UNIVERSITY OF CAPE COAST
COLLEGE OF AGRICULTURAL AND NATURAL SCIENCES
SCHOOL OF PHYSICAL SCIENCE

STABILITY ANALYSIS AND NUMERICAL SIMULATION OF A GOMPERTZ PREDATOR PREY MODEL.

BY

GBEDEH EBENEZER

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This Project work submitted to the Department of Mathematics of the School of Physical Sciences, University of Cape Coast, in partial fulfillment of the award of B.Sc. Degree in Mathematics.

AUGUST, 2023

DECLARATION

Candidate's Declaration

I hereby declare that this project work is a result of my original work and that no part of it has been presented for another degree in this University or elsewhere.

Candidate's Signature: …………………………… Date ………………

Name of Candidate: Gbedeh Ebenezer

Supervisor's Declaration

I hereby declare that the preparation and presentation of this project work were supervised in accordance with the guideline on supervision of project work laid down by the University of Cape Coast.

Supervisor's Signature…………………………. Date ………………

Name of Supervisor: Dr. Samuel Mindakifoe Naandam

ABSTRACT

In this project work, we discuss a model consisting of a population of Wolves (the predator) and deer (prey). It is assumed that the prey increase in number in the absence of predator. Mathematical models are formulated to study the dynamics of the population and analyze the stability of equilibrium points of the prey-predator model.

Numerical simulation techniques were used to find the results of stability analysis at various equilibrium points. All numerical simulations were done using the octave software. The project aims to provide insights into the Gompertz predator-prey model and its significance in studying stability analysis in predator-prey interactions.

ACKNOWLEDGEMENTS

I Wish to express my sincere gratitude to Almighty God for his love, guidance and protection given to me throughout my education.

I extend my sincere gratitude to my parents Mr. Gabriel Gbolonyo and Mrs. Hellen Gbolonyo, my siblings and to all friends and loved ones who played a vital role in my education.

Finally, I am grateful to my supervisor, Dr Samuel Mindakifoe Naandam of the Department of Mathematics UCC, for his unwavering guidance, suggestions and comments, encouragements, and expertise throughout the duration of this project.

DEDICATION

To my parents, my siblings, and those who have been a source of inspiration and encouragement throughout my academic journey.

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CHAPTER ONE
INTRODUCTION

This project focuses on studying the interactions among predators and prey in an ecosystem, specifically emphasizing the predator-prey relationships. Mathematical models, such as the Gompertz predator-prey model, have played a crucial role in this field. These models allow researchers to investigate and analyze the changes occurring in the ecosystem, particularly by examining stability analysis.

Numerical simulation techniques will be utilized to confirm the results of stability analysis, enabling some degree of accurate predictions and a better understanding of these interactions. The project aims to provide insights into the Gompertz predator-prey model and its significance in studying stability analysis in predator-prey interactions.

1.1 Background to the study

Predation is a fundamental process in which a predator captures, kills, and consumes its prey for nourishment and energy. Factors influencing predator-prey relationships include intrinsic growth rate, carrying capacity, environmental conditions, and human impacts. This study specifically focuses on the wolf-deer predator-prey relationship due to its historical, cultural, and ecological significance. Changes in population size of one species affect the other, as an increase in predator population leads to a decrease in prey population. (Alfred J. Lotka,1910)

Predators and prey have evolved specific traits and strategies over time. Predators have developed attributes like strong jaws, sharp claws, and keen senses to effectively hunt, capture, and consume their prey. Prey, in turn, have developed defensive mechanisms such as camouflage, speed, and protective armor to minimize their chances of being hunted and gain time for escape. The Lotka-Volterra equations, introduced by Lotka and Volterra, are commonly used mathematical models that describe the interactions between predator and prey populations.

1.2 Statement of the problem

The Gompertz predator-prey model is a mathematical representation of the dynamic interactions between predators and prey in an ecosystem. The wolf and deer relationship serves as a notable example, where an increase in predator population leads to a decrease in prey availability. However, controversies arise from different research results, influenced by factors such as parameter sensitivity and numerical methods according to a general observation.

To address these concerns, this study aims to enhance stability analysis and numerical simulation of the Gompertz model. By employing precise parameter values, accurate numerical methods, and differential equations, we will strive to improve prediction accuracy. Mathematical models offer a feasible approach to predict biological events, enabling the study and understanding of species interactions. Focusing on the predator-prey relationship, this study's utilization of the Gompertz model will contribute valuable insights into the stability dynamics governing these interactions in an ecosystem.

1.3 Purpose of the study

The aim of this project to perform analysis on the stability of the Gompertz predator prey model by using numerical simulation.

1.4 Research objectives

- To formulate a mathematical model of the system.
- To analyze stability at equilibrium points in the Gompertz predator-prey model.
- To investigate the influence of different parameters on the stability and behavior of the predator-prey system within the Gompertz model.

1.5 Research questions

- What are the key components and equations involved in formulating a mathematical model for the Gompertz predator-prey system?
- How can the stability of equilibrium points in the Gompertz predator-prey model be analyzed, and what are the implications of stability for the long-term behavior of the system?
- What are the effects on population dynamics and ecological interactions of different parameter values in the Gompertz predator-prey model on the stability and behavior of the predator-prey system?

1.6 Significance of the study

This study on stability analysis and numerical simulation of the Gompertz predator-prey model advances knowledge of predator-prey interactions, population stability, and model validation. It explores the impact of parameter values while providing useful applications for ecological management. The study offers insights into population dynamics, assisting in conservation planning and wildlife protection. It advances information, encourages additional study and model improvement, and encourages ecological modeling.

1.7 Limitations of the study

- Obtaining accurate parameter values for the Gompertz predator-prey model was a tedious process, as inaccurate parameter values can lead to inaccurate predictions.
- The Gompertz model used in this study has certain assumptions, which may not fully capture the complexities of real-life ecological interactions. These limitations of the model can impact the accuracy of the predictions made.
- The stability analysis of the Gompertz model in this study relied on numerical simulation, which can be subject to limitations such as hidden errors and approximations made by the computer, potentially impacting the formulation and overall performance of the model.
- The limited time available for the project had an impact on the results, as insufficient preparations could be made due to the time constraint.

1.8 Delimitations of the study

There are other models that could be used to perform this research, such as the logistic model. However, this research lays emphasis on the specific type of model to be used. In this case, the Gompertz predator prey Model.

1.9 Definition of terms

Carrying capacity

Carrying capacity is a term used in ecology to describe the maximum number of individuals of a particular species that an ecosystem can support.

Stability analysis

Stability analysis is a part of the system that is used to understand the behavior of a system at equilibrium points with the help of a mathematical model. It indicates how a model reacts to changes and perturbations. Stability analysis is a suitable way for examining dynamical systems long term behavior.

Equilibrium points

An equilibrium point is defined as the point in a dynamical system where the variables remain constant over a period of time. Thus, the point at which the rates of change is Zero.

Numerical simulation

Numerical simulation is the practice of modeling and approximating the behavior of a system or process over a period of time using software for computing and algorithms. It entails dividing the system into smaller time steps or spatial intervals and repeatedly computing and modifying the state of the system using mathematical equations or models.

