Linear Algebra II
Real Canonical Forms

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Introduction

Recall that for a real normal matrix, it can be only diagonalized under a complex basis. In practice, we would like only use real base. In this case, is there standard way to represent a linear transform?

Complex Invariant Subspaces

Let A be a real matrix with a non-real eigenvalue λ. Let $\vec{v}$ be an eigenvector corresponding to λ. Note that

$$A\vec{v} = \lambda\vec{v} \quad \Rightarrow \quad A\overline{\vec{v}} = \overline{A}\,\overline{\vec{v}} = \overline{\lambda}\,\overline{\vec{v}}.$$

Therefore, $\overline{\vec{v}}$ is an eigenvector corresponding to the eigenvalue $\overline{\lambda}$. Let W be a subspace with basis $\alpha = \{\vec{v}, \overline{\vec{v}}\}$. Then

$$\mathrm{Rep}_\alpha L_A\big|_W = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}.$$

For a real matrix, its complex eigenvalues will always occur in complex conjugate pairs.

Set $\lambda = a - bi$, $\vec{v}_1 = \mathrm{Re}(\vec{v})$, and $\vec{v}_2 = \mathrm{Im}(\vec{v})$. Then

$$A(\vec{v}_1 + i\vec{v}_2) = (a - bi)(\vec{v}_1 + i\vec{v}_2) = (a\vec{v}_1 + b\vec{v}_2) + i(-b\vec{v}_1 + a\vec{v}_2).$$

We conclude that

$$A\vec{v}_1 = a\vec{v}_1 + b\vec{v}_2 \quad \text{and} \quad A\vec{v}_2 = -b\vec{v}_1 + a\vec{v}_2.$$

Set $\beta = \{\vec{v}_1, \vec{v}_2\}$, which is another basis of W. Then

$$\mathrm{Rep}_\beta(L_A\big|_W) = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right).$$

Example

Let $A = \left(\begin{smallmatrix} 5 & -2 \\ 5 & -1 \end{smallmatrix}\right)$ which eigenvalues are $2 \pm i$. Let $\vec{v} = \left(\begin{smallmatrix} 3-i \\ 5 \end{smallmatrix}\right)$ be an eigenvector corresponding to the eigenvalue $\lambda = 2 - i$. Then, $\overline{\vec{v}} = \left(\begin{smallmatrix} 3+i \\ 5 \end{smallmatrix}\right)$ is an eigenvector corresponding to the eigenvalue $\overline{\lambda} = 2 + i$. Note that $\vec{v} = \left(\begin{smallmatrix} 3-i \\ 5 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3 \\ 5 \end{smallmatrix}\right) + i \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)$. Let $P = \left(\begin{smallmatrix} 3 & -1 \\ 5 & 0 \end{smallmatrix}\right)$. Then

$$A = P \left(\begin{smallmatrix} 2 & -1 \\ 1 & 2 \end{smallmatrix}\right) P^{-1}.$$

Real Canonical Form of Complex Diagonalizable Matrices

Let A be a complex diagonalizable real matrix with pairs of complex eigenvalues $\lambda_1, \bar{\lambda}_1, \cdots, \lambda_k, \bar{\lambda}_k$ and real eigenvalues $\lambda_{k+1}, \cdots, \lambda_{k+r}$. Let $\vec{v}_j$ be the eigenvector corresponding to λ_j. Write $\vec{v}_j = \vec{u}_j + i\vec{w}_j$ for $j = 1$ to k. Then

$$\alpha = \{\vec{u}_1, \vec{w}_1, \cdots, \vec{u}_k, \vec{w}_k, \vec{v}_{k+1}, \cdots, \vec{v}_{k+r}\}$$

forms a basis of $\mathbb{R}^{2k+r}$. Moreover, let $\lambda_j = a_j - ib_j$. Then

$$\mathrm{Rep}_\alpha(L_A) = \begin{pmatrix} \boxed{\begin{matrix} a_1 & -b_1 \\ b_1 & a_1 \end{matrix}} & & & & \\ & \ddots & & & \\ & & \boxed{\begin{matrix} a_k & -b_k \\ b_k & a_k \end{matrix}} & & & \\ & & & \lambda_{k+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_{k+r} \end{pmatrix}.$$

Real Canonical Form of Orthogonal Matrices

In the previous slide, suppose A is an orthogonal matrix. For $1 \leq j \leq k$, $\lambda_j = e^{-i\theta_j} = \cos\theta_j - i\sin\theta_j$ for some θ_j and $\lambda_{k+j} = \pm 1$ for $1 \leq j \leq r$. In this case,

$$\mathrm{Rep}_\beta(L_A) = \begin{pmatrix} \boxed{\begin{matrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{matrix}} & & & & & \\ & \ddots & & & & \\ & & \boxed{\begin{matrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{matrix}} & & & \\ & & & \pm 1 & & \\ & & & & \ddots & \\ & & & & & \pm 1 \end{pmatrix}.$$

Corollary

Every orthogonal matrix is a composition of rotations on 2-dimensional plans and reflections on 1-dimensional lines.