A1

For any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers with sum $s_A = a_1+a_2+a_3+a_4$, let p_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Among all sets of four distinct positive integers, determine those sets A for which p_A is maximal.

Answer. The sets A for which p_A is maximal are the sets the form $\{d, 5d, 7d, 11d\}$ and $\{d, 11d, 19d, 29d\}$, where d is any positive integer. For all these sets p_A is 4.

Solution. Firstly, we will prove that the maximum value of p_A is at most 4. Without loss of generality, we may assume that $a_1 < a_2 < a_3 < a_4$. We observe that for each pair of indices (i, j) with $1 \leq i < j \leq 4$, the sum $a_i + a_j$ divides s_A if and only if $a_i + a_j$ divides $s_A - (a_i + a_j) = a_k + a_l$, where k and l are the other two indices. Since there are 6 distinct pairs, we have to prove that at least two of them do not satisfy the previous condition. We claim that two such pairs are (a_2, a_4) and (a_3, a_4). Indeed, note that $a_2 + a_4 > a_1 + a_3$ and $a_3 + a_4 > a_1 + a_2$. Hence $a_2 + a_4$ and $a_3 + a_4$ do not divide s_A. This proves $p_A \leq 4$.

Now suppose $p_A = 4$. By the previous argument we have

$$\begin{array}{lcl} a_1 + a_4 \mid a_2 + a_3 & \text{and} & a_2 + a_3 \mid a_1 + a_4, \\ a_1 + a_2 \mid a_3 + a_4 & \text{and} & a_3 + a_4 \nmid a_1 + a_2, \\ a_1 + a_3 \mid a_2 + a_4 & \text{and} & a_2 + a_4 \nmid a_1 + a_3. \end{array}$$

Hence, there exist positive integers m and n with $m > n \geq 2$ such that

$$\begin{cases} a_1 + a_4 = a_2 + a_3 \\ m(a_1 + a_2) = a_3 + a_4 \\ n(a_1 + a_3) = a_2 + a_4. \end{cases}$$

Adding up the first equation and the third one, we get $n(a_1 + a_3) = 2a_2 + a_3 - a_1$. If $n \geq 3$, then $n(a_1 + a_3) > 3a_3 > 2a_2 + a_3 > 2a_2 + a_3 - a_1$. This is a contradiction. Therefore $n = 2$. If we multiply by 2 the sum of the first equation and the third one, we obtain

$$6a_1 + 2a_3 = 4a_2,$$

while the sum of the first one and the second one is

$$(m + 1)a_1 + (m - 1)a_2 = 2a_3.$$

Adding up the last two equations we get

$$(m + 7)a_1 = (5 - m)a_2.$$

It follows that $5 - m \geq 1$, because the left-hand side of the last equation and a_2 are positive. Since we have $m > n = 2$, the integer m can be equal only to either 3 or 4. Substituting $(3, 2)$ and $(4, 2)$ for (m, n) and solving the previous system of equations, we find the families of solutions $\{d, 5d, 7d, 11d\}$ and $\{d, 11d, 19d, 29d\}$, where d is any positive integer.

A2

Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers such that for every positive integer n there is an integer a with

$$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = a^{n+1} + 1.$$

Answer. The only sequence that satisfies the condition is

$$(x_1, \ldots, x_{2011}) = (1, k, \ldots, k) \quad \text{with } k = 2 + 3 + \cdots + 2011 = 2023065.$$

Solution. Throughout this solution, the set of positive integers will be denoted by $\mathbb{Z}_+$.

Put $k = 2 + 3 + \cdots + 2011 = 2023065$. We have

$$1^n + 2k^n + \cdots 2011k^n = 1 + k \cdot k^n = k^{n+1} + 1$$

for all n, so $(1, k, \ldots, k)$ is a valid sequence. We shall prove that it is the only one.

Let a valid sequence $(x_1, \ldots, x_{2011})$ be given. For each $n \in \mathbb{Z}_+$ we have some $y_n \in \mathbb{Z}_+$ with

$$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = y_n^{n+1} + 1.$$

Note that $x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n < (x_1 + 2x_2 + \cdots + 2011x_{2011})^{n+1}$, which implies that the sequence (y_n) is bounded. In particular, there is some $y \in \mathbb{Z}_+$ with $y_n = y$ for infinitely many n.

