PRACTICO 4

1

a)

$$P_n: y = a_1 x + a_0 \quad \text{, donde:}$$

$$a_0 = \frac{\sum x_i^2 \sum y_i - \sum x_i y_i \sum x_i}{m \sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a_1 = \frac{m \sum x_i y_i - \left(\sum x_i\right)\left(\sum y_i\right)}{m \sum x_i^2 - \left(\sum x_i\right)^2}$$

$\sum y_i = 45{,}19$

$\sum x_i = 45$

$\sum x^2 = 285$

$\sum x_i y_i = 286{,}7$

Luego

$$a_0 = -0{,}236\ldots$$

$$a_1 = 1{,}0093\ldots$$

b)

$$E(a_0, \ldots, a_n) = \sum \left(y - (a_2 x^2 + a_1 x + a_0)\right)^2$$

$$= \sum y^2 - 2y(a_1 x^2 + a_1 x + a_0) + a_2^2 x^4 + 2a_2 a_1 x^3 + 2a_2 a_0 x^2 + a_1^2 x^2 + 2a_1 a_0 x + a_0^2$$

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Luego:

$$\frac{\partial E}{\partial a_0} = \sum -2y + 2a_2x^2 + 2a_1x + 2a_0 = 0$$

$$\Longrightarrow a_0 \cdot 5 + a_1 \sum x + a_2 \sum x^2 = \sum y$$

$$\frac{\partial E}{\partial a_1} = \sum -2yx + 2a_2x^3 + 2a_1x^2 + 2a_0x = 0$$

$$\Longrightarrow a_0 \sum x + a_1 \sum x^2 + a_2 \sum x^3 = \sum yx$$

$$\frac{\partial E}{\partial a_2} = \sum -2yx^2 + 2a_2x^4 + 2a_1x^3 + 2a_0x^2 = 0$$

$$\Longrightarrow a_0 \sum x^2 + a_1 \sum x^3 + a_2 \sum x^4 = \sum y \cdot x^2$$

$\sum y = 44$, $\sum x^2 = 47$, $\sum x^4 = 1379$, $\sum yx^2 = 1030{,}7$

$\sum x = 9$, $\sum x^3 = 243$, $\sum yx = 175{,}5$

Entonces:

$$\begin{array}{ccc} a_0 & a_1 & a_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 5 & 9 & 47 & 44 \\ 9 & 47 & 243 & 175{,}5 \\ 47 & 243 & 1379 & 1030{,}7 \end{array}\right) \Longrightarrow \begin{array}{l} a_0 = 3{,}03 \\ a_1 = -1{,}98 \\ a_2 = 0{,}99 \end{array}$$

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2

$$E(a_0, \dots, a_n) = \sum_{i=0}^{n} (y_i - Q_n(x_i))^2 \geq 0$$

sentido de cuadrados mínimos

pues es suma de cuadrados.

Donde $Q_n(x) = a_0 x^0 + a_1 x^1 + \dots + a_n x^n$

Sea $Q_n(x) = P_n(x)$ donde P_n es el polinomio interpolante, luego:

$$E(a_0, \dots, a_n) = \sum_{i=0}^{n} (y_i - y_i)^2 = 0$$

Sea $Q_n(x) = H_n(x)$ donde H no es el polinomio interpolante, entonces existe algun i tal que $H_n(x_i) \neq y_i \Rightarrow y_i - H_n(x_i) \neq 0 \Rightarrow$

$$\sum_{i=0}^{n} (y_i - H_n(x_i))^2 > 0$$

Dado que el mínimo de $E(a_0, \dots, a_n)$ es cero, el polinomio que dé dicho valor será el mejor. Por lo anterior demostramos que el único que cumple con esto es el interpolante.

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③ Sea $P(x) = a_0$.

$$E(a_0) = \sum_{i=1}^{n} (y_i - P(x_i))^2 = \sum_{i=1}^{n} (y_i - a_0)^2 = \sum_{i=1}^{n} y_i^2 - 2y_i a_0 + a_0^2$$

Luego:

$$\frac{\partial E}{\partial a_0} = \sum_{i=1}^{n} -2y_i + 2a_0 = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} a_0 = \sum_{i=1}^{n} y_i$$

$$a_0 . n = \sum_{i=1}^{n} y_i$$

$$a_0 = \frac{\sum_{i=1}^{n} y_i}{n}$$

$$\therefore P(x) = \frac{\sum_{i=1}^{n} y_i}{n}$$

④

$$y = a.e^{bx}$$

$$\ln(y) = \ln(a) + b.x$$

$f(x) = a.e^{bx}$ es la mejor aproximacion aproximación exponencial por cuadrados mínimos a los puntos (x_i, y_i) si $\ln(a)$, b son los coeficientes de la función lineal que mejor se aproxima por cuadrados mínimos a los puntos $(x_i, \ln(y_i))$

$$y_i' = a_1 x + a_0 \quad , \quad a_1 = b, \; a = e^{a_0}, \; y_i' = \ln(y_i)$$

Sabemos:

$$a_1 = \frac{\sum x_i^2 \sum y_i' - \sum x_i y_i' \sum x_i}{m . \sum x_i^2 - (\sum x_i)^2}$$

