Linear Algebra II
Normal Linear Transformation

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We will show that the extra structure "inner product" is also very powerful in the study of linear transformation.

Symmetric and Hermitian

For $A \in M_n(\mathbb{R})$ and $\vec{v}, \vec{w} \in \mathbb{R}^n$, we have

$$\langle \vec{v}, A\vec{w} \rangle = \vec{v}^t A\vec{w} = (A^t \vec{v})^t \vec{w} = \langle A^t \vec{v}, \vec{w} \rangle.$$

Here A^t is the transpose of A. When $A^t = A$, A is called symmetric.

For $A \in M_n(\mathbb{C})$ and $\vec{v}, \vec{w} \in \mathbb{C}^n$, we have

$$\langle \vec{v}, A\vec{w} \rangle = \vec{v}^t \overline{A\vec{w}} = (\overline{A}^t \vec{v})^t \overline{\vec{w}} = \langle \overline{A}^t \vec{v}, \vec{w} \rangle = \langle A^* \vec{v}, \vec{w} \rangle.$$

Here $A^* := \overline{A}^t$ is the conjugate transpose of A. When $A^* = A$, A is called Hermitian.

Examples

Example

$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ is a real symmetric matrix.

Example

$A = \begin{pmatrix} 1+i & 1-2i \\ 1-2i & 3 \end{pmatrix}$ is a complex symmetric matrix but not a Hermitian matrix.

Example

$A = \begin{pmatrix} 1 & 1+2i \\ 1-2i & 3 \end{pmatrix}$ is a Hermitian matrix.

Self-adjoint Linera Transformation

For a linear transformation T on an inner product space $(V, \langle \cdot, \cdot \rangle)$, a linear transformation T^* on V is called the adjoint of T if

$$\langle \vec{v}, T\vec{w} \rangle = \langle T^*\vec{v}, \vec{w} \rangle$$

for all $\vec{v}, \vec{w} \in V$. When $T = T^*$, T is called self-adjoint. We will show later that the existence and uniqueness of T^* later.

Remark. We also have $\langle T\vec{v}, \vec{w} \rangle = \langle \vec{v}, T^*\vec{w} \rangle$. (Exercise)

Self-Adjoint Linear Transformation

Theorem

Let T be a self-adjoint linear transformation on a complex inner product space V. Then

1. *All eigenvalues of T are real.*
2. *T admits a set of eigenvectors which forms an orthonormal basis of V. (Especially, T is diagonalizable.)*

Let $\vec{v}$ be an eigenvector corresponding to an eigenvalue of λ of T. Then

$$\lambda\langle\vec{v},\vec{v}\rangle = \langle\lambda\vec{v},\vec{v}\rangle = \langle T\vec{v},\vec{v}\rangle = \langle\vec{v},T\vec{v}\rangle = \langle\vec{v},\lambda\vec{v}\rangle = \overline{\lambda}\langle\vec{v},\vec{v}\rangle.$$

Since $\langle\vec{v},\vec{v}\rangle \neq 0$, $\lambda = \overline{\lambda}$ and λ is real.

Proof

Let us prove the part (2) by induction on $n = \dim V$.

When $n = 1$, it is obvious.

Suppose the statement holds for the case of dimension $< n$.

Let $\vec{v}$ be an eigenvector corresponding to an eigenvalue λ and W be the orthogonal compliment of $F\vec{v}$. Then $V = F\vec{v} \oplus W$. Claim that W is T-invariant.

Proof

For $\vec{w} \in W$,

$$\langle T\vec{w}, \vec{v}\rangle = \langle \vec{w}, T\vec{v}\rangle = \langle \vec{w}, \lambda\vec{v}\rangle = \bar{\lambda}\langle \vec{w}, \vec{v}\rangle = 0.$$

Therefore, $T(W) \subseteq W$ and W is T-invariant.

Note that

$$(T\big|_W)^* = (T^*)\big|_W = T\big|_W.$$

Thus $T\big|_W$ is also self-adjoint. By induction, there exists a set of eigenvectors β which forms an orthonormal basis of W. Then $\{\vec{v}\} \cup \beta$ is the desired set.

Real Symmetric Matrix

Corollary

Let A be a real symmetric matrix.

- All eigenvalues of A are real.
- A is diagonalizable over $\mathbb{R}$.
- A admits a set of eigenvectors which forms an orthonormal basis.

Corollary

Let A be a Hermitian matrix.

