Guía 4 - Lenguajes y compiladores

Repaso. Decida si los siguiente argumentos son correctos:

(1) Sea D un dominio y $f\colon D \to D$. Si f es continua, entonces f tiene un menor punto fijo.

(2) Sea D un dominio y $f\colon D \to D$. Si f es continua, entonces $\perp$ es el menor punto fijo de f.

(3) Sea D un dominio y sea $f_1 \sqsubseteq f_2 \sqsubseteq f_3 \sqsubseteq \dots$ una cadena en $D \to D$. Si el supremo de la cadena pertenece a la cadena, entonces existe un índice k tal que $f_k = f_{k+1} = f_{k+2}$.

(1) Verdadero, faltaría justificar.

(2) Falso. Contraejemplo

$$f: \mathbb{Z}^\infty \to \mathbb{Z}^\infty \qquad f(n) = \begin{cases} \infty & \text{si } n = \infty \\ n+1 & \text{c.c.} \end{cases}$$

0 es el $\perp$

El menor (y único) punto fijo es ∞.

(3) Verdadero.

Ejercicios.

(1) Utilizar la semántica denotacional para demostrar o refutar las siguientes equivalencias:

(a) $c; \textbf{skip} \ \equiv\ c$

(b) $c_1; (c_2; c_3) \equiv (c_1; c_2); c_3$

(c) $(\textbf{if } b \textbf{ then } c_0 \textbf{ else } c_1); c_2 \ \equiv\ \textbf{if } b \textbf{ then } c_0; c_2 \textbf{ else } c_1; c_2$

(d) $c_2; (\textbf{if } b \textbf{ then } c_0 \textbf{ else } c_1) \ \equiv\ \textbf{if } b \textbf{ then } c_2; c_0 \textbf{ else } c_2; c_1$

(e) $x := y\ ;\ z := w \ \equiv\ z := w\ ;\ x := y$

(a) $[\![c;\text{skip}]\!]\sigma = [\![\text{skip}]\!]_{\perp\!\!\!\perp}([\![c]\!]\sigma) = [\![\text{skip}]\!]\sigma_c = \sigma_c$

$[\![c]\!]\sigma = \sigma_c$

$c;\text{skip} \equiv c$

(b) $[\![c_1;(c_2;c_3)]\!]\sigma = [\![c_2;c_3]\!]_{\perp\!\!\!\perp}([\![c_1]\!]\sigma)$

$= [\![c_3]\!]_{\perp\!\!\!\perp}([\![c_2]\!]_{\perp\!\!\!\perp}([\![c_1]\!]\sigma))$

$= [\![c_3]\!]_{\perp\!\!\!\perp}([\![c_1;c_2]\!]\sigma)$

$= [\![(c_1;c_2);c_3]\!]\sigma$

(c) $[\![(\text{if } b \text{ then } c_0 \text{ else } c_1);c_2]\!]\sigma =$

$[\![c_2]\!]_{\perp\!\!\!\perp}([\![\text{if } b \text{ then } c_0 \text{ else } c_1]\!]\sigma) =$

$$\begin{cases} [\![b]\!]\sigma & [\![c_2]\!]_{\perp\!\!\perp} [\![c_0]\!]\sigma \\ c.c & [\![c_2]\!]_{\perp\!\!\perp} [\![c_1]\!]\sigma \end{cases}$$

$= [\![\text{if } b \text{ then } c_0; c_2 \text{ else } c_1; c_2]\!]$

(d) $[\![c_2; (\text{if } b \text{ then } c_0 \text{ else } c_1)]\!]\sigma$

$= [\![\text{if } b \text{ then } c_0 \text{ else } c_1]\!]_{\perp\!\!\perp} ([\![c_2]\!]\sigma)$

si $[\![c_2]\!]\sigma \neq \perp$ entonces:

$$\begin{cases} [\![b]\!]\,[\![c_2]\!]\sigma & [\![c_0]\!]\,[\![c_2]\!]\sigma \\ c.c & [\![c_1]\!]\,[\![c_2]\!]\sigma \end{cases}$$

$\not\equiv$

si $[\![c_2]\!]\sigma \neq \perp$ $\begin{cases} [\![b]\!]\sigma & [\![c_0]\!][\![c_2]\!]\sigma \\ c.c & [\![c_1]\!][\![c_2]\!]\sigma \end{cases}$

$= [\![\text{if } b \text{ then } c_2; c_0 \text{ else } c_2; c_1]\!]\sigma$

e) $[\![x := y \,;\, z := w]\!]\sigma$

$= [\![z := w]\!]\,([\![x := y]\!]\sigma)$

$= [\![z := w]\!]\,\underbrace{(\sigma \mid x : [\![y]\!]\sigma)}_{\sigma'}$

$= [\![z := w]\!]\sigma'$

$= (\sigma' \mid z : [\![w]\!]\sigma')$

$= (\sigma \mid x : [\![y]\!]\sigma \mid z : [\![w]\!]\sigma)$

$= (\sigma \mid z : [\![w]\!]\sigma \mid x : [\![y]\!]\sigma)$

$= [\![z := w \,;\, x := y]\!]\sigma$

(2) Utilizar la semántica denotacional para demostrar o refutar las siguientes equivalencias:

(a) **newvar** x := e **in skip** $\equiv$ **skip**

(b) **newvar** x := e **in** y := x $\equiv$ y := e

(c) **newvar** x := e_1 **in** (**newvar** y := e_2 **in** c) $\equiv$
newvar y := e_2 **in** (**newvar** x := e_1 **in** c)

