LA HW9 SVD

Informarion

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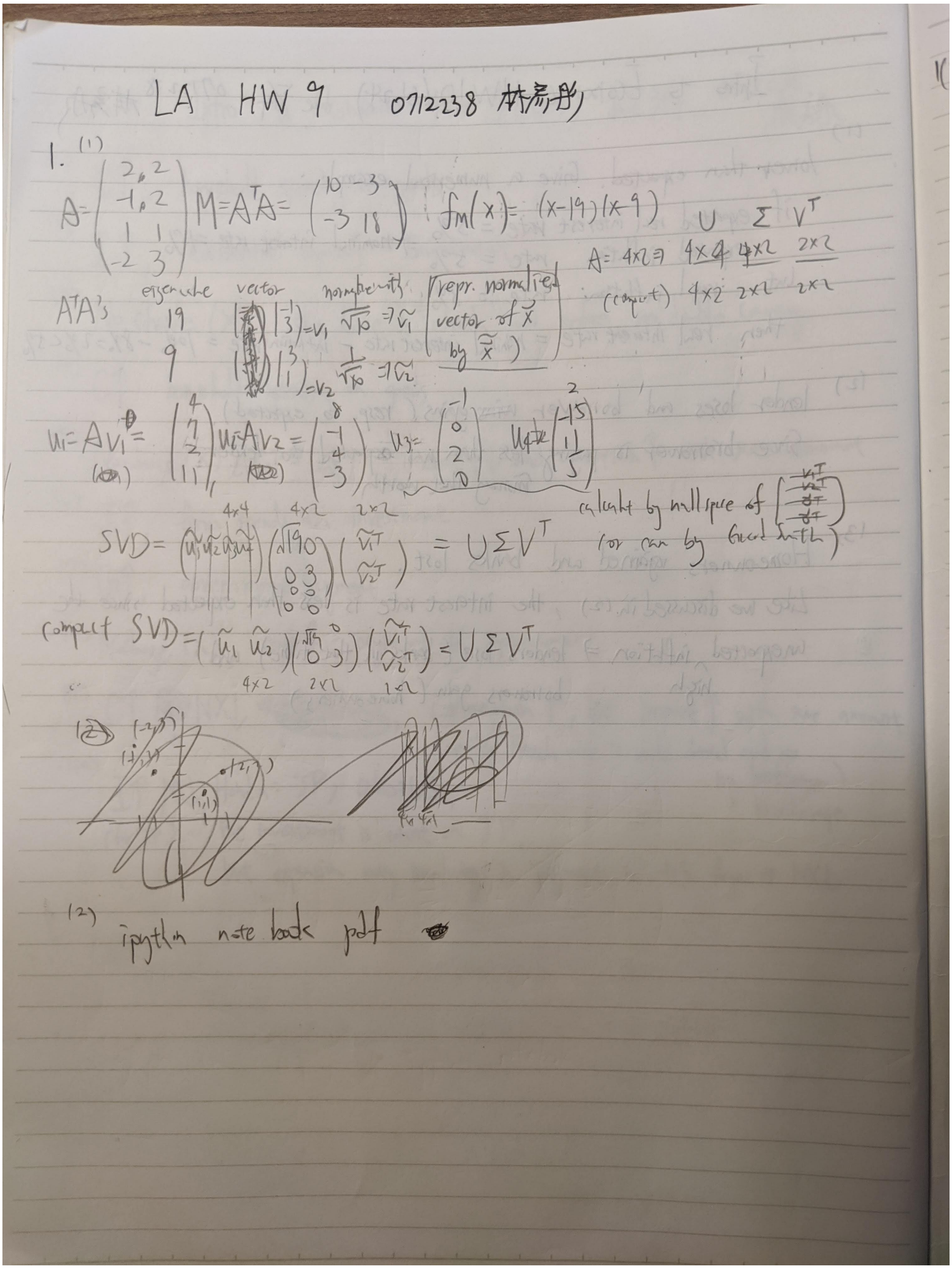
Q1



