Linear Algebra II
Nilpotent Linear Transformations

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Nilpotent Linear Transformations

Let T be a linear transformation on a vector space V over F.

Definition

A linear transform T is nilpotent if $T^k \equiv 0$ for some positive integer k. The smallest such k is called the index of T.

Proposition

Let n be the dimension of V. The following are equivalent.

1. $T^k \equiv 0$ for some positive integer k.
2. The generalized kernel of T is equal to the whole space V.
3. $T^n \equiv 0$.
4. $f_T(x) = x^n$.

Cyclic Subspace

Suppose T is nilpotent. Let $\vec{v}$ be a nonzero vector in V and

$$W = \mathrm{cyclic}(\vec{v}) = F\vec{v} \oplus \cdots \oplus FT^{m-1}(\vec{v})$$

which the cyclic subspace generated by $\vec{v}$. Recall that

$$f\big|_{T\big|_W}(x) \text{ divides } f_T(x) = x^n$$

which implie that

$$f|_{T|_W}(x) = x^m \quad \text{and} \quad T^m(\vec{v}) = \vec{0}.$$

It is convenient to express the action of T on W by the diagram

$$\vec{0} \leftarrow T^{m-1}(\vec{v}) \leftarrow \cdots \leftarrow T(\vec{v}) \leftarrow \vec{v}.$$

Matrix Representation

Under the cyclic basis $\alpha = \{\vec{v}, \cdots, T^{m-1}(\vec{v})\}$, $T\big|_W$ has the matrix representation

$$\mathrm{Rep}_\alpha(T\big|_W) = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

The goal of this lecture is to prove the following theorem.

Theorem

Let T be a nilpotent linear transformation on the vector space V. Then V can be decomposed as a direct sum of cyclic subspaces.

Example

We are looking for a decomposition as the following form.

$$
\begin{array}{ccccccc}
\vec{0} & \leftarrow & T^2(\vec{v}_1) & \leftarrow & T(\vec{v}_1) & \leftarrow & \vec{v}_1 \\
\vec{0} & \leftarrow & T^2(\vec{v}_2) & \leftarrow & T(\vec{v}_2) & \leftarrow & \vec{v}_2 \\
\vec{0} & \leftarrow & T(\vec{v}_3) & \leftarrow & \vec{v}_3 & & \\
\vec{0} & \leftarrow & \vec{v}_4 & & & &
\end{array}
$$

Find cyclic subspaces of maximal dimension first.

Example

Apply T^2 to the basis, then cyclic spaces of smaller dimension are killed. One can determine $T^2(\vec{v}_1)$ and $T^2(\vec{v}_2)$ since they form a basis of $\mathrm{I}m(T^2)$.

$$
\begin{array}{ccccccc}
\vec{0} & \leftarrow & T^2(\vec{v}_1) & \leftarrow & T(\vec{v}_1) & \leftarrow & \vec{v}_1 \\
\vec{0} & \leftarrow & T^2(\vec{v}_2) & \leftarrow & T(\vec{v}_2) & \leftarrow & \vec{v}_2 \\
\vec{0} & \leftarrow & T(\vec{v}_3) & \leftarrow & \vec{v}_3 & & \\
\vec{0} & \leftarrow & \vec{v}_4 & & & &
\end{array}
$$

Example

From $T^2(\vec{v}_1)$ and $T^2(\vec{v}_2)$, we can find $\vec{v}_1$ and $\vec{v}_2$.

$$
\begin{array}{ccccccc}
\vec{0} & \leftarrow & T^2(\vec{v}_1) & \leftarrow & T(\vec{v}_1) & \leftarrow & \vec{v}_1 \\
\vec{0} & \leftarrow & T^2(\vec{v}_2) & \leftarrow & T(\vec{v}_2) & \leftarrow & \vec{v}_2 \\
\vec{0} & \leftarrow & T(\vec{v}_3) & \leftarrow & \vec{v}_3 & & \\
\vec{0} & \leftarrow & \vec{v}_4 & & & &
\end{array}
$$

Example

Next, let us find the cyclic subspace of second largest dimension.
On the subspace $\ker(T^2)$,

$$
\begin{array}{ccccccc}
\vec{0} & \leftarrow & T^2(\vec{v}_1) & \leftarrow & T(\vec{v}_1) & \leftarrow & \vec{v}_1 \\
\vec{0} & \leftarrow & T^2(\vec{v}_2) & \leftarrow & T(\vec{v}_2) & \leftarrow & \vec{v}_2 \\
\vec{0} & \leftarrow & T(\vec{v}_3) & \leftarrow & \vec{v}_3 & & \\
\vec{0} & \leftarrow & \vec{v}_4 & & & &
\end{array}
$$

k-dimensional cyclic subspaces are of maximal dimension on $\ker(T^k)$.