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$$a_0 = \frac{m\sum x_i y_i' - \sum x_i \sum y_i'}{m\sum x_i^2 - \left(\sum x_i\right)^2}$$

$\sum y_i' = 2{,}59$ $\qquad$ $\sum x_i^2 = 6$

$\sum x_i = 2$ $\qquad$ $\sum x_i y_i = -3{,}38$

Luego:

$a_1 = -0{,}93$

$a_0 = 1{,}116$

Finalmente

$b = -0{,}93$

$a = 3{,}05$

$$\boxed{f(x) = 3{,}05 \cdot e^{-0{,}93x}}$$

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$$y_i' = -e^{ax^2+bx+c}$$

$$-y_i' = e^{ax^2+bx+c}$$

$$\underbrace{\ln(-y_i')}_{y} = ax^2+bx+c$$

$$E(a_0, a_1, a_2) = \sum \left(y - (a_2x^2 + a_1x + a_0)\right)^2$$

Análogo al 2b):

$$\begin{cases} a_0 \cdot 4 + a_1 \sum x + a_2 \sum x^2 = \sum y \\ a_0 \sum x + a_1 \sum x^2 + a_2 \sum x^3 = \sum xy \\ a_0 \sum x^2 + a_1 \sum x^3 + a_2 \sum x^4 = \sum x^2 y \end{cases}$$

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$\sum y = -1{,}6$, $\sum x^3 = 8$, $\sum x^2 y = -2{,}8$

$\sum x = 2$ $\sum x^4 = 18$

$\sum x^2 = 6$ $\sum xy = -1{,}6$

Luego:

$$\begin{array}{c} \begin{array}{ccc} a_0 & a_1 & a_2 \end{array} \\ \left(\begin{array}{ccc|c} 4 & 2 & 6 & -1{,}6 \\ 2 & 6 & 8 & -1{,}6 \\ 6 & 8 & 18 & -2{,}8 \end{array}\right) \end{array}$$

$\Longrightarrow a_0 = -0{,}42$

$a_1 = -0{,}26$

$a_2 = 0{,}1$

6

$X_i = \{1523, 880, 898, 960, 618, 709\}$

$y_i = \{1500, 900, 800, 1050, 650, 690\}$

Resolución Algebraica:

$$A = \begin{pmatrix} 1523 & 1 \\ 880 & 1 \\ 898 & 1 \\ 960 & 1 \\ 618 & 1 \\ 709 & 1 \end{pmatrix}$$

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$A^T \cdot A$

						1523	1
						880	1
						898	1
						960	1
						618	1
						709	1
1523	880	898	960	618	709	$\sum x_i^2$	$\sum X_i$
1	1	1	1	1	1	$\sum X_i$	$1.n$

$\sum x_i^2 = 5706538$

$\sum X_i = 5588$

$1.n = 6$

Luego: $A^T \cdot A = \begin{pmatrix} 5706538 & 5888 \\ 5888 & 6 \end{pmatrix}$

Por otro lado:

$A^T \cdot b$

						1500	
						900	
						800	
						1050	
						650	
						690	
1523	880	898	960	618	709	$\sum x_i y_i$	
1	1	1	1	1	1	$\sum y_i$	

$\sum y_i = 5590$

$\sum x_i y_i = 5693810$

$\Longrightarrow A^T \cdot b = \begin{pmatrix} 5693810 \\ 5590 \end{pmatrix}$

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Por lo tanto:

$$\begin{array}{cc} a_1 & a_0 \\ \left(\begin{array}{c} 5706538 \\ 5888 \end{array}\right. & \left.\begin{array}{c|c} 5888 & 5693810 \\ 6 & 5590 \end{array}\right) \end{array} \Longrightarrow \begin{cases} a_1 = 0,97094 \\ a_0 = 27,38927 \end{cases}$$

