HW1

Note - I write $[T]_{\alpha,\beta}$ to abbreviate $\mathrm{Rep}_{\alpha,\beta}[T]$

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(1)

$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, Av = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. So in fact $v \in E_1(1)$

For any $n, A^n v = v$ by induction, so $W = Z(v, T) = \mathrm{span}[v]$

Therefore, $\{v\}$ is a <u>basis</u>

The <u>matrix representation</u> is $[T\big|_W]_\alpha = ([Tv]_\alpha) = (1)$

and its <u>characteristic polynomial</u> is $\begin{vmatrix} 1-x \end{vmatrix} = -(x-1)$

(2)

$v = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix},$

$$Av = \begin{pmatrix} 19 \\ -19 \\ 0 \end{pmatrix},$$

$$A^2 v = \begin{pmatrix} 57 \\ 38 \\ -95 \end{pmatrix} = 19 \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$$

Thus $A^2 v = 19v$ so $W = Z(v, T) = \mathrm{span}[\begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 19 \\ -19 \\ 0 \end{pmatrix}]$

We can pick $\beta := \{\begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 19 \\ -19 \\ 0 \end{pmatrix}\}$ as a <u>basis</u>

The <u>matrix representation</u> is $[T\big|_W]_\beta = ([Av]_\beta \quad [A^2v]_\beta) = (\begin{matrix} 0 & 2 \\ 1 & 0 \end{matrix})$

and its <u>characteristic polynomial</u> is $\begin{vmatrix} x & -2 \\ -1 & x \end{vmatrix} = x^2 - 2$

(3)

$v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,

$Av = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$, [note: $Av - v = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$]

$A^2 v = \begin{pmatrix} 13 \\ -6 \\ -6 \end{pmatrix}$,

$A^3 v = \begin{pmatrix} 51 \\ -6 \\ -44 \end{pmatrix} = \begin{pmatrix} 13 \\ -6 \\ -6 \end{pmatrix} + \begin{pmatrix} 38 \\ 0 \\ -38 \end{pmatrix} = A^2 v + 19(Av - v) = A^2 v + 19Av - 19v$

Thus $W = Z(v, T) = \text{span}[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 13 \\ -6 \\ -6 \end{pmatrix}]$

We can pick $\gamma := \{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 13 \\ -6 \\ -6 \end{pmatrix}\}$ as a basis

The matrix representation is $[T|_W]_\gamma = ([Av]_\gamma \quad [A^2 v]_\gamma \quad [A^3 v]_\gamma) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 19 \\ 0 & 1 & -19 \end{pmatrix}$

and its characteristic polynomial is

$$\begin{vmatrix} x & 0 & -1 \\ -1 & x & -19 \\ 0 & -1 & x+19 \end{vmatrix} = x \begin{vmatrix} x & -19 \\ -1 & x+19 \end{vmatrix} + (-1) \begin{vmatrix} -1 & x \\ 0 & -1 \end{vmatrix}$$
$$= x(x^2 + 19x - 19) - (1) = x^3 + 19x^2 - 19x - 1$$

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Notice that using Laplace expansion,

$$\det xI - A = \begin{vmatrix} \underline{x} & \underline{0} & \underline{...} & \underline{...} & \underline{-a_0} \\ -1 & x & ... & ... & -a_1 \\ 0 & 1 & ... & ... & -a_2 \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & 1 & -a_{n-1}+x \end{vmatrix}$$

$$= x \begin{vmatrix} x & ... & ... & -a_1 \\ -1 & ... & ... & -a_2 \\ ... & ... & ... & ... \\ ... & 0 & -1 & -a_{n-1}+x \end{vmatrix} + (-1)^{1+n}(-a_0) \begin{vmatrix} -1 & x & ... & ... \\ 0 & -1 & ... & ... \\ ... & ... & ... & ... \\ & ... & 0 & -1 \end{vmatrix}$$

i.e.

