Introduction to Real Analysis

1 Function and ordered set

Notation 1.1. (*basic notations*)

1. ***Sum:*** $\sum_{i=m}^{n} a_i$, *where* $a_i \in \mathbb{R}$;
2. ***Factorial:*** $0! = 1$ *and* $n!$, *where* $n \in \mathbb{N}$.
3. ***Set:*** $S \subset T$, $S \supset T$, $S = T$, $S \cup T$, $S \cap T$, $S \backslash T$, $\bigcup_{i=1}^{n} S_i$, $\bigcap_{i=1}^{n} S_i$,
   and $\{x \in S : \text{ property}(x) \text{ holds}\}$.
4. ***Real numbers:*** $\mathbb{N} \subset \mathbb{N}_0 = \mathbb{N} \cup \{0\} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, *with the* ***addition*** $x + y$, *the* ***multiplication*** xy,
   the ***absolute value*** $|x|$, *and the* ***ordering:*** *exactly one of* $x < y$, $y < x$, *or* $x = y$ *holds.*
5. *The* ***extended real numbers:*** $\mathbb{R} \cup \{\pm\infty\}$.
6. ***Complex numbers:*** $\mathbb{C}$, $i = \sqrt{-1}$,
   $z = x + yi$ *where* $x, y \in \mathbb{R}$ *are called the* ***real*** *and* ***imaginary parts*** *of* z, *respectively,*
   $\overline{z} = x - yi$ *is called the* ***complex conjugate*** *of* z,
   $|z| = \sqrt{x^2 + y^2}$ *is called the* ***absolute value*** *or* ***modulus*** *of* z.
7. ***Logic symbol:*** *a statement* P *and its* ***negation*** $\sim P$, *a statement* $P \Longrightarrow Q$ *and its* ***converse*** $Q \Longrightarrow P$

Definition 1.2. (*function*)
Let A *and* B *be two sets.*
A rule f *that assigns to each element of* A *one and only one element of* B
is called a ***function from*** A ***to*** B *and is denoted by* $f : A \to B$ *or* $x \mapsto f(x)$.
The set A *is called the* ***domain*** *of* f, *the set* B *is called the* ***codomain*** *of* f,
the element $f(x)$ *is called the* ***value*** *of* f *at* x,
and the set of all values of f *on* A *is called the* ***range*** *of* f.

If $E \subset A$, *the set* $f(E) := \{f(x) \in B : x \in E\}$ *is called the* ***image*** *of* E *under* f.
We say that f *is a* ***surjective*** *function from* A *to* B (*or* f *maps* A ***onto*** B) *if* $f(A) = B$.
If $E \subset B$, *the set* $f^{-1}(E) := \{x \in A : f(x) \in E\}$ *is called the* ***pre-image*** *of* E *under* f; *although* f^{-1} *might not be a function.*
We say that f *is an* ***injective*** (*or* ***one-to-one***) *function from* A *to* B
if $f(x) = f(y)$ *implies* $x = y$.
A function $f : A \to B$ *is said to be* ***invertible*** *if there is a function* $g : B \to A$
such that $g(f(x)) = x$ *for all* $x \in A$ *and* $f(g(y)) = y$ *for all* $y \in B$.
In this case, the function g *is called the* ***inverse*** *of* f *and is denoted by* f^{-1}.

If $B = \mathbb{R}$ (*resp.* $\mathbb{R}^k$, $\mathbb{C}$), *we call* $f : A \to B$
a ***real-valued*** (*resp.* ***vector-valued***, ***complex-valued***) ***function***.

Definition 1.3. (*ordered set*)
Let S *be a set.*
An ***ordering*** *on* S *is a relation, denoted by* $<$, *with the following two properties:*

1. *If* $x, y \in S$, *then one and only one of the statements* $x < y$, $x = y$ *and* $y < x$ *is true.*
2. *If* $x, y, z \in S$ *with* $x < y$ *and* $y < z$, *then* $x < z$.

An ***ordered set*** *is a set* S *in which an ordering* $<$ *is defined.*

Proposition 1.4.
Let $a, b \in \mathbb{R}$. *Then* $a = b \Longleftrightarrow$ *for any* $\epsilon > 0$, *the inequality* $|a - b| < \epsilon$ *holds.*

Definition 1.5.
A (***Hermitian***) ***inner product space*** *is a vector space* V *over a field* F, *equal to* $\mathbb{R}$ *or* $\mathbb{C}$, *together with an* ***inner product***,
i.e., with a function $\langle \cdot, \cdot \rangle : V \times V \to F$ *satisfying for any* $u, v, w \in V$ *and* $\alpha \in F$,

1. *conjugate symmetry:* $\langle v, w \rangle = \langle w, v \rangle$ *if* $F = \mathbb{R}$, *and* $\langle v, w \rangle = \overline{\langle w, v \rangle}$ *if* $F = \mathbb{C}$.
2. *linearity in the first argument:* $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ *and* $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$.
3. *positive definiteness:* $\langle v, v \rangle \geq 0$ *and equal if and only if* $v = 0$.

2 Euclidean space

Definition 2.1. (*Euclidean space*)
Let $\mathbb{R}^k = \{(x_1, x_2, \cdots, x_k) : x_i \in \mathbb{R}$ *for all* $1 \le i \le k\}$,
that is, $\mathbb{R}^k$ *is the set of ordered k-tuples of real numbers, or* $\mathbb{R}^k$ *is the Cartesian product of* $\mathbb{R}$ *with itself k times.*
The elements of $\mathbb{R}^k$ *are called* ***points*** *or* ***vectors***.
The ***addition***, *defined by* $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \cdots, x_k + y_k)$,
and the ***scalar multiplication***, *defined by* $\alpha\mathbf{x} = (\alpha x_1, \cdots, \alpha x_k)$ *where* $\alpha \in \mathbb{R}$,
makes $\mathbb{R}^k$ *a* ***vector space over the real field*** $\mathbb{R}$.
with the standard basis $\{e_j\}_{j=1}^k$, *where* e_j *is the element of* $\mathbb{R}^k$ *with* 1 *in the* j*-th place and* 0*'s elsewhere.*
The ***inner*** *or* ***dot product*** *defined by* $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^k x_i y_i$ *turns the vector space* $\mathbb{R}^k$ *into an inner product space and the* ***Euclidean norm*** *defined by* $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{i=1}^k x_i^2}$ *turns the inner product space* $\mathbb{R}^k$ *a normed vector space.*
The set $\mathbb{R}^k$ *with all the above structure is called the* ***k-dimensional Euclidean space***.

