Linear Algebra II
Jordan Canonical Forms

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Let T be a linear transformation on a vector space V over F. Suppose $f_T(x)$ splits, so that $f_T(x) = \prod_{i=1}^{k}(x - \lambda_i)^{m_i}$ where $\lambda_1, \cdots, \lambda_k$ are all distinct.

In this case, we have the generalized eigenspace decomposition

$$V = E_\infty(\lambda_1) \oplus \cdots \oplus E_\infty(\lambda_k).$$

Since $T - \lambda_i I$ is nilpotent on $E_\infty(\lambda_i)$, $E_\infty(\lambda_i)$ can be decomposed as a direct sum of cyclic subspace.

Jordan block

On each subspace, T admits a matrix representation of the form

$$\begin{pmatrix} \lambda & 0 & \cdots & \cdots & 0 \\ 1 & \lambda & \ddots & \ddots & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda \end{pmatrix}.$$

If we reverse the order of the basis, we obtain an upper triangular matrix

$$J_{\lambda,m} := \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}.$$

called a Jordon block. (Here m stands for the size of the matrix.)

Jordan Canonical Form

Definition

A block diagonal matrix is called a Jordan matrix if every block is a Jordan block. A Jordan matrix representation of a linear transform is called the Jordan canonical form.

Immediately, we have the following result.

Theorem

For a linear transformation T on V, if $f_T(x)$ splits, then T admits a Jordan canonical form.

Uniquness of Jordan Canonical Form

For each generalized eigenspace, the dimensions in the cyclic subspace decomposition are unique. Therefore, the number of Jordan block $J_{\lambda,m}$ is unique for any given λ and m.

Definition

For a linear transformation T on V, if $f_T(x)$ splits, then T admits a unique Jordan canonical form up to permuting Jordan blocks.

To find the Jordan form of T,

1. Compute the characteristic polynomial and find all eigenvalues.
2. Find all generalized eigenspaces.
3. Decompose each generalized eigenspace as a direct sum of cyclic invariant subspace.

Example

Let $A = \begin{pmatrix} 5 & 7 & 1 & 1 & 5 \\ -2 & -3 & -1 & -1 & -5 \\ -1 & -3 & 1 & 1 & -3 \\ 0 & 0 & 0 & 3 & 1 \\ 1 & 3 & 1 & 0 & 6 \end{pmatrix}$.

1. Find the Jordan J form of A.
2. Find the matrix of change basis P such that $A = PJP^{-1}$.

Find the Jordan Form

First, we have

$$f_A(x) = \det(xI - A) = (x-2)^3(x-3).$$

For $\lambda = 2$, we have

$$\dim\ker(A - 2I) = 2 \quad \text{and} \quad \dim\ker((A-2I)^2) = 1.$$

Therefore, the dot diagram is

$$\begin{array}{ll} \bullet & \bullet \\ \bullet & \end{array}.$$

For $\lambda = 3$, we have

$$\dim\ker(A - 3I) = 1 \quad \text{and} \quad \dim\ker((A-3I)^2) = 2.$$

Therefore, the dot diagram is

$$\begin{array}{ll} \bullet & \bullet \end{array}.$$

Find the Jordan Form

First, we have

$$f_A(x) = \det(xI - A) = (x-2)^3(x-3).$$

For $\lambda = 2$, we have

$$\dim \ker(A - 2I) = 2$$

Therefore, the dot diagram is

$$\begin{array}{ll} \bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet \\ \bullet & \end{array}.$$

For $\lambda = 3$, we have

$$\dim \ker(A - 3I) = 1$$

Therefore, the dot diagram is

$$\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet.$$

Find the Jordan Form

We conclude that the Jordan form of A is

$$J = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{2} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 3 & 1 \\ 0 & 3 \end{matrix}} \end{pmatrix}$$