Let m be the maximum of all the x_i. Grouping terms with equal x_i together, the sum $x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n$ can be written as

$$x_1^n + 2x_2^n + \cdots + x_{2011}^n = a_m m^n + a_{m-1}(m-1)^n + \cdots + a_1$$

with $a_i \geq 0$ for all i and $a_1 + \cdots + a_m = 1 + 2 + \cdots + 2011$. So there exist arbitrarily large values of n, for which

$$a_m m^n + \cdots + a_1 - 1 - y \cdot y^n = 0. \tag{1}$$

The following lemma will help us to determine the a_i and y:

Lemma. Let integers $b_1, \ldots, b_N$ be given and assume that there are arbitrarily large positive integers n with $b_1 + b_2 2^n + \cdots + b_N N^n = 0$. Then $b_i = 0$ for all i.

Proof. Suppose that not all b_i are zero. We may assume without loss of generality that $b_N \neq 0$.

Dividing through by N^n gives

$$|b_N| = \left| b_{N-1} \left(\frac{N-1}{N} \right)^n + \cdots + b_1 \left(\frac{1}{N} \right)^n \right| \le (|b_{N-1}| + \cdots + |b_1|) \left(\frac{N-1}{N} \right)^n .$$

The expression $\left(\frac{N-1}{N}\right)^n$ can be made arbitrarily small for n large enough, contradicting the assumption that b_N be non-zero. $\square$

We obviously have $y > 1$. Applying the lemma to (1) we see that $a_m = y = m$, $a_1 = 1$, and all the other a_i are zero. This implies $(x_1, \ldots, x_{2011}) = (1, m, \ldots, m)$. But we also have $1 + m = a_1 + \cdots + a_m = 1 + \cdots + 2011 = 1 + k$ so $m = k$, which is what we wanted to show.

A3

Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

Answer. Either both f and g vanish identically, or there exists a real number C such that $f(x) = x^2 + C$ and $g(x) = x$ for all real numbers x.

Solution. Clearly all these pairs of functions satisfy the functional equation in question, so it suffices to verify that there cannot be any further ones. Substituting $-2x$ for y in the given functional equation we obtain

$$g(f(-x)) = f(x). \tag{1}$$

Using this equation for $-x-y$ in place of x we obtain

$$f(-x-y) = g(f(x+y)) = f(x) + (2x+y)g(y). \tag{2}$$

Now for any two real numbers a and b, setting $x = -b$ and $y = a+b$ we get

$$f(-a) = f(-b) + (a-b)g(a+b).$$

If c denotes another arbitrary real number we have similarly

$$f(-b) = f(-c) + (b-c)g(b+c)$$

as well as

$$f(-c) = f(-a) + (c-a)g(c+a).$$

Adding all these equations up, we obtain

$$\big((a+c)-(b+c)\big)g(a+b) + \big((a+b)-(a+c)\big)g(b+c) + \big((b+c)-(a+b)\big)g(a+c) = 0.$$

Now given any three real numbers x, y, and z one may determine three reals a, b, and c such that $x = b+c$, $y = c+a$, and $z = a+b$, so that we get

$$(y-x)g(z) + (z-y)g(x) + (x-z)g(y) = 0.$$

This implies that the three points $(x, g(x))$, $(y, g(y))$, and $(z, g(z))$ from the graph of g are collinear. Hence that graph is a line, i.e., g is either a constant or a linear function.

Let us write $g(x) = Ax + B$, where A and B are two real numbers. Substituting $(0, -y)$ for (x, y) in (2) and denoting $C = f(0)$, we have $f(y) = Ay^2 - By + C$. Now, comparing the coefficients of x^2 in (1) we see that $A^2 = A$, so $A = 0$ or $A = 1$.

If $A = 0$, then (1) becomes $B = -Bx + C$ and thus $B = C = 0$, which provides the first of the two solutions mentioned above.

Now suppose $A = 1$. Then (1) becomes $x^2 - Bx + C + B = x^2 - Bx + C$, so $B = 0$. Thus, $g(x) = x$ and $f(x) = x^2 + C$, which is the second solution from above.

