In matrix-vector notation, a linear program in standard form will be written as

$$\begin{array}{ll} \text{minimize} & z = c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

with $b \geq 0$. Here x and c are vectors of length n, b is a vector of length m, and A is an $m \times n$ matrix called the *constraint matrix*. The important things to notice are (i) it is a minimization problem, (ii) all the variables are constrained to be nonnegative, (iii) all the other constraints are represented as equations, and (iv) the components of the right-hand side vector b are all nonnegative. This will be the form of a linear program used within the simplex method. In other settings, other forms of a linear program may be more convenient.

Example 4.1 (Standard Form)**.** The linear program

$$\begin{array}{ll} \text{minimize} & z = 4x_1 - 5x_2 + 3x_3 \\ \text{subject to} & 3x_1 - 2x_2 + 7x_3 = 7 \\ & 8x_1 + 6x_2 + 6x_3 = 5 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

is in standard form. In terms of the matrix-vector notation,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad c = \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}, \quad A = \begin{pmatrix} 3 & -2 & 7 \\ 8 & 6 & 6 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 5 \end{pmatrix}.$$

There are $n = 3$ variables and $m = 2$ constraints. ■

All linear programs can be converted to standard form. The rules for doing this are simple and can be performed automatically by software. Most linear programming software packages allow the user to represent a linear program in any convenient way and then the software performs the conversion internally. We illustrate these techniques via examples. Justification for these rules is left to the Exercises.

If the original problem is a maximization problem:

$$\text{maximize } z = 4x_1 - 3x_2 + 6x_3 = c^T x,$$

then the objective can be multiplied by -1 to obtain

$$\text{minimize } \hat{z} = -4x_1 + 3x_2 - 6x_3 = -c^T x.$$

After the problem has been solved, the optimal objective value must be multiplied by -1, so that $z_* = -\hat{z}_*$. The optimal values of the variables are the same for both objective functions.

If any of the components of b are negative, then those constraints should be multiplied by -1. This will cause a constraint of the "$\leq$" form to be converted to a "$\geq$" constraint and vice versa.

If a variable has a lower bound other than zero, say

$$x_1 \geq 5,$$

then the variable can be replaced in the problem by

$$x_1' = x_1 - 5.$$

The constraint $x_1 \geq 5$ is equivalent to $x_1' \geq 0$. An upper bound on a variable (say, $x_1 \leq 7$) can be treated as a general constraint, that is, as one of the constraints included in the coefficient matrix A. This is inefficient but satisfactory for explaining the simplex method. More efficient techniques for handling upper bounds are described in Section 7.2.

A variable without specified lower or upper bounds, called a *free* or *unrestricted variable*, can be replaced by a pair of nonnegative variables. For example, if x_2 is a free variable, then throughout the problem it will be replaced by

$$x_2 = x_2' - x_2'' \quad \text{with} \quad x_2', x_2'' \geq 0.$$

Intuitively, x_2' will record positive values of x_2, and x_2'' will record negative values. So if $x_2 = 7$, then $x_2' = 7$ and $x_2'' = 0$, and if $x_2 = -4$, then $x_2' = 0$ and $x_2'' = 4$. The properties of the simplex method ensure that at most one of x_2' and x_2'' will be nonzero at a time (see the Exercises in Section 4.3). This is only one way of handling a free variable; an alternative is given in the Exercises; another is given in Section 7.6.6.

The remaining two transformations are used to convert general constraints into equations. A constraint of the form

$$2x_1 + 7x_2 - 3x_3 \leq 10$$

is converted to an equality constraint by including a *slack variable* s_1:

$$2x_1 + 7x_2 - 3x_3 + s_1 = 10$$

together with the constraint $s_1 \geq 0$. The slack variable just represents the difference between the left- and right-hand sides of the original constraint. Similarly a constraint of the form

$$6x_1 - 2x_2 + 4x_3 \geq 15$$

is converted to an equality by including an *excess variable* e_2:

$$6x_1 - 2x_2 + 4x_3 - e_2 = 15$$

together with the constraint $e_2 \geq 0$. (For emphasis, the slack and excess variables are labeled here as s_1 and e_2 to distinguish them from the variables used in the original formulation of the linear program. In other settings it may be more convenient to label them like the other variables, for example as x_4 and x_5. Of course, the choice of variable names does not affect the properties of the linear program.)

