Linear Algebra II
Bilinear Forms

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Bilinear Form

Let V be a vector space over F.

Defintion

A bilinear form B on V is a map from $V \times V$ to F satisfying the following conditions. For all $\vec{x}, \vec{y}, \vec{z} \in V$ and $a \in F$,

1. $B(\vec{x} + a\vec{y}, \vec{z}) = B(\vec{x}, \vec{z}) + aB(\vec{y}, \vec{z})$.
2. $B(\vec{z}, \vec{x} + a\vec{y}) = B(\vec{z}, \vec{x}) + aB(\vec{z}, \vec{y})$.

Furthermore, if $B(\vec{x}, \vec{y}) = B(\vec{y}, \vec{x})$ for all $\vec{x}, \vec{y} \in V$, then B is called a symmetric bilinear form.

Examples

Example

1. An inner product on a real vector space is a symmetric bilinear form.
2. An inner product on a complex vector space is not a bilinear form.
3. Let $V = M_n(F)$. For $X, Y \in V$,

$$B(X, Y) := \mathrm{tr}(XY^t)$$

define a bilinear form on V.

Matrix Representation of Bilinear Forms

Suppose $\dim V = 2$. Let $\alpha = \{\vec{v}_1, \vec{v}_2\}$ be a basis of V.

For any $\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2$ and $\vec{y} = y_1\vec{v}_1 + y_2\vec{v}_2$ in V,

$$
\begin{aligned}
&B(\vec{x}, \vec{y}) \\
&= x_1 y_1 B(\vec{v}_1, \vec{v}_1) + x_1 y_2 B(\vec{v}_1, \vec{v}_2) + x_2 y_1 B(\vec{v}_2, \vec{v}_1) + x_2 y_2 B(\vec{v}_2, \vec{v}_2) \\
&= (x_1\ x_2) \begin{pmatrix} B(\vec{v}_1, \vec{v}_1) & B(\vec{v}_1, \vec{v}_2) \\ B(\vec{v}_2, \vec{v}_1) & B(\vec{v}_2, \vec{v}_2) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathrm{Rep}_\alpha(\vec{x})^t \mathrm{Rep}_\alpha(B) \mathrm{Rep}_\alpha(\vec{y}).
\end{aligned}
$$

Here $\mathrm{Rep}_\alpha(B)$ is called the matrix representation of B under the basis α.

Matrix Representation of Bilinear Forms

Suppose $\dim V = n$. Let $\alpha = \{\vec{v}_1, \cdots, \vec{v}_n\}$ be a basis of V. For any $\vec{x}, \vec{y} \in V$, we also have

$$B(\vec{x}, \vec{y}) = \mathrm{Rep}_\alpha(\vec{x})^t \mathrm{Rep}_\alpha(B) \mathrm{Rep}_\alpha(\vec{y}).$$

Here $\mathrm{Rep}_\alpha(B)$ is an $n \times n$ matrices whose (i, j)-th entry is equal to $B(\vec{v}_i, \vec{v}_j)$.

Theorem

Let α be any basis of V. A bilinear form B on V is symmetric if and only if $\mathrm{Rep}_\alpha(B)$ is a symmetric matrix.

Corollary

When B is symmetric, then $Q(\vec{x}) := B(\vec{x}, \vec{x})$ defines a quadratic form on V.

Coordinate-Free Definition of Quadratic Forms

Suppose $Q(\vec{x}) = B(\vec{x}, \vec{x})$, then one can also recover the the symmetric bilinear form B from Q via the following formula.

$$B(\vec{x}, \vec{y}) = \frac{1}{2}\left(Q(\vec{x}+\vec{y}) - Q(\vec{x}) - Q(\vec{y})\right).$$

Definition

A map Q from V to F is a quadratic form if $Q(\vec{x}+\vec{y}) - Q(\vec{x}) - Q(\vec{y})$ is a bilinear form.