Theorem 2.2 (the Cauchy-Schwarz and triangle inequalities)**.**
Let $\mathbf{x} = (x_1, \cdots, x_k)$, $\mathbf{y} \in \mathbb{R}^k$ *and* $|\cdot|$ *denote the absolute value function on* $\mathbb{R}$. *Then the followings hold.*

1. $|\mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}\| \|\mathbf{y}\|$ (*called the* ***Cauchy-Schwarz inequality***);
2. $\|\cdot\| : \mathbb{R}^k \to \mathbb{R}$ *is a* ***norm***, *that is, it satisfies the followings:*
   - *if* $\|\mathbf{x}\| = 0$ *then* $\mathbf{x} = \mathbf{0}$,
   - $\|\alpha\mathbf{x}\| = |\alpha| \|\mathbf{x}\|$ *for all* $\alpha \in \mathbb{R}$, *and*
   - $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$ (*called the* ***triangle inequality***);
3. $\max\{|x_1|, \cdots, |x_k|\} \le \|\mathbf{x}\| \le \sqrt{k} \max\{|x_1|, \cdots, |x_k|\}$;
4. *Any norm* $\|\cdot\|_1 : \mathbb{R}^k \to \mathbb{R}$ *gives us that* $|\|\mathbf{x}\|_1 - \|\mathbf{y}\|_1| \le \|\mathbf{x} - \mathbf{y}\|_1$ *and* $\|\mathbf{x}\|_1 \ge 0$.

Definition 2.3. (*metric space*)
Let E *be a set and* $E \times E \equiv \{(x, y) : x, y \in E\}$.
The set E *is called a* ***metric space***
if there is a function $d : E \times E \to [0, \infty)$, *called a* ***metric***,
such that for all $x, y \in E$,

1. $d(x, x) = 0$ *and* $d(x, y) > 0$ *if* $x \ne y$; (*called the* ***positive definite property***);
2. $d(x, y) = d(y, x)$ (*called the* ***symmetric property***);
3. $d(x, y) \le d(x, z) + d(z, y)$ *for any* $z \in E$ (*called the* ***triangle inequality***).

Example 2.4.
The Euclidean space $\mathbb{R}^k$ *is a metric space with each of the following metrics:* $d_E(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^k (x_i - y_i)^2}$, $d_{max}(\mathbf{x}, \mathbf{y}) = \max\limits_{1 \le i \le k} |x_i - y_i|$, *and* $d_p(\mathbf{x}, \mathbf{y}) = \sqrt[p]{\sum_{i=1}^k |x_i - y_i|^p}$ *for all* $p \in \mathbb{N}$.

Definition 2.5. (*cross product*)
Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ *be two nonzero vectors.*
The number $\theta = \arccos\left(\dfrac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}\right)$ *in* $[0, \pi]$, *guaranteed by the Cauchy-Schwarz inequality,*
is called the ***angle*** *between* $\mathbf{x}$ *and* $\mathbf{y}$.
We say that $\mathbf{x}$ *and* $\mathbf{y}$ *are* ***orthogonal*** *to each other if* $\mathbf{x} \cdot \mathbf{y} = 0$.

Let $\mathbf{x} = (x_1, x_2, x_3)$ *and* $\mathbf{y} = (y_1, y_2, y_3)$ *be two vectors in* $\mathbb{R}^3$,
and let $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0)$, *and* $\mathbf{k} = (0, 0, 1)$ *be the standard basis vectors in* $\mathbb{R}^3$.
The ***cross product*** *of* $\mathbf{x}$ *and* $\mathbf{y}$ *is defied to be* $\mathbf{x} \times \mathbf{y} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$.

Remark 2.6.
Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.
Then $\|\mathbf{x} \times \mathbf{y}\|$ *equals to the area of the parallelogram sided by* $\mathbf{x}$ *and* $\mathbf{y}$
and $\mathbf{x} \times \mathbf{y}$ *is a vector, by the "right-hand rule", orthogonal to both* $\mathbf{x}$ *and* $\mathbf{y}$.

3 Open and closed sets

Definition 3.1. (*open and closed sets*)
The following is defined with respect to the whole space $\mathbb{R}^k$ *or a metric space* E *with easy modifications.*
The closure $\overline{S}$ *is the smallest closed set containing* S

1. *The set* $\{x \in \mathbb{R}^k : \|x - a\| = r\}$ *is called the* ***sphere*** *with center at* $\mathbf{a}$ *and radius* $r > 0$.
2. *The set* $B_r(a) := \{x \in \mathbb{R}^k : \|x - a\| < r\}$ *is called the (open)* ***ball*** *with center at* $\mathbf{a}$ *and radius* $r > 0$.
3. *The set* $S^c := \{x \in \mathbb{R}^k : x \notin S\}$ *is called the* ***complement*** *of* S.
4. *The set* S *is said to be* ***bounded*** *if there exist a point* $p \in \mathbb{R}^k$ *and a real number* $r > 0$ *such that* $S \subset B_r(p)$.
5. *A point* $x \in \mathbb{R}^k$ *is called an* ***interior point*** *of* S *if there exists* $r > 0$ *such that* $B_r(x) \subset S$.
   The set of all interior points of S *is called the* ***interior*** *of* S *and is denoted by* S^{int}.
6. *A point* $x \in \mathbb{R}^k$ *is called a* ***boundary point*** *of* S *if for any* $r > 0$, $B_r(x) \cap S \neq \varnothing$ *and* $B_r(x) \cap S^c \neq \varnothing$.
   The set of all boundary points of S *is called the* ***boundary*** *of* S *and is denoted by* ∂S.
7. *The set* S *is said to be* ***closed*** *if* $\partial S \subset S$.
8. *The set* S *is said to be* ***open*** *if* $\partial S \subset S^c$.
9. *The set* $\overline{S} := S \cup \partial S$ *is called the* ***closure*** *of* S.
10. *The set* S *is said to be* ***dense*** *in* $\mathbb{R}^k$ *if* $\overline{S} = \mathbb{R}^k$.
11. *A set* $T \subset \mathbb{R}^k$ *is called a* ***neighborhood*** *of* a *if* a *is an interior point of* T.

