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Linear Algebra II - Quiz 2

1. Let $V = \mathbb{R}^3$ and $T = L_A$ where $A = \begin{pmatrix} 5 & 1 & -1 \\ 7 & 5 & -3 \\ 11 & 4 & -2 \end{pmatrix}$. For $\vec{v} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, let W be the cyclic subspace generated by $\vec{v}$

   (a) Find a basis α of W.

   (b) Find the matrix representation of $T|_W$ with respect to the basis α.

   (c) Find the characteristic polynomial of $T|_W$

2. Let T be a reflection on $\mathbb{R}^3$. Show that T is diagonalizable.

1.

(a) By definition of cyclic subspace, W is spanned by $\{\vec{v}, A\vec{v}, A^2\vec{v}, \ldots\}$ By direct computation, we obtain that

$$A\vec{v} = \begin{pmatrix} 5 & 1 & -1 \\ 7 & 5 & -3 \\ 11 & 4 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 6 \end{pmatrix}, \qquad A^2\vec{v} = \begin{pmatrix} 5 & 1 & -1 \\ 7 & 5 & -3 \\ 11 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 24 \\ 27 \end{pmatrix} = -9\vec{v} + 6A\vec{v}$$

Hence the space spanned by $\{\vec{v}, A\vec{v}, A^2\vec{v}, \ldots\}$ is equal to the space spanned by $\{\vec{v}, A\vec{v}\} = \{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \\ 6 \end{pmatrix}\}$.

Since these two vectors are linearly independent, $\alpha := \{\vec{v}, A\vec{v}\} = \{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \\ 6 \end{pmatrix}\}$ forms a basis of W.

(b) From the computation in (a), we have

$$T(\vec{v}) = A\vec{v}, \qquad T(A\vec{v}) = A^2\vec{v} = -9\vec{v} + 6A\vec{v}$$

Thus the matrix representation of $T|_W$ with respect to the basis α. is

$$\begin{pmatrix} 0 & -9 \\ 1 & 6 \end{pmatrix}$$

(c) The characteristic polynomial of $T|_W$ is

$$\det(\begin{pmatrix} 0 & -9 \\ 1 & 6 \end{pmatrix} - xI) = \det(\begin{pmatrix} -x & -9 \\ 1 & 6-x \end{pmatrix}) = x^2 - 6x + 9$$

2. If T is a reflection on $\mathbb{R}^3$, with respect to a unit vector $\vec{v_0}$. Then for any $v \in \mathbb{R}^3$, $T(\vec{v}) = \vec{v} - 2\langle v, v_0 \rangle v_0$. In particular, $T(\vec{v_0}) = -\vec{v_0}$. Thus $\underline{-1 \text{ is an eigenvalue of } T \text{ with eigenspace is spanned by } \vec{v_0}.}$
Let H be the orthogonal complement of span$\vec{v_0}$ in $\mathbb{R}^3$ Then H is a two dimensional vector space and any element $h \in H$ satisfies $T(h) = h - 2\langle h, v_0 \rangle v_0 = h$. So $\underline{1 \text{ is an eigenvector of } T \text{ with 2 dimensional eigenspace } H.}$
This shows that T is diagonalizable. More precisely, it is possible to find a basis so that the representation of T is equal to

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$