《常微分方程》课堂作业二

时间 2 小时，满分100分

1）证明题请特别注重证明的严谨性。条理清楚、 叙述完整以及推理严格的答题**可以最高加分**50%； 简单的描述或者非严格的证明则 **最多也只能得到这道题的**50% **的分数。**
2）计算题请给出详细的计算过程，非常完美的答题可以最高加分50%。 如果仅仅给出计算结果则没有分数。
3）可以引用其它小题的结论（即使没有完成证明），但是不能循环证明。
4）可以不抄题目，但编号要写清楚，不然不予记分。
5）此课堂作业为闭卷式，不允许使用任何资料和计算器。

练习一（55分）、假设M, N 是定义在区域$D = \{(x, y) \in \mathbb{R}^2; a < x < b, c < y < d\}$上的所有一阶偏微分都是连续的函数。若

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}, \quad \forall (x,y) \in D. \tag{1.1}$$

则称方程$M(x, y)dx + N(x, y)dy = 0$ 为恰当方程。 若存在可微函数$\mu(x, y)$, 使得方程

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0, \tag{1.2}$$

为恰当方程, 则称$\mu(x, y)$为方程$M(x, y)dx + N(x, y)dy = 0$ 的积分因子。

1.1）（10分）证明$\mu(x, y)$为方程$M(x, y)dx + N(x, y)dy = 0$ 的积分因子的充分必要条件为

$$N(x,y)\frac{\partial \mu(x,y)}{\partial x} - M(x,y)\frac{\partial \mu(x,y)}{\partial y} = \left(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right)\mu(x,y).$$

证明：μ(x,y)为$M(x, y)dx + N(x, y)dy = 0$的积分因子,

$$\Longleftrightarrow \frac{\partial(\mu(x,y)M(x,y))}{\partial y} = \frac{\partial(\mu(x,y)N(x,y))}{\partial x}$$

1.2) （10分）证明方程$M(x, y)dx + N(x, y)dy = 0$ 具有形如$\mu(x)$的积分因子的充分必要条件为

$$\frac{1}{N(x,y)}\left(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right)$$

仅只是x的连续函数（与y无关）， 记为$\phi(x)$。在此情况下求$\mu(x)$.

证明：由1.1)知$\mu(x)$为$M(x, y)dx + N(x, y)dy = 0$的积分因子当且仅当

$$N(x,y)\frac{\partial \mu(x)}{\partial x} - M(x,y)\frac{\partial \mu(x)}{\partial y} = (\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}) \cdot \mu(x), \quad \frac{\partial \mu(x)}{\partial y} = 0, \frac{\partial \mu(x)}{\partial x} = \frac{d\mu(x)}{dx}$$

$$\Longleftrightarrow N(x,y)\frac{d\mu(x)}{dx} = (\frac{\partial M(x,y)}{dx} - \frac{\partial N(x,y)}{\partial x})\mu(x)$$

$$\Longleftrightarrow \frac{d\mu(x)}{\mu(x)} = \frac{1}{N(x,y)}(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x})dx$$

等式左端为只与x有关的微分, 要$\mu(x)$存在（即可解出）当且仅当$\frac{1}{N(x,y)}(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x})$仅为$x$的连续函数$\phi(x)$, 则

$$\frac{d\mu(x)}{\mu(x)} = \phi(x)dx, \quad \Rightarrow \quad \int \frac{d\mu(x)}{\mu(x)} = \int \phi(x)dx$$

解得 $\ln|\mu(x)| = \int \phi dx + c$, （$c$ 为任意常数), 即$\mu(x) = e^{\int \phi(x)dx}$, （$c$取0时).

1.3) （5分）给出方程$M(x,y)dx + N(x,y)dy = 0$ 具有形如$\mu(y)$的积分因子的充分必要条件且证明之,在此情况下求$\mu(y)$.

证明：

$$\frac{d\mu(y)}{\mu(y)} = -\frac{1}{\mu(x,y)}(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}) \cdot dy = -\psi(y)dy,$$

积分得

$$\int \frac{d\mu(y)}{\mu(y)} = \int -\psi(y)dy \quad \Rightarrow \quad \ln|\mu(y)| = \int -\psi(y)dy + c$$

其中c为任意常数, $c = 0$时有$\mu(y) = e^{\int -\psi(y)dy}$.

