# Linear Algebra II
# Inner product

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# Inner product

In $\mathbb{R}^n$, we have the standard inner product given by

$$\left\langle \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right\rangle := \begin{pmatrix} a_1, \cdots, a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i=1}^{n} a_i b_i,$$

which is a very useful extra structure of vector spaces. For instance, we define $\vec{v}$ and $\vec{w}$ are orthogonal if $\langle \vec{v}, \vec{w} \rangle = 0$.

**Theorem**

*A nonzero mutually orthogonal set of $\mathbb{R}^n$ is linearly independent.*

**Theorem**

*Let $\alpha = \{\vec{v}_1, \cdots, \vec{v}_n\}$ be an orthogonal basis of $\mathbb{R}^n$. Then for any $\vec{v} \in \mathbb{R}^n$,*

$$\mathrm{Rep}_\alpha(\vec{v}) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \text{ where } a_i = \frac{\langle \vec{v}, \vec{v}_i \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle}.$$

*Moreover, if $\alpha$ is orthonormal, then $a_i = \langle \vec{v}, \vec{v}_i \rangle$.*

# Inner product on $\mathbb{C}^n$

In $\mathbb{C}^n$, one can define the standard inner product as

$$\left\langle \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right\rangle := \sum_{i=1}^{n} a_i \bar{b}_i.$$

Here we use $\bar{b}_i$ instead of $b_i$ since we would like to have

$$\left\langle \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right\rangle = \sum_{i=1}^{n} a_i \bar{a}_i = \sum_{i=1}^{n} |a_i|^2.$$

Note that in this case, we have

$$\left\langle \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right\rangle = \overline{\left\langle \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right\rangle}$$

In $\mathbb{C}^n$, we say define $\vec{v}$ and $\vec{w}$ are orthogonal if $\langle \vec{v}, \vec{w} \rangle = 0$. As the case of $\mathbb{R}^n$, we have the following two results.

**Theorem**

*A nonzero mutually orthogonal set of $\mathbb{C}^n$ is linearly independent.*

**Theorem**

*Let $\alpha = \{\vec{v}_1, \cdots, \vec{v}_n\}$ be an orthogonal basis of $\mathbb{C}^n$. Then for any $\vec{v} \in \mathbb{C}^n$,*

$$\mathrm{Rep}_\alpha(\vec{v}) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \ \text{where } a_i = \frac{\langle \vec{v}, \vec{v}_i \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle}.$$

*Moreover, if $\alpha$ is orthonormal, then $a_i = \langle \vec{v}, \vec{v}_i \rangle$.*

A complex orthonormal basis is always called an unitary basis.

# Abstract Inner product

Let $V$ be a vector space over $F = \mathbb{R}$ or $\mathbb{C}$, an inner product $\langle \cdot, \cdot \rangle$ is a map from $V \times V$ to $F$ satisfying the following conditions.

- $\langle a\vec{v} + \vec{w}, \vec{u} \rangle = a\langle \vec{v}, \vec{u} \rangle + \langle \vec{w}, \vec{u} \rangle$ for all $a \in F$, $\vec{v}, \vec{w}, \vec{u} \in V$.
- $\langle \vec{v}, \vec{u} \rangle = \overline{\langle \vec{u}, \vec{v} \rangle}$ for all $\vec{v}, \vec{u} \in V$.
- $\langle \vec{v}, \vec{v} \rangle \geq 0$ for all $\vec{v} \in V$ and the equality holds only when $\vec{v} = 0$.

A vector space together with an inner product is called an inner product space.

**Remark**. $\langle \vec{u}, a\vec{v} + \vec{w} \rangle = \bar{a}\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$.

Inner products only exist in vector spaces over the field like $\mathbb{R}$ and $\mathbb{C}$but not over the field like $F = \{0, 1\}$.

# Inner product Space

**Definition**

An inner product space is a vector space together with an inner product.

# Properties of Inner product Space

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space of dimension $n$.

