Introduction to Analysis Final preparation

0712238 林彥彤

回應: Final Exam 候選題
由應數博 賴冠宇發表於2020年 01月 2日(Thu) 15:17

21.7

21.8

24.1

24.5

HW12全

Theorem 21.7 (the high-dimensional binomial theorem). (*exercise*)
For two vectors $x = (x_1, \cdots, x_k), y = (y_1, \cdots, y_k) \in \mathbb{R}^k$ *and a multi-index* $\gamma = (\gamma_1, \cdots, \gamma_k) \in \mathbb{N}_0^k$,
we have $(x+y)^\gamma = \sum\limits_{\alpha,\beta \in \mathbb{N}_0^k, \alpha+\beta=\gamma} \frac{\gamma!}{\alpha!\beta!} x^\alpha y^\beta = \sum\limits_{\substack{\alpha_i+\beta_i=\gamma_i \text{ for all } 1\le i\le k \\ 0\le\alpha_i,\beta_i\le\gamma_i}} \frac{\gamma_1!\cdots\gamma_k!}{\alpha_1!\cdots\alpha_k!\beta_1!\cdots\beta_k!} x_1^{\alpha_1}\cdots x_k^{\alpha_k} y_1^{\beta_1}\cdots y_k^{\beta_k}.$

Proposition 21.8 (the product rule for partial derivatives). (*exercise*)
Let $S \subset \mathbb{R}^k$ *be open,* $f, g : S \to \mathbb{R}$ *be* C^r *differentiable on* S, *and* $\gamma = (\gamma_1, \cdots, \gamma_k) \in \mathbb{N}_0^k$.
Then $\partial^\gamma(fg) = \sum\limits_{\alpha,\beta\in\mathbb{N}_0^k, \alpha+\beta=\gamma} \frac{\gamma!}{\alpha!\beta!} (\partial^\alpha f)(\partial^\beta g) = \sum\limits_{\substack{\alpha_i+\beta_i=\gamma_i \text{ for all } 1\le i\le k \\ 0\le\alpha_i,\beta_i\le\gamma_i}} \frac{\gamma_1!\cdots\gamma_k!}{\alpha_1!\cdots\alpha_k!\beta_1!\cdots\beta_k!} (\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_k^{\alpha_k} f)(\partial_1^{\beta_1}\partial_2^{\beta_2}\cdots\partial_k^{\beta_k} g).$

Theorem 24.1 (the uniqueness of the Taylor polynomial).
Let $S \subset \mathbb{R}^k$ *be an open convex set,* $c = (c_1, c_2, \cdots, c_k) \in S$, *and* $m \ge 1$ *be an integer.*
Let $f : S \to \mathbb{R}$ *be* C^m *on* S *and* $f(x) = Q(x-c) + E(x-c)$ *for all* $x \in S$,
where $Q : \mathbb{R}^k \to \mathbb{R}$ *is a polynomial in* $x - c$ *of degree* $\le m$
and $E : \mathbb{R}^k \to \mathbb{R}$ *is a function with* $\lim\limits_{x\to c} \frac{E(x-c)}{\|x-c\|^m} = 0$.
Then for any $x = (x_1, x_2, \cdots, x_k) \in S$,
$$Q(x-c) = \sum_{\substack{n=0 \\ \alpha_1+\alpha_2+\cdots+\alpha_k=n \\ 0\le\alpha_1,\alpha_2,\cdots,\alpha_k\le n}}^{m} \frac{\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_k^{\alpha_k} f(c)}{\alpha_1!\alpha_2!\cdots\alpha_k!}(x_1-c_1)^{\alpha_1}(x_2-c_2)^{\alpha_2}\cdots(x_k-c_k)^{\alpha_k},$$
i.e., $Q(x-c)$ *is the mth-order Taylor polynomial for* f *about* c *on* S.

If $k = 1$, *we may only require* f *is* C^{m-1} *on* S *and* $f^{(m)}(c)$ *exists.* (*exercise*)

Theorem 24.5 (extreme value theorem for C^r functions). (*exercise*)
Let $f : (a,b) \to \mathbb{R}$ be a C^r function on (a,b), where $r \in \mathbb{N}$, and let $c \in (a,b)$.
If $f^{(n)}(c) = 0$ for all $1 \leq n < r$ and $f^{(r)}(c) \neq 0$, then the followings hold.