Differential equations

Differential equation is one concept which has been used in mathematics over the years. The concept of differential equations is very broad and has occupied a larger part of mathematics. Any equation containing derivatives are differential equations or an equation which contains one or more derivatives of a function is referred to as a differential equation.

1.10 Organization of the study

This project is presented in five chapters with each chapter having subsections. chapter one gives the Introduction (background to the study, statement of the problem, significance of the study, research objectives, limitations of the research, and definition of terms). chapter two talks about the Literature review and the Gompertz model. chapter three talks about the Research methods employed in the study, and other calculations that were made. chapter four talks about the Results and discussions. chapter five talks about the summary, the conclusions and the recommendations that were made after the study.

CHAPTER TWO
LITERATURE REVIEW

2.1 Introduction

The Gompertz model is a mathematical model which has been used in many researches as a growth model for a longer period of time compared to another similar model known as the logistic model. The model, referred to at the time as the Gompertz theoretical law of mortality, was first suggested and applied by Mr. Benjamin Gompertz in 1825.

It is a sigmoid function which describes growth as being slowest at the start and end of a given time period. The curve approaches the function's right-side or future value asymptote considerably more gradually than it does the left-side or lower valued asymptote. However, with the simple logistic function, the curve approaches both asymptotes symmetrically.

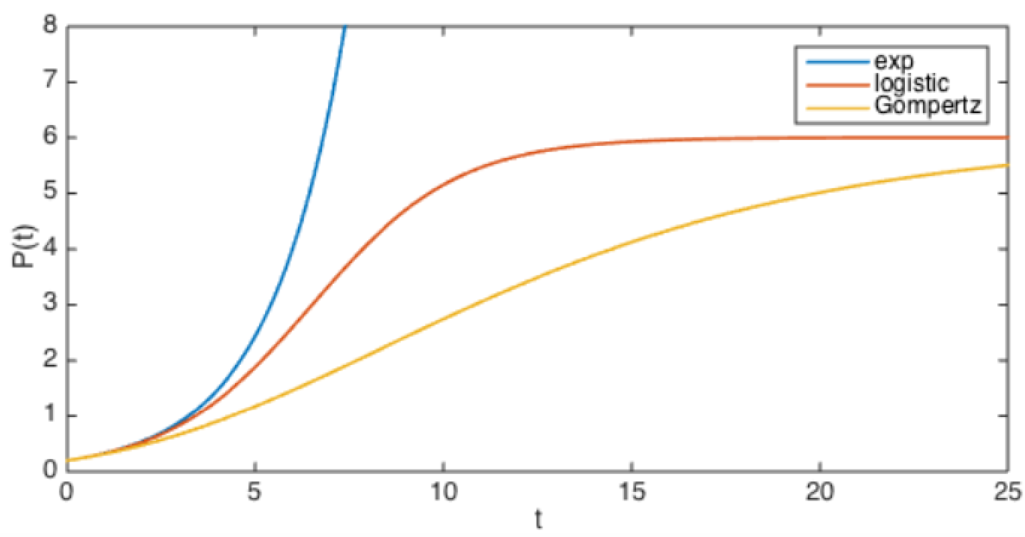

Figure 1: Graph of the exponential, logistic and Gompertz Model.

2.2 Review of literature

The Gompertz function was originally designed to describe human mortality, but has since been modified to be applied in biology, to describe and analyse populations of living organisms. Benjamin Gompertz (1779–1865) was a private-educated actuary in London. At the bottom of page 518 of his June 16, 1825 project, he made the function for the first time. It is predicated on the idea that as people age, their mortality rate rises exponentially (Gompertz, 1825).

Mr. Benjamin Gompertz in 1825, made adjustments to take into consideration the link between aging and a rise in mortality, or what he called "the average exhaustion of a man's power to avoid death." His approach to calculating death risk was quickly embraced by the insurance industry.

However, Gompertz only offered the probability density function. The well-known cumulative model Gompertz-Makeham was first introduced by Makeham.

A least-squares method was initially used to find the ideal curve of the Gompertz model. Instead of linearizing the model as is done later, they simply log transformed the data (dependent variable) to make it easier to calculate the sum of squares. In a number of fields other than human mortality, such as predicting increases in demand for goods and services, tobacco sales, rise in railway traffic, and desire for vehicles, the cumulative Gompertz-Makeham model quickly acquired prominence starting in the 1920s.

The Gompertz model of biological development was first proposed by Wright, and it is likely that Davidson utilized it to interpret biological data in his study of cow body mass rise. In 1931, Weymouth, McMillin, Rich and Thompson successfully applied the Gompertz model to explain the changes in shell size in razor clams (Siliqua patula) and Pacific cockles (Cardium corbis).

The common Gompertz model quickly became a favorite regression model for many sorts of growth of species, including those of dinosaurs, birds, and mammals, including those of marsupials. Researchers soon started to fit the model to their data via regression. The Gompertz model is very important as it is commonly used to predict growth in microbe number or density, tumor growth, and cancer patient survival.

The Lotka-Volterra predator-prey model was first presented by Alfred Lotka in his theory of autocatalytic chemical reactions in 1910 (Lotka, 1910; Goel et al., 1971). However, Pierre François Verhulst (Verhulst, 1838) arrived at this logistic equation first (Berryman, 1992).

In 1920, Lotka developed the concept of "Organic systems" utilizing a plant species and a herbivorous animal species, while Andrey Kolmogorov also contributed to this area. Lotka utilized the equations to examine predator-prey relations in his 1925 book on biomathematics. The same set of equations were first introduced in 1926 by mathematician and physicist Vito Volterra (Goel et al., 1971; Volterra, 1926), who had taken an interest in mathematical biology.

As an alternative to the Lotka-Volterra predator-prey model (and its common-prey-dependent expansions), the ratio dependent model, sometimes referred to as the Arditi and Ginzburg model, was created in the late 1980s (Arditi and Ginzburg, 1989). The validity of prey or ratio dependent models has been extensively discussed (Abrams and Ginzburg, 2000).

The Lotka-Volterra equations have been used in economic theory for a very long time; Richard Goodwin is frequently given recognition for creating them in 1965 (Gandolfo, 2008) or 1967 (Goodwin, 1967) (Desai and Ormerod, 1998), depending on the source. Numerous mathematical models have been created since the Lotka-Volterra model was first suggested, based on more realistic explicit and implicit biological assumptions; for examples, see Dennis et al. (2009), Genry (2007), Jha et al. (2017), Kar (2010), Lotka (1925), Pulley et al. (2011), Taleb (2013), and Volterra (1926).

Volterra was the first to use mathematics to model and analyze ecological problems involving several species (1927). Among the several mathematical models that delayed harvesting were age-structured models.

The French mathematician Abraham de Moivre (1667–1744) conducted earlier work on the creation of functional representations of mortality in the 1750s.De Moivre, however, believed that the death rate remained unchanged. The English actuary and mathematician William Matthew Makeham (1826-1891), who added a constant background mortality rate to Gompertz exponentially growing one, offered an addition to Gompertz work in 1860.

