Linear Algebra I - Midterm Review

Ming-Hsuan Kang

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The definition of Vector spaces

A vector space V over a field F is a set together with two binary operators: the addition from $V \times V$ to V and the scalar multiplication from $F \times V$ to V which satisfy the following properties. $\forall \vec{x}, \vec{y}, \vec{z} \in V$ and $\forall a, b \in F$:

1. $\vec{x} + \vec{y} \in V$.
2. $a\vec{x} \in V$.
3. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$.
4. $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$.
5. There exists $\vec{0} \in V$ (independent of $\vec{x}$), such that $\vec{x} + \vec{0} = \vec{x}$.
6. There exists $\vec{x}' \in V$, such that $\vec{x} + \vec{x}' = \vec{0}$. (Such $\vec{x}'$ is unique and it is denoted by $-\vec{x}$.)
7. $1\vec{x} = \vec{x}$.
8. $a(b\vec{x}) = (ab)\vec{x}$.
9. $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$.
10. $(a + b)\vec{x} = a\vec{x} + b\vec{x}$.

Subspaces

The subset W is a subspace of V if it inherits the vector space structure from V. More precisely, W is a subspace if the following conditions hold.

1. Show that $\vec{0} \in W$.
2. Show that for $k \in F$, $\vec{v_1}, \vec{v_2} \in W$, we have $k\vec{v_1} + \vec{v_2} \in W$.

Verify the structure of vector spaces

A standard way to show a given set V together with addition and scalar multiplication is a vector space over F is the following.

1. Identify the set V as a subset of a known vector space U.
2. Show that the set V is a subspace of U.

Spanning sets (generating set)

For a subset $S = \{\vec{v}_1, \cdots, \vec{v}_n\}$,

$$\text{span(S)} = \left\{ \sum_{i=1}^{n} a_i \vec{v}_i | \vec{v}_i \in S, a_i \in F \right\}.$$

The following are equivalent.

1. S spans V, which means $\text{span}(S) = V$.
2. For all $\vec{b} \in V$, the system $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{b}$ always has a solution.

Linearly independent sets

For a subset $S = \{\vec{v}_1, \cdots, \vec{v}_n\}$ of V, the following are equivalent.

1. S is linearly independent.
2. For all $\vec{v} \in S$, $\vec{v} \notin \text{span}(S \backslash \{\vec{v}\})$.
3. Whenever $\sum_{i=1}^{n} c_i \vec{v}_n = 0$, we have $c_1 = \cdots = c_n = 0$.
4. The linear system $x_1\vec{v_1} + \cdots + x_n\vec{v}_n = 0$ has the only trivial solution.

Basis

A ordered subset $S = \{\vec{v}_1, \cdots, \vec{v}_n\}$ of V is a basis if it is linearly independent and it spans V. The cardinality of the basis is called the dimension of the vector space.

The following are equivalent.

1. S is a basis.
2. S is linearly independent and S spans V.
3. S is a maximal linearly independent set.
4. S is a minimal spanning set.
5. S is linearly independent and $\dim V = n$.
6. S spans V and $\dim V = n$.
7. For all $\vec{b} \in V$, $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{b}$ always has a unique solution.
8. For all $\vec{v} \in V$, $\vec{v}$ can be written as $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ for some unique $c_i \in F$. In this case, we also write it as
$$\text{Rep}_S(\vec{v}) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

The following are the basic property of basis.

1. Every vector space has a basis.
2. Every linearly independent set can be extended to a basis.
3. Every spanning set contains a basis.
4. Any two basis have the same cardinality.

Direct Sum

For subspace $W_1, \cdots, W_k$ of V. The following are equivalent.

1. $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$.
2. Every vector $v \in V$ can be uniquely written as $w_1 + \cdots + w_n$ with $w_i \in W_i$.
3. Let α_i be a basis of W_i, then $\alpha_1 \sqcup \alpha_2 \sqcup \cdots \sqcup \alpha_k$ forms a basis of V.
4. $\dim V = \dim W_1 + \cdots + \dim W_k$ and $W_1 + \cdots + W_k = V$.

Row Rank and Column Rank

The following are some basic properties about row rank and column rank.

1. The row/column rank of a matrix is the dimension of its row/column space.
2. The row operation does not change the row space.
3. The row operation may change the column space but does not change the column rank.
4. The row rank is equal to the number of nonzero rows of the echelon form.
5. The column rank is equal to the number of leading variables.
6. The row rank is always equal to the column rank.

