GRAPHS & DIGRAPHS

FIFTH EDITION

GARY CHARTRAND
LINDA LESNIAK
PING ZHANG

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group an **informa** business
A CHAPMAN & HALL BOOK

Chapman & Hall/CRC
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

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Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2 1

International Standard Book Number-13: 978-1-4398-2629-4 (Ebook-PDF)

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In recognition of the first

275 years of Graph Theory

(1736–2011)

Table of Contents

Preface to the Fifth Edition **xi**

1. Introduction to Graphs **1**

1.1 Graphs and Subgraphs 1
1.2 Degree Sequences 22
1.3 Connected Graphs and Distance 30
1.4 Multigraphs and Digraphs 45

2. Trees and Connectivity **55**

2.1 Nonseparable Graphs 55
2.2 Trees 61
2.3 Spanning Trees 69
2.4 Connectivity and Edge-Connectivity 89
2.5 Menger's Theorem 97

3. Eulerian and Hamiltonian Graphs **107**

3.1 Eulerian Graphs 107
3.2 Hamiltonian Graphs 115
3.3 Powers of Graphs and Line Graphs 137

4. Digraphs **149**

4.1 Strong Digraphs 149
4.2 Tournaments 156
4.3 Flows in Networks 171

5. Graphs: History and Symmetry **195**

5.1 Some Historical Figures of Graph Theory 195
5.2 The Automorphism Group of a Graph 201
5.3 Cayley Color Graphs 206
5.4 The Reconstruction Problem 212

6. Planar Graphs **221**

6.1 The Euler Identity 221
6.2 Planarity versus Nonplanarity 235
6.3 The Crossing Number of a Graph 250
6.4 Hamiltonian Planar Graphs 263

7. Graph Embeddings **269**

7.1 The Genus of a Graph 269
7.2 2-Cell Embeddings of Graphs 281
7.3 The Maximum Genus of a Graph 294
7.4 The Graph Minor Theorem 307

8. Vertex Colorings **311**

8.1 The Chromatic Number of a Graph 311
8.2 Color-Critical Graphs 323
8.3 Bounds for the Chromatic Number 327
8.4 Perfect Graphs 340
8.5 List Colorings 353

9. Map Colorings **359**

9.1 The Four Color Problem 359
9.2 Colorings of Planar Graphs 376
9.3 The Conjectures of Hajós and Hadwiger 384
9.4 Chromatic Polynomials 388
9.5 The Heawood Map-Coloring Problem 395

10. Matchings, Factorization and Domination **401**

10.1 Matchings and Independence in Graphs 401
10.2 Factorization 419
10.3 Decomposition and Graceful Graphs 427
10.4 Domination 438

11. Edge Colorings **447**

11.1 Chromatic Index and Vizing's Theorem 447
11.2 Class One and Class Two Graphs 456
11.3 Tait Colorings 464
11.4 Nowhere-Zero Flows 473
11.5 List Edge Colorings and Total Colorings 484

12. Extremal Graph Theory **493**

12.1 Turán's Theorem 493
12.2 Cages 499
12.3 Ramsey Theory 508

Hints and Solutions to Odd-Numbered Exercises 523
Bibliography 547
Index of Names 565
Index of Mathematical Terms 571
List of Symbols 583

Preface to the Fifth Edition

Since graph theory was considered to have begun some 275 years ago, it has evolved into a subject with a fascinating history, a host of interesting problems and numerous diverse applications. While graph theory has developed ever-increasing connections with other areas of mathematics and a variety of scholarly fields, it is its beauty that has attracted so many to it.

As with the previous editions, the objective of this fifth edition is to describe much of the story that is graph theory – in terms of its concepts, its theorems, its applications and its history. Here too, the audience for the fifth edition is beginning graduate students and advanced undergraduate students. The main prerequisite required of students using this book is a knowledge of mathematical proofs. Some elementary knowledge of linear algebra and group theory is also useful for some topics.

Although a one-semester course in graph theory using this text can be designed by selecting topics of greatest interest to the instructor and students, there is more than ample material available for a two-semester sequence in graph theory. Our goal has been to prepare a book that is interesting, carefully written, student-friendly and consisting of clear proofs. The fifth edition is approximately 50% longer than the fourth edition. Some major changes from the fourth edition are:

- sections have been divided into subsections to make the material easier to read and locate;
- terms being defined are in bold type, making them easier to locate;
- more than 300 new exercises have been added;
- examples, figures and applications have been added to illustrate concepts and theorems;
- historical discussions of mathematicians and problems have been expanded.

There is a section at the end of the book giving hints and solutions to odd-numbered exercises, providing information on one possible approach that may be useful to solve the problem. There is expanded or new coverage of a number of topics, including

- degree sequences
- toughness
- graph minors
- perfect graphs

- chromatic polynomials
- list colorings and list edge colorings
- nowhere-zero flows
- flows in networks.

Over the years, there have been some changes in notation that a number of mathematicians now use. When certain notation appears to have been adopted by sufficiently many mathematicians working in graph theory so that this has become the norm, we have adhered to these changes. In particular,

- a path is now expressed as $P = (v_1, v_2, \ldots, v_k)$ and a cycle as $C = (v_1, v_2, \ldots, v_k, v_1)$;
- the Cartesian product of two graphs G and H is expressed as $G \,\square\, H$, rather than the previous $G \times H$;
- the union of G and H is expressed by $G + H$, rather than $G \cup H$;
- the join of two graphs G and H is expressed as $G \vee H$, rather than $G + H$.

We are grateful to Bob Stern, executive editor of CRC Press, who has been a constant source of support and assistance throughout the entire writing process. We also thank an anonymous reviewer who read an early version of the manuscript with meticulous care and who made a number of valuable suggestions.

G.C., L.L. and P.Z.

Chapter 1

Introduction to Graphs

> *The theory of graphs is one of the few fields of mathematics with a definite birth date.*

It is the subject of graph theory of course that we are about to describe. The statement above was made in 1963 by the mathematician Oystein Ore who will be encountered in Chapter 3. While graph theory was probably Ore's major mathematical area of interest during the latter part of his career, he is also known for his work and interest in number theory (the study of integers) and the history of mathematics.

While awareness of integers can be traced back for many centuries, geometry has an even longer history. Early geometry concerned distance, lengths, angles, areas and volumes, which were used for surveying, construction and astronomy. While geometry dealt with magnitudes, the German mathematician Gottfried Leibniz introduced another branch of geometry called the geometry of position. This branch of geometry did not deal with measurements and calculations, but rather with the determination of position and its properties. The famous mathematician Leonhard Euler said that it hadn't been determined what kinds of problems could be studied with the aid of the geometry of position but in 1736 he believed that he had found one, which would lead to the origin of graph theory. It is this event to which Oystein Ore was referring in his quote above. We will visit Euler again, also in Chapter 3 as well as in Chapters 6 and 7, and the problem that would mark the beginning of graph theory.

1.1 Graphs and Subgraphs

The concept of graphs can arise in many diverse situations. We look at three of these.

Example 1.1

The service department of an automobile dealer has advertised maintenance specials for a particular afternoon. One of these is a 30-minute maintenance special and the other is a 60-minute maintenance special. Eight customers, denoted by $c_1, c_2, \ldots, c_8$, have made appointments, listed below:

$$\begin{array}{llll} c_1: & 1:30-2:00 \text{ pm} & c_2: & 1:40-2:40 \text{ pm} \\ c_3: & 1:40-2:10 \text{ pm} & c_4: & 1:50-2:50 \text{ pm} \\ c_5: & 2:30-3:00 \text{ pm} & c_6: & 2:15-3:15 \text{ pm} \\ c_7: & 2:55-3:55 \text{ pm} & c_8: & 3:10-3:40 \text{ pm} \end{array}$$

In order to determine how many mechanics should be available that afternoon, the service department is interested in knowing those pairs of customers whose appointments overlap.

This situation can be modeled (or represented) by the diagram shown in Figure 1.1, where each point or small circle indicates a customer and two points are joined by a line segment if the appointment times of these customers overlap. This diagram is referred to as a *graph*. ♦

Figure 1.1: The diagram in Example 1.1

Example 1.2

There is a game that uses a board containing six squares, arranged into two rows and three columns. The game also uses three coins, where on each square is placed at most one of the three coins. A "move" in the game consists of moving one coin to an empty adjacent square (in the same row or column). There are $\binom{6}{3} = 20$ possible states for this game at any one time. Nine of these 20 states, denoted by S_i $(1 \leq i \leq 9)$, are shown in Figure 1.2. We are interested in those pairs of states where it is possible to proceed from one to the other by a single move. This situation can be represented by the diagram shown in Figure 1.2, where each small circle indicates a state and a line segment between two circles indicates that we can proceed from one state to the other by a single move. Again, this diagram is a graph. ♦

Figure 1.2: The diagram in Example 1.2

Example 1.3

A certain airline has flights into and out of a large number of cities. We denote eight of these cities by C_i $(1 \leq i \leq 8)$. This airline has direct flights between certain pairs of these cities. This information is given in the 8×8 zero-one matrix $A = [a_{ij}]$ in Figure 1.3, where $a_{ij} = 1$ if and only if there is a direct flight between the cities C_i and C_j. This information can also be obtained with the aid of the diagram (once again a graph) shown in Figure 1.3, where the eight points or small circles represent the eight cities and the line segments or curves between pairs of points represent the existence of direct flights between the corresponding cities. ♦

We now give a formal definition of the term *graph*. A **graph** G is a finite nonempty set V of objects called **vertices** (the singular is **vertex**) together with a possibly empty set E of 2-element subsets of V called **edges**. Vertices are sometimes referred to as **points** or **nodes**, while edges are sometimes called **lines** or **links**. In fact, historically, graphs were referred to as *linkages* by some. Calling these structures graphs was evidently the idea of James Joseph Sylvester (1814–1897), a well-known British mathematician who became the first mathematics professor at Johns Hopkins University in Baltimore and who founded and became editor-in-chief of the first mathematics journal in the United States

$$A: \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 1.3: The matrix and diagram in Example 1.3

(the *American Journal of Mathematics*).

To indicate that a graph G has **vertex set** V and **edge set** E, we write $G = (V, E)$. To emphasize that V and E are the vertex set and edge set of a graph G, we often write V as $V(G)$ and E as $E(G)$. Each edge $\{u, v\}$ of G is usually denoted by uv or vu. If $e = uv$ is an edge of G, then e is said to **join** u and v.

As the examples described above indicate, a graph G can be represented by a diagram, where each vertex of G is represented by a point or small circle and an edge joining two vertices is represented by a line segment or curve joining the corresponding points in the diagram. It is customary to refer to such a diagram as the graph G itself. In addition, the points in the diagram are referred to as the vertices of G and the line segments are referred to as the edges of G. For example, the graph G with vertex set $V(G) = \{u, v, w, x, y\}$ and edge set $E(G) = \{uv, uy, vx, vy, wy, xy\}$ is shown in Figure 1.4. While the edges vx and wy cross in Figure 1.4, their point of intersection is not a vertex of G.

Figure 1.4: A graph

If uv is an edge of G, then u and v are **adjacent vertices**. Two adjacent vertices are referred to as **neighbors** of each other. The set of neighbors of a vertex v is called the **open neighborhood** of v (or simply the **neighborhood** of v) and is denoted by $N_G(v)$, or $N(v)$ if the graph G is understood. The set $N[v] = N(v) \cup \{v\}$ is called the **closed neighborhood** of v. If uv and vw are distinct edges in G, then uv and vw are **adjacent edges**. The vertex u and

the edge uv are said to be **incident** with each other. Similarly, v and uv are incident.

For the graph G of Figure 1.4, the vertices u and v are therefore adjacent in G, while the vertices u and x are not adjacent. The edges uv and vx are adjacent in G, while the edges vx and wy are not adjacent. The vertex v is incident with the edge uv but is not incident with the edge wy.

The number of vertices in a graph G is the **order** of G and the number of edges is the **size** of G. The order of the graph G of Figure 1.4 is 5 and its size is 6. We typically use n and m for the order and size, respectively, of a graph. A graph of order 1 is called a **trivial graph**. A **nontrivial graph** therefore has two or more vertices. A graph of size 0 is called an **empty graph**. A **nonempty graph** then has one or more edges. In any empty graph, no two vertices are adjacent. At the other extreme is a **complete graph** in which every two distinct vertices are adjacent. The size of a complete graph of order n is $\binom{n}{2} = n(n-1)/2$. Therefore, for every graph G of order n and size m, it follows that $0 \leq m \leq \binom{n}{2}$. The complete graph of order n is denoted by K_n. The complete graphs K_n for $1 \leq n \leq 5$ are shown in Figure 1.5.

Figure 1.5: Some complete graphs

Two other classes of graphs that are often encountered are the paths and cycles. For an integer $n \geq 1$, the **path** P_n is a graph of order n and size $n-1$ whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$. For an integer $n \geq 3$, the **cycle** C_n is a graph of order n and size n whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_1 v_n$ and $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$. The cycle C_n is also referred to as an n**-cycle** and the 3**-cycle** is also called a **triangle**. The paths and cycles of order 5 or less are shown in Figure 1.6. Observe that $P_1 = K_1$, $P_2 = K_2$ and $C_3 = K_3$.

Figure 1.6: Paths and cycles of order 5 or less

The Degree of a Vertex

The **degree of a vertex** v in a graph G is the number of vertices in G that are adjacent to v. Thus the degree of v is the number of vertices in its neighborhood $N(v)$. Equivalently, the degree of v is the number of edges incident with v. The degree of a vertex v is denoted by $\deg_G v$ or, more simply, by $\deg v$ if the graph G under discussion is clear. Hence $\deg v = |N(v)|$. A vertex of degree 0 is referred to as an **isolated vertex** and a vertex of degree 1 is an **end-vertex** or a **leaf**. An edge incident with an end-vertex is called a **pendant edge**. The largest degree among the vertices of G is called the **maximum degree** of G and is denoted by $\Delta(G)$. The **minimum degree** of G is denoted by $\delta(G)$. Thus if v is a vertex of a graph G of order n, then

$$0 \leq \delta(G) \leq \deg v \leq \Delta(G) \leq n - 1.$$

For the graph G of Figure 1.4,

$$\deg w = 1,\ \deg u = \deg x = 2,\ \deg v = 3 \text{ and } \deg y = 4.$$

Thus $\delta(G) = 1$ and $\Delta(G) = 4$.

A well-known theorem in graph theory deals with the sum of the degrees of the vertices of a graph. This observation was made by Leonhard Euler in a 1736 paper [76] that is now considered the first paper ever written on graph theory – even though graphs were never mentioned in the paper. It is often referred to as the **First Theorem of Graph Theory**. Some have called Theorem 1.4 the **Handshaking Lemma**, although Euler never used this name.

Theorem 1.4 (**The First Theorem of Graph Theory**) *If G is a graph of size m, then*

$$\sum_{v \in V(G)} \deg v = 2m.$$

Proof. When summing the degrees of the vertices of G, each edge of G is counted twice, once for each of its two incident vertices. ■

The sum of the degrees of the vertices of the graph G of Figure 1.4 is 12, which is twice the size 6 of G, as was anticipated by Theorem 1.4. The **average degree** of a graph G of order n and size m is

$$\frac{\sum_{v \in V(G)} \deg v}{n} = \frac{2m}{n}.$$

For example, the average degree of the graph G of Figure 1.4 (having order $n = 5$ and size $m = 6$) is $2m/n = 12/5$. Since the average degree of this graph is strictly between 2 and 3, it follows that G must have a vertex of degree 3 or more and a vertex of degree 2 or less. This graph actually has vertices of degrees 3 and 4 as well as vertices of degrees 1 and 2.

A vertex v in a graph G is **even** or **odd**, according to whether its degree in G is even or odd. Thus the graph G of Figure 1.4 has three even vertices and two odd vertices. While a graph can have either an even or odd number of even vertices, this is not the case for odd vertices.

Corollary 1.5 *Every graph has an even number of odd vertices.*

Proof. Suppose that G is a graph of size m. By Theorem 1.4,

$$\sum_{v \in V(G)} \deg v = 2m,$$

which is, of course, an even number. Since the sum of the degrees of the even vertices of G is even, the sum of the degrees of the odd vertices of G must be even as well, implying that G has an even number of odd vertices. ■

Isomorphic Graphs

There is only one graph of order 1, two graphs of order 2, four graphs of order 3 and eleven graphs of order 4. All 18 of these graphs are shown in Figure 1.7. This brings up the question of why every two graphs in Figure 1.7 are considered different. In fact, there is the related question of what it means for two graphs to be considered the same. The technical term for this is *isomorphic graphs* (graphs having the same structure).

Figure 1.7: The (non-isomorphic) graphs of order 4 or less

Two graphs G and H are **isomorphic** if there exists a bijective function $\phi : V(G) \to V(H)$ such that two vertices u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H. The function ϕ is called an **isomorphism** from G to H. If $\phi : V(G) \to V(H)$ is an isomorphism, then the inverse function $\phi^{-1} : V(H) \to V(G)$ is an isomorphism from H to G. If G and H are isomorphic, we write $G \cong H$. If there is no such function ϕ as described above, then G and H are **non-isomorphic graphs** and we write $G \ncong H$.

The graphs G and H of Figure 1.8 (both of order 7 and size 8) are isomorphic and the function $\phi : V(G) \to V(H)$ defined by

$$\phi(u_1) = v_4,\ \phi(u_2) = v_5,\ \phi(u_3) = v_1,$$
$$\phi(u_4) = v_6,\ \phi(u_5) = v_2,\ \phi(u_6) = v_3,\ \phi(u_7) = v_7$$

is an isomorphism, although there are three other isomorphisms. The graphs F_1 and F_2 of Figure 1.8 (both of order 7 and size 10) are not isomorphic however. An explanation of this will be given shortly.

Figure 1.8: Isomorphic and non-isomorphic graphs

The graphs G_1 and G_2 in Figure 1.9 are isomorphic, while G_1 and G_3 are not isomorphic. For example, the function $\phi : V(G_1) \to V(G_2)$ defined by

$$\phi(u_1) = v_1,\ \phi(u_2) = v_3,\ \phi(u_3) = v_5,$$
$$\phi(u_4) = v_2,\ \phi(u_5) = v_4,\ \phi(u_6) = v_6$$

is an isomorphism. The graph G_3 of Figure 1.9 contains three mutually adjacent vertices w_1, w_2, w_6. If G_1 and G_3 were isomorphic, then for an isomorphism $\alpha : V(G_3) \to V(G_1)$, the vertices $\alpha(w_1), \alpha(w_2), \alpha(w_6)$ must be mutually adjacent in G_1. Since G_1 does not contain three mutually adjacent vertices, however, there is no isomorphism from G_3 to G_1 and therefore, $G_1 \ncong G_3$. Furthermore, $G_2 \ncong G_3$ as well.

Suppose that two graphs G and H are isomorphic. Then there exists an isomorphism $\phi : V(G) \to V(H)$. Since ϕ is a bijective function, $|V(G)| = |V(H)|$. Furthermore, since two vertices u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H, it follows that $|E(G)| = |E(H)|$. These facts are summarized below, together with another necessary condition for two graphs to be isomorphic.

Figure 1.9: Isomorphic and non-isomorphic graphs

Theorem 1.6 *If two graphs G and H are isomorphic, then they have the same order and the same size, and the degrees of the vertices of G are the same as the degrees of the vertices of H.*

Proof. We have already noted that isomorphic graphs have the same order and the same size. Let v be a vertex of G and suppose that $\deg v = k$. Then v is adjacent to k vertices, say $v_1, v_2, \ldots, v_k$. Suppose that v is not adjacent to $u_1, u_2, \ldots, u_\ell$. If $\phi : V(G) \to V(H)$ is an isomorphism, then $\phi(v)$ is adjacent to $\phi(v_1), \phi(v_2), \ldots, \phi(v_k)$ while $\phi(v)$ is not adjacent to $\phi(u_1), \phi(u_2), \ldots, \phi(u_\ell)$. Hence $\phi(v)$ has degree k in H. ■

The proof of Theorem 1.6 also shows that every isomorphism from a graph G to a graph H maps every vertex of G to a vertex of the same degree in H.

From Theorem 1.6, it follows that if G and H are two graphs such that (1) the orders of G and H are different, or (2) the sizes of G and H are different or (3) the degrees of the vertices of G and those of the vertices of H are different, then G and H are not isomorphic. Since the graph F_2 in Figure 1.8 contains a vertex of degree 5 and no vertex of F_1 has degree 5, it follows by Theorem 1.6 that $F_1 \ncong F_2$.

The conditions described in Theorem 1.6 are strictly necessary for two graphs to be isomorphic – they are not sufficient. Indeed, the graphs G_1 and G_3 of Figure 1.9 have the same order, the same size and the degrees of the vertices of G_1 and G_3 are the same; yet G_1 and G_3 are not isomorphic.

Next consider the four graphs H_1, H_2, H_3 and H_4 shown in Figure 1.10. The graphs H_1 and H_2 have order 7, size 7 and the degrees of the vertices of these two graphs are the same. Furthermore, each graph contains a single triangle. Nevertheless, $H_1 \ncong H_2$, for suppose that there is an isomorphism ϕ from H_1 to H_2. Since each graph has only one vertex of degree 3 and one vertex of degree 4, $\phi(u_1) = v_2$ and $\phi(y_1) = x_2$. Since v_1 is adjacent to u_1 and y_1, it follows that $\phi(v_1)$ is adjacent to v_2 and x_2. However, H_2 contains no vertex adjacent to v_2 and x_2. Thus $H_1 \ncong H_2$.

The graphs H_3 and H_4 of Figure 1.10 have order 9, size 8 and the degrees of the vertices of these two graphs are the same; yet these two graphs are not

Figure 1.10: Non-isomorphic graphs

isomorphic. Suppose that there is an isomorphism ϕ from H_3 to H_4. Consider the vertex v_3. Since $\deg_{H_3} v_3 = 2$, it follows that $\phi(v_3)$ has degree 2 in H_4. Since v_3 is adjacent to u_3 and w_3, we must have $\phi(v_3)$ adjacent to $\phi(u_3)$ and $\phi(w_3)$. Since $\deg_{H_3} u_3 = \deg_{H_3} w_3 = 2$, it follows that $\phi(u_3)$ and $\phi(w_3)$ have degree 2 in H_4. But no vertex of degree 2 in H_4 is adjacent to two vertices of degree 2. Thus $H_3 \ncong H_4$.

Subgraphs

A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, in which case we write $H \subseteq G$. If H is a subgraph of G, then G is a **supergraph** of H. If $V(H) = V(G)$, then H is a **spanning subgraph** of G. If H is a subgraph of a graph G and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a **proper subgraph** of G. Figure 1.11 shows six graphs, namely G and the graphs G_i for $i = 1, 2, \ldots, 5$. All six of these graphs are proper subgraphs of G, except G itself and G_1. Since G is a subgraph of itself, it is not a proper subgraph of G. The graph G_1 contains the edge uz, which G does not and so G_1 is not even a subgraph of G. The graph G_3 is a spanning subgraph of G since $V(G_3) = V(G)$.

For a nonempty subset S of $V(G)$, the **subgraph** $G[S]$ **of** G **induced by** S has S as its vertex set and two vertices u and v are adjacent in $G[S]$ if and only if u and v are adjacent in G. (The subgraph of G induced by S is also denoted by $\langle S \rangle_G$ or simply by $\langle S \rangle$ when the graph G is understood.) A subgraph H of a graph G is called an **induced subgraph** if there is a nonempty subset S of $V(G)$ such that $H = G[S]$. Thus $G[V(G)] = G$. For a nonempty set X of edges of a graph G, the **subgraph** $G[X]$ **induced by** X has X as its edge set and a vertex v belongs to $G[X]$ if v is incident with at least one edge in X. A subgraph H of G is **edge-induced** if there is a nonempty subset X of

Figure 1.11: Graphs and subgraphs

$E(G)$ such that $H = G[X]$. Thus $G[E(G)] = G$ if and only if G has no isolated vertices.

Once again, consider the graphs shown in Figure 1.11. Since $xy \in E(G)$ but $xy \notin E(G_4)$, the subgraph G_4 is not an induced subgraph of G. On the other hand, the subgraphs G_2 and G_5 are both induced subgraphs of G. Indeed, for $S_1 = \{v, x, y, z\}$ and $S_2 = \{u, v, y, z\}$, $G_2 = G[S_1]$ and $G_5 = G[S_2]$. The subgraph G_4 of G is edge-induced; in fact, $G_4 = G[X]$, where $X = \{uw, wx, wy, xz, yz\}$.

For a vertex v and an edge e in a nonempty graph $G = (V, E)$, the subgraph $G - v$, obtained by deleting v from G, is the induced subgraph $G[V - \{v\}]$ of G and the subgraph $G - e$, obtained by deleting e from G, is the spanning subgraph of G with edge set $E - \{e\}$. More generally, for a proper subset U of V, the graph $G - U$ is the induced subgraph $G[V - U]$ of G. For a subset X of E, the graph $G - X$ is the spanning subgraph of G with edge set $E - X$. If u and v are distinct nonadjacent vertices of G, then $G + uv$ is the graph with $V(G + uv) = V(G)$ and $E(G + uv) = E(G) \cup \{uv\}$. Thus G is a spanning subgraph of $G + uv$. For the graph G of Figure 1.12, the set $U = \{t, x\}$ of vertices and the set $X = \{tw, ux, vx\}$ of edges, the subgraphs $G - u$, $G - wx$, $G - U$ and $G - X$ of G are also shown in that figure, as is the graph $G + uv$.

If every two edges e_1 and e_2 of a graph G have the property that $G - e_1 \cong G - e_2$, then we write $G - e$ for the deletion of any edge from G. Hence for $n \geq 2$, $K_n - e$ denotes the graph obtained by deleting any edge from K_n. If $G + uv \cong G + xy$ for any two pairs $\{u, v\}$ and $\{x, y\}$ of nonadjacent vertices of G, then we write $G + e$ for the addition of any edge to G. In particular, $C_4 + e = K_4 - e$.

Figure 1.12: Deleting vertices and edges from and adding edges to a graph

Regular Graphs

There are certain classes of graphs that occur so often that they deserve special mention and in some cases, special notation. We describe some of the most prominent of these now.

A graph G is **regular** if the vertices of G have the same degree and is **regular of degree** r if this degree is r. Such graphs are also called r**-regular**. The complete graph of order n is therefore a regular graph of degree $n-1$ and every cycle is 2-regular. In Figure 1.13 are shown all (non-isomorphic) regular graphs of orders 4 and 5, including the cycles C_4 and C_5 and the complete graphs K_4 and K_5. Since no graph has an odd number of odd vertices, there is no 1-regular or 3-regular graph of order 5. Indeed, the pairs r, n of integers for which there exist r-regular graphs of order n are predictable.

Theorem 1.7 *For integers r and n, there exists an r-regular graph of order n if and only if $0 \leq r \leq n-1$ and r and n are not both odd.*

Proof. That the conditions are necessary is an immediate consequence of Corollary 1.5 and the fact that $0 \leq \deg v \leq n-1$ for every vertex v in a graph of order n. For the converse, suppose that r and n are integers where $0 \leq r \leq n-1$ and r and n are not both odd. Assume first that r is even. If $r = 0$, then the graph of order n consisting of n isolated vertices is r-regular. So we may assume that $r = 2k$ for some integer $k \geq 1$ and $n \geq 2k+1$. Let G be the graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that v_i $(1 \leq i \leq n)$ is adjacent to $v_{i\pm1}, v_{i\pm2}, \ldots, v_{i\pm k}$ (subscripts expressed modulo n). The resulting graph G

Figure 1.13: The regular graphs of orders 4 and 5

is then an r-regular graph of order n. If r is odd, then $n = 2\ell$ is even. Hence $r = 2k + 1$ for some integer k with $0 \leq k \leq \ell - 1$. For the graph G in this case, the vertex v_i is adjacent to the $2k$ vertices described above and adjacent to $v_{i+\ell}$. Again, G is an r-regular graph of order n. ■

The 4-regular and 5-regular graphs of order 10 constructed in the proof of Theorem 1.7 are shown in Figure 1.14.

Figure 1.14: 4-regular and 5-regular graphs of order 10

A 3-regular graph is also called a **cubic graph**. The graphs of Figure 1.9 are cubic as is the complete graph K_4. One of the best known cubic graphs is the **Petersen graph**, named for the Danish mathematician Julius Petersen whose 1891 research on regular graphs [171] is often credited as the beginning of the study of graphs as a theoretical subject. Three different drawings of the Petersen graph are shown in Figure 1.15. We will have many occasions to encounter this graph.

Figure 1.15: Three drawings of the Petersen graph

Bipartite Graphs

Other graphs that we often encounter are the bipartite graphs. A graph G is **bipartite** if $V(G)$ can be partitioned into two sets U and W (called **partite sets**) so that every edge of G joins a vertex of U and a vertex of W. In Figure 1.16(a) a bipartite graph G is given. Then G is redrawn in Figure 1.16(b) to emphasize that it is bipartite. If G is an r-regular bipartite graph, $r \geq 1$, with partite sets U and W, then $|U| = |W|$. This follows since its size is $r|U| = r|W|$. For two nonempty sets X and Y of vertices in a graph G, the set

$$[X, Y] = \{xy : x \in X, y \in Y\}.$$

Thus if G is a bipartite graph with partite sets U and W, then $[U, W] = E(G)$.

Figure 1.16: A bipartite graph

A graph G is a **complete bipartite graph** if $V(G)$ can be partitioned into two sets U and W (called **partite sets** again) so that uw is an edge of G if and only if $u \in U$ and $w \in W$. If $|U| = s$ and $|W| = t$, then this complete bipartite graph has order $s+t$ and size st and is denoted by $K_{s,t}$ (or $K_{t,s}$). The complete bipartite graph $K_{1,t}$ is called a **star**. The complete bipartite graphs $K_{1,3}$, $K_{2,2}$, $K_{2,3}$ and $K_{3,3}$ are shown in Figure 1.17. Observe that $K_{2,2} = C_4$. The star $K_{1,3}$ is sometimes referred to as a **claw**.

Since the size of the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil = \lfloor n^2/4 \rfloor$, there are bipartite graphs of order n and size $\lfloor n^2/4 \rfloor$. No bipartite graph of order n can have a larger size however.

Theorem 1.8 *The size of every bipartite graph of order n is at most $\lfloor n^2/4 \rfloor$.*

Figure 1.17: Complete bipartite graphs

Proof. Let G be bipartite graph of order n with partite sets U and W. Then $|U| = x$ and $|W| = n - x$ for some integer x with $1 \leq x \leq n - 1$. Hence the size of G is at most $x(n - x)$.

Since $(n - 2x)^2 \geq 0$, it follows that

$$n^2 \geq 4nx - 4x^2 = 4x(n - x)$$

and so $x(n - x) \leq n^2/4$. ■

No bipartite graph G can contain a triangle H, for otherwise, at least two vertices u and v of H must belong to the same partite set and so u and v are not adjacent. Hence if G is a graph of order $n \geq 3$ and size $m \leq \lfloor n^2/4 \rfloor$, then G need not contain a triangle. However, in 1907 the Dutch mathematician Willem Mantel [149] showed that any graph of order n with a larger size must contain a triangle.

Theorem 1.9 *Every graph of order $n \geq 3$ and size $m > \lfloor n^2/4 \rfloor$ contains a triangle.*

Proof. First observe that the result is true for $n = 3$ and $n = 4$. Suppose, however, that the statement is false. Then there is a smallest integer $n \geq 5$ and a graph G of order n and size $m > \lfloor n^2/4 \rfloor$ not containing a triangle. Let uv be an edge of G. Since G contains no triangle, there is no vertex in G adjacent to both u and v. Hence $\deg u + \deg v \leq n$. Let $G' = G - u - v$. Since G' is a subgraph of G, it follows that G' also does not contain a triangle. Furthermore, G' has order $n - 2$ and size

$$m' > m - (\deg u + \deg v) + 1$$

and so

$$m' > \lfloor n^2/4 \rfloor - n + 1.$$

Thus

$$m' > \frac{n^2}{4} - n + 1 = \frac{n^2 - 4n + 4}{4} = \frac{(n-2)^2}{4}.$$

From the defining property of the graph G, it follows that G' contains a triangle, producing a contradiction. ■

Bipartite graphs belong to a more general class of graphs. For an integer $k \geq 1$, a graph G is a **k-partite graph** if $V(G)$ can be partitioned into k subsets $V_1, V_2, \ldots, V_k$ (again called **partite sets**) such that every edge joins vertices in two different partite sets. A 1-partite graph is then an empty graph and a 2-partite graph is bipartite. A **complete k-partite graph** G is a k-partite graph such that two vertices are adjacent in G if and only if the vertices belong to different partite sets. If $|V_i| = n_i$ for $1 \leq i \leq k$, then G is denoted by $K_{n_1,n_2,\ldots,n_k}$ (the order in which the numbers $n_1, n_2, \ldots, n_k$ are written is not important). If $n_i = 1$ for all i $(1 \leq i \leq k)$, then G is the complete graph K_k. A **complete multipartite graph** is a complete k-partite graph for some integer $k \geq 2$. Some complete multipartite graphs are shown in Figure 1.18

$K_{2,4}$ $\qquad$ $K_{1,1,1,1} = K_4$ $\qquad$ $K_{2,2,2}$

Figure 1.18: Some complete multipartite graphs

The Complement of a Graph

There are many ways of producing a new graph from one or more given graphs. The most common of these is the complement of a graph. The **complement** $\overline{G}$ of a graph G is that graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. Any isomorphism from a graph G to a graph H is also an isomorphism from $\overline{G}$ to $\overline{H}$. Consequently, $G \cong H$ if and only if $\overline{G} \cong \overline{H}$. If G is a graph of order n and size m, then $\overline{G}$ is a graph of order n and size $\binom{n}{2} - m$. A graph G and its complement are shown in Figure 1.19. Thus the complement $\overline{K}_n$ of the complete graph K_n is the empty graph of order n.

Figure 1.19: A graph and its complement

A graph G is **self-complementary** if G is isomorphic to $\overline{G}$. Certainly, if G is a self-complementary graph of order n, then its size is $m = n(n-1)/4$. Since

only one of n and $n-1$ is even, either $4 \mid n$ or $4 \mid (n-1)$; that is, if G is a self-complementary graph of order n, then either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. (See Exercise 17.) The self-complementary graphs of order 5 or less are shown in Figure 1.20.

Figure 1.20: The self-complementary graphs of order 5 or less

Operations on Graphs

We next describe some common binary operations defined on graphs. This discussion introduces notation that will be especially useful in giving examples. Over the years, different authors have used different notation for the operations we are about to describe. In the following definitions, we assume that G_1 and G_2 are two graphs with disjoint vertex sets.

The **union** $G = G_1 + G_2$ of G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$. The union $G + G$ of two disjoint copies of G is denoted by $2G$. Indeed, if a graph G consists of k (≥ 2) disjoint copies of a graph H, then we write $G = kH$. The graph $2K_1 + 3K_2 + K_{1,3}$ is shown in Figure 1.21(a). The **join** $G = G_1 \vee G_2$ of G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

Using the join operation, we see that $\overline{K}_s \vee \overline{K}_t = K_{s,t}$. Another illustration is given in Figure 1.21(b).

Figure 1.21: The union and join of graphs

The **Cartesian product** G of two graphs G_1 and G_2, commonly denoted by $G_1 \square G_2$ or $G_1 \times G_2$, has vertex set

$$V(G) = V(G_1) \times V(G_2)$$

and two distinct vertices (u, v) and (x, y) of $G_1 \square G_2$ are adjacent if either

$$(1)\ u = x \text{ and } vy \in E(G_2) \text{ or } (2)\ v = y \text{ and } ux \in E(G_1).$$

A convenient way of drawing $G_1 \square G_2$ is to first place a copy of G_2 at each vertex of G_1 (see Figure 1.22(b)) and then join corresponding vertices of G_2 in those copies of G_2 placed at adjacent vertices of G_1 (see Figure 1.22(c)). Equivalently, $G_1 \square G_2$ can be constructed by placing a copy of G_1 at each vertex of G_2 and adding the appropriate edges. As expected, $G_1 \square G_2 \cong G_2 \square G_1$ for all graphs G_1 and G_2.

Figure 1.22: The Cartesian product of two graphs

An important class of graphs is defined in terms of Cartesian products. The n**-cube** Q_n is K_2 if $n = 1$, while for $n \geq 2$, Q_n is defined recursively as the Cartesian product $Q_{n-1} \square K_2$ of Q_{n-1} and K_2. The n-cube can also be defined as that graph whose vertex set is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$ or $a_1 a_2 \cdots a_n$ such that a_i is 0 or 1 for $1 \leq i \leq n$ (commonly called n**-bit strings**), such that two vertices are adjacent if and only if the corresponding ordered n-tuples differ at precisely one coordinate. The graph Q_n is an n-regular graph of order 2^n. The n-cubes for $n = 1, 2, 3$ are shown in Figure 1.23, where their vertices are labeled by n-bit strings. The graphs Q_n are often called **hypercubes**.

Figure 1.23: The n-cubes for $1 \leq n \leq 3$

Exercises for Section 1.1

1. An electronics company keeps on hand wire segments of a fixed length and of different colors for various purposes. Each wire is either colored blue (b), green (g), purple (p), red (r), silver (s), white (w) or yellow (y). The company has many wire segments of each color. All of the wire segments have been randomly stored in a large barrel. Eight handfuls of wires are removed from the barrel and each collection of wires is placed in a box. The boxes are denoted by B_i $(1 \leq i \leq 8)$. The colors of the wire segments in each box are:

$$\begin{array}{llll} B_1 = \{b, r\} & B_2 = \{p, r, s, w\} & B_3 = \{p, w, y\} & B_4 = \{g, r, y\} \\ B_5 = \{g\} & B_6 = \{b, g, y\} & B_7 = \{g, p, s, w, y\} & B_8 = \{s, w, y\}. \end{array}$$

   We are interested in those pairs of boxes containing at least one wire segment of the same color. Model this situation by a graph.

2. A graph $G = (V, E)$ of order 8 has the power set of the set $S = \{1, 2, 3\}$ as its vertex set, that is, V is the set of subsets of S. Two vertices A and B of V are adjacent if $A \cap B = \emptyset$. Draw the graph G, determine the degree of each vertex of G and determine the size of G.

3. A graph G of order 26 and size 58 has 5 vertices of degree 4, 6 vertices of degree 5 and 7 vertices of degree 6. The remaining vertices of G all have the same degree. What is this degree?

4. A graph G has order $n = 3k + 3$ for some positive integer k. Every vertex of G has degree $k + 1$, $k + 2$ or $k + 3$. Prove that G has at least $k + 3$ vertices of degree $k + 1$ or at least $k + 1$ vertices of degree $k + 2$ or at least $k + 2$ vertices of degree $k + 3$.

5. Consider the pairs G_1, G_2 and H_1, H_2 of graphs in Figure 1.24.

   (a) Determine whether $G_1 \cong G_2$.

   (b) Determine whether $H_1 \cong H_2$.

6. (a) Determine all non-isomorphic graphs of order 5.

   (b) Determine the minimum size of a graph G of order 5 such that every graph of order 5 and size 5 is isomorphic to some subgraph of G.

7. Show that if G is a nonregular graph of order n and size $rn/2$ for some integer r with $1 \leq r \leq n - 2$, then $\Delta(G) - \delta(G) \geq 2$.

8. For each integer $k \geq 2$, give an example of k non-isomorphic regular graphs, all of the same order and same size.

Figure 1.24: The graphs G_1, G_2, H_1, H_2 in Exercise 5

9. (a) Let G and H be two isomorphic graphs where one or more vertices of G (and H) have degree r. Let S be the set of vertices of degree r in G and T be the set of vertices of degree r in H. Prove that $G[S] \cong H[T]$.

   (b) Use the result in (a) to show that the graphs G and H in Figure 1.25 are not isomorphic.

Figure 1.25: The graphs G and H in Exercise 9

10. (a) Give an example of two non-isomorphic regular graphs G_1 and G_2 of the same order and same size such that (1) for every two vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, $G_1 - v_1 \ncong G_2 - v_2$ and (2) there exist 2-element subsets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$ such that $G_1 - S_1 \cong G_2 - S_2$.

    (b) Give an example of two non-isomorphic regular graphs H_1 and H_2 of the same order and same size such that (1) for every 2-element subsets $S_1 \subseteq V(H_1)$ and $S_2 \subseteq V(H_2)$, $H_1 - S_1 \ncong H_2 - S_2$ and there exist 3-element subsets $S_1' \subseteq V(H_1)$ and $S_2' \subseteq V(H_2)$ such that $H_1 - S_1' \cong H_2 - S_2'$.

11. Prove for every graph G and every integer $r \geq \Delta(G)$ that there exists an r-regular graph containing G as an induced subgraph.

12. Let G be a graph of order n all of whose vertices have degree r, where r is a positive integer, except for exactly one vertex of each of the degrees

$r-1, r-2, \ldots, r-j$, where $1 < j < r$. Show, in fact, that there exists an r-regular graph of order $2n$ containing G as an induced subgraph.

13. Determine all bipartite graphs G such that $\overline{G}$ is bipartite.

14. Show that for each integer $n \geq 2$, there is exactly one bipartite graph of order n having size $\lfloor n^2/4 \rfloor$.

15. We have seen that C_5 is a self-complementary graph. Therefore, there is a regular self-complementary graph of order 5. Show that there is a regular self-complementary graph of order 5^n for every positive integer n.

16. Let G_1 and G_2 be self-complementary graphs, where G_2 has even order n. Let G be the graph obtained from G_1 and G_2 by joining each vertex of G_2 whose degree is less than $n/2$ to every vertex of G_1. Show that G is self-complementary.

17. Prove that there exists a self-complementary graph of order n for every positive integer n with $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

18. Let G be a self-complementary graph of order n, where $n \equiv 1 \pmod 4$. Prove that G contains an odd number of vertices of degree $(n-1)/2$.

19. Let G be a graph of order n and size m such that $n = 4k+3$ for some positive integer k. Suppose that the complement $\overline{G}$ of G has size $\overline{m}$. Prove that either $m > \frac{1}{2}\binom{n}{2}$ or $\overline{m} > \frac{1}{2}\binom{n}{2}$.

20. For positive integers k and n with $n > 2k$, the graph $G_{n,k}$ is that graph whose vertices are the k-element subsets of an n-element set $S = \{1, 2, \ldots, n\}$ and where two vertices (k-element subsets) A and B are adjacent if A and B are disjoint. The graph $G_{n,k}$ is called the **Kneser graph**.

    (a) Determine the graphs $G_{6,1}$ and $G_{5,2}$.

    (b) Show that $G_{n,k}$ is an r-regular graph for some integer r.

21. For $i = 1, 2$, let u_i be a vertex in a graph G_i of order n_i and size m_i.

    (a) Determine the degree of u_1 in $G_1 + G_2$.

    (b) Determine the degree of u_1 in $G_1 \vee G_2$.

    (c) Determine the degree of (u_1, u_2) in $G_1 \,\square\, G_2$.

22. Determine the order and size of each of the graphs $P_3 \vee 2P_3$, $P_3 \,\square\, 2P_3$ and $Q_1 + Q_2 + Q_3$.

23. (a) Let n be a given positive integer, and let r and s be nonnegative integers such that $r + s = n$ and s is even. Give an example of a graph containing r even vertices and s odd vertices.

(b) Determine the minimum size of a graph G containing r even vertices and s odd vertices and satisfying the properties in (a).

(c) Determine the maximum size of a graph G containing r even vertices and s odd vertices and satisfying the properties in (a).

24. (a) Let G be a graph of order $n \geq 4$. Prove that if $\deg v \geq \frac{2n+1}{3}$ for every vertex v of G, then every edge of G belongs to a complete subgraph of order 4.

(b) Show that the result in (a) is best possible in general by showing that $\frac{2n+1}{3}$ cannot be replaced by $\frac{2n}{3}$.

1.2 Degree Sequences

We saw in the First Theorem of Graph Theory (Theorem 1.4) that the sum of the degrees of the vertices of a graph G is twice the size of G and that G must have an even number of odd vertices. We have also described conditions under which a regular graph of order n can exist. We now consider the degrees of the vertices of a graph in more detail.

A sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers is called a **degree sequence** of a graph G of order n if the vertices of G can be labeled $v_1, v_2, \ldots, v_n$ so that $\deg v_i = d_i$ for $1 \leq i \leq n$. For example, a degree sequence of the graph G of Figure 1.26 is $4, 3, 2, 2, 1$ (or $1, 2, 2, 3, 4$ or $2, 1, 4, 2, 3$, etc.).

G :

4, 3, 2, 2, 1

Figure 1.26: A degree sequence of a graph

A finite sequence s of nonnegative integers is a **graphical sequence** if s is a degree sequence of some graph. Thus $4, 3, 2, 2, 1$ is graphical. There are some obvious necessary conditions for a sequence $s : d_1, d_2, \ldots, d_n$ of n nonnegative integers to be graphical. While the conditions $d_i \leq n - 1$ for all i $(1 \leq i \leq n)$ and $\sum_{i=1}^{n} \deg v_i$ is even are necessary for s to be graphical, they are not sufficient. For example, the sequence $3, 3, 3, 1$ satisfies both conditions but it is not graphical, for if three vertices of a graph of order 4 have degree 3 then the remaining vertex must have degree 3 as well.

It is not at all unusual for a graphical sequence to be the degree sequence of more than one graph. For example, the graphical sequence $3, 2, 2, 2, 1$ is the degree sequence of the two graphs in Figure 1.27. On the other hand, each of the 18 graphs in Figure 1.7 has a degree sequence possessed by no other graph.

While two graphs with the same degree sequence need not be isomorphic, each can be obtained from the other by a sequence of edge shifts where at each step, two nonadjacent edges in some graph F are deleted and two nonadjacent

Figure 1.27: Two graphs with the same degree sequence

edges in $\overline{F}$ are added to F such that the four edges involved are incident with the same four vertices.

The Havel-Hakimi Theorem

Let H be a graph containing four distinct vertices u, v, w and x such that $uv, wx \in E(H)$ and $uw, vx \notin E(G)$. The process of deleting the edges uv and wx from H and adding uw and vx to H is referred to as a **2-switch** in H (see Figure 1.28, where a dashed line means no edge). This produces a new graph G having the same degree sequence as H.

Figure 1.28: A 2-switch in a graph

Theorem 1.10 *Let $s : d_1, d_2, \ldots, d_n$ be a graphical sequence with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$ and let $\mathcal{G}_s$ be the set of all graphs F with degree sequence s such that $V(F) = \{v_1, v_2, \ldots, v_n\}$ where $\deg v_i = d_i$ for $1 \leq i \leq n$. Then every graph $H \in \mathcal{G}_s$ can be transformed into a graph $G \in \mathcal{G}_s$ by a sequence of 2-switches such that $N_G(v_1) = \{v_2, v_3, \ldots, v_{\Delta+1}\}$.*

Proof. Suppose that this statement is false. Let $W = \{v_2, v_3, \ldots, v_{\Delta+1}\}$. Among all graphs into which H can be transformed, let G be one for which the sum of the subscripts of the vertices in $N_G(v_1)$ is minimum. Since $N_G(v_1) \neq W$, the vertex v_1 is adjacent to a vertex v_k and is not adjacent to a vertex v_j with $j < k$ and so $d_j \geq d_k$. Consequently, there is a vertex v_ℓ such that $v_j v_\ell \in E(G)$ and $v_k v_\ell \notin E(G)$. Replacing the edges $v_1 v_k$ and $v_j v_\ell$ by $v_1 v_j$ and $v_k v_\ell$ is a 2-switch in G that produces a graph $G_1 \in \mathcal{G}_s$ for which the sum of the subscripts of the vertices in $N_{G_1}(v_1)$ is less than that of $N_G(v_1)$. This is a contradiction. ■

One consequence of Theorem 1.10 is that every two graphs with the same degree sequence are related in terms of 2-switches. The following theorem appeared in the book *Graphs and Hypergraphs* by Claude Berge [18].

Theorem 1.11 *If G and H are two graphs with the same degree sequence, then H can be transformed into G by a (possibly empty) sequence of 2-switches.*

Proof. We proceed by induction on the order n of G and H. If $n \leq 4$, then the result is immediate. For a given integer $n \geq 5$, assume that every two graphs of order $n-1$ with the same degree sequence can be transformed into each other by a sequence of 2-switches. Let $s : d_1, d_2, \ldots, d_n$ be a graphical sequence with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$ and let $\mathcal{G}_s$ be the set of all graphs F with degree sequence s such that $V(F) = \{v_1, v_2, \ldots, v_n\}$ where $\deg v_i = d_i$ for $1 \leq i \leq n$. Let $W = \{v_2, v_3, \ldots, v_{\Delta+1}\}$. Let G and H be two graphs in $\mathcal{G}_s$. By Theorem 1.10, G can be transformed into a graph $G_1 \in \mathcal{G}_s$ such that $N_{G_1}(v_1) = W$ and H can be transformed into a graph $H_1 \in \mathcal{G}_s$ such that $N_{H_1}(v_1) = W$. Since $G_1 - v_1$ and $H_1 - v_1$ are two graphs of order $n-1$ with the same degree sequence, it follows by the induction hypothesis that $G_1 - v_1$ can be transformed into $H_1 - v_1$ by a sequence of 2-switches. Hence G_1 can be transformed into H_1 by a sequence of 2-switches and so G can be transformed into H by a sequence of 2-switches. ■

There are necessary and sufficient conditions for a finite sequence of nonnegative integers to be graphical. One of these is a consequence of Theorem 1.10. This theorem is due to Václav Havel [114] and S. Louis Hakimi [107] and is often referred to as the Havel-Hakimi Theorem, despite the fact that Havel and Hakimi gave independent proofs and wrote separate papers containing this theorem.

Theorem 1.12 (**Havel-Hakimi Theorem**) *A sequence $s : d_1, d_2, \ldots, d_n$ of nonnegative integers with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$ and $\Delta \geq 1$, is graphical if and only if the sequence*

$$s_1 : d_2 - 1, d_3 - 1, \ldots, d_{\Delta+1} - 1, d_{\Delta+2}, \ldots, d_n$$

is graphical.

Proof. First, assume that s_1 is graphical. Then there exists a graph G_1 of order $n-1$ such that s_1 is a degree sequence of G_1. Thus the vertices of G can be labeled as $v_2, v_3, \ldots, v_n$ so that

$$\deg_{G_1} v_i = \begin{cases} d_i - 1 & \text{if } 2 \leq i \leq \Delta + 1 \\ d_i & \text{if } \Delta + 2 \leq i \leq n. \end{cases}$$

A new graph G can now be constructed by adding a new vertex v_1 and the Δ edges $v_1 v_i$ for $2 \leq i \leq \Delta + 1$. Since $\deg_G v_i = d_i$ for $1 \leq i \leq n$, it follows that $s : \Delta = d_1, d_2, \ldots, d_n$ is a degree sequence of G and so s is graphical.

Conversely, let s be a graphical sequence. By Theorem 1.10, there exists a graph G of order n having degree sequence s with $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $\deg v_i = d_i$ for all i $(1 \leq i \leq n)$ where $N_G(v_1) = \{v_2, v_3, \ldots, v_{\Delta+1}\}$. Then $G - v_1$ has degree sequence s_1. ■

Theorem 1.12 actually provides an algorithm for determining whether a given finite sequence of nonnegative integers is graphical. If, upon repeated application of Theorem 1.12, we arrive at a sequence every term of which 0, then the original sequence is graphical. On the other hand, if we arrive at a sequence containing a negative integer, then the given sequence is not graphical.

We now illustrate Theorem 1.12 with the sequence

$$s : 5, 3, 3, 3, 3, 2, 2, 2, 1, 1, 1.$$

After one application of Theorem 1.12 (deleting 5 from s and subtracting 1 from the next five terms), we obtain

$$s_1' : 2, 2, 2, 2, 1, 2, 2, 1, 1, 1.$$

Reordering this sequence, we have

$$s_1 : 2, 2, 2, 2, 2, 2, 1, 1, 1, 1.$$

Continuing in this manner, we get

$$\begin{aligned}
&s_2' : 1, 1, 2, 2, 2, 1, 1, 1, 1\\
&s_2 : 2, 2, 2, 1, 1, 1, 1, 1, 1\\
&s_3' = s_3 : 1, 1, 1, 1, 1, 1, 1, 1\\
&s_4' : 0, 1, 1, 1, 1, 1, 1, 1\\
&s_4 : 1, 1, 1, 1, 1, 1, 0\\
&s_5' : 0, 1, 1, 1, 1, 0\\
&s_5 : 1, 1, 1, 1, 0, 0\\
&s_6' : 0, 1, 1, 0, 0\\
&s_6 : 1, 1, 0, 0, 0\\
&s_7' = s_7 : 0, 0, 0, 0
\end{aligned}$$

Therefore, s is graphical. Of course, if we observe that some sequence prior to s_7 is graphical, then we can conclude by Theorem 1.12 that s is graphical. For example, the sequence s_3 is clearly graphical since it is the degree sequence of the graph G_3 in Figure 1.29. By Theorem 1.12, each of the sequences s_2, s_1 and s is graphical. To construct a graph with degree sequence s_2, we proceed in reverse from $s_3' = s_3$ to s_2, observing that a vertex should be added to G_3 so that it is adjacent to two vertices of degree 1. We thus obtain a graph G_2 with degree sequence s_2 (or s_2'). Proceeding from s_2' to s_1, we again add a new vertex joining it to two vertices of degree 1 in G_2. This gives a graph G_1 with degree sequence s_1 (or s_1'). Finally, we obtain a graph G with degree sequence s by considering s_1'; that is, we add a new vertex to G_1 joining it to vertices of degrees $2, 2, 2, 2, 1$. This procedure is illustrated in Figure 1.29.

Figure 1.29: Construction of a graph G with a given degree sequence

It should be pointed out that the graph G in Figure 1.29 is not the only graph with degree sequence s. However, there are graphs that cannot be produced by the method used to construct the graph G in Figure 1.29. For example, the graph H of Figure 1.30 is such a graph.

Figure 1.30: A graph that cannot be constructed by the method following Theorem 1.12

The Erdős-Gallai Theorem

Suppose that $s : d_1, d_2, \ldots, d_n$ is a graphical sequence with $d_1 \geq d_2 \geq \cdots \geq d_n$. Then there exists a graph G of order n with $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $\deg v_i = d_i$ for all i $(1 \leq i \leq n)$. Of course, the sum $\sum_{i=1}^{n} d_i$ is even. Let k be integer with $1 \leq k \leq n-1$. Suppose that $V_1 = \{v_1, v_2, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$. Now the sum $\sum_{i=1}^{k} d_i$ counts every edge in $G[V_1]$ twice and counts each edge in $[V_1, V_2]$ once. The size of $G[V_1]$ is at most $\binom{k}{2} = k(k-1)/2$, while for each i $(k+1 \leq i \leq n)$ the number of edges joining v_i and V_1 is at most $\min\{k, d_i\}$. Thus

$$\sum_{i=1}^{k} d_i \leq 2\binom{k}{2} + \sum_{i=k+1}^{n} \min\{k, d_i\} = k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}. \tag{1.1}$$

Hence for every graphical sequence s, we must have both that $\sum_{i=1}^{n} d_i$ is even and (1.1) is satisfied for every integer k with $1 \leq k \leq n-1$. These conditions are not only necessary for a sequence of nonnegative integers to be graphical, they are sufficient as well. This result was established by Paul Erdős and Tibor Gallai [71], who were introduced to graph theory as youngsters by Dénes König, author of the first book on graph theory.

Theorem 1.13 (**Erdős-Gallai Theorem**) *A sequence* $s : d_1, d_2, \ldots, d_n$ $(n \geq 2)$ *of nonnegative integers with* $d_1 \geq d_2 \geq \cdots \geq d_n$ *is graphical if and only if* $\sum_{i=1}^{n} d_i$ *is even and for each integer* k *with* $1 \leq k \leq n-1$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}.$$

Irregular Graphs

It is a consequence of Theorem 1.7 that a constant sequence $s : r, r, \ldots, r$ of nonnegative integers r of length n is graphical if and only if $r \leq n-1$ and rn is even. This, of course, gives us conditions for the existence of an r-regular graph of order n. At the other extreme are nontrivial graphs, no two vertices of which have the same degree. A nontrivial graph G is **irregular** if $\deg u \neq \deg v$ for every two vertices u and v of G.

Theorem 1.14 *No graph is irregular.*

Proof. Assume, to the contrary, that there exists an irregular graph G of order $n \geq 2$. We may assume that $V(G) = \{v_1, v_2, \ldots, v_n\}$, where $\deg v_i = i-1$ for $1 \leq i \leq n$. Since $\deg v_1 = 0$, the vertex v_1 is isolated in G and since $\deg v_n = n-1$, it follows that v_n is adjacent to v_1. This is a contradiction. ■

By Theorem 1.14, for each integer $n \geq 2$, there is no graph of order n whose n vertices have distinct degrees. Since the degrees of the vertices of the graph G of Figure 1.26 are $4, 3, 2, 2, 1$, it is possible for $n-1$ vertices of a graph of order n to have distinct degrees. A graph G of order $n \geq 2$ is **nearly irregular** if exactly two vertices of G have the same degree.

Theorem 1.15 *For every integer* $n \geq 2$*, there are exactly two nearly irregular graphs of order* n.

Proof. First, observe that if G is a nearly irregular graph of order n, then G cannot contain a vertex of degree 0 *and* a vertex of degree $n-1$. Thus either each vertex of G has one of the degrees $1, 2, \ldots, n-1$ or each vertex of G has one of the degrees $0, 1, \ldots, n-2$. Furthermore, if G is nearly irregular, then so is $\overline{G}$.