(a) $[\![\text{newvar } x := e \text{ in skip}]\!]\sigma$

$= (\lambda\sigma' \in \Sigma.\ [\sigma' \mid x : \sigma x])_{\perp\!\!\perp}([\![\text{skip}]\!]\overbrace{[\sigma \mid x : [\![e]\!]\sigma]}^{\sigma'})$

$= (\lambda\sigma' \in \Sigma \cdot [\sigma' \mid x : \sigma x])\ \sigma'$

$= [\sigma' \mid x : \sigma x]$

$= [\sigma \mid x : [\![e]\!]\sigma \mid x : \sigma x]$

$= [\sigma \mid x : \sigma x]$

$= \sigma$

$= [\![skip]\!]\sigma$

(b) $[\![\text{newvar } x := e \text{ in } y := x]\!]\sigma$

$= (\lambda \sigma' \in \Sigma . [\sigma' \mid x : \sigma x])_{\perp\!\!\perp} ([\![y := x]\!] [\sigma \mid x : [\![e]\!]\sigma])$

$= (\lambda \sigma' \in \Sigma . [\sigma' \mid x : \sigma x]) [\sigma \mid x : [\![e]\!]\sigma \mid y : x]$

$= (\lambda \sigma' \in \Sigma . [\sigma' \mid x : \sigma x]) \sigma'$

$= [\sigma' \mid x : \sigma x]$

$= [\sigma \mid x : [\![e]\!]\sigma \mid y : x \mid x : \sigma x]$

$= [\sigma \mid y : [\![e]\!]\sigma]$

$= [\![y := e]\!]\sigma$

(c) $[\![\text{newvar } x := e_1 \text{ in } (\text{newvar } y := e_2 \text{ in } c)]\!]\sigma$

$= (\lambda \sigma' \in \Sigma . [\sigma' \mid x : \sigma x])_{\perp\!\!\perp} ([\![\text{newvar } y := e_2 \text{ in } c]\!] \overbrace{[\sigma : x : [\![e_1]\!]\sigma]}^{\sigma'})$

$= (\lambda \sigma' \in \Sigma . [\sigma' \mid x : \sigma x])_{\perp\!\!\perp} ([\![\text{newvar } y := e_2 \text{ in } c]\!] \sigma')$

$= (\lambda \sigma' \in \Sigma . [\sigma' \mid x : \sigma x])_{\perp\!\!\perp} ((\lambda \sigma'' \in \Sigma . [\sigma'' \mid y : \sigma' y])_{\perp\!\!\perp} [\![c]\!] \overbrace{[\sigma' \mid y : [\![e_2]\!]_{\sigma'}]}^{\sigma''})$

(Asumiendo $[\![c]\!]\sigma' \neq \perp$ ya que la equivalencia sería trivial)

$= (\lambda \sigma' \in \Sigma . [\sigma' \mid x : \sigma x])_{\perp\!\!\perp} ((\lambda \sigma'' \in \Sigma . [\sigma'' \mid y : \sigma' y])_{\perp\!\!\perp} [\![c]\!] \sigma'')$

$= (\lambda \sigma' \in \Sigma . [\sigma' \mid x : \sigma x])_{\perp\!\!\perp} ((\lambda \sigma'' \in \Sigma . [\sigma'' \mid y : \sigma' y])_{\perp\!\!\perp} \sigma''_c)$

$= (\lambda \sigma' \in \Sigma . [\sigma' \mid x : \sigma x]) [\sigma''_c \mid y : \sigma' y]$

$= [\sigma''_c \mid y : \sigma' y \mid x : \sigma x]$

No son equivalentes

$[\![\text{newvar } x := y+1 \text{ in } (\text{newvar } y := 5 \text{ in } c)]\!]\sigma$

$= (\lambda \sigma' \in \Sigma . [\sigma' \mid x : \sigma x])_{\perp\!\!\perp} ([\![\text{newvar } y := 5 \text{ in } c]\!] [\sigma \mid x : [\![y+1]\!]\sigma])$

$= (\lambda \sigma' \in \Sigma . [\sigma' \mid x : 1])_{\perp\!\!\perp} ([\![\text{newvar } y := 5 \text{ in } c]\!] [\sigma \mid x : 11])$

$= (\lambda \sigma' \in \Sigma . [\sigma' \mid x : 1])_{\perp\!\!\perp} ((\lambda \sigma'' \in \Sigma . [\sigma'' \mid y : \sigma' y])_{\perp\!\!\perp} [\![c]\!] [\sigma \mid x : 11 \mid y : 5])$

ejecuta a c con $x = 11$ $y = 5$

$[\![\text{newvar } y := 5 \text{ in } (\text{newvar } x := y+1 \text{ in } c)]\!]\sigma$

$= (\lambda\sigma' \in \Sigma. [\sigma' | y : \sigma y])_{\perp\!\!\perp} ([\![\text{newvar } x := y+1 \text{ in } c]\!] \underbrace{[\sigma | y : 5]}_{\sigma'})$

$= (\lambda\sigma' \in \Sigma. [\sigma' | y : \sigma y])_{\perp\!\!\perp} ((\lambda\sigma'' \in \Sigma. [\sigma'' | x : \underbrace{\sigma' x}_{\sigma' y = \sigma x}]) \; [\![c]\!] [\sigma' | x : \sigma' y + 1])$

$= (\lambda\sigma' \in \Sigma. [\sigma' | y : 10])_{\perp\!\!\perp} ((\lambda\sigma'' \in \Sigma. [\sigma'' | x : 9]) \; [\![c]\!] [\sigma' | x : 6])$

ejecuta a c con $x = 6$ $y = 5$

El estado que devuelve $[\![c]\!]\sigma'$ (en caso de $[\![c]\!]\sigma' \neq \perp$) puede ser distinto en ambas ejecuciones.