Comment. Another way to show that $g(x)$ is either a constant or a linear function is the following. If we interchange x and y in the given functional equation and subtract this new equation from the given one, we obtain

$$f(x) - f(y) = (2y + x)g(x) - (2x + y)g(y).$$

Substituting $(x, 0)$, $(1, x)$, and $(0, 1)$ for (x, y), we get

$$\begin{aligned}
f(x) - f(0) &= xg(x) - 2xg(0), \\
f(1) - f(x) &= (2x + 1)g(1) - (x + 2)g(x), \\
f(0) - f(1) &= 2g(0) - g(1).
\end{aligned}$$

Taking the sum of these three equations and dividing by 2, we obtain

$$g(x) = x\big(g(1) - g(0)\big) + g(0).$$

This proves that $g(x)$ is either a constant of a linear function.

A4

Determine all pairs (f, g) of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1$$

for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f}_{k}(n)\ldots))$.

Answer. The only pair (f, g) of functions that satisfies the equation is given by $f(n) = n$ and $g(n) = 1$ for all n.

Solution. The given relation implies

$$f\left(f^{g(n)}(n)\right) < f(n+1) \quad \text{for all } n, \tag{1}$$

which will turn out to be sufficient to determine f.

Let $y_1 < y_2 < \ldots$ be all the values attained by f (this sequence might be either finite or infinite). We will prove that for every positive n the function f attains at least n values, and we have (i)$_n$: $f(x) = y_n$ if and only if $x = n$, and (ii)$_n$: $y_n = n$. The proof will follow the scheme

$$\text{(i)}_1,\ \text{(ii)}_1,\ \text{(i)}_2,\ \text{(ii)}_2,\ \ldots,\ \text{(i)}_n,\ \text{(ii)}_n,\ \ldots \tag{2}$$

To start, consider any x such that $f(x) = y_1$. If $x > 1$, then (1) reads $f\left(f^{g(x-1)}(x-1)\right) < y_1$, contradicting the minimality of y_1. So we have that $f(x) = y_1$ is equivalent to $x = 1$, establishing (i)$_1$.

Next, assume that for some n statement (i)$_n$ is established, as well as all the previous statements in (2). Note that these statements imply that for all $k \geq 1$ and $a < n$ we have $f^k(x) = a$ if and only if $x = a$.

Now, each value y_i with $1 \leq i \leq n$ is attained at the unique integer i, so y_{n+1} exists. Choose an arbitrary x such that $f(x) = y_{n+1}$; we necessarily have $x > n$. Substituting $x - 1$ into (1) we have $f\left(f^{g(x-1)}(x-1)\right) < y_{n+1}$, which implies

$$f^{g(x-1)}(x-1) \in \{1, \ldots, n\} \tag{3}$$

Set $b = f^{g(x-1)}(x-1)$. If $b < n$ then we would have $x - 1 = b$ which contradicts $x > n$. So $b = n$, and hence $y_n = n$, which proves (ii)$_n$. Next, from (i)$_n$ we now get $f(k) = n \iff k = n$, so removing all the iterations of f in (3) we obtain $x - 1 = b = n$, which proves (i)$_{n+1}$.

So, all the statements in (2) are valid and hence $f(n) = n$ for all n. The given relation between f and g now reads $n + g^n(n) = n + 1 - g(n+1) + 1$ or $g^n(n) + g(n+1) = 2$, from which it immediately follows that we have $g(n) = 1$ for all n.

Comment. Several variations of the above solution are possible. For instance, one may first prove by induction that the smallest n values of f are exactly $f(1) < \cdots < f(n)$ and proceed as follows. We certainly have $f(n) \geq n$ for all n. If there is an n with $f(n) > n$, then $f(x) > x$ for all $x \geq n$. From this we conclude $f^{g(n)+1}(n) > f^{g(n)}(n) > \cdots > f(n)$. But we also have $f^{g(n)+1} < f(n+1)$. Having squeezed in a function value between $f(n)$ and $f(n+1)$, we arrive at a contradiction.

In any case, the inequality (1) plays an essential rôle.

