Parcial 1 - 2017-05-06

Lenguajes y compiladores

1. Considere la función $F : (\mathbf{Z} \to \mathbf{Z}_\perp) \to (\mathbf{Z} \to \mathbf{Z}_\perp)$ dada por:

$$F f n = \begin{cases} n & n = 0,1,2 \\ f(n-3) & n > \cancel{1}2 \\ f(-n) & n < 0 \end{cases}$$

a) ¿Cuánto vale $F^5 \perp_{Z \to Z_\perp} (-10)$?

b) ¿Cuánto vale el menor punto fijo de F en -10? Justifique su respuesta.

c) Justifique la siguiente afirmación: $F^2 \perp_{Z\to Z_\perp} \leq F^3 \perp_{Z \to Z_\perp}$.

d) Pruebe que F es continua.

(a) $F^0 \perp_{Z \to Z_\perp} n = \perp$

$$F \perp_{Z\to Z_\perp} n = \begin{cases} n & n=0,1,2 \\ \perp_{Z\to Z_\perp}(n-3) & n>2 \\ \perp_{Z\to Z_\perp}(-n) & n<0 \end{cases}$$

$$F \perp_{Z\to Z_\perp} n = \begin{cases} n & n=0,1,2 \\ \perp & n>2 \\ \perp & n<0 \end{cases}$$

$$F^2 \perp_{Z\to Z_\perp} n = \begin{cases} n & n=0,1,2 \\ (F\perp_{Z\to Z_\perp})(n-3) & n>2 \\ (F\perp_{Z\to Z_\perp})(-n) & n<0 \end{cases}$$

$$F^2 \perp_{Z\to Z_\perp} n = \begin{cases} n & n=0,1,2 \\ n-3 & n>2 \wedge n-3=0,1,2 \to n=3,4,5 \\ \perp & n>2 \wedge (n-3>2 \vee n<0) \to n>5 \\ -n & n<0 \wedge -n=0,1,2 \to n=-1,-2 \\ \perp & n<0 \wedge (-n>2 \vee -n<0) \to n<-2 \end{cases}$$

$$F^2 \perp_{Z\to Z_\perp} n = \begin{cases} n & n=0,1,2 \\ n-3 & n=3,4,5 \\ -n & n=-1,-2 \\ \perp & n<-2 \vee n>5 \end{cases}$$

$$F^3_{\perp\mathbb{Z}\to\mathbb{Z}_\perp}\,n = \begin{cases} n & n=0,1,2 \\ (F^2_{\mathbb{Z}\to\mathbb{Z}_\perp})(n-3) & n>2 \\ (F^2_{\mathbb{Z}\to\mathbb{Z}_\perp})(-n) & n<0 \end{cases}$$

$$F^3_{\perp\mathbb{Z}\to\mathbb{Z}_\perp}\,n = \begin{cases} n & n=0,1,2 \\ n-3 & n>2 \wedge n-3=0,1,2 \to n=3,4,5 \\ n-3-3 & n>2 \wedge n-3=3,4,5 \to n=6,7,8 \\ -n & n>2 \wedge n-3=-1,-2 \to n=2,1 \quad \times \\ \perp & n>2 \wedge (n-3<-2 \vee n-3>5) \to n>8 \\ -n & n<0 \wedge -n=0,1,2 \to n=-1,-2 \\ -n-3 & n<0 \wedge -n=3,4,5 \to n=-3,-4,-5 \\ n & n<0 \wedge -n=-1,-2 \quad \times \\ \perp & n<0 \wedge (-n<-2 \vee -n>5) \to n<-5 \end{cases}$$

$$F^3_{\perp\mathbb{Z}\to\mathbb{Z}_\perp}\,n = \begin{cases} n & n=0,1,2 \\ n-3 & n=3,4,5 \\ n-6 & n=6,7,8 \\ -n & n=-1,-2 \\ -n-3 & n=-3,-4,-5 \\ \perp & n<-5 \vee n>8 \end{cases}$$