1. *If r is even and $f^{(r)}(c) > 0$, then f has a local minimum at c.*
2. *If r is even and $f^{(r)}(c) < 0$, then f has a local maximum at c.*
3. *If r is odd, then f has neither a local minimum nor a local maximum at c.*

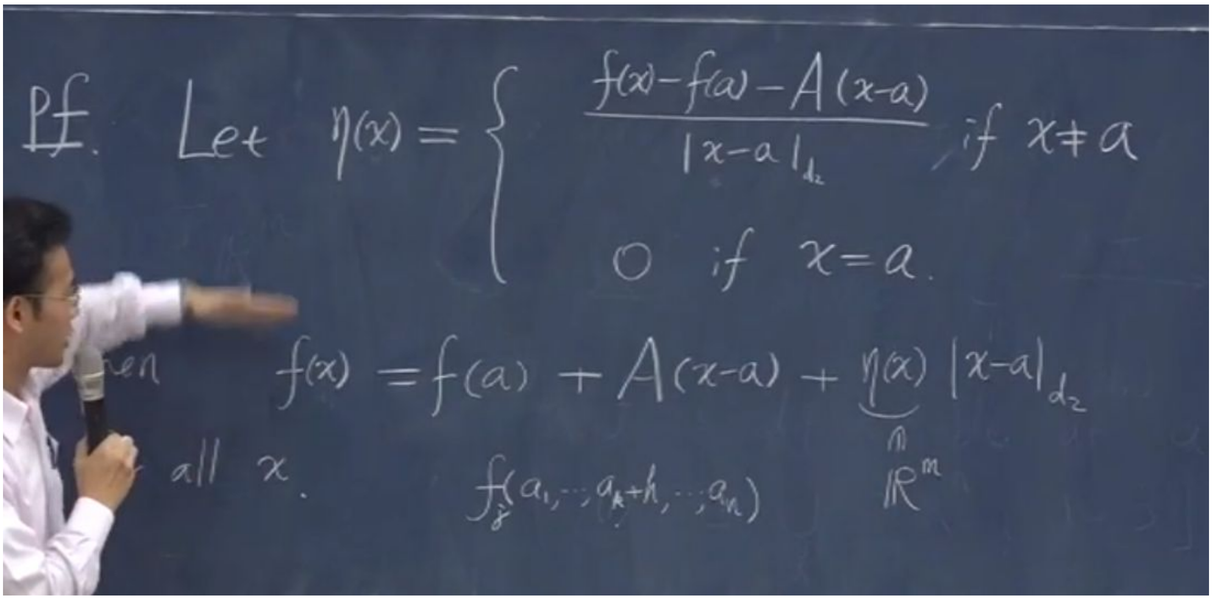
Exercise 12

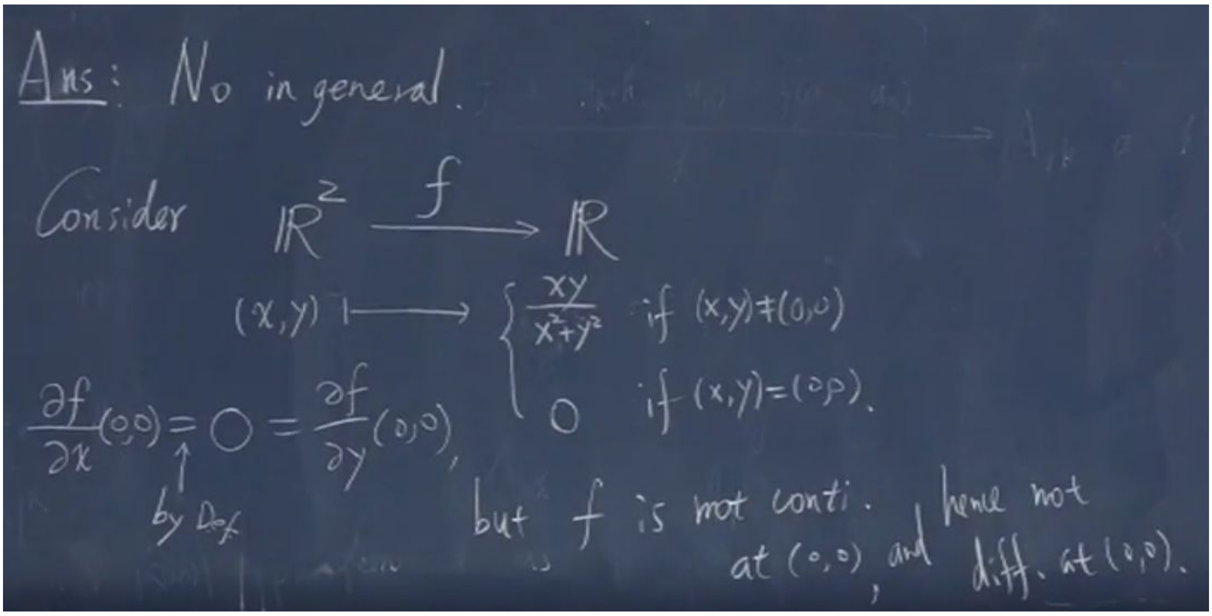
2019/12/24

- Find the maximum and minimum values of $f(x,y) = x^2 + x + 2y^2$ on the unit circle.
- Let $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ and $rank(A) = n$, $b \in \mathbb{R}^m$. Solve the following optimization problem: $x^* = \arg\min_x \|Ax - b\|^2$.
- Let $A \in \mathbb{R}^{m\times n}$ with $m \leq n$ and $rank(A) = m$, $b \in \mathbb{R}^m$. Solve the following optimization problem: $x^* = \arg\min_x \|x\|^2$ subject to $Ax = b$
- Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $f(u,v) = (u^3 + uv + v^3, u^2 - v^2)$. Show that there is a neighborhood U of $(u,v) = (1,1)$ on which $f|_U : U \to f(U)$ is diffeomorphic. Calculate $D((f|_U)^{-1})$.
- Theorem 26.7

Theorem 26.7. (C^1 *functions*)
Let $S \subset \mathbb{R}^k$ be open and $f : S \to \mathbb{R}^m$ with $f = (f_1, \cdots, f_m)$.
Then the partial derivatives $\partial_j f_i$ all exist and are continuous on S
$\Longleftrightarrow$ *f is* ***continuously differentiable*** *on S,*