A5

Prove that for every positive integer n, the set $\{2, 3, 4, \ldots, 3n+1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

Solution. Throughout the solution, we denote by $[a, b]$ the set $\{a, a+1, \ldots, b\}$. We say that $\{a, b, c\}$ is an *obtuse triple* if a, b, c are the sides of some obtuse triangle.

We prove by induction on n that there exists a partition of $[2, 3n+1]$ into n obtuse triples A_i $(2 \le i \le n+1)$ having the form $A_i = \{i, a_i, b_i\}$. For the base case $n = 1$, one can simply set $A_2 = \{2, 3, 4\}$. For the induction step, we need the following simple lemma.

Lemma. Suppose that the numbers $a < b < c$ form an obtuse triple, and let x be any positive number. Then the triple $\{a, b+x, c+x\}$ is also obtuse.

Proof. The numbers $a < b+x < c+x$ are the sides of a triangle because $(c+x) - (b+x) = c-b < a$. This triangle is obtuse since $(c+x)^2 - (b+x)^2 = (c-b)(c+b+2x) > (c-b)(c+b) > a^2$. $\square$

Now we turn to the induction step. Let $n > 1$ and put $t = \lfloor n/2 \rfloor < n$. By the induction hypothesis, there exists a partition of the set $[2, 3t+1]$ into t obtuse triples $A_i' = \{i, a_i', b_i'\}$ $(i \in [2, t+1])$. For the same values of i, define $A_i = \{i, a_i' + (n-t), b_i' + (n-t)\}$. The constructed triples are obviously disjoint, and they are obtuse by the lemma. Moreover, we have

$$\bigcup_{i=2}^{t+1} A_i = [2, t+1] \cup [n+2, n+2t+1].$$

Next, for each $i \in [t+2, n+1]$, define $A_i = \{i, n+t+i, 2n+i\}$. All these sets are disjoint, and

$$\bigcup_{i=t+2}^{n+1} A_i = [t+2, n+1] \cup [n+2t+2, 2n+t+1] \cup [2n+t+2, 3n+1],$$

so

$$\bigcup_{i=2}^{n+1} A_i = [2, 3n+1].$$

Thus, we are left to prove that the triple A_i is obtuse for each $i \in [t+2, n+1]$.

Since $(2n+i) - (n+t+i) = n-t < t+2 \le i$, the elements of A_i are the sides of a triangle. Next, we have

$$(2n+i)^2 - (n+t+i)^2 = (n-t)(3n+t+2i) \ge \frac{n}{2} \cdot (3n+3(t+1)+1) > \frac{n}{2} \cdot \frac{9n}{2} \ge (n+1)^2 \ge i^2,$$

so this triangle is obtuse. The proof is completed.

A6

Let f be a function from the set of real numbers to itself that satisfies

$$f(x+y) \leq y f(x)+f(f(x)) \tag{1}$$

for all real numbers x and y. Prove that $f(x)=0$ for all $x \leq 0$.

Solution 1. Substituting $y=t-x$, we rewrite (1) as

$$f(t) \leq t f(x)-x f(x)+f(f(x)). \tag{2}$$

Consider now some real numbers a, b and use (2) with $t=f(a)$, $x=b$ as well as with $t=f(b)$, $x=a$. We get

$$\begin{aligned} & f(f(a))-f(f(b)) \leq f(a) f(b)-b f(b), \\ & f(f(b))-f(f(a)) \leq f(a) f(b)-a f(a). \end{aligned}$$

Adding these two inequalities yields

$$2 f(a) f(b) \geq a f(a)+b f(b).$$

Now, substitute $b=2 f(a)$ to obtain $2 f(a) f(b) \geq a f(a)+2 f(a) f(b)$, or $a f(a) \leq 0$. So, we get

$$f(a) \geq 0 \quad \text{for all } a<0. \tag{3}$$

Now suppose $f(x)>0$ for some real number x. From (2) we immediately get that for every $t<\dfrac{x f(x)-f(f(x))}{f(x)}$ we have $f(t)<0$. This contradicts (3); therefore

$$f(x) \leq 0 \quad \text{for all real } x, \tag{4}$$

and by (3) again we get $f(x)=0$ for all $x<0$.

