Theoretical Problems

Numerical analysis 2022

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Chapter 1 Solving Nonlinear Equations

Problem 1.1 Consider the bisection method starting with the initial interval $[1.5, 3.5]$. In the following questions "the interval" refers to the bisection interval whose width changes across different loops.

- What is the width of the interval at the nth step?
- What is the maximum possible distance between the root r and the midpoint of the interval?

Solution *Note that the interval's width is multipled by $\frac{1}{2}$ at each step, and the initial width is 2, hence the width* ***after*** *the nth step is $\frac{1}{2^{n-1}}$.*

The maximum distance is not grater than 1 obviously.

Since the loop terminated when $|f(c)| < \varepsilon$, we could construct an increasing function f whose root is $1.5 + \delta$, and $|f(x)| < \varepsilon$ everywhere, hence the bisection loop will terminate at first step, the distance between midpoint and root is $1 - \delta$. Let $\delta \to 0^+$, we know the distance could be infynitely close to 1.

Problem 1.2 In using the bisection algorithm with its initial interval as $[a_0, b_0]$ with $a_0 > 0$, we want to determine the root with its relative error no grater than ε. Prove that this goal of accuracy is guaranteed by the following choice of the number of steps,

$$n \geq \frac{\log(b_0 - a_0) - \log\varepsilon - \log a_0}{\log 2} - 1$$

Solution *Suppose the root is $r \geq a_0$. The relative error* ***after*** *the nth step is*

$$\frac{|r - c_n|}{|r|} \tag{1.1}$$

The following inequations hold

$$\frac{|r - c_n|}{|r|} \leq \frac{\frac{1}{2}(b_n - a_n)}{r} \leq \frac{\frac{1}{2}(b_n - a_n)}{a_0} = \frac{b_0 - a_0}{a_0 2^{n+1}} \tag{1.2}$$

Hence when (1.1) holds, we have

$$\begin{aligned}
&(n+1)\log 2 \geq \log(b_0 - a_0) - \log\varepsilon - \log a_0 \\
\implies &\log 2^{n+1} \geq \log\left(\frac{b_0 - a_0}{\varepsilon a_0}\right) \\
\implies &2^{n+1} \geq \frac{b_0 - a_0}{\varepsilon a_0} \implies \frac{b_0 - a_0}{a_0 2^{n+1}} \leq \varepsilon
\end{aligned}$$

Hence the conclution is proved by (1.2).

Problem 1.3 Perform four iterations of Newton's method for the polynomial equation $p(x) = 4x^3 - 2x^2 + 3 = 0$ with the starting point $x_0 = -1$. Use a hand calculator and organize results of the iterations in a table.

Solution *Firstly we derivate $p(x)$*

$$p'(x) = 12x^2 - 4x$$

The results are shown as the following table.

n	x_n	$p(x_n)$	$p'(x_n)$	$x_n - \frac{f(x_n)}{f'(x_n)}$
0	-1	-3	16	-0.8125
1	-0.8125	-0.46582	11.1719	-0.770804
2	-0.770804	-0.0201359	10.2129	-0.768832
3	-0.768832	-3.98011e-05	10.1686	-0.768828
4	-0.768828			

Problem 1.4 Consider a variation of Newton's method in which only the derivative at x_0 is used,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)} \tag{1.3}$$

Find C and s such that

$$e_{n+1} = Ce_n^s$$

where e_n is the error of Newton's method at step n, s is a constant, and C may depend on x_n, the given function f and its derivatives.

Solution *Assume the root is r, then $e_n = x_n - r$. Let $g(x) = f(r + x)$. By (1.3), we derive*

$$e_{n+1} = e_n - \frac{g(e_n)}{g'(e_0)} = \left(1 - \frac{g(e_n)}{e_n g'(e_0)}\right) e_n$$

Let $C(n) = 1 - \frac{g(e_n)}{e_n g'(e_0)}$ and $s = 1$, we got $e_{n+1} = C(n)e_n$, and

$$\lim_{n\to\infty} C(n) = 1 - \frac{g'(0)}{g'(e_0)} = 1 - \frac{f'(r)}{f'(x_0)}$$

Problem 1.5 Within $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, will the iteration $x_{n+1} = \tan^{-1} x_n$ converge?

Solution *As we all know that $0 < \tan^{-1} x < x$ $(x > 0)$, so if $x_0 > 0$, we derive*

$$0 < x_{n+1} = \tan^{-1} x_n < x_n$$

And sequence $\{x_n\}$ has lower bound 0, so $\{x_n\}$ is convergent by monotinic sequence theorem.