We show by induction that for each integer $n \geq 2$ there are exactly two nearly irregular graphs of order n. Since K_2 and $\overline{K}_2$ are nearly irregular, the result holds for $n = 2$. Assume for an integer $n \geq 3$ that there are exactly two nearly irregular graphs of order $n-1$. Necessarily, one of these graphs is some graph F where $\Delta(F) = n-2$ and the other is $\overline{F}$ where $\delta(\overline{F}) = 0$. Then $H = \overline{F} \vee K_1$ and $\overline{H}$ are nearly irregular graphs of order n, where $\Delta(H) = n-1$ and $\delta(\overline{H}) = 0$. We claim that these are the only nearly irregular graphs of

order n. Assume, to the contrary, that there is a third graph G of order n that is nearly irregular. Then either $\Delta(G) = n - 1$ or $\delta(G) = 0$, say the former. Then $G = G_1 \vee K_1$ for some nearly irregular G_1 of order $n - 1$ with $\delta(G_1) = 0$. However then, $G_1 \cong \overline{H}$, which is a contradiction. ■

By Theorem 1.15, for each integer $n \geq 2$, there exist exactly two distinct graphical sequences of length n having exactly two equal terms.

Exercises for Section 1.2

1. Find a sequence of 2-switches that transforms the graph G of Figure 1.31 into the graph H.

Figure 1.31: The graphs G and H in Exercise 1

2. For two pairs G_1, H_1 and G_2, H_2 of graphs shown in Figure 1.32, determine the minimum number of 2-switches required to transform

   (a) G_1 into H_1 and (b) G_2 into H_2.

Figure 1.32: The graphs in Exercise 2

3. Let $s : 2, 2, 2, 2, 2, 2, 2, 2, 2$ and let $\mathcal{G}_s$ be the set of all graphs with degree sequence s. Let G be a graph with $V(G) = \mathcal{G}_s$ where two vertices F and H in G are adjacent if F can be transformed into H by a single 2-switch. Which familiar graph is G isomorphic to?

4. Give an example of a graphical sequence s (where $\mathcal{G}_s$ is the set of all graphs with degree sequence s) such that (1) the graph G has $V(G) = \mathcal{G}_s$,

(2) two vertices F and H of G are adjacent if F can be transformed into H by a single 2-switch and (3) G contains a triangle.

5. Let $s : d_1, d_2, \ldots, d_n$ be a graphical sequence with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$. Show, for each integer k with $1 \leq k \leq n$, that there exists a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ where $\deg v_i = d_i$ for $1 \leq i \leq n$ having the property that v_k is adjacent to either (1) the vertices of $\{v_1, v_2, \ldots, v_{d_k}\}$ if $k > d_k$ or (2) the vertices of $\{v_1, v_2, \ldots, v_{d_k+1}\} - \{v_k\}$ if $1 \leq k \leq d_k$.

6. Let G and H be two graphs that are neither empty nor complete. The graph H is said to be obtained from G by an **edge rotation** if G contains three vertices u, v, and w where $uv \in E(G)$ and $uw \notin E(G)$ and $H \cong G - uv + uw$.

   (a) Show that the graph G_2 of Figure 1.33 is obtained from G_1 by an edge rotation.

   (b) Show that G_3 of Figure 1.33 cannot be obtained from G_1 by an edge rotation.

   (c) Show that for every two nonempty, noncomplete graphs G and H of the same order and same size, there exists a sequence $G = G_0$, G_1, …, $G_k = H$ of graphs such that G_{i+1} is obtained from G_i by an edge rotation for $i = 0, 1, \ldots, k-1$.

Figure 1.33: The graphs in Exercise 6

7. Determine whether the following sequences are graphical. If so, construct a graph with the appropriate degree sequence.

   (a) $4, 4, 3, 2, 1$

   (b) $3, 3, 2, 2, 2, 2, 1, 1$

   (c) $7, 7, 6, 5, 4, 4, 3, 2$

   (d) $7, 6, 6, 5, 4, 3, 2, 1$

   (e) $7, 4, 3, 3, 2, 2, 2, 1, 1, 1$.

8. Prove that a sequence $d_1, d_2, \ldots, d_n$ is graphical if and only if $n - d_1 - 1, n - d_2 - 1, \ldots, n - d_n - 1$ is graphical.

9. Prove that for every integer x with $0 \leq x \leq 5$, the sequence $x, 1, 2, 3, 5, 5$ is not graphical.

10. For which integers x $(0 \leq x \leq 7)$, if any, is the sequence $7, 6, 5, 4, 3, 2, 1, x$ graphical?

11. Use Theorem 1.13 to determine whether the sequence $s : 6, 6, 5, 4, 3, 2, 2$ is graphical.

12. We have seen that there is only one graphical sequence d_1, d_2, d_3, d_4, d_5 with $4 = d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5 = 1$ such that at least one term is 3 and at least one term is 2. How many graphical sequences d_1, d_2, d_3, d_4, d_5 are there with $4 = d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5 = 2$ such that at least one term is 3?

13. Show that for every finite set S of positive integers, there exists a positive integer k such that the sequence obtained by listing each element of S a total of k times is graphical. Find the minimum such k for $S = \{2, 6, 7\}$.

14. According to Theorem 1.15, for each integer $n \geq 2$, there exist exactly two distinct graphical sequences of length n having exactly two equal terms. What terms are equal for these two sequences?

15. Two finite sequences s_1 and s_2 of nonnegative integers are called **bigraphical** if there exists a bipartite graph G with partite sets V_1 and V_2 such that s_i lists the degrees of the vertices of G in V_i for $i = 1, 2$. Prove that the sequences $s_1 : a_1, a_2, \ldots, a_r$ and $s_2 : b_1, b_2, \ldots, b_t$ of nonnegative integers with $r \geq 2$, $a_1 \geq a_2 \geq \cdots \geq a_r$, $b_1 \geq b_2 \geq \cdots \geq b_t$, $0 < a_1 \leq t$ and $0 < b_1 \leq r$ are bigraphical if and only if the sequences $s_1' : a_2, a_3, \cdots, a_r$ and $s_2' : b_1 - 1, b_2 - 1, \ldots, b_{a_1} - 1, b_{a_1+1}, \ldots, b_t$ are bigraphical.

1.3 Connected Graphs and Distance

Several problems in graph theory concern manners in which one can move about in a graph. In particular, for two vertices u and v in a graph G, common questions include:

(a) Can one proceed from u to v along edges of G?

(b) What is the fewest number of edges needed to proceed from u to v along edges of G?

(c) Starting with v, say, is it possible to proceed along all the edges (or about all the vertices) of G without repeating any and returning to the starting vertex?

For the purpose of studying questions such as these, it is useful to introduce some new concepts.

Walks, Trails and Paths

For two (not necessarily distinct) vertices u and v in a graph G, a $u-v$ **walk** W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in W are adjacent in G. Such a walk W in G can be expressed as

$$W = (u = v_0, v_1, \ldots, v_k = v), \tag{1.2}$$

where $v_i v_{i+1} \in E(G)$ for $0 \leq i \leq k-1$. (The walk W is also commonly denoted by $W : u = v_0, v_1, \ldots, v_k = v$.) Nonconsecutive vertices in W need not be distinct. The walk W is said to contain each vertex v_i $(0 \leq i \leq k)$ and each edge $v_i v_{i+1}$ $(0 \leq i \leq k-1)$. The walk W can therefore be thought of as beginning at the vertex $u = v_0$, proceeding along the edge $v_0 v_1$ to the vertex v_1, then along the edge $v_1 v_2$ to the vertex v_2, and so forth, until finally arriving at the vertex $v = v_k$. The number of edges encountered in W (including multiplicities) is the **length** of W. Hence the length of the walk W in (1.2) is k. In the graph G of Figure 1.34,

$$W_1 = (x, w, y, w, v, u, w) \tag{1.3}$$

is an $x-w$ walk of length 6. This walk encounters the vertex w three times and the edge wy twice.

Figure 1.34: Walks in a graph

A walk whose initial and terminal vertices are distinct is an **open walk**; otherwise, it is a **closed walk**. Thus the walk W_1 in (1.3) in the graph G of Figure 1.34 is an open walk. It is possible for a walk to consist of a single vertex, in which case it is a **trivial walk**. A trivial walk is therefore a closed walk.

A walk in a graph G in which no edge is repeated is a **trail** in G. For example, in the graph G of Figure 1.34, $T = (u, v, y, w, v)$ is a $u-v$ trail of length 4. While no edge of T is repeated, the vertex v is repeated, which is allowed. On the other hand, a walk in a graph G in which no vertex is repeated is called a **path**. Every nontrivial path is necessarily an open walk. Thus $P' = (u, v, w, y)$ is $u-y$ path of length 3 in the graph G of Figure 1.34. Many proofs in graph theory make use of $u-v$ walks or $u-v$ paths of minimum length (or of maximum length) for some pair u, v of vertices of a graph. The proof of the following theorem illustrates this.

Theorem 1.16 *Let u and v be two vertices of a graph G. For every $u-v$ walk W in G, there exists a $u-v$ path P such that every edge of P belongs to W.*

Proof. Among all $u-v$ walks in G, let

$$P = (u = u_0, u_1, \ldots, u_k = v)$$

be a $u-v$ walk of minimum length for which every edge of P belongs to W. Thus the length of P is k. We claim that P is a $u-v$ path. Assume, to the contrary, that this is not the case. Then some vertex of G must be repeated in P, say $u_i = u_j$ for some i and j with $0 \le i < j \le k$. If we then delete the vertices $u_{i+1}, u_{i+2}, \ldots, u_j$ from P, we arrive at the $u-v$ walk

$$W' = (u = u_0, u_1, \ldots, u_{i-1}, u_i = u_j, u_{j+1}, \ldots, u_k = v)$$

whose length is less than k and such that every edge of W' belongs to W, which is impossible. ■

The Adjacency Matrix of a Graph

We have seen that a graph can be defined or described by means of sets (the definition) or diagrams. As Example 1.3 suggests, there are also matrix representations of graphs. Suppose that G is a graph of order n, where $V(G) = \{v_1, v_2, \ldots, v_n\}$. The **adjacency matrix** of G is the $n \times n$ zero-one matrix $A(G) = [a_{ij}]$, or simply $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_iv_j \in E(G) \\ 0 & \text{if } v_iv_j \notin E(G). \end{cases}$$

Figure 1.35 shows the adjacency matrix of a graph G.

G :

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 1.35: A graph and its adjacency matrix

There are several observations that can be made about the adjacency matrix A of a graph G of order n. First, all entries along the main diagonal of A are 0 since no vertex of G is adjacent to itself. Second, A is a symmetric matrix, that is, row i of A is identical to column i of A for every integer i with $1 \le i \le n$. Also, if we were to add the entries in row i (or in column i), then we obtain the degree of v_i.

Whenever $a_{ij} = 1$, this means that G contains the edge v_iv_j and therefore a $v_i - v_j$ path of length 1 and, of course, a $v_i - v_j$ walk of length 1 as well. Not

only can the adjacency matrix of G be used to identify whether G contains a $v_i - v_j$ walk of length 1, it can be used to determine whether G contains a $v_i - v_j$ walk of length k for an arbitrary positive integer k and, in fact, the number of such walks. Before stating a theorem that provides us with this information, we need to know when two $u - v$ walks in a graph G are considered to be the same.

Two $u-v$ walks $W = (u = u_0, u_1, \ldots, u_k = v)$ and $W' = (u = v_0, v_1, \ldots, v_\ell = v)$ in a graph are **equal** if $k = \ell$ and $u_i = v_i$ for all i with $0 \le i \le k$.

Theorem 1.17 *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and adjacency matrix A. For each positive integer k, the number of different $v_i - v_j$ walks of length k in G is the (i, j)-entry in the matrix A^k.*

Proof. Let $a_{ij}^{(k)}$ denote the (i, j)-entry in the matrix A^k for a positive integer k. Thus, $A^1 = A$ and $a_{ij}^{(1)} = a_{ij}$. We proceed by induction on k. For vertices v_i and v_j of G, there can be only one $v_i - v_j$ walk of length 1 or no $v_i - v_j$ walks of length 1, and this occurs if $a_{ij} = 1$ or $a_{ij} = 0$, respectively. Therefore, the (i, j)-entry of the matrix A is the number of $v_i - v_j$ walks of length 1 in G. Thus the basis step of the induction is established.

We now verify the inductive step. Assume, for a positive integer k, that $a_{ij}^{(k)}$ is the number of different $v_i - v_j$ walks of length k in G. We show that the (i, j)-entry $a_{ij}^{(k+1)}$ in A^{k+1} gives the number of different $v_i - v_j$ walks of length $k+1$ in G. First, observe that every $v_i - v_j$ walk of length $k+1$ in G is obtained from a $v_i - v_t$ walk of length k for some vertex v_t in G that is adjacent to v_j.

Since $A^{k+1} = A^k \cdot A$, it follows that the (i, j)-entry $a_{ij}^{(k+1)}$ in A^{k+1} can be obtained by taking the inner product of row i of A^k and column j of A. That is,

$$a_{ij}^{(k+1)} = a_{i1}^{(k)} a_{1j} + a_{i2}^{(k)} a_{2j} + \ldots + a_{in}^{(k)} a_{nj} = \sum_{t=1}^{n} a_{it}^{(k)} a_{tj}. \tag{1.4}$$

By the induction hypothesis, for each integer t with $1 \le t \le n$, the integer $a_{it}^{(k)}$ is the number of different $v_i - v_t$ walks of length k in G. If $a_{tj} = 1$, then v_t is adjacent to v_j and so there are $a_{it}^{(k)}$ different $v_i - v_j$ walks of length $k + 1$ in G whose next-to-last vertex is v_t. On the other hand, if $a_{tj} = 0$, then v_t is not adjacent to v_j and there are no $v_i - v_j$ walks of length $k + 1$ in G whose next-to-last vertex is v_t. In any case, $a_{it}^{(k)} \cdot a_{tj}$ gives the number of different $v_i - v_j$ walks of length $k+1$ in G whose next-to-last vertex is v_t. Consequently, the total number of different $v_i - v_j$ walks of length $k + 1$ in G is the sum in (1.4), which is $a_{ij}^{(k+1)}$.

By the Principle of Mathematical Induction, $a_{ij}^{(k)}$ is the number of different $v_i - v_j$ walks of length k in G for every positive integer k. ■

Before giving an example to illustrate Theorem 1.17, we make a few observations. Let G be a graph of order n with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and

$A^k = \left[a_{ij}^{(k)}\right]$, where A is the adjacency matrix of G. By Theorem 1.17, $a_{ii}^{(2)}$ gives the number of different $v_i - v_i$ walks of length 2 in G. Since a $v_i - v_i$ walk of length 2 is (v_i, v_t, v_i) for some vertex v_t adjacent to v_i, it follows that $a_{ii}^{(2)} = \deg v_i$ for every vertex v_i of G. For $i \neq j$, $a_{ij}^{(2)}$ is the number of different $v_i - v_j$ *paths* of length 2 in G. Again, by Theorem 1.17, $a_{ii}^{(3)}$ gives the number of different $v_i - v_i$ walks of length 3 in G. Since a $v_i - v_i$ walk of length 3 is (v_i, v_s, v_t, v_i) for adjacent vertices v_s and v_t, each of which is adjacent to v_i, it follows that v_i must belong to a triangle. Not only is (v_i, v_s, v_t, v_i) a $v_i - v_i$ walk of length 3, so too is (v_i, v_t, v_s, v_i) a (different) $v_i - v_i$ walk of length 3 in G. Therefore, $a_{ii}^{(3)}$ is twice the number of triangles in G that contain v_i.

As an illustration, consider the graph G of Figure 1.36 having the adjacency matrix A. We can compute A^2 without matrix multiplication by observing that the (i,i) entry of A^2, $1 \leq i \leq 4$, is $\deg v_i$, and the (i,j) entry of A^2, $i \neq j$, is the number of different $v_i - v_j$ paths of length 2. We now turn to A^3. Since the different $v_1 - v_3$ walks of length 3 in G are

$$W_1 = (v_1, v_3, v_1, v_3),\ W_2 = (v_1, v_2, v_1, v_3),$$
$$W_3 = (v_1, v_3, v_2, v_3),\ W_4 = (v_1, v_3, v_4, v_3),$$

the $(1,3)$ entry of A^3 is 4. The entire matrix A^3 can be computed in this manner.

$G:$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

Figure 1.36: A graph G and powers of its adjacency matrix

Circuits and Cycles

A nontrivial closed walk in a graph G in which no edge is repeated is a **circuit** in G. For example,

$$C = (u, w, x, y, w, v, u)$$

is a circuit in the graph G of Figure 1.34. In addition to the required repetition of u in this circuit, w is repeated as well. This is acceptable provided no edge is repeated. A circuit

$$C = (v = v_0, v_1, \ldots, v_k = v),$$

$k \geq 2$, for which the vertices v_i, $0 \leq i \leq k-1$, are distinct is a **cycle** in G. Therefore,

$$C' = (u, v, y, x, w, u)$$

is a cycle of length 5 in the graph G of Figure 1.34. As in the case of the graph C_k, a cycle of length $k \geq 3$ is called a **k-cycle**. A 3-cycle is also referred to as a **triangle**. A cycle of even length is an **even cycle**, while a cycle of odd length is an **odd cycle**.

The subgraph induced by the edges in a path $(v_1, v_2, \cdots, v_k)$ or a cycle $(v_1, v_2, \cdots, v_k, v_1)$, $k \geq 3$, is itself called a **path** or **cycle**. Consequently, paths and cycles have two interpretations in graphs – as sequences of vertices and as subgraphs. This is the case with trails and circuits as well. The path P' and the cycle C' described earlier in the graph G of Figure 1.34 correspond to the subgraphs shown in Figure 1.37. We have seen that a graph of order n that is a path or a cycle is denoted by P_n and C_n, respectively.

Figure 1.37: A path and cycle in a graph

The length of a smallest cycle in a graph G (containing cycles) is the **girth** of G, denoted by $g(G)$. Thus $g(K_n) = 3$ for $n \geq 3$ and $g(K_{s,t}) = 4$ for $s, t \geq 2$, while the girth of the Petersen graph is 5.

Connected Graphs

Two vertices u and v in a graph G are **connected** if G contains a $u - v$ path. The graph G itself is **connected** if every two vertices of G are connected. By Theorem 1.16, a graph G is connected if G contains a $u - v$ walk for every two vertices u and v of G. A graph G that is not connected is a **disconnected graph**. The graph F of Figure 1.38 is connected since F contains a $u - v$ path (and a $u - v$ walk) for every two vertices u and v in F. On the other hand, the graph H is disconnected since, for example, H contains no $y_4 - y_5$ path.

A connected subgraph H of a graph G is a **component** of G if H is not a proper subgraph of any connected subgraph of G. Thus every component of G is an induced subgraph of G. The number of components in a graph G is denoted by $k(G)$. Therefore, G is connected if and only if $k(G) = 1$. For

Figure 1.38: A connected graph and disconnected graph

the sets $S_1 = \{y_1, y_2, y_3, y_4\}$ and $S_2 = \{y_5, y_6, y_7\}$ of vertices of the graph H of Figure 1.38, the induced subgraphs $H[S_1]$ and $H[S_2]$ are (the only) components of H. Therefore, $k(H) = 2$.

If u and v are distinct vertices in a connected graph G, then there is a $u-v$ path in G. In fact, there may very well be several $u-v$ paths in G, possibly of varying lengths. This information can be used to provide a measure of how close u and v are to each other or how far from each other they are. The most common definition of distance between two vertices in a connected graph is the following.

Distance in Graphs

The **distance** $d_G(u, v)$ from a vertex u to a vertex v in a connected graph G, or simply $d(u, v)$ if the graph G is clear, is the minimum of the lengths of the $u-v$ paths in G. A $u-v$ path of length $d(u, v)$ is called a $u-v$ **geodesic**. In the graph G of Figure 1.39, the path $P = (v_1, v_5, v_6, v_{10})$ is a $v_1 - v_{10}$ geodesic and so $d(v_1, v_{10}) = 3$. Furthermore,

$$d(v_1, v_1) = 0,\ d(v_1, v_2) = 1,\ d(v_1, v_6) = 2,\ d(v_1, v_7) = 3 \text{ and } d(v_1, v_8) = 4.$$

Figure 1.39: Distances in a graph

The distance d defined above satisfies each of the following properties in a connected graph G:

(1) $d(u,v) \geq 0$ for every two vertices u and v of G;

(2) $d(u,v) = 0$ if and only if $u = v$;

(3) $d(u,v) = d(v,u)$ for all $u, v \in V(G)$ (the **symmetric property**);

(4) $d(u,w) \leq d(u,v) + d(v,w)$ for all $u, v, w \in V(G)$ (the **triangle inequality**).

Since d satisfies the four properties (1)-(4), d is a **metric** on $V(G)$ and $(V(G), d)$ is a **metric space**. Since d is symmetric, we can speak of the distance between two vertices u and v rather than the distance from u to v.

We are now prepared to present a useful characterization of bipartite graphs.

Theorem 1.18 *A nontrivial graph G is a bipartite graph if and only if G contains no odd cycles.*

Proof. Suppose first that G is bipartite. Then $V(G)$ can be partitioned into partite sets U and W (and so every edge of G joins a vertex of U and a vertex of W). Let $C = (v_1, v_2, \ldots, v_k, v_1)$ be a k-cycle of G. We may assume that $v_1 \in U$. Thus $v_2 \in W$, $v_3 \in U$, and so forth. In particular, $v_i \in U$ for every odd integer i with $1 \leq i \leq k$ and $v_j \in W$ for every even integer j with $2 \leq j \leq k$. Since $v_1 \in U$, it follows that $v_k \in W$ and so k is even.

For the converse, let G be a nontrivial graph containing no odd cycles. If G is empty, then G is clearly bipartite. Hence it suffices to show that every nontrivial component of G is bipartite and so we may assume that G itself is connected. Let u be a vertex of G and let

$$U = \{x \in V(G) : \quad d(u,x) \text{ is even}\} \text{ and } W = \{x \in V(G) : \quad d(u,x) \text{ is odd}\}$$

where $u \in U$ then. We show that G is bipartite with partite sets U and W. It remains to show that no two vertices of U are adjacent and no two vertices of W are adjacent. Suppose that W contains two adjacent vertices w_1 and w_2. Let P_1 be a $u - w_1$ geodesic and P_2 a $u - w_2$ geodesic. Let z be the last vertex that P_1 and P_2 have in common (possibly $z = u$). Then the length of the $z - w_1$ subpath P_1' of P_1 and the length of the $z - w_2$ subpath P_2' of P_2 are of the same parity. Thus the paths P_1' and P_2' together with the edge w_1w_2 produce an odd cycle. This is a contradiction. The argument that no two vertices of U are adjacent is similar. ■

The **eccentricity** $e(v)$ of a vertex v in a connected graph G is the distance between v and a vertex farthest from v in G. The **diameter** $\text{diam}(G)$ of G is the greatest eccentricity among the vertices of G, while the **radius** $\text{rad}(G)$ is the smallest eccentricity among the vertices of G. The diameter of G is therefore the greatest distance between any two vertices of G. A vertex v with $e(v) = \text{rad}(G)$ is called a **central vertex** of G and a vertex v with $e(v) = \text{diam}(G)$ is called a **peripheral vertex** of G. Two vertices u and v of G with $d(u,v) = \text{diam}(G)$ are **antipodal vertices** of G. Necessarily, if u and v are antipodal vertices in G, then each of u and v is a peripheral vertex. For the graph G of Figure 1.39,

$$e(v_6) = 2,\ e(v_2) = e(v_3) = e(v_4) = e(v_5) = e(v_7) = e(v_9) = e(v_{10}) = 3,$$
$$e(v_1) = e(v_8) = 4$$

and so $\mathrm{diam}(G) = 4$ and $\mathrm{rad}(G) = 2$. In particular, v_6 is the only central vertex of G and v_1 and v_8 are the only peripheral vertices of G. Since $d(v_1, v_8) = 4 = \mathrm{diam}(G)$, it follows that v_1 and v_8 are antipodal vertices of G. It is certainly not always the case that $\mathrm{diam}(G) = 2\,\mathrm{rad}(G)$ as, for example, $\mathrm{diam}(P_4) = 3$ and $\mathrm{rad}(P_4) = 2$. Indeed, the following can be said about the radius and diameter of a connected graph.

Theorem 1.19 *For every nontrivial connected graph G,*

$$\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G).$$

Proof. The inequality $\mathrm{rad}(G) \leq \mathrm{diam}(G)$ is immediate from the definitions. Let u and w be two vertices such that $d(u, w) = \mathrm{diam}(G)$ and let v be a central vertex of G. Therefore, $e(v) = \mathrm{rad}(G)$. By the triangle inequality,

$$\mathrm{diam}(G) = d(u, w) \leq d(u, v) + d(v, w) \leq 2e(v) = 2\,\mathrm{rad}(G),$$

as desired. ■

Theorem 1.19 gives a lower bound (namely, $\mathrm{rad}(G)$) for the diameter of a connected graph G as well as an upper bound (namely, $2\,\mathrm{rad}(G)$). This is one of many results for which a question of "sharpness" is involved. These involve the question: Just how good is this result? Ordinarily, there are many interpretations of such a question. We shall consider some possible interpretations in the case of the upper bound.

Certainly, the upper bound in Theorem 1.19 would not be considered sharp if $\mathrm{diam}(G) < 2\,\mathrm{rad}(G)$ for every graph G; however, it would be considered sharp indeed if $\mathrm{diam}(G) = 2\,\mathrm{rad}(G)$ for every graph G. In the latter case, we would have a formula or an identity, not just a bound. Actually, there are graphs G for which $\mathrm{diam}(G) < 2\,\mathrm{rad}(G)$ and there are graphs H for which $\mathrm{diam}(H) = 2\,\mathrm{rad}(H)$. This alone may be a satisfactory definition of sharpness. A more likely interpretation is the existence of an infinite class $\mathcal{H}$ of graphs such that $\mathrm{diam}(H) = 2\,\mathrm{rad}(H)$ for each graph H belonging to $\mathcal{H}$. Such a class exists; for example, let $\mathcal{H}$ consist of the graphs of the type $K_t \vee \overline{K}_2$. One disadvantage of this example is that for each $H \in \mathcal{H}$, $\mathrm{diam}(H) = 2$ and $\mathrm{rad}(H) = 1$. Perhaps a more satisfactory class (which fills a more satisfactory requirement for sharpness) is the class of paths P_{2k+1}, $k \geq 1$. In this case, $\mathrm{diam}(P_{2k+1}) = 2k$ and $\mathrm{rad}(P_{2k+1}) = k$; that is, for each positive integer k, there exists a connected graph G such that $\mathrm{diam}(G) = 2\,\mathrm{rad}(G) = 2k$.

Center and Periphery

The subgraph induced by the central vertices of a connected graph G is the **center** of G and is denoted by $\mathrm{Cen}(G)$. If every vertex of G is a central

vertex, then $\mathrm{Cen}(G) = G$ and G is **self-centered**. The subgraph induced by the peripheral vertices of a connected graph G is the **periphery** of G and is denoted by $\mathrm{Per}(G)$.

For the graph G of Figure 1.39, the center of G consists of the isolated vertex v_6 and the periphery consists of the two isolated vertices v_1 and v_8. The graph H of Figure 1.40 has radius 2 and diameter 3. Therefore, every vertex of H is either a central vertex or a peripheral vertex. Indeed, the center of H is the triangle induced by the three "exterior" vertices of H, while the periphery of H is the 6-cycle induced by the six "interior" vertices of H.

Figure 1.40: The center and periphery of a graph

It is not difficult to see that $\mathrm{Cen}(P_{2k+1}) = K_1$ and $\mathrm{Cen}(P_{2k}) = K_2$ for all $k \geq 1$. Also $\mathrm{Cen}(C_n) = C_n$ for all $n \geq 3$. Hence there are many graphs that are centers of graphs. In an observation first made by Stephen Hedetniemi (see [31]), there is no restriction on which graphs can be the center of some graph.

Theorem 1.20 *Every graph is the center of some graph.*

Proof. Let G be a graph. We construct a graph H from G by first adding two new vertices u and v to G and joining them to every vertex of G but not to each other. The construction of H is completed by adding two other vertices u_1 and v_1, where u_1 is joined to u and v_1 is joined to v. (This construction is illustrated in Figure 1.41.) Since $e(u_1) = e(v_1) = 4$, $e(u) = e(v) = 3$ and $e_H(x) = 2$ for every vertex x in G, it follows that $V(G)$ is the set of central vertices of H and so $\mathrm{Cen}(H) = H[V(G)] = G$. ■

Figure 1.41: A graph with a given center

Example 1.21

Figure 1.42 shows a graph G that represents the street system of a community, where each edge is a street and each vertex is an intersection denoted by s_i $(1 \leq i \leq 20)$. The community wants to build an emergency facility at one of the intersections so that the number of blocks needed to drive from the facility to the intersection farthest from it will be as small as possible. What are the possible locations for the emergency facility?

Figure 1.42: A graph representing a street system in Example 1.21

To answer this question, we need to place an emergency facility at a central vertex of the graph G. Consequently, the eccentricity of each vertex must be computed. These are shown in Figure 1.42. Since the minimum eccentricity (radius) of G is 4, the emergency facility should be placed at s_4, s_7, s_{10} or s_{14}. ■

Not every graph is the periphery of some graph however, as Halina Bielak and Maciej Syslo [19] showed.

Theorem 1.22 *A nontrivial graph G is the periphery of some graph if and only if every vertex of G has eccentricity 1 or no vertex of G has eccentricity 1.*

Proof. If every vertex of G has eccentricity 1, then G is complete and $\mathrm{Per}(G) = G$; while if no vertex of G has eccentricity 1, then let F be the graph obtained from G by adding a new vertex w and joining w to each vertex of G. Since $e_F(w) = 1$ and $e_F(x) = 2$ for every vertex x of G, it follows that every vertex of G is a peripheral vertex of F and so $\mathrm{Per}(F) = F[V(G)] = G$.

For the converse, let G be a graph that contains some vertices of eccentricity 1 and some vertices whose eccentricity is not 1 and suppose that there exists a graph H such that $\mathrm{Per}(H) = G$. Necessarily, G is a proper induced connected subgraph of H. Thus $\mathrm{diam}(H) = k \geq 2$. Furthermore, $e_H(v) = k \geq 2$ for each $v \in V(G)$ and $e_H(v) < k$ for $v \in V(H) - V(G)$. Let u be a vertex of G such that $e_G(u) = 1$ and let w be a vertex of H such that $d(u, w) = e_H(u) = k \geq 2$. Since w is not adjacent to u, it follows that $w \notin V(G)$. On the other hand, $d(u, w) = k$ and so $e_H(w) = k$. This implies that w is a peripheral vertex of H and so $w \in V(G)$, which is impossible. ■

Exercises for Section 1.3

1. Show that for every two vertices u and v in a connected graph G, there exists a $u - v$ walk containing all vertices of G.

2. Give an example of a cubic graph of order 10 containing a k-cycle for each k with $3 \leq k \leq 10$.

3. Let G_1, G_2 and G_3 be three graphs of order n and size m having adjacency matrices A_1, A_2 and A_3, respectively.

   (a) Prove or disprove: If $A_1 = A_2$, then $G_1 \cong G_2$.

   (b) Prove or disprove: If $A_2 \neq A_3$, then $G_2 \ncong G_3$.

4. (a) Use Theorem 1.17 to compute A^4 if $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ without multiplying matrices.

   (b) Show that A^4 could be computed more easily by using Theorem 1.17 to first compute A^2.

5. Determine the adjacency matrix of the graph G_1 of Figure 1.43. Then determine A^2 and A^3 without multiplying matrices.

Figure 1.43: Graphs G_1 and G_2 in Exercises 5 and 6

6. Determine the adjacency matrix of the graph G_2 of Figure 1.43. Then determine A^2, A^3 and A^4 without multiplying matrices.

7. Determine the graph G with adjacency matrix A for which

$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } A^3 = \begin{bmatrix} 2 & 2 & 3 & 1 & 1 \\ 2 & 2 & 3 & 1 & 1 \\ 3 & 3 & 2 & 4 & 0 \\ 1 & 1 & 4 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix}.$$

8. For an $n \times n$ matrix A, it is common to define A^0 as $\mathbf{I}_n$ (the $n \times n$ identity matrix). If this is done, how is this related to Theorem 1.17?

9. Prove that "is connected to" is an equivalence relation on the vertex set of a graph.

10. (a) Let G be a graph of order n. Prove that if $\deg u + \deg v \geq n - 1$ for every two nonadjacent vertices u and v of G, then G is connected and $\operatorname{diam}(G) \leq 2$.

    (b) Examine the sharpness of the bound in (a).

11. Let $n \geq 2$ be an integer. Determine the minimum positive integer m such that *every* graph of order n and size m is connected.

12. Prove that a graph G is connected if and only if for every partition $\{V_1, V_2\}$ of $V(G)$, there exists an edge of G joining a vertex of V_1 and a vertex of V_2.

13. Prove that if G is a graph with $\delta(G) \geq 2$, then G contains a cycle.

14. Prove that every graph G has a path of length $\delta(G)$.

15. Let G be a connected graph of order n and let k be an integer such that $2 \leq k \leq n - 1$. Show that if $\deg u + \deg v \geq k$ for every pair u, v of nonadjacent vertices of G, then G contains a path of length k.

16. Let G be a disconnected graph of order $n \geq 6$ having three components. Prove that

$$\Delta(\overline{G}) \geq \frac{2n}{3}.$$

17. (a) Show that there are exactly two 4-regular graphs G of order 7. (Hint: Consider $\overline{G}$.)

    (b) How many 6-regular graphs of order 9 are there?

18. Characterize those graphs G having the property that every induced subgraph of G is a connected subgraph of G.

19. (a) Show that if G is a connected graph such that the degree of every vertex is one of three distinct numbers and each of these three numbers is the degree of at least one vertex of G, then there is a path in G containing three vertices whose degrees are distinct.

    (b) Is the statement in (a) true if "three" is replaced by "four"?

20. Let G be a nontrivial connected graph that is not bipartite. Show that G contains two adjacent vertices u and v such that $\deg u + \deg v$ is even.

21. Let k and n be integers with $2 \leq k < n$ and let G be a graph of order n. Prove that if every vertex of G has degree exceeding $(n-k)/k$, then G has fewer than k components.

22. Let $k \geq 2$ be an integer. Prove that if G is a graph of order $n \geq k+1$ and size $m \geq (k-1)(n-k-1) + \binom{k+1}{2}$, then G contains a subgraph having minimum degree k.

23. Prove that if G is a connected graph of order $n \geq 2$, then the vertices of G can be listed as $v_1, v_2, \ldots, v_n$ such that each vertex v_i $(2 \leq i \leq n)$ is adjacent to some vertex in the set $\{v_1, v_2, \ldots, v_{i-1}\}$.

24. Prove that a nontrivial graph G is bipartite if and only if G contains no induced odd cycle.

25. Let G be a connected graph of order n. For a vertex v of G and an integer k with $1 \leq k \leq n-1$, let $d_k(v)$ be the number of vertices at distance k from v.

    (a) What is $d_1(v)$?

    (b) Show that $\sum_{v \in V(G)} d_k(v)$ is even for every integer k with $1 \leq k \leq n-1$.

    (c) What is the value of $\sum_{v \in V(G)} \left(\sum_{k=1}^{n-1} d_k(v) \right)$?

26. Prove that if G is a disconnected graph, then $\overline{G}$ is connected and, in fact, $\mathrm{diam}(\overline{G}) \leq 2$.

27. Let a and b be positive integers with $a \leq b \leq 2a$. Show that there exists a connected graph G with $\mathrm{rad}(G) = a$ and $\mathrm{diam}(G) = b$.

28. Let G be a nontrivial connected graph.

    (a) Show that if u and v are adjacent vertices of G, then $|e(u) - e(v)| \leq 1$.

    (b) Show that if k is an integer with $\mathrm{rad}(G) \leq k \leq \mathrm{diam}(G)$, then there is a vertex w such that $e(w) = k$.

    (c) Show that if k is an integer with $\mathrm{rad}(G) < k \leq \mathrm{diam}(G)$, then there are at least two vertices of G with eccentricity k. (Hint: Let w be a vertex with $e(w) = k$ and let u be a vertex with $d(w,u) = e(w) = k$. For a central vertex v of G, let P be a $u-v$ path of length $d(u,v)$. Show that $e(v) < k \leq e(u)$. Then show that there is a vertex x (distinct from w) on P such that $e(x) = k$.)

29. Show that for every pair r, s of positive integers, there exists a positive integer n such that for every connected graph G of order n, either $\Delta(G) \geq r$ or $\mathrm{diam}(G) \geq s$.

30. Every complete graph is the periphery of itself. Can a complete graph be the periphery of a connected graph G with $\mathrm{diam}(G) \geq 2$?

31. A vertex w of a connected graph G is called an **eccentric vertex** of a vertex v of G if $d(v, w) = e(v)$. Show that there exists a connected graph containing four distinct vertices v_1, v_2, v_3, v_4 such that v_{i+1} is an eccentric vertex of v_i for $i = 1, 2, 3$ and the integers $e(v_1), e(v_2)$ and $e(v_3)$ are distinct.

32. The **total distance** $\mathrm{td}(u)$ of a vertex u in a connected graph G is defined by
$$\mathrm{td}(u) = \sum_{v \in V(G)} d(u, v).$$
A vertex v in G is called a **median vertex** if v has the minimum total distance among the vertices of G. Equivalently, v is a median vertex if v has the minimum average distance to all vertices of G. The **median** $\mathrm{Med}(G)$ of G is the the subgraph of G induced by its median vertices. Determine (i) the total distance $\mathrm{td}(u)$ for each vertex u of the graph G of Figure 1.44 and (ii) the median $\mathrm{Med}(G)$ of G.

G :

Figure 1.44: The graph in Exercise 32

33. Let F and H be two subgraphs in a connected graph G. Define the **distance** $d(F, H)$ between F and H as
$$d(F, H) = \min\{d(u, v) : u \in V(F), v \in V(H)\}.$$
Show that for every positive integer k, there exists a connected graph G such that $d(\mathrm{Cen}(G), \mathrm{Med}(G)) = k$ (see Exercise 32 for the definition of $\mathrm{Med}(G)$).

34. Let G and its complement $\overline{G}$ both be connected graphs of order $n \geq 5$.

    (a) Prove that if the diameter of G is at least 3, then the diameter of its complement is at most 3.

    (b) What diameters are possible for self-complementary graphs with at least 4 vertices?

    (c) If the diameter of G is 2, what are the possible diameters of its complement $\overline{G}$?

35. For a graphical sequence s, let $\mathcal{G}_s$ be the set of all graphs with degree sequence s. For $G, H \in \mathcal{G}_s$, define the **distance** $d(G, H)$ from G to H as the minimum number of 2-switches required to transform G into H. Show that $(\mathcal{G}_s, d)$ is a metric space.

36. For two vertices u and v in a connected graph G, a $u - v$ **detour** is a longest $u - v$ path in G and the length of a $u - v$ detour is the **detour distance** $D(u, v)$ between u and v. Show that $(V(G), D)$ is a metric space.

37. Show that if G is a connected graph that is not complete, then G contains two vertices u and v such that $d(u, v) = 2$.

38. Let G be a nonempty graph with the property that whenever $uv \notin E(G)$ and $vw \notin E(G)$, then $uw \notin E(G)$. Prove that G has this property if and only if G is a complete k-partite graph for some $k \geq 2$. (Consider $\overline{G}$.)

1.4 Multigraphs and Digraphs

There are occasions when a graph is not the appropriate structure to model a particular situation. For example, suppose that we are considering various locations in a certain community and there are roads between some pairs of locations that do not pass through any other location. Although this situation may be represented by a graph, there may be some characteristics in this network of roads that are not captured by a graph. For example, suppose that there are pairs of locations connected by two or more roads (not passing through any other location) and this information is important to us. Or perhaps the roads between locations are all one-way streets and this too is important to us. This leads us to different structures that are similar to but not identical to graphs.

Multigraphs

In the definition of a graph G, every two distinct vertices are joined by either one edge or no edge of G. There will be occasions when we will want to permit more than one edge to join two vertices. A **multigraph** is a nonempty set of vertices, every two of which are joined by a finite number of edges. Hence a multigraph H may be expressed as $H = (V, E)$, where E is a multiset of 2-element subsets of V. Two or more edges that join the same pair of distinct vertices are called **parallel edges**. The **underlying graph** of a multigraph H is that graph G for which $V(G) = V(H)$ and $uv \in E(G)$ if u and v are joined by at least one edge in H.

An edge joining a vertex to itself is called a **loop**. Structures that permit both parallel edges and loops (including parallel loops) are often called **pseudographs**. For emphasis then, every two vertices of a graph are joined by at most

one edge and loops are not permitted. In a multigraph, every two vertices are permitted to be joined by more than one edge but this is not required. Also, no multigraph contains a loop. In a pseudograph, every two vertices are permitted to be joined by more than one edge and loops are permitted. However, parallel edges and loops are not required in pseudographs. There are authors who refer to multigraphs or pseudographs as graphs and those who refer to what we call graphs as **simple graphs**. Consequently, when reading any material written on graph theory, it is essential that there is a clear understanding of how the term *graph* is being used. According to the terminology introduced here then, every multigraph is a pseudograph and every graph is both a multigraph and a pseudograph.

In Figure 1.45, H_1 and H_4 are multigraphs while H_2 and H_3 are pseudographs. Of course, H_1 and H_4 are also pseudographs while H_4 is the only graph in Figure 1.45. For a vertex v in a multigraph G, the **degree** $\deg v$ of v in G is the number of edges of G incident with v. In a pseudograph, each loop at a vertex contributes 2 to its degree. For the pseudograph H_3 of Figure 1.45, $\deg u = 5$ and $\deg v = 2$.

Figure 1.45: Multigraphs and pseudographs

When describing walks in multigraphs or in pseudographs, it is often necessary to list edges in the sequence as well as vertices in order to specify the edges being used in the walk. For example,

$$W = (u, e_1, u, v, e_6, w, e_6, v, e_7, w)$$

is a $u - w$ walk in the pseudograph G of Figure 1.46.

Figure 1.46: Walks in a pseudograph

Digraphs

A **directed graph** or **digraph** D is a finite nonempty set of objects called **vertices** together with a (possibly empty) set of ordered pairs of distinct vertices of D called **arcs** or **directed edges**. As with graphs, the vertex set of D is denoted by $V(D)$ and the arc set (or directed edge set) of D is denoted by $E(D)$. A digraph D with vertex set $V = \{u, v, w, x\}$ and arc set $E = \{(u, v), (v, u), (u, w), (w, v), (w, x)\}$ is shown in Figure 1.47. Observe that when a digraph is described by means of a diagram, the "direction" of each arc is indicated by an arrowhead.

Figure 1.47: A digraph

Much of the terminology used for digraphs is quite similar to that used for graphs. The cardinality of the vertex set of a digraph D is called the **order** of D and is ordinarily denoted by n, while the cardinality of its arc set is the **size** of D and is ordinarily denoted by m. If $a = (u, v)$ is an arc of a digraph D, then a is said to **join** u and v. The vertex u is said to be **adjacent to** v and v is **adjacent from** u. For a vertex v in a digraph D, the **outdegree** $\operatorname{od} v$ of v is the number of vertices of D to which v is adjacent, while the **indegree** $\operatorname{id} v$ of v is the number of vertices of D from which v is adjacent. The **out-neighborhood** $N^+(v)$ of a vertex v in a digraph D is the set of vertices adjacent from v, while the **in-neighborhood** $N^-(v)$ of v is the set of vertices adjacent to v. Thus $\operatorname{od} v = |N^+(v)|$ and $\operatorname{id} v = |N^-(v)|$. The **degree** $\deg v$ of a vertex v is defined by

$$\deg v = \operatorname{od} v + \operatorname{id} v.$$

For the vertex v in the digraph of Figure 1.48, $\operatorname{od} v = 3$, $\operatorname{id} v = 2$ and $\deg v = 5$.

Figure 1.48: The outdegree, indegree and degree of a vertex

The directed graph version of Theorem 1.4 is stated next.

Theorem 1.23 (**The First Theorem of Digraph Theory**) *If D is a digraph of size m, then*

$$\sum_{v \in V(G)} \operatorname{od} v = \sum_{v \in V(G)} \operatorname{id} v = m.$$

Proof. When the outdegrees of the vertices are summed, each arc is counted once. Similarly, when the indegrees of the vertices are summed, each arc is counted just once. ■

A digraph D_1 is **isomorphic** to a digraph D_2, written $D_1 \cong D_2$, if there exists a bijective function $\phi : V(D_1) \to V(D_2)$ such that $(u, v) \in E(D_1)$ if and only if $(\phi(u), \phi(v)) \in E(D_2)$. The function ϕ is called an **isomorphism** from D_1 to D_2.

There is only one digraph of order 1, namely the **trivial digraph**. Also, there is only one digraph of order 2 and size m for each m with $0 \leq m \leq 2$. There are four digraphs of order 3 and size 3, all of which are shown in Figure 1.49.

Figure 1.49: The digraphs of order 3 and size 3

A digraph D_1 is a **subdigraph** of a digraph D if $V(D_1) \subseteq V(D)$ and $E(D_1) \subseteq E(D)$. We use $D_1 \subseteq D$ to indicate that D_1 is a subdigraph of D. A subdigraph D_1 of D is a **spanning subdigraph** of D if $V(D_1) = V(D)$. Vertex-deleted, arc-deleted, induced and arc-induced subdigraphs are defined in the expected manner. These last two concepts are illustrated for the digraph D of Figure 1.50, where

$$V(D) = \{v_1, v_2, v_3, v_4\},\ U = \{v_1, v_2, v_3\}, \text{ and } X = \{(v_1, v_2), (v_2, v_4)\}.$$

Figure 1.50: Induced and arc-induced subdigraphs

We now consider certain types of digraphs that occur periodically. A digraph is **symmetric** if whenever (u, v) is an arc of D, then (v, u) is an arc of D as well. There is a natural one-to-one corresponding between the set of symmetric digraphs and the set of graphs. The **complete symmetric digraph** K_n^* of order n has both arcs (u, v) and (v, u) for every two distinct vertices u and v. A digraph is called an **oriented graph** if whenever (u, v) is an arc of D, then (v, u) is *not* an arc of D. Thus an oriented graph D can be obtained from a *graph* G by assigning a direction to (or by "orienting") each edge of G, thereby transforming each edge of a graph G into an arc and transforming G itself into an oriented graph. The digraph D is also called an **orientation** of G. Figure 1.51 shows three digraphs D_1, D_2 and D_3. While D_1 is a symmetric digraph and D_2 is an oriented graph, the digraph D_3 is neither. The **underlying graph** of a digraph D is that graph obtained by replacing each arc (u, v) or symmetric pair (u, v), (v, u) of arcs by the edge uv. The underlying graph of each digraph in Figure 1.51 is the graph G.

D_1 D_2 D_3 G

Figure 1.51: Digraphs with the same underlying graph

An orientation of a complete graph is called a **tournament** and will be studied in some detail in Chapter 4. A digraph D is **regular of degree** r or r**-regular** if $\operatorname{od} v = \operatorname{id} v = r$ for every vertex v of D. A 1-regular digraph D_1 and a 2-regular digraph D_2 are shown in Figure 1.52. The digraph D_2 is a tournament.

D_1 :

D_2 :

Figure 1.52: Regular digraphs

The terms walk, open and closed walk, trail, path, circuit and cycle for graphs have natural counterparts in digraph theory as well, the important difference being that the directions of the arcs must be followed in each of these

walks. In particular, when referring to digraphs, the terms **directed path**, **directed cycle** and **directed circuit** are synonymous with the terms **path**, **cycle** and **circuit**. More formally, for vertices u and v in a digraph D, a **directed** $u-v$ **walk** (or simply a $u-v$ **walk**) in D is a finite sequence

$$(u = u_0, u_1, u_2, \ldots, u_k = v)$$

of vertices, beginning with u and ending with v such that (u_i, u_{i+1}) is an arc for $0 \leq i \leq k-1$. The number k of occurrences of arcs (including repetition) in the walk is its **length**. Digraphs in which every vertex has positive outdegree must contain cycles (see Exercise 14).

Theorem 1.24 *If D is a digraph such that* $\operatorname{od} v \geq k \geq 1$ *for every vertex v of D, then D contains a cycle of length at least $k+1$.*

Connected Digraphs

A digraph D is **connected** (or **weakly connected**) if the underlying graph of D is connected. A digraph D is **strong** (or **strongly connected**) if for every pair u, v of vertices, D contains both a $u-v$ path and a $v-u$ path. While all digraphs of Figure 1.53 are connected, only D_1 is strong.

Figure 1.53: Connectedness properties of digraphs

Distance can be defined in digraphs as well. For vertices u and v in a digraph D containing a $u-v$ path, the (**directed**) **distance** $\vec{d}(u,v)$ from u to v is the length of a shortest $u-v$ path in D. Thus the distances $\vec{d}(u,v)$ and $\vec{d}(v,u)$ are defined for all pairs u, v of vertices in a digraph D if and only if D is strong. This distance is not a metric, in general. Although directed distance satisfies the triangle inequality, it is not symmetric unless D is symmetric, in which case D can be considered a graph. Eccentricity can be defined as before, as well as radius and diameter in a strong digraph D. The **eccentricity** $e(u)$ of a vertex u in D is the distance from u to a vertex farthest from u. The minimum eccentricity of the vertices of D is the **radius** $\operatorname{rad}(D)$ of D, while the **diameter** $\operatorname{diam}(D)$ is the greatest eccentricity.

Figure 1.54: Eccentricities in a strong digraph

Each vertex of the strong digraph D of Figure 1.54 is labeled with its eccentricity. Observe that $\mathrm{rad}(D) = 2$ and $\mathrm{diam}(D) = 5$, so it is not true, in general, that $\mathrm{diam}(D) \leq 2\,\mathrm{rad}(D)$, as is the case with graphs.

As with multigraphs or pseudographs, if more than one arc in the same direction is permitted to join two vertices in a digraph, then these arcs are **parallel arcs** and a **multidigraph** results. A **directed loop** in a digraph is an arc that joins a vertex to itself. In a **pseudodigraph**, parallel arcs and directed loops (including parallel directed loops) are permitted. A multidigraph D_1 and a pseudodigraph D_2 are shown in Figure 1.55.

Figure 1.55: A multidigraph and a pseudodigraph

Exercises for Section 1.4

1. Show that for every connected graph G of order 3 or more, there exists a multigraph H whose underlying graph is G and where distinct vertices of H have distinct degrees.

2. Prove for every connected graph G of order $n = 3$ or $n = 4$ and size m that it is possible to label the edges of G by $e_1, e_2, \ldots, e_m$ and replace e_i by i parallel edges for each i $(1 \leq i \leq m)$ such that the degrees of the vertices of the resulting multigraph H are distinct.

3. Determine which of the following sequences are the degree sequences of a multigraph.

(a) $s_1 : 3, 2, 1$ (b) $s_2 : 5, 2, 1$

(c) $s_3 : 6, 4, 2$ (d) $s_4 : 3, 2, 2$

(e) $s_5 : 4, 3, 2, 1$ (f) $s_6 : 5, 3, 2, 1$

(g) $s_7 : 4, 4, 4, 4$ (h) $s_8 : 7, 5, 3, 1$.

4. Prove that a sequence $s : d_1, d_2, \ldots, d_n$ $(n \geq 1)$ of nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$ is the degree sequence of a multigraph if and only if $\sum_{i=1}^{n} d_i$ is even and $d_1 \leq \frac{1}{2}\sum_{i=1}^{n} d_i$.

5. Let G be a connected graph of order n where the vertices of G are labeled as $v_1, v_2, \ldots, v_n$ in some way. A multigraph H of size m with $V(H) = V(G)$ is obtained by replacing each edge v_iv_j of G by $\min\{i, j\}$ parallel edges.

   (a) Find m if $G = K_5$.

   (b) Find sharp upper and lower bounds for m if $G = C_5$.

   (c) Find the minimum value of m if G is bipartite.

6. Determine all digraphs of order 4 and size 4.

7. Prove or disprove: For every integer $n \geq 2$, there exists a digraph D of order n such that for every two distinct vertices u and v of D, $\operatorname{od} u \neq \operatorname{od} v$ and $\operatorname{id} u \neq \operatorname{id} v$.

8. Prove or disprove: There exists a nontrivial digraph D in which no two vertices of D have the same outdegree but every two vertices of D have the same indegree.

9. Prove or disprove: No digraph contains an odd number of vertices of odd outdegree or an odd number of vertices of odd indegree.

10. Prove or disprove: If D_1 and D_2 are two digraphs with $V(D_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(D_2) = \{v_1, v_2, \ldots, v_n\}$ such that $\operatorname{id}_{D_1} u_i = \operatorname{id}_{D_2} v_i$ and $\operatorname{od}_{D_1} u_i = \operatorname{od}_{D_2} v_i$ for $i = 1, 2, \ldots, n$, then $D_1 \cong D_2$.

11. Prove that there exist regular tournaments of every odd order but there are no regular tournaments of even order.

12. Let T be a tournament with $V(T) = \{v_1, v_2, \ldots, v_n\}$. We know that

$$\sum_{i=1}^{n} \operatorname{od} v_i = \sum_{i=1}^{n} \operatorname{id} v_i = \binom{n}{2}.$$

   (a) Prove that $\sum_{i=1}^{n} (\operatorname{od} v_i)^2 = \sum_{i=1}^{n} (\operatorname{id} v_i)^2$.

   (b) Prove or disprove: $\sum_{i=1}^{n} (\operatorname{od} v_i)^3 = \sum_{i=1}^{n} (\operatorname{id} v_i)^3$.

13. The **adjacency matrix** $A(D)$ of a digraph D with $V(D) = \{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ matrix $[a_{ij}]$ defined by $a_{ij} = 1$ if $(v_i, v_j) \in E(D)$ and $a_{ij} = 0$ otherwise.

    (a) What information do the row sums and column sums of the adjacency matrix of a digraph provide?

    (b) Characterize matrices that are adjacency matrices of digraphs.

14. (a) Prove Theorem 1.24: *If D is a digraph such that* $\operatorname{od} v \geq k \geq 1$ *for every vertex v of D, then D contains a cycle of length at least $k+1$.*

    (b) Prove that if D is a digraph such that $\operatorname{id} v \geq k \geq 1$ for every vertex v of D, then D contains a cycle of length at least $k+1$.

15. Let G be a connected graph of order $n \geq 3$. Prove that there is an orientation of G in which no directed path has length 2 if and only if G is bipartite.

16. Prove that for every two positive integers a and b with $a \leq b$, there exists a strong digraph D with $\operatorname{rad}(D) = a$ and $\operatorname{diam}(D) = b$.

17. The **center** $\operatorname{Cen}(D)$ of a strong digraph D is the subdigraph induced by those vertices v with $e(v) = \operatorname{rad}(D)$. Prove that for every oriented graph D_1, there exists a strong oriented graph D such that $\operatorname{Cen}(D) = D_1$.

18. Prove that every digraph D contains a set S of vertices with the properties (1) no two vertices in S are adjacent in D and (2) for every vertex v of D not in S, there exists a vertex u in S such that $\vec{d}(u, v) \leq 2$.

19. Let S be a finite nonempty set of positive integers. The **divisor pseudodigraph** D determined by S has S as its vertex set and for two (not necessarily distinct) vertices u and v, (u, v) is a directed edge of D if $u \mid v$. The **divisor graph** G determined by S has S as its vertex set and uv $(u \neq v)$ is an edge of G if either $u \mid v$ or $v \mid u$.

    (a) Determine the divisor pseudodigraph for $S = \{1, 2, 3, 6\}$.

    (b) Prove or disprove: There exists a divisor graph G with $G \cong K_5$.

    (c) Prove or disprove: There exists a divisor graph G with $G \cong P_5$.

    (d) Prove or disprove: There exists a divisor graph G with $G \cong C_5$.

    (e) Prove or disprove: There exists a divisor graph G with $G \cong C_6$.

    (f) Prove or disprove: There exists a divisor graph G with $G \cong C_7$.

    (g) Prove or disprove: There exists a divisor graph G with $G \cong \overline{C}_7$.

Chapter 2

Trees and Connectivity

In nearly every concept, problem and theorem that we encounter, we are primarily concerned with connected graphs. There are various ways of measuring connectedness for graphs and in many instances, it will be important for us to know the degree of connectedness of the graphs being considered. While a graph may contain a $u-v$ path for every pair u, v of vertices and consequently is connected, a graph can satisfy this requirement in minimal ways or in much stronger ways.

2.1 Nonseparable Graphs

Some graphs are connected to such a small extent that the removal of a single vertex can result in a disconnected graph.

Cut-Vertices

A vertex v in a connected graph G is a **cut-vertex** if $G-v$ is disconnected. Therefore, if v is a cut-vertex of a connected graph G, then $G-v$ consists of components $G_1, G_2, \ldots, G_k$ for some $k \geq 2$. That is, $G-v = G_1+G_2+\cdots+G_k$. Necessarily, for every component G_i of $G-v$, at least one vertex of G_i is adjacent to v in G.

More generally, a vertex v is a **cut-vertex** of a graph G (connected or disconnected) if $k(G-v) > k(G)$. In the graph H_1 of Figure 2.1, the vertices u, v, w and x are cut-vertices; in H_2 every vertex of degree 4 is a cut-vertex; and in H_3 no vertex is a cut-vertex. For $n \geq 3$, no vertex of C_n is a cut-vertex, while at the other extreme, only two vertices of P_n are not cut-vertices, namely the two end-vertices of P_n. That this is the other extreme is verified in the following theorem.

Theorem 2.1 *Every nontrivial connected graph contains at least two vertices that are not cut-vertices.*

Figure 2.1: Cut-vertices in graphs

Proof. Let G be a nontrivial connected graph and let P be a longest path in G. Suppose that P is a $u-v$ path. We show that u and v are not cut-vertices. Assume, to the contrary, that u is a cut-vertex of G. Then $G-u$ is disconnected and so contains two or more components. Let w be the vertex adjacent to u on P and let P' be the $w-v$ subpath of P. Necessarily, P' belongs to a component, say G_1, of $G-u$. Let G_2 be another component of $G-u$. Then G_2 contains some vertex x that is adjacent to u. This produces an $x-v$ path that is longer than P, which is impossible. Similarly, v is not a cut-vertex of G. ■

If v is a cut-vertex of a graph G, then there exist paths in G that cannot avoid v.

Theorem 2.2 *A vertex v in a graph G is a cut-vertex of G if and only if there are two vertices u and w distinct from v such that v lies on every $u-w$ path in G.*

Proof. We may assume that G is connected, for otherwise we can consider a component of G containing v. If v is a cut-vertex in a connected graph G, then, of course, $G-v$ contains two or more components. If u and w are vertices in distinct components of $G-v$, then u and w are not connected in $G-v$. On the other hand, u and w are necessarily connected in G. Thus v lies on every $u-w$ path in G.