LA HW 9 0712238 林彥彤

1. (1)

$$A=\begin{pmatrix}2&2\\-1&2\\1&1\\-2&3\end{pmatrix}\quad M=A^TA=\begin{pmatrix}10&-3\\-3&18\end{pmatrix}\quad f_M(x)=(x-19)(x-9)$$

$A = 4\times2 \Rightarrow$ U 4×4, Σ 4×2, V^T 2×2

(compact) 4×2 2×2 2×2

A^TA's eigenvalue vector normalize it

19 $\begin{pmatrix}-1\\3\end{pmatrix}=v_1$ $\frac{1}{\sqrt{10}}\Rightarrow\tilde{v}_1$

9 $\begin{pmatrix}3\\1\end{pmatrix}=v_2$ $\frac{1}{\sqrt{10}}\Rightarrow\tilde{v}_2$

(repr. normalized vector of $\vec{x}$ by $\tilde{x}$)

$u_1=Av_1=\begin{pmatrix}4\\7\\2\\11\end{pmatrix}$, $u_2=Av_2=\begin{pmatrix}8\\-1\\4\\-3\end{pmatrix}$, $u_3=\begin{pmatrix}-1\\0\\2\\0\end{pmatrix}$, $u_4=\begin{pmatrix}-15\\1\\1\\5\end{pmatrix}$

calculate by null space of $\begin{pmatrix}\tilde{u}_1^T\\\tilde{u}_2^T\end{pmatrix}$ (or can by Gram Smith)

$$SVD=\underbrace{\begin{pmatrix}\tilde{u}_1&\tilde{u}_2&\tilde{u}_3&\tilde{u}_4\end{pmatrix}}_{4\times4}\underbrace{\begin{pmatrix}\sqrt{19}&0\\0&3\\0&0\\0&0\end{pmatrix}}_{4\times2}\underbrace{\begin{pmatrix}\tilde{v}_1^T\\\tilde{v}_2^T\end{pmatrix}}_{2\times2}=U\Sigma V^T$$

$$\text{compact } SVD=\underbrace{\begin{pmatrix}\tilde{u}_1&\tilde{u}_2\end{pmatrix}}_{4\times2}\underbrace{\begin{pmatrix}\sqrt{19}&0\\0&3\end{pmatrix}}_{2\times2}\underbrace{\begin{pmatrix}\tilde{v}_1^T\\\tilde{v}_2^T\end{pmatrix}}_{2\times2}=U\Sigma V^T$$

(2)

(2) ipython note book pdf

(3) Substruct by mean vector and do SVD $\quad \begin{pmatrix}10 & -3\\ -3 & 2\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix}\frac{1}{3} = \begin{pmatrix}27\\7\end{pmatrix}$

$$\Rightarrow A' = A - \text{mean}(A) = \begin{pmatrix}2 & 2\\ -1 & 2\\ 1 & 1\\ -2 & 3\end{pmatrix} - \begin{pmatrix}0 & 2\\ 0 & 2\\ 0 & 2\\ 0 & 2\end{pmatrix} = \begin{pmatrix}2 & 0\\ -1 & 0\\ 1 & -1\\ -2 & 1\end{pmatrix}$$

$$A'^T A' = \begin{pmatrix}10 & -3\\ -3 & 2\end{pmatrix} \qquad f_{A'^TA'}(x) = (x-11)(x-1)$$

$$\begin{pmatrix}-1 & -3\\ -3 & -9\end{pmatrix} v_1 = \vec{0} \quad v_1 = \begin{pmatrix}3\\-1\end{pmatrix} \qquad \begin{pmatrix}9 & -3\\ -3 & 1\end{pmatrix} v_2 = \vec{0}, \quad v_2 = \begin{pmatrix}1\\3\end{pmatrix}$$

