27 The mean value theorem and the implicit function theorem

Theorem 27.1 (the mean value theorem for vector-valued functions).
Let $S \subset \mathbb{R}^k$ be open and let $f : S \to \mathbb{R}^m$ a function.

1. *If $a, b \in S$ such that $L := \{a + t(b-a) : 0 \le t \le 1\} \subset S$,
and if f is continuous on L and is differentiable on $L \backslash \{a, b\}$,
then for any vector $u \in \mathbb{R}^m$ written as a column vector,
there exists $c \in L$ such that $u \cdot (f(b) - f(a)) = u \cdot (Df(c)(b-a))$, where $b - a$ is written as a column vector in $\mathbb{R}^k$.
In particular, if $m = 1$, then there exists $c \in L$ such that $f(b) - f(a) = \triangledown f(c) \cdot (b-a)^T$,
where $\triangledown f(c) := (\partial_1 f(c), \cdots, \partial_k f(c))$ and $(b-a)^T$ is rewritten as a row vector, identical to Theorem 19.1.*

2. *If S is convex, f is differentiable on S,
and there exists $M \in [0, \infty)$ such that $\|Df(x)\| \le M$ for all $x \in S$,
where $\|Df(x)\| := \max\{\|Df(x)y\| : y \in \mathbb{R}^k, \|y\| = 1\} = \sup\{\frac{\|Df(x)y\|}{\|y\|} : y \in \mathbb{R}^k, y \ne 0\}$, called the **norm** of $Df(x)$,
then $\|f(b) - f(a)\| \le M\|b - a\|$ for all $a, b \in S$.
(Note that the usual mean value theorem does not hold, e.g., the function $t \mapsto (\cos(t), \sin(t))$ on $[0, 2\pi]$.)*

Theorem 27.2 (the implicit function theorem in one variable).
*Let $f : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ be C^1 on some neighborhood of $(a, b) \in \mathbb{R}^k \times \mathbb{R}$
and write $(x, y) \mapsto f(x, y)$.
If $f(a, b) = 0$ and $\partial_{k+1} f(a, b) \ne 0$,
then there exist $r_0 > 0$ and $r_1 > 0$ such that
for any $x \in B_{r_0}(a)$ there exists a unique $y \in B_{r_1}(b)$ such that $f(x, y) = 0$,
where $B_{r_0}(a) = \{x \in \mathbb{R}^k : \|x - a\| < r_0\}$ and $B_{r_1}(b) = \{y \in \mathbb{R} : |y - b| < r_1\}$);
denote this y by $g(x)$, then $g : B_{r_0}(a) \to B_{r_1}(b)$ is a C^1 function on $B_{r_0}(a)$ with $g(a) = b$
and for all $x \in B_{r_0}(a)$, $f(x, g(x)) = 0$ and $\partial_i g(x) = -\dfrac{\partial_i f(x, g(x))}{\partial_{k+1} f(x, g(x))}$ for all $1 \le i \le k$.*

Corollary 27.3.
*Let $f : \mathbb{R}^k \to \mathbb{R}$ be C^1 on $\mathbb{R}^k$ and let $\gamma = \{x \in \mathbb{R}^k : f(x) = 0\}$.
If $a \in \gamma$ with $\triangledown f(a) \ne 0$,
then there exists a neighborhood U of a in $\mathbb{R}^k$ such that $\gamma \cap U$ is the graph of a C^1 function.*

Theorem 27.4 (the implicit function theorem in several variables).
*Let $f : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^m$ be C^1 on some neighborhood of $(a, b) \in \mathbb{R}^k \times \mathbb{R}^m$
and write $(x, y) \mapsto f(x, y)$ and $f = (f_1, \cdots, f_m)$.
If $f(a, b) = 0$ and $det(A) \ne 0$, where $A = \begin{bmatrix} \partial_{k+1} f_1(a, b) & \cdots & \partial_{k+m} f_1(a, b) \\ \vdots & \ddots & \vdots \\ \partial_{k+1} f_m(a, b) & \cdots & \partial_{k+m} f_m(a, b) \end{bmatrix}$,
then there exist $r_0 > 0$ and $r_1 > 0$ such that
for any $x \in B_{r_0}(a)$ there exists a unique $y \in B_{r_1}(b)$ such that $f(x, y) = 0$,
where $B_{r_0}(a) = \{x \in \mathbb{R}^k : \|x - a\| < r_0\}$ and $B_{r_1}(b) = \{y \in \mathbb{R}^m : \|y - b\| < r_1\}$;
denote this y by $g(x)$, then $g : B_{r_0}(a) \to B_{r_1}(b)$ is a C^1 function on $B_{r_0}(a)$ with $g(a) = b$
and for all $x \in B_{r_0}(a)$, $f(x, g(x)) = 0$ and $\partial_i g(x)$ for $1 \le i \le k$ can be computed
by differentiating $f(x, g(x)) = 0$ with respect to x_i
and solving the resulting linear system of equations for $\partial_i g_1, \cdots, \partial_i g_m$, where $g = (g_1, \cdots, g_m)$.*