Linear transformations

Let V and W be vector spaces over F. A map $T : V \to W$ is a linear transformation if

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in V.
2. $T(k\vec{u}) = kT(\vec{u})$ for all $\vec{u}$ in V and k in F.

Moreover, if T is also a bijection, then T is called a (linear) isomorphism.

Dimension Forumla

The kernel of T is

$$\ker(T) = \{\vec{v} \in V | T(\vec{v}) = \vec{0}\}.$$

The image of T is

$$\mathrm{Im}(T) = T(V) = \{T(\vec{v}) | \vec{v} \in V\}.$$

Dimension formula:

$$\dim(V) = \dim \ker(T) + \dim \mathrm{Im}(T).$$

For a linear transform $T : V \to W$, the following are equivalent

1. T is an isomorphism.
2. T is onto.
3. $\ker(T) = \{\vec{0}\}$.

Matrix Representations

Let $\alpha = \{\vec{\alpha}_1, \cdots, \vec{\alpha}_n\}$ be a basis of V and β be a basis of W. Then the matrix representing T with respect to the bases α and β is

$$\mathrm{Rep}_{\alpha,\beta}(T) = \Big(\mathrm{Rep}_\beta\big(T(\vec{\alpha}_1)\big) \ \cdots \ \mathrm{Rep}_\beta\big(T(\vec{\alpha}_n)\big)\Big).$$

Moreover, we have

$$\mathrm{Rep}_\beta\big(T(\vec{v})\big) = \mathrm{Rep}_{\alpha,\beta}(T)\mathrm{Rep}_\alpha(\vec{v}).$$

When $V = W$ and $\alpha = \beta$, we also denote $\mathrm{Rep}_{\alpha,\beta}(T)$ by $\mathrm{Rep}_\alpha(T)$ for short.

Determinants

For $A = (a_{ij}) \in M_n(F)$, let $M_{ij}(A)$ be the (i,j)-th minor of A obtained by removing the i-th row and the j-th column of A. Then for any $1 \le i \le n$, the determinant of A is given by

$$\det(A) = \sum_{j=1}^{n} a_{ij}c_{ij} = \sum_{j=1}^{n} a_{ji}c_{ji}.$$

Here

$$c_{ij} = (-1)^{i+j} \det(M_{ij}(A)),$$

which is the (i,j)-th cofactor of A.

Cramer's Ruler:
Let $C = (c_{ij})$ be the cofactor matrix and let $\mathrm{Adj}(A) = C^T$, called the adjugate matrix of A. Then

$$\mathrm{Adj}(A)A = \det(A)I_n.$$

Especially, when $\det(A) \neq 0$,

$$A^{-1} = \frac{1}{\det(A)}\mathrm{Adj}(A).$$

Theorem 1. *The following are the properties of determinants.*

1. *A square matrix A is non-singular if and only if* $\det(A) \neq 0$. *In this case,* $\det(A^{-1}) = \det(A)^{-1}$.
2. *A matrix A is of rank greater than or equal to r if and only if A contains a square submatrix B of size r with* $\det(B) \neq 0$.
3. *For two square matrices A and B of the same size,* $\det(AB) = \det(A)\det(B)$. *If B is invertible, then* $\det(BAB)^{-1} = \det(A)$.

Change of Basis

Now suppose $T : V \to V$ is a linear transform and $\alpha = \{\vec{\alpha}_1, \cdots, \vec{\alpha}_n\}$ and $\beta = \{\vec{\beta}_1, \cdots, \vec{\beta}_n\}$ are two bases of V. Then

$$\mathrm{Rep}_\alpha(T) = \mathrm{Rep}_{\beta,\alpha}(id)\mathrm{Rep}_\beta(T)\mathrm{Rep}_{\alpha,\beta}(id).$$

Especially, when $T = L_A$ (the multiplication by A from left) and α is the standard basis of F^n, we have $\mathrm{Rep}_\alpha(T) = A$. In this case, let

$$P = \mathrm{Rep}_{\beta,\alpha}(id) = \begin{pmatrix} \vec{\beta}_1, \cdots, \vec{\beta}_n \end{pmatrix}$$

be the change of basis matrix. Then

$$A = P \cdot \mathrm{Rep}_\beta(T) \cdot P^{-1}.$$

Especially, when $\vec{\beta}_1, \cdots, \vec{\beta}_n$ are all eigenvectors, then $\mathrm{Rep}_\beta(T)$ is a diagonal matrix.

