Linear Algebra II
Conic Sections

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Consider the zero set $\mathscr{Z}(g)$ of a general binary quadratic polynomial over $\mathbb{R}$

$$g(x, y) = ax^2 + bxy + cy^2 + dx + ey + f.$$

called a quadratic curve. The goal of this section is to classify the set $\mathscr{Z}(g)$.

When $g(x, y)$ is a nonzero quadratic form, we have the following result.

Theorem

$$\textit{The set } \mathcal{Z}(g) \textit{ is } \begin{cases} \{(0,0)\} & , \textit{if } b^2 - 4ac < 0; \\ \textit{the union of two lines} & , \textit{if } b^2 - 4ac > 0; \\ \textit{a line} & , \textit{if } b^2 - 4ac = 0. \end{cases}$$

When $g(x, y)$ is not a quadratic form, regard $\mathcal{Z}(g)$ as the intersection of the plane $z = 1$ and the zero set of the quadratic form

$$Q(x, y, z) = ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = (x, y, z)A \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right).$$

Here

$$A = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}.$$

The quadratic curve $\mathcal{Z}(g)$ is called non-degenerated if A has full rank. (In this case, we also call Q a non-degenerated quadratic form.)

Zero Sets of Non-degenerated Ternary Quadratic Form

When $Q(x, y, z)$ is a non-degenerated quadratic form, we have the following result.

Theorem

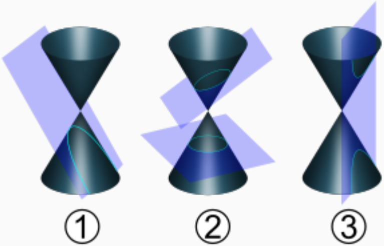
If three eigenvalues of A are of the same sign, $\mathcal{Z}(Q) = \{(0, 0, 0)\}$. Otherwise, $Z(Q)$ is a double cone (the union of two cones).

When $\mathcal{Z}(Q) = \{(0, 0, 0)\}$, $\mathcal{Z}(Q)$ and $\mathcal{Z}(z - 1)$ does not intersect, then $\mathcal{Z}(g)$ is the empty set.

When $\mathcal{Z}(Q)$ is a double cone, $\mathcal{Z}(Q) \cap \mathcal{Z}(z - 1)$ is called a conic section.

There are three possible types of conic sections:

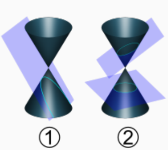
1. Parabola
2. Circle and ellipse
3. Hyperbola



Here we draw the picture under the eigenbasis of A. Therefore, the position of the plane $z = 1$ varies in different cases.

(The figure is copied from wikipedia.)

From the figures



The set $\mathcal{Z}(Q)\cap\mathcal{Z}(z)$ is

$$\begin{cases} \text{a line} & \text{, if } \mathcal{Z}(G)\cap\mathcal{Z}(z-1) \text{ is a parabola;} \\ \{(0,0)\} & \text{, if } \mathcal{Z}(G)\cap\mathcal{Z}(z-1) \text{ is an ellipse;} \\ \text{the union of two lines} & \text{, if } \mathcal{Z}(G)\cap\mathcal{Z}(z-1) \text{ is a hyperbola.} \end{cases}$$

On the other hand, $\mathcal{Z}(G)\cap\mathcal{Z}(z)$ can be identified as the zero set of $ax^2+bxy+cy^2$ by mapping $(x,y,0)$ to (x,y).

Together with the previous theorem, we obtain the following result.

Theorem

For a non-degenerated non-empty quadratic curve $\mathcal{Z}(g)$ where $g(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$,

$$\text{the curve } \mathcal{Z}(g) \text{ is } \begin{cases} \text{an ellipse} & , \text{ if } b^2 - 4ac < 0; \\ \text{a hyperbola} & , \text{ if } b^2 - 4ac > 0; \\ \text{a parabola} & , \text{ if } b^2 - 4ac = 0. \end{cases}$$