For the converse, suppose that there are two vertices u and w distinct from v such that v lies on every $u-w$ path in G. Then there is no $u-w$ path in $G-v$. Thus u and w are not connected in $G-v$, and so $G-v$ is disconnected. Therefore, v is a cut-vertex of G. ■

Thus some connected graphs may contain a vertex the removal of which separates the graph into two or more connected graphs. These vertices are, of courses, cut-vertices. As we saw, a connected graph need not contain any cut-vertices.

A nontrivial connected graph containing no cut-vertices is a **nonseparable graph**. In particular, the cycles C_n, $n \geq 3$, and the complete graphs K_n, $n \geq 2$, are nonseparable graphs. In fact, K_2 and K_3 are the only nonseparable graphs of order 3 or less. Not only do nonseparable graphs of order 3 or more contain cycles, they contain cycles possessing a rather interesting property.

Theorem 2.3 *Let G be a nonseparable graph of order 3 or more. Then every two vertices of G lie on a common cycle of G.*

Proof. Assume, to the contrary, that there are pairs of vertices of G that do not lie on a common cycle. Among all such pairs, let u, v be a pair for which $d(u, v)$ is minimum. If $d(u, v) = 1$, then $uv \in E(G)$. Since the order of G is 3 or more, either $\deg u \geq 2$ or $\deg v \geq 2$, say the former. Let w be a vertex different from v that is adjacent to u. Since $G - u$ is connected, $G - u$ contains a $w - v$ path P. Then the path P together with the path (w, u, v) produce a cycle containing u and v. Hence we may assume that $d(u, v) = k \geq 2$.

Let $P = (u = v_0, v_1, \ldots, v_{k-1}, v_k = v)$ be a $u - v$ geodesic in G. Since $d(u, v_{k-1}) = k-1 < k$, there is a cycle C containing u and v_{k-1}. By assumption, v is not on C. Since v_{k-1} is not a cut-vertex of G and u and v are distinct from v_{k-1}, it follows from Theorem 2.2 that there is a $v - u$ path Q that does not contain v_{k-1}. Since u is on C, there is a first vertex x of Q that is on C. Let Q' be the $v - x$ subpath of Q and let P' be a $v_{k-1} - x$ path on C that contains u. (If $x \neq u$, then the path P' is unique.) However, the cycle C' produced by proceeding from v to its neighbor v_{k-1}, along P' to x, and then along Q' to v contains both u and v, a contradiction. ■

This theorem has several consequences (see Exercises 1-3). For two distinct vertices u and v in a graph G, two $u - v$ paths are **internally disjoint** if they have only u and v in common.

Corollary 2.4 *A connected graph G of order 3 or more is nonseparable if and only if for every two distinct vertices u and v in G, there are two internally disjoint $u - v$ paths.*

Corollary 2.5 *Let u and w be two distinct vertices in a nonseparable graph G. If H is obtained from G by adding a new vertex v and joining v to u and w, then H is nonseparable.*

Corollary 2.6 *If U and W are disjoint sets of vertices in a nonseparable graph G of order 4 or more with $|U| = |W| = 2$, then G contains two disjoint paths connecting the vertices of U and the vertices of W.*

Blocks

Let G be a nontrivial connected graph. A **block** of G is a maximal nonseparable subgraph of G, that is, a block of G is a nonseparable subgraph of G that is not a proper subgraph of any nonseparable subgraph of G. Every two distinct blocks of G have at most one vertex in common; and if they have a vertex in common, then this vertex is a cut-vertex of G. A block of G containing exactly one cut-vertex of G is called an **end-block** of G. If B and B' are two blocks containing the cut-vertex v, while $u \in V(B)$ and $w \in V(B')$ for $u, w \neq v$, then

every $u-w$ path in G must contain the vertex v. This is basically Theorem 2.2. A graph G and its five blocks B_i, $1 \leq i \leq 5$, are shown in Figure 2.2. The end-blocks of G are B_1, B_2 and B_5. A connected graph with cut-vertices must contain two or more end-blocks.

Figure 2.2: The blocks of a graph

Theorem 2.7 *Every connected graph containing cut-vertices has at least two end-blocks.*

Proof. If G contains only one cut-vertex, then every block of G is an end-block. Hence we may assume that G contains two or more cut-vertices. Among all pairs of cut-vertices of G, let u, v be a pair for which $d(u, v)$ is maximum and let P be a $u - v$ geodesic, say

$$P = (u = u_0, u_1, \ldots, u_k = v), \text{ where } k \geq 1.$$

Then u_1 belongs to a block B and u_{k-1} belongs to a block B', where possibly $B = B'$. In fact, possibly $u_1 = v$. Since u is a cut-vertex of G, it follows that u belongs to one or more blocks different from B. Let B_0 be one of these. Similarly, let B_0' be a block different from B' that contains v. We claim that B_0 is an end-block of G. If B_0 is not an end-block, then B_0 contains a cut-vertex x different from u. Since every $x - u_1$ path must pass through u, it follows that

$$d(x, v) = d(x, u) + d(u, v) > d(u, v),$$

which is impossible. Similarly, B_0' is an end-block different from B_0. ■

The following result, which can be proved in a similar manner, is often useful as well.

Theorem 2.8 *Let G be a connected graph with at least one cut-vertex. Then G contains a cut-vertex v with the property that, with at most one exception, all blocks of G containing v are end-blocks.*

Proof. If G has only one cut-vertex, then every block of G is an end-block and contains the cut-vertex. Hence we may assume that G contains two or more cut-vertices. Among the cut-vertices of G, let u and v be two for which $d(u, v)$ is maximum and let $P = (u = u_0, u_1, \ldots, u_k = v)$, $k \geq 1$, be a $u - v$ geodesic. Then u_{k-1} belongs to a block B containing v. Let B' be a block containing v that is different from B. If B' is not an end-block, then B' contains a cut-vertex w different from v. Let P' be a $v - w$ geodesic in G. Then the path P followed by P' produces a $u - w$ geodesic whose length exceeds that of P. This is a contradiction. Thus every block containing v that is different from B is an end-block. ■

Another interesting property of blocks of graphs was observed by Frank Harary and Robert Z. Norman [113].

Theorem 2.9 *The center of every connected graph G lies in a single block of G.*

Proof. Suppose that G is a connected graph whose center $\mathrm{Cen}(G)$ does not lie within a single block of G. Then G has a cut-vertex v such that $G - v$ contains components G_1 and G_2, each of which contains vertices of $\mathrm{Cen}(G)$. Let u be a vertex such that $d(u, v) = e(v)$, and let P_1 be a $v - u$ geodesic. At least one of G_1 and G_2, say G_2, contains no vertices of P_1. Let w be a vertex of $\mathrm{Cen}(G)$ belonging to G_2, and let P_2 be a $w - v$ geodesic. The paths P_1 and P_2 together form a $u - w$ path P_3, which is necessarily a $u - w$ path of length $d(u, w)$. However, then $e(w) > e(v)$, which contradicts the fact that w is a central vertex. Thus $\mathrm{Cen}(G)$ lies in a single block of G. ■

If a graph G has components $G_1, G_2, \ldots, G_k$ and a nonempty connected graph H has blocks $B_1, B_2, \ldots, B_\ell$, then $\{V(G_1), V(G_2), \ldots, V(G_k)\}$ is a partition of $V(G)$ and $\{E(B_1), E(B_2), \ldots, E(B_\ell)\}$ is a partition of $E(H)$.

Suppose, for a cut-vertex v of a connected graph G, that the disconnected graph $G - v$ has k components $G_1, G_2, \ldots, G_k$ $(k \geq 2)$. The induced subgraphs

$$B_i = G[V(G_i) \cup \{v\}]$$

are connected and referred to as the **branches** of G at v. If a subgraph G_i contains no cut-vertices of G, then the branch B_i is a block of G, in fact, an end-block of G.

A connected graph G containing three cut-vertices u, v and w and six blocks is shown in Figure 2.3. Four of these blocks are end-blocks. The graph G has four branches at v, all of which are shown in Figure 2.3. Two of the four branches at v are end-blocks of G.

Figure 2.3: The four branches of a graph G at a cut-vertex v

Exercises for Section 2.1

1. Prove Corollary 2.4: *A connected graph G of order 3 or more is nonseparable if and only if for every two distinct vertices u and v in G, there are two internally disjoint $u - v$ paths.*

2. Prove Corollary 2.5: *Let u and w be two distinct vertices in a nonseparable graph G. If H is obtained from G by adding a new vertex v and joining v to u and w, then H is nonseparable.*

3. Prove Corollary 2.6: *If U and W are disjoint sets of vertices in a nonseparable graph G of order 4 or more with $|U| = |W| = 2$, then G contains two disjoint paths connecting the vertices of U and the vertices of W.*

4. (a) Let k be the maximum length of a cycle in a nonseparable graph G. Prove that if C and C' are any two k-cycles in G, then C and C' have at least two vertices in common.

   (b) Show that the result in (a) cannot be improved.

5. Prove that if v is a cut-vertex of a connected graph G, then v is *not* a cut-vertex of $\overline{G}$.

6. Let u and v be distinct vertices of a nonseparable graph G of order $n \geq 3$. If P is a given $u - v$ path of G, does there always exist a $u - v$ path Q such that P and Q are internally disjoint $u - v$ paths?

7. (a) An **element** of a graph G is a vertex or an edge of G. Prove that a connected graph G of order at least 3 is nonseparable if and only if every pair of elements of G lie on a common cycle of G.

(b) Let G and H be graphs with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{u_1, u_2, \ldots, u_n\}$, $n \geq 3$. Two vertices u_i and u_j are adjacent in H if and only if v_i and v_j belong to a common cycle in G. Characterize those graphs G for which H is complete.

8. Prove that if G is a graph of order $n \geq 3$ with the property that $\deg u + \deg v \geq n$ for every pair u, v of nonadjacent vertices of G, then G is nonseparable.

9. Prove or disprove: If B is a block of order 3 or more in a connected graph G, then there is a cycle in B that contains all the vertices of B.

10. A connected graph G contains k blocks and ℓ cut-vertices. What is the relationship between k and ℓ?

11. Prove or disprove: If G is a connected graph with cut-vertices and u and v are antipodal vertices of G, then no block of G contains both u and v.

12. (a) Show that for every positive integer k, there exists a connected graph G and a non-cut-vertex u of G such that $\text{rad}(G - u) = \text{rad}(G) + k$.

    (b) Prove for every nontrivial connected graph G and every non-cut-vertex v of G that $\text{rad}(G - v) \geq \text{rad}(G) - 1$.

    (c) Let G be a nontrivial connected graph with $\text{rad}(G) = r$. Among all connected induced subgraphs of G having radius r, let H be one of minimum order. Prove that $\text{rad}(H - v) = r - 1$ for every non-cut-vertex v of H.

2.2 Trees

We have seen that a connected graph may contain a vertex whose removal results in a disconnected graph and that such a vertex is a cut-vertex. There are also edges that possess this property.

Bridges

An edge $e = uv$ in a connected graph G whose removal results in a disconnected graph is a **bridge**. Necessarily, $G - e$ consists of two components, one containing u and the other containing v. More generally, an edge e is a bridge in a graph G (connected or not) if $k(G - e) > k(G)$. Then $k(G - e) = k(G) + 1$. In the graph G of Figure 2.4, v_5, v_6, v_7, v_8 and v_9 are cut-vertices, while v_4v_5, v_5v_6, v_7v_{11} and v_8v_9 are bridges.

While there are connected graphs no vertex of which is a cut-vertex (the nonseparable graphs), there is no connected graph in which every vertex is a cut-vertex (see Theorem 2.1). There are, however, connected graphs in which every edge is a bridge. It is this class of graphs that we study in the current section. First, we discuss an important property of bridges.

Figure 2.4: Bridges and cut-vertices in a graph

Theorem 2.10 *An edge e in a graph G is a bridge of G if and only if e lies on no cycle in G.*

Proof. We may assume that G is connected, for otherwise we can consider a component of G containing e.

First, suppose that $e = uv$ is an edge of G that is not a bridge. Since $G - e$ is connected, there is a $u - v$ path P in $G - e$. Then P together with e produces a cycle in G containing e.

For the converse, assume that $e = uv$ is an edge of G belonging to a cycle of G. Since e lies on a cycle of G, there is a $u - v$ path P' in G not containing e. We show that $G - e$ is connected and, consequently, that e is not a bridge. Let x and y be two vertices of G. Since G is connected, G contains an $x - y$ path Q. If e does not lie on Q, then Q is an $x - y$ path in $G - e$ as well. If, on the other hand, e lies on Q, then replacing e in Q by the $u - v$ path P' produces an $x - y$ walk in $G - e$. By Theorem 1.16, $G - e$ contains an $x - y$ path. Thus $G - e$ is connected. ■

We are now prepared to discuss one of the best known and most useful classes of graphs. An **acyclic graph** has no cycles. A **tree** is a connected acyclic graph. By Theorem 2.10, a tree is a connected graph, every edge of which is a bridge. Each graph in Figure 2.5 is a tree.

Figure 2.5: Three trees

Origin of Trees

Trees appeared implicitly in the 1847 work [133] of the German physicist Gustav Kirchhoff in his study of currents in electrical networks, while Arthur Cayley [34] used trees in 1857 to count certain types of chemical compounds. Trees are important to the understanding of the structure of graphs and are used to systematically visit the vertices of a graph. Trees are also widely used in computer science as a means to organize and utilize data.

The simplest organic chemical molecules are the *alkanes*. Alkanes are hydrocarbons and so their molecules consist only of carbon and hydrogen atoms, denoted by the symbols C and H, respectively. The valency of each carbon atom is 4 and of each hydrogen atom is 1, so if an alkane molecule has n carbon atoms, then there must be $2n + 2$ hydrogen atoms, producing the formula C_nH_{2n+2} for each alkane molecule. The five simplest alkanes along with their chemical formulas are shown in Figure 2.6. In earlier days, the degree of a vertex v in a graph was sometimes referred to as the *valency* of v.

Figure 2.6: The five simplest alkanes

All of the structures in Figure 2.6 are trees. Observe that both butane and isobutane have the same chemical formula C_4H_{10} but they have a different structure. (They are not isomorphic.) So it is possible for two different alkanes to have the same chemical formula. In fact, there are three different alkanes having the formula C_5H_{12} and 1, 117, 743, 651, 746, 953, 270 (over one quintillion) different alkanes having the formula $C_{50}H_{102}$.

Special Trees

There are several well-known classes of trees. For example, the paths P_n and stars $K_{1,n-1}$ are trees of order $n \geq 2$. For $t \geq 2$, only one vertex of a star is not a leaf; the vertex of degree t in $K_{1,t}$ is the **central vertex** of $K_{1,t}$. A tree containing exactly two vertices that are not leaves (which are necessarily adjacent) is called a **double star**. Thus a double star is a tree of diameter 3. A **caterpillar** is a tree T of order 3 or more, the removal of whose leaves produces a path (which is called the **spine** of T). Thus every path, every star (of order at least 3) and every double star is a caterpillar. Figure 2.7 shows four trees T_1, T_2, T_3 and T_4. The tree T_1 is a star, T_2 is a double star and T_3 is a caterpillar that is not a double star, while T_4 is not a caterpillar.

Figure 2.7: Four trees

Properties of Trees

Since every tree is connected, every two vertices are connected by a path. In fact, even more can be said.

Theorem 2.11 *A graph G is a tree if and only if every two vertices of G are connected by a unique path.*

Proof. First, suppose that G is a tree and that u and v are two vertices of G. Since G is connected, G contains at least one $u - v$ path. On the other hand, if G were to contain two $u - v$ paths, then G would contain a cycle, which is impossible. Therefore, G contains exactly one $u - v$ path.

Conversely, let G be a graph in which every two vertices are connected by a unique path. Certainly then, G is connected. If G were to contain a cycle C, then every two vertices on C would be connected by two paths. Thus G contains no cycle and so G is a tree. ■

While every vertex of degree 2 or more in a tree is a cut-vertex, the vertices of degree 1 (the leaves) are not. These observations provide a corollary of Theorem 2.1.

Corollary 2.12 *Every nontrivial tree contains at least two leaves.*

For a cut-vertex v of T, there are $\deg v$ branches of T at v. In the tree T of Figure 2.8, the four branches of T at v are shown in that figure. In the tree T of Figure 2.8 and in each of the trees T_1, T_2, T_3 and T_4 of Figure 2.7, the size of the tree is one less than its order. This is not only true of these trees, it is true of all trees.

Figure 2.8: The branches of a tree at a vertex

There are several ways of constructing a new tree from a given tree. For example, if v is a leaf in a tree T, then $T - v$ is also a tree. If a new vertex is added to T and joined to any vertex of T, then another tree results.

Theorem 2.13 *If T is a tree of order n and size m, then $m = n - 1$.*

Proof. We proceed by induction on the order of a tree. There is only one tree of order 1, namely K_1, and it has no edges. Thus the basis step of the induction is established. Assume that the size of every tree of order $n - 1 \geq 1$ is $n - 2$ and let T be a tree of order n and size m. By Corollary 2.12, T has at least two leaves. Let v be one of them. As we observed, $T - v$ is a tree of order $n - 1$. By the induction hypothesis, the size of $T - v$ is $n - 2$. Thus the size of T is $m = (n - 2) + 1 = n - 1$. ■

Suppose that a tree T of order $n \geq 3$, size m and maximum degree $\Delta(T) = \Delta$ has n_i vertices of degree i $(1 \leq i \leq \Delta)$. Then

$$\sum_{v \in V(T)} \deg v = \sum_{i=1}^{\Delta} i n_i = 2m = 2n - 2 = 2\sum_{i=1}^{\Delta} n_i - 2. \tag{2.1}$$

Solving (2.1) for n_1, we have the following.

Theorem 2.14 *Let T be a tree of order $n \geq 3$ having maximum degree Δ and containing n_i vertices of degree i $(1 \leq i \leq \Delta)$. Then the number n_1 of leaves of T is given by*

$$n_1 = 2 + n_3 + 2n_4 + \cdots + (\Delta - 2)n_\Delta.$$

If $s : d_1, d_2, \ldots, d_n$ is the degree sequence of a tree of order $n \geq 2$, then necessarily $\sum_{i=1}^{n} d_i = 2n - 2$. Here too, more can be said.

Theorem 2.15 *A sequence $s : d_1, d_2, \ldots, d_n$ of $n \geq 2$ positive integers is the degree sequence of a tree of order n if and only if*

$$\sum_{i=1}^{n} d_i = 2n - 2.$$

Proof. First, let T be a tree of order n and size m with degree sequence $s : d_1, d_2, \ldots, d_n$. Then, as observed,

$$\sum_{i=1}^{n} d_i = 2m = 2(n-1) = 2n - 2.$$

We verify the converse by induction on n. When $n = 2$, the only sequence of two positive integers whose sum is $2n - 2 = 2$ is $1, 1$, which is the degree sequence of the tree K_2.

Assume for a given integer $n \geq 3$ that any sequence of $n-1$ positive integers whose sum is $2(n-2) = 2n-4$ is the degree sequence of a tree of order $n-1$. Now let $d_1, d_2, \ldots, d_n$ be a sequence of n positive integers whose sum is $2n-2$. We show that this is the degree sequence of a tree of order n. Suppose that $d_1 \geq d_2 \geq \cdots \geq d_n$. Since each term is positive and $\sum_{i=1}^{n} d_i = 2n-2$, it follows that $d_{n-1} = d_n = 1$ and that $2 \leq d_1 \leq n-1$. Hence $d_1 - 1, d_2, \ldots, d_{n-1}$ is a sequence of $n-1$ positive integers whose sum is $2(n-1) - 2 = 2n-4$. By the induction hypothesis, there is a tree T' of order $n-1$ with $V(T') = \{v_1, v_2, \ldots, v_{n-1}\}$ such that $\deg_{T'} v_1 = d_1 - 1$ and $\deg_{T'} v_i = d_i$ for $2 \leq i \leq n-1$. Let T be the tree of order n obtained from T' by adding a new vertex v_n and joining it to v_1. The tree T then has the degree sequence $d_1, d_2, \ldots, d_n$. ■

A graph without cycles is a **forest**. That is, a forest is an acyclic graph. Thus each tree is a forest and every component of a forest is a tree. All of the graphs F_1, F_2 and F_3 in Figure 2.9 are forests but none are trees.

Figure 2.9: Forests

The following is an immediate corollary of Theorem 2.13.

Corollary 2.16 *The size of a forest of order n having k components is $n - k$.*

By Theorem 2.13, if G is a graph of order n and size m such that G is connected and has no cycles (that is, G is a tree), then $m = n - 1$. It is easy to see that the converse of this statement is not true. However, if we were to add to the hypothesis of the converse either of the two defining properties of a tree, then the converse would be true.

Theorem 2.17 *Let G be a graph of order n and size m. If G has no cycles and $m = n - 1$, then G is a tree.*

Proof. It remains only to show that G is connected. Suppose that the components of G are $G_1, G_2, \ldots, G_k$, where $k \geq 1$. Let n_i be the order of G_i $(1 \leq i \leq k)$ and m_i the size of G_i. Since each graph G_i is a tree, it follows by Theorem 2.13 that $m_i = n_i - 1$ and by Corollary 2.16 that $m = n - k$. Hence

$$n - 1 = m = \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} (n_i - 1) = n - k.$$

Thus $k = 1$ and so G is connected. Therefore, G is a tree. ■

Theorem 2.18 *Let G be a graph of order n and size m. If G is connected and $m = n - 1$, then G is a tree.*

Proof. Assume, to the contrary, that there exists some connected graph of order n and size $m = n - 1$ that is not a tree. Necessarily then, G contains one or more cycles. By successively deleting an edge from a cycle in each resulting subgraph, a tree of order n and size less than $n - 1$ is obtained. This contradicts Theorem 2.13. ■

Combining Theorems 2.13, 2.17 and 2.18, we have the following.

Theorem 2.19 *Let G be a graph of order n and size m. If G satisfies any two of the following three properties, then G is a tree:*

(1) *G is connected,* (2) *G has no cycles,* (3) *$m = n - 1$.*

As one would anticipate, graphs often contain many subgraphs that are trees. In fact, for each tree of a fixed order, every graph whose vertices have sufficiently large degree contains a subgraph that is isomorphic to the tree.

Theorem 2.20 *Let T be a tree of order k. If G is a graph for which $\delta(G) \geq k - 1$, then G contains a subgraph that is isomorphic to T.*

Proof. We proceed by induction on k. The result is obvious for $k = 1$ since K_1 is a subgraph of every graph and for $k = 2$ since K_2 is a subgraph of every nonempty graph.

Assume for every tree T' of order $k - 1$ with $k \geq 3$ and for every graph G' with $\delta(G') \geq k - 2$, that G' contains a subgraph that is isomorphic to T'. Now,

let T be a tree of order k and let G be a graph with $\delta(G) \geq k-1$. We show that G contains a subgraph that is isomorphic to T.

Let v be an end-vertex of T and let u be the vertex of T that is adjacent to v. Then $T - v$ is a tree of order $k-1$. Since $\delta(G) \geq k-1 > k-2$, it follows by the induction hypothesis that G contains a subgraph T' that is isomorphic to $T - v$. Let u' denote the vertex of T' corresponding to u in $T - v$. Since $\deg_G u' \geq k-1$ and the order of T' is $k-1$, the vertex u' is adjacent to a vertex v' that does not belong to T'. The tree obtained by adding v' to T' and joining it to u' is isomorphic to T, completing the proof. ■

Exercises for Section 2.2

1. Let G be a connected graph of order 3 or more. Prove that if $e = uv$ is a bridge of G, then at least one of u and v is a cut-vertex of G.

2. A graph G of order 8 has the degree sequence $s : 3, 3, 3, 1, 1, 1, 1, 1$. Prove or disprove: G is a tree.

3. (a) Give an example of a tree of order 8 containing six vertices of degree 1 and two vertices of degree 4.

   (b) Find all trees T where 75% of the vertices of T have degree 1 and the remaining 25% of the vertices have another degree (a fixed degree).

4. A certain tree T of order n contains only vertices of degree 1 and 3. Show that T contains $(n-2)/2$ vertices of degree 3.

5. Draw all forests of order 6.

6. Prove that a graph G is a forest if and only if every induced subgraph of G contains a vertex of degree at most 1.

7. Characterize those graphs with the property that every connected subgraph is an induced subgraph.

8. Prove that every connected graph all of whose vertices have even degrees contains no bridges.

9. Let G be a connected graph, and let e_1 and e_2 be two edges of G. Prove that $G - e_1 - e_2$ has three components if and only if both e_1 and e_2 are bridges in G.

10. Prove that a 3-regular graph G has a cut-vertex if and only if G has a bridge.

11. A tree is called **central** if its center is K_1 and **bicentral** if its center is K_2. Show that every tree is either central or bicentral.

12. Let T be a tree order 3 or more, and let T' be the tree obtained from T by deleting its end-vertices.

    (a) Show that $\mathrm{diam}(T) = \mathrm{diam}(T') + 2$, $\mathrm{rad}(T) = \mathrm{rad}(T') + 1$ and $\mathrm{Cen}(T) = \mathrm{Cen}(T')$.

    (b) Show that a tree T is central or bicentral (see Exercise 11) according to whether $\mathrm{diam}(T) = 2\,\mathrm{rad}(T)$ or $\mathrm{diam}(T) = 2\,\mathrm{rad}(T) - 1$.

13. Let G be a connected graph of order n and size m such that $V(G)=\{v_1, v_2, \ldots, v_n\}$, where v_i belongs to $b(v_i)$ blocks $(1 \leq i \leq n)$.

    (a) Show that $\sum_{i=1}^{n} b(v_i) \leq 2m$.

    (b) Show that $\sum_{i=1}^{n} b(v_i) = 2m$ if and only if G is a tree.

14. Determine all trees T such that $\overline{T}$ is also a tree.

15. Prove that if T is a tree of order at least 3, then T contains a cut-vertex v such that every vertex adjacent to v, with at most one exception, is an end-vertex.

16. Let T be a tree of order n with degree sequence $d_1, d_2, \ldots, d_n$ such that $d_1 \geq d_2 \geq \cdots \geq d_n$. Prove that $d_i \leq \left\lceil \frac{n-1}{i} \right\rceil$ for each integer i with $1 \leq i \leq n$.

17. Let G be a connected graph that is not a tree containing two distinct vertices u and v such that $G - u$ and $G - v$ are both trees. Show that $\deg u = \deg v$.

18. Show that there exists no tree T containing two distinct edges e_1 and e_2 such that the two components of $T - e_1$ are isomorphic *and* the two components of $T - e_2$ are isomorphic.

19. Let T be a tree of order n. Prove that T is isomorphic to a subgraph of $\overline{C}_{n+2}$.

20. Prove that if T is a tree of order $n \geq 2$ that is not a star, then T is isomorphic to a subgraph of $\overline{T}$.

21. Find all those graphs G of order $n \geq 4$ such that the subgraph induced by every three vertices of G is a tree, or show that no such graph exists.

2.3 Spanning Trees

Although there is no closed formula for the number of non-isomorphic trees of a fixed order, a formula does exist for the number of distinct labeled trees of order n (whose vertices are labeled from a fixed set of cardinality n).

Cayley's Tree Formula

Two labelings of the same graph from the same set of labels are considered **distinct labelings** if they produce different edge sets. Figure 2.10 shows three labelings of a graph of order 9 from the set $\{1, 2, \ldots, 9\}$. Since the first two labelings produce the same edge set, these two labelings are considered the same. The third labeling is different from the first two, however, since 26 is an edge there while 26 is not an edge in either of the first two labelings.

Figure 2.10: Labelings of a graph

There are three labeled trees of order 3 (whose vertices are labeled from the same set of three labels) and there are 16 labeled trees of order 4 (whose vertices are labeled from the set of four labels). These 19 trees are shown in Figure 2.11, where the vertex sets are $\{1, 2, 3\}$ and $\{1, 2, 3, 4\}$.

As we will see, the number of distinct labeled trees of order n whose vertices are labeled from the same set of n labels is n^{n-2}. Thus the number of labeled trees of order 3 is $3^1 = 3$ and the number of labeled trees of order 4 is $4^2 = 16$, as mentioned above. This formula for the number of labeled trees of order n is due to the famous British mathematician Arthur Cayley, whom we have encountered earlier and will encounter again. While Cayley discovered this result in 1889, the proof we present was given in 1918 by Heinz Prüfer [175], a German mathematician known for his work on abelian groups.

Prüfer's proof of Cayley's result (often called *Cayley's Tree Formula*) consists of establishing a one-to-one correspondence between the trees of order n with the same vertex set, say $\{1, 2, \ldots, n\}$, and the sequences (called **Prüfer codes**) of length $n - 2$ whose entries come from the set $\{1, 2, \ldots, n\}$. Since the number of such sequences is n^{n-2}, the proof is complete once the one-to-one correspondence has been verified.

Before giving a proof of Cayley's Tree Formula, we present an example to illustrate the technique employed. Consider the tree T of order $n = 8$ in Figure 2.12 whose vertices are labeled with elements of the set $\{1, 2, \ldots, 8\}$. The first term of the Prüfer code of $T_1 = T$ is the neighbor of the end-vertex of T_1 having the smallest label. This end-vertex is then deleted, producing a new tree T_2. The second term of the Prüfer code for T is the neighbor of the end-vertex of T_2 having the smallest label in T_2. We continue in this manner

Figure 2.11: Labeled trees of orders 3 and 4

until we arrive at $T_{n-2} = K_2$. The resulting sequence of length $n-2$ is the Prüfer code for T, which in this case is $(1, 8, 1, 5, 2, 5)$.

In this example, every vertex v of T appears in its Prüfer code $\deg v - 1$ times. This is true in general. Therefore, no end-vertex of T appears in the Prüfer code for T. So if T is a tree of order n and size m, then the number of terms in its Prüfer code is

$$\sum_{v \in V(T)} (\deg v - 1) = 2m - n = 2(n-1) - n = n - 2.$$

We now consider the converse question. Suppose that $s = (a_1, a_2, \ldots, a_{n-2})$ is a sequence of length $n-2$ where $a_i \in \{1, 2, \ldots, n\}$ for each i $(1 \leq i \leq n-2)$. We construct a labeled tree T of order n with vertex set $\{1, 2, \ldots, n\}$ such that the given sequence s is the Prüfer code of T. For example, suppose that the given sequence is $s = (1, 8, 1, 5, 2, 5)$. This sequence has length $n - 2 = 6$ and so $n = 8$. The smallest element of $\{1, 2, \ldots, 8\}$ not appearing in this sequence is 3. We join vertex 3 to the first term (vertex) of the sequence, that is, vertex 3 is joined to vertex 1. The first term of the sequence is then deleted, producing the reduced sequence $(8, 1, 5, 2, 5)$. Also, the element 3 is removed from $\{1, 2, \ldots, 8\}$. The smallest element of $\{1, 2, 4, 5, 6, 7, 8\}$ not appearing in $(8, 1, 5, 2, 5)$ is 4, which is joined to vertex 8, the first term of the sequence

Prüfer code for $T : (1, 8, 1, 5, 2, 5)$

Figure 2.12: Determining the Prüfer codes for a tree

$(8, 1, 5, 2, 5)$. This procedure is continued until only two elements of the set remain. These two vertices (5 and 8 in this case) are joined and a tree T has been constructed whose Prüfer code is s. This is illustrated in Figure 2.13.

In general then, to construct the Prüfer code $(a_1, a_2, \ldots, a_{n-2})$ of a tree $T = T_1$ of order n whose vertices are labeled with elements of the set $\{1, 2, \ldots, n\}$, we apply the following algorithm:

Algorithm 2.21 *Construct the Prüfer code of a labeled tree with vertex set* $\{1, 2, \ldots, n\}$.

Input: An integer $n \geq 3$ and a tree $T = T_1$ with vertex set $\{1, 2, \ldots, n\}$.

Output: The Prüfer code $(a_1, a_2, \ldots, a_{n-2})$ of T.

1. For $i = 1$ to $n - 2$
   1.1. Locate the smallest leaf v_i of T_i and let a_i be the neighbor of v_i in T_i.
   1.2. Let $T_{i+1} = T_i - v_i$.
2. Output $(a_1, a_2, \ldots, a_{n-2})$.

For a fixed set $S = S_1$ of n positive integers and for a sequence $s = s_1$ of length $n - 2$ whose terms belong to S_1, a graph G of order n with $V(G) = S_1$ is constructed by following the algorithm below. We may assume that $S_1 = \{1, 2, \ldots, n\}$ and $s_1 = (a_1, a_2, \ldots, a_{n-2})$, where $a_i \in S_1$ for $1 \leq i \leq n - 2$.

Algorithm 2.22 *Given a sequence s_1 of length $n - 2$ whose terms belong to the set $S_1 = \{1, 2, \ldots, n\}$, construct a tree with vertex set S_1 whose Prüfer code is s_1.*

Figure 2.13: Constructing a tree with a given Prüfer code

Input: An integer $n \geq 3$, $S_1 = \{1, 2, \ldots, n\}$ and a sequence $s_1 = (a_1, a_2, \ldots, a_{n-2})$, where $a_i \in \{1, 2, \ldots, n\}$ for $1 \leq i \leq n-2$.

Output: A graph G with vertex set $\{1, 2, \ldots, n\}$.

1. For $i = 1$ to $n-2$,
   1.1. Let k_i be the smallest element of S_i not appearing in s_i and let $e_i = k_i a_i$.
   1.2. Let $S_{i+1} = S_i - \{k_i\}$.
   1.3. For $i \leq n-3$, let $s_{i+1} = (a_{i+1}, a_{i+2}, \ldots, a_{n-2})$.
2. For the two elements $x, y \in S_{n-1}$, let $e_{n-1} = xy$.
3. Output $E(G) = \{e_1, e_2, \ldots, e_{n-2}\}$.

Algorithm 2.21 shows that the Prüfer code of a tree with vertex set $\{1, 2, \ldots, n\}$ is a sequence of length $n-2$ whose terms belong to $\{1, 2, \ldots, n\}$, while Algorithm 2.22 constructs a graph of order n (which will be shown to be a

tree) whose Prüfer code is a given sequence. We now use Prüfer codes to verify Cayley's Tree Formula [35].

Theorem 2.23 (**Cayley's Tree Formula**) *For each positive integer n, there are n^{n-2} distinct labeled trees of order n having the same vertex set.*

Proof. The result is obvious for $n = 1$ and $n = 2$. For $n \geq 3$, we show by induction that there is a one-to-one correspondence between the set of distinct labeled trees of order n having a fixed vertex set S of n positive integers and the set of sequences of length $n-2$ whose terms belong to S. By Algorithm 2.21, the Prüfer code of every tree of order n with vertex set S is a sequence of length $n-2$ whose terms belong to S. The proof will be complete once we show that for each such sequence only one tree of order n with vertex set S has this sequence as its Prüfer code.

For $n = 3$, let S be a set of three positive integers, say $S = \{1, 2, 3\}$. There are three trees with vertex set S, which are shown in Figure 2.11. These trees have Prüfer codes (1), (2) and (3), respectively. Thus the result is true for $n = 3$.

Assume for $n \geq 4$, that for each sequence s_0 of length $n-3$ whose terms belong to a fixed $(n-1)$-element set S_0 of positive integers that Algorithm 2.22 constructs a unique tree of order $n-1$ with vertex set S_0 whose Prüfer code is s_0.

Now let $S = \{1, 2, \ldots, n\}$ and let s be a sequence of length $n-2$ whose terms belong to S. Let G be the graph constructed by Algorithm 2.22. The goal then is to show that G is the unique tree with Prüfer code s.

Let k be the smallest element of S that does not belong to s. By Algorithm 2.22, ka_1 is an edge of the graph G constructed. By the induction hypothesis, there is a unique tree T' with vertex set $S' = S - \{k\}$ having Prüfer code $s' = (a_2, a_3, \ldots, a_{n-2})$. Since G is obtained from T' by adding a new vertex k to T' and joining k to a_1, the graph G is a tree that is uniquely determined. Furthermore, the vertex set of G is S and its Prüfer code is s. ■

A **spanning tree** of a graph G is a spanning subgraph of G that is a tree. A graph G can only contain a spanning tree if G is connected. Since the size of every tree of order n is $n-1$, it follows that if G is a connected graph of order n and size m, then $m \geq n-1$.

Spanning trees of a connected graph G can be obtained in a variety of ways. One possibility is to begin with $V(G)$. To construct the edge set $E(T)$ of a spanning tree T of G, we begin by selecting an edge e_1 of G. Next select an edge e_2 of G that is distinct from e_1. This is followed by selecting an edge e_3 distinct from those previously selected and that does not produce a cycle with those previously selected. We continue in this manner until an edge e_{n-1} is selected. Then $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$.

Another way to produce a spanning tree T of a connected graph G is to begin with G. If G is a tree, then G itself is a spanning tree. Otherwise, let f_1 be an edge on a cycle of G and remove it. Let $G_1 = G - f_1$. If G_1 is a tree,

then G_1 is a spanning tree of G. Otherwise, let f_2 be an edge on a cycle of G_1 and remove it. Let $G_2 = G_1 - f_2$ and continue in this manner until no cycle remains. The spanning subgraph T resulting in this manner is a spanning tree of G. Hence $m - (n-1) = m - n + 1$ edges must be deleted from G to obtain T. The number $m - n + 1$ is referred to as the **cycle rank** of G. Thus a tree has cycle rank 0.

If G is a connected graph of order n and size m having cycle rank 1, then $m - n + 1 = 1$ and so $n = m$. Such a graph is therefore a connected graph with exactly one cycle. These graphs are called **unicyclic graphs**. All of the graphs in Figure 2.14 are unicyclic.

Figure 2.14: Unicyclic graphs

Still another way to construct a spanning tree of G is to begin with a vertex v of G. Suppose that the eccentricity $e(v)$ of v is k and let

$$A_i = \{u \in V(G) : d(v,u) = i\} \quad \text{for } 0 \le i \le k.$$

Let T be a spanning subgraph of G such that $V(T) = V(G)$ and for each $w \in A_i$ $(1 \le i \le k)$, let $wx \in E(T)$ for exactly one vertex $x \in A_{i-1}$. Then T is a spanning tree with the property that $d_T(v,u) = d_G(v,u)$ for each $u \in V(G)$. Such a spanning tree T is called **distance-preserving** from v.

For the graph G of Figure 2.15, the trees T_1 and T_2 are spanning trees of G. The tree T_2 is distance-preserving from u. While T_1 has three leaves and T_2 has five leaves, no spanning tree of G has fewer than three or more than five leaves. According to the next result by Seymour Schuster [201] there must be a spanning tree of G with four leaves.

If u and v are two nonadjacent vertices of a tree T_0, then $T_0 + uv$ contains exactly one cycle C and the cycle C contains the edge uv. For any edge xy belonging to C, the graph $T_1 = T_0 + uv - xy$ is again a tree. The tree T_0 is said to be transformed into T_1 by an **edge exchange**. This transformation from T_0 to T_1 is illustrated in Figure 2.16.

Theorem 2.24 *Let G be a connected graph. If G contains a spanning tree with exactly r leaves and a spanning tree with exactly t leaves where $r < t$, then for every integer s with $r < s < t$, there is a spanning tree of G with exactly s leaves.*

Proof. Let T_0 be a spanning tree of G with r leaves and let T be a spanning tree of G with t leaves. Clearly, $E(T_0) \neq E(T)$. Let e be an edge of T not

Figure 2.15: Two spanning trees T_1 and T_2 in a graph

Figure 2.16: Transforming a tree T_0 into a tree T_1 by an edge exchange

belonging to T_0. Then $T_0 + e$ is a unicyclic graph whose cycle C contains an edge f not belonging to T. Thus T_0 is transformed into the spanning tree $T_1 = T_0 + e - f$ by an edge exchange where T_1 has one more edge in common with T than T_0 does. If $T_1 = T$, then the transformation is complete; otherwise, T_1 can be transformed into a spanning tree T_2 having one more edge in common with T than T_1 does. We continue this until we arrive at T, that is, we have a sequence

$$T_0, T_1, T_2, \ldots, T_k = T$$

of spanning trees of G where for each i $(1 \leq i \leq k)$, T_{i-1} is transformed into T_i by an edge exchange and T_i has one more edge in common with T than T_{i-1} does. Suppose that the tree T_i $(0 \leq i \leq k)$ has a_i leaves. Thus $a_0 = r$ and $a_k = t$.

When the tree T_{j-1} is transformed into T_j $(1 \leq j \leq k)$ by an edge exchange, suppose that the edge uv is added to T_{j-1} and the edge xy is removed from $T_{j-1} + uv$, that is, $T_j = T_{j-1} + uv - xy$. (It is possible that one of u and v is the same as one of x and y.) Depending on the degrees of u and v in T_{j-1} and the degrees of x and y in $T_{j-1} + uv$, the number $a_j - a_{j-1}$ can be any of the integers $-2, -1, 0, 1$ or 2.

If the number s does not appear in the sequence $r = a_0, a_1, \ldots, a_k = t$, then it must occur that $a_{j-1} = s - 1$ and $a_j = s + 1$ for some integer j with $1 \leq j \leq k$. Suppose that $T_j = T_{j-1} + uv - xy$. Since $a_j = a_{j-1} + 2$, it follows that the vertices u, v, x and y are distinct and that

$$\deg_{T_{j-1}} u \geq 2,\ \deg_{T_{j-1}} v \geq 2,\ \deg_{T_{j-1}} x = 2 \text{ and } \deg_{T_{j-1}} y = 2.$$

Observe that the unicyclic graph $T_{j-1} + uv$ has a_{j-1} leaves. Now both u and x lie on the cycle C in $H = T_{j-1} + uv$ and $\deg_H u \geq 3$ and $\deg_H x = 2$. Hence on a $u - x$ path on C on H, there are adjacent vertices w and z with $\deg_H w \geq 3$ and $\deg_H z = 2$. Deleting wz from H produces a spanning tree T' with $a_{j-1} + 1 = s$ leaves. ■

The Matrix-Tree Theorem

Cayley's Tree Formula, which gives the number of labeled trees of a given order, has another interpretation. As a consequence of this formula, there are n^{n-2} distinct spanning trees of the labeled graph K_n. This brings up the question of determining the number of distinct spanning trees of labeled graphs in general. An answer to this question has been given as a determinant of a matrix. This result, implicit in the work [133] of Gustav Kirchhoff, is known as the *Matrix-Tree Theorem.*

Kirchhoff is well known for his research on electrical currents, which he announced in 1845. This led to Kirchhoff's laws, the first of which states that the sum of the currents into a vertex equals the sum of the currents out of the vertex. Two years later, in 1847, he graduated from the University of Königsberg. It was during that year that he published the paper that led to his theorem on counting spanning trees. Kirchhoff spent much of his life working on experimental physics.

The proof we give of the Matrix-Tree Theorem will employ several results from matrix theory. Let M be an $r \times s$ matrix and M' an $s \times r$ matrix with $r \leq s$. The product $M \cdot M'$ is therefore an $r \times r$ matrix. Since $M \cdot M'$ is a square matrix, its determinant $\det(M \cdot M')$ exists. An $r \times r$ submatrix M_0 of M is said to correspond to the $r \times r$ submatrix M_0' of M' if the column numbers of M determining M_0 are the same as the row numbers of M' determining M_0'. For example, suppose that $r = 2$ and $s = 3$ and

$$M = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix} \text{ and } M' = \begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 0 & 2 \end{bmatrix}. \tag{2.2}$$

Then the following 2×2 matrices M_0 and M_0' of M and M', respectively, correspond to each other:

$$M_0 = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix} \text{ and } M_0' = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}.$$

A result from matrix theory states that

$$\det(M \cdot M') = \sum (\det M_0)(\det M_0'), \tag{2.3}$$

where the sum is taken over all $r \times r$ submatrices M_0 of M and where M_0' is the $r \times r$ submatrix corresponding to M_0. The numbers $\det(M_0)$ and $\det(M_0')$ are referred to as **major determinants** of M and M', respectively.

For the matrices M and M' in (2.2),

$$M \cdot M' = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 4 & 6 \end{bmatrix}.$$

Thus $\det(M \cdot M') = (-4) \cdot 6 - 4 \cdot 3 = -36$. From the result in matrix theory mentioned above, we also have

$$\begin{vmatrix} 1 & -2 \\ 2 & 0 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} -2 & 3 \\ 0 & 4 \end{vmatrix} \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = -36.$$

Suppose that A is an $n \times n$ matrix for some $n \geq 3$. Let A' be the $(n-1) \times (n-1)$ submatrix of A obtained by deleting row i and column j from A, where $1 \leq i, j \leq n$. Then $(-1)^{i+j} \det(A')$ is a **cofactor** of A. For the 3×3 matrix A in Figure 2.17, A' is the 2×2 submatrix obtained by deleting row 3 and column 3 of A, while A'' is the 2×2 submatrix obtained by deleting row 1 and column 2 of A. Then the cofactors of A' and A'' are $(-1)^{3+3} \det(A')$ and $(-1)^{1+2} \det(A'')$.

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 1 & 2 & -3 \\ -4 & -1 & 5 \end{bmatrix}$$

$$A' = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \qquad (-1)^{3+3} \det(A') = 3(2) - (-1)1 = 7$$

$$A'' = \begin{bmatrix} 1 & -3 \\ -4 & 5 \end{bmatrix} \qquad (-1)^{1+2} \det(A'') = -[1(5) - (-3)(-4)] = 7$$

Figure 2.17: Computing cofactors

Observe that both cofactors of A in Figure 2.17 have the value 7. Also observe that every row sum and column sum of A is 0. It is a theorem of matrix theory that whenever each row sum and column sum of a square matrix M is 0, then all cofactors of M have the same value. If M is a square matrix whose rows (or columns) are linearly dependent, then $\det(M) = 0$.

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The **degree matrix** $D(G) = [d_{ij}]$ is the $n \times n$ matrix with

$$d_{ij} = \begin{cases} \deg v_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Theorem 2.25 (**The Matrix-Tree Theorem**) *If G is a nontrivial labeled graph with adjacency matrix A and degree matrix D, then the number of distinct spanning trees of G is the value of any cofactor of the matrix $D - A$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. First, observe that the row sum of row i (or column sum of column i) in A is $\deg v_i$, so every row sum or column sum of $D - A$ is 0. Consequently, the cofactors of $D - A$ have the same value.

Assume first that G is a disconnected graph. Of course in this case, G has no spanning trees. Let G_1 be a component of G and suppose that $V(G_1) = \{v_1, v_2, \ldots, v_r\}$, where $1 \leq r < n$. Let M be the $(n-1) \times (n-1)$ submatrix of $D - A$ obtained by deleting row n and column n from $D - A$. Since the sum of the first r rows of M is the zero vector of length $n - 1$, the rows of M are linearly dependent and so $\det(M) = 0$, as desired.

Henceforth, we assume that G is a connected graph of order n and size m where $E(G) = \{e_1, e_2, \ldots, e_m\}$. Thus $m \geq n - 1$. Let $C = [c_{ij}]$ be an $n \times m$ matrix where $c_{ij} = 1$ or $c_{ij} = -1$ if v_i is incident with e_j and such that each column has one entry that is 1 and one entry that is -1, while all other entries in the column are 0. We show that for the transpose C^t of C, we have $C \cdot C^t = D - A$. The (i, j)-entry of $C \cdot C^t$ is

$$\sum_{k=1}^{m} c_{ik} c_{jk},$$

which has the value $\deg v_i$ if $i = j$, the value -1 if $i \neq j$ and $v_i v_j \in E(G)$ and the value 0 if $i \neq j$ and $v_i v_j \notin E(G)$. Hence, as claimed, $C \cdot C^t = D - A$.

Consider a spanning subgraph H of G containing $n - 1$ edges of G. Let C' be the $(n-1) \times (n-1)$ submatrix of C determined by the columns associated with the edges of H and by all rows of C with one exception, say row k.

We now determine the absolute value $|\det(C')|$ of the determinant of C'. If H is disconnected, then H has a component H_1 not containing v_k. The sum of the row vectors of C' corresponding to the vertices of H_1 is the zero vector of length $n - 1$. Hence the row vectors in C' are linearly dependent and so $|\det(C')| = 0$.

Next assume that H is connected. Thus H is a spanning tree of G. Let u_1 be an end-vertex of H that is distinct from v_k and let f_1 be the edge of H that is incident with u_1. In the tree $H - u_1$, let u_2 be an end-vertex distinct from v_k and let f_2 be the edge of $H - u_1$ that is incident with u_2. This procedure is continued until only the vertex v_k remains.

A matrix $C'' = \left[c''_{ij}\right]$ can now be obtained by a permutation of the rows and columns of C' such that $|c''_{ij}| = 1$ if and only if u_i and f_j are incident. From the manner in which C'' is defined, any vertex u_i is incident only with edges f_j with $j \leq i$. This, however, implies that C'' is a lower triangular matrix and since $|c''_{ii}| = 1$ for all i, we conclude that $|\det(C'')| = 1$. Consequently, $|\det(C')| = |\det(C'')| = 1$.

Since every cofactor of $D - A$ has the same value, it suffices to evaluate the determinant of the matrix obtained by deleting both row i and column i from $D - A$ for some i $(1 \leq i \leq n)$. Let C_i denote the matrix obtained from C by removing row i. Then the cofactor mentioned above equals $\det(C_i \cdot C_i^t)$, which implies by (2.3) that this number is the sum of the products of the

corresponding major determinants of C_i and C_i^t. However, these corresponding major determinants have the same value and so their product is 1 if the defining columns correspond to a spanning tree and 0 otherwise. ■

Cayley's Tree Formula (Theorem 2.23) is then a corollary of the Matrix-Tree Theorem (see Exercise 14). We illustrate the Matrix-Tree Theorem for the graph G of Figure 2.18 where the matrices D and $D - A$ are also shown.

G :

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad D - A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

Figure 2.18: Illustrating the Matrix-Tree Theorem

To calculate a cofactor of $D - A$, we delete the entries in row i and column j for some i and j with $1 \leq i, j \leq 4$ and compute the product of $(-1)^{i+j}$ and the determinant of the resulting submatrix. For example, the cofactor of the $(2, 3)$-entry in the matrix $D - A$ in Figure 2.18 is

$$(-1)^{2+3} \begin{vmatrix} 2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 2 \end{vmatrix}.$$

Expanding this determinant along the first row, we obtain

$$\begin{aligned} & -\left(2 \begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} + 0 \begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix}\right) \\ = \; & -[2(-3) + 1(-2) + 0] = 8. \end{aligned}$$

Consequently, there are eight distinct spanning trees of the labeled graph G of Figure 2.18, all of which are shown in Figure 2.19

The Minimum Spanning Tree Problem

Let G be a connected graph each of whose edges is assigned a real number (called the **cost** or **weight** of the edge). We denote the weight of an edge e of G by $w(e)$. Such a graph is called a **weighted graph**. For each subgraph H of G, the **weight** $w(H)$ of H is defined as the sum of the weights of its edges,

Figure 2.19: The distinct spanning trees of a graph

that is,

$$w(H) = \sum_{e \in E(H)} w(e).$$

We seek a spanning tree of G whose weight is minimum among all spanning trees of G. Such a spanning tree is called a **minimum spanning tree**. The problem of finding a minimum spanning tree in a connected weighted graph is called the **Minimum Spanning Tree Problem**.

The importance of the Minimum Spanning Tree Problem is due to its applications in the design of computer, communications and transportation networks. The history of this problem was researched by Ronald L. Graham and Pavol Hell [97] in 1985. They concluded that the Minimum Spanning Tree Problem was initially formulated by Otakar Borůvka [27] in 1926 because of his interest in the most economical layout of a power-line network. He also gave the first solution of the problem. Prior to Borůvka, however, the anthropologist Jan Czekanowski's work on classification schemes led him to consider ideas closely related to the Minimum Spanning Tree Problem.

Kruskal's Algorithm

Over the years, this problem has been solved in a variety of ways using a number of algorithms. One of the best known was discovered by John Bernard Kruskal [141] in 1956. It was only two years after completing his doctoral degree that the paper containing the algorithm that bears his name was published.

Kruskal's Algorithm: *For a connected weighted graph G of order n, a spanning tree T of G is constructed as follows: For the first edge e_1 of T, we select any edge of G of minimum weight. For subsequent edges $e_2, e_3, \ldots, e_{n-1}$, we select an edge of minimum weight among the remaining edges of G that does not produce a cycle with the previously selected edges.*

Figure 2.20 shows how a spanning tree T of a connected weighted graph G is constructed using Kruskal's algorithm. We now show that Kruskal's algorithm

produces a minimum spanning tree in every connected weighted graph.

Figure 2.20: Constructing a spanning tree by Kruskal's algorithm

Theorem 2.26 *Kruskal's algorithm produces a minimum spanning tree in a connected weighted graph.*

Proof. Let G be a connected weighted graph of order n and let T be a spanning tree obtained by Kruskal's algorithm, where the edges of T are selected in the order e_1, e_2, ..., e_{n-1}. Necessarily then, $w(e_1) \leq w(e_2) \leq \ldots \leq w(e_{n-1})$ and the weight of T is

$$w(T) = \sum_{i=1}^{n-1} w(e_i).$$

We show that T is a minimum spanning tree of G. Assume, to the contrary, that T is not a minimum spanning tree. Among all minimum spanning trees of G, let H be one that has a maximum number of edges in common with T. Since $H \neq T$, there is at least one edge of T that is not in H. Let e_i be the first edge of T that is not in H. Therefore, if $i > 1$, then the edges e_1, e_2, ..., e_{i-1} belong to both H and T. Now define $G_0 = H + e_i$. Then G_0 has a cycle C. Since T has no cycle, there is an edge e_0 on C that is not in T. The graph $T_0 = G_0 - e_0$ is therefore a spanning tree of G and

$$w(T_0) = w(H) + w(e_i) - w(e_0).$$

Since H is a minimum spanning tree of G, it follows that $w(H) \leq w(T_0)$. Thus, $w(H) \leq w(H) + w(e_i) - w(e_0)$ and so $w(e_0) \leq w(e_i)$. By Kruskal's algorithm, certainly $w(e_0) = w(e_i)$ if $i = 1$. Suppose then that $i > 1$. By Kruskal's algorithm, e_i is an edge of minimum weight that can be added to the edges $e_1, e_2, \ldots, e_{i-1}$ without producing a cycle. However, e_0 can also be added to $e_1, e_2, \ldots, e_{i-1}$ without producing a cycle. Thus $w(e_i) \leq w(e_0)$, which implies that $w(e_i) = w(e_0)$ when $i > 1$ as well. Therefore, $w(T_0) = w(H)$ and so T_0 is also a minimum spanning tree of G. However, T_0 has more edges in common with T than H does, which is a contradiction. ■

In fact, every minimum spanning tree in a connected weighted graph can be obtained by Kruskal's algorithm (see Exercise 26).

Prim's Algorithm

Another well-known algorithm for finding a minimum spanning tree in a connected weighted graph was developed by Robert Clay Prim [174] in 1957, although it had essentially been discovered by Vojtěch Jarník [126] in 1930.

Prim's Algorithm: *For a connected weighted graph G of order n, a spanning tree T of G is constructed as follows: For an arbitrary vertex u for G, an edge of minimum weight incident with u is selected as the first edge e_1 of T. For subsequent edges $e_2, e_3, \ldots, e_{n-1}$, we select an edge of minimum weight among those edges having exactly one of its vertices incident with an edge already selected.*

Figure 2.21 illustrates how to construct a spanning tree T of a connected weighted graph G by Prim's algorithm. Again, a tree obtained by Prim's algorithm is also a minimum spanning tree, as we show next.

Theorem 2.27 *Prim's algorithm produces a minimum spanning tree in a connected weighted graph.*

Proof. Let G be a nontrivial connected weighted graph of order n and let T be a spanning tree obtained by Prim's Algorithm, where the edges of T are selected in the order e_1, e_2, …, e_{n-1}, and where e_1 is incident with a given vertex u. Thus the weight of T is

$$w(T) = \sum_{i=1}^{n-1} w(e_i).$$

Assume, to the contrary, that T is not a minimum spanning tree. Let $\mathcal{H}$ be the set of all minimum spanning trees of G having a maximum number of edges in common with T. By assumption, $T \notin \mathcal{H}$. If no tree in $\mathcal{H}$ contains e_1, let H be any tree in $\mathcal{H}$; otherwise, let H be a tree in $\mathcal{H}$ that contains the edges e_1, e_2, …, e_k for some integer k with $1 \leq k < n-1$ but for which no tree in $\mathcal{H}$

Figure 2.21: Constructing a spanning tree by Prim's algorithm

contains all of the edges e_1, e_2, ..., e_{k+1}. Hence no tree in $\mathcal{H}$ contains all of the edges e_1, e_2, ..., e_{k+1}, where $0 \leq k < n-1$. If $k = 0$, let $U = \{u\}$; while if $k \geq 1$, let U be the set of $k+1$ vertices that are incident with one or more of the edges $e_1, e_2, \ldots, e_k$. By Prim's Algorithm, e_{k+1} joins a vertex of U and a vertex of $V(T) - U$.