$\det(xI - C(a_0, a_1, ..., a_{n-1})) = x \det(xI - C(a_1, a_2, ...a_{n-1})) + (-1)^{1+n}(-a_0)(-1)^{n-1}$

Consider the following induction

claim $\det(xI - C(a_0, ..., a_{n-1})) = x^n - a_{n-1}x^{n-1} - ... - a_0$

case n=2:

$C(a_0, a_1) = \begin{pmatrix} 0 & a_0 \\ 1 & a_1 \end{pmatrix}$ and the characteristic polynomial is

$$\begin{vmatrix} -x & a_0 \\ 1 & a_1 - x \end{vmatrix} = x(x - a_1) - a_0 = x^2 - a_1 x - a_0$$

Assume the statement is correct for n=k, claim that it is correct for n=k+1:

$$\begin{aligned} \det(xI - C(a_0, ..., a_k) - xI) &= x\det(xI - C(a_1, ..., a_k)) - a_0 \\ &= x(x^{k-1} - a_k x^{k-2} - ... - a_1) - a_0 \\ &= x^k - a_k x^{k-1} - ... - a_1 x - a_0 \end{aligned}$$

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(1)

<u>additively closed</u>

$v, w \in E_\infty(\lambda) \rightarrow \exists\, n_v, n_w$ s.t. $(T - \lambda I)^{n_v} v = 0, (T - \lambda I)^{n_w} w = 0,$

WLOG let $n_v \le n_w$ then $(T - \lambda I)^{n_w}(v + w) = (T - \lambda I)^{n_w} v + (T - \lambda I)^{n_w} w = 0 + 0$

so $v + w \in E_\infty(\lambda)$

<u>closed under scalar multiplication</u>

$v \in E_\infty(\lambda) \rightarrow \exists\ n$ s.t. $(T - \lambda I)^n v = 0$

$(T - \lambda I)^n(\alpha v) = \alpha(T - \lambda I)^n v = 0$

so $\alpha v \in E_\infty(\lambda)$

(2)

Recall that T-invariant iff $\forall\ w \in W, Tw \in W$

given $w \in E_\infty(\lambda) \to (T-\lambda I)^n w = 0$ for some n

we claim $(T-\lambda I)^m (Tw) = 0$ for some n

In fact, $(T-\lambda I)^n Tw = T(T-\lambda I)^n w = T0 = 0$

since T commutes with $f(T)$ for every $f(x) \in F[x]$ (polynomial) (while $TS \neq ST$ for general S)

thus we arrive at the conclusion that $Tw \in E_\infty(\lambda)$

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No.

Notice that $xI - A$ is not in $F^{n\times n}$ — its entries take value in $F[x]$ instead of F, thus $xI - A \in F[x]^{n\times n}$

The $\det f_A := \det(xI - A)$ is actually not $\det : F^{n\times n} \to F$ but $\overline{\det} : F[x]^{n\times n} \to F$, where $\overline{\det}$ cannot evaluate $AI - A = O$

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$x^3 + 3x^2 + 3x + 2 = (x+2)(x^2+x+1)$

$x^3 + 3x^2 + 5x + 6 = (x+2)(x^2+x+3)$

so their monic GCD is $x+2$

Alternatively, one can use the Euclidean algorithm -

$x^3 + 3x^2 + 5x + 6 - (x^3 + 3x^2 + 3x + 2) = 2x + 4 = 2(x+2)$ and notice that $f(-2) = g(-2) = 0$ so $(x+2) \mid f$ and $(x+2) \mid g$

6

$A - 2I = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ where $(A-2I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $(A-2I)^3 = O_5$

(notice that $A-2I=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\oplus\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$ so $(A-2I)^n=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}^n\oplus\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}^n$)
so the minimal polynomial is $(x-2)^3$

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If $f(x)$ splits, that is, $f(x)=\prod_{i=1}^{n}(x-\lambda_i)$ for some distinct λ_i
then for every λ_i, $E(\lambda_i)$ is strictly bigger than $\{0\}$ → $\mathrm{GM}(\lambda_i, T)=\dim E(\lambda_i)\geq 1$
on the other hand, $\mathrm{AM}(\lambda_i, T)=1$ since there are no repeated roots
so $\dim E(\lambda_i)=1$ since $\mathrm{AM}\geq\mathrm{GM}$
We have n distinct λ_i, so there are n linearly independent eigenvectors (since eigenvectors of different eigenvalues are linearly independent) →
$\dim\bigoplus_{i=1}^{n}E(\lambda_i)=n=\dim V$
thus T is diagonalizable