1.4) （10分）证明方程$M(x,y)dx + N(x,y)dy = 0$ 具有形如$\mu(x \pm y)$的积分因子的充分必要条件为

$$\frac{1}{N(x,y) \mp M(x,y)}\left(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right)$$

仅只是$x \pm y$的连续函数， 记为$\phi(x \pm y)$。在此情况下求$\mu(x \pm y)$.

证明：$M(x,y)dx + N(x,y)dy = 0$有形如$\mu(x+y)$的积分因子,

$$\iff N(x,y)\frac{\partial \mu(x+y)}{\partial x} - M(x,y)\frac{\partial \mu(x+y)}{\partial y} = (\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x})\mu(x+y),$$

令$u = x + y$, 则$\frac{\partial \mu(x+y)}{\partial x} = \frac{d\mu(u)}{du} \cdot \frac{\partial u}{\partial x} = \frac{d\mu(u)}{du}$, 同理$\frac{\partial \mu(x+y)}{\partial y} = \frac{d\mu(u)}{du}$,

$$\begin{aligned}
&\iff N(x,y)\frac{d\mu(u)}{du} - M(x,y)\frac{d\mu(u)}{du} = (\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x})\mu(u) \\
&\iff [N(x,y) - M(x,y)]\frac{d\mu(u)}{du} = (\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x})\mu(u) \\
&\iff \frac{d\mu(u)}{\mu(u)} = \frac{1}{N(x,y) - M(x,y)}(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x})du
\end{aligned}$$

左式为仅与u相关的微分, 则$\mu(u)$存在（可解）当且仅当$\frac{1}{N(x,y)-M(x,y)}(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x})$仅为$u$的连续函数$\phi(u)$（即$\phi(x+y)$).

同理可知$\mu(x-y)$为$M(x,y)dx + N(x,y)dy = 0$恰当因子的充分必要条件为

$$\frac{1}{N(x,y) + M(x,y)}(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x})$$

仅为$(x-y)$的连续函数$\phi(x-y)$.

对$u = x + y$, $\int \frac{d\mu(u)}{\mu(u)} = \int \phi(u)du \Rightarrow \ln|\mu(u)| = \int \phi(u)du + c$ （c为任意常数). c取0时, $\mu(u) = e^{\int \phi(u)du}$, 即$\mu(x+y) = e^{\int \phi(x+y)d(x+y)}$.

同理, $\mu(x-y) = e^{\int \phi(x-y)d(x-y)}$.

1.5) （20分）给出方程$M(x,y)dx + N(x,y)dy = 0$ 具有形如

$$\mu(xy), \quad \mu(x^2+y^2), \quad \mu(\frac{y}{x}), \quad \mu(x^\alpha y^\beta)$$

的积分因子的充分必要条件且证明之,在此情况下求相应的积分因子。

证明：μ为$M(x,y)dx+N(x,y)dy=0$当且仅当

$$(1)\qquad N(x,y)\frac{\partial \mu}{\partial x}-M(x,y)\frac{\partial \mu}{\partial y}=(\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x})\cdot\mu,$$

若存在$\mu(xy)$当且仅当满足(1). 令$u=xy$时,

$$\frac{\partial \mu}{\partial x}=\frac{d\mu(u)}{du}\cdot\frac{\partial u}{\partial x}=y\frac{d\mu(u)}{du}.$$

同理, $\frac{\partial\mu}{\partial y}=x\frac{d\mu(u)}{du}$.

此时(1)式变为

$$y\cdot\frac{d\mu(u)}{du}\cdot N(x,y)-x\frac{d\mu(u)}{du}M(x,y)=(\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x})\mu(u)$$
$$\Rightarrow\quad \frac{d\mu(u)}{\mu(u)}=\frac{1}{yN(x,y)-xM(x,y)}(\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x})du,$$

存在$\mu(u)$ (即$\mu(xy)$)当且仅当$\frac{1}{yN(x,y)-xM(x,y)}(\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x})$仅为$xy$的连续函数$\phi(xy)$.

此时,

$$\int\frac{d\mu(u)}{\mu(u)}=\int\phi(u)du\quad\Rightarrow\quad\ln|\mu(u)|=\int\phi(u)du+c,$$

其中c为任意常数. 当$c=0$时, $\mu(u)=e^{\int\phi(u)du}$, 即 $\mu(xy)=e^{\int\phi(xy)d(xy)}$.

同理, $\mu(x^2+y^2)$为积分因子, 当且仅当$\frac{1}{2xN(x,y)-2yM(x,y)}(\frac{\partial M(x,y)}{\partial x}-\frac{\partial N(x,y)}{\partial x})$为$x^2+y^2$的连续函数$\phi(x^2+y^2)$.