$$F^3_{\perp\mathbb{Z}\to\mathbb{Z}_\perp}\,n = \begin{cases} n \bmod 3 & n\in\{0,..,8\} \\ |n| \bmod 3 & n\in\{-5,..,-1\} \\ \perp & n<-5 \vee n>8 \end{cases}$$

$$F^4_{\perp\mathbb{Z}\to\mathbb{Z}_\perp}\,n = \begin{cases} n & n=0,1,2 \\ (F^3_{\mathbb{Z}\to\mathbb{Z}_\perp})(n-3) & n>2 \\ (F^3_{\mathbb{Z}\to\mathbb{Z}_\perp})(-n) & n<0 \end{cases}$$

$$F^4\,\perp_{\mathbb{Z}\to\mathbb{Z}_\perp} n = \begin{cases} n & n=0,1,2 \\ n-3 \bmod 3 & n-3\in\{0,\dots,8\} \wedge n>2 \to n\in\{3,\dots,11\} \\ |n-3| \bmod 3 & n-3\in\{-5,\dots,-1\} \wedge n>2 \quad \times \\ \perp & n>2 \wedge n-3>8 \to n>11 \\ -n \bmod 3 & n<0 \wedge -n\in\{0,\dots,8\} \to n\in\{-8,\dots,-1\} \\ |-n| \bmod 3 & n<0 \wedge -n\in\{-5,\dots,-1\} \quad \times \\ \perp & n<0 \wedge -n>8 \to n<-8 \end{cases}$$

$$F^4\,\perp_{\mathbb{Z}\to\mathbb{Z}_\perp} n = \begin{cases} n & n=0,1,2 \\ n \bmod 3 & n\in\{3,\dots,11\} \\ \perp & n>11 \vee n<-8 \\ -n \bmod 3 & n\in\{-8,\dots,-1\} \end{cases}$$

$$F^4\,\perp_{\mathbb{Z}\to\mathbb{Z}_\perp} n = \begin{cases} |n| \bmod 3 & n\in\{-8,\dots,11\} \\ \perp & n>11 \vee n<-8 \end{cases}$$

$$F^5\,\perp_{\mathbb{Z}\to\mathbb{Z}_\perp} n = \begin{cases} n & n=0,1,2 \\ (F^3{}_{\mathbb{Z}\to\mathbb{Z}_\perp})(n-3) & n>2 \\ (F^3{}_{\mathbb{Z}\to\mathbb{Z}_\perp})(-n) & n<0 \end{cases}$$

$$F^5\,\perp_{\mathbb{Z}\to\mathbb{Z}_\perp} n = \begin{cases} n & n=0,1,2 \\ |n-3| \bmod 3 & n>2 \wedge n-3\in\{-8,\dots,11\} \to n\in\{3,\dots,14\} \\ \perp & n>2 \wedge n-3>11 \to n>14 \\ |-n| \bmod 3 & n<0 \wedge -n\in\{-8,\dots,11\} \to n\in\{-11,\dots,-1\} \\ \perp & n<0 \wedge -n>11 \to n<-11 \end{cases}$$

$$F^5\,\perp_{\mathbb{Z}\to\mathbb{Z}_\perp} n = \begin{cases} n & n=0,1,2 \\ n \bmod 3 & n\in\{3,\dots,14\} \\ |n| \bmod 3 & n\in\{-11,\dots,-1\} \\ \perp & n>14 \vee n<-11 \end{cases}$$

$$F^3\,\perp_{\mathbb{Z}\to\mathbb{Z}_\perp} n = \begin{cases} |n| \bmod 3 & n\in\{-11,\dots,14\} \\ \perp & n>14 \vee n<-11 \end{cases}$$

$$F^5\,\perp_{\mathbb{Z}\to\mathbb{Z}_\perp}(-10) = |-10| \bmod 3 = 10 \bmod 3 = 1.$$