For $x_0 < 0$, $\{-x_n\}$ is convergent by the discussion above, hence $\{x_n\}$ is convergent.

For $x_0 = 0$, clearly $x_n = 0$ $(\forall n)$.

Problem 1.6 Let $p > 1$. What is the value of the following continued fraction?

$$x = \frac{1}{p + \frac{1}{p + \frac{1}{p + \cdots}}}$$

Prove that the sequence of values converges.

Solution *We construct a sequence by $x_1 = \frac{1}{p}$ and $x_{n+1} = \frac{1}{p+x_n}$ $(n \geq 1)$, then $x = \lim\limits_{n\to\infty} x_n$ if it exists.*

Consider function $g(x) = \frac{1}{p+x}$, clearly $g(x) \in [0, 1]$ for all $x \in [0, 1]$. And

$$\lambda = \max_{x\in[0,1]} |g'(x)| = \max_{x\in[0,1]} -\frac{1}{(x + p)^2} = \frac{1}{p^2} < 1$$

Hence g is a contraction in $[0, 1]$, and consider equation

$$x = g(x) = \frac{1}{p + x}$$

the roots are $\frac{-p\pm\sqrt{p^2+4}}{2}$, hence g has unique fixed-point $\alpha = \frac{-p+\sqrt{p^2+4}}{2}$ in $[0, 1]$.

Recall that $x_1 = \frac{1}{p} \in [0, 1]$, and $x_{n+1} = g(x_n)$. By Theorem 1.38, $\{x_n\}$ converges and $x = \lim\limits_{n\to\infty} x_n = \alpha$.

Problem 1.7 What happens in problem 1.2 if $a_0 < 0 < b_0$? Derive an inequality of the number of steps similar to that in problem 1.2. In this case, is the relative error still an appropriate measure?

Solution *In this problem we let the absolutely error $|r - c_n| < \delta$, we derive*

$$|r - c_n| \leq \frac{1}{2}(b_n - a_n) = \frac{b_0 - a_0}{2^{n+1}} \tag{1.4}$$

It is sufficient to let $\frac{b_0-a_0}{2^{n+1}} < \delta$, hence $n \geq \frac{\log(b_0-a_0)-\log\delta}{\log 2} - 1$.

We can't use relative error since r might be zero.

Chapter 2 Polynomial Interpolation

Problem 2.1 For $f \in \mathcal{C}^2[x_0, x_1]$ and $x \in (x_0, x_1)$, linear interpolation of f at x_0 and x_1 yields

$$f(x) - p_1(f;x) = \frac{f''(\xi(x))}{2}(x - x_0)(x - x_1) \tag{2.1}$$

Consider the case $f(x) = \frac{1}{x},\ x_0 = 1,\ x_1 = 2.$

- Determine $\xi(x)$ explicity.
- Extend the domain of ξ continuously from (x_0, x_1) to $[x_0, x_1]$. Find $\max \xi(x)$, $\min \xi(x)$ and $\max f''(\xi(x))$.

Solution

1. *The Lagrange's formula yields*

$$p_1(f;x) = \frac{(x-2)}{(1-2)} + \frac{1}{2} \times \frac{(x-1)}{(2-1)} = -\frac{1}{2}x + \frac{3}{2}$$

Substitute it to (2.1), with $f''(x) = 2x^{-3}$, yield

$$\frac{1}{x} + \frac{1}{2}x - \frac{3}{2} = (x-1)(x-2)\xi^{-3}(x)$$

The result follows from it:

$$\xi(x) = \sqrt[3]{2x}$$

2. *$\xi(x)$ is increasing in $[1, 2]$, hence*

$$\max \xi(x) = \xi(2) = \sqrt[3]{4}, \qquad \min \xi(x) = \xi(1) = \sqrt[3]{2}$$

Also

$$f''(\xi(x)) = 2\left(\sqrt[3]{2x}\right)^{-3} = \frac{1}{x}$$

is decreasing in $[1, 2]$, hence

$$\max f''(\xi(x)) = f''(\xi(1)) = 1$$

Problem 2.2 Let $\mathbb{P}_m^+$ be the set of all polynomials of degree $\leq m$ that are non-negative on the real line,

$$\mathbb{P}_m^+ = \{p : p \in \mathbb{P}_m,\ \forall x \in \mathbb{R},\ p(x) \geq 0\}$$

Find $p \in \mathbb{P}_{2n}^+$ such that $p(x_i) = f_i$ for $i = 0, 1, ..., n$ where $f_i \geq 0$ and x_i are distinct points on $\mathbb{R}$.