Proposition 3.2.
Let $S \subset \mathbb{R}^k$. *Then*

1. $\mathbb{R}^k$ *is decomposed into three pairwise disjoint subsets* S^{int}, ∂S, *and* $(S^c)^{int}$.
2. S *is open* $\Longleftrightarrow S \subset S^{int} \Longleftrightarrow S^c$ *is closed.*

Corollary 3.3 (exercises)**.**
The following holds.

1. *Let* $S \subset \mathbb{R}^k$. *Then* S *is open* $\Longleftrightarrow$ *for any* $x \in S$ *there exists* $r > 0$ *such that* $B_r(x) \subset S$.
2. *The union of open subsets of* $\mathbb{R}^k$ *is open in* $\mathbb{R}^k$.
3. *The intersection of finitely many open subsets of* $\mathbb{R}^k$ *is open in* $\mathbb{R}^k$.
4. *The union of finitely many closed subsets of* $\mathbb{R}^k$ *is closed in* $\mathbb{R}^k$.
5. *The intersection of closed subsets of* $\mathbb{R}^k$ *is closed in* $\mathbb{R}^k$.
6. *Let* S *and* T *be two subsets of* $\mathbb{R}^k$ *such that* $S \cap T = \varnothing$ *and* T *is open, then* $\overline{S} \cap T = \varnothing$.

For normed spaces as $\mathbb{R}^k$, norms induce metrics, and metrics induce topologies.

4 Limit and continuity

Definition 4.1. (*limit and continuity*)
Let $E \subset \mathbb{R}^m$ and $f : E \to \mathbb{R}^k$ be a function. Let $p \in \overline{E}$ satisfy $\{x \in E : 0 < \|x - p\| < r\} \neq \varnothing$ for all $r > 0$.
If $A \in \mathbb{R}^k$ with the the following property:

$$\text{for any } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that if } x \in E \text{ with } 0 < \|x - p\| < \delta \text{ then } \|f(x) - A\| < \epsilon,$$

*we say that the **limit** of f at p is A and write $\lim\limits_{x\to p} f(x) = A$ or $f(x) \to A$ as $x \to p$.*
(*The notation makes sense because such an A is unique.*)
*If $m = 1$, one can define the **one-sided limits** by $\lim\limits_{x\to p^+} f(x) := \lim\limits_{x\to p, x>p} f(x)$ and $\lim\limits_{x\to p^-} f(x) := \lim\limits_{x\to p,\ x<p} f(x)$.*
If $m = 1$, one can define $\lim_{x\to\pm\infty} f(x)$, by considering points $x \in E$ with $\pm x > \delta$, respectively.
If $k = 1$, one can define $\lim_{x\to p} f(x) = \pm\infty$ by concluding $\pm f(x) > \epsilon$, respectively.
*If $p \in E$ and $\lim_{x\to p} f(x) = f(p)$, we say that f is **continuous at** p, that is,*

$$\text{for any } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that if } x \in E \text{ with } \|x - p\| < \delta \text{ then } \|f(x) - f(p)\| < \epsilon,$$

which is equivalently to $\lim_{x\to p} f(x) = f(p)$ or $\lim_{h\to 0, h\in\mathbb{R}^m} f(p + h) = f(p)$.
*If f is continuous at every point $p \in E$, we say that f is **continuous on** E.*
*If $k = 1$, then we say that f is **upper semi-continuous at** $p \in E$ (respectively, **lower semi-continuous at** $p \in E$), if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in E$ with $\|x - p\| < \delta$ then $f(x) - f(p) < \epsilon$ (respectively, $-\epsilon < f(x) - f(p)$).*

Definition 4.2. (*components of a function*)
Let E be a subset of $\mathbb{R}^m$. Let $f_1, f_2, \cdots, f_k$ be k real-valued functions on E and
let $f : E \to \mathbb{R}^k$ be the vector-valued function defined by $f(x) = (f_1(x), f_2(x), \cdots, f_k(x))$ for all $x \in E$.
*We call $f_1, f_2, \cdots, f_k$ the **components** of f and write $f = (f_1, f_2, \cdots, f_k)$.*

Theorem 4.3. (*limit and componentwise*)
Let E be a subset of $\mathbb{R}^m$.
Let $f_1, f_2, \cdots, f_k$ be k real-valued functions on E and
let $f : E \to \mathbb{R}^k$ be the vector-valued function with $f = (f_1, f_2, \cdots, f_k)$.
Let $p \in E$ and let $A = (a_1, a_2, \cdots, a_k)$ be a point in $\mathbb{R}^k$.
Then $\lim\limits_{x\to p} f(x) = A$ if and only if $\lim\limits_{x\to p} f_i(x) = a_i$ for all $i = 1, 2, \cdots, k$.

Theorem 4.4. (*limit and continuity of a composition*)
Let $E \subset \mathbb{R}^m$, $f : E \to \mathbb{R}^k$, $A \in \mathbb{R}^k$ and $g : f(E) \cup \{A\} \to \mathbb{R}^n$, and $p \in E$.
If A is the limit of f at p and g is continuous at A then $g(A)$ is the limit of the composition $g \circ f$ at p.