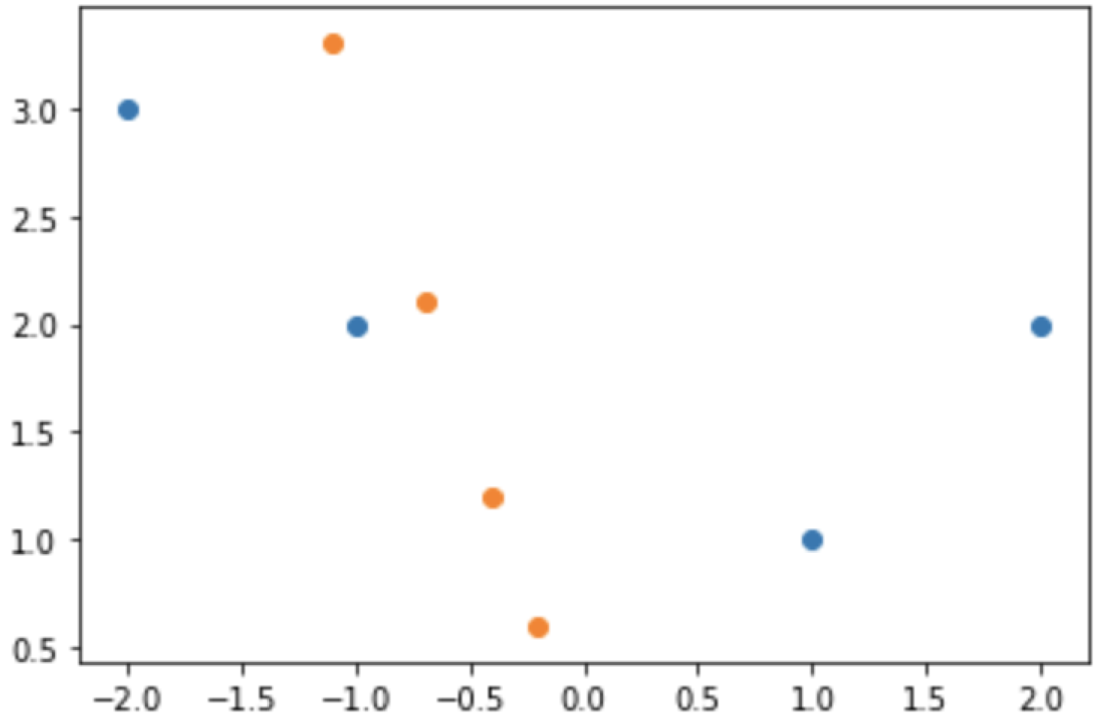
$$\cancel{u_1 = \begin{pmatrix}27\\7\end{pmatrix} \ u_2 = \begin{pmatrix}1\\3\end{pmatrix}} \quad u_1 = \begin{pmatrix}6\\-3\\2\\-5\end{pmatrix} \quad u_2 = \begin{pmatrix}2\\-1\\-2\\1\end{pmatrix} \quad u_3 = \begin{pmatrix}1\\2\\0\\0\end{pmatrix} \quad u_4 = \begin{pmatrix}-12\\6\\5\\20\end{pmatrix}$$