Costo $\approx 1143,97027\$$

7

a) Buscamos $P(x) = a_1 . x + a_0$

La solucion general:

$$\begin{cases} a_0 \int_a^b dx + a_1 \int_a^b x\,dx = \int_a^b f(x)\,dx \\ a_0 \int_a^b x\,dx + a_1 \int_a^b x^2\,dx = \int_a^b x . f(x)\,dx \end{cases}$$

$$\int_0^1 dx = 1 \quad , \quad \int_0^1 x^2\,dx = \frac{1}{3}$$

$$\int_0^1 x\,dx = \frac{1}{2} \quad , \quad \int_0^1 (x^2+3x+2)\,dx = \frac{x^3}{3} + \frac{3x^2}{2} + 2x \Big|_0^1 = \frac{1}{3} + \frac{3}{2} + 2 = \frac{23}{6}$$

$$\int_0^1 (x^3+3x^2+2x)\,dx = \frac{x^4}{4} + x^3 + x^2 \Big|_0^1 = \frac{9}{4}$$

Entonces:

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$$\begin{array}{cc} a_0 & a_1 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{23}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{9}{4} \end{array}\right) \Longrightarrow \begin{array}{l} a_0 = 1{,}97 \\ a_1 = 3{,}73 \end{array}$$

b) buscamos $P(x) = a_1 x + a_0$

Usamos la misma solucion general que en el a):

$$\int_{-1}^{1} dx = 2$$

$$\int_{-1}^{1} x\,dx = 0$$

$$\int_{-1}^{1} x^2\,dx = \frac{2}{3}$$

$$\int_{-1}^{1} (x^2+3x+2)\,dx = \frac{14}{3}$$

$$\int_{-1}^{1} (x^3+3x^2+2x)\,dx = 2$$

Luego:

$$\begin{cases} 2a_0 = \frac{14}{3} \\ \frac{2a_1}{3} = 2 \end{cases} \Longrightarrow a_0 = \frac{14}{6}, \quad a_1 = 3$$

c) $\int_0^2 dx = 2$, $\int_0^2 x^2\,dx = \frac{8}{3}$, $\int_0^2 e^x\,dx = e^2 - 1$

$\int_0^2 x\,dx = 2$, $\int_0^2 xe^x = xe^x - e^x \Big|_0^2 = (2e^2 - e^2) - (-1) = e^2 + 1$

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Luego:

$$\begin{array}{cc} a_0 & a_1 \end{array}$$
$$\left(\begin{array}{cc|c} 2 & 2 & e^2-1 \\ 2 & \frac{8}{3} & e^2+1 \end{array}\right) \Rightarrow \quad a_0 = \frac{e^2-7}{2}, \quad a_1 = 3$$

8

$$E(a,b) = \sum_{i=0}^{10} \left(y_i - (a.\cos(x_i) + b.\sin(x_i))\right)^2$$

$$= \sum_{i=0}^{10} y_i^2 - 2a y_i \cos(x_i) - 2b\, y_i \sin(x_i) + a^2\cos^2(x_i) + 2ab\cos(x_i)\sin(x_i) + b^2\sin^2(x_i)$$

Luego

$$\frac{\partial E}{\partial a} = -2\sum y_i \cos(x_i) + 2a\sum \cos^2(x_i) + 2b\sum \cos(x_i)\sin(x_i) = 0$$

$$\Rightarrow a\sum \cos^2(x_i) + b\sum \cos(x_i)\sin(x_i) = \sum y_i \cos(x_i)$$

$$\frac{\partial E}{\partial b} = -2\sum y_i \sin(x_i) + 2a\sum \cos(x_i)\sin(x_i) + 2b\sum \sin^2(x_i) = 0$$

$$\Rightarrow a\sum \cos(x_i)\sin(x_i) + b\sum \sin^2(x_i) = \sum y_i \sin(x_i)$$

$\sum \cos^2(x_i) = 5,99$ $\qquad$ $\sum y_i \sin(x_i) = 16,07$

$\sum \cos(x_i)\sin(x_i) = 0,32$

$\sum y_i \cos(x_i) = 10,75$

$\sum \sin^2(x_i) = 5$

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Luego:

$$\begin{array}{cc} a & b \\ \left(\begin{array}{cc|c} 5,99 & 0,32 & 10,75 \\ 0,32 & 5 & 16,07 \end{array}\right) \end{array} \Rightarrow \begin{array}{l} a = 1,62 \\ b = 3,1 \end{array}$$

9

Un conjunto $\{P_0, \dots, P_n\}$ es ortogonal en $[a,b]$ respecto a la función de peso w si:

$$\int_a^b w(x) \cdot P_k(x) P_j(x) = \begin{cases} 0 & \text{si } j \neq k \\ \alpha_j > 0 & \text{si } j = k. \end{cases}$$