Theorem 4.5. (*basic rules of limit and continuity*)
Let $E \subset \mathbb{R}^m$ and $p \in E$.
Let f and g be two vector-valued functions which have limits (respectively, continuous) at p.
Then $f \pm g$, $f \cdot g$, and $\dfrac{f}{g}$ provided that g is real-valued and $\lim_{x\to p} g(x) \neq 0$
all have limits (respectively, continuous) at p, and $\lim\limits_{x\to p}(f \boxdot g)(x) = \lim\limits_{x\to p} f(x) \boxdot \lim\limits_{x\to p} g(x)$,
*where $\boxdot$ denotes the **sum, difference, inner product,** and **quotient** rule of two functions defined componentwise, respectively.*

Theorem 4.6. (*continuity and openness and closeness of inverse image*)
Let $E \subset \mathbb{R}^m$ and $f : E \to \mathbb{R}^k$ be a function. Then the following holds.
f is continuous on E
$\Longleftrightarrow$ for any open subset U of $\mathbb{R}^k$, there exists an open subset V of $\mathbb{R}^m$ such that $f^{-1}(U) = E \cap V$
$\Longleftrightarrow$ for any closed subset U of $\mathbb{R}^k$, there exists a closed subset V of $\mathbb{R}^m$ such that $f^{-1}(U) = E \cap V$.

Definition 4.7.
Let $f : \mathbb{R}^m \to \mathbb{R}^k$ be a function.
*We say that f is **separately continuous** at $p = (p_1, \cdots, p_m) \in \mathbb{R}^m$*
if for each $1 \leq i \leq m$, the function $x \mapsto f(p_1, \cdots, p_{i-1}, x, p_{i+1}, \cdots, p_m)$ is continuous at p_i.

Proposition 4.8.
Continuity implies separate continuity, but the converse does not hold.

Proposition 4.9.
Let $U, V \subset \mathbb{R}^m$ be open and $f : U \to V$ be a one-to-one and onto function such that f and f^{-1} are both continuous.
Let $S \subset U$ satisfy $\partial S \subset U$. Then the following holds:

1. *$f(\partial S) \subset \partial(f(S))$;*
2. *$\partial(f(S)) \subset V \Longleftrightarrow \partial(f(S)) \subset f(\partial S)$;*
3. *it might happen that $f(\partial S) \neq \partial(f(S))$;*
4. *if $U = V = \mathbb{R}^m$, then $f(\partial S) = \partial(f(S))$.*

5 Sequence

Definition 5.1. (*sequence*)
A function f defined on $\mathbb{N}$ is called a ***sequence****;*
by letting $f(n) = x_n$ for all $n \in \mathbb{N}$, the sequence f is denoted by $\{x_n\}_{n=1}^{\infty}$.
*The element x_n is called the n-**th term** of the sequence $\{x_n\}_{n=1}^{\infty}$.*
If E is a set and $x_n \in E$ for all $n \in \mathbb{N}$, we say that $\{x_n\}_{n=1}^{\infty}$ is a ***sequence*** *in E.*

Definition 5.2. (*convergence and divergence*)
Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^m$.
We say that $\{x_n\}_{n=1}^{\infty}$ is ***convergent*** *if there exists $A \in \mathbb{R}^m$ such that*
for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $\|x_n - A\| < \epsilon$;
in this case, we say that $\{x_n\}_{n=1}^{\infty}$ ***converges*** *to A and write $\lim\limits_{n\to\infty} x_n = A$.*
We say that $\{x_n\}_{n=1}^{\infty}$ is ***divergent*** *if it is not convergent.*
For $m = 1$, we write $\lim\limits_{n\to\infty} x_n = \infty$
if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $x_n > \epsilon$,
and similarly one can define $\lim\limits_{n\to\infty} x_n = -\infty$;
however, in both cases we still say that the sequence is ***divergent****.*
In the above cases, we call $\lim\limits_{n\to\infty} x_n$ the ***limit*** *of the sequence $\{x_n\}_{n=1}^{\infty}$.*
We say that $\{x_n\}_{n=1}^{\infty}$ is ***bounded*** *if $\{x_n\}_{n=1}^{\infty}$ is a bounded subset of $\mathbb{R}^m$.*
In general, we can define the above concepts in an arbitrary metric space.

Theorem 5.3. (*sequence and closure*)
Let $S \subset \mathbb{R}^m$ and $x \in \mathbb{R}^m$.
Then $x \in \overline{S} \iff$ there is a sequence in S which converges to x.
Notice that our proof can be easily modified so that this result holds if $\mathbb{R}^m$ is replaced by an arbitrary metric space.

Theorem 5.4. (*sequence and continuity*)
Let $S \subset \mathbb{R}^m$, $p \in S$, and $f : S \to \mathbb{R}^k$.
Then f is continuous at p
$\iff$ for any sequence $\{x_n\}_{n=1}^{\infty}$ in S with $\lim\limits_{n\to\infty} x_n = p$, we have $\lim\limits_{n\to\infty} f(x_n) = f(p)$.

Theorem 5.5. (*squeeze theorem*)
Let I be an interval and let $p \in I$. Let f, g, and h be real-valued functions defined on I, except possibly at p.
If $g(x) \leq f(x) \leq h(x)$ for all $x \in I\backslash\{p\}$ and $\lim\limits_{x\to p} g(x) = \lim\limits_{x\to p} h(x) = A$ for some $A \in \mathbb{R}$,
then $\lim\limits_{x\to p} f(x) = A$.

Corollary 5.6.
Let I be an interval and let $p \in I$. Let f and g be real-valued functions defined on I, except possibly at p.
If $\lim\limits_{x\to p} f(x) = 0$ and g is ***bounded*** *on $I\backslash\{p\}$, that is, there exists $M > 0$ such that $|g(x)| \leq M$ for all $I\backslash\{p\}$,*
then $\lim\limits_{x\to p} f(x)g(x) = 0$.

