Linear Algebra II
Quadratic Forms

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Quadratic Forms on F^n

For $\vec{x} = (x_1, \cdots, x_n) \in F^n$, a quadratic form $Q(\vec{x})$ is a function from F^n to F of the form

$$Q(\vec{x}) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j.$$

Example

1. $Q(x_1, x_2) = 2x_1^2 - 3x_1x_2 + x_2^2$ is a quadratic form on $\mathbb{R}^2$.
2. $Q(x_1, x_2) = 2x_1^2 - 3x_1x_2 + x_2^2 - x_1$ is not a quadratic form on $\mathbb{R}^2$.
3. $Q(x_1, x_2, x_3) = 2x_1^2 - 3x_1x_2 + x_2^2 - x_1x_3$ is a quadratic form on $\mathbb{R}^3$.

Matrix Representation of Binary Quadratic Forms

Consider the following example.

$$Q(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 3x_2^2 = (x_1\ x_2)\begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

In general, we have

$$Q(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 = (x_1\ x_2)\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Quadratic Forms from Calculus

Let $f(x, y)$ be a smooth function on $\mathbb{R}^2$, then we have

$$\begin{aligned} f(x, y) =& f(0,0) + f_x(0,0)x + f_y(0,0)y \\ &+ \frac{1}{2}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right) + \cdots \\ =& f(0,0) + (\nabla f)(0,0)\cdot(x,y) + \frac{1}{2}(x,y)(Hf)(0,0)\begin{pmatrix} x \\ y \end{pmatrix} + \cdots. \end{aligned}$$

Here $\nabla f = (f_x, f_y)$ is the gradient of f and $Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$ is called the Hessian of $f(x)$. When $(0,0)$ is a critical point, i.e. $(\nabla f)(0,0) = (0,0)$, then

$$f(x, y) = f(0,0) + \frac{1}{2}(x,y)(Hf)(0,0)\begin{pmatrix} x \\ y \end{pmatrix} + \text{(higher order terms)}.$$

Example

Let a and b be two positive integers.

- When $f(x, y) = ax^2 + by^2$, $f(0, 0)$ is a (local) minimum.
- When $f(x, y) = -ax^2 - by^2$, $f(0, 0)$ has a (local) maximum.
- When $f(x, y) = ax^2 - by^2$, $f(0, 0)$ is neither a local minimum nor a local maximum.

In general, we can we say about $ax^2 + bxy + cy^2$?

Example

Let $f(x_1, x_2) = 3x_1^2 + 4x_1x_2 = (x_1\ x_2)A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ where $A = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$.
Since A is symmetric, there exists an orthogonal matrix P and a diagonal matrix D such that $A = PDP^t$, or equivalent $D = P^tAP$.

Let us find P and D.

- $f_A(\lambda) = \det(\lambda I_2 - A) = \lambda^2 - 3\lambda - 4 \Rightarrow \lambda = 4, -1$.
- For $\lambda = 4$, let $\vec{v}_1 = \frac{1}{\sqrt{5}}\binom{2}{1}$.
- For $\lambda = -1$, let $\vec{v}_2 = \frac{1}{\sqrt{5}}\binom{-1}{2}$.
- Let $P = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$.

Now set $\begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} P$ be the change of variables. Then

$$f(x_1, x_2) = (y_1\ y_2)P^tAP\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (y_1\ y_2)D\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 4y_1^2 - y_2^2.$$

Behaviors of Binary Quadratic Forms

Consider a general binary quadratic form

$$Q(x_1, x_2) = (x_1\ x_2) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1\ x_2) A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Let λ_1 and λ_2 be the eigenvalues of A. Recall that

$$\mathrm{tr}(A) = a + c = \lambda_1 + \lambda_2 \quad \text{and} \quad \det(A) = ac - \frac{b^2}{4} = \lambda_1 \lambda_2.$$

From the above result, we have

Theorem

- $(0,0)$ *is the absolute minimum of* $Q(x_1, x_2)$ *if* $\mathrm{tr}(A) > 0$ *and* $\det(A) > 0$.
- $(0,0)$ *is the absolute maximum of* $Q(x_1, x_2)$ *if* $\mathrm{tr}(A) < 0$ *and* $\det(A) > 0$.

Matrix Representation of Ternary Quadratic Forms

The following is a quadratic form with 3 variables.

$$\begin{aligned} Q(x_1, x_2, x_3) &= 2x_1x_2 + x_2^2 + 4x_1x_3 + 2x_2x_3 \\ &= (x_1\ x_2\ x_3) \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned}$$

Let us diagonalize this quadratic forms.

- $f_A(\lambda) = \det(\lambda I_2 - A) = \lambda(\lambda+2)(\lambda-3) \Rightarrow \lambda = 3, -2, 0.$
- $\vec{v}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, and $\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ be the eigenvectors asssociated to 3,-2,0 respectively.
- Let $P = (\vec{v}_1\ \vec{v}_2\ \vec{v}_3)$ and $D = \begin{pmatrix} 3 & & \\ & -2 & \\ & & 0 \end{pmatrix}$, then $A = PDP^t$.

Now set $\begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} P$ be the change of variables. Then $Q(x_1, x_2, x_3) = 3y_1^2 - 2y_2^2$.

Matrix Representation of Quadratic Forms

For quadratic forms with n variables, we have

$$Q(x_1, \cdots, x_n) = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Here $A = \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \cdots & \frac{1}{2}a_{1n} \\ \frac{1}{2}a_{12} & a_{22} & \cdots & \frac{1}{2}a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{2}a_{1n} & \frac{1}{2}a_{2n} & \cdots & a_{nn} \end{pmatrix}$ is a symmetric matrix.

Diagonalization of Real Quadratic Forms

When the coefficients of Q are real, A is a real symmetric matrix. Then there exists an orthogonal matrix P and a diagonal matrix D such that $A = PDP^t$. Then

$$Q(\vec{x}) = \vec{x}^t A\vec{x} = \vec{x}PDP^t\vec{x} = \vec{y}^t D\vec{y}.$$

Here $\vec{y}^t = \vec{x}^t P$ is the change of variables.

We always use $\vec{x}$ to denote a column vector!