first principal component $= v_1 = \begin{pmatrix}3\\-1\end{pmatrix}$

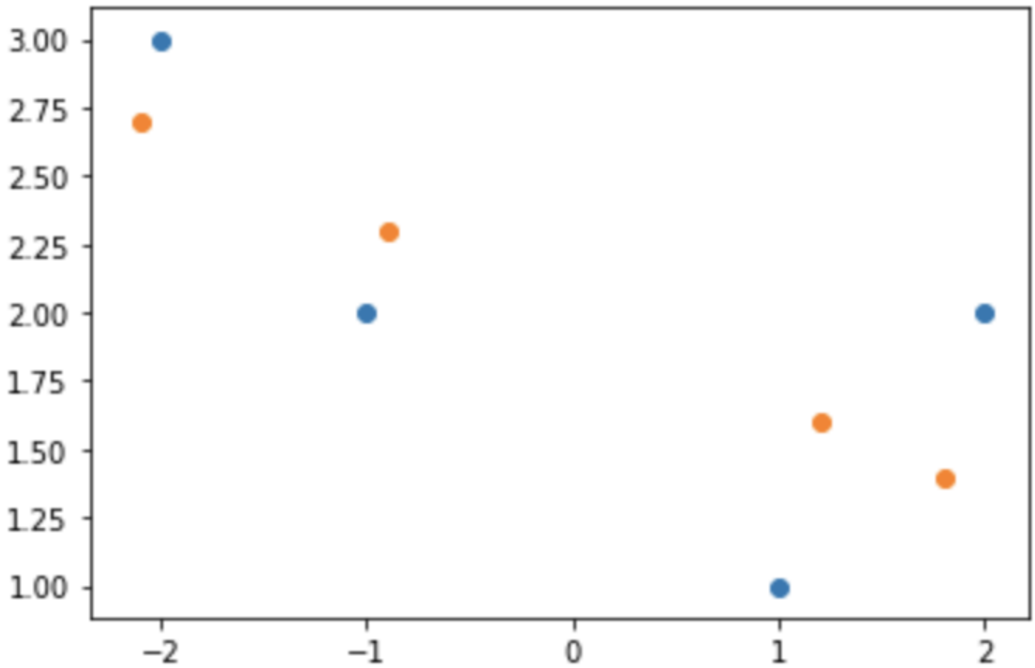
(4)

see ipython pdf

SVD projection



PCA projection



Q2

2. $X = \begin{pmatrix} 0 & -1 \\ 4 & -2 \\ 3 & -2 \\ 1 & 0 \\ 2 & 0 \end{pmatrix}$ $\quad X^T X = \begin{pmatrix} 30 & -14 \\ -14 & 9 \end{pmatrix}$

$f_{X^TX} = (x-30)(x-9) - 196 = x^2 - 39x + 74 = (x-37)(x-2)$

$\lambda = 37 \quad \begin{pmatrix} -7 & -14 \\ -14 & -28 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \hat{v}_1 = \frac{v_1}{\sqrt{5}} \quad u_1 = X v_1 = \begin{pmatrix} 1 \\ 10 \\ 8 \\ 2 \\ 4 \end{pmatrix} \quad \tilde{u}_1 = \frac{u_1}{|u_1|}$

$\lambda = 2 \quad \begin{pmatrix} 28 & -14 \\ -14 & 7 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \hat{v}_2 = \frac{v_2}{\sqrt{5}} \quad u_2 = X v_2 = \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 2 \end{pmatrix} \quad \tilde{u}_2 = \frac{u_2}{|u_2|}$

(1)

$X = \underbrace{\begin{pmatrix} \tilde{u}_1 & \tilde{u}_2 \end{pmatrix}}_{U_{5\times 2}} \underbrace{\begin{pmatrix} 37 & 0 \\ 0 & 2 \end{pmatrix}}_{\Sigma_{2\times 2}} \underbrace{\begin{pmatrix} \hat{v}_1^t \\ \hat{v}_2^t \end{pmatrix}}_{V^T_{2\times 2}}$

(2)

graph is displayed with ipython notebook

(3)

$X' = X - \begin{pmatrix} \bar{x} \\ \vdots \\ \bar{x} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 2 & -1 \\ 1 & -1 \\ -1 & 1 \\ 0 & 1 \end{pmatrix} \quad X'^T X' = \begin{pmatrix} 10 & -4 \\ -4 & 4 \end{pmatrix}$