The subgraph $H + e_{k+1}$ therefore contains a cycle C and e_{k+1} is on C. Necessarily, C contains an edge e_0 distinct from e_{k+1} such that e_0 also joins a vertex of U and a vertex of $V(T) - U$. By the construction of T from Prim's Algorithm, $w(e_{k+1}) \leq w(e_0)$. Now $T' = H + e_{k+1} - e_0$ is a spanning tree of G whose weight is

$$w(T') = w(H) + w(e_{k+1}) - w(e_0).$$

Since H is a minimum spanning tree, $w(H) \leq w(T')$ and so $w(H) \leq w(H) + w(e_{k+1}) - w(e_0)$, which implies that $w(e_0) \leq w(e_{k+1})$. Consequently, $w(e_0) = w(e_{k+1})$ and $w(H) = w(T')$. Therefore, T' is also a minimum spanning tree of G. If e_0 does not belong to T, then T' is a minimum spanning tree having more edges in common with T than H does, which is impossible since $H \in \mathcal{H}$. Hence e_0 belongs to T, which implies that T' has the same number of edges in common with T as H does and so $T' \in \mathcal{H}$. Necessarily, $e_0 = e_j$ for some $j > k+1$. Since T' contains all of the edges e_1, e_2, ..., e_{k+1}, this contradicts the defining property of H. ■

Complexity of an Algorithm

While it is essential that an algorithm solve the problem under consideration, another important characteristic of an algorithm concerns its efficiency. The **complexity** of an algorithm refers to the number of basic computational steps (ordinarily arithmetic operations and comparisons) required to execute the algorithm. This number depends on the nature of its input as well as the size of the input. For a graph of order n and size m, the complexity typically depends on n and/or m. If the complexity is bounded above by a polynomial in its input size, then it is called a **polynomial-time algorithm**. For example, if the input is a graph of order n and the complexity of an algorithm is at most some constant times n^p for some positive integer p, then the algorithm is a polynomial-time algorithm and we denote the complexity by $O(n^p)$. An algorithm of complexity $O(n)$ is a **linear-time algorithm**, while an algorithm of complexity $O(n^2)$ is a **quadratic-time algorithm**.

Polynomial-time algorithms are considered efficient as these algorithms are typically computationally feasible even when applied to graphs of large order. The complexity of Kruskal's algorithm is $O(m^2)$ and that of Prim's algorithm is $O(n^3)$.

Exercises for Section 2.3

1. Determine the Prüfer codes of the trees in Figure 2.11.

2. (a) Which trees have constant Prüfer codes?
   (b) Which trees have Prüfer codes each term of which is one of two numbers?
   (c) Which trees have Prüfer codes with distinct terms?

3. Determine the labeled tree having Prüfer code $(4, 5, 7, 2, 1, 1, 6, 6, 7)$.

4. In the Prüfer code of a certain tree T, seven numbers appear twice each, one number appears three times and no other number appears in the code. How many leaves does T contain?

5. The Prüfer code of a tree T of order n with vertex set $\{1, 2, \ldots, n\}$ is $(8, 6, 8, 8, 1, 6, 2, 2, 9)$. What is the degree sequence $d_1, d_2, \ldots, d_n$ of T where $d_1 \geq d_2 \geq \cdots \geq d_n$?

6. (a) Show that if G is a connected graph of order $n \geq 4$ having cycle rank 1, then for every two distinct vertices u and v of G, the graph G contains at most two $u - v$ paths.
   (b) Is the statement in (a) true if G has cycle rank 2?

7. Prove that a sequence $s_n : d_1, d_2, \ldots, d_n$ $(n \geq 3)$ of integers with $1 \leq d_i \leq n-1$ for $1 \leq i \leq n$ is a degree sequence of a unicyclic graph of order n if and only if at most $n-3$ terms of s_n are 1 and $\sum_{i=1}^{n} d_i = 2n$.

8. (a) Show that for every connected graph G there is a spanning tree T of G such that $\operatorname{diam}(T) \leq 2\operatorname{diam}(G)$.

   (b) Prove or disprove: For every positive integer k, there exists a connected graph G and a spanning tree T of G such that $\operatorname{diam}(T) > k\operatorname{diam}(G)$.

9. Give an example of a connected graph G and a vertex v of G for which there exist two distinct spanning trees that are distance-preserving from v.

10. (a) Let G be a connected graph. Show that if T is a spanning tree of G that is distance-preserving from some vertex of G, then

$$\operatorname{diam}(G) \leq \operatorname{diam}(T) \leq 2\operatorname{diam}(G).$$

   (b) Show that for every positive integer a, there exists a connected graph G of diameter a and a vertex v of G such that for every integer b with $a \leq b \leq 2a$, there is a spanning tree T that is distance-preserving from v and $\operatorname{diam}(T) = b$.

11. Give an example of a connected graph G that is not a tree and two vertices u and v of G such that a distance-preserving spanning tree from v is the same as a distance-preserving spanning tree from u.

12. Let G be the labeled graph in Figure 2.22.

Figure 2.22: The graph G in Exercise 12

   (a) Use the Matrix Tree Theorem to compute the number of distinct labeled spanning trees of G.

   (b) Draw all the distinct labeled spanning trees of G.

13. (a) Let $G = K_4$ with $V(G) = \{v_1, v_2, v_3, v_4\}$. Draw all labeled spanning trees of G in which v_4 is an end-vertex.

   (b) Let v be a fixed vertex of $G = K_n$. Determine the number of labeled spanning trees of G in which v is an end-vertex.

14. Prove Theorem 2.23 as a corollary of Theorem 2.25.

15. Prove that an edge e of a connected graph is a bridge if and only if e belongs to every spanning tree of G.

16. Let F be a subgraph of a connected graph G. Prove that F is a subgraph of every spanning tree of G if and only if F contains no cycles.

17. What is the maximum number of spanning trees, no two of which have an edge in common, that a complete graph of order $n \geq 4$ can have?

18. (a) Let G be a nontrivial connected graph. Prove that if v is an end-vertex of some spanning tree of G, then v is not a cut-vertex of G.

    (b) Use (a) to give an alternative proof of the fact that every nontrivial connected graph contains at least two vertices that are not cut-vertices.

    (c) Let v be a vertex in a nontrivial connected graph G. Show that there exists a spanning tree of G that contains all edges of G that are incident with v.

    (d) Prove that if a connected graph G has exactly two vertices that are not cut-vertices, then G is a path.

19. Show that there is only one positive integer k such that no graph contains exactly k spanning trees.

20. Let T and T' be two distinct spanning trees of a connected graph G of order n. Show that there exists a sequence $T = T_0, T_1, \ldots, T_k = T'$ of spanning trees of G such that T_i and T_{i+1} have $n-2$ edges in common for each i with $0 \leq i \leq k-1$.

21. Show for every two integers r and t with $2 \leq r \leq t$ that there exists a connected graph G such that r is the minimum number of end-vertices in a spanning tree of G and t is the maximum number of end-vertices in a spanning tree of G.

22. For the graph G in Figure 2.23, determine

    (a) the adjacency matrix A of G and the degree matrix D of G,

    (b) a cofactor of the matrix $D - A$,

    (c) the matrix C described in the proof of Theorem 2.25,

    (d) the matrix C_3 described in the proof of Theorem 2.25,

    (e) the matrix $C_3 \cdot C_3^t$,

    (f) the major determinants of C_3 and C_3^t.

Figure 2.23: The graph G in Exercise 22

Illustrate (2.3) in the case where $M = C_3$ and $M' = C_3^t$, and where $\det(M_0)$ and $\det(M_0')$ are the corresponding major determinants of C_3 and C_3^t. Then show that the value of $\det(C_3 \cdot C_3^t)$ obtained is the expected value.

23. Which (non-isomorphic) spanning trees does each of the graphs G and H in Figure 2.24 contain?

Figure 2.24: The graphs in Exercise 23

24. Apply both Kruskal's and Prim's algorithms to find a minimum spanning tree in the weighted graph in Figure 2.25. In each case, show how this tree is constructed, as in Figures 2.20 and 2.21.

Figure 2.25: The weighted graph in Exercise 24

25. Apply both Kruskal's and Prim's algorithms to find a minimum spanning tree in the weighted graph in Figure 2.26. In each case, show how this tree is constructed, as in Figures 2.20 and 2.21.

26. Let G be a connected weighted graph. Prove that every minimum spanning tree of G can be obtained by Kruskal's algorithm.

27. Let G be a connected weighted graph and T a minimum spanning tree of G. Show that T is a unique minimum spanning tree of G if and only

Figure 2.26: The weighted graph in Exercise 25

if the weight of each edge e of G that is not in T exceeds the weight of every other edge on the cycle in $T + e$.

2.4 Connectivity and Edge-Connectivity

Each tree of order 3 or more contains at least one vertex whose removal results in a disconnected graph. In fact, every vertex in a tree that is not a leaf has this property. Furthermore, the removal of every edge in a tree results in a disconnected graph (with exactly two components). On the other hand, no vertex or edge in a nonseparable graph of order 3 or more has this property. Hence, in this sense, nonseparable graphs possess a greater degree of connectedness than trees. We now look at the two most common measures of connectedness of graphs. In the process of doing this, we will encounter some of the best known and most useful theorems dealing with the structure of graphs.

The Connectivity of a Graph

A **vertex-cut** of a graph G is a set S of vertices of G such that $G - S$ is disconnected. A vertex-cut of minimum cardinality in G is called a **minimum vertex-cut** of G and this cardinality is called the **vertex-connectivity** (or, more simply, the **connectivity**) of G and is denoted by $\kappa(G)$. (The symbol κ is the Greek letter *kappa*.)

Let S be a minimum vertex-cut of a (noncomplete) connected graph G and let $G_1, G_2, \ldots, G_k$ $(k \geq 2)$ be the components of $G - S$. Then the subgraphs $B_i = G[V(G_i) \cup S]$ are called the **branches** of G at S or the S**-branches** of G. For the minimum vertex-cut $S = \{u, v\}$ of the graph G of Figure 2.27, the three S-branches of G are also shown in that figure.

Complete graphs do not contain vertex-cuts. Indeed, the removal of any proper subset of vertices from a complete graph results in a smaller complete graph. The connectivity of the complete graph of order n is defined as $n - 1$, that is, $\kappa(K_n) = n - 1$. In general then, the **connectivity** $\kappa(G)$ of a graph

Figure 2.27: The branches of a graph at $S = \{u, v\}$

G is the smallest number of vertices whose removal from G results in either a disconnected graph or a trivial graph. Therefore, for every graph G of order n,

$$0 \leq \kappa(G) \leq n - 1.$$

Thus a graph G has connectivity 0 if and only if either $G = K_1$ or G is disconnected; a graph G has connectivity 1 if and only if $G = K_2$ or G is a connected graph with cut-vertices; and a graph G has connectivity 2 or more if and only if G is a nonseparable graph of order 3 or more.

Often it is more useful to know that a given graph G cannot be disconnected by the removal of a certain number of vertices rather than to know the actual connectivity of G. A graph G is **k-connected**, $k \geq 1$, if $\kappa(G) \geq k$. That is, G is k-connected if the removal of fewer than k vertices from G results in neither a disconnected nor a trivial graph. In particular, to show that a graph G of order $n \geq k + 1$ is k-connected, it suffices to show that $G - S$ is connected for every set $S \subseteq V(G)$ with $|S| = k - 1$. The 1-connected graphs are then the nontrivial connected graphs, while the 2-connected graphs are the nonseparable graphs of order 3 or more.

The Edge-Connectivity of a Graph

How connected a graph G is can be measured not only in terms of the number of vertices that need to be deleted from G to arrive at a disconnected or trivial graph but in terms of the number of edges that must be deleted from G to produce a disconnected or trivial graph.

An **edge-cut** of a graph G is a subset X of $E(G)$ such that $G - X$ is disconnected. An edge-cut of minimum cardinality in G is a **minimum edge-cut** and this cardinality is the **edge-connectivity** of G, which is denoted by $\lambda(G)$. (The symbol λ is the Greek letter *lambda*.) The trivial graph K_1 does not contain an edge-cut but we define $\lambda(K_1) = 0$. Therefore, $\lambda(G)$ is the minimum number of edges whose removal from G results in a disconnected or

trivial graph. Since the set of edges incident with any vertex of a graph G of order n is an edge-cut of G, it follows that

$$0 \leq \lambda(G) \leq n - 1. \tag{2.4}$$

A graph G is **k-edge-connected**, $k \geq 1$, if $\lambda(G) \geq k$. That is, G is k-edge-connected if the removal of fewer than k edges from G results in neither a disconnected graph nor a trivial graph. Thus a 1-edge-connected graph is a nontrivial connected graph and a 2-edge-connected graph is a nontrivial connected bridgeless graph.

For the graph G of Figure 2.28, $\kappa(G) = 2$ and $\lambda(G) = 3$. Both $\{u, v_1\}$ and $\{u, v_2\}$ are minimum vertex-cuts, while $\{e_1, e_2, e_3\}$ is a minimum edge-cut.

Figure 2.28: Connectivity and edge-connectivity

The edge-connectivity of the complete graphs is given in the next theorem.

Theorem 2.28 *For every positive integer n,*

$$\lambda(K_n) = n - 1.$$

Proof. Since the edge-connectivity of K_1 is defined to be 0, we may assume that $n \geq 2$. We have observed in (2.4) that $\lambda(K_n) \leq n-1$. Let X be a minimum edge-cut of K_n. Then $|X| = \lambda(K_n)$ and $G - X$ consists of two components, say G_1 and G_2. Suppose that G_1 has order k. Then G_2 has order $n - k$. Thus $|X| = k(n-k)$. Since $k \geq 1$ and $n-k \geq 1$, it follows that $(k-1)(n-k-1) \geq 0$ and so

$$(k-1)(n-k-1) = k(n-k) - (n-1) \geq 0,$$

which implies that

$$\lambda(K_n) = |X| = k(n-k) \geq n - 1.$$

Therefore, $\lambda(K_n) = n - 1$. ■

Relationship of Connectivities and Minimum Degree

That the edge-connectivity of a graph is bounded above by its minimum degree and bounded below by its connectivity was first observed by Hassler Whitney [238].

Theorem 2.29 *For every graph G,*

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Proof. Let G be a graph of order n. If G is disconnected, then $\kappa(G) = \lambda(G) = 0$; while if G is complete, then $\kappa(G) = \lambda(G) = \delta(G) = n - 1$. Thus the desired inequalities hold in these two cases. Hence we may assume that G is a connected graph that is not complete.

Since G is not complete, $\delta(G) \leq n - 2$. Let v be a vertex of G such that $\deg v = \delta(G)$. If the edges incident with v are deleted from G, then a disconnected graph is produced. Hence $\lambda(G) \leq \delta(G) \leq n - 2$.

It remains to show that $\kappa(G) \leq \lambda(G)$. Let X be a minimum edge-cut of G. Then $|X| = \lambda(G) \leq n - 2$. Necessarily, $G - X$ consists of two components, say G_1 and G_2. Suppose that the order of G_1 is k. Then the order of G_2 is $n - k$, where $k \geq 1$ and $n - k \geq 1$. Also, every edge in X joins a vertex of G_1 and a vertex of G_2. We consider two cases.

Case 1. *Every vertex of G_1 is adjacent to every vertex of G_2.* Then $|X| = k(n - k)$. Since $k - 1 \geq 0$ and $n - k - 1 \geq 0$, it follows that

$$(k - 1)(n - k - 1) = k(n - k) - (n - 1) \geq 0$$

and so

$$\lambda(G) = |X| = k(n - k) \geq n - 1.$$

This, however, contradicts $\lambda(G) \leq n - 2$ and so Case 1 cannot occur.

Case 2. *There exist a vertex u in G_1 and a vertex v in G_2 such that $uv \notin E(G)$.* We now define a set U of vertices of G. Let $e \in X$. If e is incident with u, say $e = uv'$, then the vertex v' is placed in the set U. If e is not incident with u, say $e = u'v'$ where u' is in G_1, then the vertex u' is placed in U. Hence, for every edge $e \in X$, one of its two incident vertices belongs to U but $u, v \notin U$. Thus $|U| \leq |X|$ and U is a vertex-cut. Therefore,

$$\kappa(G) \leq |U| \leq |X| = \lambda(G),$$

as desired. ■

The connectivity of a graph G of a given order n and size m can only be so large. For example, if $m < n - 1$, then G is disconnected and so $\kappa(G) = 0$. On the other hand, if $m \geq n - 1$, then there is a sharp upper bound for $\kappa(G)$ in terms of the average degree of G, which we present next.

Theorem 2.30 *If G is a graph of order n and size $m \geq n - 1$, then*

$$\kappa(G) \leq \left\lfloor \frac{2m}{n} \right\rfloor.$$

Proof. Since the sum of the degrees of the vertices of G is $2m$, the average degree of the vertices of G is $2m/n$ and so $\delta(G) \leq 2m/n$. Since $\delta(G)$ is an integer, $\delta(G) \leq \lfloor 2m/n \rfloor$. By Theorem 2.29, $\kappa(G) \leq \lfloor 2m/n \rfloor$. ■

The observation stated in Theorem 2.30 is due to Frank Harary. In the second book ever written on graph theory, *Théorie des Graphes et Ses Applications*, the author Claude Berge [15] wrote (translated into English):

> *What is the maximum connectivity of a graph with n vertices and m edges?*

In 1962 Harary [109] answered Berge's question (see Exercise 13) by showing that for every pair n, m of integers with $2 \leq n-1 \leq m \leq \binom{n}{2}$, there exists a graph G of order n and size m with $\kappa(G) = \left\lfloor \frac{2m}{n} \right\rfloor$. Of course, if $m = n-1 \geq 2$, every tree T of order n has the desired property since

$$\kappa(T) = \left\lfloor \frac{2m}{n} \right\rfloor = \left\lfloor \frac{2n-2}{n} \right\rfloor = 1.$$

We observed that $\kappa(G) = 2$ and $\lambda(G) = 3$ for the graph G of Figure 2.28. Since $\delta(G) = 4$, this graph shows that the two inequalities stated in Theorem 2.29 can be strict. The first of these inequalities cannot be strict for cubic graphs, however.

Theorem 2.31 *For every cubic graph G,*

$$\kappa(G) = \lambda(G).$$

Proof. For a cubic graph G, it follows that $\kappa(G) = \lambda(G) = 0$ if and only if G is disconnected. If $\kappa(G) = 3$, then $\lambda(G) = 3$ by Theorem 2.29. So two cases remain, namely $\kappa(G) = 1$ and $\kappa(G) = 2$. Let U be a minimum vertex-cut of G. Then $|U| = 1$ or $|U| = 2$. So $G - U$ is disconnected. Let G_1 and G_2 be two components of $G - U$. Since G is cubic, for each $u \in U$, at least one of G_1 and G_2 contains exactly one neighbor of u.

Case 1. $\kappa(G) = |U| = 1$. Thus U consists of a cut-vertex u of G. Since some component of $G - U$ contains exactly one neighbor w of u, the edge uw is a bridge of G and so $\lambda(G) = \kappa(G) = 1$.

Case 2. $\kappa(G) = |U| = 2$. Let $U = \{u, v\}$. Assume that each of u and v has exactly one neighbor, say u' and v', respectively, in the same component of $G - U$. (This is the case that holds if $uv \in E(G)$.) Then $X = \{uu', vv'\}$ is an edge-cut of G and $\lambda(G) = \kappa(G) = 2$. (See Figure 2.29(a) for the situation when u and v are not adjacent.)

Hence we may assume that u has one neighbor u' in G_1 and two neighbors in G_2; while v has two neighbors in G_1 and one neighbor v' in G_2 (see Figure 2.29(b)). Therefore, $uv \notin E(G)$ and $X = \{uu', vv'\}$ is an edge-cut of G; so $\lambda(G) = \kappa(G) = 2$. ■

Figure 2.29: A step in the proof of Case 2

As would be expected, the higher the degrees of the vertices of a graph, the more likely it is that the graph has large connectivity. There are several sufficient conditions of this type. We present the simplest of these, due to Gary Chartrand and Frank Harary [36].

Theorem 2.32 *Let G be a graph of order n and let k be an integer with $1 \leq k \leq n-1$. If*

$$\deg v \geq \left\lceil \frac{n+k-2}{2} \right\rceil$$

for every vertex v of G, then G is k-connected.

Proof. Suppose that this theorem is false. Then there is a graph G satisfying the hypothesis of the theorem such that G is not k-connected. Certainly then, G is not a complete graph. Hence there exists a vertex-cut U of G such that $|U| = \ell \leq k-1$. The graph $G-U$ is therefore disconnected of order $n-\ell$.

Let G_1 be a component of $G-U$ of smallest order, say n_1. Thus $n_1 \leq \lfloor (n-\ell)/2 \rfloor$. Let v be a vertex of G_1. Necessarily, v is adjacent in G only to the vertices of U or to other vertices of G_1. Hence

$$\begin{aligned} \deg v &\leq \ell + (n_1 - 1) \leq \ell + \lfloor (n-\ell)/2 \rfloor - 1 \\ &= \lfloor (n+\ell-2)/2 \rfloor \leq \lfloor (n+k-3)/2 \rfloor, \end{aligned}$$

contrary to the hypothesis. ■

The class of k-edge-connected graphs is characterized in the following simple but useful theorem.

Theorem 2.33 *A nontrivial graph G is k-edge-connected if and only if there exists no nonempty proper subset W of $V(G)$ such that the number of edges joining W and $V(G)-W$ is less than k.*

Proof. First assume that there exists no nonempty proper subset W of $V(G)$ such that the number of edges joining W and $V(G)-W$ is less than k but that G is not k-edge-connected. Since G is nontrivial, there exist ℓ edges, where $0 < \ell < k$, such that their deletion from G results in a disconnected graph H.

Let H_1 be a component of H. Since the number of edges joining $V(H_1)$ and $V(G) - V(H_1)$ is at most ℓ, where $\ell < k$, this is a contradiction.

Conversely, suppose that G is k-edge-connected. If there exists a nonempty proper subset W of $V(G)$ such that j edges ($j < k$) join W and $V(G) - W$, then the deletion of these j edges produces a disconnected graph, which is impossible. ■

According to Theorem 2.29, $\lambda(G) \leq \delta(G)$ for every graph G. The following theorem of Jan Plesník [173] gives a sufficient condition for equality to hold in this case.

Theorem 2.34 *If G is a connected graph of diameter 2, then $\lambda(G) = \delta(G)$.*

Proof. Let X be an edge-cut with $|X| = \lambda(G) = \lambda$, and let H_1 and H_2 be the two components of $G - X$. Now it cannot occur that both H_1 and H_2 have vertices v_1 and v_2, respectively, that are not incident with any edge of X, for then $d(v_1, v_2) \geq 3$.

We may assume then that every vertex of H_1 is incident with an edge of X. Thus

$$n_1 = |V(H_1)| \leq |X| = \lambda(G) \leq \delta(G). \tag{2.5}$$

Since each vertex u in H_1 is incident with at most $n_1 - 1$ edges in H_1, it follows that u is incident with at least $\delta(G) - n_1 + 1$ edges in X. Consequently,

$$\lambda = |X| \geq n_1(\delta(G) - n_1 + 1). \tag{2.6}$$

If $n_1 = 1$, then (2.6) gives $\lambda(G) \geq \delta(G)$. If, however, $n_1 > 1$, then multiplying the inequality $\delta(G) \geq n_1$ in (2.5) by $n_1 - 1$ gives $\delta(G)(n_1 - 1) \geq n_1(n_1 - 1)$ and so

$$n_1(\delta(G) - n_1 + 1) \geq \delta(G).$$

Hence in both cases, we have $\lambda(G) \geq \delta(G)$. Thus $\lambda(G) = \delta(G)$. ■

If G is a graph of order $n \geq 3$ such that $\deg u + \deg v \geq n - 1$ for every two nonadjacent vertices u and v, then $\mathrm{diam}(G) = 2$. Therefore, the following result of Linda Lesniak [145] is a consequence of Theorem 2.34.

Corollary 2.35 *If G is a graph of order n such that for all distinct nonadjacent vertices u and v,*

$$\deg u + \deg v \geq n - 1,$$

then $\lambda(G) = \delta(G)$.

Exercises for Section 2.4

1. Determine the connectivity and edge-connectivity of each complete k-partite graph.

2. Let $v_1, v_2, \ldots, v_k$ be k distinct vertices of a k-connected graph G. Let H be the graph formed from G by adding a new vertex w of degree k that is adjacent to each of $v_1, v_2, \ldots, v_k$. Show that $\kappa(H) = k$.

3. (a) Prove that if G is a k-connected graph, then $G \vee K_1$ is $(k+1)$-connected.

   (b) Prove that if G is a k-edge-connected graph, then $G \vee K_1$ is $(k+1)$-edge-connected.

4. Let G be a graph with degree sequence $d_1, d_2, \ldots, d_n$ where $d_1 \geq d_2 \geq \cdots \geq d_n$. Determine $\lambda(G \vee K_1)$.

5. Show that every k-connected graph contains every tree of order $k+1$ as a subgraph.

6. (a) Let G be a noncomplete graph of order n and connectivity k such that $\deg v \geq (n+2k-2)/3$ for every vertex v of G. Show that if S is a minimum vertex-cut of G, then $G-S$ has exactly two components.

   (b) Let G be a noncomplete graph of order n and connectivity k such that $\deg v \geq (n+kt-t)/(t+1)$ for some integer $t \geq 2$. Show that if S is a vertex-cut of cardinality $\kappa(G)$, then $G-S$ has at most t components.

7. For a graph G of order $n \geq 2$, define the **k-connectivity** $\kappa_k(G)$ of G $(2 \leq k \leq n)$ as the minimum number of vertices whose removal from G results in a graph with at least k components or a graph of order less than k. (Therefore, $\kappa_2(G) = \kappa(G)$.) A graph G is defined to be **(ℓ, k)-connected** if $\kappa_k(G) \geq \ell$. Let G be a graph of order n containing a set of at least k pairwise nonadjacent vertices. Show that if

$$\deg_G v \geq \left\lceil \frac{n+(k-1)(\ell-2)}{k} \right\rceil$$

   for every $v \in V(G)$, then G is (ℓ, k)-connected.

8. Let G be a graph of diameter 2. Show that if S is a set of $\lambda(G)$ edges whose removal disconnects G, then at least one of the components of $G-S$ is isomorphic to K_1 or to $K_{\delta(G)}$.

9. Let a, b and c be positive integers with $a \leq b \leq c$. Prove that there exists a graph G with $\kappa(G) = a$, $\lambda(G) = b$ and $\delta(G) = c$.

10. Verify that Theorem 2.32 is best possible by showing that for each positive integer k, there exists a graph G of order $n \geq k+1$ such that $\delta(G) = \left\lceil \frac{n+k-3}{2} \right\rceil$ and $\kappa(G) < k$.

11. Verify that Theorem 2.34 is best possible by finding an infinite class of connected graphs G of diameter 3 for which $\lambda(G) \neq \delta(G)$.

12. The **connection number** $c(G)$ of a connected graph G of order $n \geq 2$ is the smallest integer k with $2 \leq k \leq n$ such that *every* induced subgraph of order k in G is connected. State and prove a theorem that gives a relationship between $\kappa(G)$ and $c(G)$ for a graph G of order n.

13. For an even integer $k \geq 2$, show that the minimum size of a k-connected graph of order n is $kn/2$.

14. Prove or disprove: Let G be a nontrivial graph. For every vertex v of G, $\kappa(G - v) = \kappa(G)$ or $\kappa(G - v) = \kappa(G) - 1$.

15. (a) Prove that if G is a k-connected graph and e is an edge of G, then $G - e$ is $(k - 1)$-connected.
    (b) Prove that if G is a k-edge-connected graph and e is an edge of G, then $G - e$ is $(k - 1)$-edge-connected.

16. (a) Show that if G is a 0-regular graph, then $\kappa(G) = \lambda(G)$.
    (b) Show that if G is a 1-regular graph, then $\kappa(G) = \lambda(G)$.
    (c) Show that if G is a 2-regular graph, then $\kappa(G) = \lambda(G)$.
    (d) By (a) – (c) and Theorem 2.31, if G is r-regular, where $0 \leq r \leq 3$, then $\kappa(G) = \lambda(G)$. Find the minimum positive integer r for which there exists an r-regular graph G such that $\kappa(G) \neq \lambda(G)$.
    (e) Find the minimum positive integer r for which there exists an r-regular graph G such that $\lambda(G) \geq \kappa(G) + 2$.

17. For a graph G, define $\overline{\kappa}(G) = \max\{\kappa(H)\}$ and $\overline{\lambda}(G) = \max\{\lambda(H)\}$, where each maximum is taken over all subgraphs H of G. How are $\overline{\kappa}(G)$ and $\overline{\lambda}(G)$ related to $\kappa(G)$ and $\lambda(G)$, respectively, and to each other?

18. Let G_1 and G_2 be two k-connected graphs, where $k \geq 2$, and let $\mathcal{G}$ be the set of all graphs obtained by adding k edges between G_1 and G_2. Determine $\max\{\kappa(G) : G \in \mathcal{G}\}$.

2.5 Menger's Theorem

A nontrivial graph G is connected (or, equivalently, 1-connected) if every two distinct vertices of G are connected by at least one path. This fact has been generalized in several ways, many of which involve, either directly or indirectly, a theorem [157] due to Karl Menger (1902–1985), who was born in Vienna, Austria. Menger developed a talent for mathematics and physics at an early age and entered the University of Vienna in 1920 to study physics. The following year he attended a lecture by Hans Hahn on *Neueres über den Kurvenbegriff* (*What's new concerning the concept of a curve*) and Menger's interests turned toward mathematics. In the lecture it was mentioned that (at that time) there was no satisfactory definition of a curve, despite attempts to do so by a number

of distinguished mathematicians, including Georg Cantor, Camille Jordan and Giuseppe Peano. Some mathematicians, including Felix Hausdorff and Ludwig Bieberbach, felt that it was unlikely that this problem would ever be solved. Despite being an undergraduate with limited mathematical background, Menger solved the problem and presented his solution to Hahn. This led Menger to work on curve and dimension theory. After completing his studies at Vienna, Menger left to broaden his mathematics in Amsterdam.

In 1927 Menger returned to the University of Vienna to accept the position of Chair of Geometry. It was during that year that he published the paper "Zur allgemeinen Kurventheorie" (which contained *Menger's Theorem*). Menger himself referred to this result as the "n-arc theorem" and proved it as a lemma for a theorem in curve theory.

In the spring of 1930, Karl Menger traveled to Budapest and met many Hungarian mathematicians, including Dénes König. Menger had read some of König's papers. During his visit, Menger learned that König was working on a book that would contain what was known about graph theory at that time. Menger was pleased to hear this and mentioned his theorem to König, which had only been published three years earlier. König was not aware of Menger's work and, in fact, didn't believe that the theorem was true. Indeed, the very evening of their meeting, König set out to construct a counterexample. When the two met again the next day, König greeted Menger with "A sleepless night!". König then asked Menger to describe his proof, which he did. After that, König said that he would add a final section to his book on the theorem. As a result, König added a chapter to his 1936 book *Theorie der endlichen und unendliehen Graphen*, which would become the first book written on graph theory [139]. This was a major reason why Menger's theorem became so widely known among those interested in graph theory.

Before stating and presenting a proof of Menger's theorem, some additional terminology is needed. For two nonadjacent vertices u and v in a graph G, a $u-v$ **separating set** is a set $S \subseteq V(G) - \{u, v\}$ such that u and v lie in different components of $G - S$. A $u-v$ separating set of minimum cardinality is called a **minimum** $u-v$ **separating set**.

For two distinct vertices u and v in a graph G, a collection of $u-v$ paths is **internally disjoint** if every two paths in the collection have only u and v in common. Menger's theorem states that the concepts of internally disjoint $u-v$ paths and $u-v$ separating sets are linked. In the graph G of Figure 2.30, there is a set $S = \{w_1, w_2, w_3\}$ of vertices of G that separate the vertices u and v. No set with fewer than three vertices separates u and v. According to Menger's theorem, there are three internally disjoint $u-v$ paths in G.

Theorem 2.36 (**Menger's Theorem**) *Let u and v be nonadjacent vertices in a graph G. The minimum number of vertices in a $u-v$ separating set equals the maximum number of internally disjoint $u-v$ paths in G.*

Proof. We proceed by induction on the size of graphs. The theorem is certainly true for every empty graph. Assume that the theorem holds for all graphs

Figure 2.30: A graph illustrating Menger's theorem

of size less than m, where $m \geq 1$, and let G be a graph of size m. Moreover, let u and v be two nonadjacent vertices of G. If u and v belong to different components of G, then the result follows. So we may assume that u and v belong to the same component of G. Suppose that a minimum $u - v$ separating set consists of $k \geq 1$ vertices. Then G contains at most k internally disjoint $u - v$ paths. We show, in fact, that G contains k internally disjoint $u - v$ paths. Since this is obviously true if $k = 1$, we may assume that $k \geq 2$. We now consider three cases.

Case 1. *Some minimum $u - v$ separating set X in G contains a vertex x that is adjacent to both u and v.* Then $X - \{x\}$ is a minimum $u - v$ separating set in $G - x$ consisting of $k - 1$ vertices. Since the size of $G - x$ is less than m, it follows by the induction hypothesis that $G - x$ contains $k - 1$ internally disjoint $u - v$ paths. These paths together with the path $P = (u, x, v)$ produce k internally disjoint $u - v$ paths in G.

Case 2. *For every minimum $u - v$ separating set S in G, either every vertex in S is adjacent to u and not to v or every vertex in S is adjacent to v and not to u.* Necessarily then, $d(u, v) \geq 3$. Let $P = (u, x, y, \ldots, v)$ be a $u - v$ geodesic in G, where $e = xy$. Every minimum $u - v$ separating set in $G - e$ contains at least $k - 1$ vertices. We show, in fact, that every minimum $u - v$ separating set in $G - e$ contains k vertices.

Suppose that there is some minimum $u - v$ separating set in $G - e$ with $k - 1$ vertices, say $Z = \{z_1, z_2, \ldots, z_{k-1}\}$. Then $Z \cup \{x\}$ is a $u - v$ separating set in G and therefore a minimum $u - v$ separating set in G. Since x is adjacent to u (and not to v), it follows that every vertex z_i $(1 \leq i \leq k - 1)$ is also adjacent to u and not adjacent to v.

Since $Z \cup \{y\}$ is also a minimum $u - v$ separating set in G and each vertex z_i $(1 \leq i \leq k - 1)$ is adjacent to u but not to v, it follows that y is adjacent to u. This, however, contradicts the assumption that P is a $u - v$ geodesic. Thus k is the minimum number of vertices in a $u - v$ separating set in $G - e$. Since the size of $G - e$ is less than m, it follows by the induction hypothesis that there are k internally disjoint $u - v$ paths in $G - e$ and in G as well.

Case 3. *There exists a minimum $u - v$ separating set W in G in which no vertex is adjacent to both u and v and containing at least one vertex not adjacent to u and at least one vertex not adjacent to v.* Let $W = \{w_1, w_2, \ldots, w_k\}$. Let G_u be the subgraph of G consisting of, for each i with $1 \leq i \leq k$, all $u - w_i$ paths in G in which $w_i \in W$ is the only vertex of the path belonging to W. Let G'_u be the graph constructed from G_u by adding a new vertex v' and joining v' to each vertex w_i for $1 \leq i \leq k$. The graphs G_v and G'_v are defined similarly.

Since W contains a vertex that is not adjacent to u and a vertex that is not adjacent to v, the sizes of both G'_u and G'_v are less than m. So G'_u contains k internally disjoint $u - v'$ paths A_i $(1 \leq i \leq k)$, where A_i contains w_i. Also, G'_v contains k internally disjoint $u' - v$ paths B_i $(1 \leq i \leq k)$, where B_i contains w_i. Let A'_i be the $u - w_i$ subpath of A_i and let B'_i be the $w_i - v$ subpath of B_i for $1 \leq i \leq k$. The k paths constructed from A'_i and B'_i for each i $(1 \leq i \leq k)$ are internally disjoint $u - v$ paths in G. ■

Whitney's Theorem

We mentioned that Karl Menger visited Dénes König in Budapest, Hungary, early in 1930, which led to his theorem being included in König's book. Later in 1930, Menger went to the United States and spent the period from September of 1930 to February of 1931 as a visiting lecturer at Harvard University. It was at one of his seminar talks that he presented his theorem. During that period, Hassler Whitney (1907–1989) was doing research for his doctoral thesis in graph theory at Harvard. For a short period after receiving his Ph.D. in 1932 from Harvard, Whitney worked on graph theory, making important contributions, but thereafter turned to topology when the area was just being called *topology*. Whitney was a faculty member at Harvard until 1952, when he went to the Institute for Advanced Study in Princeton. Whitney was at Harvard during the period that the United States was involved in World War II. He had great interest and ability in applied mathematics. Because of this, he was brought in as a consultant to the Applied Mathematics Group at Columbia University. That part of the Applied Mathematics Panel was primarily the responsibility of Whitney. He developed mathematical principles to discover best techniques for aerial gunnery and was involved in making improvements to weapons systems.

Although research was a large part of Whitney's professional life, he contributed to mathematics in many ways. During 1944-1949 he was editor of the *American Journal of Mathematics* and during 1949-1954 he was editor of *Mathematical Reviews*. During 1953-1956, he chaired the National Science Foundation mathematical panel. On the personal side, Whitney was an avid mountain climber. In fact, the Whitney-Gilman Ridge on Cannon Cliff in Franconia, New Hampshire, was named for his cousin and him, who were the first to climb it (on August 3, 1939).

If G is a k-connected graph $(k \geq 1)$ and v is a vertex of G, then $G - v$ is $(k - 1)$-connected. In fact, if $e = uv$ is an edge of G, then $G - e$ is also

$(k-1)$-connected (see Exercise 15 in Section 2.4). With the aid of Menger's theorem, a useful characterization of k-connected graphs, due to Whitney [238], was established. Since nonseparable graphs of order 3 or more are 2-connected, this gives a generalization of Corollary 2.4.

Theorem 2.37 (**Whitney's Theorem**) *A nontrivial graph G is k-connected for some integer $k \geq 2$ if and only if for each pair u, v of distinct vertices of G, there are at least k internally disjoint $u - v$ paths in G.*

Proof. First, suppose that G is a k-connected graph, where $k \geq 2$, and let u and v be two distinct vertices of G. Assume first that u and v are not adjacent. Let U be a minimum $u - v$ separating set. Then

$$k \leq \kappa(G) \leq |U|.$$

By Menger's theorem, G contains at least k internally disjoint $u - v$ paths.

Next, assume that u and v are adjacent, where $e = uv$. As observed earlier, $G - e$ is $(k-1)$-connected. Let W be a minimum $u - v$ separating set in $G - e$. So

$$k - 1 \leq \kappa(G - e) \leq |W|.$$

By Menger's theorem, $G - e$ contains at least $k - 1$ internally disjoint $u - v$ paths, implying that G contains at least k internally disjoint $u - v$ paths.

For the converse, assume that G contains at least k internally disjoint $u - v$ paths for every pair u, v of distinct vertices of G. If G is complete, then $G = K_n$, where $n \geq k + 1$, and so $\kappa(G) = n - 1 \geq k$. Hence G is k-connected. Thus we may assume that G is not complete.

Let U be a minimum vertex-cut of G. Then $|U| = \kappa(G)$. Let x and y be vertices in distinct components of $G - U$. Thus U is an $x - y$ separating set of G. Since there are at least k internally disjoint $x - y$ paths in G, it follows by Menger's theorem that

$$k \leq |U| = \kappa(G).$$

and so G is k-connected. ■

The following three results are consequences of Theorem 2.37 (see Exercises 3-5).

Corollary 2.38 *Let G be a k-connected graph, $k \geq 1$, and let S be any set of k vertices of G. If a graph H is obtained from G by adding a new vertex w and joining w to the vertices of S, then H is also k-connected.*

Corollary 2.39 *If G is a k-connected graph, $k \geq 2$, and $u, v_1, v_2, \ldots, v_t$ are $t + 1$ distinct vertices of G, where $2 \leq t \leq k$, then G contains a $u - v_i$ path for each i $(1 \leq i \leq t)$, every two paths of which have only u in common.*

Corollary 2.40 *A graph G of order $n \geq 2k$ is k-connected if and only if for every two disjoint sets V_1 and V_2 of k distinct vertices each, there exist k pairwise disjoint paths connecting V_1 and V_2.*

By Theorem 2.3, every two vertices in a 2-connected graph lie on a common cycle of the graph. Gabriel Dirac [61] generalized this to k-connected graphs.

Theorem 2.41 *If G is a k-connected graph, $k \geq 2$, then every k vertices of G lie on a common cycle of G.*

Proof. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of k vertices of G. Among all cycles in G, let C be one containing a maximum number ℓ of vertices of S. Then $\ell \leq k$. If $\ell = k$, then the result follows, so we may assume that $\ell < k$. Since G is k-connected, G is 2-connected and so by Theorem 2.3, $\ell \geq 2$. We may further assume that $v_1, v_2, \ldots, v_\ell$ lie on C. Let u be a vertex of S that does not lie on C. We consider two cases.

Case 1. *The cycle C contains exactly ℓ vertices, say $C = (v_1, v_2, \ldots, v_\ell, v_1)$.* By Corollary 2.39, G contains a $u - v_i$ path P_i for each i with $1 \leq i \leq \ell$ such that every two of the paths $P_1, P_2, \ldots, P_\ell$ have only u in common. Replacing the edge v_1v_2 on C by P_1 and P_2 produces a cycle containing at least $\ell + 1$ vertices of S. This is a contradiction.

Case 2. *The cycle C contains at least $\ell + 1$ vertices.* Let v_0 be a vertex on C that does not belong to S. Since $2 < \ell + 1 \leq k$, it follows by Corollary 2.39 that G contains a $u - v_i$ path P_i for each i with $0 \leq i \leq \ell$ such that every two of the paths $P_0, P_1, \ldots, P_\ell$ have only u in common. For each i $(0 \leq i \leq \ell)$, let u_i be the first vertex of P_i that belongs to C and let P_i' be the $u - u_i$ subpath of P_i. Suppose that the vertices u_i $(0 \leq i \leq \ell)$ are encountered in the order $u_0, u_1, \ldots, u_\ell$ as we proceed about C in some direction. For some i with $0 \leq i \leq \ell$ and $u_{\ell+1} = u_0$, there is a $u_i - u_{i+1}$ path P on C, none of whose internal vertices belong to S. Replacing P on C by P_i' and P_{i+1}' produces a cycle containing at least $\ell + 1$ vertices of S. Again, this is a contradiction. ■

There are analogues to Menger's theorem (Theorem 2.36) and to Whitney's theorem (Theorem 2.37) in terms of edge-cuts. For two distinct vertices u and v in a graph G, an edge-cut X of G is a $u - v$ **separating set** if u and v lie in different components of $G - X$. We begin with the edge analogue of Theorem 2.36.

Theorem 2.42 *For distinct vertices u and v in a graph G, the minimum cardinality of a $u - v$ separating set $X \subseteq E(G)$ equals the maximum number of pairwise edge-disjoint $u - v$ paths in G.*

Proof. It is convenient to actually prove a stronger result here by allowing G to be a multigraph.

If u and v are vertices in different components of a multigraph G, then the theorem is immediate. Hence we may assume that the multigraphs under

consideration are connected. If the minimum number of edges that separate u and v is 1, then G contains a bridge e so that u and v lie in different components of $G - e$. Thus every $u - v$ path in G contains e, so the maximum number of pairwise edge-disjoint $u - v$ paths in G is also 1.

Suppose that the statement is false. Then there is a smallest integer $k \geq 2$ such that there exist multigraphs containing two vertices u and v for which the minimum number of edges that separate u and v is k but there do not exist k pairwise edge-disjoint $u - v$ paths. Among all such multigraphs, let G be one of minimum size.

If every $u - v$ path of G has length 1 or 2, then since the minimum number of edges of G that separate u and v is k, it follows that there are k pairwise edge-disjoint $u - v$ paths in G, which produces a contradiction. Thus G contains a $u - v$ path P of length 3 or more.

Let e_1 be an edge of P that is incident with neither u nor v. Consider the multigraph $G - e_1$. Since the size of $G - e_1$ is less than the size of G, it is impossible for $G - e_1$ to contain k edges that separate u and v. Thus e_1 must belong to every set of k edges that separate u and v. Let $S = \{e_1, e_2, \ldots, e_k\}$ be one such set.

We now subdivide each edge of S, that is, each edge $e_i = u_i v_i$ of S is replaced by a new vertex w_i and the two new edges $u_i w_i$ and $w_i v_i$ for $1 \leq i \leq k$. The k vertices w_i $(1 \leq i \leq k)$ are now identified, producing a new vertex w and a new multigraph H. (See Figure 2.31 for a possible situation.) Observe that w is a cut-vertex in H and every $u - v$ path of H contains w.

Figure 2.31: A step in the proof of Theorem 2.42

Denote by H_u the submultigraph of H consisting of all $u - w$ paths of H and denote by H_v the submultigraph consisting of all $v - w$ paths of H. The minimum number of edges separating u and w in H_u is k and the minimum number of edges separating v and w in H_v is k. Since each of H_u and H_v has smaller size than G, it follows that H_u contains k pairwise edge-disjoint $u - w$ paths and that H_v contains k pairwise edge-disjoint $w - v$ paths.

For $i = 1, 2, \ldots, k$, we can pair off a $u - w$ path in H_u and a $w - v$ path in H_v to produce a $u - v$ path in H containing the two edges $u_i w$ and $w v_i$. This produces k pairwise edge-disjoint $u - v$ paths in H. The process of subdividing the edges $u_i v_i$ of G and identifying the vertices w_i to obtain w can now be

reversed to produce k pairwise edge-disjoint $u - v$ paths in G. This, however, produces a contradiction.

Since the theorem has been proved for multigraphs, it is valid for graphs. ■

With the aid of Theorem 2.42, it is now possible to present an edge analogue of Theorem 2.37 (see Exercise 9).

Theorem 2.43 *A nontrivial graph G is k-edge-connected if and only if G contains k pairwise edge-disjoint $u-v$ paths for each pair u, v of distinct vertices of G.*

Exercises for Section 2.5

1. Show that the converse of Theorem 2.41 is not true in general.

2. Prove that a graph G of order $n \geq k+1 \geq 3$ is k-connected if and only if for each set S of k distinct vertices of G and for each two-vertex subset T of S, there is a cycle of G that contains both vertices of T but no vertices of $S - T$.

3. Prove Corollary 2.38: *Let G be a k-connected graph, $k \geq 1$, and let S be any set of k vertices of G. If a graph H is obtained from G by adding a new vertex w and joining w to the vertices of S, then H is also k-connected.*

4. Prove Corollary 2.39: *If G is a k-connected graph, $k \geq 2$, and u, v_1, v_2, ..., v_t are $t+1$ distinct vertices of G, where $2 \leq t \leq k$, then G contains a $u - v_i$ path for each i $(1 \leq i \leq t)$, every two paths of which have only u in common.*

5. Prove Corollary 2.40: *A graph G of order $n \geq 2k$ is k-connected if and only if for every two disjoint sets V_1 and V_2 of k distinct vertices each, there exist k pairwise disjoint paths connecting V_1 and V_2.*

6. Let G be a k-connected graph and let v be a vertex of G. For a positive integer t, define G_t to be the graph obtained from G by adding t new vertices $u_1, u_2, \ldots, u_t$ and all edges of the form $u_i w$, where $1 \leq i \leq t$ and for which $vw \in E(G)$. Show that G_t is k-connected.

7. Show that if G is a k-connected graph with nonempty disjoint subsets S_1 and S_2 of $V(G)$, then there exist k internally disjoint paths $P_1, P_2, \ldots, P_k$ such that each path P_i is a $u - v$ path for some $u \in S_1$ and some $v \in S_2$ for $i = 1, 2, \ldots, k$ and $|S_1 \cap V(P_i)| = |S_2 \cap V(P_i)| = 1$.

8. Let G be a k-connected graph, $k \geq 3$, and let $v, v_1, v_2, \ldots, v_{k-1}$ be k vertices of G. Show that G has a cycle containing all of $v_1, v_2, \ldots, v_{k-1}$ but not v and $k-1$ internally disjoint $v - u_i$ paths P_i $(1 \leq i \leq k-1)$ such that for each i, the vertex u_i is the only vertex of P_i on C.

9. Prove Theorem 2.43: *A nontrivial graph G is k-edge-connected if and only if G contains k pairwise edge-disjoint $u - v$ paths for each pair u, v of distinct vertices of G.*

10. Prove or disprove: If G is a k-edge-connected graph and $v, v_1, v_2, \ldots, v_k$ are $k + 1$ vertices of G, then for $i = 1, 2, \ldots, k$, there exist $v - v_i$ paths P_i such that each path P_i contains exactly one vertex of $\{v_1, v_2, \ldots, v_k\}$, namely v_i, and for $i \neq j$, P_i and P_j are edge-disjoint.

11. Prove or disprove: If G is a k-edge-connected graph with nonempty disjoint subsets S_1 and S_2 of $V(G)$, then there exist k edge-disjoint paths $P_1, P_2, \ldots, P_k$ such that for each i, P_i is a $u - v$ path for some $u \in S_1$ and some $v \in S_2$ for $i = 1, 2, \ldots, k$ and $|S_1 \cap V(P_i)| = |S_2 \cap V(P_i)| = 1$.

12. Show that $\kappa(Q_n) = \lambda(Q_n) = n$ for all positive integer n.

13. Assume that G is a *graph* in the proof of Theorem 2.42. Does the proof go through? If not, where does it fail?

14. Let G be a graph of order n with $\kappa(G) \geq 1$. Prove that

$$n \geq k(G)[\mathrm{diam}(G) - 1] + 2.$$

15. By a **chorded cycle** is meant a cycle C (of length at least 4) together with an edge that joins two nonconsecutive vertices of C. Prove that every 3-connected graph contains a chorded cycle but that this need not be the case for a 2-connected graph.

16. (a) Show that for every two vertices u and v of a 3-connected graph G, there exist two internally disjoint $u - v$ paths of different lengths in G.

    (b) Show that the result in (a) is not true in general if G is 2-connected.

17. (a) Prove that if G is a connected graph of order n and k is an integer with $1 \leq k \leq n - 3$ such that the maximum number of internally disjoint $u - v$ paths in G is k for every pair u, v of nonadjacent vertices of G, then G contains a vertex-cut with exactly $k + 1$ vertices.

    (b) Let G be a k-connected graph of diameter k, where $k \geq 2$. Prove that G contains $k+1$ distinct vertices $v, v_1, v_2, \ldots, v_k$ and k internally disjoint $v - v_i$ paths P_i $(1 \leq i \leq k)$ such that P_i has length i.

Chapter 3

Eulerian and Hamiltonian Graphs

Many of the early concepts and theorems of graph theory came about quite indirectly, often from recreational mathematics, through puzzles or games, or problems that, as were seen later, could be phrased in terms of graphs. The very first of these was a puzzle called the *Königsberg Bridge Problem*, which was not only solved by one of the most famous mathematicians of all time but whose solution is considered the origin of graph theory. Over a century later a game called the *Icosian* was invented by a well-known mathematician, knighted for his contribution to physics, and this too would prove to have close connections to graph theory. This puzzle and game would give rise to two important concepts in graph theory. It is these concepts that are the major topics of this chapter.

3.1 Eulerian Graphs

Early in the 18th century, the East Prussian city of Königsberg (now called Kaliningrad and located in Russia) occupied both banks of the River Pregel and the island of Kneiphof, lying in the river at a point where it branches into two parts. There were seven bridges that spanned various sections of the river. (See Figure 3.1.)

The Königsberg Bridge Problem

A popular puzzle, called the **Königsberg Bridge Problem**, asked whether there was a route that crossed each of these bridges exactly once. Although such a route was long thought to be impossible, the first mathematical verification of this was presented by the famed mathematician Leonhard Euler (1707–1783) at the Petersburg Academy on 26 August 1735. Euler's proof was contained in a

paper [76] that would turn out to be the beginning of graph theory. This paper appeared in the 1736 volume of the proceedings of the Petersburg Academy. Euler's paper, written in Latin, started as follows (translated into English):

> *In addition to that branch of geometry which is concerned with magnitudes, and which has always received the greatest attention, there is another branch, previously almost unknown, which Leibniz first mentioned, calling it the geometry of position. This branch is concerned only with the determination of position and its properties; it does not involve measurements, nor calculations made with them. It has not yet been satisfactorily determined what kind of problems are relevant to this geometry of position, or what methods should be used in solving them. Hence, when a problem was recently mentioned, which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it was concerned with the geometry of position – especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this kind of problem, as an example of the geometry of position.*

Euler denoted the island Kneiphof in Königsberg by the letter A and the other three land regions by B, C and D. The seven bridges that crossed the River Pregel were denoted by a, b, c, d, e, f and g (see Figure 3.1).

Figure 3.1: The bridges of Königsberg

In his paper, Euler describes what must occur if there was a route that crossed each of the seven bridges exactly once. Such a route could be represented as a sequence of letters, each term of which is one of the letters A, B, C and D. A particular term in this sequence would indicate that the route had reached that land region and the term immediately following it would indicate the land region to which the route had progressed after crossing a bridge. Since there are seven bridges, the sequence must consist of eight terms.

Because there are five bridges leading into (or out of) land region A (the island Kneiphof), each occurrence of A must indicate that either the route began at A, ended at A or had progressed to and then exited from A. Thus A must appear three times in the sequence. In a similar manner, each of B, C and D must appear twice in the sequence. However, this implies that such a sequence contains nine terms, which is impossible. Therefore, there is no route in Königsberg that crosses each bridge exactly once.

As Euler mentioned in his paper, he also formulated a more general problem. In order to describe and present a solution to the general problem, we turn to the modern-day approach in which both the Königsberg Bridge Problem and its generalization are described in terms of graphs.

Let G be a nontrivial connected graph. A circuit C of G that contains every edge of G (necessarily exactly once) is an **Eulerian circuit**, while an open trail that contains every edge of G is an **Eulerian trail**. (Some refer to an Eulerian circuit as an **Euler tour**.) The graph G_1 of Figure 3.2 contains an Eulerian trail while G_2 contains an Eulerian circuit.

Figure 3.2: Graphs with Eulerian trails and Eulerian circuits

These terms are defined in exactly the same way if G is a nontrivial connected multigraph. In fact, the map of Königsberg in Figure 3.1 can be represented by the multigraph shown in Figure 3.3. Then the Königsberg Bridge Problem can be reformulated as follows:

> Does the multigraph shown in Figure 3.3 contain either an Eulerian circuit or an Eulerian trail?

As Euler showed (although not using this terminology, of course, nor even graphs), the answer to this question is *no*.

Characterizations of Eulerian Graphs

A connected graph G is called **Eulerian** if G contains an Eulerian circuit. Thus the graph G_2 of Figure 3.2 is Eulerian. Simple but useful characterizations of both Eulerian graphs and graphs with Eulerian trails exist; in fact, in each case the characterization is attributed to Euler [76].

Theorem 3.1 *A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.*

Figure 3.3: The multigraph of Königsberg

Proof. Assume first that G is an Eulerian graph. Then G contains an Eulerian circuit C. Let v be a vertex of G. Suppose first that v is not the initial vertex of C (and thus not the terminal vertex of C either). Since each occurrence of v in C indicates that v is both entered and exited on C and produces a contribution of 2 to the degree of v, the degree of v is even. Next, suppose that v is the initial and terminal vertex of C. As the initial vertex of C, this represents a contribution of 1 to the degree of v. There is also a contribution of 1 to the degree of v because v is the terminal vertex of C as well. Every other occurrence of v on C again represents a contribution of 2 to the degree of v. Here too v is even.

We now turn to the converse. Let G be a nontrivial connected graph in which every vertex is even. Let u be a vertex of G. First, we show that G contains a $u-u$ circuit. Construct a trail T beginning at u that contains a maximum number of edges of G. We claim that T is, in fact, a circuit; for suppose that T is a $u-v$ trail, where $v \neq u$. Then there is an odd number of edges incident with v and belonging to T. Since the degree of v in G is even, there is at least one edge incident with v that does not belong to T. Suppose that vw is such an edge. However then, T followed by w produces a trail T' with initial vertex u containing more edges than T, which is impossible. Thus T is a circuit with initial and terminal vertex u. We now denote T by C.

If C is an Eulerian circuit of G, then the proof is complete. Hence we may assume that C does not contain all edges of G. Since G is connected, there is a vertex x on C that is incident with an edge that does not belong to C. Let $H = G - E(C)$. Since every vertex on C is incident with an even number of edges on C, it follows that every vertex of H is even. Let H' be the component of H containing x. Consequently, every vertex of H' has positive even degree. By the same argument as before, H' contains a circuit C' with initial and terminal vertex x. By inserting C' at some occurrence of x in C, a $u-u$ circuit C'' in G is produced having more edges than C. This is a contradiction. ■

A characterization of connected graphs containing an Eulerian trail is also due to Euler.

Theorem 3.2 *A connected graph G contains an Eulerian trail if and only if*

exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at the other.

Proof. If G contains an Eulerian trail $u - v$ trail T, then we construct a new connected graph H from G by adding a new vertex w of degree 2 and joining it to u and v. Then T together with the two edges uw and wv form an Eulerian circuit in H. By Theorem 3.1, every vertex of H is even and so only u and v have odd degrees in $G = H - w$.

For the converse, let G be a connected graph containing exactly two vertices u and v of odd degree. We show that G contains an Eulerian trail T, where T is either a $u - v$ trail or a $v - u$ trail. Add a new vertex w of degree 2 to G and join it to u and v, calling the resulting graph H. Therefore, H is a connected graph all of whose vertices are even. By Theorem 3.1, H is an Eulerian graph containing an Eulerian circuit C. The circuit C necessarily contains uw and wv as consecutive edges. Deleting w from C results in an Eulerian trail of G that begins at u or v and terminates at the other. ■

In the next-to-last paragraph of Euler's paper, Euler wrote (again an English translation):

> *So whatever arrangement may be proposed, one can easily determine whether or not a journey can be made, crossing each bridge once, by the following rules:*
>
> *If there are more than two areas to which an odd number of bridges lead, then such a journey is impossible.*
>
> *If, however, the number of bridges is odd for exactly two areas, then the journey is possible if it starts in either of these areas.*
>
> *If, finally, there are no areas to which an odd number of bridges lead, then the required journey can be accomplished from any starting point.*
>
> *With these rules, the given problem can also be solved.*

Euler ended his paper by writing:

> *When it has been determined that such a journey can be made, one still has to find how it should be arranged. For this I use the following rule: let those pairs of bridges which lead from one area to another be mentally removed, thereby considerably reducing the number of bridges; it is then an easy task to construct the required route across the remaining bridges, and the bridges which have been removed will not significantly alter the route found, as will become clear after a little thought. I do not therefore think it worthwhile to give any further details concerning the finding of the routes.*

Consequently, in Euler's paper, he actually only verified that every vertex being even is a necessary condition for a connected graph to be Eulerian and

that exactly two vertices being odd is a necessary condition for a connected graph to contain an Eulerian trail. Euler did not show that these are sufficient conditions. The first proof that these are also sufficient conditions would not be published for another 137 years, in an 1873 paper authored by Carl Hierholzer [118]. Hierholzer received his Ph.D. in 1870 and died in 1871. Thus his paper was published two years after his death. He had told colleagues of what he had done but died before he could write a paper containing this work. His colleagues wrote the paper on his behalf and had it published for him.

