Linear Algebra I - Practice Final 2019 Fall

1. Let V and W be subspaces of the space of real functions. Suppose V admits a basis $\alpha = \{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$ and W admits a basis $\{1, x, x^2, x^3\}$. Consider the linear transform $T : V \to W$ given by

$$T(f(x)) = f(0) + f'(0)x + f''(0)x^2 + f'''(0)x^3.$$

- Find the matrix A of T with respect the bases α and β.
- Find the inverse of A.

2. Let V be the real vector space of two variable functions with the basis $B = \{x^3, x^2y, xy^2, y^3\}$. Let $T(f(x,y)) = f(y,x)$ be a function from V to V.

- Show that T is a linear transform.
- Show that T is an isomorphism.
- Find the matrix A of T with respect to the basis B.
- Find eigenvalues of T.
- Write $A = PDP^{-1}$ where D is a diagonal matrix and P is a diagonal matrix.

3. Let

$$A = \begin{pmatrix} 4 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 4 & f \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

be a real matrix. Determine for which a, b, c, d, e, f, A is diagonalizable.

4. For $V = \mathbb{R}^4$ and $W = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$. Let P be the orthogonal projection from V onto W.

- Find the orthogonal complement of W.
- Find $P(\begin{pmatrix} 2 \\ 2 \\ 3 \\ 4 \end{pmatrix})$.
- Find the matrix of P with respect to the standard basis.
- Find the least square solution of the following system.

$$\begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

5. Let V be a real vector space of functions with the basis $B = \{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$ with the inner product

$$\langle f(x), g(x) \rangle = f(0)g(0) + f'(0)g'(0) + f''(0)g''(0) + f'''(0)g'''(0).$$

- Apply the Gram-Schmidt method to B with respect to the given inner product.
- Suppose there is a function $f(x)$ in V satisfying $f(0) = 1, f'(0) = 3, f''(0) = 4, f'''(0) = 2$. Find $f(x)$.

6. Let $T : V \to V$ be a linear transformation. Suppose $T^2 = T$. Show that T is diagonalizable. (Hint: Show that $V = E(0) \oplus E(1)$)