Cyclic subspaces

Let $T : V \mapsto V$ be a linear transformation. For a nonzero vector $\vec{v}$ of V, there is a simple way to find the smallest T-invariant subspace containing $\vec{v}$ as follows. Let

$$S = \{\vec{v}, T(\vec{v}), T^2(\vec{v}), \cdots\}.$$

It is clear that every T-invariant subspace containing $\vec{v}$ must also contain the set S and hence it also contains $\text{span}(S)$. The subspace $\text{span}(S)$ is called the cyclic subspace generated by $\vec{v}$.

Theorem 1. *Let W be the cyclic subspace generated by $\vec{v}$. Then*

1. *W is T-invariant. (Especially, this implies that W is the smallest T-invariant subspace containing $\vec{v}$.)*
2. *Let m be the largest integer satisfying the condition that $\alpha = \{\vec{v}, T(\vec{v}), \cdots, T^{m-1}(\vec{v})\}$ is linearly independent. Then α forms a basis of W and $\dim W = m$.*
3. *Suppose*
$$T^m(\vec{v}) = a_0\vec{v} + a_1 T(\vec{v}) + \cdots + a_{m-1}T^{m-1}(\vec{v}).$$
Then the matrix representation of $T\big|_W$ is
$$\text{Rep}_\alpha(T\big|_W) = \begin{pmatrix} 0 & 0 & \cdots & \cdots & a_0 \\ 1 & 0 & \ddots & \ddots & a_1 \\ 0 & 1 & \ddots & \ddots & a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & a_{m-1} \end{pmatrix}$$
and the characteristic polynomial of $T\big|_W$ is
$$f_{T|_W}(x) = x^m - a_{m-1}x^{m-1} - \cdots - a_1 x - a_0.$$

Proof.
(1) It is sufficient to show that $T(S) \subseteq W$, which is trivial.

(2) We will show that every elements in S is contained in $\text{span}(\alpha)$. Then α spans W and α is a basis of W. By the definition of m, $\{\vec{v}, T(\vec{v}), \cdots, T^m(\vec{v})\}$ is linearly dependent, so there exists a non-trivial linear relation:

$$c_0\vec{v} + c_1T(\vec{v}) + \cdots + c_mT^m(\vec{v}) = 0.$$

On the other hand, the first m terms of the right hand side are linearly independent, we must have $c_m \neq 0$. We conclude that $T^m(\vec{v})$ can be written as a linear combination of $\{\vec{v}, T(\vec{v}), \cdots, T^{m-1}(\vec{v})\}$ and hence it is contained in $\text{span}(\alpha)$. Note that we have shown that $T(\alpha) = \{T(\vec{v}), \cdots, T^m(\vec{v})\} \subseteq \text{span}(\alpha)$ which means $\text{span}(\alpha)$ is T-invariant. By (1), $\text{span}(\alpha) = W$.

(3) The matrix representation can be obtained directly from the definition. For example, the first column of the matrix is given by

$$\mathrm{Rep}_\alpha\Big(T(\vec{v})\Big) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and the last column of the matrix is given by

$$\mathrm{Rep}_\alpha\Big(T\big(T^{m-1}(\vec{v})\big)\Big) = \mathrm{Rep}_\alpha\Big(T^m(\vec{v})\Big) = \mathrm{Rep}_\alpha\Big(\sum_{i=0}^{m-1} a_i T^i(\vec{v})\Big) = \begin{pmatrix} a_0 \\ \vdots \\ a_{m-1} \end{pmatrix}.$$

Finally, let us compute the characteristic polynomial

$$f_{T|_W}(x) = \det\Big(xI - \mathrm{Rep}_\alpha(T|_W)\Big) = \det \begin{pmatrix} x & 0 & \cdots & \cdots & -a_0 \\ -1 & x & \ddots & \ddots & -a_1 \\ 0 & -1 & \ddots & \ddots & -a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & x - a_{m-1} \end{pmatrix}$$

The above determinant is left for students as an exercise. $\square$