A key advantage of the method described here is that it can handle American options. The calculations are as we have described them except that we test for early exercise at each node for each of the alternative values of the path function at the node. (In practice, the early exercise decision is liable to depend on both the value of the path function and the value of the underlying asset.) Consider the American version of the average price call considered here. The value calculated using the 20-step tree and four averages at each node is 7.77; with 60 time steps and 100 averages, the value is 6.17.

The approach just described can be used in a range of different situations. The two conditions that must be satisfied were listed at the beginning of this section. Efficiency is improved somewhat if quadratic rather than linear interpolation is used at each node.

27.6 BARRIER OPTIONS

Chapter 26 presented analytic results for standard barrier options. This section considers numerical procedures that can be used for barrier options when there are no analytic results.

In principle, many barrier options can be valued using the binomial and trinomial trees discussed in Chapter 21. Consider an up-and-out option. A simple approach is to value this in the same way as a regular option except that, when a node above the barrier is encountered, the value of the option is set equal to zero.

Trinomial trees work better than binomial trees, but even for them convergence is very slow when the simple approach is used. A large number of time steps are required to obtain a reasonably accurate result. The reason for this is that the barrier being assumed by the tree is different from the true barrier.[24] Define the *inner barrier* as the barrier formed by nodes just on the inside of the true barrier (i.e., closer to the center of the tree) and *the outer barrier* as the barrier formed by nodes just outside the true barrier (i.e., farther away from the center of the tree). Figure 27.4 shows the inner and outer barrier for a trinomial tree on the assumption that the true barrier is horizontal. The usual tree calculations implicitly assume that the outer barrier is the true barrier because the barrier conditions are first used at nodes on this barrier. When the time step is Δt, the vertical spacing between the nodes is of order $\sqrt{\Delta t}$. This means that errors created by the difference between the true barrier and the outer barrier also tend to be of order $\sqrt{\Delta t}$.

One approach to overcoming this problem is to:

1. Calculate the price of the derivative on the assumption that the inner barrier is the true barrier.
2. Calculate the value of the derivative on the assumption that the outer barrier is the true barrier.
3. Interpolate between the two prices.

Another approach is to ensure that nodes lie on the barrier. Suppose that the initial stock price is S_0 and that the barrier is at H. In a trinomial tree, there are three possible

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[24] For a discussion of this, see P. P. Boyle and S. H. Lau, "Bumping Up Against the Barrier with the Binomial Method," *Journal of Derivatives*, 1, 4 (Summer 1994): 6–14.

Figure 27.4 Barriers assumed by trinomial trees.
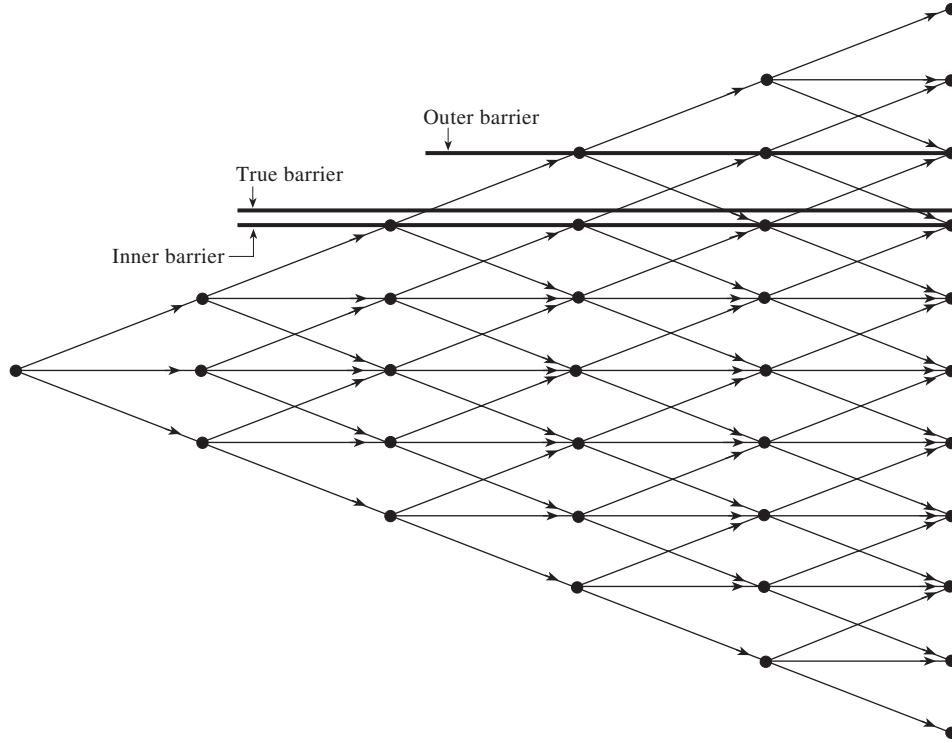
movements in the asset's price at each node: up by a proportional amount u; stay the same; and down by a proportional amount d, where $d = 1/u$. We can always choose u so that nodes lie on the barrier. The condition that must be satisfied by u is

$$H = S_0 u^N$$

or

$$\ln H = \ln S_0 + N \ln u$$

for some positive or negative N.