Example 4.2 (Transformation to Standard Form)**.** To illustrate these transformation rules, we consider the example

$$\begin{array}{ll} \text{maximize} & z = -5x_1 - 3x_2 + 7x_3 \\ \text{subject to} & 2x_1 + 4x_2 + 6x_3 = 7 \\ & 3x_1 - 5x_2 + 3x_3 \leq 5 \\ & -4x_1 - 9x_2 + 4x_3 \leq -4 \\ & x_1 \geq -2,\ 0 \leq x_2 \leq 4,\ x_3 \text{ free.} \end{array}$$

To convert to a minimization problem, we multiply the objective by -1:

$$\text{minimize } \hat{z} = 5x_1 + 3x_2 - 7x_3.$$

The third constraint is multiplied by -1 so that all the right-hand sides of the constraints are nonnegative:

$$4x_1 + 9x_2 - 4x_3 \geq 4.$$

The variable x_1 will be transformed to

$$x_1' = x_1 + 2.$$

The upper bound $x_2 \leq 4$ will be treated here as one of the general constraints and the variable x_3 will be transformed to

$$x_3 = x_3' - x_3'',$$

because it is a free variable. When these substitutions have been made we obtain

$$\begin{array}{ll} \text{minimize} & \hat{z} = 5x_1' + 3x_2 - 7x_3' + 7x_3'' - 10 \\ \text{subject to} & 2x_1' + 4x_2 + 6x_3' - 6x_3'' = 11 \\ & 3x_1' - 5x_2 + 3x_3' - 3x_3'' \leq 11 \\ & 4x_1' + 9x_2 - 4x_3' + 4x_3'' \geq 12 \\ & \qquad\qquad\qquad\qquad\quad x_2 \leq 4 \\ & x_1', x_2, x_3', x_3'' \geq 0. \end{array}$$

The constant term in the objective, "-10," is usually removed via a transformation of the form $z' = \hat{z} + 10$ so that we obtain the revised objective

$$\text{minimize } z' = 5x_1' + 3x_2 - 7x_3' + 7x_3''.$$

The final step in the conversion is to add slack and excess variables to convert the general constraints to equalities:

$$\begin{array}{lr} \text{minimize} & z' = 5x_1' + 3x_2 - 7x_3' + 7x_3'' \qquad\qquad \\ \text{subject to} & 2x_1' + 4x_2 + 6x_3' - 6x_3'' = 11 \\ & 3x_1' - 5x_2 + 3x_3' - 3x_3'' + s_2 = 11 \\ & 4x_1' + 9x_2 - 4x_3' + 4x_3'' - e_3 = 12 \\ & x_2 + s_4 = 4 \\ & x_1', x_2, x_3', x_3'', s_2, e_3, s_4 \geq 0. \end{array}$$

With this the original linear program has been converted to an equivalent one in standard form.

In matrix-vector form it would be represented as

$$\begin{array}{ll} \text{minimize} & z = c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

with $c = (5, 3, -7, 7, 0, 0, 0)^T$, $b = (11, 11, 12, 4)^T$, and

$$A = \begin{pmatrix} 2 & 4 & 6 & -6 & 0 & 0 & 0 \\ 3 & -5 & 3 & -3 & 1 & 0 & 0 \\ 4 & 9 & -4 & 4 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The vector of variables is $x = (x_1', x_2, x_3', x_3'', s_2, e_3, s_4)^T$.

It can be shown that the solution to the problem in standard form is

$$\begin{aligned} &z' = -0.12857, x_1' = 0, x_2 = 1.65714, x_3' = 0.728571, \\ &x_3'' = 0, s_2 = 17.1, e_3 = 0, s_4 = 2.34286, \end{aligned}$$

so that the solution to the original problem is

$$z = 10.12857, x_1 = -2, x_2 = 1.65714, x_3 = 0.728571. \qquad \blacksquare$$

One of the reasons that the general constraints in the problem are converted to equalities is that it allows us to use the techniques of elimination to manipulate and simplify the constraints. For example, the system

$$\begin{aligned} x_1 &= 1 \\ x_1 + x_2 &= 2 \end{aligned}$$

can be reduced to the equivalent system

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \end{aligned}$$

by subtracting the first constraint from the second. However, if we erroneously apply the same operation to

$$\begin{aligned} x_1 &\geq 1 \\ x_1 + x_2 &\geq 2, \end{aligned}$$

then it results in

$$\begin{aligned} x_1 &\geq 1 \\ x_2 &\geq 1, \end{aligned}$$

a system of constraints that defines a *different* feasible region. The two regions are illustrated in Figure 4.2. Elimination is not a valid way to manipulate systems of inequalities because it can alter the set of solutions to such systems.

It might seem that the rules for transforming a linear program to standard form could greatly increase the size of a linear program, particularly if a large number of slack and excess variables must be added to obtain a problem in standard form. However, these new variables only appear in the problem in a simple way so that the additional variables do not make the problem significantly harder to solve.