Definition 5.7. (*accumulation point*)
Let $S \subset \mathbb{R}^m$.
A point $p \in \mathbb{R}^m$ is called an ***accumulation point*** *(**cluster point** or **limit point**) of S*
if for any neighborhood T of p (i.e., $p \in T^{int}$), there exists $q \in T \cap S\backslash\{p\}$.
A point $p \in \mathbb{R}^m$ is called an ***isolated point*** *of S if $p \in S$ and p is not an accumulation point of S.*

Theorem 5.8.
Let $S \subset \mathbb{R}^m$ and $p \in \mathbb{R}^m$. Then the followings hold.

1. *The point p is an accumulation point of S*
   $\iff$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in S such that $\lim\limits_{n\to\infty} x_n = p$, and $x_n \neq p$ for all $n \in \mathbb{N}$.
2. *The closure of S is the union of S and the set of all accumulations points of S.*
3. *S is closed $\iff$ every accumulation point of S is in S.*
4. *If $T \subset S$ then $T^{int} \subset S^{int}$ and $\overline{T} \subset \overline{S}$.*
5. *The closure $\overline{S}$ is the smallest closed set containing S, while the interior S^{int} is the largest open set contained in S.*

Theorem 5.9 (Cantor's theorem)**.**
The set of all zero-one sequences and the unit interval $[0,1]$ both are ***uncountable****,*
that is, there is no surjective function from $\mathbb{N}$ to such a nonempty set.

Notice that from now on, our proof for the above results can be easily modified so that
the same results hold when the Euclidean space $\mathbb{R}^m$ is replaced by an arbitrary metric space.

6 The least-upper-bound property

Recall 6.1. (*ordered set*)
Let S be a set.
*An **ordering** on S is a relation, denoted by $<$, with the following two properties:*

1. *If $x, y \in S$, then one and only one of the statements $x < y$, $x = y$ and $y < x$ is true.*
2. *If $x, y, z \in S$ with $x < y$ and $y < z$, then $x < z$.*

*An **ordered set** is a set S in which an ordering $<$ is defined.*

Definition 6.2. (*bounded above and below*)
Let S be an ordered set and let $E \subset S$.

1. *If there exists $\alpha \in S$ such that for all $x \in E$, $x \leq \alpha$,*
   *we say that E is **bounded above** and call α an **upper bound** of E.*
2. *If there exists $\alpha \in S$ such that for all $x \in E$, $\alpha \leq x$,*
   *we say that E is **bounded below** and call α a **lower bound** of E.*

Definition 6.3. (*supremum and infimum*)
Let S be an ordered set and let $E \subset S$.

1. *Suppose there exists $\alpha \in S$ with the following two properties:*
   (a) *α is an upper bound of E, i.e., for any $x \in E$ one has $x \leq \alpha$;*
   (b) *If $\beta < \alpha$, then β is not an upper bound of E, i.e., there exists $x_0 \in E$ such that $\beta < x_0$.*

   *We call such an α the **least upper bound** of E or the **supremum** of E,*
   and write $\alpha = \sup(E)$.
2. *Suppose there exists an $\alpha \in S$ with the following two properties:*
   (a) *α is a lower bound of E.*
   (b) *If $\alpha < \beta$, then β is not a lower bound of E.*

   *We call such an α the **greatest lower bound** of E or the **infimum** of E,*
   and write $\alpha = \inf(E)$.

We often consider the ordered set $S = \mathbb{R} \cup \{\pm\infty\}$ so that both $\sup(E)$ and $\inf(E)$ are always exists;
moreover, $\sup(\varnothing) = -\infty$, $\inf(\varnothing) = +\infty$, $\sup(\mathbb{N}) = +\infty$, and $\inf(\mathbb{Z}\backslash\mathbb{N}) = -\infty$.

Definition 6.4. (*least-upper-bound property*)
Let S be an ordered set.

1. *We say that S has the **least-upper-bound property***
   if any subset of S which is non-empty and bounded above has the supremum in S.
2. *We say that S has the **greatest-lower-bound property***
   if any subset of S which is non-empty and bounded below has the infimum in S.

Theorem 6.5. (*equivalence of the LUB and GLB properties*)
Every ordered set with the least-upper-bound property
also has the greatest-lower-bound property, and vice versa.

7 Field and the completeness of the real field

Definition 7.1. (*field*)
A ***field*** *is a set F with two operations, called* ***addition*** $(+)$ *and* ***multiplication*** $(\cdot)$, *which satisfy the following so-called* ***field axioms****:*

1. *Axioms for addition:*
   (a) *closeness: $x + y \in F$ for all $x, y \in F$.*
   (b) *commutativity: $x + y = y + x$ for all $x, y \in F$.*
   (c) *associativity: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.*
   (d) *existence of unit: $0 \in F$ and $0 + x = x$ for all $x \in F$.*
   (e) *inverse: for any $x \in F$, there exists $(-x) \in F$ such that $x + (-x) = 0$.*
2. *Axioms for multiplication:*
   (a) *closeness: $x \cdot y \in F$ for all $x, y \in F$.*
   (b) *commutativity: $x \cdot y = y \cdot x$ for all $x, y \in F$.*
   (c) *associativity: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in F$.*
   (d) *existence of unit: $1 \in F$ and $1 \cdot x = x$ for all $x \in F$.*
   (e) *inverse: for any $x \in F \setminus \{0\}$, there exists $\frac{1}{x} \in F$ such that $x \cdot \frac{1}{x} = 1$.*
3. *The distribution law: $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in F$.*

Definition 7.2. (*ordered field*)
An ***ordered field*** *is a field F which is an ordered set with the following two properties:*

1. *If $x, y, z \in F$ and $x < y$, then $x + z < y + z$.*
2. *If $x, y \in F$ with $0 < x$ and $0 < y$, then $0 < x \cdot y$.*

For an element x in an ordered field, we said that x is ***positive*** *if $0 < x$ and that x is* ***negative*** *if $x < 0$.*

Example 7.3. (*a field but not an ordered field*)
The sets $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields.
The set $\mathbb{C}$ is a field with the operators: $(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + y_1) + (x_2 + y_2)i$ and $(x_1 + y_1 i) \cdot (x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i$, but is not an ordered field.