b) Claramente $F^i \bot_{\mathbb{Z}\to\mathbb{Z}_\bot} n = |n| \bmod 3$

Luego $\bigsqcup_{i=0}^{\infty} F^i \bot_{\mathbb{Z}\to\mathbb{Z}_\bot} n = |n| \bmod 3$

y el menor punto fijo de F vale 1 en -10

(c)

$$F^2 \bot_{\mathbb{Z}\to\mathbb{Z}_\bot} n = \begin{cases} |n| \bmod 3 & n \in \{-2, \dots, 5\} \\ \bot & n < -2 \vee n > 5 \end{cases}$$

$$F^3 \bot_{\mathbb{Z}\to\mathbb{Z}_\bot} n = \begin{cases} |n| \bmod 3 & n \in \{-5, \dots, 8\} \\ \bot & n < -5 \vee n > 8 \end{cases}$$

Para $n \in \{-2, \dots, 5\}$ $F^2 \bot_{\mathbb{Z}\to\mathbb{Z}_\bot} n = |n| \bmod 3 = F^3 \bot_{\mathbb{Z}\to\mathbb{Z}_\bot} n$

pero para $n \in \{-5, \dots, -3\} \cup \{6, \dots, 8\}$

$$F^2 \bot_{\mathbb{Z}\to\mathbb{Z}_\bot} n = \bot \leq |n| \bmod 3 = F^3 \bot_{\mathbb{Z}\to\mathbb{Z}_\bot} n$$

(d) Primero veamos que es monótona, sean $f, g \in \mathbb{Z} \to \mathbb{Z}_\bot$ tales que $f \sqsubseteq g$

Si $n = 0, 1, 2$

$$F f n = n = F g n$$

Si $n > 2$

$$F f n = f(n-3) \leq g(n-3) = F g n$$

Si $n < 0$

$$F f n = f(-n) \leq g(-n) = F g n$$

Ahora veamos que es continua, sea $f_1 \leq f_2 \leq \dots$ una cadena interesante

$$F(\sup\{f_i\}_{i\in N}) n = \begin{cases} n & n = 0, 1, 2 \\ (\sup\{f_i\}_{i\in N})(n-2) & n > 2 \\ (\sup\{f_i\}_{i\in N})(-n) & n < 0 \end{cases}$$

Si $n = 0, 1, 2$

$$F(\sup'\{f_i\}_{i\in N}) n = n$$
$$= \sup\{n, n, \dots\}$$

$= \sup \{F f_1 n, F f_2 n, \dots\}$

$= \sup \{F f_i n\}_{i \in \mathbb{N}}$

$= \sup' \{F f_i\}_{i \in \mathbb{N}}\ n$

Si $n > 2$

$F(\sup' \{f_i\}_{i \in \mathbb{N}})n = (\sup' \{f_i\}_{i \in \mathbb{N}})(n-2)$

$= \sup \{f_i(n-2)\}_{i \in \mathbb{N}}$

$= \sup \{F f_i n\}_{i \in \mathbb{N}}$

$= \sup' \{F f_i\}_{i \in \mathbb{N}}\ n$

Si $n < 0$

$F(\sup' \{f_i\}_{i \in \mathbb{N}})\ n = (\sup' \{f_i\}_{i \in \mathbb{N}})(-n)$

$= \sup \{f_i(-n)\}_{i \in \mathbb{N}}$

$= \sup \{F f_i n\}_{i \in \mathbb{N}}$

$= \sup' \{F f_i\}_{i \in \mathbb{N}}\ n$

3. *a*) Complete las siguientes igualdades, expresando de la forma más sencilla posible el resultado, sin efectuar ningún cálculo. Considere el lenguaje que corresponde en cada caso.
   1) $[\![\forall x. \exists y.\ y + y = y]\!]\sigma =$
   2) $[\![?x;\ \textbf{while true do skip};\ !x]\!]\sigma =$
   3) $[\![x := 1;\ \textbf{newvar}\ x := 0\ \textbf{in}\ (!x;\ \textbf{fail};\ !x)]\!]\sigma =$

   b) Calcule la semántica denotacional del programa del item a) 3).

