Linear Algebra II
Positive Definite Quadratic Forms

Ming-Hsuan Kang

Positive Definite Quadratic Forms

Let $Q(\vec{x}^t)$ be a real quadratic form on $\mathbb{R}^n$.

- $Q(\vec{x}^t)$ is called positive semi-definite if $Q(\vec{x}^t) \geq 0$ for all $\vec{x}$;
- $Q(\vec{x}^t)$ is called positive definite if $Q(\vec{x}^t) > 0$ for all nonzero $\vec{x}$.
- $Q(\vec{x}^t)$ is called negative semi-definite if $Q(\vec{x}^t) \leq 0$ for all $\vec{x}$;
- $Q(\vec{x}^t)$ is called negative definite if $Q(\vec{x}^t) < 0$ for all $\vec{x}$;

Example

For any inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$, $Q(\vec{x}^t) := \langle \vec{x}, \vec{x} \rangle$ is a positive definite quadratic form.

Positive Definite Quadratic Forms

Theorem

Let $Q(\vec{x}^t) = \vec{x}^t A\vec{x}$ be a real quadratic form on $\mathbb{R}^n$ where A is a symmetric matrix. The following are equivalent.

(a) *$Q(\vec{x}^t)$ is positive definite.*

(b) *All eigenvalues of A are positive. (Such A is also called positive definite.)*

(c) *$Q(\vec{x}^t)$ has the unique absolute minimum at $\vec{0}^t$.*

Proof

(a) $Q(\vec{x}^t)$ is positive definite. $\Rightarrow$ (b) All eigenvalues of A are positive.

Suppose $Q(\vec{x}^t)$ is positive definite. Let $\vec{v}$ be an eigenvector corresponding to a given eigenvalue λ of A. Then

$$0 < Q(\vec{v}^t) = \vec{v}^t A \vec{v} = \vec{v}^t(\lambda \vec{v}) = \lambda \|\vec{v}\|^2.$$

We conclude that $\lambda > 0$.

Proof

(b) All eigenvalues of A are positive. $\Rightarrow$ (c) $Q(\vec{x}^t)$ has the unique absolute minimum at $\vec{0}^t$.

Suppose all eigenvalues of A are positive. There there exists an orthonormal eigenbasis $\alpha = \{\vec{v}_1, \cdots, \vec{v}_n\}$ such that $A\vec{v}_i = \lambda_i \vec{v}_i$ and λ_i is positive for all i. For any non-zero vector $\vec{x}$ in $\mathbb{R}^n$, let $\vec{x} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$. Then some of $\{a_i\}$ are non-zero and

$$\begin{aligned} Q(\vec{x}) &= (a_1\vec{v}_1 + \cdots + a_n\vec{v}_n)^t A(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) \\ &= (a_1\vec{v}_1 + \cdots + a_n\vec{v}_n)^t(\lambda_1 a_1\vec{v}_1 + \cdots + \lambda_n a_n\vec{v}_n) \\ &= \lambda_1 a_1^2 + \cdots + \lambda_n a_n^2 > 0 = Q(\vec{0}^t). \end{aligned}$$

Therefore, $Q(\vec{x}^t)$ has the unique absolute minimum at $\vec{0}^t$.

Proof

(c) $Q(\vec{x}^t)$ has the unique absolute minimum at $\vec{0}^t$. $\Rightarrow$ (a) $Q(\vec{x}^t)$ is positive definite.

Suppose $Q(\vec{x}^t)$ has the unique absolute minimum at $\vec{0}^t$. Then for any non-zero $\vec{x}$ in $\mathbb{R}^n$,

$$Q(\vec{x}^t) > Q(\vec{0}^t) = 0.$$

Hence, $Q(\vec{x}^t)$ is positive definite.

Negative Definite Quadratic Forms

Theorem

Let $Q(\vec{x}^t) = \vec{x}^t A \vec{x}$ be a real quadratic form on $\mathbb{R}^n$ where A is a symmetric matrix. The following are equivalent.

(a) *$Q(\vec{x}^t)$ is negative definite.*

(b) *All eigenvalues of A are negative. (Such A is also called negative definite.)*

(c) *$Q(\vec{x}^t)$ has the unique absolute maximum at $\vec{0}^t$.*

Theorem

Let $Q(\vec{x}^t) = \vec{x}^t A\vec{x}$ *be a real quadratic form on* $\mathbb{R}^n$ *where* A *is a symmetric matrix. The following are equivalent.*

(a) $Q(\vec{x}^t)$ *is positive semi-definite.*

(b) *All eigenvalues of* A *are non-negative. (Such* A *is also called positive semi-definite.)*

(c) $Q(\vec{x}^t)$ *has the absolute minimum at* $\vec{0}^t$*.*

Negative Semi-definite Quadratic Forms

Theorem

Let $Q(\vec{x}^t) = \vec{x}^t A \vec{x}$ be a real quadratic form on $\mathbb{R}^n$ where A is a symmetric matrix. The following are equivalent.

(a) $Q(\vec{x}^t)$ *is negative semi-definite.*

(b) *All eigenvalues of A are non-positive. (Such A is also called negative semi-definite.)*

(c) $Q(\vec{x}^t)$ *has the absolute maximum at $\vec{0}^t$.*

Let n_1 (resp. n_{-1}) be the number of positive (resp. negative) eigenvalues of $Q(\vec{x}^t)$. Recall that $n_1 - n_{-1}$ is the signature of Q and $n_1 + n_{-1}$ is the rank of Q.

Theorem

The number n_1 equals $\max\{\dim W | Q \text{ is positive definite on } W\}$.