When Holling (1959) investigated how tiny mammals preyed on pine sawflies, he found that predation rates increased in direct proportion to the size of the prey population. This was the result of two factors: (1) each predator increased its consumption rate when exposed to a higher prey density, and (2) prey density increased with rising predator density.

2.3 Chapter summary

The gompertz model has a lot of useful applications as listed above. The gompertz model was first developed by Mr. Benjamin Gompertz in 1825.The well-known cumulative model Gompertz-Makeham was first introduced by Makeham. The Gompertz-Makeham model started to gain prominence in the 1920s as it was used in various fields including predicting increases in demand for goods and services.

Other researchers such as Alfred Lotka who developed the Lotka-Volterra predator-prey model, Wright who first proposed the Gompertz model of biological development , Abraham de Moivre (1667–1744), Weymouth, McMillin, Rich and Weymouth and Thompson are known to contribute a lot to the development of the Gompertz Model.

CHAPTER THREE
METHODOLOGY

3.1 Introduction

This chapter focuses on some applications of the Gompertz model, assumptions for the Gompertz model, modelling the predator-prey equation, and analyzing stability locally and globally. This study focuses on the Lotka-Volterra model to formulate the Gompertz predator-prey model.

3.2 Applications of the Gompertz model

- **Population dynamics**

  Using the Gompertz model, one may predict population increase and calculate future population sizes. This mathematical approach is a useful tool for understanding population patterns and predicting demographic changes.

- **Disease modeling**

  Epidemiology and disease modeling have both made substantial use of the Gompertz model. Researchers can calculate how contagious diseases spread and develop.

- **Cancer research and tumor growth**

  The Gompertz model is frequently employed in the field of cancer research. It takes into account factors such as the initial tumor size, the availability of nutrients, and cell division rates. Researchers can measure therapy response, and gain insights into the development of tumors by fitting the Gompertz curve to data on tumor growth.

3.3 Assumptions for the Gompertz predator-prey model

In constructing the gompertz predator-prey model, the following assumptions are made.

- Predator birth rate is proportional to the size of both the predator and prey population.
- Prey death rate is proportional to the size of both the predator and prey population.
- Prey birth rate is proportional to the size of the population.
- Predator death rate is proportional to the size of the predator population.
- The species coexist with other species in an ecosystem where the effects of external disruptions like epidemics, fires, and droughts are uniform or comparable.
- The rate of increase of the predator population depends on the amount of prey biomass it converts as food.
- One prey is easy to capture by the predator, while the other prey has adopted to capture it .
- In the absence of a predator or human interference with the prey, there is logistic growth of the prey. In other words, the population of the prey would grow or shrink exponentially until it reached the park's maximum density.

3.4 Mathematical modelling

Let x(t) and y(t) represent the wolf and deer populations at any given time t. Assume that at time t, the non-dimensional population density of the prey is x and that of the predator is y. We must proportionally apply the previously mentioned assumption when formulating the model**.**

- The growth rate of any species at a given time is proportional to the number of species present at that time.
- The species are living in a homogeneous environment and age structures are not taken into consideration
- In the absence of the predators, the prey population would grow at logistic growth (natural rate),say r with $\frac{dx}{dt} = rx\,ln\left(\frac{k}{x}\right), r > 0$
- When both prey and predator are present ,the specific growth rate of prey is diminished by an amount population to the predator population and the growth rate of population enhanced proportional to prey population, consequently the effect of predator eating prey is an interaction rate of decline ($-mxy$)in the prey population x and an interaction rate of growth $(-mxy)$ of the predator population y with n and m positive constants.
- In the absence of the prey, the predator population would decline at natural rate, then we will obtain $\frac{dy}{dt} = -sy, s > 0$.

Under the above assumption proportionality, we modeled the prey-predator equations as

$$\frac{dx}{dt} = rx\,ln\left(\frac{k}{x}\right) - mxy$$

$$\frac{dy}{dt} = nxy - sy$$

$$x(0) \geq 0 \; and \; y(0) \geq 0$$

Table 2. Definition of state variables

State Variables	Definition
$x(t)$	Population size of prey at time (t)
$y(t)$	Population size of predator at time (t)

Table 2. Definition of parameters

Parameters	Definition
r	Prey birth rate
k	Carrying capacity
s	Predator death rate
$m \; and \; n$	Effect of the interaction between the predator and prey .

3.5 Equilibrium points

The equilibrium points of the model occur when $\frac{dx}{dt} = \frac{dy}{dt} = 0$ .That is to say that

$$rx \, ln\left(\frac{k}{x}\right) - mxy = 0 \qquad (1)$$

$$nxy - sy = 0 \qquad (2)$$

If $y = 0$

Then from equation (1) $$rx\, ln\left(\frac{k}{x}\right) = 0$$

$$ln\left(\frac{k}{x}\right) = 0$$

$$e^{In\frac{k}{x}} = e^{0}$$

$$\frac{k}{x} = 1 \ , \qquad x = k$$

$$E_1 = (k, 0)$$

From equation (2) $$nxy = sy$$

$$x = \frac{Sy}{ny}$$

$$x = \frac{s}{n} \qquad (3)$$

Substitute equation (3) into equation (1)

Then $$\frac{S}{n} r\left(\ln\left(\frac{kn}{S}\right)\right) - m\left(\frac{S}{n}\right) y = 0$$

$$\frac{msy}{n} = \frac{s}{n} r\, ln\left(\frac{kn}{s}\right)$$

$$my = r\ln\left(\frac{kn}{S}\right)$$

$$y = \frac{r}{m} ln\left(\frac{kn}{S}\right)$$

$$E_2 = \left(\frac{s}{n}\ ,\ \frac{r}{m} ln\left(\frac{nk}{s}\right)\right)$$

Hence $E_2 = (\frac{s}{n}, \frac{r}{m} ln\left(\frac{nk}{s}\right))$ provided that $s < nk$. This implies the equilibrium points are $E_1 = (k, 0)$, that is at predator free equilibrum, and $E_2 = (\frac{s}{n}, \frac{r}{m} ln\left(\frac{nk}{s}\right))$ provided that $s < nk$.

3.6 Local stability analysis

The aim of local stability analysis is to ascertain if minor variations from this equilibrium point will cause the system to stabilize or diverge from the equilibrium. The analysis is referred to as "local" because it concentrates on the behavior of the system just around the equilibrium point, usually by observing the behavior of linearized equations in that region.

In this study, the Jacobian matrix method is used to linearize the non-linear system of prey-predator equations. The equilibrium points of the system are necessary for the purpose of studying the local stability nature of the ecological model. The system, under investigation has the following equilibrium points to be considered. $E_0 = (0,0)$, $E_1 = (k, 0)$ and $E_2 = (\frac{s}{n}, \frac{r}{m} ln\left(\frac{nk}{s}\right))$, exist if $s < nk$.