that is, f is differentiable on S and $Df : S \to L(\mathbb{R}^k, \mathbb{R}^m)$ is continuous on S
(in this case we say that f is C^1 on S or $f \in C^1(S)$). (*advanced exercise*)

https://www.youtube.com/watch?time_continue=247&v=et2sTg9qslw&feature=emb_logo
CORRESPONDING TO CHAPTER 26





Theorem (the chain rule)

Given
$$\begin{array}{ccccc}
\mathbb{R}^n & & \mathbb{R}^m & & \\
\cup\!\!\shortmid & & \cup\!\!\shortmid & & \\
S & \xrightarrow{\ f\ } & T & \xrightarrow{\quad g \quad} & \mathbb{R}^\ell \\
\Downarrow\!\!\!\shortmid\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} & & \\
a & \longmapsto & f(a) & &
\end{array}$$
and $A \in M_{m,n}(\mathbb{R})$, $B \in M_{\ell,m}(\mathbb{R})$,

if
$$\begin{cases} f(x) \underset{1}{\sim} f(a) + A(x-a) & \text{as } x \to a \\ g(y) \underset{1}{\sim} g(f(a)) + B(y - f(a)) & \text{as } y \to f(a) \end{cases} \Longrightarrow g(f(x)) \underset{1}{\sim} g(f(a)) + BA(x-a) \quad \text{as } x \to a.$$

More explicitly,

$$\left.\frac{\partial g_j(f_1(x_1,\cdots,x_n),\cdots,f_m(x_1,\cdots,x_n))}{\partial x_k}\right|_{(x_1,\cdots,x_n)=(a_1,\cdots,a_n)}$$

$$= \sum_{s=1}^{m} \frac{\partial g_r}{\partial y_s}(f_1(a_1,\cdots,a_n),\cdots,f_m(a_1,\cdots,a_n))\ \frac{\partial f_s}{\partial x_k}(a_1,\cdots,a_n)$$

Ex f is continuously diff on S and $a \in S \Longrightarrow$ f is diff at a.