We are left to find $f(0)$. Setting $t=x<0$ in (2) we get

$$0 \leq 0-0+f(0),$$

so $f(0) \geq 0$. Combining this with (4) we obtain $f(0)=0$.

Solution 2. We will also use the condition of the problem in form (2). For clarity we divide the argument into four steps.

Step 1. We begin by proving that f attains nonpositive values only. Assume that there exist some real number z with $f(z) > 0$. Substituting $x = z$ into (2) and setting $A = f(z)$, $B = -zf(z) - f(f(z))$ we get $f(t) \leq At + B$ for all real t. Hence, if for any positive real number t we substitute $x = -t$, $y = t$ into (1), we get

$$\begin{aligned} f(0) &\leq tf(-t) + f(f(-t)) \leq t(-At + B) + Af(-t) + B \\ &\leq -t(At - B) + A(-At + B) + B = -At^2 - (A^2 - B)t + (A + 1)B. \end{aligned}$$

But surely this cannot be true if we take t to be large enough. This contradiction proves that we have indeed $f(x) \leq 0$ for all real numbers x. Note that for this reason (1) entails

$$f(x + y) \leq yf(x) \tag{5}$$

for all real numbers x and y.

Step 2. We proceed by proving that f has at least one zero. If $f(0) = 0$, we are done. Otherwise, in view of Step 1 we get $f(0) < 0$. Observe that (5) tells us now $f(y) \leq yf(0)$ for all real numbers y. Thus we can specify a positive real number a that is so large that $f(a)^2 > -f(0)$. Put $b = f(a)$ and substitute $x = b$ and $y = -b$ into (5); we learn $-b^2 < f(0) \leq -bf(b)$, i.e. $b < f(b)$. Now we apply (2) to $x = b$ and $t = f(b)$, which yields

$$f(f(b)) \leq \big(f(b) - b\big)f(b) + f(f(b)),$$

i.e. $f(b) \geq 0$. So in view of Step 1, b is a zero of f.

Step 3. Next we show that if $f(a) = 0$ and $b < a$, then $f(b) = 0$ as well. To see this, we just substitute $x = b$ and $y = a - b$ into (5), thus getting $f(b) \geq 0$, which suffices by Step 1.

Step 4. By Step 3, the solution of the problem is reduced to showing $f(0) = 0$. Pick any zero r of f and substitute $x = r$ and $y = -1$ into (1). Because of $f(r) = f(r - 1) = 0$ this gives $f(0) \geq 0$ and hence $f(0) = 0$ by Step 1 again.

Comment 1. Both of these solutions also show $f(x) \leq 0$ for all real numbers x. As one can see from Solution 1, this task gets much easier if one already knows that f takes nonnegative values for sufficiently small arguments. Another way of arriving at this statement, suggested by the proposer, is as follows:

Put $a = f(0)$ and substitute $x = 0$ into (1). This gives $f(y) \leq ay + f(a)$ for all real numbers y. Thus if for any real number x we plug $y = a - x$ into (1), we obtain

$$f(a) \leq (a - x)f(x) + f(f(x)) \leq (a - x)f(x) + af(x) + f(a)$$

and hence $0 \leq (2a - x)f(x)$. In particular, if $x < 2a$, then $f(x) \geq 0$.

Having reached this point, one may proceed almost exactly as in the first solution to deduce $f(x) \leq 0$ for all x. Afterwards the problem can be solved in a few lines as shown in steps 3 and 4 of the second solution.

Comment 2. The original problem also contained the question whether a nonzero function satisfying the problem condition exists. Here we present a family of such functions.

Notice first that if $g : (0, \infty) \longrightarrow [0, \infty)$ denotes any function such that

$$g(x + y) \geq yg(x) \tag{6}$$

for all positive real numbers x and y, then the function f given by

$$f(x) = \begin{cases} -g(x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{7}$$

automatically satisfies (1). Indeed, we have $f(x) \leq 0$ and hence also $f(f(x)) = 0$ for all real numbers x. So (1) reduces to (5); moreover, this inequality is nontrivial only if x and y are positive. In this last case it is provided by (6).