Solution *Let $q(x) \in \mathbb{P}_n$ be the unique interpolation polynomial satisfies*

$$q(x_i) = \sqrt{f_i}, \qquad i = 0, 1, ..., n$$

Let $p(x) = q^2(x)$, then $p(x) \in \mathbb{P}_{2n}^+$ and

$$p(x_i) = q^2(x_i) = f_i, \qquad i = 0, 1, ..., n$$

Hence $p(x)$ is what we need. The Lagrange's interpolation formula yields:

$$p(x) = \left(\sum_{i=0}^{n} \sqrt{f_i} \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}\right)^2$$

Problem 2.3 Cnosider $f(x) = e^x$.

- Prove by induction that

$$\forall t \in \mathbb{R}, \qquad f[t, t+1, ..., t+n] = \frac{(e-1)^n}{n!}e^t \tag{2.2}$$

- From Corollary 2.22 we know

$$\exists \xi \in (0, n) \text{ s.t. } f[0, 1, ..., n] = \frac{1}{n!} f^{(n)}(\xi) \tag{2.3}$$

Determine ξ from the above two equations. Is ξ located to the left or to the right of the midpoint $n/2$.

Solution

1. *The Lagrange's formuula yields*

$$p(f; x) = \sum_{k=0}^{n} e^{t+k} \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j} = e^t \sum_{k=0}^{n} e^k \frac{(-1)^{n-k} \prod_{j=0,j\neq k}^{n} x - x_j}{k!(n-k)!}$$

Hence

$$f[t, t+1, ..., t+n] = e^t \sum_{k=0}^{n} \frac{(-1)^{n-k} e^k}{k!(n-k)!} = \frac{e^t}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} e^k = \frac{(e-1)^n}{n!} e^t$$

2. *Let $t = 0$ in (2.2) and yield*

$$f[0, 1, ..., n] = \frac{(e-1)^n}{n!}$$

Substitute it to (2.3), with $f^{(n)}(x) = e^x$, yield

$$\frac{(e-1)^n}{n!} = \frac{e^\xi}{n!}$$

The result follows from it:

$$\xi = n \ln(e-1) > \frac{n}{2}$$

Hence ξ is located to the right of the midpoint.

Problem 2.4 Consider $f(0) = 5,\ f(1) = 3,\ f(3) = 5,\ f(4) = 12.$

- Use the Newton's formula to obtain $p_3(f; x)$;
- The data suggests that f has a minimum in $x \in (1, 3)$. Find an approximate value for the location $x_{\min}$ of the minimum.

Solution

1. *The result follows from Newton's interpolation formula:*

$$p_3(f; x) = 5 - 2x + x(x-1) + \frac{1}{4} x(x-1)(x-3)$$

Transform it into the canonical form:

$$p_3(f; x) = \frac{1}{4} x^3 - \frac{9}{4} x + 5$$

2. *Firstly, calculate the derivative of $p_3(f; x)$:*

$$p_3'(f; x) = \frac{3}{4} x^2 - \frac{9}{4}$$

The first-order necessary condition $p_3'(f; x) = 0$ yields that

$$x_{extreame} = \pm\sqrt{3}$$

In $x \in (1, 3)$, the extreame point might be $x^* = \sqrt{3}$. The second-order condition shows that

$$p_3''(f; x^*) = \frac{3}{2} x^* = \frac{3\sqrt{3}}{2} > 0$$

Hence x^* is the minimum, and $x_{min} = \sqrt{3} \approx 1.73205$.

Problem 2.5 Consider $f(x) = x^7$.

- Compute $f[0, 1, 1, 1, 2, 2]$.

- We konw that this devided difference is expressible in terms of the 5th derivative of f evaluated at some $\xi \in (0,2)$. Determine ξ.

Solution

1. *Solve the Hermite's interpolation with a difference table. The result of Newton's form follows:*

$$p(x) = x + 6x(x-1) + 15x(x-1)^2 + 42x(x-1)^3 + 30x(x-1)^3(x-2)$$

Hence

$$f[0,1,1,1,2,2] = 30$$

2. *The 5th derivate of f is*

$$f^{(5)}(x) = 2520x^2$$

Then $f^{(5)}(x) = f[0,1,1,1,2,2]$ yields

$$2520\xi^2 = 30 \qquad \Longrightarrow \qquad \xi = \sqrt{\frac{1}{84}} = \frac{1}{2\sqrt{21}} \approx 0.1091 \in (0,2)$$

Problem 2.6 f is a function on $[0,3]$ for which one knows that

$$f(0) = 1, \quad f(1) = 2, \quad f'(1) = -1, \quad f(3) = f'(3) = 0$$

- Estimate $f(2)$ using Hermite's interpolation.
- Estimate the maximum possible error of the above answer if one konws, in addition, that $f \in \mathcal{C}^5[0,3]$ and $|f^{(5)}(x)| \leq M$ on $[0,3]$. Express the answer in terms of M.