(3) Teniendo en cuenta los ejercicios anteriores, discuta en grupo las siguientes afirmaciones:
  (a) El parser puede eliminar toda ocurrencia de **skip**.
  (b) El parser puede elegir inclinar las secuencias de más de dos comandos hacia la derecha o hacia la izquierda.

(a) No, por ejemplo en el caso de:
while True do skip.

(b) Sí, la asociatividad no cambia el resultado de la ejecución

(4) Considere el comando **while true do** $x := x - 1$
  (a) Dar la función F que define su semántica. Calcular la expresión más sencilla que pueda para F.
  (b) Existe algún n tal que $F^n \perp_{\Sigma \to \Sigma_\perp}$ no sea idénticamente $\perp$?
  (c) Considere la cadena en $\Sigma \to \Sigma_\perp$ dada por

$$\omega_i \; \sigma \;=\; \begin{cases} \sigma & \text{si } 0 \leq \sigma x \leq i \\ \perp & \text{caso contrario} \end{cases}$$

Es sabido que la continuidad de F garantiza la igualdad $F(\bigsqcup \omega_i) = \bigsqcup F\omega_i$. Compruebe la misma calculando cada miembro de la igualdad para el caso de la cadena dada.

(a) $F\,\omega\,\sigma = \begin{cases} \omega_{\perp\!\!\perp} [\![x := x-1]\!]\sigma & \text{si } [\![\text{true}]\!]\sigma \\ \sigma & \text{si no} \end{cases}$

como $[\![x := x-1]\!]\sigma \neq \perp$ siempre entonces

$$F\,w\,\sigma = \begin{cases} w\,[\![x := x-1]\!]\sigma & \text{si } [\![true]\!]\sigma \\ \sigma & \text{si no} \end{cases}$$

(b) No, comprobemoslo

$$F^0 \perp_{\Sigma \to \Sigma_\perp} = \perp_{\Sigma \to \Sigma_\perp}$$

$$F^1 \perp_{\Sigma \to \Sigma_\perp} = F(F^0 \perp_{\Sigma \to \Sigma_\perp})$$

$$= F \perp_{\Sigma \to \Sigma_\perp}$$

$$= \begin{cases} \perp_{\Sigma \to \Sigma_\perp} [\![x := x-1]\!]\sigma & \text{si } [\![true]\!]\sigma \\ \sigma & \text{si no} \end{cases}$$

$$= \begin{cases} \perp & \text{si } [\![true]\!]\sigma \\ \sigma & \text{si no} \end{cases}$$

$$= \perp$$

$$F^2 \perp_{\Sigma \to \Sigma_\perp} = F(F \perp_{\Sigma \to \Sigma_\perp})$$

$$= \begin{cases} F \perp_{\Sigma \to \Sigma_\perp} [\![x := x+1]\!]\sigma & \text{si } [\![true]\!]\sigma \\ \sigma & \text{si no} \end{cases}$$

$$= \begin{cases} \perp & \text{si } [\![true]\!]\sigma \\ \sigma & \text{si no} \end{cases}$$

$$= \perp$$

Luego para cada $i \geq 0$ $F^i \perp_{\Sigma \to \Sigma_\perp} = \perp$

(c) $$w_i\,\sigma = \begin{cases} \sigma & \text{si } 0 \leq \sigma x \leq i \\ \perp & \text{c.c} \end{cases}$$

$$\left(\bigsqcup_{i=0}^{\infty} w_i\right)\sigma = \begin{cases} \sigma & \text{si } 0 \leq \sigma x \leq i \\ \perp & \text{c.c} \end{cases}$$

$$F\left(\bigsqcup_{i=0}^{\infty} w_i\right)\sigma = \begin{cases} \left(\bigsqcup_{i=0}^{\infty} w_i\right) [\![x := x-1]\!]\sigma & \text{si } [\![true]\!]\sigma \\ \sigma & \neg[\![true]\!]\sigma \end{cases}$$

$$= \begin{cases} \left(\bigsqcup_{i=0}^{\infty} w_i\right)[\sigma \mid x : \sigma x - 1] & \text{si } [\![true]\!]\sigma \\ \sigma & \text{si } \neg[\![true]\!]\sigma \end{cases}$$

$$= \left(\bigsqcup_{i=0}^{\infty} w_i\right) [\sigma | x : \sigma x - 1]$$

$$= \bigsqcup_{i=0}^{\infty} w_i \, [\sigma | x : \sigma x - 1]$$

$$= \begin{cases} [\sigma | x : \sigma x - 1] & 0 \leq [\sigma | x : \sigma x - 1] x < i \\ \perp & \text{c.c} \end{cases}$$

$$F w_i \, \sigma = \begin{cases} w_{i \perp\!\!\perp} \, [\![x : x - 1]\!] \sigma & \text{si } [\![\text{true}]\!] \sigma \\ \sigma & \text{si } \neg [\![\text{true}]\!] \sigma \end{cases}$$

$$= w_i \, [\sigma | x : \sigma x - 1]$$

$$= \begin{cases} [\sigma | x : \sigma x - 1] & \text{si } 0 \leq [\sigma | x : \sigma x - 1] x < i \\ \perp & \text{c.c} \end{cases}$$

$$\left(\bigsqcup_{i=0}^{\infty} F w_i\right) \sigma = \begin{cases} [\sigma | x : \sigma x - 1] & \text{si } 0 \leq [\sigma | x : \sigma x - 1] x < i \\ \perp & \text{c.c} \end{cases}$$

Entonces $F\left(\bigsqcup_{i=0}^{\infty} w_i\right) = \left(\bigsqcup_{i=0}^{\infty} F w_i\right)$

