1

(1)

$\ker(A^3)$ = generalized kernel of $A$

$$= \ker \begin{pmatrix} 3 & 9 & 0 & 6 & 12 \\ 0 & 3 & 0 & 3 & 3 \\ -1 & -4 & 0 & -3 & -5 \\ 1 & 2 & 0 & 1 & 3 \\ -1 & -4 & 0 & -3 & -5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 9 & 0 & 6 & 12 \\ 0 & 3 & 0 & 3 & 3 \\ -1 & -4 & 0 & -3 & -5 \\ 1 & 2 & 0 & 1 & 3 \\ -1 & -4 & 0 & -3 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 3a+9b+6d+12e=0 \\ 3b+3d+3e=0 \\ -a-4b-3d-5e=0 \\ a+2b+d+3e=0 \\ -a-4b-3d-5e=0 \end{cases} \Rightarrow \begin{cases} a+3b+2d+4e=0 \\ b+d+e=0 \\ -2b-2d-2e=0 \\ a+2b+d+3e=0 \\ -2b-2d-2e=0 \end{cases} \Rightarrow \begin{cases} a+b+2e=0 \\ b+d+e=0 \\ a+b+2e=0 \end{cases}$$

$$\Rightarrow \ker A^3 = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} \middle| A^3 \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = 0 \right\} = \left\{ \begin{pmatrix} a \\ -a-2e \\ c \\ a+e \\ e \end{pmatrix} \middle| a, c, e \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

generalized image of $A = \operatorname{Im}(A^3) = \operatorname{span}\{\text{columns of } A^3\} = \operatorname{span}\left\{ \begin{pmatrix} 3 \\ 0 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 9 \\ 3 \\ -4 \\ 2 \\ -4 \end{pmatrix} \right\}$,

(2)

$$E_\infty(0) = \ker A^3 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$E_\infty(1) = \ker(A-I)^2 = \ker \begin{pmatrix} -4 & 2 & 0 & 9 & -3 \\ -2 & 0 & 0 & 3 & -3 \\ 5 & -3 & 1 & -12 & 2 \\ -1 & 1 & 0 & 3 & 0 \\ 3 & -1 & 0 & -6 & 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -4 & 2 & 0 & 9 & -3 \\ -2 & 0 & 0 & 3 & -3 \\ 5 & -3 & 1 & -12 & 2 \\ -1 & 1 & 0 & 3 & 0 \\ 3 & -1 & 0 & -6 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -4a+2b+9d-3e=0 \\ -2a+3d-3e=0 \\ 5a-3b+c-12d+2e=0 \\ -a+b+3d=0 \\ 3a-b-6d+3e=0 \end{cases} \Rightarrow \begin{cases} -2b-3d-3e=0 \\ -2b-3d-3e=0 \\ 2b+c+3d+2e=0 \\ 2b+3d+3e=0 \\ -a+b+3d=0 \end{cases} \Rightarrow \begin{cases} 2b+3d+3e=0 \\ c-e=0 \\ -a+b+3d=0 \end{cases}$$

$$\Rightarrow \ker(A-I)^2 = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} \middle| (A-I)^2 \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = 0 \right\} = \left\{ \begin{pmatrix} a \\ b \\ -\left(\frac{a+b}{3}\right) \\ \frac{a-b}{3} \\ -\left(\frac{a+b}{3}\right) \end{pmatrix} \middle| a, b \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} \right\}$$

(3)

Let $m_A(x) = x^{r_1}(x-1)^{r_2}$, $1 \le r_1 \le 3$, $1 \le r_2 \le 2$, where

$r_1 = \min\{m \in \mathbb{N} \mid \operatorname{rank} A^m = 5-3=2\}$

$r_2 = \min\{m \in \mathbb{N} \mid \operatorname{rank} (A-I)^m = 5-2=3\}$

we have $\operatorname{rank} A = 3$. $\operatorname{rank} A^2 = 2 \Rightarrow r_1 = 2$

$\operatorname{rank}(A - I) = 4 \Rightarrow r_2 = 2$

#

2.

$f_A(x) = (x^2 - 3x + 1)^2 = \left(x - \frac{3+\sqrt{5}}{2}\right)^2 \left(x - \frac{3-\sqrt{5}}{2}\right)^2$ splits

$x^2 - 3x + 1$

Let $f(x) = \left(x - \frac{3+\sqrt{5}}{2}\right)^{''} \left(x - \frac{3-\sqrt{5}}{2}\right)$, By Corollary 13 in Lecture note week 2,

$A$ is diagonalizable iff $f(A) \equiv 0$. Now

$$f(A) = A^2 - 3A + I = \begin{pmatrix} -1 & 6 & 3 & 6 \\ 3 & 5 & 0 & -3 \\ -3 & 0 & 2 & 6 \\ -3 & 3 & 3 & 8 \end{pmatrix} - \begin{pmatrix} 0 & 6 & 3 & 6 \\ 3 & 6 & 0 & -3 \\ -3 & 0 & 3 & 6 \\ -3 & 3 & 3 & 9 \end{pmatrix} + \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$= 0.$$

$\Rightarrow A$ is diagonalizable. #