Now it is not hard to come up with a nonzero function g obeying (6). E.g. $g(z) = Ce^z$ (where C is a positive constant) fits since the inequality $e^y > y$ holds for all (positive) real numbers y. One may also consider the function $g(z) = e^z - 1$; in this case, we even have that f is continuous.

A7

Let a, b, and c be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2+b^2+c^2 = 3$. Prove that

$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}. \tag{1}$$

Throughout both solutions, we denote the sums of the form $f(a,b,c) + f(b,c,a) + f(c,a,b)$ by $\sum f(a,b,c)$.

Solution 1. The condition $b + c > \sqrt{2}$ implies $b^2 + c^2 > 1$, so $a^2 = 3 - (b^2 + c^2) < 2$, i.e. $a < \sqrt{2} < b + c$. Hence we have $b + c - a > 0$, and also $c + a - b > 0$ and $a + b - c > 0$ for similar reasons.

We will use the variant of HÖLDER's inequality

$$\frac{x_1^{p+1}}{y_1^p} + \frac{x_1^{p+1}}{y_1^p} + \ldots + \frac{x_n^{p+1}}{y_n^p} \geq \frac{(x_1 + x_2 + \ldots + x_n)^{p+1}}{(y_1 + y_2 + \ldots + y_n)^p},$$

which holds for all positive real numbers $p, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$. Applying it to the left-hand side of (1) with $p = 2$ and $n = 3$, we get

$$\sum \frac{a}{(b+c-a)^2} = \sum \frac{(a^2)^3}{a^5(b+c-a)^2} \geq \frac{(a^2+b^2+c^2)^3}{\left(\sum a^{5/2}(b+c-a)\right)^2} = \frac{27}{\left(\sum a^{5/2}(b+c-a)\right)^2}. \tag{2}$$

To estimate the denominator of the right-hand part, we use an instance of SCHUR's inequality, namely

$$\sum a^{3/2}(a-b)(a-c) \geq 0,$$

which can be rewritten as

$$\sum a^{5/2}(b+c-a) \leq abc(\sqrt{a} + \sqrt{b} + \sqrt{c}).$$

Moreover, by the inequality between the arithmetic mean and the fourth power mean we also have

$$\left(\frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{3}\right)^4 \leq \frac{a^2 + b^2 + c^2}{3} = 1,$$

i.e., $\sqrt{a} + \sqrt{b} + \sqrt{c} \leq 3$. Hence, (2) yields

$$\sum \frac{a}{(b+c-a)^2} \geq \frac{27}{\left(abc(\sqrt{a} + \sqrt{b} + \sqrt{c})\right)^2} \geq \frac{3}{a^2b^2c^2},$$

thus solving the problem.

Comment. In this solution, one may also start from the following version of HÖLDER's inequality

$$\left(\sum_{i=1}^{n} a_i^3\right)\left(\sum_{i=1}^{n} b_i^3\right)\left(\sum_{i=1}^{n} c_i^3\right) \geq \left(\sum_{i=1}^{n} a_i b_i c_i\right)^3$$

applied as

$$\sum \frac{a}{(b+c-a)^2} \cdot \sum a^3(b+c-a) \cdot \sum a^2(b+c-a) \geq 27.$$

After doing that, one only needs the slightly better known instances

$$\sum a^3(b+c-a) \leq (a+b+c)abc \qquad \text{and} \qquad \sum a^2(b+c-a) \leq 3abc$$

of SCHUR's Inequality.

Solution 2. As in Solution 1, we mention that all the numbers $b+c-a$, $a+c-b$, $a+b-c$ are positive. We will use only this restriction and the condition

$$a^5+b^5+c^5 \geq 3, \tag{3}$$

which is weaker than the given one. Due to the symmetry we may assume that $a \geq b \geq c$.

