Linear Algebra II - Annihilator of A Linear Transformation

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Recall that for a linear transformation $T : V \mapsto V$, its annihilator is

$$\operatorname{Ann}(T) = \left\{ f(x) \in F[x] \middle| f(T) \equiv 0 \right\}$$

which always contains the characteristic polynomial of T by the Cayley-Hamilton Theorem. This set has some nice properties which are very useful for studying linear transforms.

First, let us recall some properties of polynomials, which you should already know in high school. (They will be proved again in the course of Abstract Algebra.) For two non-zero polynomials $f(x)$ and $g(x)$,

1. $h(x)$ is called a common divisor of $f(x)$ and $g(x)$ if $h(x)|f(x)$ and $h(x)|g(x)$;
2. there exists a unique monic (,i.e. its leading coefficient equals to one) greatest common divisor (GCD) $d(x)$ such that for all common divisor $h(x)$, we have $\deg h(x) \leq \deg d(x)$;
3. there exist two polynomials $a(x)$ and $b(x)$ such that $a(x)f(x) + b(x)g(x) = d(x)$. Especially, when $d(x) = 1$, $f(x)$ and $g(x)$ are called coprime.

Now for $f(T) \in \operatorname{Ann}(T)$, suppose $f(x) = f_1(x)f_2(x)$ such that $f_1(x)$ and $f_2(x)$ are coprime. Then there exist two polynomials $a_1(x)$ and $a_2(x)$ such that $a_1(x)f_2(x) + a_2(x)f_1(x) = 1$. Let $P_1 = a_1(T)f_2(T)$ and $P_2 = a_2(T)f_1(T)$, then we have

$$P_1P_2 = a_2(T)f_1(T)a_1(T)f_2(T) \equiv 0 \qquad \text{and} \qquad P_1 + P_2 = I.$$

In this case, we have the following result.

Theorem 1. *Let $V_1 = \ker(f_1(T))$ and $V_2 = \ker(f_2(T))$.*

1. *V_1 and V_2 are T-invariant.*
2. *$V = V_1 \oplus V_2$.*
3. *P_1 is a projection onto V_1 along V_2.*
4. *P_2 is a projection onto V_2 along V_1.*

Proof. The part (1) is left to readers as exercise. For $v \in V$, let $\vec{v}_1 = P_1(\vec{v})$ and $\vec{v}_2 = P_2(\vec{v})$. First, we show that $\vec{v}_1 \in V_1$ and $\vec{v}_2 \in V_2$. Note that

$$f_1(T)(\vec{v}_1) = f_1(T)(P_1(\vec{v})) = f_1(T)\Big(a_1(T)f_2(T)(\vec{v})\Big) = a_1(T)\Big(f_1(T)f_2(T)\Big)(\vec{v}) = \vec{0}$$

which means $\vec{v}_1 \in V_1$. Similarly, we also have $\vec{v}_2 \in V_2$. Suppose $\vec{v} \in V_1 \cap V_2$, which means $f_1(T)(\vec{v}) = f_2(T)(\vec{v}) = \vec{0}$. Then

$$\vec{v} = (P_1 + P_2)(\vec{v}) = \Big(a_1(T)f_2(T) + a_2(T)f)_1(T)\Big)(\vec{v}) = a_1(T)f_2(T)(\vec{v}) + a_2(T)f_1(T)(\vec{v}) = \vec{0}.$$

Combining the above two results, we conclude that $V = V_1 \oplus V_2$.

We have shown that $\vec{v} = \vec{v}_1 + \vec{v}_2$ where $\vec{v}_1 = P_1(\vec{v}) \in V_1$ and $\vec{v}_2 = P_2(\vec{v}) \in V_2$. By the definition of projections, (3) and (4) hold. □

Corollary 2. *We have* $\ker(f_2(T)) = \operatorname{Im}(f_1(T))$ *and* $V = \ker(f_1(T)) \oplus \operatorname{Im}(f_1(T))$.