a) 1) $[\![\forall x. \exists y.\ y + y = y]\!]\sigma = V$ (para $y = 0$)

2) $[\![?x; \text{while true do skip}; !x]\!]\sigma = \iota_{in}(\lambda n.\ \iota_{out}(n, \iota_{term}[\sigma | x : n]))$

3) $[\![x := 1; \text{newvar } x := 0 \text{ in } (!x; \text{fail}; !x)]\!]\sigma = \iota_{out}(0, \iota_{abort}[\sigma | x : 1])$

b) $[\![x := 1; \text{newvar } x := 0 \text{ in } (!x; \text{fail}; !x)]\!]\sigma =$

$[\![\text{newvar } x := 0 \text{ in } (!x; \text{fail}; !x)]\!]_* ([\![x := 1]\!]\sigma) =$

$[\![\text{newvar } x := 0 \text{ in } (!x; \text{fail}; !x)]\!][\sigma | x : 1] =$

$(\lambda \sigma'.\ [\sigma' | x : [\sigma | x : 1]x])_\dagger ([\![!x; \text{fail}; !x]\!][\sigma | x : 1 | x : 0]) =$

$(\lambda \sigma'.\ [\sigma' | x : 1])_\dagger ([\![\text{fail}; !x]\!]_* [\![!x]\!][\sigma | x : 0]) =$

$(\lambda \sigma'.\ [\sigma' | x : 1])_\dagger ([\![\text{fail}; !x]\!]\ \iota_{out}([\sigma | x : 0]x, \iota_{term}[\sigma | x : 0])) =$

$(\lambda \sigma'.\ [\sigma' | x : 1])_\dagger ([\![!x]\!]_* [\![\text{fail}]\!]\ \iota_{out}(0, \iota_{term}[\sigma | x : 0])) =$

$(\lambda\sigma'. [\sigma'|x:1])_\dagger (\llbracket !x \rrbracket_* \iota_{out}(0, \llbracket fail \rrbracket \iota_{term}[\sigma|x:0])) =$

$(\lambda\sigma'. [\sigma'|x:1])_\dagger (\llbracket !x \rrbracket_* \iota_{out}(0, \iota_{abort}[\sigma|x:0])) =$

$(\lambda\sigma'. [\sigma'|x:1])_\dagger (\iota_{out}(0, \llbracket !x \rrbracket_* \iota_{abort}[\sigma|x:0])) =$

$(\lambda\sigma'. [\sigma'|x:1])_\dagger \iota_{out}(0, \iota_{abort}[\sigma|x:0]) =$

$\iota_{out}(0, (\lambda\sigma'. [\sigma'|x:1])_\dagger \iota_{abort}[\sigma|x:0]) =$

$\iota_{out}(0, \iota_{abort}(\lambda\sigma'. [\sigma'|x:1])[\sigma|x:0]) =$

$\iota_{out}(0, \iota_{abort}[\sigma|x:0|x:1]) =$

$\iota_{out}(0, \iota_{abort}[\sigma|x:1])$

4. Considere el dominio Ω del lenguaje con fallas, input y output.
   a) ¿Qué relaciones de orden encuentra entre los siguientes elementos?
      $\iota_{in}(\perp_{\mathbf{Z}\to\Omega})$, $\iota_{out}(n, \iota_{term}\ \sigma)$, $\iota_{in}(\lambda n \in \mathbf{Z}.\ \iota_{term}\ \sigma)$, $\iota_{out}(n, \perp)$,
      $\iota_{out}(n, \iota_{out}(n, \iota_{term}\ \sigma))$, $\iota_{in}(\lambda n \in \mathbf{Z}.\ \iota_{abort}\ \sigma)$,
   b) Dé un ejemplo de una cadena interesante cuyo primer elemento sea $\iota_{in}(\perp_{\mathbf{Z}\to\Omega})$.
   c) ¿Puede encontrar un programa que tenga como semántica al supremo de la cadena? Si la respuesta es sí, muétrelo.(No calcule nada!)