In view of (3), it suffices to prove the inequality

$$\sum \frac{a^3b^2c^2}{(b+c-a)^2} \geq \sum a^5,$$

or, moving all the terms into the left-hand part,

$$\sum \frac{a^3}{(b+c-a)^2}\left((bc)^2-(a(b+c-a))^2\right) \geq 0. \tag{4}$$

Note that the signs of the expressions $(yz)^2-(x(y+z-x))^2$ and $yz-x(y+z-x)=(x-y)(x-z)$ are the same for every positive x, y, z satisfying the triangle inequality. So the terms in (4) corresponding to a and c are nonnegative, and hence it is sufficient to prove that the sum of the terms corresponding to a and b is nonnegative. Equivalently, we need the relation

$$\frac{a^3}{(b+c-a)^2}(a-b)(a-c)(bc+a(b+c-a)) \geq \frac{b^3}{(a+c-b)^2}(a-b)(b-c)(ac+b(a+c-b)).$$

Obviously, we have

$$a^3 \geq b^3 \geq 0, \quad 0 < b+c-a \leq a+c-b, \quad \text{and} \quad a-c \geq b-c \geq 0,$$

hence it suffices to prove that

$$\frac{ab+ac+bc-a^2}{b+c-a} \geq \frac{ab+ac+bc-b^2}{c+a-b}.$$

Since all the denominators are positive, it is equivalent to

$$(c+a-b)(ab+ac+bc-a^2)-(ab+ac+bc-b^2)(b+c-a)\geq 0,$$

or

$$(a-b)(2ab-a^2-b^2+ac+bc)\geq 0.$$

Since $a \geq b$, the last inequality follows from

$$c(a+b)>(a-b)^2$$

which holds since $c>a-b\geq 0$ and $a+b>a-b\geq 0$.

Claim 3. $f(n) \neq f(1)$ if and only if $a \mid n$.

Proof. Since $f(1) = \cdots = f(a-1) < f(a)$, the claim follows from the fact that

$$f(n) = f(1) \iff f(n+a) = f(1).$$

So it suffices to prove this fact.

Assume that $f(n) = f(1)$. Then $f(n+a) \mid f(a) - f(-n) = f(a) - f(n) > 0$, so $f(n+a) \leq f(a) - f(n) < f(a)$; in particular the difference $f(n+a) - f(n)$ is stricly smaller than $f(a)$. Furthermore, this difference is divisible by $f(a)$ and nonnegative since $f(n) = f(1)$ is the least value attained by f. So we have $f(n+a) - f(n) = 0$, as desired. For the converse direction we only need to remark that $f(n+a) = f(1)$ entails $f(-n-a) = f(1)$, and hence $f(n) = f(-n) = f(1)$ by the forward implication. □

We return to the induction step. So let us take two arbitrary integers m and n with $f(m) \leq f(n)$. If $a \nmid m$, then we have $f(m) = f(1) \mid f(n)$. On the other hand, suppose that $a \mid m$; then by Claim 3 $a \mid n$ as well. Now define the function $g(x) = f(ax)$. Clearly, g satisfies the conditions of the problem, but $N_g < N_f - 1$, since g does not attain $f(1)$. Hence, by the induction hypothesis, $f(m) = g(m/a) \mid g(n/a) = f(n)$, as desired.

Comment. After the fact that f attains a finite number of values has been established, there are several ways of finishing the solution. For instance, let $f(0) = b_1 > b_2 > \cdots > b_k$ be all these values. One may show (essentially in the same way as in Claim 3) that the set $S_i = \{n : f(n) \geq b_i\}$ consists exactly of all numbers divisible by some integer $a_i \geq 0$. One obviously has $a_i \mid a_{i-1}$, which implies $f(a_i) \mid f(a_{i-1})$ by Claim 1. So, $b_k \mid b_{k-1} \mid \cdots \mid b_1$, thus proving the problem statement.

Moreover, now it is easy to describe all functions satisfying the conditions of the problem. Namely, all these functions can be constructed as follows. Consider a sequence of nonnegative integers $a_1, a_2, \ldots, a_k$ and another sequence of positive integers $b_1, b_2, \ldots, b_k$ such that $|a_k| = 1$, $a_i \neq a_j$ and $b_i \neq b_j$ for all $1 \leq i < j \leq k$, and $a_i \mid a_{i-1}$ and $b_i \mid b_{i-1}$ for all $i = 2, \ldots, k$. Then one may introduce the function

$$f(n) = b_{i(n)}, \qquad \text{where} \quad i(n) = \min\{i : a_i \mid n\}.$$

These are all the functions which satisfy the conditions of the problem.