\

Example Consider $f(x) = x^{x^3}$, $x > 0$

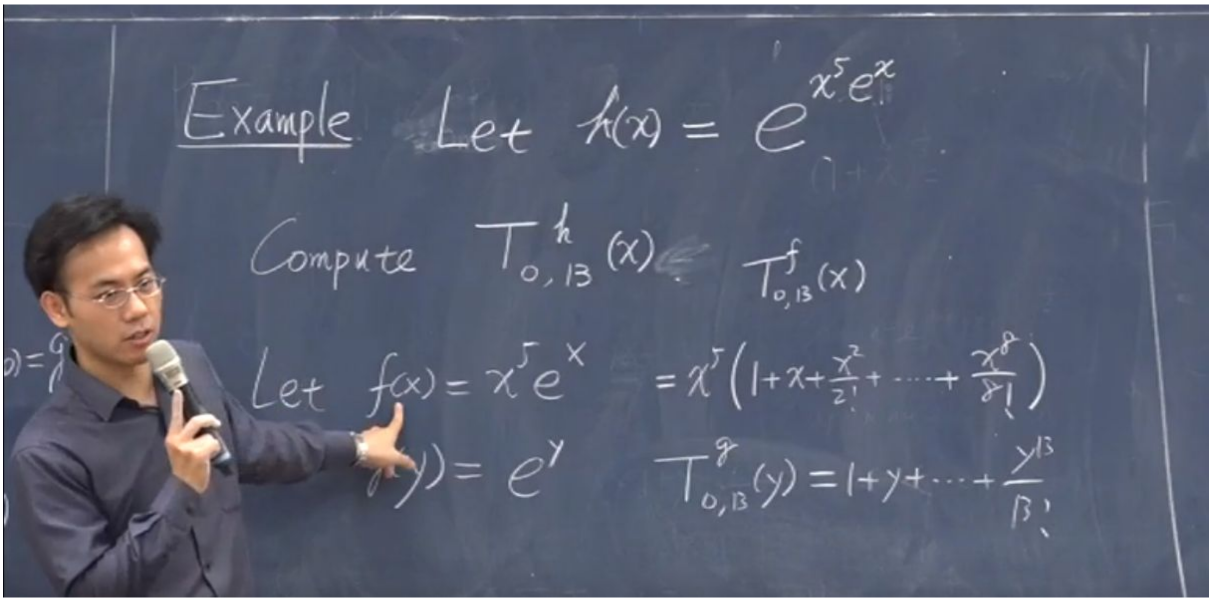
$f'(x)$. $\quad g(y_1, y_2) = y_1^{y_2} \quad (y_1 > 0)$

$$\frac{\partial g}{\partial y_1} = y_2\, y_1^{y_2 - 1}, \quad \frac{\partial g}{\partial y_2} = y_1^{y_2} \ln y_1 \Longrightarrow$$

Both are conti.

$\Longrightarrow g$: conti.ly diff.

$$f(x) = g(x, x^3) \qquad f'(x) = \frac{\partial g}{\partial y_1}(x, x^3)\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial g}{\partial y_2}(x, x^3)\underbrace{\frac{dx^3}{dx}}_{=3x^2}$$



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To simplify life, use the $\|\cdot\|_1$ norm. To reduce clutter, let $\phi_k(h) = (h_1, \ldots, h_k, 0, \ldots 0)$. Let $\phi_0(h) = 0$, and note that $\|\phi_k(h) - \phi_j(h)\|_1 \leq \|h\|_1$.

First suppose $m = 1$.

Let $\epsilon > 0$.

By continuity, we can choose $\delta > 0$ such that $|D_k f(a+h) - D_k f(a)| < \epsilon$ for all k and $\|h\| < \delta$.

Let $A = (D_1 f(a), \ldots, D_n f(a))$ and suppose $\|h\| < \delta$, then we have

$$\begin{aligned} |f(a+h) - f(a) - Ah| &= |\sum_{k=1}^{n} (f(a+\phi_k(h)) - f(a+\phi_{k-1}(h)) - D_k f(a) h_k)| \\ &\leq \sum_{k=1}^{n} |f(a+\phi_k(h)) - f(a+\phi_{k-1}(h)) - D_k f(a) h_k| \end{aligned}$$

By the mean value theorem, there are $c_k \in [a+\phi_{k-1}(h), a+\phi_k(h)]$ (that is, each c_k lies on the line segment) such that $f(a+\phi_k(h)) - f(a+\phi_{k-1}(h)) = D_k f(c_k) h_k$. Note that $\|c_k - a\|_1 \leq \|h\|_1$. Continuing:

$$\begin{aligned} |f(a+h) - f(a) - Ah| &\leq \sum_{k=1}^{n} |D_k f(c_k) h_k - D_k f(a) h_k| \\ &= \sum_{k=1}^{n} |D_k f(c_k) - D_k f(a)||h_k| \\ &< \epsilon \sum_{k=1}^{n} |h_k| \\ &= \epsilon \|h\|_1 \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this shows that f is differentiable at a and $Df(a)h = Ah$.