Theorems 3.1 and 3.2 hold for multigraphs as well as graphs. By these theorems then, the multigraph of Figure 3.3 contains neither an Eulerian trail nor an Eulerian circuit.

As we have now seen, if G is a nontrivial connected graph with no odd vertices, then G contains an Eulerian circuit. If G contains exactly two odd vertices, then G contains an Eulerian trail. If G contains more than two odd vertices, then G contains neither an Eulerian circuit nor an Eulerian trail. However, the following result by Gary Chartrand, Albert D. Polimeni and M. James Stewart [38] shows that every connected graph with odd vertices must contain a certain number of trails of a certain type and of certain lengths.

The **distance between two subgraphs** F **and** H in a connected graph G is $\min\{d(u,v) : u \in V(F), v \in V(H)\}$.

Theorem 3.3 *If G is a connected graph containing $2k$ odd vertices ($k \geq 1$), then G contains k pairwise edge-disjoint open trails connecting odd vertices and such that every edge of G lies on one of these trails and at most one of these trails has odd length.*

Proof. Let $y_1, y_2, \ldots, y_k$ and $z_1, z_2, \ldots, z_k$ be the odd vertices of G. We construct a new graph H from G by adding k new vertices $x_1, x_2, \ldots, x_k$ to G and joining x_i to y_i and z_i for $i = 1, 2, \ldots, k$. Thus H is Eulerian and therefore contains an Eulerian circuit C. Since $y_i x_i$ and $x_i z_i$ are consecutive on C for $i = 1, 2, \ldots, k$, deleting the k vertices x_i ($1 \leq i \leq k$) from H results in k pairwise edge-disjoint trails in G connecting odd vertices such that every edge of G lies on one of these trails.

It remains to show that there are k such trails, at most one of which has odd length. Assume, to the contrary, that in any collection of k pairwise edge-disjoint trails of G connecting odd vertices and such that every edge of G lies on one of these trails, there are at least two trails of odd length. Among all such collections of k trails, consider those collections containing a minimum number of trails of odd length; and, among those, consider one, say $\{T_1, T_2, \ldots, T_k\}$, where the distance between some pair of trails of odd length is minimum. If two trails T_a and T_b of odd length have a vertex in common, then they may be replaced by two trails T_a^* and T_b^* of even length connecting odd vertices such that

$$E(T_a) \cup E(T_b) = E(T_a^*) \cup E(T_b^*).$$

Otherwise, let T_r and T_s be two trails of odd length where the distance between T_r and T_s attains this minimum distance. Suppose that P is a path of minimum length connecting a vertex w_r in T_r and a vertex w_s in T_s, and let $w_r x$ be the edge of P incident with w_r. Then $w_r x$ belongs to a trail T_p among $T_1, T_2, \ldots, T_k$. Necessarily, T_p has even length. However, T_r and T_p have the vertex w_r in common. So T_r and T_p may be replaced by trails T'_r and T'_p connecting odd vertices such that T'_r has even length, T'_p has odd length,

$$E(T_r) \cup E(T_p) = E(T'_r) \cup E(T'_p)$$

and $w_r x$ belongs to T'_p. Since the distance between T'_p and T_s is less than the distance between T_r and T_s, this is a contradiction. ■

Eulerian graphs have several useful characterizations. We present two of these, the first of which is due to Oswald Veblen [226]. Veblen's result characterizes Eulerian graphs in terms of their cycle structure.

Theorem 3.4 *A nontrivial connected graph G is Eulerian if and only if G contains pairwise edge-disjoint cycles such that every edge of G lies on one of these cycles.*

Proof. First suppose that G is Eulerian. We proceed by induction on the size m of G. If $m = 3$, then $G = K_3$ has the desired property. Assume, then, that every Eulerian graph of size less than m, where $m \geq 4$, contains pairwise edge-disjoint cycles such that every edge of the graph lies on one of these cycles. Let G be an Eulerian graph of size m. Since G is Eulerian, every vertex of G has even degree. Thus G is not a tree and so G contains at least one cycle C. If $G = C_m$, then the proof is complete. Otherwise, there are edges of G not in C. Removing the edges of C from G produces a graph G' in which every vertex is even. Thus each nontrivial component of G' is Eulerian and has fewer than m edges. By the induction hypothesis, each nontrivial component of G' contains pairwise edge-disjoint cycles such that every edge of this component lies on one of these cycles. Now all of these cycles of the components of G', together with C, give the desired result.

For the converse, suppose that G contains pairwise edge-disjoint cycles such that every edge of G lies on one of these cycles. Then every vertex G is even and so G is Eulerian by Theorem 3.1. ■

The next characterization of Eulerian graphs also involves cycles. The necessity of this result is due to Shunichi Toida [218] and the sufficiency to Terry A. McKee [156].

Theorem 3.5 *A nontrivial connected graph G is Eulerian if and only if every edge of G lies on an odd number of cycles in G.*

Proof. First, let G be an Eulerian graph and let $e = uv$ be an edge of G. Then $G - e$ is connected. Consider the set of all $u - v$ trails for which v appears

exactly once, namely as the terminal vertex. There is an odd number of edges possible for the initial edge of such a trail. Once the initial edge has been chosen and the trail has then proceeded to the next vertex, say w, then again there is an odd number of choices for the edges that are incident with w but different from uw. We continue this process until we arrive at vertex v. At each vertex different from v in such a trail, there is an odd number of edges available for a continuation of the trail. Hence there is an odd number of these trails.

Suppose that T_1 is a $u-v$ trail that is not a $u-v$ path and T_1 contains v only once. Then some vertex v_1 ($\neq v$) occurs at least twice on T_1, implying that T_1 contains a $v_1 - v_1$ circuit, say $C = (v_1, v_2, \ldots, v_k, v_1)$. Hence, there exists a $u-v$ trail T_2 identical to T_1 except that C is replaced by the "reverse" circuit $C' = (v_1, v_k, v_{k-1}, \ldots, v_2, v_1)$. This implies that the $u-v$ trails that are not $u-v$ paths occur in pairs. Therefore, there is an even number of such $u-v$ trails that are not $u-v$ paths and, consequently, there is an odd number of $u-v$ paths in $G-e$. This, in turn, implies that there is an odd number of cycles containing e.

For the converse, suppose that G is a nontrivial connected graph that is not Eulerian. We show that some edge of G lies on an even number of cycles in G. If G contains a bridge f, then f lies on no cycles. So we may assume that G contains no bridges. Since G is not Eulerian, G contains a vertex v of odd degree. For each edge e incident with v, denote by $c(e)$ the number of cycles of G containing e. Since each such cycle contains two edges incident with v, it follows that the sum $\sum c(e)$, taken over all edges e incident with v, equals twice the number of cycles containing v. Because there is an odd number of terms in this sum, some $c(e)$ is even. ■

Exercises for Section 3.1

1. In present-day Königsberg (Kaliningrad), there are two additional bridges, one between regions B and C and one between regions B and D. Is it now possible to devise a route over all bridges of Königsberg without recrossing any of them?

2. In Euler's solution of the Königsberg Bridge Problem, he observed that if there was a route that crossed each bridge exactly once, then this route could be represented by a sequence of eight letters, each of which is one of the four land regions A, B, C and D shown in Figure 3.1. Show that it is impossible for any of these letters to appear only among the middle six terms of the sequence. What conclusion can be made from this observation?

3. Let F and H be two disjoint connected non-Eulerian regular graphs and let $G = (F + H) \vee K_1$, that is, G is obtained from F and H by adding a new vertex v and joining v to each vertex in F and H. Prove that G is Eulerian.

4. Let G be a connected graph of order $n \geq 6$ that has neither an Eulerian circuit nor an Eulerian trail. A graph H is constructed by adding a new vertex v to G and joining v to every odd vertex of G. Prove or disprove: H is Eulerian.

5. Find a necessary and sufficient condition for the Cartesian product $G \square H$ of two nontrivial connected graphs G and H to be Eulerian.

6. Prove that if a graph of order $n \geq 6$ has an Eulerian $u - v$ trail such that $\deg u - \deg v \geq n - 2$, then n must be even.

7. Suppose that G is an r-regular graph of order n such that both G and its complement $\overline{G}$ are connected. Is it possible that neither G nor $\overline{G}$ is Eulerian?

8. Show that if T is a tree containing at least one vertex of degree 2, then $\overline{T}$ is not Eulerian.

9. Prove that an Eulerian graph G has even size if and only if G has an even number of vertices v for which $\deg v \equiv 2 \pmod 4$.

10. (a) Prove that every Eulerian graph of odd order has three vertices of the same degree.
    (b) Prove that for each odd integer $n \geq 3$, there exists exactly one Eulerian graph of order n containing exactly three vertices of the same degree and at most two vertices of any other degree.

11. Let G be a connected graph with exactly two odd vertices u and v, where $\deg u \geq 3$ and $\deg v \geq 3$. Prove or disprove:
    (a) There exist two edge-disjoint $u - v$ trails in G.
    (b) If G is 2-edge-connected, then there exist two edge-disjoint $u - v$ trails in G such that every edge of G lies on one of these trails.

3.2 Hamiltonian Graphs

William Rowan Hamilton (1805–1865) was gifted even as a child and his numerous interests and talents ranged from languages (having mastered many by age 10) to mathematics and physics. In 1832 he predicted that a ray of light passing through a biaxial crystal would be refracted into the shape of a cone. When this was experimentally confirmed, it was considered a major discovery and led to his being knighted in 1835, thereby becoming *Sir* William Rowan Hamilton. Even today, Hamilton is regarded as one of the leading mathematicians and physicists of the 19th century.

Although Hamilton's accomplishments were many, one of his best known in mathematics was his creation of a new algebraic system called *quaternions*. On 16 October 1843, while walking with his wife along the Royal Canal in

Dublin, Hamilton suddenly discovered a collection of 4-dimensional numbers $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = (a, b, c, d)$, where a, b, c and d are real numbers, that formed a division algebra. Furthermore,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Hamilton carved these equations into the stone of the Brougham Bridge. In the quaternions, $\mathbf{ij} = \mathbf{k}$ and $\mathbf{ji} = -\mathbf{k}$; so the quaternions are not commutative.

The Icosian Game

In 1856 Hamilton developed another example of a non-commutative algebraic system in a game he called the *Icosian Game*, initially exhibited by Hamilton at a meeting of the British Association in Dublin. The Icosian Game (the prefix *icos* is from the Greek for *twenty*) consisted of a board on which were placed twenty holes and some lines between certain pairs of holes. The diagram for this game is shown in Figure 3.4, where the holes are designated by the twenty consonants of the English alphabet.

Figure 3.4: Hamilton's Icosian Game

Hamilton later sold the rights of his game for 25 pounds to John Jacques & Son, a game manufacturer especially well known as a dealer in chess sets. The preface to the instruction pamphlet for the Icosian Game, prepared by Hamilton for marketing the game in 1859, read as follows:

> *In this new Game (invented by Sir WILLIAM ROWAN HAMILTON, LL.D., & c., of Dublin, and by him named Icosian from a Greek word signifying 'twenty') a player is to place the whole or part of a set of twenty numbered pieces or men upon the points or in the holes of a board, represented by the diagram above drawn, in such a manner as always to proceed along the lines of the figure, and*

also to fulfill certain other conditions, which may in various ways be assigned by another player. Ingenuity and skill may thus be exercised in proposing as well as in resolving problems of the game. For example, the first of the two players may place the first five pieces in any five consecutive holes, and then require the second player to place the remaining fifteen men consecutively in such a manner that the succession may be cyclical, that is, so that No. 20 *may be adjacent to No.* 1*; and it is always possible to answer any question of this kind. Thus, if B C D F G be the five given initial points, it is allowed to complete the succession by following the alphabetical order of the twenty consonants, as suggested by the diagram itself; but after placing the piece No.* 6 *in hole H, as above, it is also allowed* (*by the supposed conditions*) *to put No.* 7 *in X instead of J, and then to conclude with the succession, W R S T V J K L M N P Q Z. Other Examples of Icosian Problems, with solutions of some of them, will be found in the following page.*

Of course, the diagram of Hamilton's Icosian game shown in Figure 3.4 can be immediately interpreted as a graph (see Figure 3.5), where the lines in the diagram have become the edges of the graph and the holes have become the vertices. Indeed, the graph of Figure 3.5 can be considered as the graph of the geometric solid called the **dodecahedron** (where the prefix *dodec* is from the Greek for *twelve*, pertaining to the twelve faces of the solid). This subject will be discussed in more detail in Chapter 6.

Figure 3.5: The graph of the dodecahedron

The problems proposed by Hamilton in his Icosian Game gave rise to concepts in graph theory, which eventually became a popular subject of study by mathematicians. Let G be a graph. A path in G that contains every vertex of G is called a **Hamiltonian path** of G, while a cycle in G that contains every vertex of G is called a **Hamiltonian cycle** of G. A graph that contains

a Hamiltonian cycle is itself called **Hamiltonian**. Certainly, the order of every Hamiltonian graph is at least 3 and every Hamiltonian graph contains a Hamiltonian path. On the other hand, a graph with a Hamiltonian path need not be Hamiltonian. The graph G_1 of Figure 3.6 is Hamiltonian and therefore contains both a Hamiltonian cycle and a Hamiltonian path. The graph G_2 contains a Hamiltonian path but is not Hamiltonian; while G_3 contains neither a Hamiltonian cycle nor a Hamiltonian path.

Figure 3.6: Hamiltonian paths and cycles in graphs

As implied by Hamilton's remarks, the graph of the dodecahedron in Figure 3.5 is Hamiltonian. Indeed, Hamilton's statement implies that this graph has a much stronger property. If one begins with any path of order 5 in the graph of Figure 3.5, then the path can be extended to a Hamiltonian cycle in this graph.

In 1855 (the year before Hamilton developed his Icosian Game) the Reverend Thomas Penyngton Kirkman (1806–1895) studied such questions as whether it is possible to visit all corners (vertices) of a polyhedron exactly once by moving along edges of the polyhedron and returning to the starting vertex. He observed that this could be done for some polyhedra but not all. While Kirkman had studied *Hamiltonian cycles* on general polyhedra and had preceded Hamilton's work on the dodecahedron by several months, it is Hamilton's name that became associated with spanning cycles in graphs, not Kirkman's.

Ore's Theorem

Since the concepts of a circuit that contains every edge of a graph and a cycle that contains every vertex seem so very similar and since there is a simple and useful characterization of graphs that are Eulerian, one might very well anticipate the existence of such a characterization of graphs that are Hamiltonian. However, no such theorem has ever been discovered. On the other hand, it is much more likely that a graph is Hamiltonian if the degrees of its vertices are large. It wasn't until 1952 that a general theorem by Gabriel Andrew Dirac appeared, giving a sufficient condition for a graph to be Hamiltonian. However, in 1960 an even more general theorem [166], due to Oystein Ore (1899–1968), would be discovered and lead to a host of other sufficient conditions.

Theorem 3.6 (**Ore's Theorem**) *Let G be a graph of order $n \geq 3$. If*

$$\deg u + \deg v \geq n$$

for each pair u, v of nonadjacent vertices of G, then G is Hamiltonian.

Proof. Suppose that the statement is false. Then for some integer $n \geq 3$, there exists a graph H of order n such that $\deg u + \deg v \geq n$ for each pair u, v of nonadjacent vertices of H but yet H is not Hamiltonian. Add as many edges as possible between pairs of nonadjacent vertices of H so that the resulting graph G is still not Hamiltonian. Hence G is a maximal non-Hamiltonian graph. Certainly, G is not a complete graph. Furthermore, $\deg_G u + \deg_G v \geq n$ for every pair u, v of nonadjacent vertices of G.

If the edge xy were to be added between two nonadjacent vertices x and y of G, then necessarily $G + xy$ is Hamiltonian and so $G + xy$ contains a Hamiltonian cycle C. Since C must contain the edge xy, the graph G contains a Hamiltonian $x - y$ path $(x = v_1, v_2, \ldots, v_n = y)$. If $xv_i \in E(G)$, where $2 \leq i \leq n-1$, then $yv_{i-1} \notin E(G)$; for otherwise,

$$C' = (x = v_1, v_2, \ldots, v_{i-1}, y = v_n, v_{n-1}, v_{n-2}, \ldots, v_i, x)$$

is a Hamiltonian cycle of G, which is impossible. Hence for each vertex of G adjacent to x, there is a vertex of $V(G) - \{y\}$ not adjacent to y. However then,

$$\deg_G y \leq (n-1) - \deg_G x,$$

that is, $\deg_G x + \deg_G y \leq n - 1$, producing a contradiction. ■

The aforementioned 1952 paper of Dirac [59] contained the following sufficient condition for a graph to be Hamiltonian, which is then a consequence of Theorem 3.6.

Corollary 3.7 (**Dirac's Theorem**) *If G is a graph of order $n \geq 3$ such that*

$$\deg v \geq n/2$$

for each vertex v of G, then G is Hamiltonian.

We have now seen fundamental results by Dirac on both connectivity (Theorem 2.41) and on Hamiltonian graphs. We will encounter him often again later in the book. Gabriel Andrew Dirac (1925–1984) was born in Budapest, Hungary. He was the stepson of Paul Adrien Maurice Dirac and the nephew of Eugene Paul Wigner, both recipients of the Nobel Prize in physics. Dirac became interested in graph theory early in his mathematical career, at a time when the respectability of graph theory occurred primarily within Hungary. However, Dirac was attracted to graph theory and became one of those mathematicians whose contributions to graph theory brought it to a place of prominence. In fact, the mathematician Carsten Thomassen wrote:

He (Dirac) was more interested in the intrinsic beauty of graph theory than its applications.

If the degree of every vertex of a graph of order $n \geq 3$ and size m is at least $n/2$, then $m \geq n^2/4$. That is, the hypothesis of Corollary 3.7 not only implies that the graph G is Hamiltonian, it implies that the size of G is at least $n^2/4$. The same can be said of graphs satisfying the hypothesis of Theorem 3.6.

Theorem 3.8 *Let G be a graph of order $n \geq 3$ and size m. If $\deg u + \deg v \geq n$ for each pair u, v of nonadjacent vertices of G, then $m \geq n^2/4$.*

Proof. As we have already noted, if the degree of every vertex of G is at least $n/2$, then $m \geq n^2/4$. Hence we may assume that the minimum degree of G is k where $k < n/2$. Thus $k \leq (n-1)/2$ and so $n - 1 \geq 2k$. Let S be the set of vertices of degree k in G. Since the sum of the degrees of every two nonadjacent vertices of G is at least n, it follows that $G[S]$ is a complete subgraph of G. Since G is Hamiltonian by Theorem 3.6, G is connected. Thus some vertex in S is adjacent to a vertex not in S. Hence $|S| \leq k$. Let $u \in S$. Thus $\deg_G u = k$. Therefore, u is adjacent to $k-1$ vertices having degree k or more and adjacent to a vertex of degree $k+1$ or more. Let w be a vertex distinct from u that is not adjacent to u. Then $\deg u + \deg w \geq n$ and so $\deg w \geq n - k$. Therefore,

$$\begin{aligned} 2m &= \sum_{v \in V(G)} \deg v = \deg u + \sum_{v \in N(u)} \deg v + \sum_{w \notin N[u]} \deg w \\ &\geq k + [(k-1)k + (k+1)] + (n-k-1)(n-k) \\ &= 2k(k-n+1) + (n^2 - n + 1). \end{aligned}$$

Since $n - 1 \geq 2k$, it follows that

$$\begin{aligned} 4k(k-n+1) + (n-1)^2 &= 4k^2 + (n-1)[(n-1) - 4k] \\ &\geq 4k^2 + (2k)(-2k) = 0. \end{aligned} \tag{3.1}$$

From (3.1), we have

$$4k(k-n+1) + 2(n^2 - n + 1) \geq n^2 + 1$$

and so

$$2k(k-n+1) + (n^2 - n + 1) \geq (n^2+1)/2.$$

Consequently, $m \geq (n^2+1)/4$. Hence in any case, $m \geq n^2/4$. ■

With the aid of Theorem 3.6, a sufficient condition for a graph to have a Hamiltonian path can also be given.

Corollary 3.9 *Let G be a graph of order $n \geq 2$. If*

$$\deg u + \deg v \geq n - 1$$

for each pair u, v of nonadjacent vertices of G, then G contains a Hamiltonian path.

Proof. Let $H = G \vee K_1$ be the join of G and K_1, where w is the vertex of H that does not belong to G. Then $\deg u + \deg v \geq n + 1$ for each pair u, v of nonadjacent vertices of H. Since the order of H is $n + 1$, it follows by Theorem 3.6 that H is Hamiltonian. Let C be a Hamiltonian cycle of H. Deleting w from C produces a Hamiltonian path in G. ■

The Closure of a Graph

J. Adrian Bondy and Vašek Chvátal [26] observed that the proof of Ore's theorem (Theorem 3.6) neither uses nor needs the full strength of the requirement that the degree sum of each pair of nonadjacent vertices is at least the order of the graph being considered. Initially, Bondy and Chvátal observed the following result.

Theorem 3.10 *Let u and v be nonadjacent vertices in a graph G of order n such that $\deg u + \deg v \geq n$. Then $G + uv$ is Hamiltonian if and only if G is Hamiltonian.*

Proof. Certainly if G is Hamiltonian, then $G + uv$ is Hamiltonian. For the converse, suppose that $G + uv$ is Hamiltonian but G is not. Then every Hamiltonian cycle in $G + uv$ contains the edge uv, implying that G contains a Hamiltonian $u - v$ path. We can now proceed exactly as in the proof of Theorem 3.6 to produce a contradiction. ■

The preceding result inspired a definition. Let G be a graph of order n. The **closure** $CL(G)$ of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n (in the resulting graph at each stage) until no such pair remains. A graph G and its closure are shown in Figure 3.7.

Figure 3.7: Constructing the closure of a graph

First, we show that the closure is a well-defined operation on graphs, that is, the same graph is obtained regardless of the order in which edges are added.

Theorem 3.11 *Let G be a graph of order n. If G_1 and G_2 are graphs obtained by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair remains, then $G_1 = G_2$.*

Proof. Suppose that G_1 is obtained by adding the edges $e_1, e_2, \ldots, e_r$ to G in the given order and G_2 is obtained from G by adding the edges $f_1, f_2, \ldots, f_s$ in the given order. Assume, to the contrary, that $G_1 \neq G_2$. Then $E(G_1) \neq E(G_2)$. Thus we may assume that there is a first edge $e_i = xy$ in the sequence $e_1, e_2, \ldots, e_r$ that does not belong to G_2. If $i = 1$, then let $G = H$; otherwise, let $H = G + \{e_1, e_2, \ldots, e_{i-1}\}$. Then H is a subgraph of G_2 and x and y are nonadjacent vertices of H. Since $\deg_H x + \deg_H y \geq n$, it follows that $\deg_{G_2} x + \deg_{G_2} y \geq n$, which produces a contradiction. ■

Repeated applications of Theorem 3.10 give us the following result.

Theorem 3.12 *A graph is Hamiltonian if and only if its closure is Hamiltonian.*

Since each complete graph with at least three vertices is Hamiltonian, we obtain the following sufficient condition for a graph to be Hamiltonian due to Bondy and Chvátal [26].

Theorem 3.13 *Let G be a graph with at least three vertices. If $CL(G)$ is complete, then G is Hamiltonian.*

If a graph G satisfies the conditions of Theorem 3.6, then $CL(G)$ is complete and so, by Theorem 3.13, G is Hamiltonian. Thus, Ore's theorem is an immediate corollary of Theorem 3.13 (although chronologically it preceded the theorem of Bondy and Chvátal by several years). In fact, many sufficient conditions for a graph to be Hamiltonian based on the degrees of the vertices of a graph can be deduced from Theorem 3.13. The following result of Chvátal [44] is an example of one of the strongest of these.

Theorem 3.14 *Let G be a graph of order $n \geq 3$, the degrees d_i of whose vertices satisfy $d_1 \leq d_2 \leq \cdots \leq d_n$. If there is no integer $k < n/2$ for which $d_k \leq k$ and $d_{n-k} \leq n - k - 1$, then G is Hamiltonian.*

Proof. Let $H = CL(G)$. We show that H is complete which, by Theorem 3.13, implies that G is Hamiltonian. Assume, to the contrary, that H is not complete. Let u and w be nonadjacent vertices of H for which $\deg_H u + \deg_H w$ is as large as possible. Since u and w are nonadjacent vertices of H, it follows that $\deg_H u + \deg_H w \leq n - 1$. Assume, without loss of generality, that $\deg_H u \leq \deg_H w$. Thus if $k = \deg_H u$, we have that $k \leq (n-1)/2 < n/2$ and $\deg_H w \leq n - 1 - k$. Let W denote the vertices other than w that are not adjacent to w in H. Then $|W| = n - 1 - \deg_H w \geq k$. Also, by the choice of u and w, every vertex $v \in W$ satisfies

$$\deg_G v \leq \deg_H v \leq \deg_H u = k.$$

Thus, G has at least k vertices of degree at most k and so $d_k \leq k$. Similarly, let U denote the vertices other than u that are not adjacent to u in H. Then $|U| = n - 1 - \deg_H u = n - k - 1$. Every vertex $v \in U$ satisfies

$$\deg_G v \leq \deg_H v \leq \deg_H w \leq n - 1 - k,$$

implying that $d_{n-k-1} \leq n - k - 1$. However,

$$\deg_G u \leq \deg_H u \leq \deg_H w \leq n - 1 - k,$$

so $d_{n-k} \leq n - k - 1$. This, however, contradicts the hypothesis of the theorem. Thus, $H = CL(G)$ is complete. ■

The sufficient conditions for Hamiltonicity that we have presented so far all involve the degrees of the vertices of the graph. Our next result, however, involves the cardinality of sets of pairwise nonadjacent vertices of a graph and its connectivity. A set U of vertices in a graph G is **independent** if no two vertices in U are adjacent. (Some refer to an independent set of vertices as a **stable set**.) The maximum number of vertices in an independent set of vertices of G is called the **vertex independence number** or, more simply, the **independence number** of G and is denoted by $\alpha(G)$. For example, $\alpha(K_{r,s}) = \max\{r, s\}$, $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$ and $\alpha(K_n) = 1$. Vašek Chvátal and Paul Erdős [48] showed that if G is a graph of order at least 3 whose connectivity is at least as large as its independence number, then G must be Hamiltonian.

Theorem 3.15 *Let G be a graph of order at least 3. If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.*

Proof. If $\alpha(G) = 1$, then G is complete and therefore Hamiltonian. Hence we may assume that $\alpha(G) = k \geq 2$. Since $\kappa(G) \geq 2$, it follows that G is 2-connected and so G contains a cycle by Theorem 2.41. Let C be a longest cycle in G. By Theorem 2.41, C contains at least k vertices. We show that C is a Hamiltonian cycle. Assume, to the contrary, that C is not a Hamiltonian cycle. Then there is some vertex w of G that does not lie on C. Since G is k-connected, it follows with the aid of Corollary 2.39 that G contains k paths $P_1, P_2, \cdots, P_k$ such that P_i is a $w - v_i$ path where v_i is the only vertex of P_i on C and such that the paths are pairwise-disjoint except for w.

In some cyclic ordering of the vertices of C, let u_i be the vertex that follows v_i on C for each i $(1 \leq i \leq k)$. No vertex u_i is adjacent to w, for otherwise, replacing the edge v_iu_i by P_i and wu_i produces a cycle whose length exceeds that of C. Let $S = \{w, u_1, u_2, \ldots, u_k\}$. Since $|S| = k + 1 > \alpha(G)$ and $wu_i \notin E(G)$ for each i $(1 \leq i \leq k)$, there are distinct integers r and s such that $1 \leq r, s \leq k$ and $u_ru_s \in E(G)$. Replacing the edges u_rv_r and u_sv_s by the edge u_ru_s and the paths P_r and P_s produces a cycle that is longer than C. This is a contradiction. ■

A Necessary Condition for Hamiltonian Graphs

While we have described several sufficient conditions for a graph to be Hamiltonian, there is also a useful necessary condition. Recall that $k(H)$ denotes the number of components in a graph H.

Theorem 3.16 *If G is a Hamiltonian graph, then*

$$k(G - S) \leq |S|$$

for every nonempty proper subset S of $V(G)$.

Proof. Let S be a nonempty proper subset of $V(G)$. If $G - S$ is connected, then certainly, $k(G-S) \leq |S|$. Hence we may assume that $k(G-S) = k \geq 2$ and that $G_1, G_2, \ldots, G_k$ are the components of $G - S$. Let $C = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle of G. Without loss of generality, we may assume that $v_1 \in V(G_1)$. For $1 \leq j \leq k$, let v_{i_j} be the last vertex of C that belongs to G_j. Necessarily then, $v_{i_j+1} \in S$ for $1 \leq j \leq k$ and so $|S| \geq k = k(G - S)$. ■

Because Theorem 3.16 presents a necessary condition for a graph to be Hamiltonian, it is most useful in its contrapositive formulation:

> *If there exists a nonempty proper subset S of the vertex set $V(G)$ of a graph G such that $k(G - S) > |S|$, then G is not Hamiltonian.*

Certainly every Hamiltonian graph is connected. As a consequence of Theorem 3.16, no graph with a cut-vertex is Hamiltonian, that is, every Hamiltonian graph is 2-connected. The graph G of Figure 3.8 is not Hamiltonian either for if we let $S = \{w, x\}$, then $k(G - S) = 3$ and so $k(G - S) > |S|$.

Figure 3.8: A non-Hamiltonian graph

We have seen that many sufficient conditions for a graph of order n to be Hamiltonian require that the graph contains some vertices of degree at least $n/2$. In the case of regular graphs, however, this situation can be improved. Bill Jackson [124] showed that every 2-connected r-regular graph of order at most $3r$ is Hamiltonian. As we are about to see, the Petersen graph shows that $3r$ cannot be replaced by $3r + 1$. Even though the Petersen graph is not Hamiltonian, Theorem 3.16 cannot be used to verify this. Recall that the girth of the Petersen graph (the length of a smallest cycle) is 5.

Theorem 3.17 *The Petersen graph is not Hamiltonian.*

Proof. Assume, to the contrary, that the Petersen graph P is Hamiltonian. Then P has a Hamiltonian cycle

$$C = (v_1, v_2, \ldots, v_{10}, v_1).$$

Since P is cubic, v_1 is adjacent to exactly one of the vertices $v_3, v_4, \ldots, v_9$. However, since P contains neither a 3-cycle nor a 4-cycle, v_1 is adjacent to exactly one of v_5, v_6, and v_7. Because of the symmetry of v_5 and v_7, we may assume that v_1 is adjacent either to v_5 or to v_6.

Case 1. *v_1 is adjacent to v_5 in P.* Then v_{10} is adjacent to exactly one of v_4, v_5 and v_6, which results in a 4-cycle, a 3-cycle or a 4-cycle, respectively, each of which is impossible.

Case 2. *v_1 is adjacent to v_6 in P.* Again, v_{10} is adjacent to exactly one of v_4, v_5 and v_6. Since P does not contain a 3-cycle or a 4-cycle, v_{10} must be adjacent to v_4. However, then this returns us to Case 1, where v_1 and v_5 are replaced by v_{10} and v_4, respectively. ■

The Toughness of a Graph

We saw from Theorem 3.16 that for every Hamiltonian graph G,

$$k(G - S) \leq |S|$$

for every nonempty proper set S of vertices of G. So if G is a Hamiltonian graph, then

$$\frac{|S|}{k(G - S)} \geq 1$$

for every nonempty proper set S of vertices of G.

For a nonnegative real number t, a noncomplete graph G is **t-tough** if

$$\frac{|S|}{k(G - S)} \geq t$$

for every vertex-cut S of G. If G is a t-tough graph and s is a nonnegative real number such that $s < t$, then G is also s-tough.

Consequently, every Hamiltonian graph is 1-tough. The converse is not true, however. By Theorem 3.17, the Petersen graph is not Hamiltonian. It is 1-tough though.

Theorem 3.18 *The Petersen graph is* 1*-tough.*

Proof. The Petersen graph P shown in Figure 3.9 is 3-connected; in fact, $\kappa(P) = 3$. We show that $|S|/k(P - S) \geq 1$ for every vertex-cut S of P.

Figure 3.9: The Petersen graph

Indeed, we show for every nonempty set S of vertices of P with $|S| \geq 3$ that $|S|/k(P-S) \geq 1$. This is obvious if $|S| \geq 5$, so it remains only to show that this is the case for $|S| = 3$ and $|S| = 4$.

Let $C = (u_1, u_2, \ldots, u_5, u_1)$ be the exterior cycle of P and C'=(v_1, v_3, v_5, v_2, v_4, v_1) the interior cycle. Assume, without loss of generality, that at least as many vertices of S lie on C as on C'. We consider three cases.

Case 1. *No vertex of S lies on C'.* Then $P - S$ is connected and $|S|/k(P - S) > 1$.

Case 2. *One vertex of S lies on C'.* Then $P - S$ contains a component of order at least 5 and so $|S|/k(P - S) \geq 1$.

Case 3. *Two vertices of S lie on C'.* Then $|S| = 4$ and $P - S$ contains at most four components. Thus $|S|/k(P - S) \geq 1$. ■

While not every 1-tough graph is Hamiltonian, it was conjectured by Chvátal that there is some fixed number t such that every t-tough graph is Hamiltonian. Indeed, for quite some time it was believed that every 2-tough graph is Hamiltonian. In 2000, however, Douglas Bauer, Hajo Broersma and Henk Jan Veldman [7] showed that the graph G of Figure 3.10 (the **Bauer-Broersma-Veldman graph**), which is the join of F and K_2, is 2-tough but not Hamiltonian.

Theorem 3.19 *The Bauer-Broersma-Veldman graph* (*of Figure* 3.10) *is a* 2-*tough graph that is not Hamiltonian.*

Proof. Let G be the Bauer-Broersma-Veldman graph. Then $G = F \vee K_2$ is the join of the graph F of Figure 3.10 and K_2. For $1 \leq i \leq 5$, let $V_i = \{s_i, t_i, u_i, v_i, w_i, x_i, y_i, z_i\}$, $F_i = F[V_i]$ and $G_i = F_i - \{y_i, z_i\}$, where the graph G_i is shown in Figure 3.11. Also, let H be the subgraph of F where $H = K_{10}$.

First, we show that G is not Hamiltonian. Assume, to the contrary, that G is Hamiltonian. Then G contains a Hamiltonian cycle C. At least one of the five pairs $\{s_i, t_i\}$, $1 \leq i \leq 5$, of vertices contains no vertex that is adjacent to either p or q on C, say $\{s_1, t_1\}$ has this property. Then $s_1u_1, s_1w_1, t_1v_1, t_1x_1 \in$

Figure 3.10: The Bauer-Broersma-Veldman graph:
A 2-tough graph that is not Hamiltonian

$E(C)$. Necessarily, $u_1w_1, u_1x_1, v_1x_1, v_1w_1 \notin E(C)$. This, however, implies that $v_1y_1, w_1y_1, u_1z_1, x_1z_1 \in E(C)$, which is impossible. Thus G is not Hamiltonian.

It remains to show that G is 2-tough. First, observe that every vertex-cut of G must contain both p and q. Let $S \cup \{p, q\}$ be a vertex-cut of G, where $p, q \notin S$. Then S is a vertex-cut of F. Suppose first that $V(H) \subseteq S$. If $S = V(H)$, then $|S|/k(G - S) = |V(H)|/k(G - V(H)) = 10/5 = 2$. We may assume then that $S \neq V(H)$. Furthermore, we may assume that $V(F_i) \not\subseteq S$ for any i $(1 \leq i \leq 5)$; for otherwise, for each i, $|S|/k(G - S)$ contributes 6 to the numerator of $|V(H)|/k(G - V(H))$ and -1 to its denominator. Since G_i is 2-connected for each i $(1 \leq i \leq 5)$ and $\alpha(G_i) = 2$, at least two vertices must be removed from G_i to produce a subgraph with at most two components. If this is done with k of the five subgraphs G_i $(1 \leq i \leq 5)$ for $1 \leq k \leq 5$, then

$$\frac{|S|}{k(G - S)} \geq \frac{10 + 2k}{5 + k} = 2.$$

Hence we may assume that $V(H) - S \neq \emptyset$. Thus some component of $G - S$ contains all vertices of $V(H) - S$. For each i $(1 \leq i \leq 5)$, one of the following must occur: (1) $y_i, z_i \in S$, (2) either $y_i \in S$ or $z_i \in S$ but not both, (3) $y_i, z_i \notin S$. We say that the pair y_i, z_i is of *type* 1, 2 or 3, according to whether (1), (2) or (3) occurs. We consider these three cases.

Figure 3.11: The subgraph G_i of the graph G of Figure 3.10

Case 1. *y_i, z_i is of type* 1 *for some i* $(1 \leq i \leq 5)$. At least two additional vertices must be removed to increase the number of components by 1 and four additional vertices must be removed to increase the number of components by 2.

Case 2. *y_i, z_i is of type* 2 *for some i* $(1 \leq i \leq 5)$. At least two additional vertices must be removed to increase the number of components by 1 and three additional vertices must be removed to increase the number of components by 2.

Case 3. *y_i, z_i is of type* 3 *for some i* $(1 \leq i \leq 5)$. At least two additional vertices must be removed to increase the number of components by 1 and four additional vertices must be removed to increase the number of components by 2.

Let $S' = S \cap V(H)$. Among the pairs y_i, z_i, suppose that there are a of type 1, b of type 2 and c of type 3. Then $a + b + c = 5$ and $0 \leq a \leq 4$. Suppose then that we first remove the vertices of S' from G. Therefore,

$$\frac{|S'|}{k(G - S')} = \frac{2a + b + 2}{a + 1} \geq 2.$$

We then successively remove from $G - S'$ the vertices of $S - S'$ that belong to each subgraph F_i. For each of the a pairs y_i, z_i of the vertices of type 1, we either remove at least two vertices from $G - S'$ and increase the number of components of $G - S'$ by at least 1 or remove four vertices from $G - S'$ and increase the number of components $G - S'$ by 2. Thus the minimum contribution to $|S'|/k(G - S')$ from these a pairs of vertices of type 1 adds $2a$ to the numerator and a to the denominator. For the b pairs y_i, z_i of vertices of type 2, the minimum contribution to $|S'|/k(G - S')$ adds $3b$ to the numerator and $2b$ to the denominator. For the c pairs of y_i, z_i of vertices of type 3, the minimum contribution to $|S'|/k(G - S')$ adds $2c$ to the numerator and c to the denominator. Thus

$$\begin{aligned} \frac{|S|}{k(G - S)} &\geq \frac{|S'| + 2a + 3b + 2c}{k(G - S') + a + 2b + c} \\ &= \frac{(2a + b + 2) + (2a + 3b + 2c)}{(a + 1) + (a + 2b + c)} = 2. \end{aligned}$$

Therefore, G is 2-tough. ■

The maximum real number t for which a graph G is t-tough is called the **toughness** of G and is denoted by $t(G)$. Since complete graphs do not contain vertex-cuts, this definition does not apply to such graphs. Certainly, the toughness of a noncomplete graph is a rational number. Also $t(G) = 0$ if and only if G is disconnected. Indeed, it follows that if G is a noncomplete graph, then

$$t(G) = \min\left\{\frac{|S|}{k(G-S)}\right\}, \tag{3.2}$$

where the minimum is taken over all vertex-cuts S of G. In fact, the toughness of the Bauer-Broersma-Veldman graph shown in Figure 3.10 is 2. These three went a step further by describing a sequence $\{G_k\}$ of non-Hamiltonian graphs for which $\lim_{k\to\infty} t(G_k) = 9/4$.

For the graph G of Figure 3.12, $S_1 = \{u, v, w\}$, $S_2 = \{w\}$ and $S_3 = \{u, v\}$ are three (of many) vertex-cuts. Observe that

$$\frac{|S_1|}{k(G-S_1)} = \frac{3}{8}, \quad \frac{|S_2|}{k(G-S_2)} = \frac{1}{3} \quad \text{and} \quad \frac{|S_3|}{k(G-S_3)} = \frac{2}{7}.$$

Since there is no vertex-cut S of G with $|S|/k(G-S) < \frac{2}{7}$, it follows that $t(G) = \frac{2}{7}$.

Figure 3.12: A graph with toughness $\frac{2}{7}$

The toughness of a graph G can be considered as a measure of how tightly the subgraphs of G are held together. Thus the smaller the toughness the more vulnerable the graph is. A 1-tough graph, for example, has the property that breaking the graph into k components (if this is possible) requires the removal of at least k vertices; while breaking a 2-tough graph into k components requires the removal of at least $2k$ vertices.

A parameter that plays an important role in the study of toughness is the independence number. The independence number is related to toughness in the sense that among all the vertex-cuts S of a noncomplete graph G, the maximum value of $k(G-S)$ is the independence number $\alpha(G)$ of G; so for every vertex-cut S of G, we have that $\kappa(G) \le |S|$ and $k(G-S) \le \alpha(G)$. This leads to bounds for the toughness of a graph obtained by Chvátal [45].

Theorem 3.20 *For every noncomplete graph G,*

$$\frac{\kappa(G)}{\alpha(G)} \leq t(G) \leq \frac{\kappa(G)}{2}.$$

Proof. According to (3.2),

$$t(G) = \min\left\{\frac{|S|}{k(G-S)}\right\} \geq \frac{\kappa(G)}{\alpha(G)}.$$

Let S' be a vertex-cut with $|S'| = \kappa(G)$. Thus $k(G - S') \geq 2$, so

$$t(G) = \min\left\{\frac{|S|}{k(G-S)}\right\} \leq \frac{|S'|}{k(G-S')} \leq \frac{\kappa(G)}{2},$$

as desired. ■

For a given graph F, a graph G is **F-free** if G contains no induced subgraph isomorphic to F. A K_3-free graph is called a **triangle-free** graph. In this context, a graph of particular interest is $K_{1,3}$, which, as mentioned earlier, is referred to as a claw. A $K_{1,3}$-free graph is then referred to as a **claw-free graph**. The following result by Manton Matthews and David Sumner [151] provides a class of graphs for which the upper bound in Theorem 3.20 is attained.

Theorem 3.21 *If G is a noncomplete claw-free graph, then $t(G) = \frac{1}{2}\kappa(G)$.*

Proof. If G is disconnected, then $t(G) = \kappa(G) = 0$ and the result follows. So we assume that $\kappa(G) = r \geq 1$. Let S be a vertex-cut such that $t(G) = |S|/k(G - S)$. Suppose that $k(G - S) = k$ and that $G_1, G_2, \ldots, G_k$ are the components of $G - S$.

Let $u_i \in V(G_i)$ and $u_j \in V(G_j)$, where $i \neq j$. Since G is r-connected, it follows by Theorem 2.37 that G contains at least r internally disjoint $u_i - u_j$ paths. Each of these paths contains a vertex of S. Consequently, there are at least r edges joining the vertices of S and the vertices of G_i for each i $(1 \leq i \leq k)$ such that no two of these edges are incident with the same vertex of S.

Hence there is a set X containing at least kr edges between S and $G - S$ such that any two edges incident with a vertex of S are incident with vertices in distinct components of $G - S$. However, since G is claw-free, no vertex of S is joined to vertices in three components of $G - S$. Therefore,

$$kr \leq |X| \leq 2|S| = 2kt(G),$$

so $kr \leq 2kt(G)$. Thus $t(G) \geq \frac{r}{2} = \frac{1}{2}\kappa(G)$. By Theorem 3.20, $t(G) = \frac{1}{2}\kappa(G)$. ■

In defining the toughness of a graph we were in some sense fine-tuning the idea of connectivity. For example, if a graph G is 2-connected, then the removal of one vertex from G does not result in a disconnected graph. The removal of

two vertices, however, may not only result in a disconnected graph but in fact may result in a graph with many components. If, however, we know that G is 1-tough, then not only is G a 2-connected graph but the removal of any two vertices of G can result in a graph with at most two components.

Hamiltonian-Connected Graphs

As we have already remarked, obtaining an applicable characterization of Hamiltonian graphs remains an unsolved problem in graph theory. In view of the lack of success in developing such a characterization, it is not surprising that special subclasses of Hamiltonian graphs have been singled out for investigation. We now discuss several types of *highly Hamiltonian* graphs.

A graph G is **Hamiltonian-connected** if for every pair u, v of vertices of G, there is a Hamiltonian $u - v$ path in G. Necessarily, every Hamiltonian-connected graph of order 3 or more is Hamiltonian but the converse is not true. The cubic graph $G_1 = C_3 \square K_2$ of Figure 3.13 is Hamiltonian-connected, while the cubic graph $G_2 = C_4 \square K_2 = Q_3$ is not Hamiltonian-connected. The graph G_2 contains no Hamiltonian $u - v$ path, for example. (See Exercise 30.)

Figure 3.13: Hamiltonian-connected and non-Hamiltonian-connected graphs

There is a sufficient condition for a graph to be Hamiltonian-connected that is similar in statement to the sufficient condition for a graph to contain a Hamiltonian cycle presented in Theorem 3.6. The following theorem is also due to Oystein Ore [168] and provides a sufficient condition for a graph to be Hamiltonian-connected.

Theorem 3.22 *Let G be a graph of order $n \geq 4$. If*

$$\deg u + \deg v \geq n + 1$$

for every pair u, v of nonadjacent vertices of G, then G is Hamiltonian-connected.

Proof. Let u and v be two vertices of G and let H be the graph of order $n+1$ obtained from G by adding a new vertex w that is joined to u and v. We now construct the closure $F = CL(H)$ of H. Since $\deg_H x + \deg_H y \geq n + 1$ for every two nonadjacent vertices x and y of H that belong to G, it follows that $F[V(G)] = K_n$. Furthermore, if $x \in V(G) - \{u, v\}$, then $\deg_F x + \deg_F w \geq$

$(n-1)+2 = n+1$ and so $xw \in E(F)$. Therefore, $F = CL(H) = K_{n+1}$, which implies by Theorem 3.13 that H is Hamiltonian. Since $\deg_H w = 2$, every Hamiltonian cycle C of H must contain the edges uw and vw. Removing w from C produces a Hamiltonian $u-v$ path in G. ■

There is now an immediate corollary, similar in statement to the sufficient condition given in Corollary 3.7 for a graph to be Hamiltonian.

Corollary 3.23 *Let G be a graph of order n such that*

$$\deg v \geq (n+1)/2$$

for every vertex v of G, then G is Hamiltonian-connected.

Panconnected and Pancyclic Graphs

We now consider a property that a graph may possess that is even stronger than being Hamiltonian-connected. A connected graph G of order n is said to be **panconnected** if for each pair u, v of distinct vertices of G, there exists a $u-v$ path of length ℓ for each integer ℓ satisfying $d(u,v) \leq \ell \leq n-1$. If a graph is panconnected, then it is Hamiltonian-connected; the next example indicates that these concepts are not equivalent.

For $k \geq 3$, let G_k be that graph such that $V(G_k) = \{v_1, v_2, \ldots, v_{2k}\}$ and

$$\begin{aligned} E(G_k) &= \{v_i v_{i+1} : \ i = 1, 2, \ldots, 2k\} \cup \{v_i v_{i+3} : \ i = 2, 4, \ldots, 2k-4\} \cup \\ &\quad \{v_1 v_3, v_{2k-2} v_{2k}\}, \end{aligned}$$

where all subscripts are expressed modulo $2k$. Although for each pair u, v of distinct vertices and for each integer ℓ satisfying $k \leq \ell \leq 2k-1$, the graph G_k contains a $u-v$ path of length ℓ, there is no $v_1 - v_{2k}$ path of length ℓ if $1 < \ell < k$. Since $d(v_1, v_{2k}) = 1$, it follows that G_k is not panconnected.

James E. Williamson [240] obtained a sufficient condition for a graph G to be panconnected in terms of the minimum degree of G.

Theorem 3.24 *If G is a graph of order $n \geq 4$ such that*

$$\deg v \geq (n+2)/2$$

for every vertex v of G, then G is panconnected.

Proof. If $n = 4$, then $G = K_4$ and the statement is true.

Suppose that the theorem is not true. Thus there exists a graph G of order $n \geq 5$ with $\delta(G) \geq (n+2)/2$ such that G is not panconnected; that is, there are vertices u and v of G and an integer ℓ with $d(u,v) < \ell < n-1$ such that there is no $u-v$ path of length ℓ. Let $G^* = G - \{u, v\}$. Then G^* has order

$n^* = n - 2 \geq 3$ and $\delta(G^*) \geq (n+2)/2 - 2 = n^*/2$. Therefore by Corollary 3.7, the graph G^* contains a Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{n^*}, v_1)$.

If $uv_i \in E(G)$, $1 \leq i \leq n^*$, then $vv_{i+\ell-2} \notin E(G)$, where the subscripts are expressed modulo n^*; for otherwise,

$$(u, v_i, v_{i+1}, \ldots, v_{i+\ell-2}, v)$$

is a $u-v$ path of length ℓ in G. Thus for each vertex of C that is adjacent to u in G, there is a vertex of C that is not adjacent to v in G. Since $\deg_G u \geq (n+2)/2$, we conclude that u is adjacent to at least $n/2$ vertices of C, so

$$\deg_G v \leq 1 + n^* - \frac{n}{2} = \frac{n}{2} - 1.$$

This, however, produces a contradiction. ■

The result presented in Theorem 3.24 cannot be improved in general. Let $n = 2k+1 \geq 7$, and consider the graph $K_{k,k+1}$ with partite sets V_1 and V_2, where $|V_1| = k$ and $|V_2| = k+1$. The graph G is obtained from $K_{k,k+1}$ by constructing a path P_{k-1} on $k-1$ vertices of V_2. Join the remaining two vertices x and y of V_2 by an edge (see Figure 3.14 for $k = 4$). Then $\deg v \geq (n+1)/2$ for every vertex v but G is not panconnected since G contains no $x - y$ path of length 3.

Figure 3.14: A graph that is not panconnected

Perhaps surprisingly it is not the case for a graph G of order $n \geq 4$ that $\deg u + \deg v \geq n + 2$ for each pair u, v of nonadjacent vertices of G implies that G is panconnected (see Exercise 28). We have, however, seen a number of results concerning a graph G of order n and size m such that if $\deg u + \deg v$ is at least some expression $f(n)$ involving n for all pairs u, v of nonadjacent vertices of G, then G has a certain property. In particular, if $\deg u + \deg v \geq n$, then by Theorems 3.8 and 3.6, $m \geq n^2/4$ and G is Hamiltonian. J. Adrian Bondy [25] showed that under these conditions G has an even stronger property.

A graph G of order $n \geq 3$ is called **pancyclic** if G contains a cycle of every possible length, that is, G contains a cycle of length ℓ for each ℓ with $3 \leq \ell \leq n$.

Theorem 3.25 *If G is a graph of order $n \geq 3$ such that*

$$\deg u + \deg v \geq n$$

for all distinct nonadjacent vertices u and v, then either G is pancyclic or n is even and $G = K_{\frac{n}{2},\frac{n}{2}}$.

NP-Completeness

As we have now seen, while the problem of determining whether a graph is Eulerian is quite easy to solve, the problem of determining whether a graph is Hamiltonian is, in general, difficult to solve. In fact, this Hamiltonian problem belongs to a much studied class of mathematical problems, both within and outside of graph theory.

Most problems in graph theory apply to all members of some family of graphs (such as all connected graphs or all graphs of order n the degrees of whose vertices are at least $n/2$) or to all members of some family of digraphs (such as strong digraphs or tournaments). When referring to an instance of a problem we mean that the problem is being applied to a particular member of the family. For example, an instance of the Minimum Spanning Tree Problem is the problem of finding a minimum spanning tree in a specific weighted connected graph.

The class of all problems solvable by polynomial-time algorithms is denoted by **P**. For example, the Minimum Spanning Tree Problem belongs to **P**. Determining the problems belonging to **P** is a question of great interest to many. There is a related class of problems denoted by **NP**, which stands for *nondeterministic polynomial-time*.

A **decision problem** is a question that can be answered *yes* or *no*. A decision problem belongs to the class **P** if there is a polynomial-time algorithm that solves every instance of the problem in polynomial time. There are decision problems that are difficult to solve but once a solution is revealed, it is easy to verify that this is in fact a solution. For example, it is difficult in general to determine whether a given graph G is Hamiltonian but it is easy to verify that a cyclic sequence of vertices of G constitutes a Hamiltonian cycle of G. The class of such difficult-to-solve easy-to-verify problems is denoted by **NP**.

The problems in **NP** and the problems in **P** have one property in common, namely: Given a solution to a problem in either class, the solution can be verified in polynomial-time. Thus $\mathbf{P} \subseteq \mathbf{NP}$. One of the best known problems in mathematics asks whether every problem in **NP** is also in **P**. This is called the **P = NP Problem** and is considered by many as the most important problem in theoretical computer science. Its importance and fame have only been magnified because of a million dollar prize offered by the Clay Mathematics Institute for its solution.

A problem in the set **NP** is called **NP-complete** if a polynomial-time algorithm for a solution to the problem would result in polynomial-time solutions for all problems in **NP**. The **NP**-complete problems are among the most difficult in the set **NP** and can be reduced from and to all other **NP**-complete problems in polynomial time. The concept of **NP**-completeness was initiated in 1971 by Stephen Cook [51] who gave an example of the first **NP**-complete problem. The following year Richard M. Karp [130] described some twenty diverse **NP**-complete problems. There are now known to be thousands of **NP**-

complete problems. For example, the problem of determining whether a graph is Hamiltonian is **NP**-complete.

Exercises for Section 3.2

1. Show that the graph of the dodecahedron is Hamiltonian.

2. Show that if G is a 2-connected graph containing a vertex that is adjacent to at least three vertices of degree 2, then G is not Hamiltonian.

3. (a) Prove that $K_{r,2r,3r}$ is Hamiltonian for every positive integer r.
   (b) Prove that $K_{r,2r,3r+1}$ is Hamiltonian for no positive integer r.
   (c) Let $G = K_{n_1,n_2,\ldots,n_k}$ be the complete k-partite graph of order at least 3, where $n_1 \leq n_2 \leq \cdots \leq n_k$. Find a necessary and sufficient condition for the graph G to be Hamiltonian.

4. (a) Prove that if G and H are Hamiltonian graphs, then $G \,\square\, H$ is Hamiltonian.
   (b) Prove that the n-cube Q_n, $n \geq 2$, is Hamiltonian.

5. Show that the bound presented in Theorem 3.6 is sharp, that is, show that for infinitely many integers $n \geq 3$ there are non-Hamiltonian graphs G of order n such that $\deg u + \deg v \geq n - 1$ for all distinct nonadjacent vertices u and v.

6. A Hamiltonian graph G of order n is **k-ordered Hamiltonian** for an integer k with $1 \leq k \leq n$ if for every ordered set $S = \{v_1, v_2, \ldots, v_k\}$ of k vertices of G, there is a Hamiltonian cycle of G encountering these k vertices of S in the order listed.
   (a) Let G be a graph of order $n \geq 3$ such that $\deg v \geq n/2$ for every vertex v of G. Show that G is 3-ordered Hamiltonian.
   (b) Let G be a graph of order $n \geq 4$ such that $\deg v \geq n/2$ for every vertex v of G. Show that G need not be 4-ordered Hamiltonian.
   (c) Show that if G is a 4-ordered Hamiltonian graph, then G is 3-connected.

7. Let G be a graph of order $n \geq 3$ having the property that for each vertex v of G, there is a Hamiltonian path with initial vertex v. Show that G is 2-connected but not necessarily Hamiltonian.

8. (a) A connected graph G of order $n = 2k+1$ has $k+1$ vertices of degree 2, no two of which are adjacent, while the remaining k vertices have degree 3 or more. Show that G is not Hamiltonian.

(b) Give an example of a Hamiltonian graph H of order $n = 2k$ for some $k \geq 2$, where k vertices have degree 2, no two vertices of which are adjacent, while the remaining vertices have degree 3 or more.

9. Show that the bound presented in Theorem 3.15 is sharp, that is, show that for infinitely many integers $n \geq 3$ there are non-Hamiltonian graphs G of order n such that $\kappa(G) \geq \alpha(G) - 1$.

10. (a) Prove that if G is a graph of order $n \geq 3$ and size $m \geq \binom{n-1}{2} + 2$, then G is Hamiltonian.

    (b) Prove that if G is a graph of order $n \geq 3$ and size $m \geq \binom{n-1}{2} + 1$, then G has a Hamiltonian path.

11. Let G be a bipartite graph with partite sets U and W such that $|U| = |W| = k \geq 2$. Prove that if $\deg v > k/2$ for every vertex v of G, then G is Hamiltonian.

12. (a) Let G be a graph of order $n \geq 2$, the degrees d_i of whose vertices satisfy $d_1 \leq d_2 \leq \cdots \leq d_n$. Show that if there is no integer $k < (n+1)/2$ for which $d_k \leq k - 1$ and $d_{n+1-k} \leq n - k - 1$, then G has a Hamiltonian path.

    (b) Show that every self-complementary graph has a Hamiltonian path.

13. Show that if G is a graph of order at least 2 for which $\kappa(G) \geq \alpha(G) - 1$, then G has a Hamiltonian path.

14. Prove that if T is a tree of order at least 4 that is not a star, then $\overline{T}$ contains a Hamiltonian path.

15. (a) Give an example of a graph G containing a Hamiltonian path for which $k(G - S) > |S|$ for some nonempty proper subset S of $V(G)$.

    (b) State and prove a result analogous to Theorem 3.16 that gives a necessary condition for a graph to contain a Hamiltonian path.

16. Let $K_{s,t}$ be the complete bipartite graph where $2 \leq s \leq t$. Prove that $K_{s,t}$ is Hamiltonian if and only if $s = t$.

17. (a) Prove that if G is a graph of order 101 and $\delta(G) = 51$, then every vertex of G lies on a cycle of length 27.

    (b) State and prove a generalization of (a).