Solution

1. *The Hermite's interpolation gives that*

$$p(x) = 1 + x - 2x(x-1) + \frac{2}{3}x(x-1)^2 - \frac{5}{36}x(x-1)^2(x-3)$$

Hence, estimate $f(2)$ as

$$f(2) \approx p(2) = \frac{11}{18} \approx 0.611111$$

2. *Theorem 2.35 gives that*

$$f(x) - p(x) = \frac{f^{(5)}(\xi)}{120}x(x-1)^2(x-3)^2$$

The result follows directly:

$$|f(2) - p(2)| = \left|\frac{f^{(5)}(\xi)}{60}\right| \leq \frac{M}{60}$$

Problem 2.7 Define foward difference by

$$\Delta f(x) = f(x+h) - f(x), \qquad \Delta^{k+1} f(x) = \Delta\Delta^k f(x) = \Delta^k f(x+h) - \Delta^k f(x)$$

and backward difference by

$$\nabla f(x) = f(x) - f(x-h), \qquad \nabla^{k+1} f(x) = \nabla\nabla^k f(x) = \nabla^k f(x) - \nabla^k f(x-h)$$

Prove

$$\Delta^k f(x) = k! h^k f[x_0, x_1, ..., x_k] \tag{2.4}$$

$$\nabla^k f(x) = k! h^k f[x_0, x_{-1}, ..., x_{-k}] \tag{2.5}$$

where $x_j = x + jh$.

Solution *The Lagrange's interpolation formula yields*

$$f[x_0, x_1, ..., x_k] = \sum_{i=0}^{k} f(x_i) \frac{1}{\prod_{j=1,j\neq i}^{k}(x_i - x_j)} = \sum_{i=0}^{k} \frac{(-1)^{k-i} f(x+ih)}{h^k i!(k-i)!}$$

It yields an equivalent form of (2.4):

$$\Delta^k f(x) = k! h^k f[x_0, x_1, ..., x_k] = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x+ih) \tag{2.6}$$

Now prove (2.6) by an induction. For $k = 1$, it could be verified directly:

$$\binom{1}{0}(-1)^{1-0} f(x) + \binom{1}{1}(-1)^{1-1} f(x+h) = f(x+h) - f(x) = \Delta f(x)$$

Suppose (2.6) holds for some $k \geq 1$, then

$$\begin{aligned}
\Delta^{k+1} f(x) &= \Delta \left(\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x+ih) \right) \\
&= \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x+(i+1)h) - \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x+ih) \\
&= f(x+(k+1)h) - (-1)^k f(x) + \sum_{i=1}^{k} \left(\binom{k}{i-1} (-1)^{k+1-i} f(x+ih) - \binom{k}{i} (-1)^{k-i} f(x+ih) \right) \\
&= f(x+(k+1)h) + (-1)^{k+1} f(x) + \sum_{i=1}^{k} (-1)^{k+1-i} f(x+ih) \left(\binom{k}{i-1} + \binom{k}{i} \right) \\
&= f(x+(k+1)h) + (-1)^{k+1} f(x) + \sum_{i=1}^{k} \binom{k+1}{i} (-1)^{k+1-i} f(x+ih) \\
&= \sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^{k+1-i} f(x+ih)
\end{aligned}$$

It shows that (2.6) holds for $(k+1)$. Hence (2.4) is proved by induction. Now we prove that

$$\Delta^k f(x) = \nabla^k f(x+kh) \tag{2.7}$$

by an induction. For $k = 1$, it could be verified directly:

$$\Delta f(x) = f(x+h) - f(x) = \nabla f(x+h)$$

Suppose (2.7) holds for some $k \geq 1$, then

$$\begin{aligned}
\Delta^{k+1} f(x) &= \Delta \left(\Delta^k f(x) \right) = \Delta \left(\nabla^k f(x+kh) \right) = \nabla^k f(x+(k+1)h) - \nabla^k f(x+kh) \\
&= \nabla \left(\nabla^k f(x+(k+1)h) \right) = \nabla^{k+1} f(x+(k+1)h)
\end{aligned}$$

Hence (2.7) is proved by induction. Finally, (2.5) follows immediately from (2.4),(2.7) and Corollary 2.15.