Proposition

Given a vector space V, there is a one-to-one correspondence between bilinear symmetric forms and quadratic forms on V.

Positive definite Symmetric Bilinear Forms

A symmetric bilinear form B on V is positive definite if the corresponding quadratic form is positive definite.

Theorem

Given a real vector space V, there is a bijection between positive definite symmetric bilinear forms and inner products on V.

Non-degenerated Bilinear Forms

A bilinear form B on V is non-degenerated if for any nonzero $\vec{y} \in V$, there exists some $\vec{x} \in V$, such that $B(\vec{x}, \vec{y}) \neq 0$.

Theorem

The following are equivalent.

- *The bilinear form B is non-degenerated.*
- *The matrix $\mathrm{Rep}_\alpha(B)$ is invertiable for any basis α.*

Multi-Linear Forms

Let $V_1, \cdots, V_m$ be vector spaces over F. A map M from $V_1 \times \cdots \times V_m$ to F is called a multi-linear form if it is separately F-linear in each of its arguments.

Example

- The determinant is a multi-linear form on V^n where $n = \dim V$.

Tensor Products

Given two vector spaces V and W over F with the base $\alpha = \{\vec{v}_1, \cdots, \vec{v}_n\}$ and $\beta = \{\vec{w}_1, \cdots, \vec{w}_m\}$, the tensor product of V and W, denoted by $V \otimes W$, is a vector space over F spanned by the following set of elements

$$\{\vec{v}_i \otimes \vec{w}_j | 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Moreover, for $\vec{v} = \sum a_i \vec{v}_i \in V$ and $\vec{w} = \sum b_j \vec{w}_j \in W$, define

$$\vec{v} \otimes \vec{w} = \sum_{i,j} a_i b_j (\vec{v}_i \otimes \vec{w}_j).$$

Tensor product is a kind of product on vector spaces!!

Example

Let $V = \mathbb{R}\vec{e}_1 \oplus \vec{\mathbb{R}}e_2$. Then

$$V \otimes V = \mathbb{R}(\vec{e}_1 \otimes \vec{e}_1) \oplus \mathbb{R}(\vec{e}_1 \otimes \vec{e}_2) \oplus \mathbb{R}(\vec{e}_2 \otimes \vec{e}_1) \oplus \mathbb{R}(\vec{e}_2 \otimes \vec{e}_2).$$

Moreover, we have

$$(2\vec{e}_1 + 3\vec{e}_2) \otimes (\vec{e}_1 - 5\vec{e}_2) = 2(\vec{e}_1 \otimes \vec{e}_1) - 10(\vec{e}_1 \otimes \vec{e}_2) + 3(\vec{e}_2 \otimes \vec{e}_1) - 15(\vec{e}_2 \otimes \vec{e}_2).$$

Theorem

There is a bijection between multilinear forms $V_1 \times \cdots \times V_m$ on and linear functions on $V_1 \otimes \cdots \otimes V_m$.

Example

Given a bilinear form B on $\mathbb{R}^2 \times \mathbb{R}^2$, write

$$B(\vec{x}, \vec{y}) = \vec{x}^t \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \vec{y}.$$

Let $\alpha = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis of $\mathbb{R}^2$. Then $\beta = \{\vec{e}_1 \otimes \vec{e}_1, \vec{e}_1 \otimes \vec{e}_2, \vec{e}_2 \otimes \vec{e}_1, \vec{e}_2 \otimes \vec{e}_2\}$ forms a basis of $\mathbb{R}^2 \otimes \mathbb{R}^2$.

Define a linear function T on $\mathbb{R}^2 \otimes \mathbb{R}^2$ characterized by

$$T(\vec{e}_i \otimes \vec{e}_j) = B(\vec{e}_i, \vec{e}_j) \quad \forall i, j.$$

Then

$$\mathrm{Rep}_\beta(T) := (a\, b\, c\, d).$$