Find Generalized Eigenspaces

Next, let us find an explicit Jordan basis. To do so, first we need find

$$\ker(A-2I)=\operatorname{span}\left\{\begin{pmatrix}2\\-1\\1\\0\\0\end{pmatrix},\begin{pmatrix}8\\-4\\0\\-1\\1\end{pmatrix}\right\}$$

$$\ker(A-2I)^2=\operatorname{span}\left\{\begin{pmatrix}2\\-1\\1\\0\\0\end{pmatrix},\begin{pmatrix}-4\\1\\0\\1\\0\end{pmatrix},\begin{pmatrix}4\\-3\\0\\0\\1\end{pmatrix}\right\}$$

$$\ker(A-3I)=\operatorname{span}\left\{\begin{pmatrix}-1\\0\\1\\1\\0\end{pmatrix}\right\}$$

$$\ker(A-3I)^2=\operatorname{span}\left\{\begin{pmatrix}-1\\0\\1\\1\\0\end{pmatrix},\begin{pmatrix}1\\-1\\-1\\0\\1\end{pmatrix}\right\}.$$

Decompose Generalized Eigenspaces

For $\lambda = 2$, $E_\infty(2) = \ker(A-2I)^2$ admits a basis of the form

$$\begin{array}{ccccc} \vec{0} & \leftarrow & (A-2I)(\vec{v}_1) & \leftarrow & \vec{v}_1 \\ \vec{0} & \leftarrow & \vec{v}_2. & & \end{array}$$

Note that

$$\begin{aligned}
\operatorname{span}\{(A-2I)(\vec{v}_1)\} &= (A-2I)\Big(\ker(A-2I)^2\Big) \\
&= (A-2I)\left(\operatorname{span}\left\{\begin{pmatrix}4\\-3\\0\\0\\1\end{pmatrix}, \begin{pmatrix}-4\\1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\-1\\1\\0\\0\end{pmatrix}\right\}\right) \\
&= \operatorname{span}\left\{\begin{pmatrix}-4\\2\\2\\1\\-1\end{pmatrix}, \begin{pmatrix}-4\\2\\2\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}\right\}.
\end{aligned}$$

Therefore, we can choose $(A-2I)(\vec{v}_1) = \begin{pmatrix}-4\\2\\2\\1\\-1\end{pmatrix}$, $\vec{v}_1 = \begin{pmatrix}4\\-3\\0\\0\\1\end{pmatrix}$.

Decompose Generalized Eigenspaces

To find $\vec{v}_2$, note that

$$\ker(A-2I)=\mathrm{span}\left\{\begin{pmatrix}2\\-1\\1\\0\\0\end{pmatrix},\begin{pmatrix}8\\-4\\0\\-1\\1\end{pmatrix}\right\}=\mathrm{span}\left\{(A-2I)(\vec{v}_1),\vec{v}_2\right\}.$$

Therefore, we can choose $\vec{v}_2=\begin{pmatrix}2\\-1\\1\\0\\0\end{pmatrix}$.

Decompose Generalized Eigenspaces

For $\lambda = 3$, $E_\infty(3)$ admits a basis of the form

$$\vec{0} \quad \leftarrow \quad (A - 3I)(\vec{v}_3) \quad \leftarrow \quad \vec{v}_3$$

Note that we can choose v_3 to be any vector in $\ker(T - 3I)^2$ not in $\ker(T - 3I)$. For instance, let $\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$. Finally, let

$$\alpha = \left\{(A - 2I)(\vec{v}_1), \vec{v}_1, v_2, (A - 3I)(\vec{v}_3), \vec{v}_3\right\}$$

and P be the corresponding matrix of change basis, then $A = PJP^{-1}$.

Example

Let V be a subspace of the vector space of real functions which has a basis $\alpha = \{x^2e^x, xe^x, e^x\}$. Let $T(f(x)) := f'(x)$ be the differential operator on V.

1. Find the Jordan J form of T.
2. Find a Jordan basis of T.

Example

By direct computation, we have

$$A := \mathrm{Rep}_\alpha(T) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and $f_A(x) = (x-1)^3$. Since $\dim \ker(A - I) = 1$, the dot digram must be

and its Jordan form is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Example

To find the Jordan basis, we have to find

$$\vec{0} \;\leftarrow\; (A-I)^2(\vec{v}) \;\leftarrow\; (A-I)(\vec{v}) \;\leftarrow\; \vec{v}.$$

Since

$$Im(A-I)^2 = Im\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} = \mathrm{span}\{(A-I)^2\vec{v}\},$$

we can choose $(A-I)^2\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (which implies that $(A-I)(\vec{v}) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

