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Linear Algebra II - Homework 1

1. Let $V = \mathbb{R}^3$ and $T = L_A$ where $A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & -2 & 3 \\ -2 & 3 & 0 \end{pmatrix}$. For

$$(1)\ \vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad (2)\ \vec{v} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} \qquad (3)\ \vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

let W be the cyclic subspace generated by $\vec{v}$. Solve the following problems for each given $\vec{v}$.

- Find the basis α of W.
- Find the matrix representation of $T\big|_W$ with respect to the basis α.
- Find the characteristic polynomial of $T\big|_W$

2. Let

$$A = \begin{pmatrix} 0 & 0 & \cdots & \cdots & a_0 \\ 1 & 0 & \ddots & \ddots & a_1 \\ 0 & 1 & \ddots & \ddots & a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & a_{n-1} \end{pmatrix}$$

Show that $f_A(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0$.

3. Let $T : V \mapsto V$ be a linear transform. For $\lambda \in F$, define

$$E_\infty(\lambda) = \bigcup_{n=1}^{\infty} \left\{ \vec{v} \in V \middle| (T - \lambda I)^n(\vec{v}) = \vec{0} \right\}.$$

Show that (1) $E_\infty(\lambda)$ is a subspace of V. (2) Show that $E_\infty(\lambda)$ is a T-invariant subspace.

4. In Cayley-Hamilton Theorem, recall that $f_T(x) := \det(xI - T)$. Can we just say that $f_T(T) = \det(TI - T) = \det(T - T) = 0$?.

5. Let $f(x) = x^3 + 3x^2 + 3x + 2$ and $g(x) = x^3 + 3x^2 + 5x + 6$. Find the monic GCD of $f(x)$ and $g(x)$.

6. Let $A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$. Find the minimal polynomial of A. (Hint: this polynomial is of the form $(x-2)^k$ Find the minimal k.)

7. Let $T : V \mapsto V$ be a linear transform. Suppose there exists some split polynomial $f(x)$ without repeated roots such that $f(T) \equiv 0$. Show that T is diagonalizable.