Given that $$f(x, y) = rx\, ln\left(\frac{k}{x}\right) - mxy$$

$$g(x, y) = nxy - sy$$

$$\frac{\partial}{\partial x}(f) = rx\, ln\left(\frac{k}{x}\right) - mxy$$

$$= r\, ln\left(\frac{k}{x}\right) + rx\left(\frac{1}{\left(\frac{k}{x}\right)}\right)\left(\frac{-k}{x^2}\right) - my$$

$$= r\, ln\left(\frac{k}{x}\right) + rx\left(\frac{x}{k}\right)\left(\frac{-k}{x^2}\right) - my$$

$$= r\, ln\left(\frac{k}{x}\right) + rx\left(\frac{-1}{x}\right) - my$$

$$= r\, ln\left(\frac{k}{x}\right) - r - my$$

Then $$\frac{\partial}{\partial x}(f) = r\left(In\frac{k}{x} - 1\right) - my$$

$$\frac{\partial}{\partial y}(f) = -mx$$

$$\frac{\partial}{\partial x}(g) = ny$$

$$\frac{\partial}{\partial y}(g) = nx - s$$

Then the Jacobian matrix for the equilibrium point $E = (x, y)$ is given by

$$J_E = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

$$J_E = \begin{bmatrix} r\left(In\frac{k}{x} - 1\right) - my & -mx \\ ny & nx - s \end{bmatrix}$$

At $E_1 = (k, 0)$, it is clear that the equilibrium point E_1 exists and its corresponding Jacobian matrix is given by

$$J_{E_1} = \begin{bmatrix} r\left(In\frac{k}{k} - 1\right) - m(0) & -m(k) \\ n(0) & n(k) - s \end{bmatrix}$$

$$J_{E_1} = \begin{bmatrix} r(In\,(1) - 1) & -mk \\ 0 & nk - s \end{bmatrix}$$

$$J_{E_1} = \begin{bmatrix} -r & -mk \\ 0 & nk - s \end{bmatrix}$$

Finding the eigenvalues, let $\quad I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

Such that $\quad J_{E_1} - I = 0$

Then we obtain $\quad \begin{bmatrix} -r & -mk \\ 0 & nk - s \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$

$$\begin{bmatrix} r - \lambda & -mk - 0 \\ 0 - 0 & nk - S - \lambda \end{bmatrix}$$

$$\begin{bmatrix} r - \lambda & -mk \\ 0 & nk - s - \lambda \end{bmatrix}$$

$$(r - \lambda)(nk - S - \lambda) = 0$$

This implies that $\quad (r - \lambda) = 0$

$$\lambda_1 = -r$$

Also, for the second eigenvalue , $\quad (nk - S - \lambda) = 0$

$$\lambda_2 = nk - s$$

From J_{E_2} ,we have the eigenvalues $\lambda_1 = -r < 0$ and $\lambda_2 = nk - s$.

For stability, $\quad \lambda_2 < 0$

$$nk - S < 0$$

$$k < \frac{S}{n}$$

Hence the dynamical system is stable if $k < \frac{S}{n}$ and E_1 is a stable node.

The equilibrium points at $E_2 = (\frac{s}{n}, \frac{r}{m} ln\left(\frac{nk}{s}\right))$ is stable.

At $E_2 = \left(\frac{s}{n}, \frac{r}{m} ln\left(\frac{nk}{s}\right)\right)$,it is clear that the equilibrium point E_2 exists.

Calculating the corresponding Jacobian matrix at E_2

$$J_E = \begin{bmatrix} r\left(In\frac{k}{x} - 1\right) - my & -mx \\ ny & nx - s \end{bmatrix}$$

$$J_E = \begin{bmatrix} r\left(In\frac{k}{x}\right) - r - my & -mx \\ ny & nx - s \end{bmatrix} \quad (4)$$

Substituting $E_2 = \left(\frac{s}{n}, \frac{r}{m} ln\left(\frac{nk}{s}\right)\right)$ *into equation* 4

$$J_{E_2} = \begin{bmatrix} r\left(In\frac{k}{(\frac{S}{n})}\right) - r - m\left(\frac{r}{m} ln\left(\frac{nk}{s}\right)\right) & -m\left(\frac{S}{n}\right) \\ n\left(\frac{r}{m} ln\left(\frac{nk}{s}\right)\right) & n\left(\frac{S}{n}\right) - s \end{bmatrix}$$

$$J_{E_2} = \begin{bmatrix} r\,ln\left(\frac{nk}{s}\right) - r - m\left(\frac{r}{m} ln\left(\frac{nk}{s}\right)\right) & -m\left(\frac{S}{n}\right) \\ n\left(\frac{r}{m} ln\left(\frac{nk}{s}\right)\right) & n\left(\frac{S}{n}\right) - s \end{bmatrix}$$

$$J_{E_2} = \begin{bmatrix} r\,ln\left(\frac{nk}{s}\right) - r - r\,ln\left(\frac{nk}{s}\right) & -m\left(\frac{S}{n}\right) \\ n\left(\frac{r}{m} ln\left(\frac{nk}{s}\right)\right) & s - s \end{bmatrix}$$

$$J_{E_2} = \begin{bmatrix} -r & \frac{-sm}{n} \\ \frac{ln\left(\frac{nk}{s}\right) rn}{m} & 0 \end{bmatrix}$$

To determine whether the equilibrium point E_2 is saddle and the dynamical system is stable, we compute the trace and the determinant of the Jacobian matrix.

$$\text{Let} \quad a_{11} = -r \qquad a_{21} = \frac{ln\left(\frac{nk}{s}\right)rn}{m}$$

$$a_{12} = \frac{-sm}{n} \qquad a_{22} = 0$$

$$\text{Trace } (J_{E_2}) = a_{11} + a_{22}$$

$$= -r + 0$$

$$= -r < 0$$

$$Det\left(J_{E_2}\right) = a_{11}a_{22} - a_{12}a_{21}$$

$$= -r(0) - \left(\frac{-sm}{n}\left(\frac{ln\left(\frac{nk}{s}\right)rn}{m}\right)\right)$$

$$= rsln\left(\frac{nk}{s}\right) > 0$$

It can be observed that trace $(J_{E3}) < 0$ and $det\left(J_{E_3}\right) > 0$.This implies that the equilibrium point E_3 is saddle and the dynamical system is stable. In a dynamical system, a saddle point is a form of equilibrium point where certain trajectories move away from the point in a particular direction while others move toward it in a different direction.

3.7 Global stability analysis

Global stability analysis is an important component of mathematical modeling. The primary goal of a global stability analysis is to identify the long-term behavior and stability characteristics of a system across its entire state, rather than just around specific equilibrium points.