Finally, let $\beta = \{x^2e^x, 2xe^x, 2e^x\}$, then

$$\mathrm{Rep}_\beta(T) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this first example, we have

$$J = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{2} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 3 & 1 \\ 0 & 3 \end{matrix}} \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{2} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 3 & 0 \\ 0 & 3 \end{matrix}} \end{pmatrix} + \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{0} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} \end{pmatrix}$$

$$= D + N.$$

Here D is a diagonal matrix and N is a nilpotent matrix. In this case,

$$A = PJP^{-1} = PDP^{-1} + PNP^{-1}.$$

In this case, PDP^{-1} is a diagonalizable matrix, called the semisimple part of A; PNP^{-1} is a nilpotent matrix, called the nilpotent part of A.

Jordan–Chevalley Decomposition

Note that

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + 3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= 2P_1 + 3P_2.$$

Here P_1 and P_2 are the projections onto $E_\infty(2)$ and $E_\infty(3)$ with respect to the eignespace decomposition respectively. By the previous homework, there exist $g_1(x)$ and $g_{(}x)$ such that

$$D = 2P_1 + 3P_2 = 2g_1(A) + 3g_2(A).$$

Jordan–Chevalley Decomposition

Theorem

For $A \in M_n(\mathbb{C})$, there exist a unqiue diagonalizable matrix D and a unique unipotent matrix N such that

$$A = D + N, \quad \text{and} \quad DN = ND.$$

Proof(Existence)

From the theory of Jordan form, there exists some D and N diagonalizable matrix D and a unique unipotent matrix N such that

$$A = D + N.$$

Suppose $\mathbb{C}^n = \oplus_{i=1}^{k} E_{\infty}(\lambda_i)$ be the generalized eigenspace decomposition and P_i be the projection onto $E_{\infty}(\lambda_i)$ with respect to this decomposition. By the previous homework, $Pi = g_i(A)$ for some $g_i(x) \in \mathbb{C}[x]$. Let $g(x) = \sum \lambda_i g_i(x)$. Then

$$D = \sum_{i=1}^{k} \lambda_i P_i = \sum_{i=1}^{k} \lambda_i g_i(A) = g(A).$$

Note that $N = D - A = g(A) - A$. Therefore

$$DN = g(A)\Big(A - g(A)\Big) = \Big(A - g(A)\Big)g(A) = ND.$$

Theorem

For $A, B \in M_n(F)$, suppose A and B are both diagonalizable and $AB = BA$. Then there exists an invertible matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal matices. Moreover, $A + B$ is also diagonalizable.

Proof.

Exercise. $\square$

Theorem

For $A, B \in M_n(F)$, suppose A and B are both nilpotent and $AB = BA$. Then $A + B$ is also nilpotent.

Proof(Uniqueness)

Suppose D' and N' be another decomposition of A. Then we have

$$D - D' = N' - N.$$

Since $D'N' = N'D'$ and $A = D' + N'$, we have that A commutes with D' and N'. Therefore, $D = g(A)$ and $N = g(A) - A$ both commute with D' and N'. By the previous two theorems, $D - D'$ is diagoanlizalbe and $N' - N$ is nilptoent. Since a diagoanlizalbe nilpotent matrix must be equal the zero matrix, we have $D = D'$ and $N = N'$.

Powers of a Matrix

Let A be a $n \times n$ matrix over $\mathbb{C}$ and

$$A = PJP^{-1} = P(D+N)P^{-1}$$

where J is its Jordan form; D is the semisimple part of the Jordan form; N is the nilpotent part of the Jordan form. Note that

$$A^m = PJ^mP^{-1}$$

and

$$J^m = (D+N)^m = D^m + mD^{m-1}N + \binom{m}{2}D^{m-2}N^2 + \cdots$$

Here we use the property that D and N commute. Note that $N^k \equiv 0$ where k is the largest dimension of cyclic subspaces of N.