$f_{X'^TX'} = (x-12)(x-2)$

$\lambda = 12 \quad \begin{pmatrix} -2 & -4 \\ -4 & -8 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \hat{v}_1 = \frac{v_1}{\sqrt{5}} \quad \cancel{u_1 = X' v_1 = (\;)}$

$\lambda = 2 \quad \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \hat{v}_2 = \frac{v_2}{\sqrt{5}} \quad \cancel{u_2 = X v_2 = (\;)}$

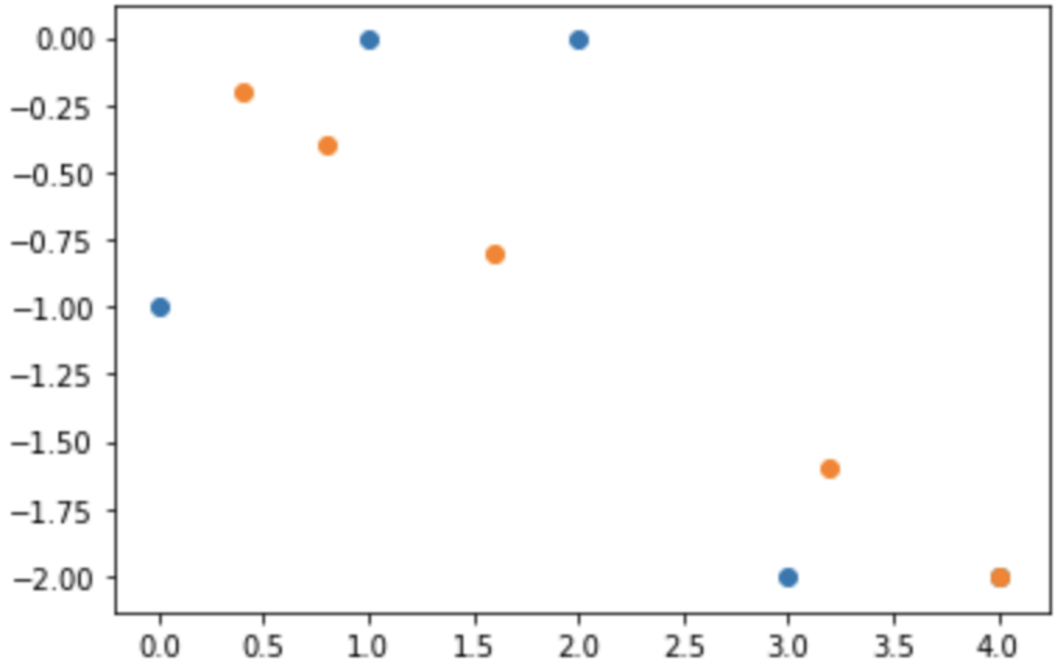
first PC $= v_1$

(4)