From the model equations, let

$$f(x,y) = rx\,ln\left(\frac{k}{x}\right) - mxy$$

$$g(x,y) = nxy - sy$$

By the Bendixion - Dulac criterion,

$$\phi(x,y) = \frac{1}{xy} > 0 \text{ then}$$

$$D = \frac{\partial}{\partial x}(\phi f) + \frac{\partial}{\partial y}(\phi g)$$

$$\phi f = \frac{1}{xy}\left(rx\,ln\left(\frac{k}{x}\right) - mxy\right)$$

$$= \frac{-rx}{xy} ln\left(\frac{x}{y}\right) - \frac{mxy}{xy}$$

$$\phi f = \frac{r}{y} ln\left(\frac{k}{x}\right) - m$$

$$\frac{\partial}{\partial x}(\phi f) = \frac{\partial}{\partial x}\left(\frac{r}{y} ln\left(\frac{k}{x}\right) - m\right)$$

$$= \frac{r}{y}\left(\frac{1}{\left(\frac{k}{x}\right)}\right)\left(\frac{-k}{x^2}\right) + 0\left(ln\left(\frac{k}{x}\right)\right) - 0$$

$$= \frac{r}{y}\left(\frac{x}{k}\right)\left(\frac{-k}{x^2}\right)$$

$$\frac{\partial}{\partial x}(\phi f) = -\frac{r}{xy} < 0$$

Also $\phi g = \frac{1}{xy}(nxy - sy)$

$$= \frac{nxy}{xy} - \frac{sy}{xy}$$

$$= n - \frac{s}{x}$$

$$\frac{\partial}{\partial y}(\phi g) = \frac{\partial}{\partial y}\left(n - \frac{s}{x}\right)$$

$$\frac{\partial}{\partial y}(\phi g) = 0$$

Therefore

$$D = \frac{\partial}{\partial x}(\phi f) + \frac{\partial}{\partial y}(\phi g)$$

$$D = -\frac{r}{xy} + 0 < 0$$

$$D = -\frac{r}{xy} < 0$$

$$D < 0$$

It is clearly observed that D < 0 for all (x, y) in $\mathbb{R}^2$. Therefore, by the Dulac criterion, the system has no non-trivial solution periodic solutions. Since D < 0,the local asymptotical stability of the system ensures its global asymptotical stability around the positive equilibrium $E_2(x, y)$

CHAPTER FOUR
RESULTS AND DISCUSSION

4.1 Introduction

Numerical simulation on a predator-prey model involves using computational methods to analyze the dynamics of interacting predator and prey populations over time. This chapter focuses on solving the system of equations given numerically, by fixing some of the parameter values in the model. All numerical simulations performed in this chapter is done by using the OCTAVE software.

4.2 Simulation of model

We will be looking at analyzing stability at the equilibrium points we obtained in chapter three. Also, we will be looking at how different parameters and different parameter values, influence the stability and behavior of the predator-prey system within the Gompertz model.

The equilibrium points obtained in chapter three were $E_0 = (0,0)$, $E_1 = (k, 0)$ and $E_2 = (\frac{s}{n}, \frac{r}{m} ln\left(\frac{nk}{s}\right))$. In analyzing stability at these points, we will be selecting parameter values to perform the numerical simulation.

At the equilibrium point $E_0 = (0,0)$, both the prey and predator populations have reached extinction. In this state, there are no individuals of either species present in the system, indicating a complete collapse of the ecological interaction. Hence, we cannot perform numerical simulation at this equilibrium point as both the prey and predator populations have become extinct.

In selecting the parameter values for $E_1 = (k, 0)$, we have r = 0.25; K = 100; m = 0.002; s = 0.4 ; n = 0. 005

At $the\ equilibrum\ point\ E_1 = (k, 0)$,it is observed that there are preys and there are no predators in the system. The value of $k = 100$.This implies the numerical simulation will be performed at the equilibrium point is $E_1 = (100,0)$.

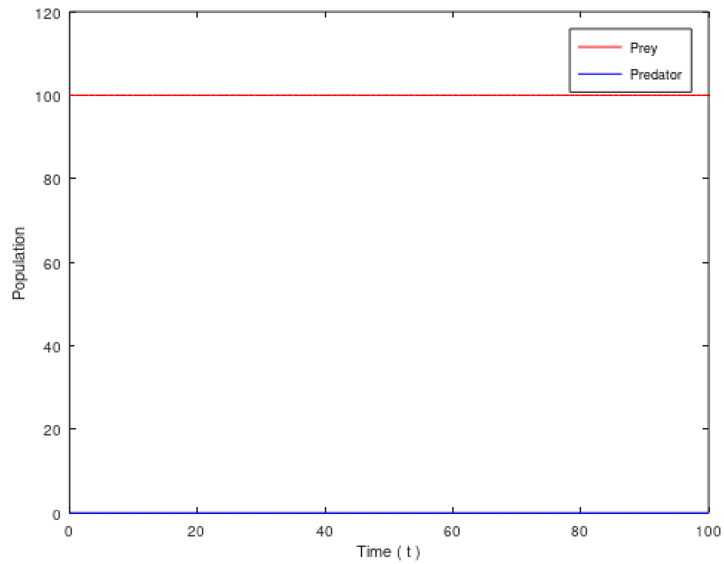

Figure 2: Time series plot predator prey model for E_1.

From figure 2, the prey population, initially set at a value of 100, consistently maintains its population size at the carrying capacity of 100 individuals throughout the simulation period.

At this equilibrium, it is evident that the predator-prey interactions have reached a state of balance. The prey population neither exhibits significant population growth nor experiences a decline, remaining in a stable state due to the absence of predators.

This stability implies that the ecosystem's resources can sustain a population of 100 prey individuals over an extended period.

At $the\ equilibrum\ point$, $E_2 = (\frac{s}{n}, \frac{r}{m} ln\left(\frac{nk}{s}\right))$, the equilibrium values can be calculated by using the following parameter values. Let r = 0.45; K = 100; m = 0.002 ; s = 0.4 ; n = 0.005 By substituting the values of s, n, r, m, and k into $E_2 = (\frac{s}{n}, \frac{r}{m} ln\left(\frac{nk}{s}\right))$, we obtained $E_2 = (90\ , 50.20729905\ )$.

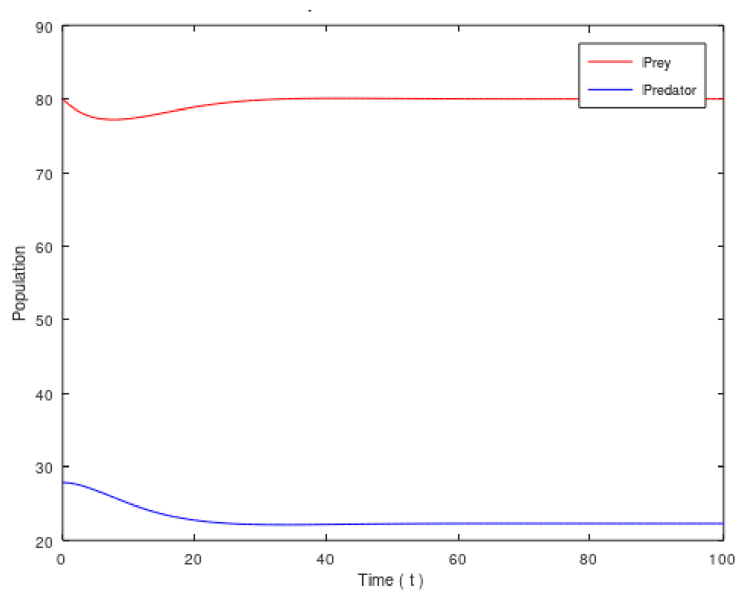

Figure 3: Time series plot predator prey model for E_2.