- All eigenvalues of A are real.
- A is diagonalizable.
- A admits a set of eigenvectors which forms an orthonormal(unitary) basis.

Let α be an orthonormal basis of V and $A = \mathrm{Rep}_\alpha(T)$. Then for $\vec{w}_1 = \mathrm{Rep}_\alpha(\vec{v}_1)$ and $\vec{w}_2 = \mathrm{Rep}_\alpha(\vec{v}_2)$,

$$\begin{aligned}\langle \vec{v}_1, T\vec{v}_2 \rangle &= \langle \vec{w}_1, A\vec{w}_2 \rangle_{F^n} \\ &= \langle A^*\vec{w}_1, \vec{w}_2 \rangle_{F^n} = \langle T^*\vec{v}_1, \vec{v}_2 \rangle\end{aligned}$$

Here T^* is the linear transformation with the matrix representation A^* under the basis α, which is also the adjoint of T.

Under an orthonormal basis, the conjugate transpose of the matrix representation of T is equal to the matrix representation of T^*.

A symmetric/Hermitian matrix is a matrix representation of a self-adjoint linear transform under an orthonormal basis.

Uniqueness of Adjoint Linear Transformation

Suppose T' is also an adjoint of T. For any $\vec{v} \in V$, we will show that $T^*\vec{v} = T'\vec{v}$. Let α be an orthonormal basis of V. Then

$$T^*\vec{v} = \sum_{\vec{w}\in\alpha} \langle T^*\vec{v}, \vec{w}\rangle \vec{w} = \sum_{\vec{w}\in\alpha} \langle \vec{v}, T\vec{w}\rangle \vec{w} = \sum_{\vec{w}\in\alpha} \langle T'\vec{v}, \vec{w}\rangle \vec{w} = T'\vec{v}.$$

Therefore, $T^* = T'$.

A linear transformation T on an inner product space V is normal if T^* and T commute. The main theorem of this section is the following.

Theorem

A complex linear transformation is diagonalizable under some orthonormal basis if and only if it is normal.

Before proving the theorem, we need some preparations.

Theorem

Let T be a normal operator. Let $\vec{v}$ be the eigenvector corresponding to the eigenvalue λ of T. Then $\vec{v}$ be the eigenvector corresponding to the eigenvalue $\bar{\lambda}$ of T^*.

We shall show that $T^*\vec{v} = \bar{\lambda}\vec{v}$. Consider

$$
\begin{aligned}
&\langle T^*\vec{v} - \bar{\lambda}\vec{v}, T^*\vec{v} - \bar{\lambda}\vec{v}\rangle \\
&= \langle T^*\vec{v}, T^*\vec{v}\rangle - \langle \bar{\lambda}\vec{v}, T^*\vec{v}\rangle - \langle T^*\vec{v}, \bar{\lambda}\vec{v}\rangle + \langle \bar{\lambda}\vec{v}, \bar{\lambda}\vec{v}\rangle \\
&= \langle T\vec{v}, T\vec{v}\rangle - \bar{\lambda}\langle T\vec{v}, \vec{v}\rangle - \lambda\, \langle \vec{v}, T\vec{v}\rangle + |\lambda|^2\langle \vec{v}, \vec{v}\rangle \\
&= \langle \lambda\vec{v}, \lambda\vec{v}\rangle - \bar{\lambda}\langle \lambda\vec{v}, \vec{v}\rangle - \lambda\, \langle \vec{v}, \lambda\vec{v}\rangle + |\lambda|^2\langle \vec{v}, \vec{v}\rangle = 0.
\end{aligned}
$$

Here the red part is the only place that we use the condition that T is normal. The proof is complete.

Proposition

Let T be a normal complex linear transformation T, then

1. $\ker(T) = \ker(T^*T) = \ker(TT^*) = ker(T^*)$.
2. $\ker(T) = \mathrm{I}m(T)^{\perp}$ and $V = \ker(T) \oplus \mathrm{I}m(T)$.
3. $\mathrm{I}m(T) = \mathrm{I}m_{\infty}(T)$.
4. $\ker(T) = \ker_{\infty}(T)$.
5. For two distinct eigenvalues λ_1 and λ,
   $\ker(T - \lambda_1 I) \perp \ker(T - \lambda_2 I)$.

Proof

1. $\ker(T) = \ker(T^*T) = \ker(TT^*) = ker(T^*)$.