28 Curves in the plane and surfaces in the space

Definition 28.1 (C^1 curve and surface).

1. *A set $\gamma \subset \mathbb{R}^2$ is called a C^1* ***curve*** *if every $a \in \gamma$ has a neighborhood U such that $\gamma \cap U$ is the graph of a C^1 function g (either $y = g(x)$ or $x = g(y)$).*

2. *A set $\gamma \subset \mathbb{R}^3$ is called a C^1* ***surface*** *if every $a \in \gamma$ has a neighborhood U such that $\gamma \cap U$ is the graph of a C^1 function g (either $z = g(x,y)$, $y = g(x,z)$, or $x = g(y,z)$).*

Theorem 28.2 (sufficient conditions for C^1 curves in $\mathbb{R}^2$).

1. *(a non-parametric form)*
   Let $S \subset \mathbb{R}^2$ be open, $f : S \to \mathbb{R}$ be C^1 on S, and $\gamma = \{(x,y) \in S : f(x,y) = 0\}$.
   If $a \in \gamma$ and $\triangledown f(a) \neq 0$,
   then there exists a neighborhood U of a in $\mathbb{R}^2$
   such that $\gamma \cap U$ is the graph of a C^1 function g (either $y = g(x)$ or $x = g(y)$).

2. *(a parametric form)*
   Let $f : (a,b) \to \mathbb{R}^2$ be C^1 on (a,b).
   If $t_0 \in (a,b)$ and $Df(t_0) = f'(t_0) \neq 0$,
   then there exists an open interval I of t_0 in (a,b)
   such that the set $\{f(t) : t \in I\}$ is the graph of a C^1 function g
   (either $y = g(x)$ or $x = g(y)$).

Theorem 28.3 (sufficient conditions for C^1 surface in $\mathbb{R}^3$).

1. *(a non-parametric form)*
   Let $S \subset \mathbb{R}^3$ be open, $f : S \to \mathbb{R}$ be C^1 on S
   which is denoted by $(x,y,z) \mapsto f(x,y,z)$, and let $\gamma = \{(x,y,z) \in S : f(x,y,z) = 0\}$.
   If $a \in \gamma$ and $\triangledown f(a) \neq 0$,
   then there exists a neighborhood U of a in $\mathbb{R}^3$
   such that $\gamma \cap U$ is the graph of a C^1 function g
   (either $z = g(x,y)$, $y = g(x,z)$, or $x = g(y,z)$).

2. *(a parametric form)*
   Let $S \subset \mathbb{R}^2$ be open and $f : S \to \mathbb{R}^3$ be C^1 on S, denoted by $(s,t) \mapsto f(s,t)$.
   If $(s_0,t_0) \in S$ and the vectors $\frac{\partial f}{\partial s}(s_0,t_0)$ and $\frac{\partial f}{\partial t}(s_0,t_0)$ are linearly independent
   (equivalently, if $(s_0,t_0) \in S$ and the cross product $\frac{\partial f}{\partial s}(s_0,t_0) \times \frac{\partial f}{\partial t}(s_0,t_0) \neq 0$),
   then there exists a neighborhood U of (s_0,t_0) in S
   such that the set $\{f(s,t) : (s,t) \in U\}$ is the graph of a C^1 function g
   (either $z = g(x,y)$, $y = g(x,z)$, or $x = g(y,z)$)..