From figure 3, it can be observed that the predators exerted significant pressure on the prey, causing a decline in the prey population. Once the prey population decreased, the predators' hunting pressure may have reduced, allowing the predator population to stabilize or remain constant. This dynamic interaction shows how changes in prey populations can influence the behavior and population dynamics of their predators in ecological systems.

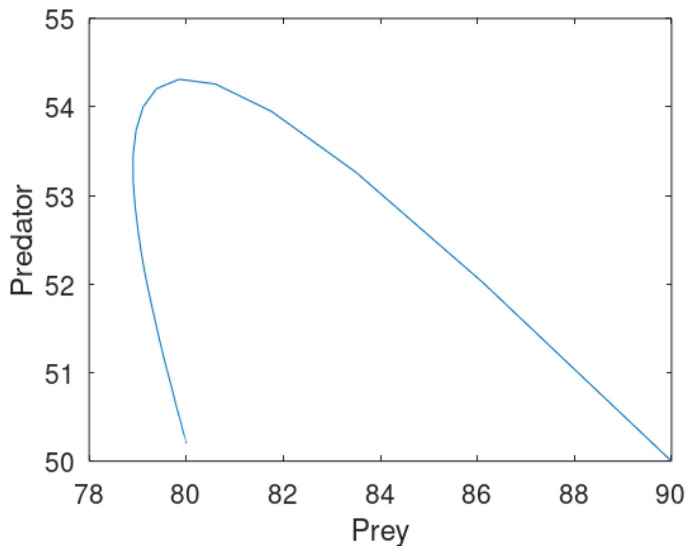

Figure 4: Phase portrait predator prey model for E_2.

From figure 4, it can be observed that the predator's population rises, gets to a peak and it falls whereas the prey population reduces and increases when the predator population gets to its peak. It can also be observed that the predator population decreases, suggesting that the predators are struggling to find enough prey to sustain their numbers. With the predator population decreasing, the prey population may experience relief from predation pressure, This leads to an increase in prey population as depicted in the phase portrait.

The above numerical simulations are for $E_1 = (k, 0)$ and $E_2 = (\frac{s}{n}, \frac{r}{m} ln\left(\frac{nk}{s}\right))$ .To perform further analysis, we will assume some populations of prey and predators. This will help us make further observations and deductions in our research.

We want to consider a population where there are no predators and there are some preys. We will choose E_3= (50,0).That is we select a population where there are 50 preys and there are no predators. We use the parameter values given by r = 0.25; K = 100; m = 0.002; s = 0.4 ; n = 0. 005

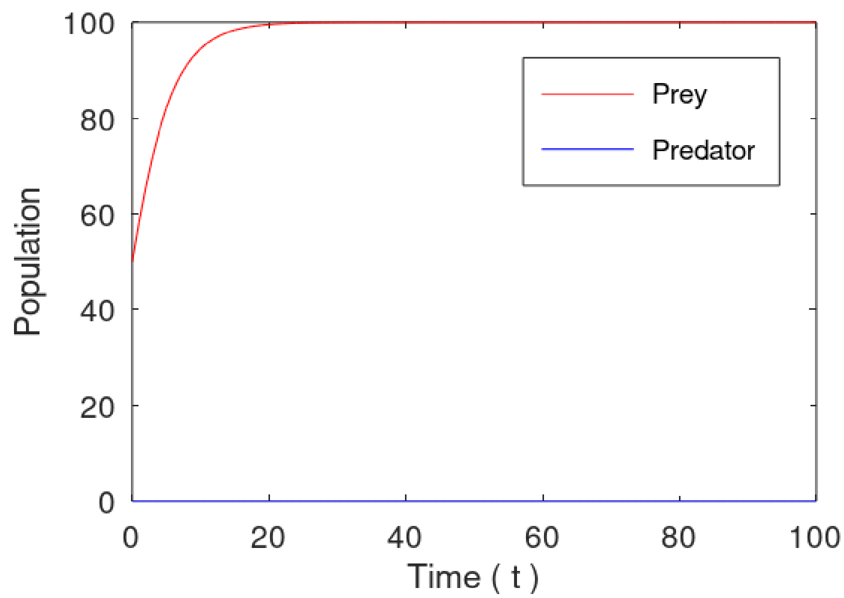

Figure 5: Time series plot predator prey model for E_3.

From figure 2, it can be observed that the prey population approaches the carrying capacity from the initial point. Also, the predator population remains zero over time. This means that, when there are no predators in a population, the prey population grows and slows down as its approaches it carrying capacity. In this case, the prey population will follow the Gompertz curve.

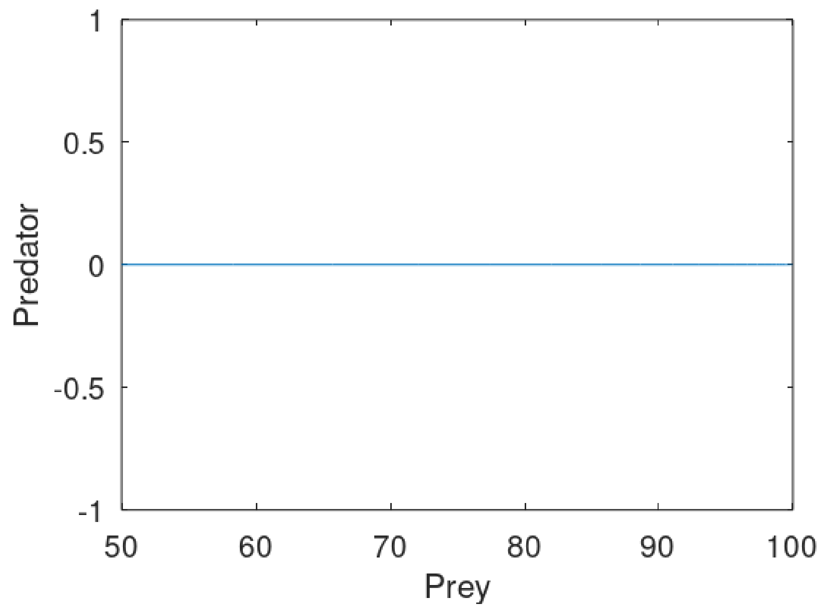

Figure 6: Phase portrait predator prey model for E_3.

From Figure 3, it is evident that the prey population exhibits a growth trend when no predators are present in the population. The initial population, set at fifty (50), serves as the starting point for this growth trajectory. The absence of predators eliminates predation pressures, allowing the prey population to experience unhindered growth.

Additionally, Figure 3 reveals a lack of interaction between the prey and predator populations during this scenario. With the predator population absent, there is no predation-related impact on the prey population.