(5) Calcule la semántica denotacional de los siguientes comandos:

(a) **while** x < 2 **do if** x < 0 **then** x := 0 **else** x := x + 1

(b) **while** x < 2 **do if** y = 0 **then** x := x + 1 **else skip**

(a)

$$[\![\text{while } x < 2 \text{ do if } x < 0 \text{ then } x := 0 \text{ else } x := x + 1]\!] \sigma = \bigsqcup_{i=0}^{\infty} F^i \perp_{\Sigma \to \Sigma_\perp}$$

con

$$F \, w_i \sigma = \begin{cases} w_{\perp\!\!\perp} ([\![\text{if } x < 0 \text{ then } x := 0 \text{ else } x := x + 1]\!] \sigma) & \text{si } \sigma x < 2 \\ \sigma & \text{si no} \end{cases}$$

$$F^0 \perp_{\Sigma \to \Sigma_\perp} \sigma = \perp_{\Sigma \to \Sigma_\perp}$$

$$F^1 \perp_{\Sigma \to \Sigma_\perp} \sigma = F (F^0 \perp_{\Sigma \to \Sigma_\perp})$$

$$= \begin{cases} (\perp_{\Sigma \to \Sigma_\perp})_{\perp\!\!\perp} ([\![\text{if } x < 0 \text{ then } x := 0 \text{ else } x := x + 1]\!] \sigma) & \text{si } \sigma x < 2 \\ \sigma & \text{si no} \end{cases}$$

$$= \begin{cases} (\perp_{\Sigma\to\Sigma_\perp})_{\perp\!\!\perp}(\text{if } \sigma x<0 \text{ then } [\sigma|x:0] \text{ else } [\sigma|x:\sigma x+1]) & \text{si } \sigma x<2 \\ \sigma & \text{si no} \end{cases}$$

$$= \begin{cases} (\perp_{\Sigma\to\Sigma_\perp})_{\perp\!\!\perp}\,[\sigma|x:0] & \text{si } \sigma x<2 \wedge \sigma x<0 \\ (\perp_{\Sigma\to\Sigma_\perp})_{\perp\!\!\perp}\,[\sigma|x:\sigma x+1] & \text{si } \sigma x<2 \wedge \neg\sigma x<0 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} \perp & \text{si } \sigma x<0 \vee 0\le\sigma x<2 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} \sigma & \text{si } \sigma x\ge 2 \\ \perp & \text{c.c} \end{cases}$$

$$F^2\perp_{\Sigma\to\Sigma_\perp}\sigma = F(F\perp_{\Sigma\to\Sigma_\perp}\sigma)$$

$$= \begin{cases} (F\perp_{\Sigma\to\Sigma_\perp})_{\perp\!\!\perp}([\![\text{if } x<0 \text{ then } x:=0 \text{ else } x:=x+1]\!]\sigma) & \text{si } \sigma x<2 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} (F\perp_{\Sigma\to\Sigma_\perp})_{\perp\!\!\perp}(\text{if } \sigma x<0 \text{ then } [\sigma|x:0] \text{ else } [\sigma|x:\sigma x+1]) & \text{si } \sigma x<2 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} (F\perp_{\Sigma\to\Sigma_\perp})_{\perp\!\!\perp}\,[\sigma|x:0] & \text{si } \sigma x<2 \wedge \sigma x<0 \\ (F\perp_{\Sigma\to\Sigma_\perp})_{\perp\!\!\perp}\,[\sigma|x:\sigma x+1] & \text{si } \sigma x<2 \wedge \sigma x\ge 0 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} [\sigma|x:0] & \text{si } \sigma x<2 \wedge \sigma x<0 \wedge [\sigma|x:0]x\ge 2 \quad \times \\ \perp & \text{si } \sigma x<2 \wedge \sigma x<0 \wedge \neg[\sigma|x:0]x\ge 2 \quad \checkmark \\ [\sigma|x:\sigma x+1] & \text{si } \sigma x<2 \wedge \sigma x\ge 0 \wedge [\sigma|x:x+1]x\ge 2 \quad \checkmark \\ \perp & \text{si } \sigma x<2 \wedge \sigma x\ge 0 \wedge \neg[\sigma|x:x+1]x\ge 2 \quad \checkmark \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} \perp & \text{si } \sigma x<0 \\ [\sigma|x:\sigma x+1] & \text{si } 0\le\sigma x<2 \wedge [\sigma|x:\sigma x+1]x\ge 2 \\ \perp & \text{si } 0\le\sigma x<2 \wedge \underbrace{[\sigma|x:\sigma x+1]x<2}_{\sigma x<1} \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} \perp & \text{si } \sigma x<1 \\ [\sigma|x:\sigma x+1] & \text{si } 1\le\sigma x<2 \\ \sigma & \text{c.c} \end{cases}$$

$$F^3 \perp_{\Sigma \to \Sigma_\perp} \sigma = F(F^2 \perp_{\Sigma \to \Sigma_\perp})\sigma$$

