# QUESTION 1:

What proportion of students who answer this question quickly will pass the class? Assume the probability of passing the class is 0.9. Assume the probability of answering this question quickly is 0.6 if you'll pass the class, while the probability drops to 0.3 if you'll not pass the class.

$$P(Pass|Quick) = \frac{P(Quick|Pass)P(Pass)}{P(Quick)} = \frac{P(Quick|Pass)P(Pass)}{P(Quick|Pass)P(Pass) + P(Quick|Fail)P(Fail)} = \frac{0.6 \times 0.9}{0.6 \times 0.9 + 0.3 \times 0.1}$$

**Conclusion:** The proportion of students who pass given that they answered the question quickly is **0.9474**.

# QUESTION 2:

Find the posterior distribution of a dirichlet prior and a multinomial likelihood. What are the parameters? Note, The posterior is always proportional to the joint $p(\theta \mid y) = \frac{1}{c} \cdot p(\theta, y) \propto p(\theta, y)$ and the posterior is always a proper distribution if the prior is.

Let $Y \sim Mk(n, \boldsymbol{\theta})$, where $Mk$ is a multinomial distribution with parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, ...\theta_k)$. Let $\boldsymbol{y} = (y_1, y_2, ..., y_k)$, where $\boldsymbol{y}$ is a realization from the multinomial distribution.

**Prior:** $\Theta \sim Dir(\boldsymbol{\alpha})$ such that $P(\Theta = \boldsymbol{\theta}; \boldsymbol{\alpha}) = p(\boldsymbol{\theta}; \boldsymbol{\alpha}) = \left( \frac{\Gamma\left(\Sigma_{i=1}^k \alpha_i\right)}{\prod_{i=1}^k \Gamma\left(\alpha_i\right)} \right) \prod_{i=1}^k \theta_i^{\alpha_i - 1}$

**Likelihood:** $P(Y = \boldsymbol{y}|\Theta = \boldsymbol{\theta}) = p(\boldsymbol{y}|\boldsymbol{\theta}) = \left( \frac{n!}{\prod_{i=1}^k y_i!} \right) \prod_{i=1}^k \theta_i^{y_i} = \left( \frac{n!}{y_1! \times y_2! \times ... \times y_k!} \right) \theta_1^{y_1} \times \theta_2^{y_2} \times ... \times \theta_k^{y_k}$

**Posterior:** $P(\Theta = \boldsymbol{\theta}|Y = \boldsymbol{y}) = p(\boldsymbol{\theta}|\boldsymbol{y}) = \frac{p(\boldsymbol{\theta}, \boldsymbol{y})}{p(\boldsymbol{y})} = \frac{p(\boldsymbol{y}|\boldsymbol{\theta})p(\boldsymbol{\theta}; \boldsymbol{\alpha})}{p(\boldsymbol{y})} = \left( \frac{\left( \frac{n!}{\prod_{i=1}^k y_i!} \right) \left( \frac{\Gamma\left(\Sigma_{i=1}^k \alpha_i\right)}{\prod_{i=1}^k \Gamma\left(\alpha_i\right)} \right)}{p(\boldsymbol{y})} \right) \prod_{i=1}^k \theta^{y_i + \alpha_i - 1}$

**Conclusion:** Since the posterior distribution is proportional to the dirichlet distribution, we conclude that $\Theta|Y \sim Dir(\boldsymbol{y} + \boldsymbol{\alpha})$. This means that the dirichlet distribution is a **conjugate prior** for the multinomial distribution.