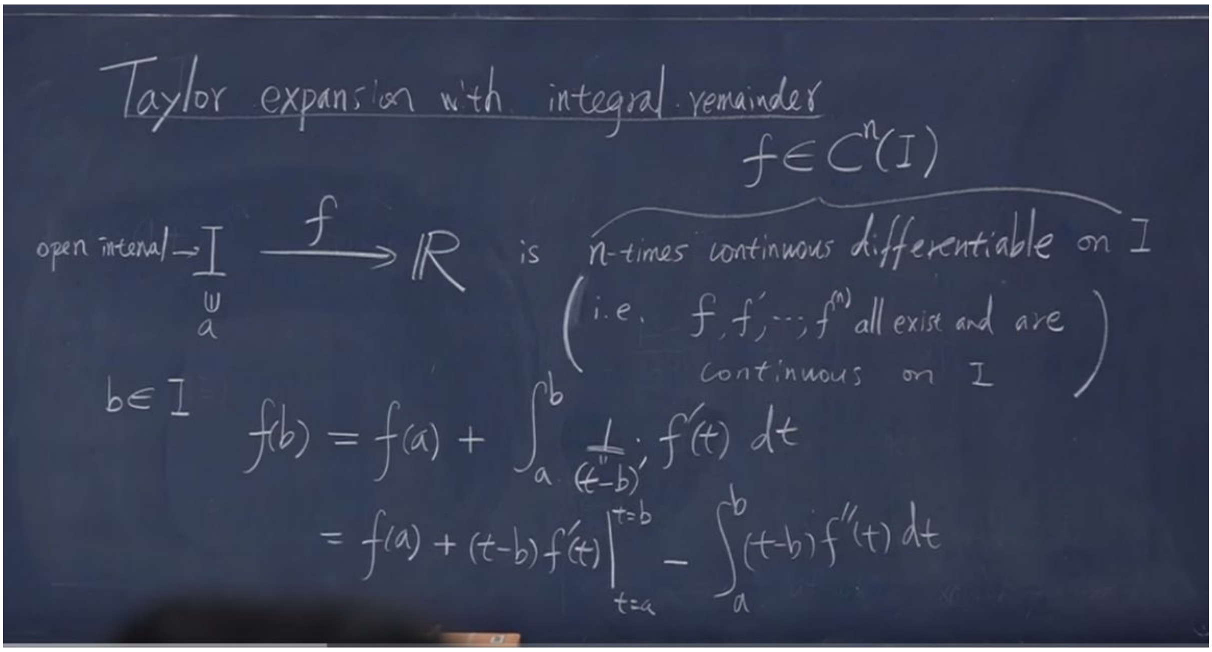
It is straightforward to show that if $f_1, \ldots f_m$ are differentiable, then so is $f(x) = (f_1(x), \ldots, f_n(x))$, and $Df(x)h = (Df_1(x)h, \ldots, Df_m(x)h)$.