$$= \begin{cases} (F^2 \perp_{\Sigma \to \Sigma_\perp})_{\perp\!\!\perp}([\![\text{if } x<0 \text{ then } x:=0 \text{ else } x:=x+1]\!]\sigma) & \text{si } \sigma x<2 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} (F^2 \perp_{\Sigma \to \Sigma_\perp})_{\perp\!\!\perp}(\text{if } \sigma x<0 \text{ then } [\sigma|x:0] \text{ else } [\sigma|x:\sigma x+1]) & \text{si } \sigma x<2 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} (F^2 \perp_{\Sigma \to \Sigma_\perp})[\sigma|x:0] & \text{si } \sigma x<0 \\ (F^2 \perp_{\Sigma \to \Sigma_\perp})[\sigma|x:\sigma x+1] & \text{si } 0 \le \sigma x < 2 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} \perp & \text{si } \sigma x<0 \wedge [\sigma|x:0]x<1 \\ [\sigma|x:0|x:\sigma x+1] & \text{si } \sigma x<0 \wedge 1 \le [\sigma|x:0]x<2 \\ [\sigma|x:0] & \text{si } \sigma x<0 \wedge [\sigma|x:0]x \ge 2 \\ \perp & \text{si } 0 \le \sigma x<2 \wedge [\sigma|x:\sigma x+1]x<1 \\ [\sigma|x:\sigma x+1|x:\sigma x+1] & \text{si } 0 \le \sigma x<2 \wedge 1 \le [\sigma|x:\sigma x+1]x<2 \\ [\sigma|x:\sigma x+1] & \text{si } 0 \le \sigma x<2 \wedge [\sigma|x:\sigma x+1] \ge 2 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} \perp & \text{si } \sigma x<0 \wedge 0<1 \\ [\sigma|x:1] & \text{si } \sigma x<0 \wedge 1 \le 0 \le 2 \quad \times \\ [\sigma|x:0] & \text{si } \sigma x<0 \wedge 0 \ge 2 \quad \times \\ \perp & \text{si } 0 \le \sigma x<2 \wedge \sigma x<0 \quad \times \\ [\sigma|x:\sigma x+2] & \text{si } 0 \le \sigma x<2 \wedge 0 \le \sigma x<1 \\ [\sigma|x:\sigma x+1] & \text{si } 0 \le \sigma x<2 \wedge \sigma x \ge 1 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} \perp & \text{si } \sigma x<0 \\ [\sigma|x:\sigma x+2] & \text{si } 0 \le \sigma x<1 \\ [\sigma|x:\sigma x+1] & \text{si } 1 \le \sigma x<2 \\ \sigma & \text{c.c} \end{cases}$$

Solo se toma como variables a enteros, podemos simplificar

$$= \begin{cases} \perp & \text{si } \sigma x < 0 \\ [\sigma \mid x:2] & \text{si } \sigma x \in \{0,1\} \\ \sigma & \text{si } \sigma x \geq 2 \end{cases}$$

$$F^4 \perp_{\Sigma \to \Sigma_\perp} = F(F^3 \perp_{\Sigma \to \Sigma_\perp})$$

$$= \begin{cases} (F^3 \perp_{\Sigma \to \Sigma_\perp})_{\perp\!\!\perp}(\llbracket \text{if } x<0 \text{ then } x:=0 \text{ else } x:=x+1 \rrbracket \sigma) & \text{si } \sigma x < 2 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} (F^3 \perp_{\Sigma \to \Sigma_\perp})\,[\sigma \mid x:0] & \text{si } \sigma x < 0 \\ (F^3 \perp_{\Sigma \to \Sigma_\perp})\,[\sigma \mid x:\sigma x+1] & \text{si } 0 \leq \sigma x < 2 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} \perp & \text{si } \sigma x < 0 \wedge [\sigma \mid x:0]x < 0 \quad \times \\ [\sigma \mid x:0 \mid x:2] & \text{si } \sigma x < 0 \wedge [\sigma \mid x:0]x \in \{0,1\} \quad \checkmark \\ [\sigma \mid x:0] & \text{si } \sigma x < 0 \wedge [\sigma \mid x:0]x \geq 2 \quad \times \\ \perp & \text{si } 0 \leq \sigma x < 2 \wedge [\sigma \mid x:\sigma x+1]x < 0 \quad \times \\ [\sigma \mid x:\sigma x+1 \mid x:2] & \text{si } 0 \leq \sigma x < 2 \wedge [\sigma \mid x:\sigma x+1]x \in \{0,1\} \quad \checkmark \\ [\sigma \mid x:\sigma x+1] & \text{si } 0 \leq \sigma x < 2 \wedge [\sigma \mid x:\sigma x+1]x \geq 2 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} [\sigma \mid x:2] & \text{si } \sigma x < 0 \\ [\sigma \mid x:2] & \text{si } 0 \leq \sigma x < 2 \wedge [\sigma \mid x:\sigma x]x \in \{-1,0\} \\ [\sigma \mid x:\sigma x+1] & \text{si } 0 \leq \sigma x < 2 \wedge \sigma x \geq 1 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} [\sigma \mid x:2] & \text{si } \sigma x \leq 0 \\ [\sigma \mid x:\sigma x+1] & \text{si } \sigma x = 1 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} [\sigma \mid x:2] & \text{si } \sigma x \leq 1 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} [\sigma \mid x:2] & \text{si } \sigma x < 2 \\ \sigma & \sigma x \geq 2 \end{cases}$$

Es claro que $(\sqcup_{i=0}^{\infty} F^i \bot_{\Sigma\to\Sigma_\bot})\sigma = \begin{cases} [\sigma | x:2] & \text{si } \sigma x < 2 \\ \sigma & \sigma x \geq 2 \end{cases}$

(b) $[\![\text{while } x<2 \text{ do if } y=0 \text{ then } x:=x+1 \text{ else skip}]\!]\sigma = (\sqcup_{i=0}^{\infty} F^i \bot_{\Sigma\to\Sigma_\bot})\sigma$