Example

On the subspace $\ker(T^2)$, apply T to its basis, then cyclic spaces of smaller dimension are killed. One can determine $T(\vec{v}_3)$ since it together with $T^2(\vec{v}_1)$ and $T^2(\vec{v}_2)$ form a basis of $\mathrm{I}m(T)$ on $\ker(T^2)$.

$$\begin{array}{ccccccc}
\vec{0} & \leftarrow & T^2(\vec{v}_1) & \leftarrow & T(\vec{v}_1) & \leftarrow & \vec{v}_1 \\
\vec{0} & \leftarrow & T^2(\vec{v}_2) & \leftarrow & T(\vec{v}_2) & \leftarrow & \vec{v}_2 \\
\vec{0} & \leftarrow & T(\vec{v}_3) & \leftarrow & \vec{v}_3 & & \\
\vec{0} & \leftarrow & \vec{v}_4 & & & &
\end{array}$$

Example

From $T(\vec{v}_3)$, we can find $\vec{v}_3$.

$$\begin{array}{ccccccc}
\vec{0} & \leftarrow & T^2(\vec{v}_1) & \leftarrow & T(\vec{v}_1) & \leftarrow & \vec{v}_1 \\
\vec{0} & \leftarrow & T^2(\vec{v}_2) & \leftarrow & T(\vec{v}_2) & \leftarrow & \vec{v}_2 \\
\vec{0} & \leftarrow & T(\vec{v}_3) & \leftarrow & \vec{v}_3 & & \\
\vec{0} & \leftarrow & \vec{v}_4 & & & &
\end{array}$$

Repeat this process, we can find a cyclic decomposition.

Theorem

Let T be a nilpotent linear transformation on V of index $k+1$. Let $\vec{v}_1, \cdots, \vec{v}_m$ be vectors in V such that $T^k(\vec{v}_1), \cdots, T^k(\vec{v}_m)$ form a basis of $\mathrm{Im}(T^k)$.

(1) $V = \ker(T^k) \bigoplus_{i=1}^m F\vec{v}_i$.

(2) $\sum_{i=1}^m \mathrm{cyclic}(\vec{v}_i) = \bigoplus_{i=1}^m \mathrm{cyclic}(\vec{v}_i)$.

(3) *There exists some T-invariant subspace W such that* $V = W \bigoplus_{i=1}^m \mathrm{cyclic}(\vec{v}_i)$.

Review of Direct Sum

1. For subspaces $W_1, \cdots, W_r$ of V, the following are equivalent.
   a) $W_1 + \cdots + W_r = W_1 \oplus \cdots \oplus W_r$.
   b) Whenever $\vec{w}_1 + \cdots + \vec{w}_r = \vec{0}$ with $\vec{w}_i \in W_i$ for all i, we have $\vec{w}_1 = \cdots = \vec{w}_r = \vec{0}$.
   c) Let α_i be a basis of W_i, then $\alpha_1 \cup \cdots \cup \alpha_m$ is linearly independent.
2. $W_1 \oplus \cdots \oplus W_r = V$ if $\dim W_1 + \cdots + \dim W_r = \dim V$.

Proof of (1) $V = \ker(T^k) \bigoplus F\vec{v}_i$.

First, we claim that the right hand side of (1) is a direct sum. In other words, we have to show

$$\ker(T^k) + \sum_{i=1}^{m} F\vec{v}_i = \ker(T^k) \bigoplus_{i=1}^{m} F\vec{v}_i.$$

Suppose $\vec{0} = \vec{v} + \sum_{i=1}^{m} a_i\vec{v}_i$ for some $\vec{v} \in \ker(T^k)$ and $a_i \in F$. Apply T^k to the both hand sides, we obtain

$$\vec{0} = \vec{0} + \sum_{i=1}^{m} a_i T^k(\vec{v}_i).$$

Since $\{T^k(\vec{v}_i)\}$ is linearly independent, we have $a_1 = \cdots = a_m = 0$ which implies that $\vec{v} = 0$. The claim is proved.

Proof of (1) $V = \ker(T^k) \bigoplus F\vec{v}_i$.

By the dimension formula, we have

$$\dim V = \dim \ker(T^k) + \dim \mathrm{I}m(T^k) = \dim \ker(T^k) + m.$$

The statement (1) is proved.