18. Determine the toughness of the regular complete 3-partite graph $K_{r,r,r}$ $(r \geq 2)$.

19. Show that if H is a spanning subgraph of a noncomplete graph G, then $t(H) \leq t(G)$.

20. (a) Show that if G is a noncomplete graph of order n, then $t(G) \leq (n - \alpha(G))/\alpha(G)$.

(b) Show that the order of every noncomplete connected graph G is at least $\alpha(G)(1 + t(G))$.

21. Show that for positive integers r and s with $r + s \geq 3$,

$$t(K_{r,s}) = \min\{r, s\}/\max\{r, s\}.$$

22. Show that every 1-tough graph is 2-connected.

23. (a) Determine the toughness of the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ where $n_1 \leq n_2 \leq \cdots \leq n_k$.

    (b) Show that for every nonnegative rational number r, there exists a graph G with $t(G) = r$.

24. Determine a formula for the toughness of a tree.

25. Show that if G is a graph of order $n \geq 4$ and size $m \geq \binom{n-1}{2} + 3$, then G is Hamiltonian-connected.

26. Prove that every Hamiltonian-connected graph of order 4 or more is 3-connected.

27. Give an example of a graph G that is pancyclic but not panconnected.

28. Give an example of a graph G of order $n \geq 4$ that is not panconnected but $\deg u + \deg v \geq n + 2$ for all distinct nonadjacent vertices u and v.

29. Give a proof by contradiction of Theorem 3.22:

    Let G be a graph of order $n \geq 4$. If $\deg u + \deg v \geq n+1$ for every pair u, v of nonadjacent vertices of G, then G is Hamiltonian-connected.

    by first observing that $G - v$ is Hamiltonian for every vertex v of G.

30. Prove that no bipartite graph of order 3 or more is Hamiltonian-connected.

3.3 Powers of Graphs and Line Graphs

A number of graph operations have been defined and studied that have led to several results dealing with Hamiltonian and Eulerian properties. One of the simplest operations is that of the subdivision graph of a graph. The **subdivision graph** $S(G)$ of a graph G is that graph obtained from G by replacing each edge $e = uv$ of G by a new vertex w_e and the two new edges uw_e and vw_e. The subdivision graph of K_4 is shown in Figure 3.15.

If G is a graph of order n and size m, then the order of $S(G)$ is $n + m$ and its size is $2m$. Furthermore, $S(G)$ is a bipartite graph with partite sets $V(G)$ and $V(S(G)) - V(G)$.

Figure 3.15: The subdivision graph of a graph

The Square and Cube of a Graph

Associated with each connected graph of order n and diameter d is a class of graphs defined in terms of distance. For each positive integer k, the **kth power** G^k of a graph G is that graph with $V(G^k) = V(G)$ and $uv \in E(G^k)$ if and only if $1 \leq d_G(u, v) \leq k$. Thus $G^1 = G$ and $G^k = K_n$ if $k \geq d$. The graph G^2 is also called the **square** of G while G^3 is called the **cube** of G. A graph with its square and cube are shown in Figure 3.16.

Figure 3.16: A graph and its square and cube

If G is a nontrivial connected graph of order n and k is an integer with $1 \leq k \leq n-1$, then G^k is k-connected (see Exercise 4). Since the kth power G^k ($k \geq 2$) of a connected graph G contains G as a subgraph (as a proper subgraph if G is not complete), it follows that G^k is Hamiltonian if G is Hamiltonian. Whether G is Hamiltonian or not, for a connected graph G of order at least 3 and for a sufficiently large integer k, the graph G^k is Hamiltonian since G^d is complete if G has diameter d. It is therefore natural to ask for the minimum k for which G^k is Hamiltonian. Certainly, for connected graphs in general, $k = 2$ will not suffice since if $G = S(K_{1,3})$ is the graph of Figure 3.16, then G^2 is not Hamiltonian. The graph G^3 *is* Hamiltonian, however.

In fact, the cube of every connected graph of order at least 3 is Hamiltonian. Indeed, a stronger result exists, discovered by Milan Sekanina [202] and later, but independently, by Jerome Karaganis [129].

Theorem 3.26 *If G is a connected graph, then G^3 is Hamiltonian-connected.*

Proof. If H is a spanning tree of G and H^3 is Hamiltonian-connected, then G^3 is Hamiltonian-connected. Hence it suffices to prove that the cube of every tree is Hamiltonian-connected. To show this, we proceed by induction on n, the order of the tree. For small values of n the result is obvious.

Assume for all trees H of order less than n that H^3 is Hamiltonian-connected and let T be a tree of order n. Let u and v be any two vertices of T. We consider two cases.

Case 1. *u and v are adjacent in T.* Let $e = uv$, and consider the forest $T-e$. This forest has two components, one tree T_u containing u and the other tree T_v containing v. By hypothesis, T_u^3 and T_v^3 are Hamiltonian-connected. Let u_1 be any vertex of T_u adjacent to u, and let v_1 be any vertex of T_v adjacent to v. If T_u or T_v is trivial, we define $u_1 = u$ or $v_1 = v$, respectively. Note that u_1 and v_1 are adjacent in T^3 since $d_T(u_1, v_1) \leq 3$. Let P_u be a Hamiltonian $u - u_1$ path (which may be trivial) of T_u^3 and let P_v be a Hamiltonian $v_1 - v$ path of T_v^3. The path formed by beginning with P_u followed by the edge u_1v_1 and then the path P_v is a Hamiltonian $u - v$ path of T^3.

Case 2. *u and v are not adjacent in T.* Since T is a tree, there exists a unique path between every two of its vertices. Let P be the unique $u - v$ path of T, and let $f = uw$ be the edge of P incident with u. The graph $T - f$ consists of two trees, one tree T_u containing u and the other tree T_w containing w. By hypothesis, there exists a Hamiltonian $w - v$ path P_w in T_w^3. Let u_1 be a vertex of T_u adjacent to u, or let $u_1 = u$ if T_u is trivial, and let P_u be a Hamiltonian $u - u_1$ path in T_u^3. Because $d_T(u_1, w) \leq 2$, the edge u_1w is present in T^3. Hence the path formed by starting with P_u followed by u_1w and then P_w is a Hamiltonian $u - v$ path of T^3. ■

It is, of course, an immediate consequence of Theorem 3.26 that for any connected graph G of order at least 3, its cube G^3 is Hamiltonian. Although it is not true that the square of every connected graph of order at least 3 is Hamiltonian, it was conjectured independently by Crispin Nash-Williams and Michael D. Plummer that for 2-connected graphs this is the case. In 1974, Herbert Fleischner [82] verified this conjecture. More recent proofs of this theorem have been given by by Stanislav Řiha [183] in 1991 and Agelos Georgakopoulos [93] in 2009.

Theorem 3.27 *If G is a 2-connected graph, then G^2 is Hamiltonian.*

A variety of results strengthening (but employing) Fleischner's work have been obtained. For example, Gary Chartrand, Arthur Hobbs, Heinz Jung, S. F. Kapoor and Crispin Nash-Williams [37] showed that the square of a 2-connected graph is Hamiltonian-connected.

Theorem 3.28 *If G is a 2-connected graph, then G^2 is Hamiltonian-connected.*

Proof. Since G is 2-connected, G has order at least 3. Let u and v be any two vertices of G. Let $G_1, G_2, \ldots, G_5$ be five distinct copies of G and let u_i and

v_i $(i = 1, 2, \ldots, 5)$ be the vertices in G_i corresponding to u and v in G. Form a new graph F by adding to the union $G_1 + G_2 + \cdots + G_5$ two new vertices w_1 and w_2 and ten new edges w_1u_i and w_2v_i $(i = 1, 2, \ldots, 5)$. The graph F is shown in Figure 3.17. Clearly, neither w_1 nor w_2 is a cut-vertex of F. Furthermore, since each graph G_i is 2-connected and contains two vertices adjacent to vertices in $V(F) - V(G_i)$, no vertex of G_i is a cut-vertex of F. Hence F is 2-connected.

Figure 3.17: The graph F in the proof of Theorem 3.28

By Theorem 3.27, F^2 has a Hamiltonian cycle C, which, of course, contains w_1 and w_2. At least one of the graphs G_i, say G_1, contains no vertex adjacent to w_1 or w_2 on C. Since u_1 and v_1 are the only vertices of G_1 adjacent on C to vertices not in G_1, it follows that C has a $u_1 - v_1$ path containing all vertices of G_1. Thus G_1^2 has a Hamiltonian $u_1 - v_1$ path, which implies that G^2 contains a Hamiltonian $u - v$ path. ■

The Line Graph of a Graph

The most familiar graph operation of a graph is that of the line graph. The **line graph** $L(G)$ of a graph G is that graph whose vertices can be put in one-to-one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. A graph and its line graph are shown in Figure 3.18 where the vertex u_i $(1 \leq i \leq 6)$ of $L(G)$ corresponds to the edge e_i of G.

It is relatively easy to determine the number of vertices and the number of edges in the line graph $L(G)$ of a graph G in terms of easily computed quantities in G. Indeed, if G is a graph of order n and size m with degree sequence $d_1, d_2, \ldots, d_n$ and $L(G)$ is a graph of order n' and size m', then $n' = m$ and

$$m' = \sum_{i=1}^{n} \binom{d_i}{2}$$

Figure 3.18: A graph and its line graph

since each edge of $L(G)$ corresponds to a pair of adjacent edges of G. For each nontrivial connected graph G, its line graph $L(G)$ is also connected (see Exercise 10). For the set $\mathcal{G}$ of all connected graphs and the set $\mathcal{G}'$ of all nonempty connected graphs, we may think of L as a function then, namely $L : \mathcal{G}' \to \mathcal{G}$. Hassler Whitney [238] showed that this function is very nearly injective.

Theorem 3.29 *Let G_1 and G_2 be nontrivial connected graphs. If $L(G_1) \cong L(G_2)$, then $G_1 \cong G_2$ unless one of G_1 and G_2 is K_3 and the other is $K_{1,3}$.*

The function $L : \mathcal{G}' \to \mathcal{G}$ is not close to being surjective, however. The graph $K_{1,3}$ is one of many graph that is not isomorphic to the line graph of any graph. Suppose that there is a graph H such that $L(H) = K_{1,3}$. Then H is a graph of size 4 containing an edge that is adjacent to the other three edges, no two of which are adjacent to each other. Such a graph H does not exist and so $K_{1,3}$ is not the line graph of any graph. Indeed, Lowell W. Beineke [10] obtained the following result.

Theorem 3.30 *A graph G is isomorphic to the line graph of some graph if and only if none of the graphs of Figure* 3.19 *is isomorphic to an induced subgraph of G.*

We turn to the problem of determining characteristics possessed by a graph that yield certain Hamiltonian properties of its line graph. Frank Harary and Crispin Nash-Williams [112] characterized those graphs having a Hamiltonian line graph. A circuit C in a graph G is called a **dominating circuit** if every edge of G either belongs to C or is adjacent to an edge of C. Equivalently, a circuit C in a graph G is a dominating circuit if every edge of G is incident with a vertex of C.

Theorem 3.31 *Let G be a graph without isolated vertices. Then $L(G)$ is Hamiltonian if and only if $G = K_{1,\ell}$ for some $\ell \geq 3$ or G contains a dominating circuit.*

Proof. If $G = K_{1,\ell}$ for some $\ell \geq 3$, then $L(G)$ is Hamiltonian since $L(G) = K_\ell$. Suppose, then, that G contains a dominating circuit

$$C = (v_1, v_2, \ldots, v_t, v_1).$$

Figure 3.19: The induced subgraphs not contained in any line graph

It suffices to show that there exists an ordering $s : e_1, e_2, \ldots, e_m$ of the m edges of G such that e_i and e_{i+1} are adjacent edges of G, for $1 \leq i \leq m-1$, as are e_1 and e_m, since such an ordering s corresponds to a Hamiltonian cycle of $L(G)$. Begin the ordering s by selecting, in any order, all edges of G incident with v_1 that are not edges of C, followed by the edge v_1v_2. At each successive vertex v_i, $2 \leq i \leq t-1$, select, in any order, all edges of G incident with v_i that are neither edges of C nor previously selected edges, followed by the edge v_iv_{i+1}. This process terminates with the edge $v_{t-1}v_t$. The ordering s is completed by adding the edge v_tv_1. Since C is a dominating circuit of G, every edge of G appears exactly once in s. Furthermore, consecutive edges of s as well as the first and last edges of s are adjacent in G.

Conversely, suppose that G is not a star but $L(G)$ is Hamiltonian. We show that G contains a dominating circuit. Since $L(G)$ is Hamiltonian, there is an ordering $s : e_1, e_2, \ldots, e_m$ of the m edges of G such that e_i and e_{i+1} are adjacent edges of G for $1 \leq i \leq m-1$, as are e_1 and e_m. For $1 \leq i \leq m-1$, let v_i be the vertex of G incident with both e_i and e_{i+1}. (Note that $1 \leq k \neq q \leq m-1$ does not necessarily imply that $v_k \neq v_q$.) Since G is not a star, there is a smallest integer j_1 exceeding 1 such that $v_{j_1} \neq v_1$. The vertex v_{j_1-1} is incident with e_{j_1}, the vertex v_{j_1} is incident with e_{j_1} and $v_{j_1-1} = v_1$. Thus, $e_{j_1} = v_1v_{j_1}$. Next, let j_2 (if it exists) be the smallest integer exceeding j_1 such that $v_{j_2} \neq v_{j_1}$. The vertex v_{j_2-1} is incident with e_{j_2}, the vertex v_{j_2} is incident with e_{j_2} and $v_{j_2-1} = v_{j_1}$. Thus, $e_{j_2} = v_{j_1}v_{j_2}$. Continuing in this fashion, we finally arrive at a vertex v_{j_t} such that $e_{j_t} = v_{j_t-1}v_{j_t}$, where $v_{j_t} = v_{m-1}$. Since every edge of G appears exactly once in s and since $1 < j_1 < j_2 < \cdots < j_t \leq m-1$, this

construction yields a trail

$$T = (v_1, e_{j_1}, v_{j_1}, e_{j_2}, v_{j_2}, \ldots, v_{j_{t-1}}, e_{j_t}, v_{j_t} = v_{m-1})$$

in G with the properties that (i) every edge of G is incident with a vertex of T and (ii) neither e_1 nor e_m is an edge of T.

Let w be the vertex of G incident with both e_1 and e_m. We consider four possible cases.

Case 1. $w = v_1 = v_{m-1}$. Then T itself is a dominating circuit of G.

Case 2. $w = v_1$ *and* $w \neq v_{m-1}$. Since e_m is incident with both w and v_{m-1}, it follows that $e_m = v_{m-1}w = v_{m-1}v_1$. Thus $C = (T, e_m, v_1)$ is a dominating circuit of G.

Case 3. $w = v_{m-1}$ *and* $w \neq v_1$. Since e_1 is incident with both w and v_1, we have that $e_1 = wv_1 = v_{m-1}v_1$. Thus $C = (T, e_1, v_1)$ is a dominating circuit of G.

Case 4. $w \neq v_{m-1}$ *and* $w \neq v_1$. Since e_m is incident with both w and v_{m-1}, it follows that $e_m = v_{m-1}w$. Since e_1 is incident with both w and v_1, we have that $e_1 = wv_1$. Thus $v_1 \neq v_{m-1}$, and $C = (T, e_m, w, e_1, v_1)$ is a dominating circuit of G. ■

As a consequence of Theorem 3.31, if G is either Eulerian or Hamiltonian, then $L(G)$ is Hamiltonian. In fact, successively taking the line graph of a connected graph has some interesting consequences.

For a nonempty graph G, we write $L^0(G)$ to denote G and $L^1(G)$ to denote $L(G)$. For an integer $k \geq 2$, the **iterated line graph** $L^k(G)$ is defined as $L(L^{k-1}(G))$, where $L^{k-1}(G)$ is assumed to be nonempty. The following result is due to Gary Chartrand and Curtiss E. Wall [41].

Theorem 3.32 *If G is a connected graph such that* $\deg v \geq 3$ *for every vertex v of G, then $L^2(G)$ is Hamiltonian.*

Proof. Let v be a vertex of G, where $\deg v = r \geq 3$. Then in $L(G)$, the edges incident with v give rise to a subgraph H_v, where $H_v \cong K_r$. Let C_v be a Hamiltonian cycle in H_v. Let H be the spanning subgraph of $L(G)$ defined by

$$V(H) = V(L(G)) \text{ and } E(H) = \bigcup_{v \in V(G)} E(C_v).$$

Then H is connected and every vertex of H is even. Consequently, H is Eulerian and so H is a dominating circuit of $L(G)$. By Theorem 3.31, $L^2(G)$ is Hamiltonian. ■

If $G = P_n$, where $n \geq 2$, then $L(G) = P_{n-1}$. Thus $L^{n-1}(G) = P_1$ and $L^k(G)$ is not defined for $k \geq n$. If $G = K_{1,3}$, then $L(G) = C_3$. If $G = C_n$ for some $n \geq 3$, then $L(G) = C_n$ and $L^k(C_n) = C_n$ for every nonnegative integer

k. Suppose then that G is a connected graph that is not a path, $K_{1,3}$ or a cycle. Then $\Delta(G) \geq 3$. Suppose that G contains a vertex u of degree 1. Then G contains a $u - v$ path $P = (u = u_0, u_1, \ldots, u_\ell = v)$ of minimum length $\ell \geq 1$ where $\deg v \geq 3$. This gives rise to an $x_0 - x$ path P' of length ℓ in $L(G)$, where x_0 corresponds to u_0u_1, x corresponds to $u_{\ell-1}u_\ell$, $\deg x_0 = 1$ if $\ell \geq 2$, and $\deg x \geq 3$. By successively taking the line graph of G, $L(G)$, $L^2(G)$ and so on, we eventually arrive at a graph $L^k(G)$ for some k where there are no end-vertices. If a connected graph H contains no end-vertices but does contain vertices of degree 2, then G contain a $y - z$ path (or $y - z$ cycle if $y = z$) $Q = (y = y_0, y_1, \ldots, y_t = z)$ where $t \geq 2$, $\deg y_i = 2$ for $1 \leq i \leq t-1$, $\deg y \geq 3$ and $\deg z \geq 3$. This gives rise to a $w - x$ path $Q' = (w = w_1, w_2, \ldots, w_t = x)$ of length $t-1$ in $L(G)$ where w_1 corresponds to the edge y_0y_1 in G, w_t corresponds to the edge $y_{t-1}y_t$, $\deg w_i = 2$ for $2 \leq i \leq t-1$, $\deg w_1 \geq 3$ and $\deg w_t \geq 3$. By successively taking the line graph of H, $L(H)$, $L^2(H)$ and so on, we arrive at a graph $L^r(H)$ in which all vertices have degree at least 3. This is illustrated for the graph G of Figure 3.20. It then follows that there is a sufficiently large integer s such that the degree of every vertex of $L^s(G)$ is at least 3. By the discussion above and Theorem 3.32, we then have the following.

Figure 3.20: A graph and iterated line graphs

Theorem 3.33 *If G is a connected graph that is not a path, then there exists an integer n_0 such that $L^k(G)$ is Hamiltonian for every integer $k \geq n_0$.*

The Total Graph of a Graph

A graph operation related to the line graph is the total graph. The **total graph** $T(G)$ of a graph G is that graph whose vertices can be put in one-to-one correspondence with the elements of the set $V(G) \cup E(G)$ such that two vertices of $T(G)$ are adjacent if the corresponding elements in G are two adjacent vertices, two adjacent edges or an incident vertex and edge. A graph H and its total graph $T(H)$ are shown in Figure 3.21.

Figure 3.21: A graph and its total graph

One might observe that $T(H) \cong G^2$ for the graph H of Figure 3.21 and the graph G of Figure 3.16. We saw that G^2 is not Hamiltonian. Thus, for the graph H of Figure 3.21, $T(H)$ is not Hamiltonian. This, in fact, is not surprising for if F is any graph, then $T(F) \cong [S(F)]^2$, that is, the total graph of F is the square of the subdivision graph of F. Since the graph G of Figure 3.16 is isomorphic to $S(K_{1,3})$ and the graph H of Figure 3.21 is isomorphic to $K_{1,3}$, it follows that $T(H) \cong G^2$, where G is the graph of Figure 3.16.

While the total graph of a nontrivial connected graph G need not be Hamiltonian, the same cannot be said for $T(T(G))$.

Theorem 3.34 *If G is a nontrivial connected graph, then $T(T(G))$ is Hamiltonian.*

Proof. First, we consider $T(G)$. Since $S(G)$ is a nontrivial connected graph, $[S(G)]^2$ is 2-connected and $T(G) \cong [S(G)]^2$, it follows that $H = T(G)$ is 2-connected. Hence $S(H)$ is 2-connected. By Theorem 3.27, $[S(H)]^2$ is Hamiltonian. Thus $T(H) \cong [S(H)]^2$ and so $T(T(G))$ is Hamiltonian. ■

Exercises for Section 3.3

1. Determine all those connected graphs G for which $S(G)$ is Eulerian.
2. Determine all those connected graphs G for which $S(G)$ is Hamiltonian.
3. Show that if G is a connected graph of order n and size m, then $\alpha(S(G)) = m$ unless G belongs to a familiar class of graphs.

4. Show that if G is a connected graph of order $n \geq 2$ and k is an integer with $1 \leq k \leq n-1$, then G^k is k-connected.

5. Show that if G is a connected graph of diameter ℓ and $1 \leq k \leq \ell$, then $\operatorname{diam}(G^k) = \lceil \ell/k \rceil$.

6. A graph H is called a **square root** of a connected graph G if $H^2 = G$.

   (a) Give an example of a connected graph with two non-isomorphic square roots.

   (b) Give an example of a connected graph with a unique square root.

7. Show that the graph G^2 of Figure 3.16 is not Hamiltonian.

8. Prove that if v is any vertex of a connected graph G of order at least 4, then $G^3 - v$ is Hamiltonian.

9. Prove that if G is a self-complementary graph of order at least 5, then G^2 is Hamiltonian-connected.

10. Prove that the line graph of every nontrivial connected graph is connected.

11. Determine a formula for the number of triangles in the line graph $L(G)$ in terms of quantities in G.

12. Prove that $L(G)$ is Eulerian if G is Eulerian.

13. (a) Find a necessary and sufficient condition for a graph G to have the property that $G \cong L(G)$.

    (b) Find a necessary and sufficient condition for a graph G to have the property that $L(G) \cong L^2(G)$.

14. For each of the following, prove or disprove.

    (a) If G is Hamiltonian, then G^2 is Hamiltonian-connected.

    (b) If G is connected and $L(G)$ is Eulerian, then G is Eulerian.

    (c) If G is Hamiltonian, then $L(G)$ is Hamiltonian-connected.

    (d) If G has a dominating circuit, then $L(G)$ has a dominating circuit.

15. Prove that if G is a connected graph and $L^3(G)$ is Eulerian, then $L^2(G)$ is Eulerian.

16. Give an example of a connected graph G such that $\deg v \geq 3$ for every vertex v of G but $L(G)$ is not Hamiltonian.

17. (a) Show that if G is a k-edge-connected graph, $k \geq 2$, then $L(G)$ is k-connected.

(b) Show that if G is a k-edge-connected graph, $k \geq 2$, then $L(G)$ is $(2k-2)$-edge-connected.

18. Show that there exists a graph G that is not isomorphic to the total graph of any graph.

19. (a) Prove that if G is a nontrivial connected graph, then $T(G^2)$ is Hamiltonian.

    (b) Prove that if G is a nontrivial connected graph, then $(T(G))^2$ is Hamiltonian.

Chapter 4

Digraphs

While there are many concepts in digraphs that are analogues to concepts we have encountered in graphs, there are also concepts that are quite unique to digraphs. We consider many of these in this current chapter, beginning with the most studied type of connectedness for digraphs.

4.1 Strong Digraphs

In Chapter 1 we described two types of connectedness for digraphs, namely weakly connected (or, more simply, connected) digraphs and strongly connected (or simply strong) digraphs. In this section we explore strong digraphs in more detail. Recall that a digraph D is strong if for every pair u, v of vertices of D, there is both a $u-v$ path and a $v-u$ path. Recall, also, that when dealing with digraphs, the terms path and cycle refer to directed path and directed cycle. The same is true of the terms walk, trail and circuit.

The following theorem is the digraph analogue of Theorem 1.16 and its proof is analogous as well (see Exercise 1).

Theorem 4.1 *Let u and v be two vertices in a digraph D. For every $u-v$ walk W in D, there exists a $u-v$ path P such that every arc of P belongs to W.*

Strong digraphs are characterized in the following theorem.

Theorem 4.2 *A digraph D is strong if and only if D contains a closed spanning walk.*

Proof. Assume that $W = (u_1, u_2, \ldots, u_k, u_1)$ is a closed spanning walk in D. Let $u, v \in V(D)$. Then $u = u_i$ and $v = u_j$ for some i, j with $1 \leq i, j \leq k$ and $i \neq j$. Without loss of generality, assume that $i < j$. Then $W_1 = (u_i, u_{i+1}, \ldots, u_j)$ is a $u_i - u_j$ walk in D and $W_2 = (u_j, u_{j+1}, \ldots, u_k, u_1, \ldots, u_i)$ is a $u_j - u_i$ walk

in D. By Theorem 4.1, D contains both a $u_i - u_j$ path and a $u_j - u_i$ path in D, and so D is strong.

Conversely, assume that D is a nontrivial strong digraph with $V(D) = \{v_1, v_2, \ldots, v_n\}$ and consider the cyclic sequence $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$. Since D is strong, D contains a $v_i - v_{i+1}$ path P_i for $i = 1, 2, \ldots, n$. Then the sequence $P_1, P_2, \ldots, P_n$ of paths produces a closed spanning walk in D. ■

The **converse** $\vec{D}$ of a digraph D is obtained from D by reversing the direction of every arc of D. Thus D is strong if and only if its converse $\vec{D}$ is strong (see Exercise 2).

Robbins' Theorem

Recall that an orientation of a graph G is a digraph obtained by assigning a direction to each edge of G. Herbert E. Robbins (1922–2001) studied those graphs having a strong orientation. Certainly, if G has a strong orientation, then G must be connected. Also, if G has a bridge, then it is impossible to produce a strong orientation of G. Robbins [190] showed that if G is a bridgeless connected graph, then G always has a strong orientation.

Theorem 4.3 (**Robbins' Theorem**) *A nontrivial graph G has a strong orientation if and only if G is 2-edge-connected.*

Proof. We have already observed that if a graph G has a strong orientation, then G is 2-edge-connected. Suppose that the converse is false. Then there exists a 2-edge-connected graph G that has no strong orientation. Among the subgraphs of G, let H be one of maximum order that has a strong orientation. Such a subgraph exists since for each $v \in V(G)$, the subgraph $G[\{v\}]$ trivially has a strong orientation. Thus $|V(H)| < |V(G)|$, since, by assumption, G has no strong orientation.

Assign directions to the edges of H so that the resulting digraph D is strong, but assign no directions to the edges of $G - E(H)$. Let $u \in V(H)$ and let $v \in V(G) - V(H)$. Since G is 2-edge-connected, it follows by Theorem 2.43 that G contains two edge-disjoint $u - v$ paths. Let P be one of these $u - v$ paths and let Q be the $v - u$ path that results from the other $u - v$ path. Further, let u_1 be the last vertex of P that belongs to H, and let v_1 be the first vertex of Q belonging to H. Next, let P_1 be the $u_1 - v$ subpath of P and let Q_1 be the $v - v_1$ subpath of Q. Direct the edges of P_1 from u_1 toward v, producing the directed path P_1' and direct the edges of Q_1 from v toward v_1, producing the directed path Q_1'.

Define the digraph D' by

$$V(D') = V(D) \cup V(P_1') \cup V(Q_1') \text{ and } E(D') = E(D) \cup E(P_1') \cup E(Q_1').$$

Since D is strong, so too is D', contradicting the choice of H. ■

As we mentioned, Theorem 4.3 is due to Robbins. The paper in which this theorem appears is titled "A theorem on graphs, with an application to a problem of traffic control" and was published in 1939 in the *American Mathematical Monthly*, only a year after Robbins received his Ph.D. from Harvard University in topology, under the direction of Hassler Whitney. This was only Robbins' second publication of what would become a long and impressive list. Also in 1939, at age 24, Robbins began work on the classic book *What is Mathematics?* with Richard Courant. Robbins classified this book as a literary work rather than a scientific work. This book discussed mathematics as it existed at that time. A few years later Robbins became interested in and devoted his research to mathematical statistics, in which he made major contributions.

Eulerian Digraphs

Eulerian and Hamiltonian graphs have natural analogues for digraphs. In both instances, these are strong digraphs. An **Eulerian circuit** in a connected digraph D is a circuit that contains every arc of D (necessarily exactly once); while an **Eulerian trail** in D is an open trail that contains every arc of D. A connected digraph that contains an Eulerian circuit is an **Eulerian digraph**.

The next theorem gives a characterization of Eulerian digraphs whose statement and proof are similar to that of Theorem 3.1 (see Exercise 7).

Theorem 4.4 *Let D be a nontrivial connected digraph. Then D is Eulerian if and only if* $\operatorname{od} v = \operatorname{id} v$ *for every vertex v of D.*

With the aid of Theorem 4.4, a characterization of digraphs containing an Eulerian trail can be given (see Exercise 9).

Theorem 4.5 *Let D be a nontrivial connected digraph. Then D contains an Eulerian trail if and only if D contains two vertices u and v such that*

$$\operatorname{od} u = \operatorname{id} u + 1 \quad \textit{and} \quad \operatorname{id} v = \operatorname{od} v + 1,$$

while $\operatorname{od} w = \operatorname{id} w$ *for all other vertices w of D. Furthermore, each Eulerian trail of D begins at u and ends at v.*

Thus the digraph D_1 of Figure 4.1 contains an Eulerian circuit, D_2 contains an Eulerian $u - v$ trail and D_3 contains neither an Eulerian circuit nor an Eulerian trail.

Suppose that D is a connected digraph containing two vertices u and v such that $\operatorname{od} u - \operatorname{id} u = \operatorname{id} v - \operatorname{od} v = k$ for some nonnegative integer k and $\operatorname{od} w = \operatorname{id} w$ for all $w \in V(D) - \{u, v\}$. If $k = 0$, then D is Eulerian; while if $k = 1$, then D contains an Eulerian $u - v$ trail. More generally, for $k \geq 1$, we have the following.

Theorem 4.6 *If D is a connected digraph containing two vertices u and v such that*

Figure 4.1: Eulerian circuits and trails in digraphs

$$\operatorname{od} u = \operatorname{id} u + k \quad \textit{and} \quad \operatorname{id} v = \operatorname{od} v + k$$

for some positive integer k and $\operatorname{od} w = \operatorname{id} w$ for all other vertices w of D, then D contains k arc-disjoint $u - v$ paths.

Proof. Let D' be the digraph constructed from D by adding k new vertices $w_1, w_2, \ldots, w_k$ and the $2k$ arcs (v, w_i) and (w_i, u) for $1 \leq i \leq k$. Since $\operatorname{od} x = \operatorname{id} x$ for every vertex x of D', the digraph D' is Eulerian and therefore contains an Eulerian circuit C. Suppose that C begins (and ends) at u. Since $\operatorname{od} u - \operatorname{id} u = k$ and $\operatorname{id} v - \operatorname{od} v = k$, it follows that C contains k arc-disjoint $u - v$ trails. By Theorem 4.1, for each such $u - v$ trail T, there is a $u - v$ path each of whose arcs belongs to T. Since none of these paths contains any of the vertices w_i $(1 \leq i \leq k)$, D contains k arc-disjoint $u - v$ paths. ■

Hamiltonian Digraphs

A digraph D is **Hamiltonian** if D contains a spanning cycle. Such a cycle is called a **Hamiltonian cycle** of D. As with Hamiltonian graphs, no characterization of Hamiltonian digraphs exists. Indeed, if anything, the situation for Hamiltonian digraphs is even more complex than it is for Hamiltonian graphs. Where there are sufficient conditions for a digraph to be Hamiltonian, these are analogues of the simpler sufficient conditions for Hamiltonian graphs.

The following result of Henri Meyniel [158] gives a sufficient condition (much like that in Theorem 3.6 for graphs) for a digraph to be Hamiltonian.

Theorem 4.7 (**Meyniel's Theorem**) *If D is a nontrivial strong digraph of order n such that*

$$\deg u + \deg v \geq 2n - 1$$

for every pair u, v of nonadjacent vertices, then D is Hamiltonian.

Theorem 4.7 has a large number of consequences. We consider these now, beginning with a result originally discovered by Douglas Woodall [242].

Corollary 4.8 *If D is a nontrivial digraph of order n such that*

$$\operatorname{od} u + \operatorname{id} v \geq n$$

whenever u and v are distinct vertices and $(u, v) \notin E(D)$, then D is Hamiltonian.

The following theorem is due to Alain Ghouila-Houri [94]. The proof is an immediate consequence of Theorem 4.7.

Corollary 4.9 *If D is a strong digraph of order n such that $\deg v \geq n$ for every vertex v of D, then D is Hamiltonian.*

Corollary 4.9 also has a corollary. We provide a proof of this result.

Corollary 4.10 *If D is a digraph of order n such that*

$$\operatorname{od} v \geq n/2 \quad \textit{and} \quad \operatorname{id} v \geq n/2$$

for every vertex v of D, then D is Hamiltonian.

Proof. Suppose that the theorem is false. Since the theorem is clearly true for $n = 2$ and $n = 3$, there exists some integer $n \geq 4$ and a digraph D of order n that satisfies the hypothesis but which is not Hamiltonian. Let C be a cycle in D of maximum length k. It follows from Theorem 1.24 and the assumption that D is not Hamiltonian that $1 + n/2 \leq k < n$. Also, let P be a path of maximum length such that no vertex of P lies on C. Suppose that P is a $u - v$ path of length $\ell \geq 0$. Therefore, $k + \ell + 1 \leq n$. (See Figure 4.2.)

Figure 4.2: A step in the proof of Corollary 4.10

Since

$$\ell \leq n - k - 1 \leq n - \left(1 + \frac{n}{2}\right) - 1 = \frac{n}{2} - 2,$$

it follows that $\ell \leq n/2 - 2$ and that there are at least two vertices adjacent to u which lie on C and at least two vertices adjacent from v which lie on C.

Let a denote the number of vertices on C that are adjacent to u. Thus $a \geq 2$. For every vertex x on C that is adjacent to u, the $\ell + 1$ vertices immediately following x on C are not adjacent from v, for otherwise, D has a cycle of length exceeding k. Since C contains vertices adjacent from v, there must be a vertex y on C that is adjacent to u such that none of the $\ell + 1$ vertices immediately following y on C are adjacent to u or adjacent from v.

For each of the $a - 1$ vertices on C that are distinct from y and adjacent to u, the vertex immediately following it cannot be adjacent from v. Hence at least $(a - 1) + (\ell + 1) = a + \ell$ vertices on C are not adjacent from v, for otherwise again, D has a cycle of length exceeding k. Since P is a longest path in D containing no vertices of C, every vertex adjacent to u is either on C or on P.

Because $\operatorname{id} u \geq n/2$ and the only vertices of D that can be adjacent to u belong to C or P, it follows that $a+\ell \geq n/2$. Therefore, v is adjacent to at most $(n-1)-(a+\ell) \leq (n-1)-n/2 = n/2-1$ vertices, producing a contradiction. ■

Line Digraphs

In Section 3.3, the line graph of a nonempty graph is defined. There is an analogous concept for digraphs. An arc $e = (u, v)$ is **adjacent to** an arc $f = (x, y)$ in a digraph D if $v = x$. In the **line digraph** $L(D)$ of a nonempty digraph D, the vertices of $L(D)$ can be put in one-to-one correspondence with the arcs of D in such a way that vertex u is adjacent to vertex v in $L(D)$ if the arc corresponding to u in D is adjacent to the arc corresponding to v. A digraph D and its line digraph are shown in Figure 4.3, where v_i in $L(D)$ corresponds to e_i in D $(i = 1, 2, \ldots, 7)$.

Figure 4.3: A digraph and its line digraph

While the line graph of an Eulerian graph is both Eulerian and Hamiltonian and the line graph of a Hamiltonian graph is Hamiltonian, this is not true in general for line digraphs (see Exercises 17 and 18). We do have the following result however (see Exercise 20).

Theorem 4.11 *Let D be a nontrivial connected digraph. Then $L(D)$ is Hamiltonian if and only if D is Eulerian.*

Exercises for Section 4.1

1. Prove Theorem 4.1: *Let u and v be two vertices in a digraph D. For every $u - v$ walk W in D, there exists a $u - v$ path P such that every arc of P belongs to W.*

2. Show that a digraph D is strong if and only if its converse $\vec{D}$ is strong.

3. Let G be a nontrivial connected graph without bridges.

   (a) Show that for every edge e of G and for every orientation of e, there exists an orientation of the remaining edges of G such that the resulting digraph is strong.

   (b) Show that (a) need not be true if we begin with an orientation of two edges of G.

4. Let G be a connected graph with cut-vertices. Show that an orientation D of G is strong if and only if the subdigraph of D induced by the vertices of each block of G is strong.

5. According to Theorem 4.3, a nontrivial graph G has a strong orientation if and only if G is connected and contains no bridges.

   (a) Prove that if G is a nontrivial connected graph with at most two bridges, then there exists an orientation D of G having the property that if u and v are any two vertices of D, there is either a $u - v$ path or a $v - u$ path.

   (b) Show that the statement (a) is false if G contains three bridges.

6. Prove or disprove: There exists a 4-regular graph G of order 7 and an orientation D of G such that for each vertex u of D, there exists either a $u - v$ path of length 1 or a $u - v$ path of length 2 but not both for every vertex v of D with $v \neq u$.

7. Prove Theorem 4.4: *Let D be a nontrivial connected digraph. Then D is Eulerian if and only if* $\operatorname{od} v = \operatorname{id} v$ *for every vertex v of D.*

8. Prove that a graph G has an Eulerian orientation if and only if G is Eulerian.

9. Prove Theorem 4.5: *Let D be a nontrivial connected digraph. Then D contains an Eulerian trail if and only if D contains two vertices u and v such that* $\operatorname{od} u = \operatorname{id} u + 1$ *and* $\operatorname{id} v = \operatorname{od} v + 1$, *while* $\operatorname{od} w = \operatorname{id} w$ *for all other vertices w of D. Furthermore, each Eulerian trail of D begins at u and ends at v.*

10. Prove that a nontrivial connected digraph D is Eulerian if and only if $E(D)$ can be partitioned into subsets E_i, $1 \leq i \leq k$, where the subdigraph $D[E_i]$ induced by the set E_i is a cycle for each i.

11. Prove that if D is a connected digraph such that $\sum_{v \in V(D)} |\operatorname{od} v - \operatorname{id} v| = 2t$, where $t \geq 1$, then $E(D)$ can be partitioned into subsets E_i, $1 \leq i \leq t$, so that the subgraph $G[E_i]$ induced by E_i is an open trail for each i.

12. Let D be a connected digraph of order n with $V(D) = \{v_1, v_2, \ldots, v_n\}$. Prove that if $\operatorname{od} v_i \geq \operatorname{id} v_i$ for $1 \leq i \leq n$, then D is Eulerian.

13. A vertex v in a digraph D is said to be **reachable** from a vertex u in D if D contains a $u - v$ path. Let D be a digraph and for each vertex u of D, let $R(u)$ be the set of vertices reachable from u and let $r(u) = |R(u)|$. Since $u \in R(u)$ for every vertex u of D, it follows that $r(u) \geq 1$. Prove that if $r(x) \neq r(y)$ for every two distinct vertices x and y of D, then D contains a Hamiltonian path.

14. Corollary 4.9 states: *If D is a strong digraph of order n such that* $\deg v \geq n$ *for every vertex v of D, then D is Hamiltonian.* Show that if the digraph D is not required to be strong, then D need not be Hamiltonian.

15. Show for infinitely many positive integers n that there exists a digraph D of order n such that $\operatorname{od} v \geq (n-1)/2$ and $\operatorname{id} v \geq (n-1)/2$ for every vertex v of D but D is not Hamiltonian.

16. Prove that if D is a nontrivial strong digraph, then $L(D)$ is strong.

17. Give an example of a Hamiltonian digraph whose line digraph is not Hamiltonian.

18. Give an example of an Eulerian digraph whose line digraph is not Eulerian.

19. Give an example of a digraph D such that D is not Eulerian and $L(L(D))$ is Hamiltonian.

20. Prove Theorem 4.11: *Let D be a nontrivial connected digraph. Then $L(D)$ is Hamiltonian if and only if D is Eulerian.*

4.2 Tournaments

There are sporting events involving teams or individuals that require every two teams or individuals to compete against each other exactly once. This is referred to as a round robin tournament. Men's soccer has been part of the Summer Olympic Games since 1900. Teams from 16 countries participate, divided into four pools of four teams each. In each pool, a round robin tournament takes place, in which the top two teams in each pool advance to play for Olympic medals. This also occurs during the World Cup for soccer supremacy when 32 countries participate, divided into eight pools of four teams each.

Round robin tournaments give rise quite naturally to a class of digraphs introduced in Chapter 1. A **tournament** is an orientation of a complete graph. Therefore, a tournament can be defined as a digraph such that for every pair

u, v of distinct vertices, exactly one of (u, v) and (v, u) is an arc. A tournament T then models a round robin tournament in which no ties are permitted. The vertices of T are the teams in the round robin tournament and (u, v) is an arc in T if team u defeats team v.

Figure 4.4 shows two tournaments of order 3. In fact, these are the only two tournaments of order 3. The number of non-isomorphic tournaments increases sharply with their orders, however. For example, there is only one tournament of order 1 and one of order 2. As we just observed, the tournaments T_1 and T_2 in Figure 4.4 are the only two tournaments of order 3. There are four tournaments of order 4, 12 of order 5, 56 of order 6 and over 154 billion of order 12.

Figure 4.4: The tournaments of order 3

Since the size of a tournament of order n is $\binom{n}{2}$, it follows from Theorem 1.23 that

$$\sum_{v \in V(T)} \operatorname{od} v = \sum_{v \in V(T)} \operatorname{id} v = \binom{n}{2}.$$

Transitive Tournaments

A tournament T is **transitive** if whenever (u, v) and (v, w) are arcs of T, then (u, w) is also an arc of T. The tournament T_2 of Figure 4.4 is transitive while T_1 is not. The following result gives an elementary property of transitive tournaments. An **acyclic digraph** has no cycles.

Theorem 4.12 *A tournament is transitive if and only if it is acyclic.*

Proof. Let T be an acyclic tournament and suppose that (u, v) and (v, w) are arcs of T. Since T is acyclic, $(w, u) \notin E(T)$. Therefore, $(u, w) \in E(T)$ and T is transitive.

Conversely, suppose that T is a transitive tournament and assume that T contains a cycle, say $C = (v_1, v_2, \ldots, v_k, v_1)$, where $k \geq 3$. Since (v_1, v_2) and (v_2, v_3) are arcs of the transitive tournament T, it follows that (v_1, v_3) is also an arc of T. Similarly, (v_1, v_4), (v_1, v_5), $\ldots$, (v_1, v_k) are arcs of T. However, this contradicts the fact that (v_k, v_1) is an arc of T. Thus, T is acyclic. ∎

Suppose that a tournament T of order n with vertex set $V(T)=\{v_1, v_2, \ldots, v_n\}$ represents a round robin tournament involving competition among n teams $v_1, v_2, \ldots, v_n$. If team v_i defeats team v_j, then (v_i, v_j) is an arc of T. The number of victories by team v_i is the outdegree of v_i. For this reason, the

outdegree of the vertex v_i in a tournament is also referred to as the **score** of v_i. A sequence $s_1, s_2, \ldots, s_n$ of nonnegative integers is called a **score sequence of a tournament** if there exists a tournament T of order n whose vertices can be labeled $v_1, v_2, \ldots, v_n$ such that $\operatorname{od} v_i = s_i$ for $i = 1, 2, \ldots, n$.

Figure 4.5 shows transitive tournaments of order n for $n = 3, 4, 5$. The score sequence of every transitive tournament has an interesting property. The following result describes precisely which sequences are score sequences of transitive tournaments.

Figure 4.5: Transitive tournaments of orders 3, 4, 5

Theorem 4.13 *A nondecreasing sequence π of n nonnegative integers is a score sequence of a transitive tournament of order n if and only if π is the sequence $0, 1, \ldots, n-1$.*

Proof. First we show that $\pi : 0, 1, \ldots, n-1$ is a score sequence of a transitive tournament of order n. Let T be the tournament with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(T) = \{(v_i, v_j) : \ 1 \leq i < j \leq n\}$. We claim that T is transitive. Let (v_i, v_j) and (v_j, v_k) be arcs of T. Then $i < j < k$. Since $i < k$, (v_i, v_k) is an arc of T and so T is transitive. For $1 \leq i \leq n$, $\operatorname{od} v_i = n - i$. Therefore, a score sequence of T is $\pi : 0, 1, \ldots, n-1$.

Next, we show that if T is a transitive tournament of order n, then $0, 1, \ldots, n-1$ is a score sequence of T. This is equivalent to showing that every two vertices of T have distinct scores. Let u and w be two vertices of T. Assume, without loss of generality, that (u, w) is an arc of T. Let W be the set of vertices of T to which w is adjacent. Therefore $\operatorname{od} w = |W|$. For each $x \in W$, (w, x) is an arc of T. Since T is transitive, (u, x) is also an arc of T. However then, $\operatorname{od} u \geq |W| + 1$ and so $\operatorname{od} u \neq \operatorname{od} w$. ■

The proof of Theorem 4.13 shows that the structure of a transitive tournament is uniquely determined.

Corollary 4.14 *For every positive integer n, there is exactly one transitive tournament of order n.*

Combining this corollary with Theorem 4.12, we arrive at yet another corollary.

Corollary 4.15 *For every positive integer n, there is exactly one acyclic tournament of order n.*

Although there is only one transitive tournament of each order n, in a certain sense, which we now explore, every tournament has the structure of a transitive tournament. Let T be a tournament. We define a relation on $V(T)$ by u is related to v if there is both a $u-v$ path and a $v-u$ path in T. This relation is an equivalence relation and, as such, this relation partitions $V(T)$ into equivalence classes $V_1, V_2, \ldots, V_k$ $(k \geq 1)$. Let $S_i = T[V_i]$ for $i = 1, 2, \ldots, k$. Then each subdigraph S_i is a strong tournament and, indeed, is maximal with respect to the property of being strong. The subdigraphs $S_1, S_2, \ldots, S_k$ are called the **strong components** of T. So the vertex sets of the strong components of T produce a partition of $V(T)$.

Let T be a tournament with strong components $S_1, S_2, \ldots, S_k$, and let $\widetilde{T}$ denote that digraph whose vertices $u_1, u_2, \ldots, u_k$ are in one-to-one correspondence with these strong components (where u_i corresponds to S_i, $i = 1, 2, \ldots, k$) such that (u_i, u_j) is an arc of $\widetilde{T}$, $i \neq j$, if and only if some vertex of S_i is adjacent to some vertex of S_j. If (u_i, u_j) is an arc of $\widetilde{T}$, then because S_i and S_j are distinct strong components of T, it follows that every vertex of S_i is adjacent to *every* vertex of S_j. Hence, $\widetilde{T}$ is obtained by identifying the vertices of S_i for $i = 1, 2, \ldots, k$. A tournament T and its associated digraph $\widetilde{T}$ are shown in Figure 4.6.

Figure 4.6: A tournament T and its associated transitive tournament $\widetilde{T}$

Observe that for the tournament T of Figure 4.6, $\widetilde{T}$ is itself a tournament, indeed a transitive tournament. That this always occurs follows from Theorem 4.16. (See Exercise 5.)

Theorem 4.16 *If T is a tournament with exactly k strong components, then $\widetilde{T}$ is the transitive tournament of order k.*

Since for every tournament T, the tournament $\widetilde{T}$ is transitive, it follows that if T is a tournament that is not strong, then $V(T)$ can be partitioned as $\{V_1, V_2, \ldots, V_k\}$ $(k \geq 2)$ such that $T[V_i]$ is a strong tournament for each i, and if $v_i \in V_i$ and $v_j \in V_j$, where $i > j$, then $(v_i, v_j) \in E(T)$. This decomposition is often useful when studying the properties of tournaments that are not strong.

We already noted that there are four tournaments of order 4. Of course, one of these is transitive, which consists of four trivial strong components

S_1, S_2, S_3, S_4, where the vertex of S_i is adjacent to the vertex of S_j if and only if $i > j$. There are two tournaments of order 4 containing two strong components S_1 and S_2, depending on whether S_1 or S_2 is the strong component of order 3. (No strong component has order 2.) Since there are four tournaments of order 4, there is exactly one strong tournament of order 4. These tournaments are depicted in Figure 4.7. The arcs not drawn in the tournaments T_1, T_2 and T_3 that are not strong are all directed downward, as indicated by the double arrow.

Figure 4.7: The four tournaments of order 4

We also stated that there are 12 tournaments of order 5. There are six tournaments of order 5 that are not strong, shown in Figure 4.8. Again all arcs that are not drawn are directed downward. Thus there are six strong tournaments of order 5.

Figure 4.8: The six tournaments of order 5 that are not strong

Score Sequences of Tournaments

Theorem 4.13 characterizes score sequences of transitive tournaments. We next investigate score sequences of tournaments in general. We begin with a theorem similar to Theorem 1.12.

Theorem 4.17 *A nondecreasing sequence $\pi : s_1, s_2, \ldots, s_n$ $(n \geq 2)$ of nonnegative integers is a score sequence of a tournament if and only if the sequence $\pi_1 : s_1, s_2, \ldots, s_{s_n}, s_{s_n+1} - 1, \ldots, s_{n-1} - 1$ is a score sequence of a tournament.*

Proof. Assume that π_1 is a score sequence of a tournament. Then there exists a tournament T_1 of order $n-1$ having π_1 as a score sequence. Hence the vertices of T_1 can be labeled as $v_1, v_2, \ldots, v_{n-1}$ such that

$$\operatorname{od} v_i = \begin{cases} s_i & \text{for } 1 \le i \le s_n \\ s_i - 1 & \text{for } i > s_n. \end{cases}$$

We construct a tournament T by adding a vertex v_n to T_1 where v_n is adjacent to v_i if $1 \le i \le s_n$ and v_n is adjacent from v_i otherwise. The tournament T then has π as a score sequence.

For the converse, we assume that π is a score sequence. Hence there exist tournaments of order n whose score sequence is π. Among all such tournaments, let T be one such that $V(T) = \{v_1, v_2, \ldots, v_n\}$, $\operatorname{od} v_i = s_i$ for $i = 1, 2, \ldots, n$ and the sum of the scores of the vertices adjacent from v_n is minimum. We claim that v_n is adjacent to vertices having scores $s_1, s_2, \ldots, s_{s_n}$. Assume, to the contrary, that v_n is not adjacent to vertices having scores $s_1, s_2, \ldots, s_{s_n}$. Necessarily, then, there exist vertices v_j and v_k with $j < k$ and $s_j < s_k$ such that v_n is adjacent to v_k and v_n is adjacent from v_j. Since the score of v_k exceeds the score of v_j, there exists a vertex v_t such that v_k is adjacent to v_t, and v_t is adjacent to v_j (Figure 4.9(a)). Thus, a 4-cycle $C = (v_n, v_k, v_t, v_j, v_n)$ is produced. If we reverse the directions of the arcs of C, a tournament T' is obtained also having π as a score sequence (Figure 4.9(b)). However, in T', the vertex v_n is adjacent to v_j rather than v_k. Hence the sum of the scores of the vertices adjacent from v_n is smaller in T' than in T, which is impossible. Thus, as claimed, v_n is adjacent to vertices having scores $s_1, s_2, \ldots, s_{s_n}$. Then $T - v_n$ is a tournament having score sequence π_1. ■

Figure 4.9: A step in the proof of Theorem 4.22

As an illustration of Theorem 4.17, we consider the sequence

$$\pi : 1, 2, 2, 3, 3, 4.$$

In this case, s_n (actually s_6) has the value 4; thus, we delete the last term, repeat the first $s_n = 4$ terms, and subtract 1 from the remaining terms, obtaining

$$\pi_1' : 1, 2, 2, 3, 2.$$

Rearranging, we have

$$\pi_1 : 1, 2, 2, 2, 3.$$

Repeating this process twice more, we have

$$\begin{aligned} \pi_2' &: 1, 2, 2, 1 \\ \pi_2 &: 1, 1, 2, 2 \\ \pi_3 &: 1, 1, 1. \end{aligned}$$

The sequence π_3 is clearly a score sequence of a tournament. By Theorem 4.17, π_2 is as well, as are π_1 and π. We can use this information to construct a tournament with score sequence π. The sequence π_3 is the score sequence of the tournament T_3 of Figure 4.10. Proceeding from π_3 to π_2, we add a new vertex to T_3 and join it *to* two vertices of T_3 and *from* the other, producing a tournament T_2 with score sequence π_2. To proceed from π_2 to π_1, we add a new vertex to T_2 and join it *to* vertices having scores 1, 2 and 2 and *from* the remaining vertex of T_2, producing a tournament T_1 with score sequence π_1. Continuing in the same fashion, we finally produce a desired tournament T with score sequence π by adding a new vertex to T_1 and joining it *to* vertices having scores 1, 2, 2 and 3, and joining it *from* the other vertex.

Figure 4.10: Construction of a tournament with a given score sequence

The sociologist Hyman Garshin Landau [143] characterized those sequences of nonnegative integers that are score sequences of tournaments. The proof we present of his theorem is due to Carsten Thomassen [215].

Theorem 4.18 *A nondecreasing sequence $\pi : s_1, s_2, \ldots, s_n$ of nonnegative integers is a score sequence of a tournament if and only if for each integer k with $1 \leq k \leq n$,*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2}, \tag{4.1}$$

with equality holding when $k = n$.

Proof. Suppose first that $\pi : s_1, s_2, \ldots, s_n$ is a score sequence of a tournament of order n. Then there exists a tournament T with $V(T) = \{v_1, v_2, \ldots, v_n\}$ such that $\operatorname{od} v_i = s_i$ for $i = 1, 2, \ldots, n$. For an integer k with $1 \leq k \leq n$ and $S = \{v_1, v_2, \ldots, v_k\}$, the subdigraph $T_1 = T[S]$ induced by S is a tournament of order k and size $\binom{k}{2}$. Since $\operatorname{od}_T v_i \geq \operatorname{od}_{T_1} v_i$ for $1 \leq i \leq k$, it follows that

$$\sum_{i=1}^{k} s_i = \sum_{i=1}^{k} \operatorname{od}_T v_i \geq \sum_{i=1}^{k} \operatorname{od}_{T_1} v_i = \binom{k}{2}.$$

We now verify the converse. Suppose that the converse is false. Then among all counterexamples for which n is minimum, let $\pi : s_1, s_2, \ldots, s_n$ be one for which s_1 is minimum. Suppose first that there exists an integer k with $1 \leq k \leq n-1$ such that

$$\sum_{i=1}^{k} s_i = \binom{k}{2}. \tag{4.2}$$

Since $k < n$, it follows that $\pi_1 : s_1, s_2, \ldots, s_k$ is a score sequence of a tournament T_1 of order k.

Let $\tau : t_1, t_2, \ldots, t_{n-k}$ be the sequence, where $t_i = s_{k+i} - k$ for $i = 1, 2, \ldots, n-k$. Since

$$\sum_{i=1}^{k+1} s_i \geq \binom{k+1}{2},$$

it follows from (4.2) that

$$s_{k+1} = \sum_{i=1}^{k+1} s_i - \sum_{i=1}^{k} s_i \geq \binom{k+1}{2} - \binom{k}{2} = k.$$

Since π is a nondecreasing sequence,

$$t_i = s_{k+i} - k \geq s_{k+1} - k \geq 0$$

for $i = 1, 2, \ldots, n-k$ and so τ is a nondecreasing sequence of nonnegative integers. We now show that τ satisfies (4.1).

For each integer r with $1 \leq r \leq n-k$, we have

$$\sum_{i=1}^{r} t_i = \sum_{i=1}^{r} (s_{k+i} - k) = \sum_{i=1}^{r} s_{k+i} - rk = \sum_{i=1}^{r+k} s_i - \sum_{i=1}^{k} s_i - rk.$$

Since

$$\sum_{i=1}^{r+k} s_i \geq \binom{r+k}{2}$$

and

$$\sum_{i=1}^{k} s_i = \binom{k}{2},$$

it follows that

$$\sum_{i=1}^{r} t_i \geq \binom{r+k}{2} - \binom{k}{2} - rk = \binom{r}{2},$$

with equality holding for $r = n - k$. Thus τ satisfies (4.1). Since $n - k < n$, there is a tournament T_2 of order $n - k$ having score sequence τ.

Let T be the tournament with $V(T) = V(T_1) \cup V(T_2)$ and

$$E(T) = E(T_1) \cup E(T_2) \cup \{(u, v) : u \in V(T_2), v \in V(T_1)\}.$$

Then π is a score sequence for T, contrary to our assumption. Consequently,

$$\sum_{i=1}^{k} s_i > \binom{k}{2}$$

for $k = 1, 2, \ldots, n - 1$. In particular, $s_1 > 0$.

We now consider the sequence $\pi' : s_1 - 1, s_2, s_3, \ldots, s_{n-1}, s_n + 1$. Then π' is a nondecreasing sequence of nonnegative integers satisfying (4.1). By the minimality of s_1, there is a tournament T' of order n having score sequence π'. Let x and y be vertices of T' such that $\mathrm{od}_{T'}\, x = s_n + 1$ and $\mathrm{od}_{T'}\, y = s_1 - 1$. Since $\mathrm{od}_{T'}\, x \geq \mathrm{od}_{T'}\, y + 2$, there is a vertex $w \neq x, y$ such that $(x, w) \in E(T')$ and $(w, y) \in E(T')$. Thus $P = (x, w, y)$ is a path in T'.

Let T be a tournament obtained from T' by reversing the directions of the arcs in P. Then π is a score sequence for T, producing a contradiction. ■

Frank Harary and Leo Moser [111] obtained a related characterization of sequences of nonnegative integers that are score sequences of strong tournaments (see Exercise 10).

