3, and the standard form of Q_1 is $y_1^2 - y_2^2 - y_3^2$ which can be obtained by letting $\vec{y}^t = \vec{x}^t \tilde{P}$.

Example 2

Let

$$Q_2(x_1, x_2, x_3) = (x_1\, x_2\, x_3) A_2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \text{where } A_2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 4 \\ 2 & 4 & -1 \end{pmatrix}.$$

Let $\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, and $\vec{u}_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ be the eigenvectors of A corresponding to the eigenvalues $5, -5, -1$. Let $P_2 = (\vec{v}_1\, \vec{v}_2\, \vec{v}_3)$. Then

$$A_2 = P_2 \begin{pmatrix} \frac{5}{\|\vec{u}_1\|^2} & & \\ & \frac{-5}{\|\vec{u}_2\|^2} & \\ & & \frac{-1}{\|\vec{u}_3\|^2} \end{pmatrix} P_2^t = P_2 \begin{pmatrix} \frac{5}{3} & & \\ & -\frac{5}{2} & \\ & & -\frac{1}{6} \end{pmatrix} P_2^t.$$

If we let $\tilde{P}_2 = (\sqrt{\frac{5}{3}}\vec{u}_1\, \sqrt{\frac{5}{2}}\vec{u}_2\, \frac{1}{\sqrt{6}}\vec{u}_3)$, then

$$A_2 = \tilde{P}_2 \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \tilde{P}_2^t.$$

We conclude that the signature of Q_2 is -1, the rank of Q_2 is 3, and the standard form of Q_2 is $y_1^2 - y_2^2 - y_3^2$.

For the previous two examples, two quadratic forms have the same signature and the same rank, so they are equivalent. In fact, we have

$$(\tilde{P}_1)^{-1}A_1(\tilde{P}_1^t)^{-1} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} = (\tilde{P}_2)^{-1}A_2(\tilde{P}_2^t)^{-1}.$$

Therefore,

$$A_2 = (\tilde{P}_2)(\tilde{P}_1)^{-1}A_1(\tilde{P}_1^t)^{-1}(\tilde{P}_2^t) = (\tilde{P}_2\tilde{P}_1^{-1})A_1(\tilde{P}_2\tilde{P}_1^{-1})^t$$

In other words,

$$Q_2(\vec{x}^t) = Q_1(\vec{x}^t\tilde{P}_2\tilde{P}_1^{-1}).$$