Theoretical Problems

Numerical analysis 2022

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Chapter 1 Solving Nonlinear Equations

Problem 1.1 Consider the bisection method starting with the initial interval $[1.5, 3.5]$. In the following questions "the interval" refers to the bisection interval whose width changes across different loops.

- What is the width of the interval at the nth step?
- What is the maximum possible distance between the root r and the midpoint of the interval?

Solution *Note that the interval's width is multipled by $\frac{1}{2}$ at each step, and the initial width is 2, hence the width* ***after*** *the nth step is $\frac{1}{2^{n-1}}$.*

The maximum distance is not grater than 1 obviously.

Since the loop terminated when $|f(c)| < \varepsilon$, we could construct an increasing function f whose root is $1.5 + \delta$, and $|f(x)| < \varepsilon$ everywhere, hence the bisection loop will terminate at first step, the distance between midpoint and root is $1 - \delta$. Let $\delta \to 0^+$, we know the distance could be infynitely close to 1.

Problem 1.2 In using the bisection algorithm with its initial interval as $[a_0, b_0]$ with $a_0 > 0$, we want to determine the root with its relative error no grater than ε. Prove that this goal of accuracy is guaranteed by the following choice of the number of steps,

$$n \geq \frac{\log(b_0 - a_0) - \log \varepsilon - \log a_0}{\log 2} - 1$$

Solution *Suppose the root is $r \geq a_0$. The relative error* ***after*** *the nth step is*

$$\frac{|r - c_n|}{|r|} \tag{1.1}$$

The following inequations hold

$$\frac{|r - c_n|}{|r|} \leq \frac{\frac{1}{2}(b_n - a_n)}{r} \leq \frac{\frac{1}{2}(b_n - a_n)}{a_0} = \frac{b_0 - a_0}{a_0 2^{n+1}} \tag{1.2}$$

Hence when (1.1) holds, we have

$$\begin{aligned}
&(n+1)\log 2 \geq \log(b_0 - a_0) - \log \varepsilon - \log a_0 \\
\Longrightarrow\ &\log 2^{n+1} \geq \log\left(\frac{b_0 - a_0}{\varepsilon a_0}\right) \\
\Longrightarrow\ &2^{n+1} \geq \frac{b_0 - a_0}{\varepsilon a_0} \Longrightarrow \frac{b_0 - a_0}{a_0 2^{n+1}} \leq \varepsilon
\end{aligned}$$

Hence the conclution is proved by (1.2).

Problem 1.3 Perform four iterations of Newton's method for the polynomial equation $p(x) = 4x^3 - 2x^2 + 3 = 0$ with the starting point $x_0 = -1$. Use a hand calculator and organize results of the iterations in a table.

Solution *Firstly we derivate $p(x)$*

$$p'(x) = 12x^2 - 4x$$

The results are shown as the following table.

n	x_n	$p(x_n)$	$p'(x_n)$	$x_n - \frac{f(x_n)}{f'(x_n)}$
0	-1	-3	16	-0.8125
1	-0.8125	-0.46582	11.1719	-0.770804
2	-0.770804	-0.0201359	10.2129	-0.768832
3	-0.768832	-3.98011e-05	10.1686	-0.768828
4	-0.768828			

Problem 1.4 Consider a variation of Newton's method in which only the derivative at x_0 is used,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)} \tag{1.3}$$

Find C and s such that

$$e_{n+1} = Ce_n^s$$

where e_n is the error of Newton's method at step n, s is a constant, and C may depend on x_n, the given function f and its derivatives.

Solution *Assume the root is r, then $e_n = x_n - r$. Let $g(x) = f(r+x)$. By (1.3), we derive*

$$e_{n+1} = e_n - \frac{g(e_n)}{g'(e_0)} = \left(1 - \frac{g(e_n)}{e_n g'(e_0)}\right) e_n$$

Let $C(n) = 1 - \frac{g(e_n)}{e_n g'(e_0)}$ and $s = 1$, we got $e_{n+1} = C(n)e_n$, and

$$\lim_{n\to\infty} C(n) = 1 - \frac{g'(0)}{g'(e_0)} = 1 - \frac{f'(r)}{f'(x_0)}$$

Problem 1.5 Within $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, will the iteration $x_{n+1} = \tan^{-1} x_n$ converge?

Solution *As we all know that $0 < \tan^{-1} x < x$ $(x > 0)$, so if $x_0 > 0$, we derive*

$$0 < x_{n+1} = \tan^{-1} x_n < x_n$$

And sequence $\{x_n\}$ has lower bound 0, so $\{x_n\}$ is convergent by monotinic sequence theorem.

For $x_0 < 0$, $\{-x_n\}$ is convergent by the discussion above, hence $\{x_n\}$ is convergent.

For $x_0 = 0$, clearly $x_n = 0$ $(\forall n)$.

Problem 1.6 Let $p > 1$. What is the value of the following continued fraction?

$$x = \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p + \cdots}}}$$

Prove that the sequence of values converges.

Solution *We construct a sequence by $x_1 = \frac{1}{p}$ and $x_{n+1} = \frac{1}{p+x_n}$ $(n \geq 1)$, then $x = \lim\limits_{n\to\infty} x_n$ if it exists.*

Consider function $g(x) = \frac{1}{p+x}$, clearly $g(x) \in [0,1]$ for all $x \in [0,1]$. And

$$\lambda = \max_{x\in[0,1]} |g'(x)| = \max_{x\in[0,1]} -\frac{1}{(x+p)^2} = \frac{1}{p^2} < 1$$

Hence g is a contraction in $[0,1]$, and consider equation

$$x = g(x) = \frac{1}{p+x}$$

the roots are $\frac{-p\pm\sqrt{p^2+4}}{2}$, hence g has unique fixed-point $\alpha = \frac{-p+\sqrt{p^2+4}}{2}$ in $[0,1]$.

Recall that $x_1 = \frac{1}{p} \in [0,1]$, and $x_{n+1} = g(x_n)$. By Theorem 1.38, $\{x_n\}$ converges and $x = \lim\limits_{n\to\infty} x_n = \alpha$.

Problem 1.7 What happens in problem 1.2 if $a_0 < 0 < b_0$? Derive an inequality of the number of steps similar to that in problem 1.2. In this case, is the relative error still an appropriate measure?

Solution *In this problem we let the absolutely error $|r - c_n| < \delta$, we derive*

$$|r - c_n| \leq \frac{1}{2}(b_n - a_n) = \frac{b_0 - a_0}{2^{n+1}} \tag{1.4}$$

It is sufficient to let $\frac{b_0-a_0}{2^{n+1}} < \delta$, hence $n \geq \frac{\log(b_0-a_0)-\log\delta}{\log 2} - 1$.

We can't use relative error since r might be zero.