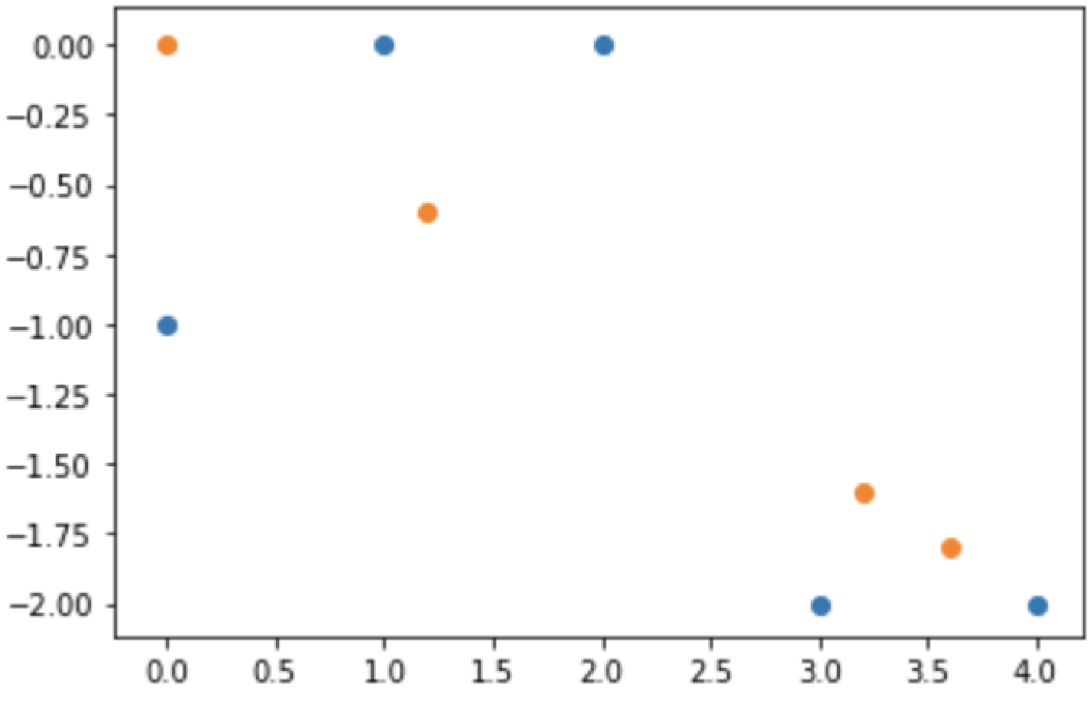
graph is as displayed in ipython notebook

- Note: the $\sum$ diagonal should be $\sqrt{32}$, $\sqrt{2}$

SVD projection



PCA projection



Q3

3. (1) want to show $\forall\ \alpha = \{\vec{w_1}, \vec{w_2}\}$, $\beta = \{\vec{v_1}, \vec{v_2}\}$, α, β are orthogonal basis of W

$Q(W) = Q(\vec{w_1}) + Q(\vec{w_2}) = Q(\vec{v_1}) + Q(\vec{v_2})$ since $w_1^t Q w_1$ 2-D subspace of $\mathbb{R}^n$

where $Q(\vec{x}) = \vec{x}^t A \vec{x}$, A is real symmetric ($A^t = A$)

(A is real symmetric then is orthogonal diagonizable let $A = P \Lambda P^T$)
in which we chose a Λ that eigenvalues are decreasing from Λ_{11} to $\Lambda_{n,n}$

let's repr. $\vec{v_1}, \vec{v_2}$ in α, let ~~$\vec{v_1} = a\vec{w_1} + b\vec{w_2}$~~ ~~in which $|\vec{v_1}|^2 = a^2 + b^2 = 1$~~
~~$\vec{v_2} = c\vec{w_1} + d\vec{w_2}$~~ ~~$|\vec{v_2}|^2 = c^2 + d^2 = 1$~~
~~$|\vec{v_1}||\vec{v_2}| = ac + bd = 0$~~
$\Rightarrow \vec{v_1} = \cos\theta\, \vec{w_1} + \sin\theta\, \vec{w_2}$ for some θ
$\vec{v_2} = -\sin\theta\, \vec{w_1} + \cos\theta\, \vec{w_2}$

$$Q(\vec{v_1}) + Q(\vec{v_2}) = (a\vec{w_1} + b\vec{w_2})^t A (a\vec{w_1} + b\vec{w_2}) + (c\vec{w_1} + d\vec{w_2})^t A (c\vec{w_1} + d\vec{w_2})$$

linearality

$$= a\vec{w_1}^t A a\vec{w_1} + b\vec{w_2} A a\vec{w_1} + a\vec{w_1} A b\vec{w_2} + b\vec{w_2} A b\vec{w_2}$$
$$+ c\vec{w_1}^t A c\vec{w_1} + d\vec{w_2} A c\vec{w_1} + c\vec{w_1} A d\vec{w_2} + d\vec{w_2} A d\vec{w_2}$$
$$= (a^2 + c^2)\, \vec{w_1}^t A \vec{w_1} + (b^2 + d^2)\, \vec{w_2}^t A \vec{w_2} + (ab + cd)(\vec{w_2} A \vec{w_1} + \vec{w_1} A \vec{w_2})$$
$$= 1\, \vec{w_1}^t A \vec{w_1} + 1\, \vec{w_2}^t A \vec{w_2} + 0(\ldots)$$
$$= Q(\vec{w_1}) + Q(\vec{w_2})$$

Q.E.D.