Example 1

Let $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$\begin{aligned} A^m &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}^m + m \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}^{m-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda^m & m\lambda^{m-1} \\ 0 & \lambda^m \end{pmatrix} = \lambda^m \begin{pmatrix} 1 & \frac{m}{\lambda} \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Example 2

Let $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Then

$$A^m = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}^m + m \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}^{m-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \binom{m}{2} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}^{m-2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^m & m\lambda^{m-1} & \binom{m}{2}\lambda^{m-2} \\ 0 & \lambda^m & m\lambda^{m-1} \\ 0 & 0 & \lambda^m \end{pmatrix} = \lambda^m \begin{pmatrix} 1 & \frac{m}{\lambda} & \frac{m(m-1)}{2\lambda^2} \\ 0 & 1 & \frac{m}{\lambda} \\ 0 & 0 & 1 \end{pmatrix}$$

Example 3

Let $A = \begin{pmatrix} 12 & 25 \\ -4 & -8 \end{pmatrix}$. Then $\det(A - xI) = (x-2)^2$ and $\mathrm{I}m(A - 2I) = \begin{pmatrix} 10 & 25 \\ -4 & -10 \end{pmatrix}$.

Therefore, the Jordan form of A is $J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. To find the Jordan basis, choose $(A-2I)(\vec{v}) = \begin{pmatrix} 10 \\ -4 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We obtain

$$\begin{aligned} A^m = PJ^mP^{-1} &= \begin{pmatrix} 10 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} 2^m & m2^{m-1} \\ 0 & 2^m \end{pmatrix} (\frac{1}{4}) \begin{pmatrix} 0 & -1 \\ 4 & 10 \end{pmatrix} \\ &= \frac{2^m}{4} \begin{pmatrix} 10 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{m}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 4 & 10 \end{pmatrix} \\ &= 2^{m-2} \begin{pmatrix} 20m+4 & 50m \\ -8m & 4-20m \end{pmatrix}. \end{aligned}$$

Example 3

Let $A = \left(\begin{smallmatrix} 12 & 25 \\ -4 & -8 \end{smallmatrix}\right)$. Then $\det(A - xI) = (x-2)^2$ and $\mathrm{I}m(A - 2I) = \left(\begin{smallmatrix} 10 & 25 \\ -4 & -10 \end{smallmatrix}\right)$.

Therefore, the Jordan form of A is $J = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)$. To find the Jordan basis, choose $(A-2I)(\vec{v}) = \left(\begin{smallmatrix} 10 \\ -4 \end{smallmatrix}\right)$ and $\vec{v} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. We obtain

$$
\begin{aligned}
A^m = PJ^mP^{-1} &= \left(\begin{smallmatrix} 10 & 1 \\ -4 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 2^m & m2^{m-1} \\ 0 & 2^m \end{smallmatrix}\right) (\frac{1}{4}) \left(\begin{smallmatrix} 0 & -1 \\ 4 & 10 \end{smallmatrix}\right) \\
&= \frac{2^m}{4} \left(\begin{smallmatrix} 10 & 1 \\ -4 & 0 \end{smallmatrix}\right) \begin{pmatrix} 1 & \frac{m}{2} \\ 0 & 1 \end{pmatrix} \left(\begin{smallmatrix} 0 & -1 \\ 4 & 10 \end{smallmatrix}\right) \\
&= 2^{m-2} \left(\begin{smallmatrix} 20m+4 & 50m \\ -8m & 4-20m \end{smallmatrix}\right).
\end{aligned}
$$

Example 3

Let $A = \begin{pmatrix} 21 & 43 & 5 \\ -8 & -16 & -2 \\ 4 & 9 & 3 \end{pmatrix}$. Then $\det(A - xI) = (x-3)^2(x-2)$. For $\lambda = 3$, $\dim \ker(A - 3I) = 1$, so its Jordan block is $J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. To find the Jordan basis, consider

$$(A - 3I)(\ker(A - 3I)^2) = \operatorname{span}\left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix} \right\}.$$

Therefore, one can choose $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $(A - 3I)\vec{v}_1 = \begin{pmatrix} 18 \\ -8 \\ 4 \end{pmatrix}$. For λ_2,

$$\ker(A - 2I) = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Now set

$$P = \begin{pmatrix} 18 & 1 & 2 \\ -8 & 0 & -1 \\ 4 & 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Example 3

Then

$$A^m = PJ^mP^{-1} = \begin{pmatrix} 18 & 1 & 2 \\ -8 & 0 & -1 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3^m & m3^{m-1} & 0 \\ 0 & 3^m & 0 \\ 0 & 0 & 2^m \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ 4 & 10 & 2 \\ 0 & 4 & 8 \end{pmatrix}$$