Proposition 7.4. (*the non-completeness of* $\mathbb{Q}$)
The rational field $\mathbb{Q}$ does NOT have the least-upper bound property.

Theorem 7.5 (the least-upper-bound property of the real field or the completeness of $\mathbb{R}$)**.**
(***Existence***) *There exists an ordered filed $(\mathbb{R}, +, \cdot, <)$ such that $\mathbb{Q} \subset \mathbb{R}$, $(\mathbb{Q}, +, \cdot, <)$ is an ordered field, and $\mathbb{R}$ has the least-upper-bounded property.*
A ***real number*** *can be defined as a* ***Dedekind cut*** *on $(\mathbb{Q}, +, \cdot, <)$, denoted as A, that is, $A \subset \mathbb{Q}$ such that (i) $A \neq \varnothing$ and $A \neq \mathbb{Q}$, (ii) if $a \in A$ and $b \in \mathbb{Q} \backslash A$, then $a < b$, and (iii) A contains no largest element.*
Furthermore, on real numbers, we define the ordering as $A < B$ if and only if $A \subsetneq B$;
the addition as $A + B = \{a + b : a \in A, b \in B\}$; and
the multiplication on positives as $AB = \{x \in \mathbb{Q} : x \leq ab \text{ for some } a \in A \text{ and } b \in B \text{ with } a > 0 \text{ and } b > 0\}$.
(***Uniqueness***) *Moreover, given any two such ordered fields, there exists a unique field isomorphism among them; this allows us to think of them as essentially the same mathematical object preserving operators,*
From now one, $\mathbb{R}$ is called the ***ordered filed of real numbers****.*

Proposition 7.6.
Any subset S of $\mathbb{R}$ which is non-empty and bounded above (resp. bounded below) has the supremum (resp. infimum) in $\overline{S}$.

8 The bounded monotonic sequence theorem and Cauchy's sequence

Definition 8.1. (*monotonicity and boundedness of sequences*)
A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is said to be
increasing *if $x_n \le x_{n+1}$ for all $n \in \mathbb{N}$,*
decreasing *if $x_{n+1} \le x_n$ for all $n \in \mathbb{N}$, and*
monotonic *if it is either increasing or decreasing.*

A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is said to be
bounded above *if there exists $M \in \mathbb{R}$ such that $x_n \le M$ for all $n \in \mathbb{N}$,*
bounded below *if there exists $m \in \mathbb{R}$ such that $m \le x_n$ for all $n \in \mathbb{N}$, and*
bounded *if it is both bounded above and bounded below (same as that the subset $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is bounded defined in L5).*

Theorem 8.2 (the bounded monotonic sequence theorem)**.**
A monotonic sequence in $\mathbb{R}$ is bounded $\Longleftrightarrow$ it is convergent.
More precisely, an increasing and bounded above sequence (resp. a decreasing and bounded below sequence) converges to its supremum (resp. infimum) as a finite number.

Theorem 8.3 (the nested intervals theorem)**.**
Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of closed, bounded and nonempty intervals in $\mathbb{R}$.

1. *If $I_n \supset I_{n+1}$ for all $n \ge 1$, then $\bigcap_{n=1}^{\infty} I_n$ is a closed, bounded and nonempty interval.*
2. *If, in addition, the length of I_n tends to 0 as $n \to \infty$,*
   then $\bigcap_{n=1}^{\infty} I_n$ consists of exactly one point.

Definition 8.4. (*subsequence*)
Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Let $\{n_i\}_{i=1}^{\infty}$ be a sequence in $\mathbb{N}$ with $n_i < n_{i+1}$ for all $i \ge 1$.
*We call the sequence $\{x_{n_i}\}_{i=1}^{\infty}$ a **subsequence** of $\{x_n\}_{n=1}^{\infty}$.*

Theorem 8.5 (the Bolzano-Weierstrass theorem I)**.**
Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.
Notice that the truth of the above statement is based on the least upper bound property of $\mathbb{R}$.

Theorem 8.6 (Cauchy's convergence criterion *or* the completeness of $\mathbb{R}^k$)**.**
Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a metric space E with the metric d.
*If $\{x_n\}_{n=1}^{\infty}$ is convergent then $\{x_n\}_{n=1}^{\infty}$ is a **Cauchy's sequence***
that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, n \ge N$ then $d(x_m, x_n) < \epsilon$.
*The metric space E is said to be **complete** if every Cauchy's sequence in E is convergent.*
For instance, $\mathbb{Q}$ is NOT complete but $\mathbb{R}^k$ is complete.
On $\mathbb{R}$, the least-upper bounded property is equivalent to the completeness.

Definition 8.7. (*limit superior and limit inferior*)
Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$.
Then the sequence $\{\sup\{x_n : n \ge m\}\}_{m=1}^{\infty}$ is decreasing and the sequence $\{\inf\{x_n : n \ge m\}\}_{m=1}^{\infty}$ is increasing.
The limits $\lim_{m\to\infty}(\sup\{x_n : n \ge m\})$ and $\lim_{m\to\infty}(\inf\{x_n : n \ge m\})$, which might be $\pm\infty$,
*are called the **limit superior** and **limit inferior** of $\{x_n\}_{n=1}^{\infty}$, respectively,*
and denoted by $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$, respectively.
*The **twin prime conjecture** states that $\liminf_{n\to\infty}(p_{n+1} - p_n) = 2$, where $\{p_n\}_{n=1}^{\infty}$ denotes all the primes with $p_n < p_{n+1}$.*

Proposition 8.8.
Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, and let $A \in \mathbb{R}$ and $B \in \mathbb{R} \cup \{\pm\infty\}$. Then the followings hold.