$\cos\theta \quad \sin\theta$
$\cos\phi \quad \sin\phi$ (2)

$$Q(W) = \vec{w_1}^t A \vec{w_1} + \vec{w_2}^t A \vec{w_2}$$

let W be any subspace of 2-d

$$= \vec{w_1}^t P \Lambda P^T \vec{w_1} + \vec{w_2}^t P \Lambda P^t \vec{w_2}$$
$$\le \underbrace{\vec{v_1}^t P \lambda_1 P^T \vec{v_1}}_{\lambda_1} + \vec{w_2} P \Lambda P^t \vec{w_2}$$

cannot pick λ_1, v_1

$$\le \vec{v_1}^t P \lambda_1 P^t \vec{v_1} + \underbrace{\vec{v_2}^t P \lambda_2 P^T \vec{v_2}}_{\lambda_2}$$

, where $P^t v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $P^t v_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$

note that Λ eigenvalue $\Lambda_{11} \ge \Lambda_{22} \ge \ldots \ge \Lambda_{n,n}$

$$= Q(V)$$

where V be the subspace spanned by v_1, v_2, eigenvector corresponding to λ_1, λ_2

Q4

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Compression with first 50 singular values
Compression rate = 0.09765625
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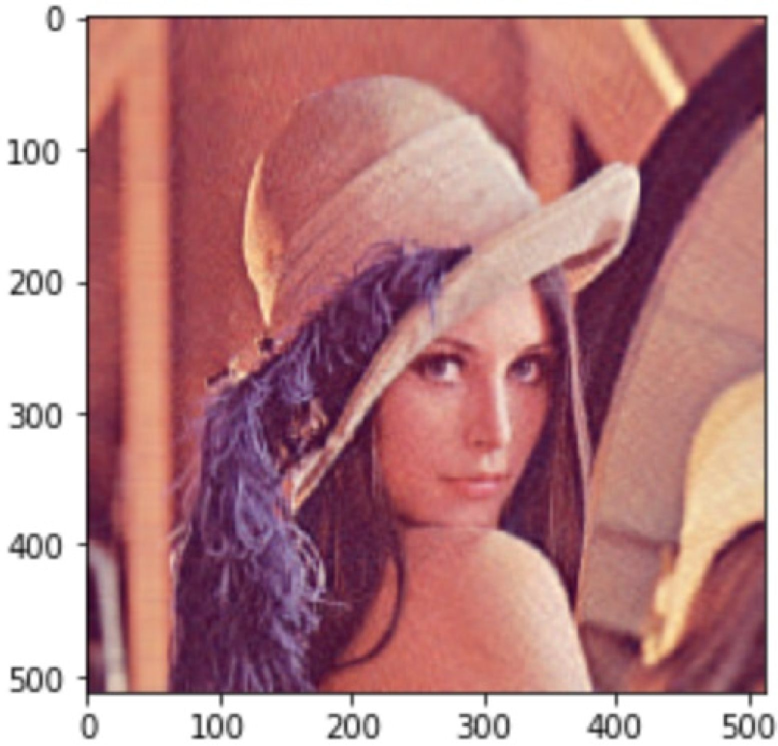

Code

- Code for Q1-2
  - “Homework 9 Calculation and Plotting.pdf” in LA_HW9_0712238.zip
- Code for Q4
  - “Homework 9 SVD compression.pdf” in LA_HW9_0712238.zip