N6

Let $P(x)$ and $Q(x)$ be two polynomials with integer coefficients such that no nonconstant polynomial with rational coefficients divides both $P(x)$ and $Q(x)$. Suppose that for every positive integer n the integers $P(n)$ and $Q(n)$ are positive, and $2^{Q(n)}-1$ divides $3^{P(n)}-1$. Prove that $Q(x)$ is a constant polynomial.

Solution. First we show that there exists an integer d such that for all positive integers n we have $\gcd\big(P(n),Q(n)\big)\le d$.

Since $P(x)$ and $Q(x)$ are coprime (over the polynomials with rational coefficients), EUCLID's algorithm provides some polynomials $R_0(x), S_0(x)$ with rational coefficients such that $P(x)R_0(x)-Q(x)S_0(x)=1$. Multiplying by a suitable positive integer d, we obtain polynomials $R(x)=d\cdot R_0(x)$ and $S(x)=d\cdot S_0(x)$ with integer coefficients for which $P(x)R(x)-Q(x)S(x)=d$. Then we have $\gcd\big(P(n),Q(n)\big)\le d$ for any integer n.

To prove the problem statement, suppose that $Q(x)$ is not constant. Then the sequence $Q(n)$ is not bounded and we can choose a positive integer m for which

$$M=2^{Q(m)}-1\ge 3^{\max\{P(1),P(2),\dots,P(d)\}}. \tag{1}$$

Since $M=2^{Q(n)}-1\,\big|\,3^{P(n)}-1$, we have $2,3\nmid M$. Let a and b be the multiplicative orders of 2 and 3 modulo M, respectively. Obviously, $a=Q(m)$ since the lower powers of 2 do not reach M. Since M divides $3^{P(m)}-1$, we have $b\,\big|\,P(m)$. Then $\gcd(a,b)\le\gcd\big(P(m),Q(m)\big)\le d$. Since the expression $ax-by$ attains all integer values divisible by $\gcd(a,b)$ when x and y run over all nonnegative integer values, there exist some nonnegative integers x,y such that $1\le m+ax-by\le d$.

By $Q(m+ax)\equiv Q(m)\pmod a$ we have

$$2^{Q(m+ax)}\equiv 2^{Q(m)}\equiv 1\pmod M$$

and therefore

$$M\,\big|\,2^{Q(m+ax)}-1\,\big|\,3^{P(m+ax)}-1.$$

Then, by $P(m+ax-by)\equiv P(m+ax)\pmod b$ we have

$$3^{P(m+ax-by)}\equiv 3^{P(m+ax)}\equiv 1\pmod M.$$

Since $P(m+ax-by)>0$ this implies $M\le 3^{P(m+ax-by)}-1$. But $P(m+ax-by)$ is listed among $P(1),P(2),\dots,P(d)$, so

$$M<3^{P(m+ax-by)}\le 3^{\max\{P(1),P(2),\dots,P(d)\}}$$

which contradicts (1).

Comment. We present another variant of the solution above.

Denote the degree of P by k and its leading coefficient by p. Consider any positive integer n and let $a = Q(n)$. Again, denote by b the multiplicative order of 3 modulo $2^a - 1$. Since $2^a - 1 \mid 3^{P(n)} - 1$, we have $b \mid P(n)$. Moreover, since $2^{Q(n+at)} - 1 \mid 3^{P(n+at)} - 1$ and $a = Q(n) \mid Q(n + at)$ for each positive integer t, we have $2^a - 1 \mid 3^{P(n+at)} - 1$, hence $b \mid P(n + at)$ as well.

Therefore, b divides $\gcd\{P(n + at) : t \geq 0\}$; hence it also divides the number

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} P(n + ai) = p \cdot k! \cdot a^k.$$

Finally, we get $b \mid \gcd\bigl(P(n), k! \cdot p \cdot Q(n)^k\bigr)$, which is bounded by the same arguments as in the beginning of the solution. So $3^b - 1$ is bounded, and hence $2^{Q(n)} - 1$ is bounded as well.