Theorem 4.19 *A nondecreasing sequence $\pi : s_1, s_2, \ldots, s_n$ of nonnegative integers is a score sequence of a strong tournament if and only if*

$$\sum_{i=1}^{k} s_i > \binom{k}{2}$$

for $1 \leq k \leq n - 1$ and

$$\sum_{i=1}^{n} s_i = \binom{n}{2}.$$

Furthermore, every tournament whose score sequence satisfies these conditions is strong.

Distance in Tournaments

Recall that if u and v are vertices of a digraph D such that D contains at least one $u - v$ path, then the length of a shortest $u - v$ path is called the directed distance from u to v and is denoted by $\vec{d}(u, v)$.

Theorem 4.20 *If u is a vertex of maximum score in a nontrivial tournament T, then for every vertex v of T, $\vec{d}(u, v) \leq 2$.*

Proof. Assume that $\operatorname{od} u = k$. Necessarily, $k \geq 1$. Let $v_1, v_2, \ldots, v_k$ denote the vertices of T adjacent from u. Then $\vec{d}(u, v_i) = 1$ for $i = 1, 2, \ldots, k$. If $V(T) = \{u, v_1, v_2, \ldots, v_k\}$, then the proof is complete.

Assume, then, that $V(T) - \{u, v_1, v_2, \ldots, v_k\}$ is nonempty, and let $v \in V(T) - \{u, v_1, v_2, \ldots, v_k\}$. If v is adjacent from some vertex v_i, $1 \leq i \leq k$, then $\vec{d}(u, v) = 2$, producing the desired result. Suppose that this is not the case. Then v is adjacent to all of the vertices $v_1, v_2, \ldots, v_k$, as well as to u, so $\operatorname{od} v \geq 1 + k = 1 + \operatorname{od} u$. However, this contradicts the fact that u is a vertex of maximum score. ■

If a tournament represents a competition among teams where ties are not permitted, then Theorem 4.20 says that if A is a team with the greatest number of victories and B is any other team, then either A has defeated B or A has defeated a team that has defeated B.

Landau discovered Theorem 4.20 first during a study of pecking orders and domination among chickens. In the case of chickens, the theorem says that if chicken c pecks the largest number of chickens, then for every other chicken d, either c pecks d, or c pecks some chicken that pecks d. Thus c dominates every other chicken either directly or indirectly in two steps. Such a chicken is called a king chicken. In a nontrivial tournament T, a vertex v is called a **king** if for every vertex u distinct from v, either v is adjacent to u or v is adjacent to a vertex that is adjacent to u. By Theorem 4.20, every nontrivial tournament contains a king.

In the case of nontrivial strong tournaments, there are always at least three kings, as we are about to see. Let D be a strong digraph. Recall that the eccentricity $e(v)$ of a vertex v of D is defined as

$$e(v) = \max\{\vec{d}(v, w) : \ w \in V(D)\}.$$

The radius of D is

$$\operatorname{rad} D = \min\{e(v) : \ v \in V(D)\}$$

and the center $\operatorname{Cen}(D)$ of D is defined as the induced subdigraph

$$D[\{v : \ e(v) = \operatorname{rad}(D)\}].$$

Theorem 4.20 provides an immediate result dealing with the radius of a strong tournament (see Exercise 12).

Corollary 4.21 *Every nontrivial strong tournament has radius* 2.

We now show that every nontrivial strong tournament contains at least three kings.

Theorem 4.22 *The center of every nontrivial strong tournament contains at least three vertices.*

Proof. Let T be a nontrivial strong tournament. Then by Corollary 4.21, $\operatorname{rad} T = 2$. Let w be a vertex having eccentricity 2. Since T is strong, there are vertices adjacent to w; let v be one of these having maximum score. Among the vertices adjacent to v, let u be one of maximum score. We show that both u and v have eccentricity 2, which will complete the proof.

Assume, to the contrary, that one of the vertices u and v does not have eccentricity 2. Suppose, then, that $x \in \{u, v\}$ and $e(x) \geq 3$. Hence, there exists a vertex y in T such that $\vec{d}(x, y) \geq 3$. Thus, y is adjacent to x. Moreover, y is adjacent to every vertex adjacent from x. These observations imply that $\operatorname{od} y > \operatorname{od} x$.

Suppose that $x = v$. Since x is adjacent to w, it follows that y is adjacent to w. However, $\operatorname{od} y > \operatorname{od} v$, which contradicts the defining property of v. Therefore, $x \neq v$ and so $x = u$. Here x is adjacent to v so that y is adjacent to v, but $\operatorname{od} y > \operatorname{od} u$. This contradicts the defining property of u. Hence, $x \neq u$ and the proof is complete. ■

Hamiltonian Tournaments

The large number of arcs that a tournament has often produces a variety of paths and cycles. We now investigate paths and cycles in tournaments. We begin with perhaps the most basic result of this type, a property of tournaments first observed by László Rédei [181] in 1934. A path in a digraph D containing every vertex of D is a **Hamiltonian path**.

Theorem 4.23 *Every tournament contains a Hamiltonian path.*

Proof. Let T be a tournament of order n and let $P = (v_1, v_2, \ldots, v_k)$ be a longest path in T. If P is not a Hamiltonian path of T, then $1 \leq k < n$ and there is a vertex v of T not on P. Since P is a longest path, $(v, v_1), (v_k, v) \notin E(T)$, and so $(v_1, v), (v, v_k) \in E(T)$. This implies that there is an integer i $(1 \leq i < k)$ such that $(v_i, v) \in E(T)$ and $(v, v_{i+1}) \in E(T)$ (see Figure 4.11). But then

$$(v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_k)$$

is a path whose length exceeds that of P, producing a contradiction. ■

A simple but useful consequence of Theorem 4.23 concerns transitive tournaments.

Figure 4.11: A step in the proof of Theorem 4.23

Corollary 4.24 *Every transitive tournament contains exactly one Hamiltonian path.*

The preceding corollary is a special case of a result found independently by Rédei [181] and Tibor Szele [213], who showed that every tournament contains an odd number of Hamiltonian paths.

While every tournament contains a Hamiltonian path, certainly not every tournament contains a Hamiltonian cycle. Indeed, by Theorem 4.12, every transitive tournament is acyclic. If a tournament T contains a Hamiltonian cycle, then T is strong by Theorem 4.2. Paul Camion [32] showed that the converse is true as well.

Theorem 4.25 *A nontrivial tournament T is Hamiltonian if and only if T is strong.*

Proof. We have already seen that every Hamiltonian tournament is strong. For the converse, assume that T is a nontrivial strong tournament. Thus T contains cycles. Let C be a cycle of maximum length in T. If C contains all of the vertices of T, then C is a Hamiltonian cycle. So assume that C is not Hamiltonian, say

$$C = (v_1, v_2, \ldots, v_k, v_1),$$

where $3 \leq k < n$. If T contains a vertex v that is adjacent to some vertex of C and adjacent from some vertex of C, then there must be a vertex v_i of C that is adjacent to v such that v_{i+1} is adjacent from v. In this case,

$$C' = (v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_k, v_1)$$

is a cycle whose length is greater than that of C, producing a contradiction. Hence, every vertex of T that is not on C is either adjacent to every vertex of C or adjacent from every vertex of C. Since T is strong, there must be vertices of each type.

Let U be the set of all vertices of T that are not on C and such that each vertex of U is adjacent from every vertex of C, and let W be the set of those vertices of T that are not on C such that every vertex of W is adjacent to each vertex of C (see Figure 4.12). Then $U \neq \emptyset$ and $W \neq \emptyset$.

Since T is strong, there is a path from every vertex of C to every vertex of W. Since no vertex of C is adjacent to any vertex of W, there must be a vertex $u \in U$ that is adjacent to a vertex $w \in W$. However then,

Figure 4.12: A step in the proof of Theorem 4.25

$$C'' = (v_1, v_2, \ldots, v_k, u, w, v_1)$$

is a cycle whose length is greater than the length of C, a contradiction. ■

If T is a Hamiltonian tournament, then of course every vertex of T lies on every Hamiltonian cycle of T. Actually, every vertex of T lies on a triangle of T as well.

Theorem 4.26 *Every vertex in a nontrivial strong tournament belongs to a triangle.*

Proof. Let v be a vertex in a nontrivial strong tournament T. By Theorem 4.25, T is Hamiltonian. Thus T contains a Hamiltonian cycle ($v = v_1, v_2, \ldots, v_n, v_1$). Since v is adjacent to v_2 and adjacent from v_n, there is a vertex v_i with $2 \leq i < n$ such that (v, v_i) and (v_{i+1}, v) are arcs of T. Thus (v, v_i, v_{i+1}, v) is a triangle of T containing v. ■

It is perhaps surprising that if a tournament is Hamiltonian, then it must possess significantly stronger properties. A digraph D of order $n \geq 3$ is **pancyclic** if it contains a cycle of every possible length, that is, D contains a cycle of length ℓ for each $\ell = 3, 4, \ldots, n$ and is **vertex-pancyclic** if each vertex v of D lies on a cycle of every possible length. Frank Harary and Leo Moser [111] showed that every nontrivial strong tournament is pancyclic, while John W. Moon [161] went one step further by obtaining the following result. The proof given here is due to Carsten Thomassen.

Theorem 4.27 *Every nontrivial strong tournament is vertex-pancyclic.*

Proof. Let T be a strong tournament of order $n \geq 3$, and let v_1 be a vertex of T. We show that v_1 lies on an ℓ-cycle for each $\ell = 3, 4, \ldots, n$. We proceed by induction on ℓ.

Since T is strong, it follows by Theorem 4.26 that v_1 lies on a 3-cycle. Assume that v_1 lies on an ℓ-cycle $C = (v_1, v_2, \ldots, v_\ell, v_1)$, where $3 \leq \ell \leq n-1$. We show that v_1 lies on an $(\ell+1)$-cycle.

Case 1. *There is a vertex v not on C that is adjacent to at least one vertex of C and is adjacent from at least one vertex of C.* This implies that for some

i $(1 \leq i \leq \ell)$, both (v_i, v) and (v, v_{i+1}) are arcs of T (where all subscripts are expressed modulo ℓ). Thus, v_1 lies on the $(\ell+1)$-cycle

$$(v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_\ell, v_1).$$

Case 2. *No vertex v exists as in Case* 1. Let U denote the set of all vertices in $V(T) - V(C)$ that are adjacent from every vertex of C, and let W be the set of all vertices in $V(T) - V(C)$ that are adjacent to every vertex of C. Then $U \cup W = V(T) - V(C)$. Since T is strong, neither U nor W is empty and there is a vertex u in U and a vertex w in W such that $(u, w) \in E(T)$. Thus, v_1 lies on the $(\ell+1)$-cycle

$$(u, w, v_1, v_2, \ldots, v_{\ell-1}, u),$$

completing the proof. ■

Corollary 4.28 *Every nontrivial strong tournament is pancyclic.*

Exercises for Section 4.2

1. Give an example of two non-isomorphic strong tournaments of order 5.

2. How many tournaments of order 7 are there that are not strong?

3. Determine those positive integers n for which there exist regular tournaments of order n.

4. Give an example of two non-isomorphic regular tournaments of the same order.

5. Prove Theorem 4.16. *If T is a tournament with exactly k strong components, then $\widetilde{T}$ is the transitive tournament of order k.*

6. (a) Show that if two vertices u and v have the same score in a tournament T, then u and v belong to the same strong component of T.
   (b) Prove that every regular tournament is strong.

7. Which of the following sequences are score sequences of tournaments? For each sequence that is a score sequence, construct a tournament having the given sequence as a score sequence.
   (a) $0, 1, 1, 4, 4$
   (b) $1, 1, 1, 4, 4, 4$
   (c) $1, 3, 3, 3, 3, 3, 5$
   (d) $2, 3, 3, 4, 4, 4, 4, 5$.

8. Show that if $\pi : s_1, s_2, \ldots, s_n$ is a score sequence of a tournament, then $\pi_1 : n-1-s_1, n-1-s_2, \ldots, n-1-s_n$ is a score sequence of a tournament.

9. What tournament T of order n has a score sequence $s_1, s_2, \ldots, s_n$ such that equality holds in (4.1) for every integer k with $1 \leq k \leq n$?

10. Prove Theorem 4.19: *A nondecreasing sequence $\pi : s_1, s_2, \ldots, s_n$ of nonnegative integers is a score sequence of a strong tournament if and only if $\sum_{i=1}^{k} s_i > \binom{k}{2}$ for $1 \leq k \leq n-1$ and $\sum_{i=1}^{n} s_i = \binom{n}{2}$. Furthermore, every tournament whose score sequence satisfies these conditions is strong.*

11. Prove that if T is a nontrivial regular tournament, then $\operatorname{diam} T = 2$.

12. Prove Corollary 4.21: *Every nontrivial strong tournament has radius* 2.

13. (a) Show that no tournament has exactly two kings.
    (b) Show that if $n \geq 3$ is an integer with $n \neq 4$, then there is a tournament of order n in which every vertex is a king.

14. A vertex z in a nontrivial tournament is called a **serf** if for every vertex x distinct from z, either x is adjacent to z or x is adjacent to a vertex that is adjacent to z. Prove that every nontrivial tournament has at least one serf.

15. Prove that if T is a tournament that is not transitive, then T has at least three Hamiltonian paths.

16. Prove or disprove: If every vertex of a tournament T belongs to a cycle in T, then T is strong.

17. (a) Prove or disprove: Every arc of a nontrivial strong tournament T lies on a Hamiltonian cycle of T.
    (b) A digraph D is **Hamiltonian-connected** if for every pair u, v of vertices of D, there exists a Hamiltonian $u - v$ path. Prove or disprove: Every vertex-pancyclic tournament is Hamiltonian-connected.

18. Show that if a tournament T has an ℓ-cycle, then T has an s-cycle for $s = 3, 4, \ldots, \ell$.

19. We have seen that there is exactly one transitive tournament of each order. A tournament of order $n \geq 3$ is defined to be **circular** if whenever (u, v) and (v, w) are arcs of T, then (w, u) is an arc of T.
    (a) How many circular tournaments of order 3 are there?
    (b) Show that in a tournament of order 3 or more, every vertex, with at most two exceptions, has positive outdegree and positive indegree.
    (c) How many circular tournaments of order 4 or more are there?

20. For each positive integer k, there exist round robin tournaments containing $2k$ teams with no ties permitted in which k of these teams win r games and the remaining k of these teams win s games for some r and s with $r \neq s$. What is the minimum value of s for which this is possible?

21. Prove that if u and v are vertices of a tournament such that $\vec{d}(u,v) = k$, then $\operatorname{id} u \geq k-1$.

22. For a tournament T of order n, let

$$\Delta = \max\{\operatorname{od} v : v \in V(T)\} \text{ and } \delta = \min\{\operatorname{od} v : v \in V(T)\}.$$

Prove that if $\Delta - \delta < \frac{n}{2}$, then T is strong.

23. Let (u,v) be an arc of a tournament T. Show that if $\operatorname{od} v > \operatorname{od} u$, then (u,v) lies on a triangle of T.

24. Show that a tournament can contain three vertices of outdegree 1 but can never contain four vertices of outdegree 1.

25. Let T be a tournament of order $n \geq 10$. Suppose that T contains two vertices u and v such that when the arc joining u and v is removed, the resulting digraph D contains neither a $u-v$ path nor a $v-u$ path. Show that $\operatorname{od}_D u = \operatorname{od}_D v$.

26. Let u and v be two vertices in a tournament T. Prove that if $\vec{d}(u,v) = k \geq 2$, then T contains a cycle of length ℓ for each integer ℓ with $3 \leq \ell \leq k+1$.

27. Let u and v be two vertices in a tournament T. Prove that if u and v do not lie on a common cycle, then $\operatorname{od} u \neq \operatorname{od} v$.

28. Let T be a tournament with the property that every vertex of T belongs to a directed 3-cycle. Let u and v be distinct vertices of T. Prove that if $|\operatorname{od} u - \operatorname{od} v| \leq 1$, then T contains both a directed $u-v$ path and a directed $v-u$ path.

4.3 Flows in Networks

In the first section of this chapter we considered classes of strongly connected digraphs, while in the second section we discussed the class of digraphs called tournaments, primarily concentrating on questions of connectedness. In this section, problems of connectedness will again be central as we discuss a class of digraphs possessing additional properties.

A **network** N is a digraph D with two distinguished vertices u and v, called the **source** and **sink**, respectively, together with a nonnegative real-valued function c on $E(D)$. The digraph D is called the **underlying digraph** of N and the function c is called the **capacity function** of N. The value

$c(a) = c(x, y)$ of an arc $a = (x, y)$ of D is called the **capacity** of a. Any vertex of N distinct from u and v is called an **intermediate vertex** of N.

The source u of N can be thought of as the location from which material is shipped and then transported through N, eventually reaching its destination, namely the sink v of N. The capacity of an arc (x, y) in N may be thought of as the maximum amount of material that can be transported from x to y along (x, y) per unit time. For example, the capacity of the arc (x, y) may represent the number of seats available on a direct flight from city x to city y in some airline network, or perhaps $c(x, y)$ is the capacity of a pipeline from city x to city y in some oil network, or perhaps $c(x, y)$ is the maximum weight of items that can be transported by truck from city x to city y in some highway network. The problem then is to maximize the *flow* of material that can be transported from the source u to the sink v without exceeding the capacity of any arc.

A network N can be represented by drawing the underlying digraph D of N and labeling each arc of D with its capacity. A network is shown in Figure 4.13. In this network, the capacity of the arc (x, y) is then $c(x, y) = 4$. While in general there may be more than one source from which material originates and more than one sink providing destinations of the material, it suffices to consider a network with a single source and a single sink.

Figure 4.13: A network

For a vertex x in a digraph D, recall that the out-neighborhood $N^+(x)$ and the in-neighborhood $N^-(x)$ of x are defined by

$$\begin{aligned} N^+(x) &= \{y \in V(D) : (x, y) \in E(D)\} \text{ and} \\ N^-(x) &= \{y \in V(D) : (y, x) \in E(D)\}. \end{aligned}$$

Thus $|N^+(x)| = \operatorname{od} x$ and $|N^-(x)| = \operatorname{id} x$.

For a digraph D and a real-valued function g defined on $E(D)$, it is convenient to introduce some notation. For subsets X and Y of $V(D)$, define the set $[X, Y]$ and the number $g(X, Y)$ by

$$[X, Y] = \{(x, y) : x \in X, y \in Y\}$$

and

$$g(X, Y) = \sum_{(x,y)\in[X,Y]} g(x, y),$$

where $g(X,Y) = 0$ if $[X,Y] = \emptyset$. For $x \in V(D)$,

$$g^+(x) = \sum_{y \in N^+(x)} g(x,y) \text{ and } g^-(x) = \sum_{y \in N^-(x)} g(y,x). \tag{4.3}$$

More generally, for $X \subseteq V(D)$,

$$g^+(X) = \sum_{x \in X} g^+(x) \text{ and } g^-(X) = \sum_{x \in X} g^-(x).$$

A **flow** in a network N with underlying digraph D, source u, sink v and capacity function c is a real-valued function f on $E(D)$ satisfying

$$0 \leq f(a) \leq c(a) \text{ for every arc } a \text{ of } D \tag{4.4}$$

such that

$$f^+(x) = f^-(x) \text{ for each intermediate vertex } x \text{ of } D. \tag{4.5}$$

If f is a function on $E(D)$ defined by $f(a) = 0$ for every arc a of D, then f satisfies both (4.4) and (4.5) and so is a flow, called the **zero flow**.

The value $f(a) = f(x,y)$ of an arc $a = (x,y)$ is called the **flow along the arc** a and can be interpreted as the rate at which material is transported along a under the flow f. Condition (4.4) requires that the flow along a cannot exceed the capacity of a. Condition (4.5) is referred to as the **conservation equation** and states that the rate at which material is transported into an intermediate vertex x equals the rate at which material is transported out of x.

For a flow f in a network N, the **net flow out of a vertex** x is defined by

$$f^+(x) - f^-(x),$$

while the **net flow into** x is

$$f^-(x) - f^+(x).$$

By the conservation equation (4.5), it follows that for every intermediate vertex x of D, the net flow out of x equals the net flow into x and this common value is zero.

If $f(a) = c(a)$ for an arc a in a network N, then the arc a is said to be **saturated** with respect to the flow f. On the other hand, if $f(a) < c(a)$, then the arc a is **unsaturated**. An example of a flow in a network is shown in Figure 4.14. The first number associated with an arc is its capacity and this is a fixed number for each arc of N, while the second number is the flow along the arc. In general, many flows are possible for a given network. While the arc (x,t) is saturated for the flow f shown in the network in Figure 4.14 since $f(x,t) = c(x,t)$, the arc (w,y) is unsaturated since $f(w,y) < c(w,y)$. In this example, the net flow out of the source u is 3 the net flow into the sink v is also 3. As we will soon see, that these two numbers are equal is true in general.

Figure 4.14: A flow in a network

Theorem 4.29 *Let u and v be the source and sink, respectively, of a network N with underlying digraph D and let f be a flow defined on N. Then the net flow out of u equals the net flow into v, that is,*

$$f^+(u) - f^-(u) = f^-(v) - f^+(v).$$

Proof. Since $f^+(V(D)) = f^-(V(D))$, it follows that

$$\sum_{x \in V(D)} f^+(x) = \sum_{x \in V(D)} f^-(x). \tag{4.6}$$

By (4.5),

$$f^+(x) = f^-(x) \text{ when } x \neq u, v. \tag{4.7}$$

By (4.6) then,

$$f^+(u) + f^+(v) = f^-(u) + f^-(v),$$

giving the desired result. ■

The **value of a flow** f in a network N, denoted by $\text{val}(f)$, is defined as the net flow out of the source of N. By Theorem 4.29, $\text{val}(f)$ is also the net flow into the sink of N. For the flow f defined on the network in Figure 4.14, we have $\text{val}(f) = 3$.

There are certain flows of particular and obvious interest to us. A flow in a network N whose value is maximum among all flows that can be defined on N is called a **maximum flow**. Thus, a flow f defined on N is a maximum flow if $\text{val}(f) \geq \text{val}(f')$ for every flow f' defined on N. For a given network, a major goal is to find a maximum flow. For the purpose of doing this, it will be convenient to introduce another concept.

Let N be a network with underlying digraph D, source u, sink v and capacity function c. For a set X of vertices in D, let $\overline{X} = V(D) - X$. A **cut** in N is a set of arcs of the form $[X, \overline{X}]$, where $u \in X$ and $v \in \overline{X}$. If $K = [X, \overline{X}]$ is a cut in N, then the **capacity** of K, denoted by $\text{cap}(K)$, is

$$\text{cap}(K) = c(X, \overline{X}) = \sum_{(x,y) \in [X, \overline{X}]} c(x, y).$$

For the network N of Figure 4.14 and $X = \{u, x\}$, the cut

$$K = [X, \overline{X}] = \{(u, z), (x, y), (x, t)\}$$

in N (see Figure 4.15) has capacity

$$\text{cap}(K) = c(u, z) + c(x, y) + c(x, t) = 4 + 4 + 3 = 11.$$

Figure 4.15: The cut $[X, \overline{X}]$ in the network N of Figure 4.14 for $X = \{u, x\}$

If K is a cut in a network N, then any path from the source u to the sink v must contain at least one arc of K. Consequently, if all arcs of K were removed from the underlying digraph D of N, then there would be no path from u to v. So just as a vertex-cut in a graph G separates some pair of vertices in G and an edge-cut in G separates some pair of vertices in G, a cut in the underlying digraph D of a network N separates u and v in a certain sense.

Let N be a network with underlying digraph D, source u and sink v. For a set X of vertices of D with $u \in X$ and $v \in \overline{X}$ and a flow f defined on N, the **net flow out of** X is $f^+(X) - f^-(X)$ and the **net flow into** X is $f^-(X) - f^+(X)$. It then follows (see Exercise 8) that

$$f^+(X) - f^-(X) = f(X, \overline{X}) - f(\overline{X}, X). \tag{4.8}$$

For the set $X = \{u, x, t\}$ in the network N in Figure 4.14, $f^+(X) = 10$ and $f^-(X) = 7$. For this network then, the net flow out of X is $f^+(X) - f^-(X) = 10 - 7 = 3 = \text{val}(f)$. We now show that for a network N and any cut $K = [X, \overline{X}]$ in N, the value of any flow in N is the net flow out of X and that this value never exceeds the capacity of K.

Theorem 4.30 *Let f be a flow in a network N and let $K = [X, \overline{X}]$ be a cut in N. Then*

$$\text{val}(f) = f^+(X) - f^-(X) \leq \text{cap}(K).$$

Proof. Let D be the underlying digraph of N, let u and v be the source and sink, respectively, of N and let c be the capacity function of N.

Since $f^+(x) - f^-(x) = 0$ for every $x \in X - \{u\}$, it follows that

$$\sum_{x \in X} (f^+(x) - f^-(x)) = f^+(u) - f^-(u) = \text{val}(f).$$

Furthermore,

$$\sum_{x \in X} (f^+(x) - f^-(x)) = \sum_{x \in X} f^+(x) - \sum_{x \in X} f^-(x) = f^+(X) - f^-(X)$$

and so $\text{val}(f) = f^+(X) - f^-(X)$. Since $0 \leq f(a) \leq c(a)$ for every arc a of N, it follows that

$$\begin{aligned} \text{val}(f) &= f^+(X) - f^-(X) = f(X, \overline{X}) - f(\overline{X}, X) \\ &\leq f(X, \overline{X}) \leq c(X, \overline{X}) = \text{cap}(K), \end{aligned}$$

giving the desired result. ■

There are in general many cuts in a network N and each cut has a capacity. Any cut in N whose capacity is minimum among all cuts in N is called a **minimum cut**. That is, a cut K in N is a minimum cut if $\text{cap}(K) \leq \text{cap}(K')$ for every cut K' in N. The following two corollaries provide some important information about minimum cuts and maximum flows.

Corollary 4.31 *If f is a flow in a network N and K is a cut in N such that* $\text{val}(f) = \text{cap}(K)$*, then f is a maximum flow and K is a minimum cut in N.*

Proof. If f^* is a maximum flow in N and K^* is a minimum cut, then $\text{val}(f^*) \leq \text{cap}(K^*)$ by Theorem 4.30. Consequently,

$$\text{val}(f) \leq \text{val}(f^*) \leq \text{cap}(K^*) \leq \text{cap}(K). \tag{4.9}$$

Since $\text{val}(f) = \text{cap}(K)$, it follows that there is equality throughout (4.9) and so $\text{val}(f) = \text{val}(f^*)$ and $\text{cap}(K^*) = \text{cap}(K)$, that is, f is a maximum flow and K is a minimum cut. ■

Corollary 4.32 *If f is flow in a network N with capacity function c and $[X, \overline{X}]$ is a cut in N such that*

$$f(a) = c(a) \text{ for all } a \in [X, \overline{X}]$$

and

$$f(a) = 0 \text{ for all } a \in [\overline{X}, X],$$

then f is a maximum flow in N and $[X, \overline{X}]$ is a minimum cut.

Corollary 4.32 (see Exercise 9) suggests how the values of a flow f should be defined on the arcs of a minimum cut in order for f to be a maximum flow.

A network N with source u and sink v is shown in Figure 4.16 together with a flow f defined on N. As always, the first number associated with an arc a is its capacity $c(a)$ and the second number is the flow $f(a)$. If $X = \{u, x, y\}$, then $K = [X, \overline{X}]$ is a cut in N. Since $f(a) = c(a)$ for all $a \in [X, \overline{X}]$ and $f(a) = 0$ for all $a \in [\overline{X}, X]$, it follows by Corollary 4.32 that f is a maximum flow and K is a minimum cut. Since $\text{cap}(K) = 4$, the value of the maximum flow f is 4.

Figure 4.16: A cut $K = [X, \overline{X}]$ in a network, where $X = \{u, x, y\}$

According to Corollary 4.31, if it should ever occur that the value of some flow f in a network N equals the capacity of some cut K in N, then f must be a maximum flow and K is a minimum cut. We next show that for any maximum flow f and any minimum cut K, it must occur that $\text{val}(f) = \text{cap}(K)$. In preparation for proving this, some additional terminology is useful.

For a digraph D, an $x - y$ **semipath** in D is an alternating sequence

$$P = (x = w_0, a_1, w_1, a_2, w_2, \ldots, w_{k-1}, a_k, w_k = y)$$

of distinct vertices and arcs of D beginning with x and ending with y such that either $a_i = (w_{i-1}, w_i)$ or $a_i = (w_i, w_{i-1})$ for each i $(1 \leq i \leq k)$. In this case, (w_{i-1}, w_i) is called a **forward arc** of P and (w_i, w_{i-1}) is a **backward arc** of P. Hence, when proceeding from x to y along the semipath P in D, we move in the direction of a forward arc on P and move opposite to the direction of a backward arc on P.

f-Augmenting Semipaths

Let N be a network with underlying digraph D and capacity function c and on which is defined a flow f. Recall that an arc a is unsaturated if $f(a) < c(a)$. A semipath

$$P = (w_0, a_1, w_1, a_2, \ldots, w_{k-1}, a_k, w_k)$$

in D is said to be **f-unsaturated** if for each i $(1 \leq i \leq k)$,

(i) a_i is unsaturated whenever a_i is a forward arc and

(ii) $f(a_i) > 0$ whenever a_i is a backward arc.

A trivial semipath in D is vacuously f-unsaturated. If P is an f-unsaturated $u-v$ semipath in D where u and v are the source and sink, respectively, then P is called an **f-augmenting semipath**. As we will see, given an f-augmenting semipath in the underlying digraph D of a network N, it is possible to augment (alter) the values of the flow f on each arc of P to obtain a new flow f' whose value exceeds that of f. For example, consider the flow f in the network N in Figure 4.14, shown again in Figure 4.17. Then

$$P = (u, (t, u), t, (x, t), x, (x, y), y, (y, v), v)$$

is an f-augmenting semipath.

Figure 4.17: An f-augmenting semipath in a network

Recall that the value of the flow f defined on the network N in Figure 4.17 is 3. Let f' be the function defined on $E(D)$ by

$$f'(a) = \begin{cases} f(a) + 1 & \text{if } a \in \{(x, y), (y, v)\} \\ f(a) - 1 & \text{if } a \in \{(t, u), (x, t)\} \\ f(a) & \text{otherwise.} \end{cases}$$

Then f' is also a flow in N and $\text{val}(f') = 4$. Consequently, f is not a maximum flow. In fact, f' is not a maximum flow either. That f is not a maximum flow and that N contains an f-augmenting semipath is not a coincidence, as Lester Randolph Ford, Jr. and Delbert Ray Fulkerson [83] showed.

Theorem 4.33 *Let N be a network with underlying digraph D. A flow f in N is a maximum flow if and only if there is no f-augmenting semipath in D.*

Proof. Let u and v be the source and sink, respectively, of N and let c be the capacity function. Suppose first that D contains an f-augmenting semipath

$$P = (u = w_0, a_1, w_1, a_2, w_2, \ldots, w_{k-1}, a_k, w_k = v).$$

Then, for $1 \leq i \leq k$,

(a) $f(a_i) < c(a_i)$ whenever a_i is a forward arc and

(b) $f(a_i) > 0$ whenever a_i is a backward arc.

If there is at least one forward arc $a_i = (w_{i-1}, w_i)$ on P, let p_1 be the minimum value of $c(a_i) - f(a_i)$ over all forward arcs a_i. If there is at least one backward arc $a_i = (w_i, w_{i-1})$ on P, let p_2 be the minimum value of $f(a_i)$ over all such backward arcs a_i. If only one of p_1 and p_2 is defined, then denote this number by p; otherwise, let $p = \min(p_1, p_2)$.

Define a function f' on $E(D)$ by

$$f'(a) = \begin{cases} f(a) + p & \text{if } a \text{ is a forward arc on } P \\ f(a) - p & \text{if } a \text{ is a backward arc on } P \\ f(a) & \text{if } a \notin E(P). \end{cases}$$

Then f' is also a flow. Since $\text{val}(f') = \text{val}(f) + p$, it follows that f is not a maximum flow.

We now turn to the converse. Assume that there is no f-augmenting semipath in D. Let X be the set of all vertices x in D for which there exists an f-unsaturated $u - x$ semipath. Then $u \in X$ and, by assumption, $v \notin X$. Thus $K = [X, \overline{X}]$ is a cut in N.

Let (y, z) be an arc in K. Since $y \in X$, there exists an f-unsaturated $u - y$ semipath P in D. Since $z \in \overline{X}$, there is no f-unsaturated $u - z$ semipath in D, which implies that $f(y, z) = c(y, z)$. Similarly, if $(w, x) \in [\overline{X}, X]$, then $f(w, x) = 0$. Since $K = [X, \overline{X}]$ is a cut such that $f(a) = c(a)$ for every arc $a \in [X, \overline{X}]$ and $f(a) = 0$ for every arc $a \in [\overline{X}, X]$, it follows by Corollary 4.32 that f is a maximum flow. ■

The Max-Flow Min-Cut Theorem

With the aid of the preceding theorem, Ford and Fulkerson [83] proved a famous result in 1956 that is known as the *Max-Flow Min-Cut Theorem.* Independently, and also in 1956, Peter Elias, Amiel Feinstein and Claude Elwood Shannon [68] discovered and proved the very same result.

Theorem 4.34 (**The Max-Flow Min-Cut Theorem**) *In any network, the value of a maximum flow equals the capacity of a minimum cut.*

Proof. Let f be a maximum flow in a network N having capacity function c. By Theorem 4.33, there is no f-augmenting semipath in the underlying digraph D of N. By the proof of Theorem 4.33, since D contains no f-augmenting semipath, there is a minimum cut $K = [X, \overline{X}]$ such that

$$f(a) = \begin{cases} c(a) & \text{if } a \in K \\ 0 & \text{if } a \in [\overline{X}, X]. \end{cases}$$

Therefore, by (4.8),

$$\begin{aligned} \mathrm{val}(f) &= f^+(X) - f^-(X) = f(X, \overline{X}) - f(\overline{X}, X) \\ &= c(X, \overline{X}) - 0 = \mathrm{cap}(K), \end{aligned}$$

as desired. ■

The proof of Theorem 4.33 provides the basis of an algorithm, also due to Ford and Fulkerson [84], for finding a maximum flow and a minimum cut in a network. A refinement of this algorithm, due to the Russian scientist Efim A. Dinic [58], was first published in 1970 and independently by Jack Edmonds and Richard M. Karp [65] in 1972, although they discovered this earlier. The Edmonds-Karp algorithm, which we describe next, searches for an f-augmenting semipath in a network with a given flow f. This is an efficient algorithm, having complexity $O(nm^2)$, where the underlying digraph of the network has order n and size m.

Let u and v be the source and sink of a network N with underlying digraph D and let f be a given flow in N (perheps the zero flow). The algorithm proceeds by constructing a sequence of labelings of the vertices of D. A vertex w in D is assigned a label only if there is an f-unsaturated $u - w$ semipath P in D. The label assigned to w is an ordered pair. If x is the vertex immediately preceding w on P, then the first coordinate of the label is either $x+$ or $x-$, according to whether the arc immediately preceding w is (x, w) or (w, x). The second coordinate of the label is a positive number that reflects the potential change in f along P as we will soon describe. As we proceed through the algorithm, a list of labeled vertices is created. At a certain stage of the algorithm, a list L consisting of some labeled vertices of D is formed. At some point, the first vertex on L is examined (scanned) and removed from L to determine whether this vertex is adjacent to certain unlabeled vertices possessing a particular property in which case all such vertices are then labeled and added to L. If the sink v is labeled, then a new flow of greater value can be obtained. This process is then repeated. If, however, the sink v is not labeled, then f is a maximum flow. In this case, the labels can be used to find a minimum cut.

Throughout the course of the algorithm, each vertex of D is considered to be in one of the following three states:

(1) unlabeled, (2) labeled and unscanned, (3) labeled and scanned.

Prior to the implementation of the algorithm, all vertices are unlabeled. Once a vertex is assigned a label, it is placed at the end of a list L consisting of the labeled and unscanned vertices. The vertices of L are scanned on a *first-labeled first-scanned* basis, which will guarantee the selection of a shortest f-augmenting semipath. The Edmonds-Karp algorithm is now stated.

Algorithm 4.35 *Given a network N with underlying digraph D, source u, sink v and capacity function c.*

1. Let f be a flow on D and label each arc of D with the value of f.
2. Label the source u with the ordered pair $(-, \infty)$ and add u to the list L of labeled and unscanned vertices.
3. If L is empty, then stop. Otherwise, scan and remove the first element of L, say x, having the label $(z+, \epsilon(x))$ or $(z-, \epsilon(x))$.

   3.1 Assign to each unlabeled vertex y for which
   $$(x, y) \in E(D) \text{ and } f(x, y) < c(x, y)$$
   the label $(x+, \epsilon(y))$, where
   $$\epsilon(y) = \min\{\epsilon(x), c(x, y) - f(x, y)\}$$
   and add y to the end of L.

   3.2 Assign to each unlabeled vertex y for which
   $$(y, x) \in E(D) \text{ and } f(y, x) > 0$$
   the label $(x-, \epsilon(y))$, where
   $$\epsilon(y) = \min\{\epsilon(x), f(y, x)\}$$
   and add y to the end of L.
4. If v has been labeled, go to Step 5; otherwise return to Step 3.
5. The labels describe an f-augmenting semipath
   $$(u = w_0, a_1, w_1, a_2, w_2, \ldots, w_{k-1}, a_k, w_k = v)$$
   where, for $1 \leq i \leq k$, w_i is labeled
   $$(w_{i-1}+, \epsilon(w_i)) \text{ if } a_i = (w_{i-1}, w_i) \text{ is a forward arc}$$
   or w_i is labeled
   $$(w_{i-1}-, \epsilon(w_i)) \text{ if } a_i = (w_i, w_{i-1}) \text{ is a backward arc.}$$
   In the first case, replace $f(w_{i-1}, w_i)$ by $f(w_{i-1}, w_i) + \epsilon(v)$; while in the second case, replace $f(w_i, w_{i-1})$ by $f(w_i, w_{i-1}) - \epsilon(v)$.
6. Discard all labels, remove all vertices from L and return to Step 2.

If the capacity of every arc in a network is an integer, then the value of an integer-valued flow increases by 1 or more with each iteration of Algorithm 4.35 and the algorithm terminates after finitely many iterations. This is also the case when the capacity of every arc is a rational number and a given flow is rational-valued. Since networks commonly encountered in discrete mathematics have integer capacities, it is integer-valued flows that we seek. For the remainder of our discussion on flows in networks, we assume that the capacities and flows in networks are integer-valued.

Theorem 4.36 *Algorithm* 4.35 *terminates with a maximum flow f in N. Furthermore, if X is the set of labeled vertices upon termination, then $[X, \overline{X}]$ is a minimum cut.*

Proof. By Theorem 4.33, each time that Step 5 is completed, a new flow f having a larger value is constructed. If, in Step 3, the list L is empty, then there is no f-augmenting semipath and so, by Theorem 4.33, f is a maximum flow and $[X, \overline{X}]$ is a minimum cut. This process must terminate since for every flow f' in N,

$$\text{val}(f') \leq c(u, V(D))$$

and so Step 5 can be repeated at most $c(u, V(D))$ times. ■

We now illustrate Algorithm 4.35 by applying it to the network shown in Figure 4.18. It is not difficult to find the flow f shown in Figure 4.18, which we take as the initial flow. As always, each arc a is labeled with the pair $c(a)$, $f(a)$, where $c(a)$ is the capacity of a and $f(a)$ is the flow along a.

Figure 4.18: A flow f in a network

Since an initial flow has been given to the network N, Step 1 of Algorithm 4.35 has been completed and we have the situation shown in Figure 4.18. Initially, the source u is assigned the label $(-, \infty)$ and the list L of labeled but unscanned vertices now consists of the vertex u only. Thus Step 2 of Algorithm 4.35 is now completed.

We now move on to Step 3. Since L is not empty, we do not stop. We now scan and remove the first element of L from this list. In this case, only u belongs to L, so u is now removed from L. Thus L is now empty. We search for all vertices x of N such that either $(u, x) \in E(D)$ and $f(u, x) < c(u, x)$ or $(x, u) \in E(D)$ and $f(x, u) > 0$. There are two vertices with one of these properties, namely r and s. We consider r first. Since $(u, r) \in E(D)$, the first coordinate of the label of r is $u+$. Since $c(u, r) - f(u, r) = 3$, the second coordinate of the label of r is 3 and so $\epsilon(r) = 3$. That is, r is assigned the label $(u+, 3)$. The vertex r is then placed at the end of L. Since L was previously empty, we now have $L : r$. We now turn to the vertex s. Because $(s, u) \in E(D)$, the first coordinate of the label of s is $u-$. Because $f(s, u) = 2$, the second coordinate of the label of s is 2 and so $\epsilon(s) = 2$. Thus s is assigned the label

$(u-, 2)$. The vertex s is then added to the end of L (and so s follows r in the list L). At this moment, we now have three labeled vertices, while the list L consists of two vertices, so

$$\text{labeled vertices: } u\ (-, \infty),\ r\ (u+, 3),\ s\ (u-, 2) \ \text{ and } \ \text{list } L : r, s.$$

Since we have now completed Step 3, we move on to Step 4. Since v has not been labeled, we return to Step 3.

Since L is not empty, we do not stop and, instead, we scan and remove the first element of L, namely r, from this list and search for all unlabeled vertices x of N such that either $(r, x) \in E(D)$ and $f(r, x) < c(r, x)$ or $(x, r) \in E(D)$ and $f(x, r) > 0$. Only the vertex y has this property, namely $(r, y) \in E(D)$ and $f(r, y) < c(r, y)$. Therefore,

$$\epsilon(y) = \min\{\epsilon(r),\ c(r, y) - f(r, y)\} = \min\{3, 2\} = 2$$

and so y is assigned the label $(r+, 2)$ and y is placed at the end of L, resulting in $L : s, y$. Since Step 3 has been completed, we move on to Step 4. Since v has not been labeled, we return to Step 3.

The list L is not empty, so we scan and remove the first vertex, namely s, from L and search for all unlabeled vertices x of N such that either $(s, x) \in E(D)$ and $f(s, x) < c(s, x)$ or $(x, s) \in E(D)$ and $f(x, s) > 0$. The vertices w and t satisfy these properties. Since $(w, s) \in E(D)$ and $f(w, s) > 0$, it follows that

$$\epsilon(w) = \min\{\epsilon(s),\ f(w, s)\} = \min\{2, 3\} = 2$$

and so the vertex w is assigned the label $(s-, 2)$ and placed at the end of L. Since $(s, t) \in E(D)$ and $f(s, t) < c(s, t)$, it follows that

$$\epsilon(t) = \min\{\epsilon(s),\ c(s, t) - f(s, t)\} = \min\{2, 3\} = 2$$

and so the vertex t is assigned the label $(s+, 2)$ and placed at the end of L. Thus we currently have the following

> labeled vertices: $u\ (-, \infty)$, $r\ (u+, 3)$, $s\ (u-, 2)$, $y\ (r+, 2)$, $w\ (s-, 2)$, $t\ (s+, 2)$
>
> list $L : y, w, t$.

Since Step 3 has been completed, we turn to Step 4. Since v has not been labeled, we once again return to Step 3. The list L is not empty, so the vertex y on L is removed and scanned. Only one unlabeled vertex is adjacent to or from y, namely the sink v. Because $(y, v) \in E(D)$ and $f(y, v) < c(y, v)$, the vertex v is assigned a label. The first coordinate is $y+$ and the second coordinate is

$$\epsilon(v) = \min\{\epsilon(y),\ c(y, v) - f(y, v)\} = \min\{2, 3\} = 2.$$

Thus the vertex v is assigned the label $(y+, 2)$. This completes Step 3, so we move on to Step 4. Since v is labeled, we now move on to Step 5. Working in reverse and beginning with the sink v, we observe that the first coordinate of the label of v is $y+$. This tells us that the f-augmenting semipath being constructed ends with $y, (y, v), v$. Because the first coordinate of the label of y is $r+$, the semipath being constructed ends with $r, (r, y), y, (y, v), v$. Since the first coordinate of the label of r is $u+$, we obtain the following f-augmenting semipath:

$$P = (u, (u, r), r, (r, y), y, (y, v), v).$$

Since $\epsilon(v) = 2$, a new flow f' (see Figure 4.19) is obtained from f by augmenting the flow of each arc of P by 2.

Figure 4.19: The f-augmenting semipath and the new flow f' in the network

Since Step 5 has been completed, we go to Step 6, when the labels of all vertices of D are removed the vertices on the list L are removed. The table below summarizes what has transpired.

u is labeled $(-, \infty)$	$L : u$
u is scanned	$L :$
r is labeled $(u+, 3)$	$L : r$
s is labeled $(u-, 2)$	$L : r, s$
r is scanned	$L : s$
y is labeled $(r+, 2)$	$L : s, y$
s is scanned	$L : y$
w is labeled $(s-, 2)$	$L : y, w$
t is labeled $(s+, 2)$	$L : y, w, t$
y is scanned	$L : w, t$
v is labeled $(y+, 2)$	$L : w, t, v$

We now apply Algorithm 4.35 to the flow f' in the network N shown in Figure 4.19. The table below summarizes what occurs in this case.

u is labeled $(-,\infty)$	$L:u$
u is scanned r is labeled $(u+,1)$ s is labeled $(u-,2)$	$L:$ $L:r$ $L:r,s$
r is scanned	$L:s$
s is scanned w is labeled $(s-,2)$ t is labeled $(s+,2)$	$L:$ $L:w$ $L:w,t$
w is scanned v is labeled $(w+,2)$	$L:t$ $L:t,v$

In this application of Algorithm 4.35, the f'-augmenting semipath

$$P' = (u,(s,u),s,(w,s),w,(w,v),v)$$

is obtained. Since $\epsilon(v) = 2$, a new flow f'' is obtained from f' by augmenting the flow of each arc of P' by 2, as shown in Figure 4.20.

Figure 4.20: The f'-augmenting semipath and the new flow f'' in the network

Once again Algorithm 4.35 is applied, this time to the flow f'' in the network N shown in Figure 4.20. The table below shows what happens here.

u is labeled $(-,\infty)$	$L:u$
u is scanned r is labeled $(u+,1)$	$L:$ $L:r$
r is scanned	$L:$

The algorithm stops and the current flow f'' is a maximum flow with $\text{val}(f'') = 8$. Let X consist of the labeled vertices of D, namely $X = \{u,r\}$. So $K = [X,\overline{X}]$, $\text{cap}(K) = 8$ and K is a minimum cut in the network.

Menger Theorems for Digraphs

Since Menger's theorem (Theorem 2.36) and its edge analogue (Theorem 2.42) both deal with sets that separate two vertices in a graph, it is perhaps not surprising that these two theorems are closely related to the Max-Flow Min-Cut

Theorem which deals with deleting the arcs of a cut, thereby "separating" the source and sink of a network

Recall that Menger's theorem states that if u and v are nonadjacent vertices of a graph G, then the maximum number of internally disjoint $u - v$ paths in G equals the minimum number of vertices that separate u and v. As it turns out, there are other forms of Menger's theorem. We saw one of these in Theorem 2.42, which is the edge form of Menger's theorem. The vertex and edge forms of Menger's theorem have natural analogues for digraphs. All four of these results can be proved, either directly or indirectly, using the Max-Flow Min-Cut Theorem. In each case, the object is to construct an appropriate network from the given graph or digraph. For example, for two nonadjacent vertices u and v in a given graph G, Algorithm 4.35 can be used to determine the minimum number of vertices that separate u and v as well as such a set of vertices.

While we have defined separating sets of vertices and separating sets of edges in a graph, we need analogous terminology for digraphs. Let D be a digraph and let u and v be two nonadjacent vertices of D. A set $S \subseteq V(D) - \{u, v\}$ is said to be a $u - v$ **separating set of vertices** if every $u - v$ path in P contains at least one vertex of S. A set S of arcs in D is a $u - v$ **separating set of arcs** if every $u - v$ path in P contains at least one arc of S. We now present the arc form of Menger's theorem.

Theorem 4.37 (**The Arc Form of Menger's Theorem**) *For distinct vertices u and v in a digraph D, the maximum number of pairwise arc-disjoint $u - v$ paths in D equals the minimum number of arcs in a $u - v$ separating set of arcs.*

Proof. Suppose that the maximum number of $u - v$ paths in a collection of pairwise arc-disjoint $u - v$ paths in D is k and the minimum number of arcs in a $u - v$ separating set S is ℓ. Since each of these $u - v$ paths contains at least one arc of S and no arc in S belongs to more than one such $u - v$ path, it follows that $k \leq \ell$. It remains to show that $\ell \leq k$.

We now construct a network N with underlying digraph D, source u and sink v by defining a capacity function c on $E(D)$ such that $c(a) = 1$ for each arc a of D. By the Max-Flow Min-Cut Theorem (Theorem 4.34), the value of a maximum flow in N equals the capacity of a minimum cut. Let f be a maximum flow in N and K a minimum cut. Thus $\text{cap}(K) = \text{val}(f)$. We next show that $\ell \leq \text{cap}(K)$ and $\text{val}(f) \leq k$, from which it will follow that $\ell \leq k$ and so $\ell = k$.

Since K is a minimum cut in N, the set K is a $u - v$ separating set of arcs in D. Therefore, $\ell \leq |K| = \text{cap}(K)$.

Since f is an integer-valued function defined on $E(D)$ such that $0 \leq f(a) \leq c(a)$ for every arc a of D, it follows that either $f(a) = 0$ or $f(a) = 1$ for every arc a of D. Let D_1 be the digraph obtained from D by deleting all arcs a from D for which $f(a) = 0$. Consequently, $f(a) = 1$ for each arc a of D_1. Since f is

a flow in D, it follows by (4.7) that

$$f^+(x) = f^-(x) \text{ for each } x \in V(D) - \{u, v\}$$

and that

$$f^+(u) - f^-(u) = \text{val}(f) = f^-(v) - f^+(v).$$

However, for each vertex x in D,

$$f^+(x) = \text{od}_{D_1} x \text{ and } f^-(x) = \text{id}_{D_1} x.$$

Therefore,

$$\text{od}_{D_1} w = \text{id}_{D_1} w \text{ if } w \in V(D) - \{u, v\}$$

and

$$\text{od}_{D_1} u - \text{id}_{D_1} u = \text{val}(f) = \text{id}_{D_1} v - \text{od}_{D_1} v.$$

Since $\text{od}_{D_1} u = \text{id}_{D_1} u + \text{val}(f)$ and $\text{id}_{D_1} v = \text{od}_{D_1} v + \text{val}(f)$, it follows by Theorem 4.6 that the digraph D_1 and D as well contain $\text{val}(f)$ arc-disjoint $u - v$ paths and so $k \geq \text{val}(f)$. Therefore, $\ell = k$. ■

With the aid of the proof of Theorem 4.37, we use Algorithm 4.35 to determine the maximum number of arc-disjoint $u - v$ paths in the digraph D of Figure 4.21(a). Define a capacity function c on $E(D)$ such that $c(a) = 1$ for each arc a of D. We then have a network N with underlying digraph D, source u, sink v and capacity function c. Applying Algorithm 4.35, we obtain a maximum flow f and minimum cut $K = [X, \overline{X}]$, where $X = \{u, u_4, u_5\}$, indicated in Figure 4.21(b), so that $\text{val}(f) = \text{cap}(K) = 3$. Hence

$$K = \{(u, u_1), (u, u_2), (u_5, v)\}.$$

Consequently, the maximum number of arc-disjoint $u - v$ paths in D is 3. Three such paths are

$$P = (u, u_1, u_2, u_3, v),\ P' = (u, u_2, v) \text{ and } P'' = (u, u_5, v).$$

We now present the digraph analogue of Menger's theorem itself, which we refer to as the directed vertex form of Menger's theorem.

Theorem 4.38 (**The Directed Vertex Form of Menger's Theorem**) *Let D be a digraph and let u and v be distinct vertices of D such that $(u, v) \notin E(D)$. Then the maximum number of internally disjoint $u - v$ paths in D equals the minimum number of vertices in a $u - v$ separating set of vertices of D.*

Proof. Suppose that the maximum number of $u - v$ paths in a collection of internally disjoint $u - v$ paths in D is k and the minimum number of vertices in a $u - v$ separating set S of vertices of D is ℓ. Since each of these $u - v$ paths contains a vertex of S and no vertex of S lies on more than one such path, it follows that $k \leq \ell$. It remains to show that $\ell \leq k$.

Figure 4.21: Determining the maximum number of internally disjoint $u - v$ paths in a digraph

A new digraph D' is constructed from D by replacing each vertex $t \neq u, v$ by two new vertices t' and t'' and the arc (t', t''). Suppose that x and y are two vertices of D such that neither x nor y is u or v, that is, $x, y \notin \{u, v\}$. If (x, y) is an arc of D, then (x, y) is replaced by (x'', y'). If (u, x) is an arc of D, then (u, x) is replaced by (u, x'). If (x, u) is an arc of D, then (x, u) is replaced by (x'', u). If (x, v) is an arc of D, then (x, v) is replaced by (x'', v); while if (v, x) is an arc of D, then (v, x) is replaced by (v, x'). This is summarized in the table below.

For $x, y \in V(D)$ with $\{x, y\} \cap \{u, v\} = \emptyset$,

arcs in D	replaced by	arcs in D'
(x, y)		(x'', y')
(u, x)		(u, x')
(x, u)		(x'', u)
(x, v)		(x'', v)
(v, x)		(v, x')

This is illustrated for the digraph D shown in Figure 4.22.

Figure 4.22: Constructing a digraph D' from a digraph D

Suppose that k' is the maximum number of pairwise arc-disjoint $u - v$ paths in D' and ℓ' is the minimum number of arcs in a $u - v$ separating set of arcs in

D'. By Theorem 4.37, $\ell' = k'$. We show that $\ell \leq \ell'$ and $k' \leq k$, which will give us the desired inequality $\ell \leq k$.

Let A be a $u-v$ separating set of arcs in D' with $|A| = \ell'$. Thus A contains no arc of the form (z, u) or (v, z) where $z \neq u, v$. Each arc $a \in A$ is either of the form

$$a = (u, x'),\ a = (x', x''),\ a = (y'', x') \text{ or } a = (x'', v)$$

for some vertices $x, y \in V(D) - \{u, v\}$. Regardless of what the arc a is, denote the vertex x involved in a by w_a. Furthermore, let

$$W = \{w_a :\ a \in A\} \text{ so that } W \subseteq V(D) - \{u, v\}$$

and $|W| \leq |A| = \ell'$. Since A is a $u-v$ separating set of arcs in D', the set W is a $u-v$ separating set of vertices in D. Thus $\ell \leq |W| \leq \ell'$ and so $\ell \leq \ell'$.

Next, let $P'_1, P'_2, \ldots, P'_{k'}$ be a collection of k' pairwise arc-disjoint $u-v$ paths in D'. Then each path P'_i $(1 \leq i \leq k)$ is of the form

$$P'_i = (u, x'_1, x''_1, x'_2, x''_2, \ldots, x'_r, x''_r, v)$$

and gives rise to the path

$$P_i = (u, x_1, x_2, \ldots, x_r, v).$$

From the manner in which the digraph D' is constructed and the fact that $P'_1, P'_2, \ldots, P'_{k'}$ are pairwise arc-disjoint, it follows that $P_1, P_2, \ldots, P_{k'}$ are internally disjoint $u-v$ paths in D. Thus $k \geq k'$. Hence $\ell \leq \ell' = k' \leq k$ and so $\ell = k$. ■

As an illustration of the proof of Theorem 4.38, consider the digraph D of Figure 4.23. We seek the maximum number of internally disjoint $u-v$ paths in D. Although this is rather easy to do in this case, we construct the digraph D' described in the proof of Theorem 4.38 and determine the maximum number of pairwise arc-disjoint $u-v$ paths in D'. This number is 2 and two such paths are

$$P'_1 = (u, w', w'', z', z'', v) \text{ and } P'_2 = (u, y', y'', x', x'', v).$$

As described in the proof of Theorem 4.38, this gives rise to the paths

$$P_1 = (u, w, z, v) \text{ and } P_2 = (u, y, x, v),$$

which constitute a maximum set of internally disjoint $u-v$ paths in D.

For a given graph G, an alternative proof of the edge form of Menger's theorem (Theorem 2.43) can be obtained by applying Theorem 4.37 to the symmetric digraph D with underlying graph G. For vertices u and v in G, it remains only to observe that there is a one-to-one correspondence between the $u-v$ paths in G and the (directed) $u-v$ paths in D. A proof of Menger's theorem itself (the vertex form for graphs in Theorem 2.36) for a graph G can

Figure 4.23: Illustrating the proof of Theorem 4.38

also be obtained by applying Theorem 4.38 to the symmetric digraph D whose underlying graph is G.

Exercises for Section 4.3

1. Let N be the network with source u and sink v shown in Figure 4.24, where each arc is labeled with its capacity. A function f is defined on the arcs of N as follows:

$$\begin{array}{llll} f(u,s)=3 & f(s,t)=3 & f(t,v)=4 & f(u,x)=3 \\ f(x,y)=3 & f(y,v)=1 & f(x,t)=1 & f(w,u)=0 \\ f(y,w)=2 & f(w,v)=2. & & \end{array}$$

Is f a flow?

Figure 4.24: The network N in Exercise 1

2. Let N be the network with source u and sink v shown in Figure 4.25, where each arc is labeled with its capacity.

   (a) Show that no flow can have a value exceeding 9.

(b) Give an example of a flow f on N such that $\mathrm{val}(f) = 9$.

Figure 4.25: The network N in Exercise 2

3. For the network N shown in Figure 4.26 with source u and sink v, each arc has unlimited capacity. A flow f in the network is indicated by the labels on the arcs.

   (a) Determine the missing flows a, b and c.

   (b) Determine $\mathrm{val}(f)$.

Figure 4.26: The network N in Exercise 3

4. Assume that the network N shown in Figure 4.27 has unlimited capacities. Give an example of a flow f on N where the flow along each arc is a positive integer and where the maximum of the flows along the arcs is as small as possible.

5. Let N be a network with underlying digraph D, source u, sink v, capacity function c and let f be a flow on N. Suppose that $t \in V(D) - \{u, v\}$ such that $\mathrm{id}\, t = 0$ and that N' is obtained from N by deleting t. Define a function f' on $E(D-t)$ by $f'(x, y) = f(x, y)$ for all arcs $(x, y) \in E(D-t)$. Show that f' is a flow on N' and that $\mathrm{val}(f) = \mathrm{val}(f')$. Show that the same conclusion holds if $\mathrm{od}\, t = 0$.