此时

$$\int\frac{d\mu(u)}{\mu(u)}=\int\phi(u)du\quad\Rightarrow\quad\mu(u)=e^{\int\phi(u)du}\quad\Rightarrow\quad\mu(x^2+y^2)=e^{\int\phi(x^2+y^2)d(x^2+y^2)}.$$

同理, $\mu(\frac{y}{x})$为积分因子, 当且仅当$\frac{1}{\frac{-y}{x^2}M(x,y)-\frac{1}{x}M(x,y)}(\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x})$仅为$\frac{y}{x}$的连续函数$\phi(\frac{y}{x})$, 此时同上面证明可知$\mu(u)=e^{\int\phi(u)du}$, 即$\mu(\frac{y}{x})=e^{\int\phi(\frac{y}{x})d\frac{y}{x}}$.

同理, $\mu(x^\alpha y^\beta)$为积分因子, 当且仅当

$$\frac{1}{\alpha x^{\alpha-1}y^\beta N(x,y)-\beta y^{\beta-1}x^\alpha M(x,y)}(\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x})$$

仅为$x^\alpha y^\beta$的连续函数.

此时同上面的证明可知$\mu(u)=e^{\int\phi(u)du}$, 即$\mu(x^\alpha y^\beta)=e^{\int\phi(x^\alpha y^\beta)d(x^\alpha y^\beta)}$.

练习二（45分）、

2.1) (15分)验证下列方程

$$(2.1a)\qquad(3x^2+6xy^2)dx+(6x^2y+4y^3)dy=0,$$

$$(2.1b)\qquad\left(\cos x+\frac{1}{y}\right)dx+\left(\frac{1}{y}-\frac{x}{y^2}\right)dy=0,$$

$$(2.1c)\qquad(y\cos x+2xe^y)dx+(\sin x+x^2e^y+2)dy=0$$

是恰当方程,并求它们的通解.

解：2.1a) 令$M(x,y)=3x^2+6xy^2$, $N(x,y)=6x^2y+4y^3$, 有

$$\frac{\partial M(x,y)}{\partial y}=12xy,\qquad \frac{\partial N(x,y)}{\partial x}=12xy,$$

则$\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}=12xy$, 因此$(3x^2+6xy^2)dx+(6x^2y+4y^3)dy=0$为恰当方程.

该式可重新组合为

$$\begin{aligned} &3x^2dx+4y^3dy+3y^2\cdot 2xdx+x^2\cdot 6ydy=0,\\ \Rightarrow\quad &dx^3+dy^4+3y^2dx^2+x^2d(3y^2)=0,\\ \Rightarrow\quad &dx^3+dy^4+d(3y^2x^2)=0,\end{aligned}$$

即 $d(x^3+y^4+3y^2x^2)=0$. 通解为$x^3+y^4+3y^2x^2=c$, c为任意常数.

2.1b) $d(\sin x+ln|y|+\frac{x}{y})=0$通解为$\sin x+\ln|y|+\frac{x}{y}=c$, c为任意常数.

2.1c) $d(y\sin x+x^2e^y+2y)=0$. 通解为$y\sin x+x^2e^y+2y=c$, c为任意常数.

2.2) （6分）利用求积分因子方法求方程

$$(p(x)y-q(x))dx+dy=0$$

的通解。

解：令$M(x,y)=p(x,y)-q(x), N(x,y)=1$, 则

$$\frac{\partial M(x,y)}{\partial x}=p(x),\quad \frac{\partial N(x,y)}{\partial x}=0,\quad \frac{\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x}}{N(x,y)}=p(x),$$

仅为x的连续函数. 有恰当因子$\mu(x)$使得$\frac{d\mu(x)}{\mu(x)}=p(x)dx$, 因此$\int\frac{d\mu(x)}{\mu(x)}=\int p(x)dx$, 即, $\ln|\mu(x)|=\int p(x)dx+c$, c为任意常数. 取$c=0$, 则$\mu(x)=e^{\int p(x)dx}$.

则原方程左右均乘$\mu(x)$后为

$$e^{\int p(x)dx}(p(x)y-q(x))dx+e^{\int p(x)dx}dy=0,$$

$$\Rightarrow\qquad d(ye^{\int p(x)dx})-d(\int q(x)e^{\int p(x)dx})=0.$$

通解为

$$ye^{\int p(x)dx}-\int q(x)e^{\int p(x)dx}dx=c,$$

其中c为任意常数. 因为$e^{\int p(x)dx}\neq 0$, 因此$y=e^{-\int p(x)dx}(c+\int q(x)e^{\int p(x)dx})$, c为任意常数.