$w(x) = 1$

$$\int_{-1}^{1} 1 \cdot P_2(x) P_0(x) dx = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right) dx = \frac{x^3}{3} - \frac{1}{3}x \Big|_{-1}^{1} = 0 \quad ✓$$

$$\int_{-1}^{1} 1 \cdot P_2(x) P_1(x) dx = \int_{-1}^{1} x^3 - \frac{1}{3}x = \frac{x^4}{4} - \frac{1}{6}x^2 \Big|_{-1}^{1} = 0 \quad ✓$$

$$\int_{-1}^{1} 1 \cdot P_0(x) P_1(x) dx = \int_{-1}^{1} x \, dx = \frac{x^2}{2} \Big|_{-1}^{1} = 0 \quad ✓$$

$$\int_{-1}^{1} 1 \, P_0(x) P_0(x) = \int_{-1}^{1} dx = 2 > 0 \quad ✓$$

$$\int_{-1}^{1} 1 \cdot P_1(x) P_1(x) = \int_{-1}^{1} x^2 dx = \frac{x^3}{3} \Big|_{-1}^{1} = \frac{1}{3} + \frac{1}{3} > 0 \quad ✓$$

$$\int_{-1}^{1} 1 \cdot P_2(x) P_2(x) = \int_{-1}^{1} \left(x^4 - \frac{2x^2}{3} + \frac{1}{9}\right) dx = \frac{x^5}{5} - \frac{2x^3}{9} + \frac{1}{9}x \Big|_{-1}^{1} = \frac{2}{5} + \frac{2}{9} > 0 \quad ✓$$

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$\therefore$ Es ortogonal.

10

Lineal: $P_1(x) = \sum_{i=0}^{1} a_k \phi_k(x)$

Cuadrático: $P_2(x) = \sum_{i=0}^{2} a_k \phi_k(x)$

donde:

$$a_k = \frac{\int_a^b w(x) f(x) \phi_k(x)\, dx}{\int_a^b w(x) \phi_k(x)^2\, dx}$$

Luego:

$$a_0 = \frac{\int_{-1}^{1} 1 . e^x . 1 . dx}{\int_{-1}^{1} 1 . 1^2 dx} = \frac{e^x \Big|_{-1}^{1}}{x \Big|_{-1}^{1}} = \frac{e - \frac{1}{e}}{2} = \frac{e^2 - 1}{2e}$$

$$a_1 = \frac{\int_{-1}^{1} 1 . e^x . x . dx}{\int_{-1}^{1} 1 . x^2 dx} = \frac{x . e^x - e^x \Big|_{-1}^{1}}{\frac{x^3}{3} \Big|_{-1}^{1}} = \frac{\frac{2}{e}}{\frac{2}{3}} = \frac{3}{e}$$

$$a_2 = \frac{\int_{-1}^{1} 1 . e^x (x^2 - \frac{1}{3}) dx}{\int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx} = \frac{\frac{2e^2 - 14}{3e}}{\frac{8}{45}} = \frac{45(2e^2 - 14)}{24e}$$

Entonces:

$$P_1(x) = \frac{3}{e} x + \frac{e^2 - 1}{2e}$$

$$P_2(x) = \frac{45(2e^2 - 14)}{24e} \left(x^2 - \frac{1}{3}\right) + \frac{3}{e} x + \frac{e^2 - 1}{2e}$$

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$$\begin{cases} \phi_0(x) = 1 \\ \phi_1(x) = x - B_1 \\ \phi_k = (x - B_k)\cdot\phi_{k-1} - C_k\,\phi_{k-2}(x) \end{cases} \quad , \; k \geq 2$$

$$\phi_2(x) = (x - B_2)(x - B_1) - C_2$$

$$B_1 = \frac{\int_a^b x\,w(x)\,\phi_0(x)^2\,dx}{\int_a^b w(x)\,\phi_0(x)^2\,dx} = \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx} = 0$$

$$C_2 = \frac{\int_a^b x\,w(x)\,\phi_1(x)\,\phi_0(x)\,dx}{\int_a^b w(x)\,\phi_0(x)^2\,dx} = \frac{\int_{-1}^{1} x^3(x - B_1)\,dx}{\int_{-1}^{1} x^2\,dx} = \frac{\int_{-1}^{1} x^4\,dx}{\int_{-1}^{1} x^2\,dx} = \frac{\frac{2}{5}}{\frac{2}{3}} = \frac{3}{5}$$

$$B_2 = \frac{\int_{-1}^{1} x\,w(x)\,\phi_1(x)^2\,dx}{\int_{-1}^{1} w(x)\,\phi_{k-1}(x)^2\,dx} = \frac{\int x^3(x - B_1)^2\,dx}{\int_{-1}^{1} x^2(x - B_1)^2\,dx} = \frac{\int_{-1}^{1} x^5\,dx}{\int_{-1}^{1} x^4\,dx} = 0$$

Luego:

$$\phi_0(x) = 1, \quad \phi_1(x) = x, \quad \phi_2(x) = x^2 - \frac{3}{5}$$

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