differentiable but not continuous differentiable

A good example of such a function is $x_2(\sin(1/x_2))$

which has a finite derivative at x=0,

but the derivative is essentially discontinuous at x=0



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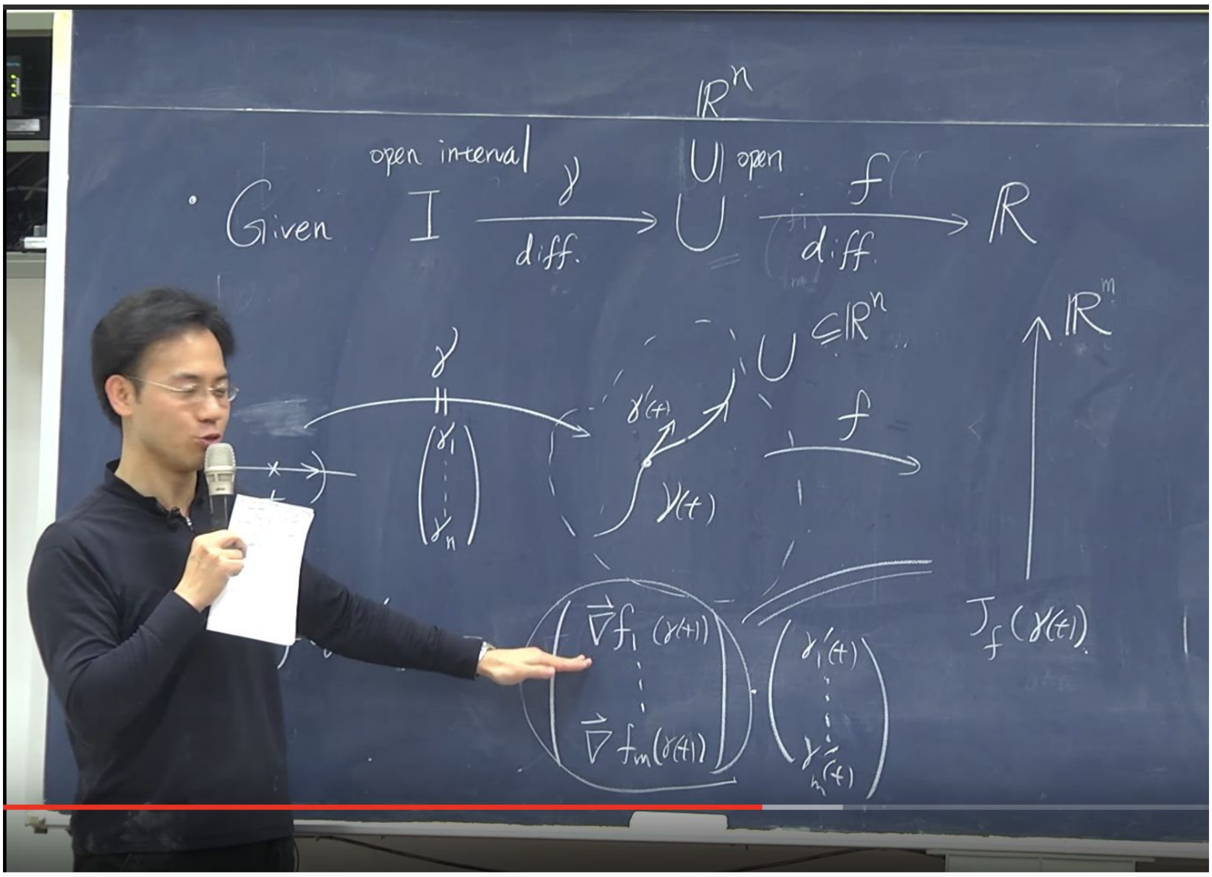
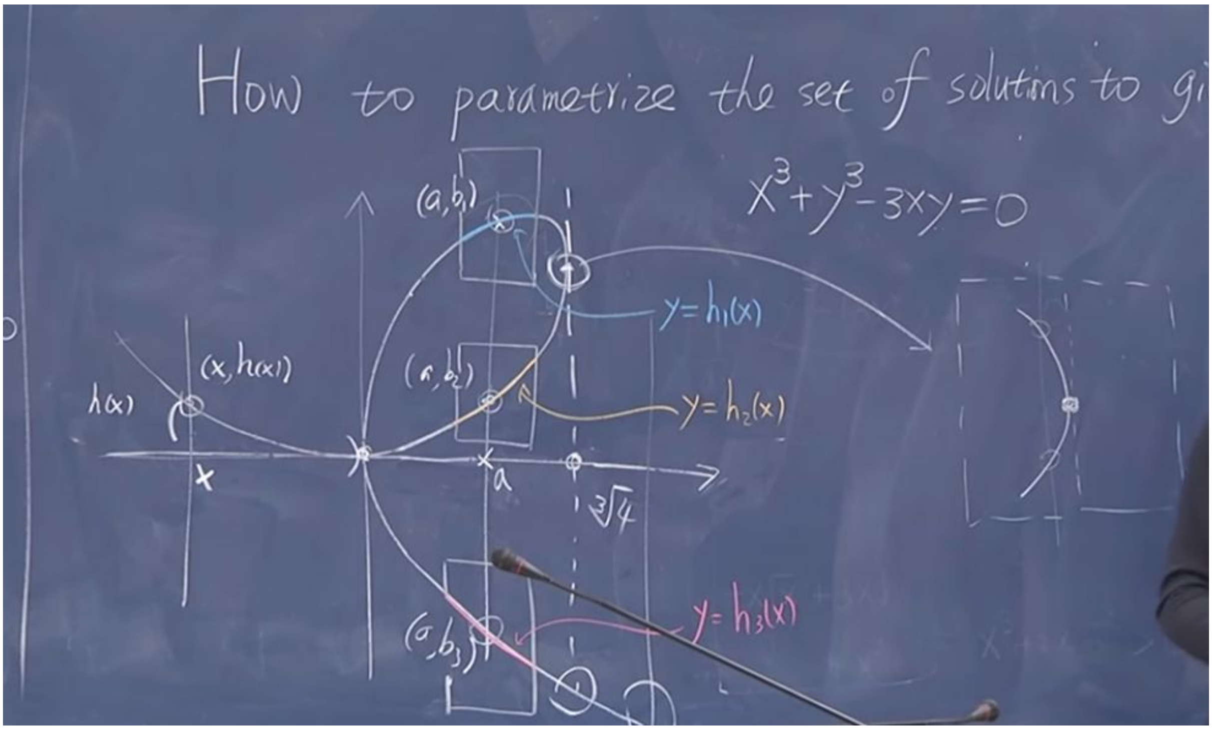
積分餘項
多變數泰勒
【微積分一/齊震宇老師2015/12/22A】Taylor展開式餘項的積分表達