We want to look at a population where there are less preys and more predators. In particular we select E_4= (30,70).That is we select a population where there are 30 preys and there are 70 predators. We use the parameter values given by r = 0.25; K = 100; m = 0.002; s = 0.4 ; n = 0. 005

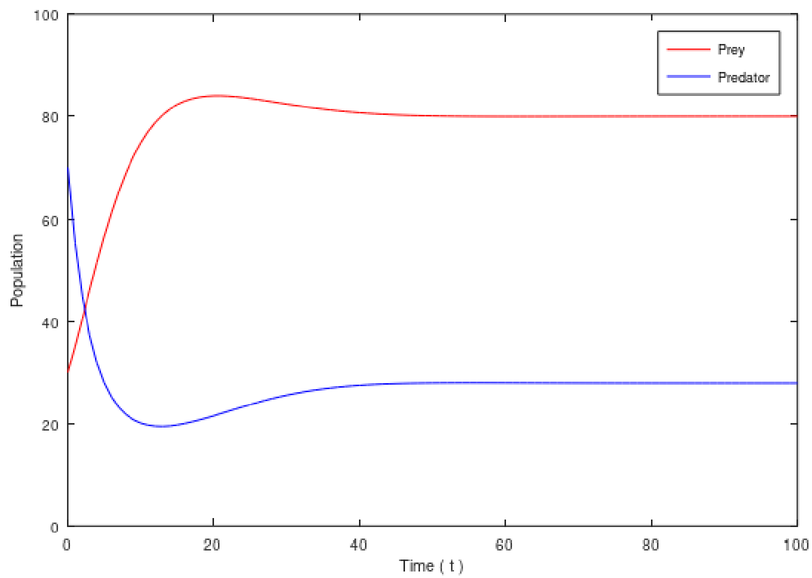

Figure 7: Time series predator prey model for E_4.

From figure 4, it can be observed that there is a decrease in predator population and an increase in prey population. This indicates that the predator population has declined over time. This might be due to certain factors such as shortage of prey, increased competition or other factors that affect the predator’s survival.

Additionally, an increase in prey population suggests that the prey population has thrived over time. This could be due to certain factors such as reduced predation pressure, environmental conditions that support growth of the prey population, migration, reproduction rates and adaptation.

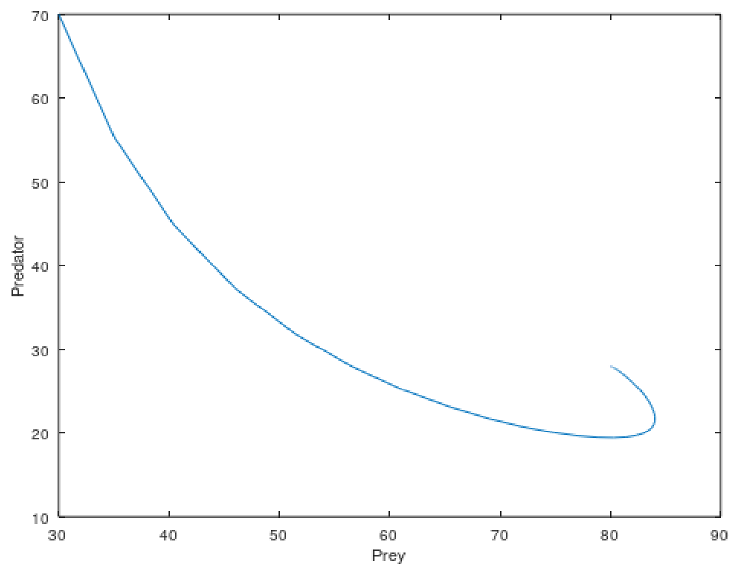

Figure 8: Phase portrait predator prey model for E_4.

From figure 5, it can be observed that the predator population decreases, suggesting that the predators are struggling to find enough prey to sustain their numbers. With the predator population decreasing, the prey population may experience relief from predation pressure, this leads to an increase in prey population as depicted in the phase portrait.

In summary, figure 5 illustrates a scenario where the predator population is declining while the prey population is increasing, which is a common pattern in predator-prey dynamics.

4.3 Discussion

The behavior of the system and the results of numerical simulations are greatly influenced by the parameters in a system of equations, especially in ecological models like predator-prey equations. Based on the parameter values we chose for our numerical simulation, we made the following observations

- **Equilibrium points**

  We observed that the positions and stability of equilibrium points in a system are affected by parameter values. The equilibrium points varied as a result of changes in parameter values.

- **Sensitivity analysis**

  We observed that small changes in our parameter values affected the behavior of the system, helping to understand its robustness and sensitivity to external factors. This shows that adjusting parameter values in numerical simulations allows for sensitivity analysis.

- **Numerical simulations**

  Our results of numerical simulations were greatly influenced by the parameter values we chose. We can examine alternative scenarios and determine how the system would behave under different circumstances by changing parameter values in numerical simulations.

- **Population Dynamics**

  We observed that the parameters greatly affected our results. Birth rate, death rate and the other parameters. They contributed to the populations we obtained for both the prey and the predator.

CHAPTER FIVE
SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 Summary

The study of the Gompertz predator-prey model, which was intended to explain the complex dynamics of ecological systems, comes to a conclusion in this Chapter. The main objectives were to develop a precise mathematical model that captured the essence of predator-prey interactions, to carefully examine the stability of equilibrium points within this Gompertz model, and to explore the interesting field of parameter influence on system behavior.

Our numerical simulations were done using octave software only. We carefully studied the subtle effects of various parameters and their values on the behavior of the predator-prey system. With insights into how changes in parameters might affect stability and equilibrium points within the model, this investigation revealed the complex interactions between predator and prey populations. In essence, our research has improved our understanding of ecological dynamics and highlighted the value of stability analysis and mathematical modeling. It highlights how even little changes in parameters can have a significant impact on the stability and behavior of predator-prey systems.

5.2 Conclusions

In Conclusion, the Gompertz predator-prey model has provided us with important new understandings about the complexity of ecological systems. We discovered the dynamic character of predator-prey interactions through a carefully mathematical formulation and an accurate stability analysis. Different parameters greatly affect the behavior and stability of the model, according to the Octave numerical simulations.

By showing how sensitive predator-prey systems are to parameter changes, this study adds to existing bodies of knowledge in the fields of ecology and mathematical modeling. The significance of taking ecological aspects such predator effectiveness, carrying capacity, mortality rates, and growth rates into account when researching these systems is emphasized.

5.3 Recommendations

In view of our findings, we provide a few recommendations for additional research and practical application. To begin, researchers should continue to improve their work on the Gompertz predator-prey model by incorporating more ecological features and taking real-world data into account for validation. Secondly, the model's sensitivity to parameter changes emphasizes the need of monitoring and regulating critical ecological parameters in conservation and wildlife management operations. Finally, authorities, policy makers and conservationists should examine the potential implications of our research when formulating methods for preserving the delicate equilibrium of predator-prey systems, particularly in the face of environmental changes and human interventions. These recommendations can pave the path for more thorough ecological understanding and informed ecosystem decision-making.

