Linear Algebra II
Matrix Exponential

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Motivation

Recall that for a linear differential equation

$$x'(t) = ax(t),$$

its solution is $x(t) = c\exp(at)$ where $c = x(0)$ is a constant. For the coupled system of linear equations

$$\begin{cases} x_1'(t) = ax_1(t) + bx_2(t) \\ x_2'(t) = cx_1(t) + dx_2(t) \end{cases}$$

One can rewrite the system as

$$X'(t) = AX(t), \text{ where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

In this case, can we obtain a solution in terms of $\exp(At)$?

First, let us try to define $\exp(A)$. Recall that for all $x \in \mathbb{R}$, we define $\exp(x)$ using the power series expression

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{m \to \infty} \sum_{n=0}^{m} \frac{x^n}{n!}.$$

Therefore, a nature way to define $\exp(A)$ is

$$\exp(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!} = \lim_{m \to \infty} \sum_{n=0}^{m} \frac{A^n}{n!}.$$

To do so, we need know the meaning of matrix limit.

Convergence of Complex Sequences

For a complex sequence $\{a_1 + b_1 i, a_2 + b_2 i, \cdots\}$,

$$\sum_{n=1}^{\infty} a_n + ib_n \ \text{ converges} \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n \ \text{ both converge.}$$

Note that

$$\begin{aligned}
\sum_{n=1}^{\infty} |a_n + b_n i| \text{ converges} \quad &\Rightarrow \quad \sum_{n=1}^{\infty} |a_n| \text{ and } \sum_{n=1}^{\infty} |b_n| \text{ both converge} \\
&\Rightarrow \quad \sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n \text{ both converge} \\
&\Rightarrow \quad \sum_{n=1}^{\infty} a_n + ib_n \ \text{ converges}
\end{aligned}$$

Complex Exponential Function

Recall that all real number x, the power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

always converges.

For a complex number z, define the complex exponential function as

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Since $\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = \exp(|z|)$ converges, the above summation always converges.

Matrix Limit

Let $A_1, A_2, \cdots$ be a sequence of 2×2 complex matrices. Let

$$A_i = \begin{pmatrix} A_{i,11} & A_{i,12} \\ A_{i,21} & A_{i,22} \end{pmatrix}.$$

For $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$,

$$\lim_{i \to \infty} A_i = B \qquad \text{if} \qquad \lim_{i \to \infty} A_{i,jk} = B_{jk} \quad \forall j, k.$$

In other words, when the limit of each entries exists, we have

$$\lim_{i \to \infty} A_i = \begin{pmatrix} \lim_{i \to \infty} A_{i,11} & \lim_{i \to \infty} A_{i,12} \\ \lim_{i \to \infty} A_{i,21} & \lim_{i \to \infty} A_{i,22} \end{pmatrix}.$$

Matrix Limit

In general, let $A_1, A_2, \cdots$ be a sequence of $n \times n$ complex matrices. Write $A_i = (A_{i,jk})$. For another $n \times n$ complex matrix $B = (B_{jk})$, we say

$$\lim_{i \to \infty} A_i = B \qquad \text{if} \qquad \lim_{i \to \infty} A_{i,jk} = B_{jk} \quad \forall j, k.$$

Example

Let $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} t$ and $A_n = \sum_{m=1}^{n} \frac{A^m t^m}{m!}$. Then

$$A_n = \sum_{m=0}^{n} \frac{t^m}{m!} \begin{pmatrix} \lambda_1^m & 0 \\ 0 & \lambda_2^m \end{pmatrix} = \begin{pmatrix} \sum_{m=0}^{n} \frac{(\lambda_1 t)^m}{m!} & 0 \\ 0 & \sum_{m=0}^{n} \frac{(\lambda_2 t)^m}{m!} \end{pmatrix}.$$

Therefore, for all t,

$$\exp(At) = \lim_{n \to \infty} A_n = \begin{pmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2 t) \end{pmatrix}.$$

Example

Let $A = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right) t$ and $A_n = \sum_{m=0}^{n} \frac{A^m t^m}{m!}$. Then

$$A_n = \sum_{m=0}^{n} \frac{1}{m!} \begin{pmatrix} \lambda^m t^m & m\lambda^{m-1} t^m \\ 0 & \lambda^m \end{pmatrix} = \begin{pmatrix} \sum_{m=0}^{n} \frac{(\lambda t)^m}{m!} & t \sum_{m=1}^{n} \frac{(\lambda t)^{m-1}}{(m-1)!} \\ 0 & \sum_{m=0}^{n} \frac{(\lambda t)^m}{m!} \end{pmatrix}.$$

Therefore, for all $t \in \mathbb{R}$,

$$\exp(At) = \lim_{n\to\infty} A_n = \begin{pmatrix} \exp(\lambda t) & t \exp(\lambda t) \\ 0 & \exp(\lambda t) \end{pmatrix}.$$

Product of Matrix Limit

To study the exponential of a general matrix, recall that if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then

$$\lim_{n\to\infty} a_n + b_n = a + b \quad \text{and} \quad \lim_{n\to\infty} a_n b_n = ab.$$

Theorem

Suppose A and B are two square complex matrices of the same size such that $\lim_{n\to\infty} A_n = A$, and $\lim_{n\to\infty} B_n = B$. Then $\lim_{n\to\infty} A_n B_n = AB$.

Exercise

Suppose A and P are two square complex matrices of the same size and P is invertible. If $\lim_{n\to\infty} A_n = A$, then $\lim_{n\to\infty} PA_nP^{-1} = PAP^{-1}$.

Matrix Exponential

Theorem

For any complex $m \times m$ matrix A,

$$\exp(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

always converges. Moreover, we have

$$\frac{d}{dt} \exp(At) = A \exp(At).$$

For the coupled system of linear equations

$$\begin{cases} x_1'(t) = ax_1(t) + bx_2(t) \\ x_2'(t) = cx_1(t) + dx_2(t) \end{cases} \quad \Rightarrow \quad X'(t) = AX(t).$$

where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Let $X(t) = \exp(At)\left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right)$ where c_1 and c_2 for some constants. Then

$$X'(t) = A\exp(At)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = AX'(t).$$

We conclude that $X(t)$ is a solution and we have $X(0) = \left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right)$.

Example

Solve the coupled system of linear equations

$$\begin{cases} x_1'(t) = x_1(t) + 4x_2(t) \\ x_2'(t) = x_1(t) + x_2(t) \end{cases} \quad \text{with the initial condition} \quad \begin{cases} x_1(0) = 1 \\ x_2(0) = 1 \end{cases}$$

Let $A = \left(\begin{smallmatrix} 1 & 4 \\ 1 & 1 \end{smallmatrix}\right)$. Then $\left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ be eigenvectors of A corresponding the eigenvalues -1 and 3 respectively. Therefore

$$A = \left(\begin{smallmatrix} 2 & 2 \\ -1 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} -1 & 0 \\ 0 & 3 \end{smallmatrix}\right) \left(\begin{smallmatrix} 2 & 2 \\ -1 & 1 \end{smallmatrix}\right)^{-1}.$$

Example

Now we have

$$\begin{aligned}\exp(At) &= \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{4} \begin{pmatrix} 2e^{-t}+2e^{3t} & -4e^{-t}+4e^{3t} \\ -e^{-t}+e^{3t} & 2e^{-t}+2e^{3t} \end{pmatrix}.\end{aligned}$$

Therefore, the solution is

$$X(t) = \exp(At) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -2e^{-t} + 6e^{3t} \\ e^{-t} + 3e^{3t} \end{pmatrix}.$$