Proof of (2) $\sum \mathrm{cyclic}(\vec{v}_i) = \bigoplus \mathrm{cyclic}(\vec{v}_i)$.

Note that the subspace $\mathrm{cyclic}(\vec{v}_i)$ admits a basis

$$\alpha_i = \{\vec{v}_i, T(\vec{v}_i) \cdots, T^k(\vec{v}_i)\}.$$

It remains to show that $\alpha_1 \cup \cdots \cup \alpha_m$ is linearly independent. Suppose not, then there exists some non-trivial linear relation, namely,

$$\vec{0} = \sum_{i=0}^{k} \sum_{j=1}^{m} a_{ij} T^i(\vec{v}_j).$$

Proof of (2) $\sum \mathrm{cyclic}(\vec{v}_i) = \bigoplus \mathrm{cyclic}(\vec{v}_i)$.

Let r be the smallest integer satisfying $a_{rj} \neq 0$ for some j.
Applying T^{k-r} to the above equation, we obtain

$$\vec{0} = \sum_{i=r}^{k} \sum_{j=1}^{m} a_{ij} T^{k-r+i}(\vec{v}_j) = \sum_{j=1}^{m} a_{rj} T^k(\vec{v}_j) \quad (\text{since } T^{k+1} \equiv 0)$$

Thus, we obtain a non-trivial linear relation of $T^k(\vec{v}_1), \cdots, T^k(\vec{v}_m)$, which is a contradiction.

Proof of (3)

Theorem

(3) *There exists some T-invariant subspace W such that* $V = W \bigoplus_{i=1}^{m} \mathrm{cyclic}(\vec{v}_i)$.

We will prove the statement by induction on the index $k+1$.

When $k = 0$, T^0 is the identity map and $\mathrm{I}m(T^0) = V$. In this case, $\bigoplus_{i=1}^{m} \mathrm{cyclic}(\vec{v}_i) = V$, so we can just let $W = \{\vec{0}\}$.

Suppose the statement (3) holds for all nilpotent linear transformation of index $< k+1$.

Proof of (3)

Let $\tilde{T}$ be the restriction of T on $\tilde{V} = \ker(T^k)$. Then $\tilde{T}$ is a nilpotent linear transformation on $\tilde{V}$ of index k and the statement (3) holds for $\tilde{T}$. First, let us find a special basis of $\mathrm{I}m(\tilde{T}^{k-1})$.

Note that $T(\vec{v}_1), \cdots, T(\vec{v}_m) \in \tilde{V}$ and

$$\left\{ \tilde{T}^{k-1}(T(\vec{v}_1)), \cdots, \tilde{T}^{k-1}(T(\vec{v}_m)) \right\} = \left\{ T^k(\vec{v}_1), \cdots, T^k(\vec{v}_m) \right\}$$

is a linearly independent subset of $\mathrm{I}m(\tilde{T}^{k-1})$. One can extend this subset to a basis of $\mathrm{I}m(\tilde{T}^{k-1})$, namely

$$\left\{ \tilde{T}^{k-1}(T(\vec{v}_1)), \cdots, \tilde{T}^{k-1}(T(\vec{v}_m)) \right\} \bigcup \left\{ \tilde{T}^{k-1}(\vec{u}_1), \cdots, \tilde{T}^{k-1}(\vec{u}_r) \right\}$$

for some $\vec{u}_1, \cdots, \vec{u}_r$ in $\tilde{V}$.

Proof of (3)

By (3), there exists some $\tilde{T}$-invariant space W_0 (which is also T-invariant) such that

$$\ker(T^k) = \tilde{V} = W_0 \bigoplus_{j=1}^{r} \mathrm{cyclic}(\vec{u}_j) \bigoplus_{i=1}^{m} \mathrm{cyclic}\left(T(\vec{v}_i)\right)$$

Finally, we have

$$\begin{aligned} V &= \ker(T^k) \bigoplus F\vec{v}_i \\ &= W_0 \bigoplus_{j=1}^{r} \mathrm{cyclic}(\vec{u}_j) \bigoplus_{i=1}^{m} \left(\mathrm{cyclic}\left(T(\vec{v}_i)\right) \oplus F\vec{v}_i\right) \\ &= W \bigoplus_{i=1}^{m} \mathrm{cyclic}\left(\vec{v}_i\right) \end{aligned}$$

Theorem

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(3) *There exists some T-invariant subspace W such that $V = W \bigoplus_{i=1}^{m} \mathrm{cyclic}(\vec{v}_i)$.*

Theorem

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