(a) $\iota_{in}(\lambda n \in \mathbf{Z}\ \iota_{term}\ \sigma)$ $\iota_{in}(\lambda n \in \mathbf{Z}\ \iota_{abort}\ \sigma)$ $\iota_{out}(n, \iota_{term}\ \sigma)$ $\iota_{out}(n, \iota_{out}(n, \iota_{term}\ \sigma))$

$\iota_{in}(\perp_{\mathbf{Z}\to\Omega})$ $\iota_{out}(n, \perp)$

(b) $\iota_{in}(\perp_{\mathbf{Z}\to\Omega}) \leq \iota_{in}(\iota_{in}(\perp_{\mathbf{Z}\to\Omega})) \leq \iota_{in}(\iota_{in}(\iota_{in}(\perp_{\mathbf{Z}\to\Omega}))) \ldots$

(c) while true do ?x

5. Determinar si son equivalentes. Si lo son probarlo utilizando semántica denotacional, si no lo son, dar un contraejemplo.
   newvar $v := e$ **in catchin** c_0 **with** c_1
   catchin newvar $v := e$ **in** c_0 **with** c_1

Falso, supongamos $c_0 = fail$ $c_1 = x := x+1$ $v := x$ $e := 0$

$\llbracket newvar\ x := 0 \text{ in catchin } c_0 \text{ with } c_1 \rrbracket \sigma =$

$(\lambda\sigma'. [\sigma'|x:\sigma x])_\dagger (\llbracket catchin\ c_0 \text{ with } c_1 \rrbracket [\sigma|x:0]) =$

$$(\lambda\sigma'.[\sigma'|x:\sigma x])_+(\llbracket c_1\rrbracket_+\llbracket \mathrm{fail}\rrbracket[\sigma|x:0]) =$$
$$(\lambda\sigma'.[\sigma'|x:\sigma x])_+(\llbracket c_1\rrbracket_+\langle \mathrm{abort}, [\sigma|x:0]\rangle) =$$
$$(\lambda\sigma'.[\sigma'|x:\sigma x])_+(\llbracket x:=x+1\rrbracket[\sigma|x:0]) =$$
$$(\lambda\sigma'.[\sigma'|x:\sigma x])_+[[\sigma|x:0]|x:\sigma x+1] =$$
$$(\lambda\sigma'.[\sigma'|x:\sigma x])_+[\sigma|x:1] =$$
$$[[\sigma|x:1]|x:\sigma x] = \sigma$$

$$\llbracket \mathrm{catchin\ newvar}\ x:=0\ \mathrm{in}\ c_0\ \mathrm{with}\ c_1\rrbracket\sigma =$$
$$\llbracket x:=x+1\rrbracket_+(\llbracket \mathrm{newvar}\ x:=0\ \mathrm{in\ fail}\rrbracket\sigma) =$$
$$\llbracket x:=x+1\rrbracket_+((\lambda\sigma'.[\sigma'|x:\sigma x])_+\llbracket \mathrm{fail}\rrbracket[\sigma|x:0]) =$$
$$\llbracket x:=x+1\rrbracket_+((\lambda\sigma'.[\sigma'|x:\sigma x])_+\langle \mathrm{abort}, [\sigma|x:0]\rangle) =$$
$$\llbracket x:=x+1\rrbracket_+\langle \mathrm{abort}, [[\sigma|x:0]|x:\sigma x]\rangle =$$
$$\llbracket x:=x+1\rrbracket_+\langle \mathrm{abort}, \sigma\rangle =$$
$$[\sigma|x:\sigma x+1].$$