$F^0 \bot_{\Sigma\to\Sigma_\bot}\sigma = \bot_{\Sigma\to\Sigma_\bot}$

$$\begin{aligned}
F \bot_{\Sigma\to\Sigma_\bot}\sigma &= F(F^0\bot_{\Sigma\to\Sigma_\bot})\sigma \\
&= \begin{cases} (F^0\bot_{\Sigma\to\Sigma_\bot})_{\bot\!\!\bot}([\![\text{if } y=0 \text{ then } x:=x+1 \text{ else skip}]\!]\sigma) & \text{si } \sigma x<2 \\ \sigma & \text{c.c} \end{cases} \\
&= \begin{cases} \bot & \text{si } \sigma x<2 \\ \sigma & \text{c.c} \end{cases} \\
&= \begin{cases} \sigma & \text{si } \sigma x \geq 2 \\ \bot & \text{c.c} \end{cases}
\end{aligned}$$

$$\begin{aligned}
F^2\bot_{\Sigma\to\Sigma_\bot}\sigma &= F(F\bot_{\Sigma\to\Sigma_\bot})\sigma \\
&= \begin{cases} (F\bot_{\Sigma\to\Sigma_\bot})_{\bot\!\!\bot}([\![\text{if } y=0 \text{ then } x:=x+1 \text{ else skip}]\!]\sigma) & \text{si } \sigma x<2 \\ \sigma & \text{c.c} \end{cases} \\
&= \begin{cases} (F\bot_{\Sigma\to\Sigma_\bot})[\sigma | x:\sigma x+1] & \text{si } \sigma x<2 \wedge \sigma y=0 \\ (F\bot_{\Sigma\to\Sigma_\bot})\sigma & \text{si } \sigma x<2 \wedge \sigma y\neq 0 \\ \sigma & \text{c.c} \end{cases} \\
&= \begin{cases} [\sigma | x:\sigma x+1] & \text{si } \sigma x<2 \wedge \sigma y=0 \wedge [\sigma|x:\sigma x+1]\geq 2 \\ \bot & \text{si } \sigma x<2 \wedge \sigma y=0 \wedge [\sigma|x:\sigma x+1]<2 \\ \sigma & \text{si } \sigma x<2 \wedge \sigma y\neq 0 \wedge \sigma x\geq 2 \quad \times \quad \vee \text{ c.c} \\ \bot & \text{si } \sigma x<2 \wedge \sigma y\neq 0 \wedge \sigma x<2 \end{cases} \\
&= \begin{cases} [\sigma | x:\sigma x+1] & \text{si } 1\leq \sigma x<2 \wedge \sigma y=0 \\ \bot & \text{si } \sigma x<1 \wedge \sigma y=0 \\ \bot & \text{si } \sigma x<2 \wedge \sigma y\neq 0 \\ \sigma & \sigma x\geq 2 \end{cases}
\end{aligned}$$

$$= \begin{cases} [\sigma \mid x:2] & \text{si } \sigma x = 1 \wedge \sigma y = 0 \\ \perp & \text{si } (\sigma x < 1 \wedge \sigma y = 0) \vee (\sigma x < 2 \wedge \sigma y \neq 0) \\ \sigma & \sigma x \geq 2 \end{cases}$$

$$F^3 \perp_{\Sigma \to \Sigma_\perp} \sigma = F(F^2 \perp_{\Sigma \to \Sigma_\perp})\sigma$$

$$= \begin{cases} (F^2 \perp_{\Sigma \to \Sigma_\perp})[\sigma \mid x:\sigma x+1] & \text{si } \sigma x < 2 \wedge \sigma y = 0 \\ (F^2 \perp_{\Sigma \to \Sigma_\perp})\sigma & \text{si } \sigma x < 2 \wedge \sigma y \neq 0 \\ \sigma & \sigma x \geq 2 \end{cases}$$

$$= \begin{cases} [\sigma \mid x:\sigma x+1 \mid x:2] & \text{si } \sigma x < 2 \wedge \sigma y = 0 \wedge \sigma x+1 = 1 \\ \perp & \text{si } \sigma x < 2 \wedge \sigma y = 0 \wedge \sigma x+1 < 1 \\ [\sigma \mid x:\sigma x+1] & \text{si } \sigma x < 2 \wedge \sigma y = 0 \wedge \sigma x+1 \geq 2 \\ [\sigma \mid x:2] & \text{si } \sigma x < 2 \wedge \sigma y \neq 0 \wedge \sigma x = 1 \wedge \sigma y = 0 \quad \times \\ \perp & \text{si } \sigma x < 2 \wedge \sigma y \neq 0 \quad \checkmark \\ \sigma & \text{si } \sigma x < 2 \wedge \sigma y \neq 0 \wedge \sigma x \geq 2 \quad \times \\ \sigma & \text{si } \sigma x \geq 2 \quad \checkmark \end{cases}$$

$$= \begin{cases} [\sigma \mid x:2] & \text{si } \sigma x = 0 \wedge \sigma y = 0 \\ \perp & \text{si } \sigma x < 0 \wedge \sigma y = 0 \\ [\sigma \mid x:2] & \text{si } \sigma x = 1 \wedge \sigma y = 0 \\ \perp & \text{si } \sigma x < 2 \wedge \sigma y \neq 0 \\ \sigma & \text{si } \sigma x \geq 2 \end{cases}$$

$$= \begin{cases} [\sigma \mid x:2] & \text{si } \sigma x \in \{0,1\} \wedge \sigma y = 0 \\ \perp & \text{si } (\sigma x < 0 \wedge \sigma y = 0) \vee (\sigma x < 2 \wedge \sigma y \neq 0) \\ \sigma & \sigma x \geq 2 \end{cases}$$

$$F^i \perp_{\Sigma \to \Sigma_\perp} = \begin{cases} [\sigma \mid x:2] & \text{si } \sigma x \in \{-i+3, \dots, 1\} \wedge \sigma y = 0 \\ \perp & \text{si } \sigma x < 2 \wedge \sigma y \neq 0 \\ \sigma & \sigma x \geq 2 \end{cases}$$

(Faltaría probarlo por inducción)

$$\left(\bigsqcup_{i=0}^{\infty} F^i \omega\right)\sigma = \begin{cases} [\sigma \mid x:2] & \text{si } \sigma x < 2 \wedge \sigma y = 0 \\ \perp & \text{si } \sigma x < 2 \wedge \sigma y \neq 0 \\ \sigma & \sigma x \geq 2 \end{cases}$$

(6) Suponga que $[\![\textbf{while } b \textbf{ do } c]\!]\sigma \neq \perp$.