1. *The limit superior $\limsup_{n\to\infty} x_n = A \Longleftrightarrow$ the sequence $\{\sup\{x_n : n \ge m\}\}_{m=1}^{\infty}$ is bounded below with the supremum A*
   $\Longleftrightarrow$ for any $\epsilon > 0$, there are infinitely many n's such that $A - \epsilon < x_n$, but only finitely many n's such that $A + \epsilon < x_n$.
2. *The limit inferior $\liminf_{n\to\infty} x_n = A \Longleftrightarrow$ the sequence $\{\inf\{x_n : n \ge m\}_{m=1}^{\infty}$ is bounded above with the infimum A*
   $\Longleftrightarrow$ for any $\epsilon > 0$, there are infinitely many n's such that $x_n < A + \epsilon$, but only finitely many n's such that $x_n < A - \epsilon$.
3. *The limit superior $\limsup_{n\to\infty} x_n$ is the limit of a subsequence of $\{x_n\}_{n=1}^{\infty}$ so is $\liminf_{n\to\infty} x_n$.*
4. *If B is the limit of a subsequence of $\{x_n\}_{n=1}^{\infty}$, then $\liminf_{n\to\infty} x_n \le B \le \limsup_{n\to\infty} x_n$.*
5. *If B is the limit of the sequence $\{x_n\}_{n=1}^{\infty} \Longleftrightarrow \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = B$.*

9 Compactness and the Heine-Borel and Bolzano-Weierstrass theorems

Definition 9.1. (*open covering*)
Let E be a metric space and $S \subset E$, and let A be a set.
A collection $\{G_\alpha : \alpha \in A\}$ of subsets of E
*is called an **open covering** of S if each G_α is an open subset of E and $S \subset \bigcup_{\alpha\in A} G_\alpha$;*
*and in addition, is said to have a **finite subcovering** if there exist $N \in \mathbb{N}$ and $\{\alpha_i\}_{i=1}^N \subset A$ such that $S \subset \bigcup_{i=1}^N G_{\alpha_i}$.*

Theorem 9.2 (a generalization of the Heine-Borel-Bolzano-Weierstrass theorem)**.**
Let E be a metric space and $S \subset E$.
Then we have $(a) \Longleftrightarrow (b) \Longrightarrow (c)$, where

(*a*) *S is **compact**, that is, every open covering of S has a finite subcovering;*

(*b*) *S is **sequentially compact**, that is, every sequence in S has a convergent subsequence with the limit in S; and*

(*c*) *S is closed and bounded.*

If $E = \mathbb{R}^k$, then $(a) \Longleftrightarrow (b) \Longleftrightarrow (c)$;
in this case, $(a) \Longleftrightarrow (c)$ is called the Heine-Borel theorem and $(b) \Longleftrightarrow (c)$ is called the Bolzano-Weierstrass theorem.
*In some textbooks, a subset S of $\mathbb{R}^k$ with property (c) is said to be **compact**.*

(***Lebesgue number lemma***)
*$(b) \Longrightarrow$ Every open covering $\{G_\alpha : \alpha \in A\}$ of S has a **Lebesgue number** r_0,*
that is, there exists $r_0 > 0$ such that for any $p \in S$ there exists $\alpha \in A$ such that $B_{r_0}(p) \subset G_\alpha$;
notice that the converse is not true, for instance, $S = E = \mathbb{Z}$ with $r_0 = 1/2$.

Theorem 9.3.

1. *A closed subset of a compact set is compact.*
2. *A continuous image of a compact set is compact.*
3. (***Tychnoff theorem***) *An arbitrary produc of compact spaces is compact in the product topology.*

Theorem 9.4 (the extreme value theorem)**.**
A continuous real-valued function on a compact nonempty set always
attains both the global maximum and the global minimum there.

Definition 9.5.
Let $U, V \subset \mathbb{R}^k$ be nonempty.
*The **distance** of U and V is defined to be $d(U,V) = \inf\{\|x-y\| : x \in U, y \in V\}$.*
*The **diameter** of U is defined to be $diam(U) = \sup\{\|x-y\| : x \in U, y \in U\}$, possibly infinite.*

Theorem 9.6 (the nested compact sets theorem)**.**
Let $\{S_n\}_{n=1}^\infty$ be a sequence of compact and nonempty subsets in a metric space E. Then the followings hold.

1. *If $S_n \supset S_{n+1}$ for all $n \geq 1$, then $\bigcap_{n=1}^\infty S_n \neq \varnothing$ and is compact.*
2. *If, in addition, $f : E \to E$ is a function, then $f(\bigcap_{n=1}^\infty S_n) \subset \bigcap_{n=1}^\infty f(S_n)$.*
3. *If, in addition, f is continuous, then $f(\bigcap_{n=1}^\infty S_n) = \bigcap_{n=1}^\infty f(S_n)$.*
4. *If, in addition, $E = \mathbb{R}^k$ and the diameter of S_n tends to 0 as $n \to \infty$,*
   then $\bigcap_{n=1}^\infty S_n$ consists of exactly one point.

Theorem 9.7.
Every compact metric space is complete.

Example 9.8.
The interval $(0,1)$, as a metric space itself, is closed and bounded, but is not compact nor complete.
Let ℓ^∞ be the set of all bounded sequences of real numbers.
Then ℓ^∞ is a metric space with the metric defined by $d(\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty) = \sup\{|a_n - b_n| : n \in \mathbb{N}\}$.
Let $S = \{\{a_n\}_{n=1}^\infty \in \ell^\infty : d(\{a_n\}_{n=1}^\infty, \{0\}_{n=1}^\infty) \leq 1\}$.
Then S is closed and bounded in ℓ^∞.
Let $\{x_m\}_{m=1}^\infty \subset S$ with each $x_m \in S$ has one in the m-th place and zero in all other places.
Then $\{x_m\}_{m=1}^\infty$ has no convergent subsequence.

Theorem 9.9 (advanced exercise)**.**
*For a metric space E, one has that E is compact $\Longleftrightarrow$ E is complete and and **totally bounded**,*
that is, for any $r > 0$, the space is the union of a finite collection of open balls with radius r.

