



Theoretical Problems

Numerical analysis 2022

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Chapter 1 Solving Nonlinear Equations

Problem 1.1 Consider the bisection method starting with the initial interval $[1.5, 3.5]$. In the following questions "the interval" refers to the bisection interval whose width changes across different loops.

- What is the width of the interval at the n th step?
- What is the maximum possible distance between the root r and the midpoint of the interval?

Solution Note that the interval's width is multiplied by $\frac{1}{2}$ at each step, and the initial width is 2, hence the width after the n th step is $\frac{1}{2^{n-1}}$.

The maximum distance is not greater than 1 obviously.

Since the loop terminated when $|f(c)| < \varepsilon$, we could construct an increasing function f whose root is $1.5 + \delta$, and $|f(x)| < \varepsilon$ everywhere, hence the bisection loop will terminate at first step, the distance between midpoint and root is $1 - \delta$. Let $\delta \rightarrow 0^+$, we know the distance could be infinitely close to 1.

Problem 1.2 In using the bisection algorithm with its initial interval as $[a_0, b_0]$ with $a_0 > 0$, we want to determine the root with its relative error no greater than ε . Prove that this goal of accuracy is guaranteed by the following choice of the number of steps,

$$n \geq \frac{\log(b_0 - a_0) - \log \varepsilon - \log a_0}{\log 2} - 1$$

Solution Suppose the root is $r \geq a_0$. The relative error after the n th step is

$$\frac{|r - c_n|}{|r|} \quad (1.1)$$

The following inequations hold

$$\frac{|r - c_n|}{|r|} \leq \frac{\frac{1}{2}(b_n - a_n)}{r} \leq \frac{\frac{1}{2}(b_n - a_n)}{a_0} = \frac{b_0 - a_0}{a_0 2^{n+1}} \quad (1.2)$$

Hence when (1.1) holds, we have

$$\begin{aligned} (n+1) \log 2 &\geq \log(b_0 - a_0) - \log \varepsilon - \log a_0 \\ \implies \log 2^{n+1} &\geq \log \left(\frac{b_0 - a_0}{\varepsilon a_0} \right) \\ \implies 2^{n+1} &\geq \frac{b_0 - a_0}{\varepsilon a_0} \implies \frac{b_0 - a_0}{a_0 2^{n+1}} \leq \varepsilon \end{aligned}$$

Hence the conclusion is proved by (1.2).

Problem 1.3 Perform four iterations of Newton's method for the polynomial equation $p(x) = 4x^3 - 2x^2 + 3 = 0$ with the starting point $x_0 = -1$. Use a hand calculator and organize results of the iterations in a table.

Solution Firstly we derivate $p(x)$

$$p'(x) = 12x^2 - 4x$$

The results are shown as the following table.

n	x_n	$p(x_n)$	$p'(x_n)$	$x_n - \frac{f(x_n)}{f'(x_n)}$
0	-1	-3	16	-0.8125
1	-0.8125	-0.46582	11.1719	-0.770804
2	-0.770804	-0.0201359	10.2129	-0.768832
3	-0.768832	-3.98011e-05	10.1686	-0.768828
4	-0.768828			

Problem 1.4 Consider a variation of Newton's method in which only the derivative at x_0 is used,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)} \quad (1.3)$$

Find C and s such that

$$e_{n+1} = C e_n^s$$

where e_n is the error of Newton's method at step n , s is a constant, and C may depend on x_n , the given function f and its derivatives.

Solution Assume the root is r , then $e_n = x_n - r$. Let $g(x) = f(r + x)$. By (1.3), we derive

$$e_{n+1} = e_n - \frac{g(e_n)}{g'(e_0)} = \left(1 - \frac{g(e_n)}{e_n g'(e_0)}\right) e_n$$

Let $C(n) = 1 - \frac{g(e_n)}{e_n g'(e_0)}$ and $s = 1$, we got $e_{n+1} = C(n) e_n$, and

$$\lim_{n \rightarrow \infty} C(n) = 1 - \frac{g'(0)}{g'(e_0)} = 1 - \frac{f'(r)}{f'(x_0)}$$

Problem 1.5 Within $(-\frac{\pi}{2}, \frac{\pi}{2})$, will the iteration $x_{n+1} = \tan^{-1} x_n$ converge?

Solution As we all know that $0 < \tan^{-1} x < x$ ($x > 0$), so if $x_0 > 0$, we derive

$$0 < x_{n+1} = \tan^{-1} x_n < x_n$$

And sequence $\{x_n\}$ has lower bound 0, so $\{x_n\}$ is convergent by monotonic sequence theorem.

For $x_0 < 0$, $\{-x_n\}$ is convergent by the discussion above, hence $\{x_n\}$ is convergent.

For $x_0 = 0$, clearly $x_n = 0$ ($\forall n$).

Problem 1.6 Let $p > 1$. What is the value of the following continued fraction?

$$x = \frac{1}{p + \frac{1}{p + \frac{1}{p + \dots}}}$$

Prove that the sequence of values converges.

Solution We construct a sequence by $x_1 = \frac{1}{p}$ and $x_{n+1} = \frac{1}{1+x_n}$ ($n \geq 1$), then $x = \lim_{n \rightarrow \infty} x_n$ if it exists.

Consider function $g(x) = \frac{1}{1+x}$, clearly $g(x) \in [0, 1]$ for all $x \in [0, 1]$. And

$$\lambda = \max_{x \in [0, 1]} |g'(x)| = \max_{x \in [0, 1]} \log(x+1) = \log 2 < 1$$

Hence g is a contraction in $[0, 1]$, and consider equation

$$x = g(x) = \frac{1}{1+x}$$

the roots are $\frac{-1 \pm \sqrt{5}}{2}$, hence $\alpha = \frac{-1 + \sqrt{5}}{2}$ is the unique root in $[0, 1]$, i.e. g has unique fixed-point in $[0, 1]$.

Recall that $x_1 = \frac{1}{p} \in [0, 1]$, and $x_{n+1} = g(x_n)$. By Theorem 1.38, $\{x_n\}$ converges and $x = \lim_{n \rightarrow \infty} x_n = \alpha$.

Problem 1.7 What happens in problem 1.2 if $a_0 < 0 < b_0$? Derive an inequality of the number of steps similar to that in problem 1.2. In this case, is the relative error still an appropriate measure?

Solution In this problem we let the absolutely error $|r - c_n| < \delta$, we derive

$$|r - c_n| \leq \frac{1}{2}(b_n - a_n) = \frac{b_0 - a_0}{2^{n+1}} \quad (1.4)$$

It is sufficient to let $\frac{b_0 - a_0}{2^{n+1}} < \delta$, hence $n \geq \frac{\log(b_0 - a_0) - \log \delta}{\log 2} - 1$.

We can't use relative error since r might be zero.