29 Curves in $\mathbb{R}^3$ and n-dimensional manifolds in $\mathbb{R}^k$

Definition 29.1. (*curves in* $\mathbb{R}^3$)
A C^1 ***curve*** γ *in* $\mathbb{R}^3$ *is defined to be one of the following:*

1. (*the non-parametric form*) *the set* $\gamma = \{x \in U : f(x) = 0\}$,
where $U \subset \mathbb{R}^3$ *be open and* $f : U \to \mathbb{R}^2$ *is a* C^1 *function with* $f = (f_1, f_2)$
such that the vectors $\triangledown f_1(x)$ *and* $\triangledown f_2(x)$ *are* ***linearly independent*** *at each* $x \in \gamma$,
or equivalently, the matrix $Df(x)$ *has rank* 2 *at every* $x \in \gamma$.

2. (*the parametric form*) *the set* $\gamma = \{f(t) : t \in V\}$,
where $V \subset \mathbb{R}$ *is open and* $f : V \to \mathbb{R}^3$ *is a* C^1 *function*
such that $f'(t) \neq 0$ *at each* $t \in V$.

Definition 29.2. (*manifold in* $\mathbb{R}^k$)
A C^1 n***-dimensional manifold*** γ *in* $\mathbb{R}^k$ *with* $n < k$ *is defined to be one of the following:*

1. (*the non-parametric form*) *the set* $\gamma = \{x \in U : f(x) = 0\}$,
where $U \subset \mathbb{R}^k$ *be open and* $f : U \to \mathbb{R}^{k-n}$ *is a* C^1 *function with* $f = (f_1, \ldots, f_{k-n})$
such that the vectors $\triangledown f_1(x), \ldots, \triangledown f_{k-n}(x)$ *are* ***linearly independent*** *at each* $x \in \gamma$,
or equivalently, the matrix $Df(x)$ *has rank* $k - n$ *at every* $x \in \gamma$.

2. (*the parametric form*) *the set* $\gamma = \{f(t) : t \in V\}$,
where $V \subset \mathbb{R}^n$ *is open and* $f : V \to \mathbb{R}^k$ *is a* C^1 *function*
such that $\frac{\partial f(t)}{\partial t_i}$, $i = 1, \ldots, n$, *are* ***linearly independent*** *at each* $t \in V$,
or equivalently, the matrix $Df(t)$ *has rank* n *at each* $t \in V$.

In fact, in the theory of Differential Geometry,
we define a C^1 n***-dimensional manifold*** γ *by using local charts so that* γ *is locally like* $\mathbb{R}^n$,
then the above becomes sufficient conditions for γ *being a* C^1 n*-dimensional manifold*
and the tangent space of γ *at each point on it is well defined.*

30 Transformation and the inverse function theorem

Recall 30.1. (*transformation*)
A function $f : \mathbb{R}^k \to \mathbb{R}^k$ is called a ***transformation****.*

Theorem 30.2 (the inverse function theorem)**.**
Let $U, V \subset \mathbb{R}^k$ be open and $f : U \to V$ be C^1 on U.
If $a \in U$ and the Jacobian $det(Df(a)) \neq 0$,
then there exist open neighborhoods $U_1 \subset U$ of a and $V_1 \subset V$ of $f(a)$
such that f is one-to-one from U_1 onto V_1,
the inverse function $f^{-1} : V_1 \to U_1$ is C^1,
and $Df^{-1}(f(x)) = [Df(x)]^{-1}$ for all $x \in U_1$.

Remark 30.3. (*globally one-to-one*)
Let $U, V \subset \mathbb{R}^k$ be open. Suppose $f : U \to V$ is C^1 and $det(Df(x)) \neq 0$ for all $x \in U$.
Is f one-to-one on U?

1. *If $k = 1$, the answer is YES.*
2. *If $k > 1$, the answer is NO.*

Problem 30.4 (Jacobian conjecture)**.**
Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a map whose component functions are all polynomials.
Suppose $det(Df(x)) = 1$ for all $x \in \mathbb{R}^k$.
Is f one-to-one on $\mathbb{R}^k$?
(*If yes, then one can prove that the inverse of f is a map defined on $\mathbb{R}^k$*
whose component functions are also all polynomials.)
The conjecture that the answer is yes is called ***Jacobian conjecture****;*
this is a famous unsolved problem.