2.3) （6分）利用变量变换求Bernoulli方程

$$\frac{dy}{dx}=p(x)y+q(x)y^n,\qquad n\neq 0,1$$

的积分因子。

解：$y\neq 0$, 上式左右同乘y^{-n}得

$$y^{-n}\frac{dy}{dx}=p(x)y^{1-n}+q(x),$$

因此

$$\frac{1}{1-n}\frac{dy^{1-n}}{dx}=p(x)y^{1-n}+q(x).$$

令$y^{1-n}=z$, 则

$$\Rightarrow \qquad [(n-1)p(x)z+(n-1)q(x)]dx+dz=0.$$

令$(n-1)p(x)=M(x), -(n-1)q(x)=N(x)$, 则上式化为

$$[M(x)z-N(x)]dx+dz=0,$$

利用2.2)证明中的积分因子形式, 得该式积分因子为

$$\mu(x)=e^{\int M(x)dx}=e^{\int (n-1)p(x)dx}.$$

原方程的积分因子为$y^{-n}e^{\int (n-1)p(x)dx}$.

2.4) （6分）求方程

$$\left(\frac{y^2}{2}+2ye^x\right)dx+(y+e^x)dy=0$$

的形如$\mu(x)$的积分因子。然后求其通解。

解：$\frac{\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x}}{N(x,y)}=\frac{y+2e^x-e^x}{y+e^x}=1$ 仅为x的连续函数, 积分因子为$\mu(x)=e^{\int dx}=e^x$.
对题目中的式子同乘e^x，重组后为

$$\frac{y^2}{2}\cdot e^x dx+e^x\cdot ydy+e^{2x}dy+y\cdot 2e^{2x}dx=0,$$

因此$d(\frac{y^2}{2}e^x)+d(ye^{2x})=0$的通解为$\frac{y^2}{2}e^x+ye^{2x}=c$, $\quad$(c为任意常数).

2.5) （6分）求方程

$$ydx+(y-x)dy=0$$

的形如$\mu(y)$的积分因子。然后求其通解。

解：

$$\frac{\partial M(x,y)}{\partial y}=1,\quad \frac{\partial N(x,y)}{\partial x}=-1,\quad \frac{\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x}}{-M(x,y)}=\frac{2}{-y}$$

仅为y的连续函数. 因此

$$\mu(y)=e^{\int -\frac{2}{y}dy}=\frac{1}{y^2}.$$

对题目中的式子同乘$\frac{1}{y^2}$得

$$\frac{1}{y^2}\cdot ydx+\frac{1}{y^2}(y-x)dy=0 \quad\Rightarrow\quad \frac{1}{y}dx+(\frac{1}{y}-\frac{x}{y^2})dy=0,$$

重组后得

$$\frac{1}{y}dy+\frac{1}{y}dx-\frac{x}{y^2}dy=0,$$
$$\Rightarrow \qquad d(\ln|y|+\frac{x}{y})=0.$$

通解为 $\ln|y| + \frac{x}{y} = c$, （c为任意常数).

2.6) （6分）求方程

$$(2x^3 + 3x^2y + y^2 - y^3)dx + (2y^3 + 3xy^2 + x^2 - x^3)dy = 0$$

的形如$\mu(x+y)$的积分因子。

解：计算 $\frac{\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}}{N(x,y-M(x,y))} = \frac{(3x^2+2y-3y^2)-(3y^2+2x-3x^2)}{(2y^3+3xy^2+x^2-x^3)-(2x^3+3x^2y+y^2-y^3)} = \frac{-6(x+y)+2}{3(x+y)^2-(x+y)}$, 仅为$(x+y)$的连续函数.

由1.4)题的结论, 令$u = x + y$, 则

$$\frac{-6(x+y)+2}{3(x+y)^2-(x+y)} = -\frac{6u+2}{3u^2-u},$$

因此

$$\mu(u) = e^{\int \frac{-6u+2}{3u^2-u}du} = e^{\int \frac{-2(3u-1)}{u(3u-1)}du} = e^{\int \frac{-2}{u}du} = \frac{1}{u^2} = \frac{1}{(x+y)^2}.$$

即积分因为$\mu(x+y) = \frac{1}{(x+y)^2}$.