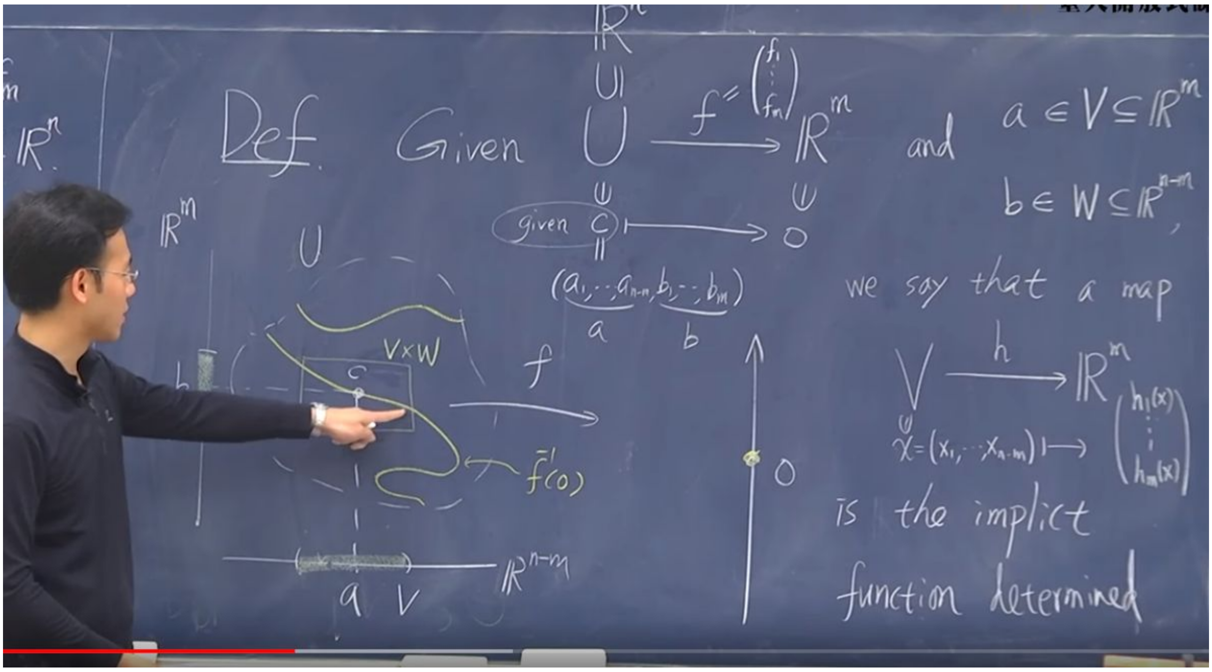
https://www.youtube.com/watch?v=O4n9laJ2mR8
微積分二:【多變數微分理論2】梯度向量；(積分與偏導數混合版本的)均值定理