**Theorem**

*A nonzero mutually orthogonal set of $V$ is linearly independent.*

**Theorem**

*There exists an orthogonal/orthonormal basis of $V$ (which can be constructed by Gram-Schmidt method).*

**Theorem**

*Let $\alpha = \{\vec{v}_1, \cdots, \vec{v}_n\}$ be an orthogonal basis of $V$. Then for any $\vec{v} \in V$,*

$$\mathrm{Rep}_\alpha(\vec{v}) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \text{ where } a_i = \frac{\langle \vec{v}, \vec{v}_i \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle}.$$

*Moreover, if $\alpha$ is orthonormal, then $a_i = \langle \vec{v}, \vec{v}_i \rangle$.*

# Example of Inner product Spaces

Let $V$ be a vector space over $F(= \mathbb{C}$ or $\mathbb{R})$ of dimension $n$. Let $\rho$ be a linear isomorphism from $V$ to $F^n$. Then we can define an inner product on $V$ via $\rho$ as follows. For $\vec{v}, \vec{w} \in V$,

$$\langle \vec{v}, \vec{w} \rangle := \langle \rho(\vec{v}), \rho(\vec{w}) \rangle_{F^n} = \rho(\vec{v})^t \overline{\rho(\vec{w})}.$$

**Example**

Let $V$ be a subspace of real functions spanned by $e^x, xe^x, x^2e^x$ and let $\rho(a_1e^x + a_2xe^x + a_3x^2e^x) = (a_1, a_2, a_3)$. Then

$$\langle a_1e^x + a_2xe^x + a_3x^2e^x, b_1e^x + b_2xe^x + b_3x^2e^x \rangle := a_1b_1 + a_2b_2 + a_3b_3$$

defines an inner product on $V$.

# Example of Inner product Spaces

In general, let $\alpha = \{\vec{v}_1, \cdots, \vec{v}_n\}$. Then we can define

$$\langle \vec{v}, \vec{w} \rangle_\alpha = \mathrm{Rep}_\alpha(\vec{v})^t \overline{\mathrm{Rep}_\alpha(\vec{w})}.$$

In other words, if $\vec{v} = \sum a_i \vec{v}_i$ and $\vec{w} = \sum b_i \vec{v}_i$, then

$$\langle \vec{v}, \vec{w} \rangle_\alpha = \sum a_i \bar{b}_i.$$

# Example of Inner product Spaces

Let $V$ be the set of real continuous functions on $[0, 1]$. For $f(x), g(x) \in V$, define

$$\langle f(x), g(x) \rangle := \int_0^1 f(x)g(x)dx.$$

It is clear that $\langle \cdot, \cdot \rangle$ satisfies the first two conditions of the inner product. For the third one, we pnly need to show that if $f(x) \not\equiv 0$, $\langle f(x), f(x) \rangle > 0$.

## The Inner Product Space of Continuous Functions

Since $f(x) \not\equiv 0$, we have $a = |f(x_0)| \neq 0$ for some $x_0$. Since $f(x)$ is continuous, we may assume that $x_0 \in (0, 1)$. Moreover, for $\epsilon = a/2 > 0$, there exists some $\delta > 0$ such that

- $(x_0, x_0 + \epsilon) \subset [0, 1]$;
- whenever $|x - x_0| < \delta$, $|f(x) - f(x_0)| < a/2$.

Note that whenever $|x - x_0| < \delta$, by the triangle inequality,

$$|f(x)| \geq |f(x_0)| - |f(x) - f(x_0)| > a/2.$$

Therefore,

$$\langle f(x), f(x) \rangle = \int_0^1 f(x)^2 dx \geq \int_{x_0}^{x_0+\epsilon} |f(x)|^2 dx \geq \frac{\epsilon a^2}{4} > 0$$

## The Inner Product Space of Discrete Signals

Fix a positive integer $N > 0$, let $V$ be the set of complex discrete signals consisting all complex functions on

$$\left\{0, \frac{1}{N}, \cdots, \frac{N-1}{N}\right\}.$$

For $f(x), g(x) \in V$, define

$$\langle f(x), g(x) \rangle := \frac{1}{N} \sum_{k=0}^{N-1} f(\frac{k}{N}) \overline{g(\frac{k}{N})}.$$

Then $\langle \cdot, \cdot \rangle$ defines an inner product on $V$.

