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Linear Algebra II - Homework 3

In this homework, T is a linear transformation from V to V.

1. For

(1) $A = \begin{pmatrix} 0 & 9 & 0 & 0 \\ 2 & 0 & 5 & -1 \\ 0 & -4 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}$ (2) $A = \begin{pmatrix} 2 & 5 & 1 & 0 \\ -1 & -2 & 0 & -1 \\ 1 & 0 & -2 & 5 \\ 0 & -1 & -1 & 2 \end{pmatrix}$ (3) $A = \begin{pmatrix} 8 & 12 & -1 & 5 \\ -4 & -6 & 1 & -2 \\ 4 & 4 & 1 & 7 \\ -3 & -4 & 0 & -3 \end{pmatrix}$

- Find the dot diagram.
- Find the explicit cyclic subspace decomposition corresponding to A.

2. Let $A = \begin{pmatrix} -10 & 26 & 6 & -19 & -12 & 23 \\ -7 & 12 & 6 & -12 & 4 & 21 \\ -2 & -4 & 3 & -1 & 16 & 11 \\ 0 & -6 & 1 & 2 & 12 & 4 \\ -4 & 8 & 3 & -7 & 0 & 11 \\ 2 & -2 & -2 & 3 & -4 & -7 \end{pmatrix}$, $A^2 = \begin{pmatrix} 0 & 0 & 13 & -13 & 0 & 13 \\ 0 & 0 & 6 & -6 & 0 & 6 \\ 0 & 0 & -2 & 2 & 0 & -2 \\ 0 & 0 & -3 & 3 & 0 & -3 \\ 0 & 0 & 4 & -4 & 0 & 4 \\ 0 & 0 & -1 & 1 & 0 & -1 \end{pmatrix}$, and $\ker(A) =$ $\operatorname{Span}\left\{ \begin{pmatrix} 6 \\ 1 \\ -2 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\}$. Find the explicit cyclic subspace decomposition corresponding to A.

3. Let T be a nilpotent linear transform on $\mathbb{R}^6$.
   - Suppose the index of T is equal to 4, i.e. $T^4 \equiv 0$ and $T^3 \not\equiv 0$. Find all possible dot diagrams of T.
   - Suppose $T^3 \equiv 0$. Find all possible dot diagrams of T.

4. Let T be a nilpotent linear transformation on V of index k, i.e. $T^k \equiv 0$ and $T^{k-1} \not\equiv 0$. Show that
   (1) $\dim \ker(T^i) < \dim \ker(T^{i+1})$ for all $i < k$.
   (2) $\dim \ker(T^i) = \dim \ker(T^{i+1})$ for all $i \geq k$.

5. Without using the cyclic subspace decomposition, show that for every nilpotent linear transformation T on V, there exists a basis α of V such that $\operatorname{Rep}_\alpha(T)$ is an upper triangular matrix.

6. For a linear transformation T on V, if $f_T(x)$ splits, use Problem 4. to show that there exists a basis α such that $\operatorname{Rep}_\alpha(T)$ is an upper triangular matrix.