(a) Demuestre que existe $n \geq 0$ tal que $F^n \perp \sigma \neq \perp$.

(b) Demuestre que si $\sigma' = [\![\textbf{while } b \textbf{ do } c]\!]\sigma$ entonces $\neg[\![b]\!]\sigma'$

(a) $[\![\text{while } b \text{ do } c]\!]\sigma = \left(\bigsqcup_{i=0}^{\infty} F^i \perp_{\Sigma \to \Sigma_\perp} \sigma\right)$

$$F\, w\, \sigma = \begin{cases} w_{\perp\!\!\perp} [\![c]\!]\sigma & \text{si } [\![b]\!]\sigma \\ \sigma & \text{c.c} \end{cases}$$

$\left(\bigsqcup_{i=0}^{\infty} F^i \perp_{\Sigma \to \Sigma_\perp} \sigma\right) \neq \perp$

$\sup(F^0 \perp_{\Sigma \to \Sigma_\perp} \sigma, F \perp_{\Sigma \to \Sigma_\perp} \sigma, \ldots) \neq \perp$

Supongamos que no existe $n \geq 0$ tal que $F^n \perp \sigma \neq \perp$

entonces $\left(\bigsqcup_{i=0}^{\infty} F^i \perp_{\Sigma \to \Sigma_\perp} \sigma\right) =$

$\sup(F^0 \perp_{\Sigma \to \Sigma_\perp} \sigma, F \perp_{\Sigma \to \Sigma_\perp} \sigma, \ldots) =$

$\sup(\perp, \perp, \perp, \ldots) = \perp$

pero $[\![\text{while } b \text{ do } c]\!]\sigma = \left(\bigsqcup_{i=0}^{\infty} F \perp_{\Sigma \to \Sigma_\perp} \sigma\right) = \perp$

lo cual contradice la hipótesis.

(7) Demostrar o refutar las siguientes equivalencias usando semántica denotacional:

(a) **while false do** $c \equiv$ **skip**

(b) **while** b **do** $c \equiv$ **while** b **do** $(c; c)$

(c) (**while** b **do** c) ; **if** b **then** c_0 **else** $c_1 \equiv$ (**while** b **do** c) ; c_1

(a) $[\![\text{while false do } c]\!] = (\bigsqcup_{i=0}^{\infty} F^i \perp_{\Sigma \to \Sigma_\perp} \sigma)$

$F^0 \perp_{\Sigma \to \Sigma_\perp} \sigma = \perp_{\Sigma \to \Sigma_\perp}$

$$F \perp_{\Sigma \to \Sigma_\perp} \sigma = \begin{cases} (\perp_{\Sigma \to \Sigma_\perp})_{\perp\!\!\perp}[\![c]\!]\sigma & \text{si } [\![\text{false}]\!]\sigma \\ \sigma & \text{c.c} \end{cases}$$

$= \sigma$

$$F^2 \perp_{\Sigma \to \Sigma_\perp} \sigma = \begin{cases} \sigma & \text{si } [\![\text{false}]\!]\sigma \\ \sigma & \text{si } \neg[\![\text{false}]\!]\sigma \end{cases}$$

$= \sigma$

$(\bigsqcup_{i=0}^{\infty} F^i \perp_{\Sigma \to \Sigma_\perp} \sigma) = \sigma = [\![\text{skip}]\!]\sigma$

(b) $[\![\text{while } x<1 \text{ do } (x := x+1)]\!]\sigma$ $\quad \sigma x = 0$

$= (\bigsqcup_{i=0}^{\infty} F^i \perp_{\Sigma \to \Sigma_\perp})$

$F^0 \perp_{\Sigma \to \Sigma_\perp} \sigma = \Sigma \to \Sigma_\perp$

$$F \perp_{\Sigma \to \Sigma_\perp} \sigma = \begin{cases} \perp & \sigma x < 3 \\ \sigma & \text{c.c} \end{cases}$$

$$F^2 \perp_{\Sigma \to \Sigma_\perp} \sigma = \begin{cases} (F \perp_{\Sigma \to \Sigma_\perp})_{\perp\!\!\perp} [\sigma | x : \sigma x + 1] & \sigma x < 1 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} \perp & \sigma x < 1 \wedge \sigma x + 1 < 1 \\ [\sigma | x : \sigma x + 1] & \sigma x < 1 \wedge \sigma x + 1 \geq 1 \\ \sigma & \text{c.c} \end{cases}$$

$$= \begin{cases} \perp & \sigma x < 0 \\ [\sigma | x : 1] & \sigma x = 0 \\ \sigma & \text{c.c} \end{cases}$$

$$F^i \bot_{\Sigma \to \Sigma_\bot} \sigma = \begin{cases} [\sigma | x : 1] & \sigma x < 1 \\ \sigma & \sigma x \geq 1 \end{cases}$$

$$[\![\text{while } x < 1 \text{ do } (x := x+1; x := x+1)]\!] \sigma = (\sqcup_{i=0}^{\infty} F^i \bot_{\Sigma \to \Sigma_\bot})$$