Proof. Note that we have

$$\operatorname{Im}(f_1(T)) \supseteq \operatorname{Im}\big(f_1(T)a_2(T)\big) = \operatorname{Im}(P_2) = V_2 = \ker(f_2(T)).$$

By the dimension formula and the part (2) in the above theorem, we have

$$\dim \operatorname{Im}(f_1(T)) = \dim V - \dim \ker(f_1(T)) = \dim \ker(f_2(T)).$$

We conclude that $\operatorname{Im}(f_1(T)) = \ker(f_2(T))$ and the part (2) in the above theorem can be rewritten as $V = \ker(f_1(T)) \oplus \operatorname{Im}(f_1(T))$. □

0.1 Genearalized kernel and generalized image

Now suppose $f_1(x) = x^m$ for some m. By the previous corollary, we have the following nice decomposition:

$$V = \ker(T^m) \oplus \operatorname{Im}(T^m).$$

Remark. The above result can be applied to any T, since we can choose $f(x) = f_T(x)$ and m be the multiplicity of the zero eigenvalue.
Remark. In general, $\dim V = \dim \ker(T) + \dim \operatorname{Im}(T)$ always holds but $\ker(T) + \operatorname{Im}(T)$ may not be a direct sum.

Theorem 3. *Under the above condition, we have*

1. $\ker(T^m) = \bigcup_{i=1}^{\infty} \ker(T^i)$.
2. $\operatorname{Im}(T^m) = \bigcap_{i=1}^{\infty} \operatorname{Im}(T^i)$.

Proof. It suffices to show that $\ker(T^i) \subseteq \ker(T^m)$ and $\operatorname{Im}(T^i) \supseteq \operatorname{Im}(T^m)$. If $i \leq m$, then it is obvious. Suppose $i > m$. Since $f(x) = x^m f_2(x) \in \operatorname{Ann}(T)$, we also have $x^i f_2(x) \in \operatorname{Ann}(T)$. Applying both polynomials $x^m f_2(x)$ and $x^i f_2(x)$ to Corollary 2, we obtain

$$\ker(T^i) = \operatorname{Im}(f_2(T)) = \ker(T^m) \quad \text{and} \quad \operatorname{Im}(T^i) = \ker(f_2(T)) = \operatorname{Im}(T^m).$$

□

Definition The subspace $\bigcup_{i=1}^{\infty} \ker(T^i)$ is called the generalized kernel of T, denoted by $\ker_\infty(T)$ and $\bigcap_{i=1}^{\infty} \operatorname{Im}(T^i)$ is called the generalized image of T, denoted by $\operatorname{Im}_\infty(T)$.

Finally, let us summarize the above discussion as the following theorem.

Theorem 4. *The following holds.*

$$V = \ker_\infty(T) \oplus \operatorname{Im}_\infty(T).$$

Now Suppose $f(T) \equiv 0$ and $f(x) = f_1(x) \cdots f_k(x)$ such that $f_i(x)$ and $f_j(x)$ are coprime for all $i \neq j$. For $i = 1$ to k, set

$$g_i(x) = f(x)/f_i(x).$$

One can show that $g_1(x), \cdots, g_2(x)$ are coprime, so there exist $a_1(x), \cdots, a_k(x)$ such that

$$a_1(x)g_1(x) + \cdots + a_k g_k(x) = 1.$$

Set $P_i = a_i(T)g_1(T)$, then we still have

$$P_1 \cdots P_k = 0 \qquad \text{and} \qquad P_1 + \cdots + P_k = I.$$

Theorem 5. *Let $V_i = \ker(f_i(T))$.*

1. $V = V_1 \oplus \cdots \oplus V_k$.
2. *P_i is a projection onto V_i along $V_1 \oplus \cdots \oplus \hat{V_i} \oplus \cdots \oplus V_k$.*

Proof. Exercise. □

Applying the above theorem, we obtain the following theorem immediately.

Corollary 6. *If there exists a split polynomial $f(x)$ without repeated roots in* $\operatorname{Ann}(T)$, *(In this case, $f(x) = \prod_{i=1}^{n}(x - \lambda_i)$ for some $\lambda_i \in F$.) Then T is diagonalizable.*

Proof. Exercise. □