When discussing trinomial trees in Section 21.4, the value suggested for u was $e^{\sigma\sqrt{3\Delta t}}$, so that $\ln u = \sigma\sqrt{3\Delta t}$. In the situation considered here, a good rule is to choose $\ln u$ as close as possible to this value, consistent with the condition given above. This means that
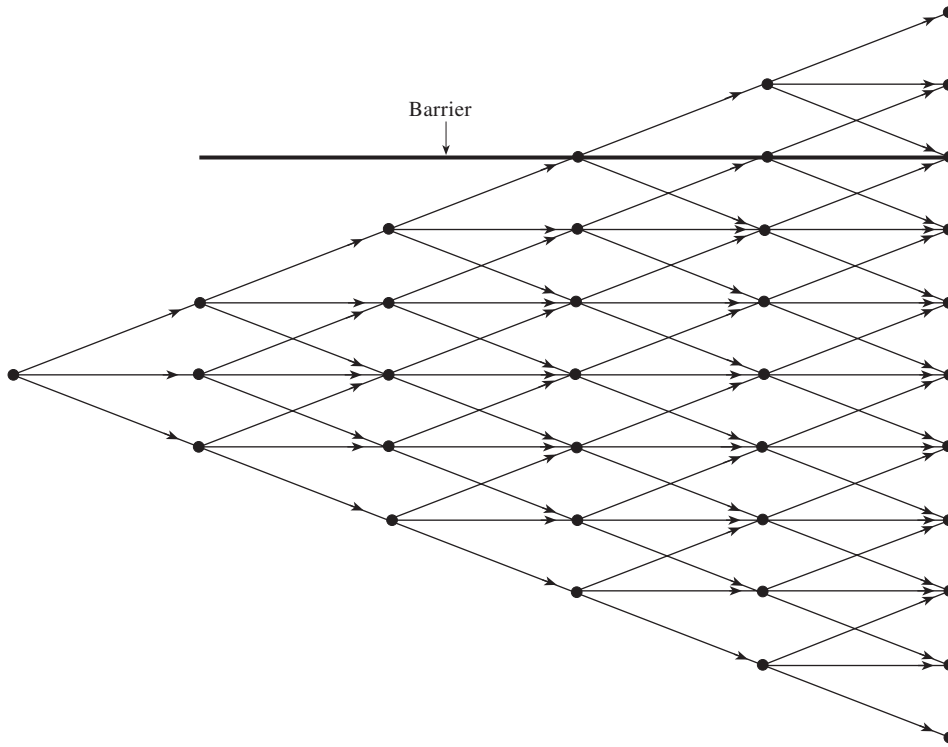
$$\ln u = \frac{\ln H - \ln S_0}{N}$$

where

$$N = \text{int}\left[\frac{\ln H - \ln S_0}{\sigma\sqrt{3\Delta t}} + 0.5\right]$$

and $\text{int}(x)$ is the integral part of x.

This leads to a tree of the form shown in Figure 27.5. The probabilities p_u, p_m, and p_d on the upper, middle, and lower branches of the tree are chosen to match the first

Figure 27.5 Tree with nodes lying on barrier.

two moments of the return, so that

$$p_d = -\frac{(r-q-\sigma^2/2)\Delta t}{2\ln u} + \frac{\sigma^2\Delta t}{2(\ln u)^2}, \; p_m = 1 - \frac{\sigma^2\Delta t}{(\ln u)^2}, \; p_u = \frac{(r-q-\sigma^2/2)\Delta t}{2\ln u} + \frac{\sigma^2\Delta t}{2(\ln u)^2}$$

The methods presented so far work reasonably well when the initial asset price is not close to the barrier. When the initial asset price is close to a barrier, the adaptive mesh model, which was introduced in Section 21.4, can be used.[25] The idea behind the model is that computational efficiency can be improved by grafting a fine tree onto a coarse tree to achieve a more detailed modeling of the asset price in the regions of the tree where it is needed most. To value a barrier option, it is computationally efficient to have a fine tree close to barriers with nodes on the barriers.

27.7 OPTIONS ON TWO CORRELATED ASSETS

Another tricky numerical problem is that of valuing American options dependent on two assets whose prices are correlated. A number of alternative approaches have been suggested. This section will explain three of these.

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[25] See S. Figlewski and B. Gao, "The Adaptive Mesh Model: A New Approach to Efficient Option Pricing," *Journal of Financial Economics*, 53 (1999): 313–51.