$$F^0 \bot_{\Sigma \to \Sigma_\bot} \sigma = \Sigma \to \Sigma_\bot$$

$$F \bot_{\Sigma \to \Sigma_\bot} \sigma = \begin{cases} \bot & \sigma x < 1 \\ \sigma & c.c \end{cases}$$

$$F^2 \bot_{\Sigma \to \Sigma_\bot} \sigma = \begin{cases} (F \bot_{\Sigma \to \Sigma_\bot})_{\perp\!\!\perp} [\sigma | \cancel{x : \sigma x+1} \; x : \sigma x + 1] & \sigma x < 1 \\ \sigma & c.c \end{cases}$$

$$\begin{cases} \bot & \sigma x < 1 \wedge \sigma x + 2 < 1 \\ [\sigma | x : \sigma x + 2] & \sigma x < 1 \quad \sigma x + 2 \geq 1 \\ \sigma & \sigma x \geq 1 \end{cases}$$

$$\begin{cases} \bot & \sigma x < -1 \\ [\sigma | x : \sigma x + 2] & -1 \leq \sigma x < 1 \\ \sigma & \sigma x \geq 1 \end{cases}$$

$$(\sqcup_{i=0}^{\infty} F^i \bot_{\Sigma \to \Sigma_\bot}) \sigma = \begin{cases} [\sigma | x : 2] & \sigma x = 0 \\ [\sigma | x : 1] & \sigma x < 0 \\ \sigma & \sigma x \geq 1 \end{cases}$$

Entonces $[\![\text{while } x < 1 \text{ do } x := x+1]\!] \sigma = [\sigma | x : 1]$

con $\sigma x = 0$ $\quad [\![\text{while } x < 1 \text{ do } (x := x+1; x := x+1)]\!] \sigma = [\sigma | x : 2]$

No son equivalentes.

(c) $[\![(\text{while } b \text{ do } c); \text{ if } b \text{ then } c_0 \text{ else } c_1]\!] \sigma$

$= [\![\text{if } b \text{ then } c_0 \text{ else } c_1]\!]_{\perp\!\!\perp} [\![\text{while } b \text{ do } c]\!] \sigma$

Supongamos $[\![\text{while } b \text{ do } c]\!] \sigma \neq \bot$ (si no la equivalencia es trivial)

$= [\![\text{if } b \text{ then } c_0 \text{ else } c_1]\!] \sigma'$

$= \text{if } [\![b]\!] \sigma' \text{ then } [\![c_0]\!] \sigma' \text{ else } [\![c_1]\!] \sigma'$

claramente $\neg [\![b]\!] \sigma'$ ya que sino no hubiéramos salido del bucle o

$[\![\text{while } b \text{ do } c]\!] \sigma = \bot$

$= [\![c_1]\!]\sigma'$

$= [\![c_1]\!] [\![\text{while } b \text{ do } c]\!]\sigma$

$= [\![(\text{while } b \text{ do } c) ; c_1]\!]\sigma$

(8) Considerar las siguientes definiciones como syntactic sugar del comando **for** $v := e_0$ **to** e_1 **do** c:

(a) $v := e_0$; **while** $v \leq e_1$ **do** $c; v := v + 1$.

(b) **newvar** $v := e_0$ **in while** $v \leq e_1$ **do** $c;\ v := v + 1$.

(c) **newvar** $w := e_1$ **in newvar** $v := e_0$ **in while** $v \leq w$ **do** $c; v := v + 1$

¿Hay alguna que pueda considerarse satisfactoria? Justificar.

Ninguna. En un for la variable v sería local, o sea, a pesar de que cambiara su valor durante la ejecución del for queremos que al terminar retome su valor original, ninguna de las opciones restaura el valor inicial de v.

Incluso si quisieramos que v quedara con el último valor obtenido durante la ejecución en ninguna de las opciones se aumenta el valor de v en cada paso de la operación, solo se aumenta 1 vez al final.

(9) Enunciar de manera completa el teorema de coincidencia y demostrar el caso **while**.

(10) Usando el Teorema de coincidencia para comandos probar que para todo par de comandos c_0, c_1, si

$$FV\ c_0 \cap FA\ c_1 = FV\ c_1 \cap FA\ c_0 = \emptyset,$$

entonces $[\![c_0; c_1]\!] = [\![c_1; c_0]\!]$.

(11) Considere los siguientes comandos:

$$
\begin{aligned}
c_0 &\doteq \textbf{newvar}\ \mathsf{x} := \mathsf{x} + \mathsf{y}\ \textbf{in} \\
&\qquad\quad c; \textbf{while}\ \mathsf{x} > 0\ \textbf{do}\ \mathsf{y} := \mathsf{y} + 1;\ \mathsf{x} := \mathsf{x} - 1 \\
c_1 &\doteq \textbf{newvar}\ \mathsf{y} := \mathsf{x} + \mathsf{z}\ \textbf{in} \\
&\qquad\quad c; \textbf{while}\ \mathsf{y} > 0\ \textbf{do}\ \mathsf{z} := \mathsf{z} + 1;\ \mathsf{y} := \mathsf{y} - 1
\end{aligned}
$$

Asuma que c es un comando que satisface $(FV\ c) \cap \{\mathsf{x}, \mathsf{y}, \mathsf{z}\} = \emptyset$. Formule la relación que existe entre estos comandos (vistos como funciones que transforman estados), y pruebe tal relación sin calcular semántica.

(12) Considere la siguiente cadena en $\Sigma \to \Sigma_\perp$. Decida si existe un programa cuya semántica sea el supremo de la cadena. $f_i\,\sigma = \begin{cases} \sigma & \text{si } \sigma\,\mathsf{x} \leqslant \sigma\,\mathsf{y} \\ \perp & \text{en caso contrario} \end{cases}$