# Discrete Foureir Series

Let

$$e_m(x) = e^{2\pi imx} = \cos(2\pi mx) + i\sin(2\pi mx).$$

**Proposition**

$\{e_0(x), \cdots, e_{N-1}(x)\}$ forms an orthonormal basis of $V$.

**Proof.** Let $m, n$ be two integers between 0 and $N-1$.

If $m = n$, it is clear that $\langle e_n(x), e_n(x)\rangle = 1$.

If $m \neq n$, then

$$\langle e_m(x), e_n(x)\rangle = \frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi ik(m-n)}{N}} = \frac{1}{N}\frac{1-e^{\frac{2\pi iN(m-n)}{N}}}{1-e^{\frac{2\pi i(m-n)}{N}}} = 0.$$

# Discrete Foureir Series

## Discrete Fourier Series

For $f(x) \in V$, we have

$$f(x) = \sum_{k=0}^{N-1} a_m e_m(x), \quad \text{where} \quad a_m = \frac{1}{N} \sum_{k=0}^{N-1} f(\frac{k}{N}) e^{-\frac{2\pi ikmx}{N}}.$$

# Discrete Fourier Transform

Let $x_k = f(\frac{k}{N})$ and $X_m = N \cdot a_m$. Then we have

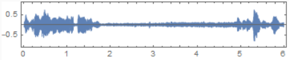
$$X_m = \sum_{k=0}^{N-1} x_m e^{-\frac{2\pi ikmx}{N}},$$

which is called the Discrete Fourier Transform(DFT).

DFT of the signal is the coordinate vector with respec to simple periodic functions $\{e_m(x)\}$.

# Example

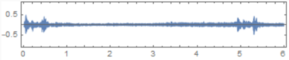
Consider the following sound example from Apollo 13 in 1970.



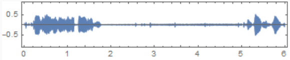

It has (11025 sample/sec )* (6 sec) = 66150 samples.

Taking the DFT, we obtain 66150 complex coefficients. We divides these coefficients into 3 parts and recontruct the sound.

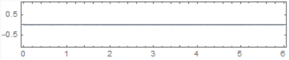
$< 300$ HZ




300-3000 HZ




$> 3000$ HZ



# The Concept behind DFT

- Coordinate vectors with respect to some basis are more meaningful.
- Coordinate vectors with respect to an orthonormal basis are easy to compute.

Recall that for a basis $\alpha = \{\vec{v}_1, \cdots, \vec{v}_n\}$, it induces a basis given by

$$\langle \vec{v}, \vec{w} \rangle_\alpha = \left\langle \sum a_i \vec{v}_i, \sum b_j \vec{v}_j \right\rangle = \sum a_i \bar{b}_i.$$

Conversely, given an inner product $\langle \cdot, \cdot \rangle$, let $\alpha$ be an orthonormal basis with respect to this inner product. Then for $\vec{v}, \vec{w} \in V$,

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= \left\langle \sum_i a_i \vec{v}_i, \sum_j b_j \vec{v}_j \right\rangle = \sum_{i,j} a_i \bar{b}_j \langle \vec{v}_i, \vec{v}_j \rangle \\ &= \sum_i a_i \bar{b}_i = \langle \vec{v}, \vec{w} \rangle_\alpha = \mathrm{Rep}_\alpha(\vec{v})^t \overline{\mathrm{Rep}_\alpha(\vec{w})}. \end{aligned}$$

In other words, the inner product $\langle \cdot, \cdot \rangle$ is equal to the inner product induced by the basis $\alpha$.

# Basis and Inner product

**Theorem**

*Every inner product of a vector space is induced from some basis.*

In particle, we do not interest in all possible inner products. We only care about those inner products which can be computed directly.