Projection onto a line

Let $V = \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ be the standard inner product. For two nonzero vectors $\vec{v}, \vec{w}$ in V, set

$$\mathrm{proj}_{\vec{w}}(\vec{v}) = \frac{\langle \vec{w}, \vec{v} \rangle}{\langle \vec{w}, \vec{w} \rangle}\vec{w}.$$

Orthogonal and Orthonormal

A set of vectors $\{\vec{v}_1, \cdots, \vec{v}_m\}$ in V is orthogonal if

$$\langle \vec{v}_i, \vec{v}_j \rangle = 0$$

for all $i \neq j$. If it also satisfies the extra condition: $\langle \vec{v}_i, \vec{v}_i \rangle = 1$ for all i, it is orthonormal.

Theorem 2. *Every orthogonal α set of nonzero vectors is linearly independent. Especially, α forms a basis of V if* $|\alpha| = \dim V$.

Theorem 3. *Let $\alpha = \{\vec{\alpha}_1, \cdots, \vec{\alpha}_n\}$ be an orthogonal basis of V. For $\vec{v} \in V$,*

$$\vec{v} = \sum_{i=1}^{n} \mathrm{proj}_{\alpha_i}(\vec{v}) = \sum_{i=1}^{n} \frac{\langle \vec{\alpha}_i, \vec{v} \rangle}{\langle \vec{\alpha}_i, \vec{\alpha}_i \rangle}\vec{\alpha}_i$$

Gram-Schmidt method

For a (linearly independent) set of vectors $\{\vec{v}_1, \cdots, \vec{v}_m\}$, set

$$\vec{w}_1 = \vec{v}_1, \quad \text{and} \quad \vec{w}_i = \vec{v}_i - \sum_{k=1}^{i-1} \text{proj}_{\vec{w}_k}(\vec{v}_i) \quad \text{for all } i \geq 2.$$

Then

1. $\{\vec{w}_1, \cdots, \vec{w}_m\}$ is orthogonal.
2. $\{\vec{v}_1, \cdots, \vec{v}_k\}$ and $\{\vec{w}_1, \cdots, \vec{w}_k\}$ span the same space for all k.

Invariant subspaces

Let $T : V \mapsto V$ be a linear transform. A subspace W is called a T-invariant subspace if $T(W) \subseteq W$. In other words, the restriction $T\big|_W$ is a linear transform from W to W.

In this case, if α is a basis of W and $\alpha \sqcup \beta$ is a basis of V, then $\text{Rep}_{\alpha\sqcup\beta}(T)$ is a block upper triangular matrix of the fomr $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Here $A = \text{Rep}_\alpha(T\big|_W)$. In this case, $\det(T) = \det(A)\det(B)$.

Direc Sum of Invariant subspaces

Suppose $V = W_1 \oplus W_2$ and W_1 and W_2 are both T- invariant subspace. Let α and β be bases of W_1 and W_2 respectively. Then $\text{Rep}_{\alpha\sqcup\beta}(T)$ is a block diagonal matrix of the fomr $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Here $A = \text{Rep}_\alpha(T\big|_{W_1})$ and $B = \text{Rep}_\beta(T\big|_{W_2})$. In this case, $\det(T) = \det(A)\det(B)$.

Eigenspace, Eigenvalues and Eigenvectors

For $\lambda \in F$, the set

$$E(\lambda) = \{\vec{v} \in V | T(\vec{v}) = \lambda\vec{\}}$$

is always a T-invariant subspace. It is called the eigenspace corresponding to λ if it contains a non-zero element. In this case, λ is called an eigenvalue and non-zero elements of $E(\lambda)$ are called eigenvectors corresponding to λ.

Eigenvalues λ are zeros of the characteristic polynomial

$$f_T(x) = \det(xI - T).$$

which is of degree $n = \dim(V)$. We say that the polynomial $f_T(x)$ splits over F, if $f_T(x)$ has n zeros (counting multiplicity) in F.

When W is a T-invariant subspace of V, we also have $f_{T\big|_W}(x)$ divides $f_T(x)$.

Diagonalization

Theorem 4. *Let $\lambda_1, \cdots, \lambda_k$ be the all distinct eigenvalues of T. The following are equivalent.*

1. *T is diagonalizable.*
2. *There exists a set of eigenvectors of T which forms a basis of V.*
3. *$V = E(\lambda_1) \oplus \cdots \oplus E(\lambda_n)$.*
4. *The polynomial $f_T(x)$ splits and $\dim E(\lambda_i)$ is equal to the multiplicity of λ_i as a zero of $f_T(x)$ for all i.*