REFERENCES

Ardit, R.; Ginzburg, L. R. (1989). Coupling in predator-prey dynamics: ratio Dependence. *Journal of Theoretical Biology*. 139(3): 311-326.

Berryman, A.(1992). *The origins and evolutions of predator-prey theory.* Ecology; 73:1530-5.

Berryman, E.& Kuang, Y. (2018). *Geometric stability switch criteria in delay Differential systems with delay dependent parameters.* SIAM J.Math.Anal., 33, 1144-1165. https://doi.org/10.1137/s0036141000376086

Gompertz B. On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. *Philosophical Transactions of the Royal Society of London B: Biological Sciences.* 1825; 182:513-85

Hochberg, M. E., & Holt, R. D. (1995). *Refuge evolution and the population Dynamics of coupled host – parasitoid associations.* Evolutionary Ecology, 9(6), 633-661.

Kar, T. K. (2005). *Stability analysis of a prey – predator model incorporating a prey refuge*. Communications in Nonlinear Science and Numerical Simulation 10(6), 681-691.

Kin, C. S., Ut, I. M., Hang, L., Hou, U. I., Weng, N. k., Ha, U. S., & Shan, L. W. (2019).*Predicting Earth's Carrying Capacity of Human Population as the Predator and The Natural Resources as the Prey in the Modified Lotka – Volterra Equations with Time – dependant Parameters.* https://arxiv.org/abs/1904.05002

LVE. “Lotka – Volterra equations”.

https://en.wikipedia.org/wiki/Lotka%E2%80%93Volterra_equations

Sih, A. (1987). “*Prey refuges and predator – prey stability”. Theoretical Biology, vol 31.Sunitha G, Komal G (2016).* A three species dynamical system involving prey- predator, competition and commensalism. Applied Mathematics and Computation, 273: 54-67

Taleb A. S. Obaid (2013). *The Predator – Prey Model Simulation.* Basrah Journal of Science. 31 (2): 103-109.

APPENDIX

```
clc
%numerical simulation for the gompertz predator prey model
function f = numerical_simulation_2023(t,x)
  r = 0.25;
  K = 100;
  m = 0.002;
  s = 0.4;
  n = 0.005;
  f=zeros(2,1);
  f(1)= r*x(1)*log(K/x(1))-m*x(1)*x(2);
  f(2)= n*x(1)*x(2) - s*x(2);
end
t0 = 0:100;
x0 = [100 0];
[t, x] = ode45(@numerical_simulation_2023, t0, x0);

% Time series plot
figure;
plot(t, x(:, 1), 'r', t, x(:, 2), 'b');
xlabel('Time ( t )');
ylabel('Population');
legend('Prey', 'Predator');
```

Code for the Time series plot predator prey model for E_1

```
clc
%numerical simulation for the gompertz predator prey model
function f = numerical_simulation_2023(t,x)
  r = 0.45;
  K = 100;
  m = 0.002;
  s = 0.4;
  n = 0.005;
  f=zeros(2,1);
  f(1)= r*x(1)*log(K/x(1))-m*x(1)*x(2);
  f(2)= n*x(1)*x(2) - s*x(2);
end
t0 = 0:100;
x0 = [90 50];
[t, x] = ode45(@numerical_simulation_2023, t0, x0);
% Time series plot
figure;
plot(t, x(:, 1), 'r', t, x(:, 2), 'b');
xlabel('Time ( t )');
ylabel('Population');
legend('Prey', 'Predator');
```

Code for the Time series plot predator prey model for E_2

```
clc
% Numerical simulation for the Gompertz predator-prey model
function f = numerical_simulation_2023(t,x)
    r = 0.45;
    K = 100;
    m = 0.002;
    s = 0.4;
    n = 0.005;
    f = zeros(2,1);
    f(1) = r*x(1)*log(K/x(1)) - m*x(1)*x(2);
    f(2) = n*x(1)*x(2) - s*x(2);
end
t0 = 0:100;
x0 = [50 0];
[t, x] = ode45(@numerical_simulation_2023, t0, x0);
figure;
% Phase Portrait plot
plot(x(:, 1), x(:, 2));
xlabel('Prey');
ylabel('Predator');
```

Code for the Phase portrait predator prey model for E_2

```
clc
%numerical simulation for the gompertz predator prey model
function f = numerical_simulation_2023(t,x)
  r = 0.45;
  K = 100;
  m = 0.002;
  s = 0.4;
  n = 0.005;
  f=zeros(2,1);
  f(1)= r*x(1)*log(K/x(1))-m*x(1)*x(2);
  f(2)= n*x(1)*x(2) - s*x(2);
end
t0 = 0:100;
x0 = [50 0];
[t, x] = ode45(@numerical_simulation_2023, t0, x0);
% Time series plot
figure;
plot(t, x(:, 1), 'r', t, x(:, 2), 'b');
xlabel('Time ( t )');
ylabel('Population');
legend('Prey', 'Predator');
```

Code for the Time series plot predator prey model for E_3

```
clc
%numerical simulation for the gompertz predator prey model
function f = numerical_simulation_2023(t,x)
  r = 0.45;
  K = 100;
  m = 0.002;
  s = 0.4;
  n = 0.005;
  f=zeros(2,1);
  f(1)= r*x(1)*log(K/x(1))-m*x(1)*x(2);
  f(2)= n*x(1)*x(2) - s*x(2);
end
t0 = 0:100;
x0 = [50 0];
[t, x] = ode45(@numerical_simulation_2023, t0, x0);
figure;
% Phase Portrait plot
plot(x(:, 1), x(:, 2));
xlabel('Prey');
ylabel('Predator');
```

Code for the Phase portrait predator prey model for E_3.

```
clc
%numerical simulation for the gompertz predator prey model
function f = numerical_simulation_2023(t,x)
  r = 0.45;
  K = 100;
  m = 0.002;
  s = 0.4;
  n = 0.005;
  f=zeros(2,1);
  f(1)= r*x(1)*log(K/x(1))-m*x(1)*x(2);
  f(2)= n*x(1)*x(2) - s*x(2);
end
t0 = 0:100;
x0 = [30 70];
[t, x] = ode45(@numerical_simulation_2023, t0, x0);
% Time series plot
figure;
plot(t, x(:, 1), 'r', t, x(:, 2), 'b');
xlabel('Time ( t )');
ylabel('Population');
legend('Prey', 'Predator');
```

Code for the Time series predator prey model for E_4.

```
clc
%numerical simulation for the gompertz predator prey model
function f = numerical_simulation_2023(t,x)
  r = 0.45;
  K = 100;
  m = 0.002;
  s = 0.4;
  n = 0.005;
  f=zeros(2,1);
  f(1)= r*x(1)*log(K/x(1))-m*x(1)*x(2);
  f(2)= n*x(1)*x(2) - s*x(2);
end
t0 = 0:100;
x0 = [30 70];
[t, x] = ode45(@numerical_simulation_2023, t0, x0);
figure;
% Phase Portrait plot
plot(x(:, 1), x(:, 2));
xlabel('Prey');
ylabel('Predator');
```

Code for the Phase portrait predator prey model for E_4.