(1) For any $\vec{v} \in \ker(T^*T)$, we have

$$0 = \langle \vec{v}, T^*T\vec{v} \rangle = \langle T\vec{v}, T\vec{v} \rangle.$$

Therefore, $T\vec{v} = \vec{0}$, which implies that $\ker(T) = \ker(T^*T)$. By the same argument and the fact $T^{**} = T$, we also have $\ker(T^*) = \ker(TT^*)$. Together with the assumption $TT^* = T^*T$,, we complete the proof.

(2) $\ker(T) = \mathrm{I}m(T)^\perp$ and $V = \ker(T) \oplus \mathrm{I}m(T)$.

For any $\vec{v} \in V$ and $\vec{w} \in \mathrm{I}m(T)^\perp$,

$$\vec{0} = \langle \vec{w}, T(\vec{v}) \rangle = \langle T^* \vec{w}, \vec{v} \rangle$$

which implies that $\vec{w} \in \ker(T^*) = \ker(T)$. Therefore, $\mathrm{I}m(T)^\perp \subseteq \ker(T)$. On the other hand, by dimension formula,

$$\dim \ker(T) = n - \dim \mathrm{I}m(T) = \dim \mathrm{I}m(T)^\perp.$$

Therefore, $\mathrm{I}m(T)^\perp = \ker(T)$ and

$$V = \mathrm{I}m(T) \oplus \mathrm{I}m(T)^\perp = \mathrm{I}m(T) \oplus \ker(T).$$

Proof

(3) $\mathrm{Im}(T) = \mathrm{Im}_\infty(T)$.

Note that

$$\mathrm{Im}(T) = T(V) = T\Big(\ker(T) \oplus \mathrm{Im}(T)\Big) = \mathrm{Im}(T^2).$$

Therefore, for all $k \geq 2$,

$$\mathrm{Im}(T^k) = T^{k-2}\Big(\mathrm{Im}(T^2)\Big) = T^{k-2}\Big(\mathrm{Im}(T)\Big) = \mathrm{Im}(T^{k-1}).$$

We conclude that $\mathrm{Im}(T) = \mathrm{Im}(T^2) = \mathrm{Im}(T^3) = \cdots = \mathrm{Im}_\infty(T)$.

Proof

(4) $\ker(T) = \ker_\infty(T)$.

By the dimension formula and (3), we have

$$\dim \ker_\infty(T) = n - \dim \mathrm{I}m_\infty(T) = n - \dim \mathrm{I}m(T) = \dim \ker(T).$$

Together with the fact $\ker(T) \subseteq \ker_\infty(T)$, we have $\ker(T) = \ker_\infty(T)$.

Proof

(5) For two distinct eigenvalues λ_1 and λ_2, $\ker(T - \lambda_1 I) \perp \ker(T - \lambda_2 I)$.

For any $\vec{v}_1 \in \ker(T - \lambda_1 I)$ and $\vec{v}_2 \in \ker(T - \lambda_2 I)$,

$$\begin{aligned} \lambda_1 \langle \vec{v}_1, \vec{v}_2 \rangle &= \langle \lambda_1 \vec{v}_1, \vec{v}_2 \rangle = \langle T\vec{v}_1, \vec{v}_2 \rangle \\ &= \langle \vec{v}_1, T^* \vec{v}_2 \rangle = \langle \vec{v}_1, \bar{\lambda}_2 \vec{v}_2 \rangle = \lambda_2 \langle \vec{v}_1, \vec{v}_2 \rangle. \end{aligned}$$

Here "$=$" is followed by the previous theorem. We conclude that $\langle \vec{v}_1, \vec{v}_2 \rangle = 0$.

Note that if T is normal, $T - \lambda I$ is also normal. (Here $(T - \lambda I)^* = T^* - \bar{\lambda} I$.) By (4) in the previous proposition, every generalized eigenspace $\ker_\infty(T - \lambda I)$ is equal to the eigenspace $\ker(T - \lambda I)$. Therefore, V can be written as a direct sum of eigenspaces.

By (5) in the previous proposition, eigenspaces are mutually orthogonal. Therefore, we can choose an orthonormal basis in each eigenspace and their union will give an orthonormal basis of V which diagonalizes T.

Diagonalizable under Orthonormal Basis $\Rightarrow$ Normal

Let α be an orthonormal basis such that $A = \mathrm{Rep}_\alpha(T)$ is a diagonal matrix. Then $A^* = \mathrm{Rep}_\alpha(T^*)$ is also a diagonal matrix. Since any two diagonal matrices commute, A and A^* commute. Therefore T and T^* commute.