Corollary

If two quadratic forms are equivalent, then they have the same number of positive eigenvalues.

Corollary

Two quadratic forms are equivalent if and only if they have the same signature and the same rank.

Proof

Suppose Q is positive definite on a subspace W of $\mathbb{R}^n$. Let U be the subspace spanned by eigenvectors corresponding to non-positive eigenvalues. Then $\dim U = n - n_1$ and $Q\big|_U$ is negative semi-definite. Since $Q\big|_{W\cap U}$ is both positive and semi-definite negative, $W \cap U = \{\vec{0}\}$, which implies that $W + U = W \oplus U$. Therefore,

$$\dim W \leq \dim \mathbb{R}^n - \dim U = n - (n - n_1) = n_1.$$

Conversely, let W_0 be the subspace spanned by eigenvectors corresponding to positive eigenvalues. Then Q is positive definite on W_0 and

$$n_1 = \dim W_0 \leq \max\{\dim W | Q \text{ is positive definite on } W\}.$$

Criterion of Positive Definiteness

Given a real symmetric matrix A of size n, let A_k be a submatrix of A of size k, such that the (i,j)-th entry of A and A_k are the same for all $1 \leq i, j \leq k$. The matrix A_k is also called the leading principal minor of order k. Note that every A_k is still a real symmetric matrix.

Theorem (Sylvester's criterion)

A real symmetric matrix is positive definite if and only if the determinant of each leading principal minor is positive.

Remark: A real symmetric matrix A is negative definite if and only if $-A$ is positive definite.

Proof

Let A be a real symmetric matrix of size n and $Q(\vec{x}^t) = \vec{x}^t A\vec{x}$ where $\vec{x}^t = (x_1, \cdots, x_n)$. Let $\mathbb{R}^k$ be the subspace of $\mathbb{R}^n$ spanned by the first k standard basis. Note that

$$\begin{aligned}
Q(x_1, \cdots, x_k, 0, \cdots) &= (x_1 \cdots x_k\, 0 \cdots)A \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ \vdots \end{pmatrix} \\
&= (x_1, \cdots, x_k)A_k \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}.
\end{aligned}$$

Therefore, the restriction of Q on $\mathbb{R}^k$ can be identified with $Q_k(x_1, \cdots, x_k) = (x_1 \cdots x_k)A_k \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$.

Proof

Suppose A is positive-definite. Then $Q(\vec{x})$ is positive-definite and so is $Q_k(x)$. Therefore, A_k is also positive-definite. Especially, $\det(A_k)$ is equal to the product of all eigenvalues, which is positive.

Conversely, suppose $\det(A_k)$ is positive for all k. We shall prove A is positive-definite by induction of n. When $n = 1$, it is trivial. Now suppose the statement holds for matrices of size less then n. Then A_{n-1} is positive-definite by induction. Thus $Q_{n-1}(x)$ is positive definite, which means that $Q(x)$ is positive definite on $\mathbb{R}^{n-1}$. By the previous theorem, A has at least $n - 1$ many positive eigenvalues. Together with the condition that $\det(A_n) = \det(A)$ is positive, we conclude that all eigenvalues are positive and A is positive-definite.

Example

For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, A is positive definite if and only if

$$a_{11} > 0 \quad \text{and} \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0.$$

For $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, A is positive definite if and only if

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0 \quad \text{and} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0.$$

Example

For $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$, A is negative definite if and only if

$$-a_{11} > 0 \quad \text{and} \quad \left|\begin{smallmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{smallmatrix}\right| > 0,$$

or equivalently

$$a_{11} < 0 \quad \text{and} \quad \left|\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right| > 0.$$

For $A = \left(\begin{smallmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{smallmatrix}\right)$, A is negative definite if and only if

$$-a_{11} > 0, \quad \left|\begin{smallmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{smallmatrix}\right| > 0 \quad \text{and} \quad \left|\begin{smallmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{smallmatrix}\right| > 0,$$

or equivalently

$$a_{11} < 0, \quad \left|\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right| > 0 \quad \text{and} \quad \left|\begin{smallmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{smallmatrix}\right| < 0.$$

Example

Example

Find all local minimum and local maximum of
$f(x, y, z) = x^2 + 4y^2 + z^2 - 2xyz$.

Step 1. Find all critical points. Solve
$(0, 0, 0) = \nabla f = (2x - yz, 8y - 2xz, 2z - 2xy)$, we obtain
$(x, y, z) = (0, 0, 0), (\pm 2, -1, \mp 2)$, and $(\pm 2, 1, \pm 2)$.

Step 2. Compute the Hessian for each critical points. We have

$$H(f) = \begin{pmatrix} 2 & -2z & -2y \\ -2z & 8 & -2x \\ -2y & -2x & 2 \end{pmatrix}.$$

Example

- For $(x, y, z) = (0, 0, 0)$, $H(f) = \begin{pmatrix} 2 & & \\ & 8 & \\ & & 2 \end{pmatrix}$, so $f(0, 0, 0)$ is a local minimum.
- For $(x, y, z) = (-2, -1, 2)$, $H(f) = \begin{pmatrix} 2 & -4 & 2 \\ -4 & 8 & 4 \\ 2 & 4 & 2 \end{pmatrix}$, the determinants of the leading principal minors are

$$2, \quad \begin{vmatrix} 2 & -4 \\ -4 & 8 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & -4 & 2 \\ -4 & 8 & 4 \\ 2 & 4 & 2 \end{vmatrix} = -128.$$

  Therefore, $f(-2, -1, 2)$ is neither a local minimum nor a local maximum.
- Similarly, the other three critical points do not give local minima nor local maxima.