10 Connectedness and the intermediate value theorem

Definition 10.1. (*connectedness and arcwise connectedness*)
A set $S \subset \mathbb{R}^k$ is said to be ***disconnected***
if there are $S_1, S_2 \subset \mathbb{R}^k$ such that $S_1 \neq \varnothing$, $S_2 \neq \varnothing$, $S = S_1 \cup S_2$, $\overline{S_1} \cap S_2 = \varnothing$ and $S_1 \cap \overline{S_2} = \varnothing$.
A set $S \subset \mathbb{R}^k$ is said to be **connected** *if it is not disconnected.*
A set $S \subset \mathbb{R}^k$ is said to be ***arcwise connected*** (*or* ***pathwise connected***)
if any two points in S can be jointed by a continuous curve in S,
that is, for any $a, b \in S$, there exists a continuous function $f : [0,1] \to \mathbb{R}^k$
such that $f(0) = a$, $f(1) = b$, and $f(t) \in S$ for all $t \in [0,1]$.

Theorem 10.2. (*connectedness in* $\mathbb{R}$)
Let $S \subset \mathbb{R}$.
Then S is connected
$\Longleftrightarrow$ *S is an* ***interval*** (*open, half-open, or closed; bounded and unbounded*),
that is, for any $a, b \in S$ with $a \leq b$, the set $\{x \in \mathbb{R} : a \leq x \leq b\} \subset S$.

Theorem 10.3. (*continuity and connectedness*)
A continuous image of a connected set is connected.

Corollary 10.4 (the intermediate value theorem, also called the Darboux property)**.**
Let $S \subset \mathbb{R}^k$ be connected and $f : S \to \mathbb{R}$ be continuous on S.
If $a, b \in S$ with $f(a) < f(b)$,
then for any d with $f(a) < d < f(b)$, there exists $c \in S$ such that $f(c) = d$.

Remark 10.5. (*transitivity of arcwise connectedness*)
The relation of being arcwise connected is ***transitive***,
that is, if there is a continuous curve from a to b and one from b to c, then there is one from a to c.

Theorem 10.6. (*connectedness and arcwise connectedness*)
Let $S \subset \mathbb{R}^k$. Then the followings hold.

1. *If S is arcwise connected, then S is connected.*
2. *If S is open and connected, then S is arcwise connected.*
3. *If S is connected and $S \subset T \subset \overline{S}$, then T is connected.* (*exercise*)
4. *There is a subset of $\mathbb{R}^k$ which is connected but not arcwise connected.* (*exercise*)

Theorem 10.7 (the Peano curve and Netto's theorem)**.**
There are continuous surjective functions from the unit interval $[0,1]$ to the unit square $[0,1]^2$,
but any of them can not be injective.

11 Uniform and Hölder continuities

Definition 11.1. (*uniform continuity*)
Let $S \subset \mathbb{R}^m$ and $f : S \to \mathbb{R}^k$.
*We say that f is **uniformly continuous on** S*
if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $x, y \in S$ with $\|x - y\| < \delta$ then $\|f(x) - f(y)\| < \epsilon$.
*In general, "**uniform**" means that the selection (of δ) DOSE NOT depend on points in the domain, and so is a uniform control.*
*By contrast, the ordinary continuity is "**pointwise**", that is, the selection DOSE depends on points.*

Remark 11.2.
Uniform continuity implies continuity.

Theorem 11.3. (*continuity+compactness implies uniform continuity*)
Let $S \subset \mathbb{R}^m$ and $f : S \to \mathbb{R}^k$ be continuous on S.
If S is compact, then f is uniformly continuous on S.

Definition 11.4. (*Hölder continuity*)
Let $S \subset \mathbb{R}^m$ and $f : S \to \mathbb{R}^k$.
*We say that f is **Hölder continuous** on S*
if there exist $C, \lambda \in \mathbb{R}$ with $C > 0$ and $\lambda > 0$ such that $\|f(x) - f(y)\| \le C\|x - y\|^\lambda$ for all $x, y \in S$.
*In this case, if $\lambda = 1$, then we say that f is **Lipschitz** on S with **a Lipschitz constant** C.*

Remark 11.5 (exercise)**.**
The Hölder continuity implies uniform continuity.

Theorem 11.6 (exercise)**.**
The image of a Cauchy sequence under a uniformly continuous function is a Cauchy sequence.

Theorem 11.7.

1. *Let $S \subset \mathbb{R}^m$ be bounded and $f : \mathbb{R}^m \to \mathbb{R}^k$ be continuous on $\mathbb{R}^m$.*
   Then $f(S)$ is bounded in $\mathbb{R}^k$.
2. *Let $S \subset \mathbb{R}^m$ be bounded and $f : S \to \mathbb{R}^k$ be uniformly continuous on S.*
   Then $f(S)$ is bounded in $\mathbb{R}^k$.
   Moreover, here "uniform continuity" can not be replaced by "continuity".
3. *If the above Euclidean spaces are replaced by abstract metric spaces, the same results hold as well.*
4. *The following metrics are all uniformly continuous from $\mathbb{R}^k$ to $\mathbb{R}$:*
   $d_{max}(\mathbf{x}, \mathbf{y}) = \max\limits_{1 \le i \le k} |x_i - y_i|$, *and* $d_p(\mathbf{x}, \mathbf{y}) = \sqrt[p]{\sum_{i=1}^k |x_i - y_i|^p}$ *for all $p \in \mathbb{N}$.*

Theorem 11.8 (unique continuous extension)**.**
There exists a unique extension of a uniformly continuous function from a dense subset of a metric space into a complete metric space such that the extension is continuous on the whole metric space.
Moreover, the extension is also uniformly continuous (for advanced exercise).
As an application, one may define the exponential function 2^x by knowing the values on rationals in each compact interval.
Notice that the function $x \mapsto \frac{1}{x - \sqrt{2}}$ is continuous (not uniformly!) on rationals and can not be extended to the real line.