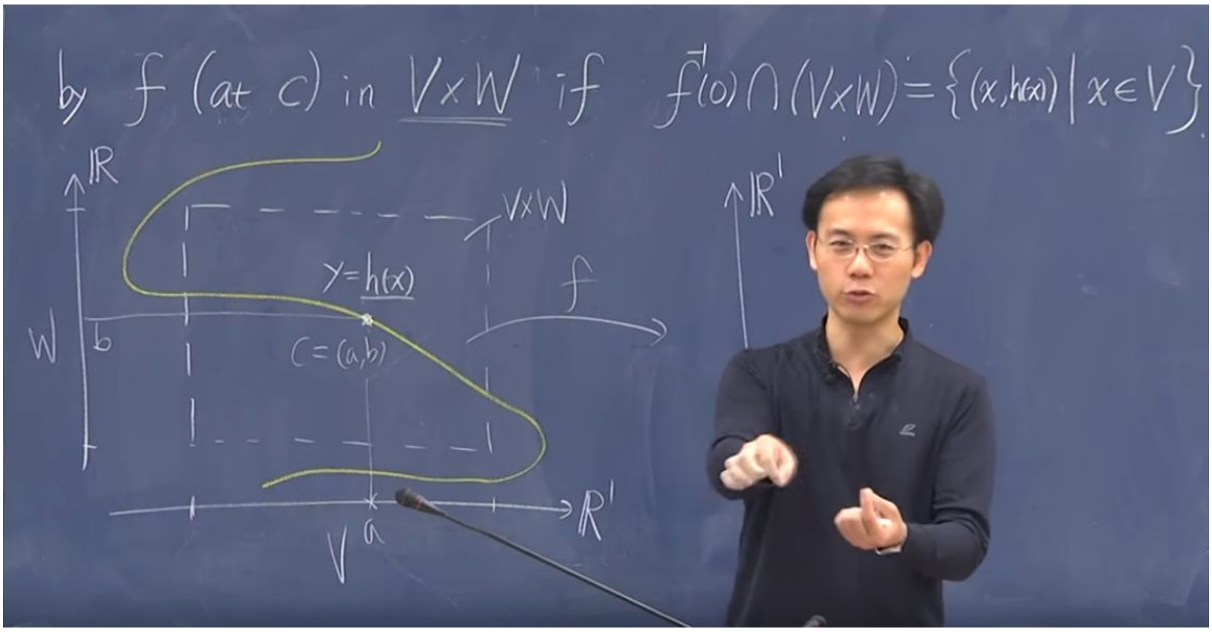


Given
open interval
diff.
U open
diff.



How to parametrize the set of solutions to gi
x3+y3-3xy=0
y=h1(x)
y=h2(x)
y=h3(x)
(x,h(x))
h(x)





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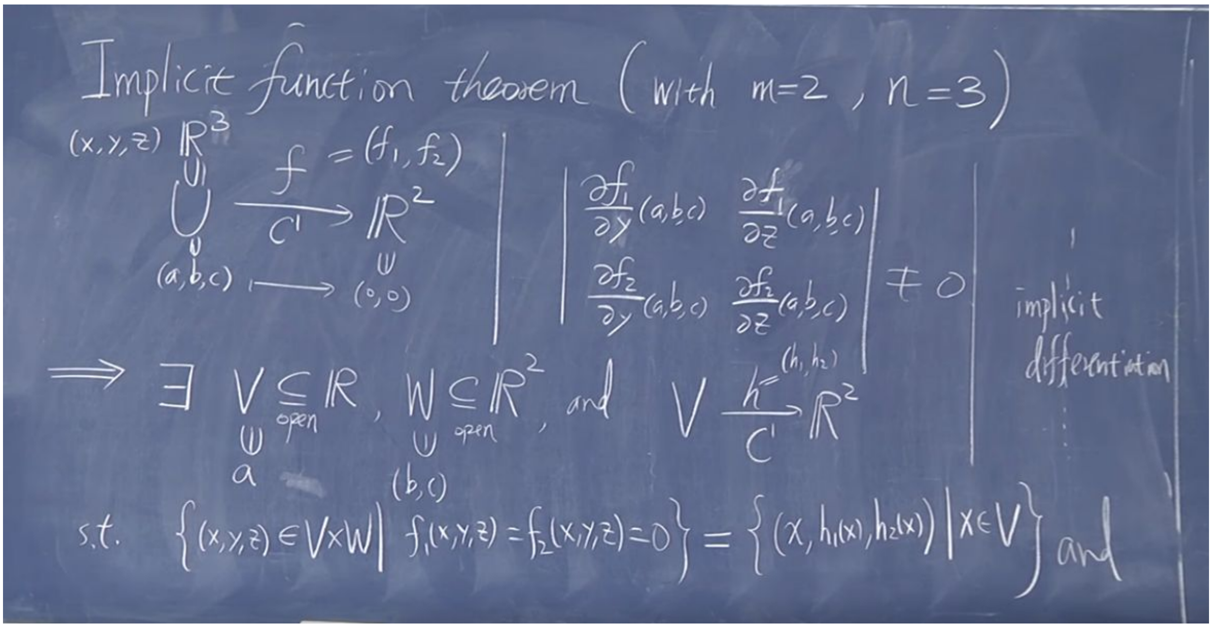

教授您好:
注意到課程網站說1/9號13:00會送學期成績至註冊組
但1/10期末考卷才能確認
是否應該在課堂確認考卷沒問題之後再送成績比較好
由於期末考很多科忘記提早寄信很抱歉

彥彤 敬上