6. A network N with source u and sink v is shown in Figure 4.28, where each arc is labeled with its capacity. Describe a flow on N where the flow along each arc is a positive integer.

Figure 4.27: The network N in Exercise 4

(a) Determine the net flow out of each vertex.

(b) What is the maximum value of a flow f in the network N?

(c) Give an example of a minimum cut $[X, \overline{X}]$ and determine $f(X, \overline{X}) - f(\overline{X}, X)$.

Figure 4.28: The network N in Exercise 6

7. Let u and v be two vertices of a digraph D and let A be a set of arcs of D such that every $u - v$ path in D contains at least one arc of A.

   (a) Show that there exists a set of arcs of the form $[X, \overline{X}]$ where $u \in X$ and $v \in \overline{X}$ and $[X, \overline{X}] \subseteq A$.

   (b) Show that $[X, \overline{X}]$ may be a proper subset of A.

8. Let N be a network with underlying digraph D, source u and sink v. For a set X of vertices of D with $u \in X$ and $v \in \overline{X}$ and a flow f defined on N, prove that $f^+(X) - f^-(X) = f(X, \overline{X}) - f(\overline{X}, X)$.

9. (a) Prove Corollary 4.32: *If f is flow in a network N with capacity function c and $[X, \overline{X}]$ is a cut in N such that $f(a) = c(a)$ for all $a \in [X, \overline{X}]$ and $f(a) = 0$ for all $a \in [\overline{X}, X]$, then f is a maximum flow in N and $[X, \overline{X}]$ is a minimum cut.*

   (b) Show that the converse of Corollary 4.32 is true.

10. Let N be a network with capacity function c and suppose that $[X, \overline{X}]$ is a minimum cut in N. Prove or disprove:

    (a) If f_1 and f_2 are flows in N that agree on $[X, \overline{X}]$ and $[\overline{X}, X]$, then f_1 and f_2 are maximum flows in N.

    (b) If f_1 and f_2 are maximum flows in N, then f_1 and f_2 agree on $[X, \overline{X}]$ and $[\overline{X}, X]$.

11. Use Algorithm 4.35 to find a maximum flow f and a minimum cut K in the network N in Figure 4.29.

Figure 4.29: The network N in Exercise 11

12. Use Algorithm 4.35 to find a maximum flow f and a minimum cut K in the network N in Figure 4.30.

Figure 4.30: The network N in Exercise 12

13. Use the proof of Theorem 4.38 to find the maximum number of internally disjoint $u - v$ paths in the digraph D shown in Figure 4.31.

Figure 4.31: The digraph D in Exercise 13

14. Find the maximum number of internally disjoint $u-v$ paths in the digraph D shown in Figure 4.32.

Figure 4.32: The digraph D in Exercise 14

Chapter 5

Graphs: History and Symmetry

We have encountered several famous mathematicians whose connections with graph theory were brief and indirect but whose work led to important and much studied areas of graph theory. Leonhard Euler's solution of the Königsberg Bridge Problem in 1736 is credited with beginning graph theory and Eulerian graphs. William Rowan Hamilton's Icosian Game dealt with finding cycles passing through all 20 vertices of a dodecahedron and is considered the beginning of Hamiltonian graphs. Arthur Cayley used trees to count certain chemical compounds and trees became one of the most important classes of graphs.

During graph theory's first 150 years or so the results in this field were isolated, often dealing with games and puzzles, and rarely making use of previously known facts. Shortly afterwards though, this began to change. In fact, 200 years after the origin of graph theory, the first book on the subject was written (in 1936). In that book and five years later, two problems were posed, one soon solved and the other still unsolved, but both dealing with the structure and symmetry of graphs.

In this chapter, several other mathematicians are discussed who had a profound effect on the development of graph theory. In addition, we discuss these two symmetry problems.

5.1 Some Historical Figures of Graph Theory

As we have seen, graph theory had the good fortune of having some of the great mathematicians of the past display an interest in some of the problems of graph theory – often without knowing that they had become involved in this new area of mathematics. It wasn't until later, however, that many mathematicians started to become aware of this fledgling new subject. The scattering of graph theory results, often as solutions to games, puzzles or isolated problems, and

appearing in diverse mathematical journals throughout Europe, made it difficult to know of any connections that might exist among these results.

Dénes König

It took some five years of work by the Hungarian mathematician Dénes König (1884–1944) to locate and organize the material that had been written to that point on graph theory. König, who was the son of a mathematics professor, spent nine semesters doing university work, the first four in Budapest and the last five in Göttingen. König studied under the famous geometer Hermann Minkowski and attended his lectures on analysis situs, which is what topology was called in its early days. Minkowski was also interested in graph theory, which no doubt played an important role in what König decided to work on.

König received his doctorate in 1907 with a dissertation in geometry. He obtained a faculty position in Budapest, where he would stay until his death. Among the courses König taught was graph theory, although the name "graph theory" never appeared in the catalogue at the university until 1927. Prior to that time, it appeared under the heading of analysis situs.

In 1935 König became a full professor. König was well known at the university for his enthusiastic lectures despite the fact that his lectures were not always well attended. Because of him, however, a number of dedicated students were introduced to this new area of mathematics. Although König was an excellent mathematician, he will probably be best known for his efforts to popularize graph theory. Because of him, graph theory grew from being a collection of unrelated results in a branch of recreational mathematics to a recognized area of the mathematical sciences. Because of his interest in graph theory, he was belittled by some mathematicians but he was never discouraged. He believed in the future of graph theory. In fact, König would often begin a lecture on graph theory by saying:

> *Graph theory is one of the most interesting of mathematical disciplines.*

In 1936, König's book *Theorie der endlichen und unendlichen Graphen* (*Theory of Finite and Infinite Graphs*) was published in Leipzig. This became the first book ever written that was entirely devoted to graph theory, although Oswald Veblen had discussed graph theory in his book *Analysis Situs*, the first book written on topology. König's book introduced graph theory to many young mathematicians but the book would have little impact until after World War II. In 1944, after the occupation of Hungary by the Nazis, König worked to assist persecuted mathematicians. Rather than face persecution himself, he committed suicide on October 19, 1944.

Claude Berge

For more than two decades, König's book would remain the sole book on graph theory – until 1958 when the book *Théorie des Graphes et Ses Applications*, authored by the French mathematician Claude Berge (1926–2002), was published. Berge had a passion for games throughout his life – both playing them (especially chess, backgammon and hex) and investigating their theoretical aspects. When graph theory was introduced to Berge, it was a subject that was unknown in France. It was Berge's goal to make some sense of this new area of mathematics. Initially, he worked on graph theory just for himself. He found the subject interesting and alive with many applications. It wasn't only the French to whom Berge introduced graph theory upon the publication of his book. He traveled extensively and lectured on graph theory.

Berge made contributions to other areas of mathematics, including game theory and topology. He introduced an alternative to the Nash equilibrium (named after John Nash whose life was chronicled in the 2002 Academy Award winning motion picture *A Beautiful Mind*) called the Berge equilibrium. Berge also coined the term *hypergraph*, a generalization of *graphs* and a concept of interest to many. He worked on game theory in 1951 and spent 1957 at the Institute of Advanced Study at Princeton and produced the book *Théorie générate des jeux á n personnes* (*General Theory of n-Person Games*). Following the publication of his book on graph theory, Berge would go on to author several more books, writing on such topics as combinatorics and economics. He developed an international reputation as a lecturer and an author of books.

Berge's writing interests were not limited to scientific subjects. Indeed, Berge was a member of OuLiPo, a group of French writers. His best known work for this organization was the short story *Who Killed the Duke of Densmore?* This is a crime story whose solution requires a knowledge of graph theory. Berge describes a scenario among characters in the story that leads to a contradiction thereby indicating that at least one of the characters is lying. The Danish mathematician Bjarne Toft wrote:

> *Claude Berge's greatest scientific achievement is that he gave graph theory a place in mathematics at large by revealing and emphasizing its connections to set theory, topology, game theory, operations research, mathematical programming, economics and other applications .... He generalized, unified, simplified, and combined in a unique way the general and the concrete. He will for a long time remain an inspirational force.*

While the graph theory books by König and Berge, written in German and French respectively, would later be translated into English, the first major book on graph theory written in English was *Theory of Graphs* published in 1962 by the American Mathematical Society and authored by Oystein Ore, whom we encountered earlier.

Alexander Zykov and Frank Harary

Although some books were written on special topics in graph theory and its applications during 1959–1969, only two other major books on graph theory were written before 1970, both in 1969. One of these was *Teoriia Konechnykn Grafov* (*Theory of Finite Graphs*) by the Russian mathematician Alexander A. Zykov, who had the greatest influence on the development of graph theory in Russia.

The second book on graph theory published in 1969 was *Graph Theory* by Frank Harary (1921–2005), who spent much of his life traveling and lecturing on graph theory, thereby introducing graph theory to many mathematicians around the world. Along the way, he acquired numerous co-authors, which led to hundreds of publications on all aspects of graph theory. Harary, known for his lucid writing style and his enthusiasm for the subject, has been referred to as the "Ambassador of Graph Theory." Harary founded the *Journal of Graph Theory* and with the famous combinatorialist Gian-Carlo Rota, founded the *Journal of Combinatorial Theory*.

Zykov and Harary, the two authors of 1969 books on graph theory, knew each other. On one occasion Zykov gave Harary a framed set of three pictures of lions: (1) a lion sleeping, (2) a lion awakening and (3) a lion roaring to represent (1) graph theory yesterday, (2) graph theory today and (3) graph theory tomorrow. Although meant as a comical gift, Zykov interpreted these pictures to mean (1) graph theory before 1936, (2) graph theory during 1936–1970 and (3) graph theory after 1970.

William Tutte

William T. Tutte and Paul Erdős were two other 20th century mathematicians who made a major impact on the development of graph theory. Born in England, Tutte (1917–2002) entered Trinity College, Cambridge, in 1935 to study chemistry. While chemistry was Tutte's major area of study, it was not his favorite subject – mathematics was. He was an active member of the Trinity Mathematical Society, where he met three undergraduates who became lifelong friends of his: Cedric Smith, Rowland Leonard Brooks and Arthur Stone. The four students wrote a paper using electrical networks to solve a geometric problem, which became a standard reference in the field.

These four students were sometimes referred to as the "Trinity Four." Smith learned of a group of mathematicians who called themselves "Hector Pétard" and had written a mathematics paper under that name. Smith thought that was an interesting idea and the Trinity Four set about finding a name for themselves. The first letters of their first names (Bill, Leonard, Arthur and Cedric) gave BLAC. The idea of using "The Reverend Cornelius Black" occurred to them but Brooks objected. The color BLACk became the more cheerful color BLAnChe (*white* in French). The phrase "Carte Blanche" then suggested the

name "Blanche Descartes" (as, of course, René Descartes was a famous mathematician). So over the years Blanche Descartes became the name of various subsets of the Trinity Four. Under that name, some thirty papers were published, some lighthearted, some serious mathematics. Even though many knew who Ms. Descartes was, Tutte would never acknowledge this. Writing under the name Blanche Descartes permitted Tutte to conceal his identity and say whatever he imagined. In 1969 Blanche Descartes reflected on graph theory, its origin and its future, in the 1969 poem [57] titled "The Expanding Unicurse":

Some citizens of Königsberg
Were walking on the strand
Beside the river Pregel
With its seven bridges spanned.

"O Euler, come and walk with us,"
Those burghers did beseech.
"We'll roam the seven bridges o'er,
And pass but once by each."

"It can't be done," thus Euler cried.
"Here comes the Q. E. D.
Your islands are but vertices,
And four have odd degree."

From Königsberg to König's book,
So runs the graphic tale,
And still it grows more colorful,
In Michigan and Yale.

The mention of Michigan and Yale in the last line refers to the universities of Harary and Ore, respectively.

Tutte's academic career was interrupted by World War II. At the invitation of his tutor in January 1941, Tutte went to Bletchley Park, the organization of code-breakers in Great Britain, the most prominent member of which was Alan Turing. In October 1941, Tutte encountered the first of a set of machine-coded messages from Berlin named Fish. The Fish code was used for high-level communications between Berlin and Nazi commanders. Using only samples of messages, Tutte was able to discover the structure of machines that generated these codes. Tony Sale, who first described the work of Tutte and his colleagues, called this the "greatest intellectual feat of the whole war."

After World War II, Tutte returned to Cambridge, this time as a mathematics major. While a graduate student, Tutte published some of his work on graph theory. After completing his Ph.D. in 1948, Tutte joined the faculty at the University of Toronto at the invitation of H. S. M. (Donald) Coxeter, one of the great geometers of the 20th century. While in Toronto, Tutte became one of the preeminent researchers in graph theory and combinatorics.

In 1957 the University of Waterloo in Canada was founded and five years later Ralph G. Stanton, the Chair of the Department of Mathematics there and

well known for his work in combinatorial designs, was able to convince Tutte to join the department. Although Tutte retired in 1984, he continued to work in graph theory and related areas of discrete mathematics. In his 1998 book *Graph Theory As I Have Known It*, Tutte described how he had arrived at many of his results.

Paul Erdős

Undoubtedly one of the best known mathematicians of the 20th century was Paul Erdős (1913–1996), who was born in Budapest. Both parents of Erdős were high school mathematics teachers and Erdős received much of his early education from them. He became fascinated with mathematics as a young child. By the time he was 21, he had already received a doctorate in mathematics. Because Erdős came from a Jewish family and because of the atmosphere that prevailed in Hungary at that time, he moved away from Hungary to England in 1934 as a guest lecturer and then to Princeton University in 1938. This began his habit of moving from one university to another, which continued throughout his life.

Possessions and money meant very little to Erdős. He would typically visit mathematicians for a short period, only long enough to work on research projects, and then move on to another location. His life was chronicled in the film *N is a Number: A Portrait of Paul Erdős*.

There is no mathematician who published more papers than Erdős, although Euler's papers contained more pages. Most of his 1500 or so papers were co-authored. He had more than 500 collaborators. Erdős was primarily interested in solving mathematical problems rather than developing theories. Among the many areas of mathematics in which Erdős was interested were number theory, probability, set theory, analysis and combinatorics – and graph theory. Erdős would often offer a cash prize for solving certain problems that he found interesting and challenging.

A somewhat humorous concept associated with Erdős is the Erdős number, probably introduced in a note (consisting only of 14 sentences) in the *American Mathematical Monthly* written by Casper Goffman [95] and titled "And what is your Erdős number?" The vertex set of the **collaboration graph** G is the set of all mathematicians, past and present. Two vertices A and B are adjacent in G if A and B are co-authors of at least one joint paper. Let PE be the vertex Paul Erdős. The **Erdős number** of a mathematician M is $d_G(M, PE)$. Thus Erdős himself is the only mathematician with Erdős number 0. Any mathematician who has co-authored a paper with Erdős has Erdős number 1, while any mathematician who does not have Erdős number 1 but has co-authored a paper with a person who has Erdős number 1 has Erdős number 2, and so on.

Erdős died in Warsaw, Poland, doing what he did so often – attending a mathematics conference.

5.2 The Automorphism Group of a Graph

Determining whether a given graph possesses a property of interest depends on its structure. One way of studying the structure of graphs is by investigating their symmetries. A common method of doing this is by means of groups.

An **automorphism** of a graph G is an isomorphism from G to itself. Thus an automorphism of G is a permutation of $V(G)$ that preserves adjacency (and nonadjacency). Of course, the identity function ϵ on $V(G)$ is an automorphism of G. The inverse of an automorphism of G is also an automorphism of G, as is the composition of two automorphisms of G. These observations lead us to the fact that the set of all automorphisms of a graph G form a group (under the operation of composition), called the **automorphism group** or simply the **group** of G and denoted by $\mathrm{Aut}(G)$.

The automorphism group of the graph G_1 of Figure 5.1 is cyclic of order 2, which we write as $\mathrm{Aut}(G_1) \cong \mathbb{Z}_2$. In addition to the identity permutation on $V(G_1)$, the group $\mathrm{Aut}(G_1)$ contains the *reflection* $\alpha = (uy)(vx)$, where α is expressed as *permutation cycles*. The graph G_2 of Figure 5.1 of order 6 has only the identity automorphism, so $\mathrm{Aut}(G_2) \cong \mathbb{Z}_1$. In fact, 6 is the smallest order of a nontrivial graph whose automorphism group consists only of the identity automorphism.

Figure 5.1: Graphs with cyclic automorphism groups of orders 2 and 1

Every permutation of the vertex set of K_n is an automorphism and so $\mathrm{Aut}(K_n)$ is the symmetric group S_n of order $n!$. On the other hand, the automorphism group of C_n, $n \geq 3$, is the dihedral group D_n of order $2n$, consisting of n rotations and n reflections. The 4-cycle C_4 and its automorphism group are illustrated in Figure 5.2.

Automophisms:

ϵ, $\alpha_1 = (u\ v\ w\ x)$

$\alpha_2 = (u\ w)(v\ x)$, $\alpha_3 = (u\ x\ w\ v)$

$\phi_1 = (u\ w)$, $\phi_2 = (v\ x)$

$\phi_3 = (u\ v)(w\ x)$, $\phi_4 = (u\ x)(v\ w)$

Figure 5.2: The 4-cycle and its automorphism group

Next we present a few basic facts concerning automorphism groups of graphs. We have already noted that every automorphism of a graph preserves both adjacency and nonadjacency. This leads to the following observation.

Theorem 5.1 *For every graph G,* $\mathrm{Aut}(G) \cong \mathrm{Aut}(\overline{G})$.

We mentioned previously that $\mathrm{Aut}(K_n) \cong S_n$ for every positive integer n. Certainly, if G is a graph of order n containing adjacent vertices as well as nonadjacent vertices, then $\mathrm{Aut}(G)$ is isomorphic to a proper subgroup of the symmetric group S_n. Combining this observation with Theorem 5.1 and Lagrange's Theorem on the order of a subgroup of a finite group, we arrive at the following.

Theorem 5.2 *The order* $|\mathrm{Aut}(G)|$ *of the automorphism group of a graph G of order n is a divisor of $n!$ and equals $n!$ if and only if $G = K_n$ or $G = \overline{K}_n$.*

Recall that two labelings of a graph G of order n from the same set of n labels are considered distinct if they do not produce the same edge set. With the aid of the automorphism group of a graph G of order n, it is possible to determine the number of distinct labelings of G.

Theorem 5.3 *The number of distinct labelings of a graph G of order n from a set of n labels is* $n!/|\mathrm{Aut}(G)|$.

Proof. Let S be a set of n labels. Certainly, there exist $n!$ labelings of G using the elements of S without regard to which labelings are distinct. For a given labeling of G, each automorphism of G gives rise to an identical labeling of G; that is, each labeling of G from S determines $|\mathrm{Aut}(G)|$ identical labelings of G. Hence there are $n!/|\mathrm{Aut}(G)|$ distinct labelings of G. ■

As an illustration of Theorem 5.3, consider the graph $G = P_3$ of Figure 5.3 and the set $S = \{1, 2, 3\}$. Since $\mathrm{Aut}(G) \cong \mathbb{Z}_2$, the number of distinct labelings of G is $3!/2 = 3$. The three distinct labelings of Theorem 5.3 are shown in Figure 5.3.

Figure 5.3: Distinct labelings of graphs

Similar Vertices

If a relation R is defined on the vertex set of a graph G by $u\ R\ v$ if $\phi(u) = v$ for some automorphism ϕ of G, then R is an equivalence relation. The equivalence classes of R are referred to as **orbits**. Two vertices belonging to the same orbit are called **similar vertices**. Then two similar vertices have the same degree. The automorphism group of the graph G of Figure 5.4 is cyclic of order 3 and G has four orbits.

Orbits:
$\{v_1\}$
$\{v_2, v_3, v_4\}$
$\{v_5, v_7, v_9\}$
$\{v_6, v_8, v_{10}\}$

Figure 5.4: Orbits of a graph

In Chapter 1 we defined the eccentricity $e(v)$ of a vertex v in a connected graph G as the distance from v to a vertex farthest from v. For the graph G of Figure 5.5, we compute the eccentricity of each vertex of G. The eccentricity of each vertex is shown in Figure 5.5 as well. The orbits of this graph are $\{r_1, r_2\}$, $\{s\}$, $\{t_1, t_2\}$, $\{u\}$, $\{v\}$, $\{w_1, w_2\}$, $\{x\}$, $\{y\}$ and $\{z\}$. Since r_1 and r_2 are similar vertices, they necessarily have the same eccentricity, namely 7. The same can be said of t_1 and t_2, as well as of w_1 and w_2. Of course, it is possible for vertices that are not similar to have the same eccentricity, as is the case with u and x. For the graph G of Figure 5.5, $\mathrm{Aut}(G)$ is the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ of three cyclic groups of order 2.

Figure 5.5: Similar vertices in a graph

Vertex-Transitive Graphs

A graph that contains a single orbit is called **vertex-transitive**. Thus a graph G is vertex-transitive if and only if for every two vertices u and v of G, there exists an automorphism ϕ of G such that $\phi(u) = v$. Necessarily, every vertex-transitive graph is regular. The graphs K_n $(n \geq 1)$, C_n $(n \geq 3)$ and $K_{r,r}$ $(r \geq 1)$ are all vertex-transitive. Also the regular graphs $G_1 = C_5 \,\square\, K_2$ and $G_2 = K_{2,2,2}$ of Figure 5.6 are vertex-transitive, while the regular graphs G_3 and G_4 are not vertex-transitive (see Exercise 7).

Figure 5.6: Vertex-transitive graphs and regular graphs that are not vertex-transitive

The two vertex-transitive graphs G_1 and G_2 of Figure 5.6 are Hamiltonian. In fact, there are many examples of vertex-transitive Hamiltonian graphs. Indeed, other than K_1 and K_2, there are only four known connected vertex-transitive graphs that are not Hamiltonian, namely the Petersen graph and the **Coxeter graph** (both shown in Figure 5.7) and the two graphs obtained from these by replacing each vertex by a triangle. These are called the **truncated Petersen graph** and the **truncated Coxeter graph**, also shown in Figure 5.7. Gordon Royle conjectured that these are the only connected vertex-transitive graphs that are not Hamiltonian.

All four graphs in Figure 5.7 fail to contain a Hamiltonian cycle; yet all four contain a Hamiltonian path. Indeed László Lovász conjectured that every connected vertex-transitive graph contains a Hamiltonian path. The computer scientist Donald Knuth reported that the problem of searching for Hamiltonian paths in highly symmetric graphs is old and originated in British campanology (bell-ringing).

Every digraph also has an automorphism group. An **automorphism** of a digraph D is an isomorphism from D to itself, that is, an automorphism of D is a permutation α on $V(D)$ such that (u, v) is an arc of D if and only if $(\alpha u, \alpha v)$

Figure 5.7: Connected vertex-transitive graphs that are not Hamiltonian

is an arc of D. The set of all automorphisms under composition forms a group, called the **automorphism group** of D, and is denoted by $\mathrm{Aut}(D)$. While $\mathrm{Aut}(G) \cong \mathbb{Z}_3$ for the graph G of Figure 5.4, there is an even simpler digraph D having $\mathrm{Aut}(D) \cong \mathbb{Z}_3$. In particular, for the digraphs D_1 and D_2 of Figure 5.8, $\mathrm{Aut}(D_1) \cong \mathbb{Z}_3$ and $\mathrm{Aut}(D_2) \cong \mathbb{Z}_5$.

Exercises for Section 5.2

1. For the graphs G_1 and G_2 in Figure 5.9, describe the automorphisms of G_1 and of G_2 in terms of permutation cycles.

2. Describe the elements of $\mathrm{Aut}(C_5)$.

3. Find a nonseparable graph G whose automorphism group is isomorphic to the cyclic group of order 4.

Figure 5.8: Digraphs with cyclic automorphism groups

Figure 5.9: The graphs G_1 and of G_2 in Exercise 1

4. Determine the number of distinct labelings of $K_{r,r}$.

5. For which pairs k, n of positive integers with $k \leq n$ does there exist a graph G of order n having k orbits?

6. For which pairs k, n of positive integers does there exist a graph G of order n and a vertex v of G such that there are exactly k vertices similar to v?

7. Show that the graphs G_3 and G_4 in Figure 5.6 are not vertex-transitive.

8. Describe the automorphism groups of the digraphs in Figure 5.10.

Figure 5.10: The digraphs in Exercise 8

5.3 Cayley Color Graphs

We have seen that a finite group can be associated with every graph or digraph. We now consider the reverse question of associating a graph and a digraph with a given finite group. A nontrivial group Γ is said to be **generated** by nonidentity elements $h_1, h_2, \ldots, h_k$ (and these elements are called **generators**) of Γ if every element of Γ can be expressed as a (finite) product of generators.

Every nontrivial finite group has a finite **generating set** (often several such sets) since the set of all nonidentity elements of the group is always a generating set for Γ.

Let Γ be a given nontrivial finite group having the generating set $\Delta = \{h_1, h_2, \ldots, h_k\}$. We associate a digraph with Γ and Δ called the **Cayley color graph of Γ with respect to Δ** and denoted by $D_\Delta(\Gamma)$. The vertex set of $D_\Delta(\Gamma)$ is the set of group elements of Γ; therefore, $D_\Delta(\Gamma)$ has order $|\Gamma|$. Each generator h_i is now regarded as a color. For $g_1, g_2 \in \Gamma$, there exists an arc (g_1, g_2) colored h_i in $D_\Delta(\Gamma)$ if $g_2 = g_1 h_i$. If h_i is a group element of order 2 (and is therefore self-inverse) and $g_2 = g_1 h_i$, then necessarily $g_1 = g_2 h_i$. When a Cayley color graph $D_\Delta(\Gamma)$ contains each of the arcs (g_1, g_2) and (g_2, g_1), both colored h_i, then it is customary to represent this symmetric pair of arcs by the single edge $g_1 g_2$ colored h_i. As we have now seen, a Cayley color graph is actually a digraph, each arc of which is assigned a color where a color is a generator in Δ. That this digraph is named for the mathematician Arthur Cayley is because of a famous theorem of his from group theory: *Every finite group is isomorphic to a group of permutations.*

We now illustrate the concepts just introduced. Let Γ denote the symmetric group S_3 of all permutations on the set $\{1, 2, 3\}$, and let $\Delta = \{a, b\}$, where $a = (123)$ and $b = (12)$. The six elements of Γ can then be expressed as a product of the elements a and b of Γ, namely the identity $e = b^2$, $a = a$, $b = b$, $(132) = a^2$, $(13) = ba$ and $(23) = ab$. The Cayley color graph $D_\Delta(\Gamma)$ in this case is shown in Figure 5.11.

Figure 5.11: A Cayley color graph

If the generating set Δ of a given nontrivial finite group Γ with n elements is chosen to be the set of all nonidentity group elements, then for every two vertices g_1, g_2 of $D_\Delta(\Gamma)$, both (g_1, g_2) and (g_2, g_1) are arcs (although not necessarily of the same color) and $D_\Delta(\Gamma)$ is the complete symmetric digraph of order n in this case.

Color-Preserving Automorphisms

Let Γ be a nontrivial finite group with generating set Δ. An element $\alpha \in \mathrm{Aut}(D_\Delta(\Gamma))$ is said to be **color-preserving** if for every arc (g_1, g_2) of $D_\Delta(\Gamma)$, the arcs (g_1, g_2) and $(\alpha(g_1), \alpha(g_2))$ have the same color. For a given nontrivial finite group Γ with generating set Δ, the set of all color-preserving automorphisms of $D_\Delta(\Gamma)$ forms a subgroup of $\mathrm{Aut}(D_\Delta(\Gamma))$.

For example, let $\Gamma = \mathbb{Z}_4$ be a cyclic group of order 4 generated by the element a. Then we can write $\Gamma = \{g_1, g_2, g_3, g_4\}$ where $g_i = a^{i-1}$ for $1 \le i \le 4$. Thus $g_1 = a^0 = e$ is the identity element of Γ. The set

$$\Delta = \{g_2, g_3, g_4\} = \{a, a^2, a^3\}$$

of the nonidentity elements of Γ is a generating set for Γ. For this group Γ and this generating set Δ, the Cayley color graph $D_\Delta(\Gamma)$ is shown in Figure 5.12(a). Since there are symmetric pairs of arcs of the same color joining the pairs g_1, g_3 and g_2, g_4, we have said that this Cayley color graph could be drawn as in Figure 5.12(b). In this case, $D_\Delta(\Gamma)$ is the complete symmetric digraph K_4^* and so $\mathrm{Aut}(D_\Delta(\Gamma))$ is the symmetric group S_4.

Figure 5.12: The Cayley color graph $D_\Delta(\Gamma)$ of a cyclic group Γ of order 4 with generating set $\Delta = \{a, a^2, a^3\}$

Since every permutation of the elements of Γ is an automorphism of $D_\Delta(\Gamma)$, the permutation $\alpha = (g_1\ g_2\ g_4)$ is an automorphism. Hence α maps the arc (g_1, g_2) into the the arc (g_2, g_4). Since the color of (g_1, g_2) is a and the color of (g_2, g_4) is α^2, it follows that α is not color-preserving. In this case, the subgroup of color-preserving automorphisms of $D_\Delta(\Gamma)$ consists of

$$\epsilon,\ (g_1\ g_2\ g_3\ g_4),\ \ (g_1\ g_3)(g_2\ g_4),\ \ (g_1\ g_4\ g_3\ g_2),$$

which is a cyclic group of order 4.

A useful characterization of color-preserving automorphisms is given in the next result (see Exercise 2).

Theorem 5.4 *Let Γ be a nontrivial finite group with generating set Δ and let α be a permutation of $V(D_\Delta(\Gamma))$. Then α is a color-preserving automorphism of $D_\Delta(\Gamma)$ if and only if*

$$\alpha(gh) = (\alpha(g))h$$

for every $g \in \Gamma$ and $h \in \Delta$.

We observed for a cyclic group Γ of order 4 and a generating set Δ for Γ that the subgroup of color-preserving automorphisms of $D_\Delta(\Gamma)$, shown in Figure 5.12, is also a cyclic group of order 4. This is not a coincidence.

Theorem 5.5 *Let Γ be a nontrivial finite group with generating set Δ. Then the group of color-preserving automorphisms of $D_\Delta(\Gamma)$ is isomorphic to Γ.*

Proof. Let $\Gamma = \{g_1, g_2, \ldots, g_n\}$. For $i = 1, 2, \ldots, n$, define $\alpha_i : V(D_\Delta(\Gamma)) \to V(D_\Delta(\Gamma))$ by $\alpha_i(g_s) = g_i g_s$ for $1 \le s \le n$. Since Γ is a group, the mapping α_i is one-to-one and onto. Let $h \in \Delta$. Then for each i $(1 \le i \le n)$ and for each s $(1 \le s \le n)$,

$$\alpha_i(g_s h) = g_i(g_s h) = (g_i g_s)h = (\alpha_i(g_s))h.$$

Hence, by Theorem 5.4, α_i is a color-preserving automorphism of $D_\Delta(\Gamma)$.

Let α be an arbitrary color-preserving automorphism of $D_\Delta(\Gamma)$ and let g_1 be the identity of Γ. Suppose that $\alpha(g_1) = g_r$. Let $g_s \in \Gamma$. Since g_s can be expressed as a product of generators, $g_s = h_1 h_2 \cdots h_t$, where $h_j \in \Delta$ and $1 \le j \le t$. Therefore,

$$\begin{aligned} \alpha(g_s) &= \alpha(g_1 h_1 h_2 \cdots h_t) = \alpha(g_1 h_1 h_2 \cdots h_{t-1})h_t \\ &= \alpha(g_1 h_1 \cdots h_{t-2})h_{t-1}h_t = \cdots = \alpha(g_1)h_1 h_2 \cdots h_t = g_r g_s. \end{aligned}$$

Thus $\alpha = \alpha_r$.

We now show that the mapping ϕ defined by $\phi(g_i) = \alpha_i$ is an isomorphism from Γ to the group of color-preserving automorphisms of $D_\Delta(\Gamma)$. The mapping ϕ is already one-to-one and onto. It remains to show that ϕ is operation-preserving, namely that $\phi(g_i, g_j) = \phi(g_i)\phi(g_j)$ for $g_i, g_j \in \Gamma$. Let $g_i g_j = g_k$. Then $\phi(g_i g_j) = \phi(g_k) = \alpha_k$ and $\phi(g_i)\phi(g_j) = \alpha_i \alpha_j$. Now

$$\alpha_k(g_s) = g_k g_s = (g_i g_j)g_s = g_i(g_j g_s) = \alpha_i(g_j g_s) = \alpha_i(\alpha_j(g_s)) = (\alpha_i \alpha_j)g_s$$

and so $\alpha_k = \alpha_i \alpha_j$. ■

Frucht's Theorem

On page 5 of the first section (Basic Concepts) of the first chapter (Foundations) of the first book written on graph theory, the author Dénes König [139, p. 5] wrote (translated into English):

When can a given abstract group be interpreted as the group of a graph and if this is the case, how can the corresponding graph be constructed?

In the early years of the 20th century, Germany had been known for its mathematicians who excelled in group theory. One of these was Issai Schur. While at the University of Berlin, Schur supervised several doctoral students. One of Schur's students was Roberto Frucht (1906–1997), who received his Ph.D. in 1930 in the area of group theory.

Finding a job as a mathematician in Germany was very difficult during those days. Frucht wasn't even able to be hired as a high school teacher since Frucht was a Czechoslovakian citizen and German citizenship was required. So Frucht moved to Italy to work for an insurance company. Frucht stayed there until 1938. During the time he was in Italy, he had essentially become mathematically inactive. However, one day in 1936 he received a catalogue advertising a book in graph theory (by König). Frucht ordered the book and became an enthusiastic graph theorist the very day that the book arrived.

König's question on automorphism groups immediately caught Frucht's attention. After several months of being unsuccessful trying to solve the problem, he found a solution that seemed rather easy (after he had found it). Frucht [87] proved that *every* finite group has this property. We are now in a position to present a proof of this result.

If Γ is the trivial group, then $\mathrm{Aut}(G) \cong \Gamma$ for $G = K_1$. Therefore, let $\Gamma = \{g_1, g_2, \ldots, g_n\}$, $n \geq 2$, be a given finite group, and let $\Delta = \{h_1, h_2, \ldots, h_t\}$, $1 \leq t \leq n$, be a generating set for Γ. We first construct the Cayley color graph $D_\Delta(\Gamma)$ of Γ with respect to Δ; the Cayley color graph is actually a digraph, of course. By Theorem 5.5, the group of color-preserving automorphisms of $D_\Delta(\Gamma)$ is isomorphic to Γ. We now transform the digraph $D_\Delta(\Gamma)$ into a graph G by the following technique. Let (g_i, g_j) be an arc of $D_\Delta(\Gamma)$ colored h_k. Delete this arc and replace it by the *graphical* path $g_i, u_{ij}, u'_{ij}, g_j$. At the vertex u_{ij} we construct a new path P_{ij} of length $2k-1$ and at the vertex u'_{ij} a path P'_{ij} of length $2k$. This construction is now performed with every arc of $D_\Delta(\Gamma)$, and is illustrated in Figure 5.13 for $k = 1, 2, 3$.

The addition of the paths P_{ij} and P'_{ij} in the formation of G is, in a sense, equivalent to the direction and the color of the arcs in the construction of $D_\Delta(\Gamma)$.

It now remains only to observe that every color-preserving automorphism of $D_\Delta(\Gamma)$ induces an automorphism of G, and conversely. Thus we have a proof of Frucht's theorem.

Theorem 5.6 (**Frucht's Theorem**) *For every finite group Γ, there exists a graph G such that* $\mathrm{Aut}(G) \cong \Gamma$.

The condition of having a given group prescribed is not a particularly stringent one for graphs. For example, Herbert Izbicki [123] showed that for every finite group Γ and integer $r \geq 3$, there exists an r-regular graph G with $\mathrm{Aut}(G) \cong \Gamma$.

Figure 5.13: Constructing a graph G from a given group Γ

We have now seen that for every finite group Γ and generating set Δ there is an associated digraph, namely the Cayley color graph $D_\Delta(\Gamma)$. The underlying graph of a Cayley color graph $D_\Delta(\Gamma)$ is called a **Cayley graph** and is denoted by $G_\Delta(\Gamma)$. Thus a graph G is a Cayley graph if and only if there exists a finite group Γ and a generating set Δ for Γ such that $G \cong G_\Delta(\Gamma)$, that is, the vertices of G are the elements of Γ and two vertices g_1 and g_2 of G are adjacent if and only if either $g_1 = g_2h$ or $g_2 = g_1h$ for some $h \in \Delta$.

As observed earlier, the complete symmetric digraph K_n^* is a Cayley color graph; consequently, every complete graph is a Cayley graph. Since $K_2 \,\square\, K_3$ is the underlying graph of the Cayley color graph of Figure 5.11, $K_2 \,\square\, K_3$ is a Cayley graph. Every Cayley graph is regular. Indeed every Cayley graph is vertex-transitive. The converse is not true, however. For example, the Petersen graph (Figure 5.7) is vertex-transitive but it is not a Cayley graph.

Exercises for Section 5.3

1. Construct the Cayley color graph of the cyclic group of order 4 when the generating set Δ has (a) one element and (b) three elements.

2. Prove Theorem 5.4: *Let Γ be a nontrivial finite group with generating set Δ and let α be a permutation of $V(D_\Delta(\Gamma))$. Then α is a color-preserving*

automorphism of $D_\Delta(\Gamma)$ *if and only if* $\alpha(gh) = (\alpha(g))h$ *for every* $g \in \Gamma$ *and* $h \in \Delta$.

3. Determine the group of color-preserving automorphisms for the Cayley color graph $D_\Delta(\Gamma)$ of Figure 5.14.

Figure 5.14: The Cayley color graph $D_\Delta(\Gamma)$ in Exercise 3

4. For a given finite group Γ, determine an infinite number of mutually nonisomorphic graphs whose groups are isomorphic to Γ.

5. Show that every n-cycle is a Cayley graph.

6. Show that the cube Q_3 is a Cayley graph.

5.4 The Reconstruction Problem

If ϕ is an automorphism of a nontrivial graph G and u is a vertex of G, then $G - u \cong G - \phi(u)$, that is, if u and v are similar vertices of a graph G, then $G - u \cong G - v$. The converse of this statement is not true, however. Indeed, the vertices u and v of the graph G of Figure 5.15 are not similar, yet $G - u \cong G - v$.

Figure 5.15: A graph with nonsimilar vertices whose vertex-deleted subgraphs are isomorphic

The comment above brings up a fact concerning isomorphism and vertex-deleted subgraphs. Suppose that G and H are two graphs such that $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{u_1, u_2, \ldots, u_n\}$. Just because the graphs $G - v_1$ and $H - u_1$, say, may be isomorphic does not mean that G and H are isomorphic. But what if, in addition to having $G - v_1 \cong H - u_1$, we also know that $G - v_2 \cong H - u_2$, $G - v_3 \cong H - u_3$, and so on, up to $G - v_n \cong H - u_n$? Can we then conclude that $G \cong H$?

Reconstructible Graphs

This question is related to the problem of determining how much structure of a graph G can be recovered from its vertex-deleted subgraphs. This, in fact, brings us to a famous problem in graph theory.

A graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 2$, is said to be **reconstructible** if for every graph H having $V(H) = \{u_1, u_2, \ldots, u_n\}$, $G - v_i \cong H - u_i$ for $i = 1, 2, \ldots, n$ implies that $G \cong H$. Hence, if G is a reconstructible graph, then the subgraphs $G - v$, $v \in V(G)$, uniquely determine G. It is believed by many but has never been verified that every graph of order at least 3 is reconstructible.

The Reconstruction Conjecture *Every graph of order at least* 3 *is reconstructible.*

This conjecture is believed to have been made in 1941 and is often attributed jointly to Paul J. Kelly and Stanislaw M. Ulam. Kelly, who spent many years as a faculty member at the University of California at Santa Barbara, obtained a number of results on this topic. Born in Poland in 1909, Ulam became interested in astronomy, physics and mathematics while a teenager and taught himself calculus. He entered the Polytechnic Institute (now in Lvov, Ukraine) in 1927. One of his professors there was Kazimierz Kuratowski, whom we will encounter in Chapter 6. Ulam received his Ph.D. in 1933.

In 1940, Ulam acquired a faculty position at the University of Wisconsin where Kelly was studying for his Ph.D. In 1943 John von Neumann asked to meet Ulam at a railroad station in Chicago. This resulted in Ulam going to the Los Alamos National Laboratory in New Mexico to work on the hydrogen bomb with the physicist Edward Teller. While at Los Alamos, Ulam developed the well-known Monte Carlo method for solving mathematical problems using a statistical sampling method with random numbers. Throughout his life, he made important contributions in many areas of mathematics.

The **Reconstruction Problem** is the problem of determining the truth or falsity of the Reconstruction Conjecture. The condition on the order in the Reconstruction Conjecture is necessary, for if $G_1 = K_2$, then G_1 is not reconstructible. If $G_2 = 2K_1$, then the subgraphs $G_1 - v$, $v \in V(G_1)$, and the subgraphs $G_2 - v$, $v \in V(G_2)$, are precisely the same. Thus G_1 is not uniquely determined by its subgraphs $G_1 - v$, $v \in V(G_1)$. By the same reasoning, $G_2 = 2K_1$ is also not reconstructible. The Reconstruction Conjecture claims that K_2 and $2K_1$ are the only non-reconstructible graphs.

If there is a counterexample to the Reconstruction Conjecture, then it must have order at least 12, for, with the aid of computers, Brendan McKay [154] and Albert Nijenhuis [163] have shown that all graphs of order less than 12 (and greater than 2) are reconstructible. The graph G of Figure 5.16 is therefore reconstructible since its order is less than 12. Hence the graphs $G - v_i$ $(1 \leq$

$i \leq 6$) uniquely determine G. However, there exists a graph H with $V(H) = \{v_1, v_2, \ldots, v_6\}$ such that $G - v_i \cong H - v_i$ for $1 \leq i \leq 5$, but $G - v_6 \ncong H - v_6$. Therefore, the graphs $G - v_i$ ($1 \leq i \leq 5$) do not uniquely determine G. On the other hand, the graphs $G - v_i$ ($4 \leq i \leq 6$) do uniquely determine G.

Figure 5.16: A reconstructible graph

Digraphs are not reconstructible, however. The vertex-deleted subdigraphs of the tournaments D_1 and D_2 of Figure 5.17 are the same; yet $D_1 \ncong D_2$. Indeed, Paul K. Stockmeyer [209] showed that there are infinitely many pairs of counterexamples for digraphs (see [136, 210] as well).

Figure 5.17: Two non-reconstructible digraphs

Recognizable Properties

There are several properties of a graph G that can be found by considering the subgraphs $G - v$, $v \in V(G)$. We begin with the most elementary of these.

Theorem 5.7 *If G is a graph of order $n \geq 3$ and size m, then n and m as well as the degrees of the vertices of G are determined from the n subgraphs $G - v$, $v \in V(G)$.*

Proof. It is trivial to determine the number n, which is necessarily one greater than the order of any subgraph $G - v$. Also, n is equal to the number of subgraphs $G - v$. To determine m, label these subgraphs by G_i, $i = 1, 2, \ldots, n$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and suppose that $G_i = G - v_i$, where $v_i \in V(G)$. Let m_i denote the size of G_i. Consider an arbitrary edge e of G, say $e = v_j v_k$. Then e belongs to $n - 2$ of the subgraphs G_i, namely all except G_j and G_k. Since $\sum_{i=1}^{n} m_i$ counts each edge $n-2$ times, it follow that $\sum_{i=1}^{n} m_i = (n-2)m$ and so

$$m = \frac{\sum_{i=1}^{n} m_i}{n-2}. \tag{5.1}$$

The degrees of the vertices of G can be determined by simply noting that $\deg v_i = m - m_i$, $i = 1, 2, \ldots, n$. ■

We illustrate Theorem 5.7 with the six subgraphs $G - v$ shown in Figure 5.18 of some unspecified graph G. From these subgraphs we determine n, m and $\deg v_i$ for $i = 1, 2, \ldots, 6$. Clearly, $n = 6$. By calculating the integers m_i $(1 \le i \le 6)$, we find that $m = 9$. Thus, $\deg v_1 = \deg v_2 = 2$, $\deg v_3 = \deg v_4 = 3$ and $\deg v_5 = \deg v_6 = 4$.

Figure 5.18: The subgraphs $G - v$ of a graph G

We say that a graphical parameter or graphical property is **recognizable** if, for each graph G of order at least 3, it is possible to determine the value of the parameter for G or whether G has the property from the subgraphs $G - v$, $v \in V(G)$. Theorem 5.7 thus states that for a graph of order at least 3, the order, the size and the degrees of its vertices are recognizable parameters. From Theorem 5.7, it also follows that the property of graph regularity is recognizable; indeed, the degree of regularity is a recognizable parameter. For regular graphs, much more can be said.

Theorem 5.8 *Every regular graph of order at least* 3 *is reconstructible.*

Proof. As we have already mentioned, regularity and the degree of regularity are recognizable. Thus, without loss of generality, we may assume that G is an r-regular graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$. It remains to show that G is uniquely determined by its subgraphs $G - v_i$, $i = 1, 2, \ldots, n$. Consider $G - v_1$, say. The graph $G - v_1$ then has order $n - 1$, where r vertices have degree $r - 1$ and the remaining $n - r - 1$ vertices have degree r. Adding the vertex v_1 to $G - v_1$ together with all those edges $v_1 v$ where $\deg_{G-v_1} v = r - 1$ produces the graph G. ■

If G has order $n \geq 3$, then it can be determined whether G is connected from the n subgraphs $G - v$, $v \in V(G)$.

Theorem 5.9 *For graphs of order at least* 3*, connectedness is a recognizable property. In particular, if G is a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$, then G is connected if and only if at least two of the subgraphs $G - v_i$ are connected.*

Proof. Let G be a connected graph. By Theorem 2.1, G contains at least two vertices that are not cut-vertices, implying the results.

Conversely, assume that there exist vertices $v_1, v_2 \in V(G)$ such that both $G - v_1$ and $G - v_2$ are connected. Thus, in $G - v_1$ and also in G, the vertex v_2 is connected to each vertex v_i for $i \geq 3$. Moreover, in $G - v_2$ (and thus in G), v_1 is connected to each vertex v_i for $i \geq 3$. Hence every pair of vertices of G are connected and so G is connected. ■

Since connectedness is a recognizable property, it is possible to determine from the subgraphs $G - v$, $v \in V(G)$, whether a graph G of order at least 3 is disconnected. We now show that disconnected graphs are reconstructible. There have been several proofs of this fact. The proof given here is due to Bennet Manvel [150].

Theorem 5.10 *Disconnected graphs of order at least* 3 *are reconstructible.*

Proof. We have already noted that disconnectedness in graphs of order at least 3 is a recognizable property. Thus, we assume without loss of generality that G is a disconnected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 3$. Further, let $G_i = G - v_i$ for $i = 1, 2, \ldots, n$. From Theorem 5.7, the degrees of the vertices v_i, $i = 1, 2, \ldots, n$, can be determined from the graphs $G - v_i$. Hence, if G contains an isolated vertex, then G is reconstructible. Assume then that G has no isolated vertices.

Since every component of G is nontrivial, it follows that $k(G_i) \geq k(G)$ for $i = 1, 2, \ldots, n$ and that $k(G_j) = k(G)$ for some integer j satisfying $1 \leq j \leq n$. Hence the number of components of G is $\min\{k(G_i) : i = 1, 2, \ldots, n\}$. Suppose that F is a component of G of maximum order. Necessarily, F is a component of maximum order among the components of the graphs G_i, that is, F is recognizable. Delete a vertex that is not a cut-vertex from F, obtaining F'.

Assume that there are r (≥ 1) components of G isomorphic to F. The number r is recognizable, as we shall see. Let

$$S = \{G_i : \ k(G_i) = k(G)\}$$

and let S' be the subset of S consisting of all those graphs G_i having a minimum number ℓ of components isomorphic to F. (Observe that if $r = 1$, then there exist graphs G_i in S containing no components isomorphic to F, that is, $\ell = 0$.) In general, then, $r = \ell + 1$. Next let S'' denote the set of those graphs G_i in S' having a maximum number of components isomorphic to F'.

Assume that $G_1, G_2, \ldots, G_t$ ($t \geq 1$) are the elements of S''. Each graph G_i in S'' has $k(G)$ components. Since each graph G_i ($1 \leq i \leq t$) has a minimum number of components isomorphic to F, each vertex v_i ($1 \leq i \leq t$) belongs to a component F_i of G isomorphic to F, where the components F_i of G ($1 \leq i \leq t$) are not necessarily distinct. Further, since each graph G_i ($1 \leq i \leq t$) has a maximum number of components isomorphic to F', it follows that $F_i - v_i = F'$ for each $i = 1, 2, \ldots, t$. Hence, every two of the graphs $G_1, G_2, \ldots, G_t$ are isomorphic and G can be produced from G_1, say, by replacing a component of G_1 isomorphic to F' by a component isomorphic to F. ■

The Reconstruction Problem is often described in a somewhat playful manner as a problem involving a deck of cards. Suppose that we begin with two people, say you and a friend, and a graph G. However, only your friend knows what the graph G is. For each vertex v of G, the graph $G - v$ (unlabeled!) is drawn on a card. The set of all such cards is then referred to as a **deck**. This deck is then given to you. Your job is to find all graphs H whose set of vertex-deleted subgraphs are those appearing on the deck of cards. Any graph H satisfying this information is referred to as a **solution** to the deck. Of course, we know that there is at least one solution to the deck, namely the graph G. Furthermore, if the Reconstruction Conjecture is true, then the graph G is the only solution to the deck. However, knowing that such a deck has a solution (or maybe more than one solution) doesn't mean that it is easy to find a solution.

To illustrate this idea, consider the deck of cards shown in Figure 5.19. The problem is to find all solutions of this deck. Each deck of cards uniquely determines the order n, size m and the degrees of the vertices of any solution of the deck.

G_1 G_2 G_3 G_4 G_5 G_6

Figure 5.19: The subgraphs $G - v$ for one or more graphs G

Let m_i denote the size of $G_i = G - v_i$ ($1 \leq i \leq 6$). Since there are six cards

in the deck, $n = 6$. Furthermore, by (5.1)

$$m = \frac{\sum_{i=1}^{n} m_i}{n-2} = \frac{36}{4} = 9.$$

Therefore, the degrees $m - m_i$ $(1 \leq i \leq 6)$ of the vertices of a solution G of the deck are $2, 2, 3, 3, 4, 4$. Observe, for example, that the subgraph G_6 has size 5 and degree sequence $3, 2, 2, 2, 1$. Hence any solution G can be obtained by adding a vertex v_6 of degree 4 to G_6. Since each solution to the deck must have degree sequence $4, 4, 3, 3, 2, 2$, the vertex v_6 must be joined to all vertices of G_6 except one vertex of degree 2. If v_6 is joined to the two adjacent vertices of degree 2 in G_6, then any solution of the deck must contain K_4 as a subgraph. However, none of the graphs G_i $(1 \leq i \leq 6)$ contains K_4. Therefore, v_6 must be adjacent to only one of these vertices of degree 2 in G_6. Because these two vertices are similar, we conclude that the deck of cards shown in Figure 5.19 has the unique solution shown in Figure 5.20.

Figure 5.20: The solution of the deck of cards in Figure 5.19

It can be shown that (connected) graphs of order at least 3 whose complements are disconnected are reconstructible (Exercise 5). However, it remains to be shown that *all* connected graphs of order at least 3 are reconstructible.

Exercises for Section 5.4

1. Reconstruct the graph G whose subgraphs $G - v, v \in V(G)$, are given in Figure 5.18.

2. Reconstruct the graph G whose subgraphs $G - v, v \in V(G)$, are given in Figure 5.21.

Figure 5.21: The subgraphs $G - v$ of the graph G in Exercise 2

3. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_7\}$ such that $G - v_i = K_{2,4}$ for $i = 1, 2, 3$ and $G - v_i = K_{3,3}$ for $i = 4, 5, 6, 7$. Show that G is reconstructible.

4. Show that the tournaments of Figure 5.17 are not isomorphic.

5. (a) Prove that if G is reconstructible, then $\overline{G}$ is reconstructible.

   (b) Prove that every graph of order n (≥ 3) whose complement is disconnected is reconstructible.

6. Prove that the property of being bipartite for a graph is recognizable.

7. Reconstruct the graph G whose subgraphs $G - v, v \in V(G)$, are given in Figure 5.22.

Figure 5.22: The subgraphs $G - v$ of the graph G in Exercise 7

8. Show that no graph of order at least 3 can be reconstructed from exactly two of the subgraphs $G - v, v \in V(G)$.

9. In the process of solving the deck of cards in Figure 5.19, we learned that the order of any solution is $n = 6$, the size is $m = 9$ and the degree sequence of a solution is $4, 4, 3, 3, 2, 2$. Was this enough information to obtain the solution shown in Figure 5.20?

10. Find all solutions to the deck of cards shown in Figure 5.23.

Figure 5.23: The deck of cards in Exercise 10

G_1 G_2 G_3 G_4 G_5 G_6

Figure 5.24: The deck of cards in Exercise 11

11. Find all solutions to the deck of cards shown in Figure 5.24.

12. For the deck of cards shown in Figure 5.25, determine, without finding a solution, whether any solution G is Hamiltonian.

G_1 G_2 G_3 G_4 G_5 G_6

Figure 5.25: The deck of cards in Exercise 12

Chapter 6

Planar Graphs

Of the methods we have described to represent a graph G, probably the most common and the most useful for determining whether G possesses particular properties of interest is that of presenting G by means of a drawing. There are occasions when the edges in a diagram may cross and other occasions when the edges in a diagram do not cross. If some pairs of edges in a diagram of a graph cross, then it may be that there are other drawings of this same graph when no edges cross – or perhaps this is not possible. It is the class of graphs that can be drawn in the plane without their edges crossing that will be of interest to us in this chapter. We will see that a result of Euler plays a central role in this study.

6.1 The Euler Identity

A **polyhedron** is a 3-dimensional object whose boundary consists of polygonal plane surfaces. These surfaces are typically called the **faces** of the polyhedron. The boundary of a face consists of the edges and vertices of the polygon. In this setting, the total number of faces in the polyhedron is commonly denoted by F, the total number of edges in the polyhedron by E and the total number of vertices by V. The best known polyhedra are the so-called **Platonic solids**: the **tetrahedron**, **cube** (**hexahedron**), **octahedron**, **dodecahedron** and **icosahedron**. These are shown in Figure 6.1, together with the values of V, E and F for these polyhedra.

During the 18th century, many letters (over 160) were exchanged between Leonhard Euler (who, as we saw in Chapter 3, essentially introduced graph theory to the world when he solved and then generalized the Königsberg Bridge Problem) and Christian Goldbach (well known for stating the conjecture that every even integer greater than or equal to 4 can be expressed as the sum of two primes). In a letter that Euler wrote to Goldbach on 14 November 1750, he stated a relationship that existed among the numbers V, E and F for a polyhedron and which would later become known as:

Platonic solid	V	E	F
tetrahedron	4	6	4
cube	8	12	6
octahedron	6	12	8
dodecahedron	20	30	12
icosahedron	12	30	20

Figure 6.1: The five Platonic solids

The Euler Polyhedral Formula *If a polyhedron has V vertices, E edges and F faces, then*

$$V - E + F = 2.$$

That Euler was evidently the first mathematician to observe this formula (which is actually an identity rather than a formula) may be somewhat surprising in light of the fact that Archimedes and René Descartes both studied polyhedra long before Euler. A possible explanation as to why others had overlooked this identity might be due to the fact that geometry had primarily been a study of distances.

The Euler Polyhedral Formula appeared in print two years later (in 1752) in two papers by Euler [77, 78]. In the first of these two papers, Euler stated that he had been unable to prove the formula. However, in the second paper, he presented a proof by dissecting polyhedra into tetrahedra. Although his proof was clever, he nonetheless made some missteps. The first generally accepted proof was obtained by the French mathematician Adrien-Marie Legendre [144].

Each polyhedron can be converted into a map and then into a graph by

inserting a vertex at each meeting point of the map (which is actually a vertex of the polyhedron). This is illustrated in Figure 6.2 for the cube.

Figure 6.2: From a polyhedron to a map to a graph

The graphs obtained from the five Platonic solids are shown in Figure 6.3. These graphs have a property in which we will be especially interested: No two edges cross (intersect each other) in the graph.

Figure 6.3: The graphs of the five Platonic solids

Planar Graphs

A graph G is called a **planar graph** if G can be drawn in the plane without any two of its edges crossing. Such a drawing is also called an **embedding of G in the plane**. In this case, the embedding is a **planar embedding**. A graph G that is already drawn in the plane in this manner is a **plane graph**. Certainly then, every plane graph is planar and every planar graph can be drawn as a plane graph. In particular, all five graphs of the Platonic solids are planar.

When those points in the plane that correspond to the vertices and edges of a plane graph G are removed from the plane, the resulting connected pieces of the plane are the **regions** of G. One of the regions is unbounded and is called the **exterior region** of G. For every planar embedding of a planar graph G and every region R in this planar embedding, there exists a planar embedding of G in which R is the exterior region. When considering a plane graph G of a polyhedron, the faces of the polyhedron become the regions of G, one of which is the exterior region of G. On the other hand, a planar graph need not be the graph of any polyhedron. The plane graph H of Figure 6.4 is not the graph of any polyhedron. This graph has five regions, denoted by R_1, R_2, R_3, R_4 and R_5, where R_5 is the exterior region.

Figure 6.4: The boundaries of the regions of a plane graph

For a region R of a plane graph G, the vertices and edges incident with R form a subgraph of G called the **boundary** of R. Every edge of G that lies on a cycle belongs to the boundary of two regions of G, while every bridge of G belongs to the boundary of a single region. In Figure 6.4, the boundaries of the five regions of H are shown as well.

The five graphs G_1, G_2, G_3, G_4 and G_5 shown in Figure 6.5 are all planar, although G_1 and G_3 are not plane graphs. The graph G_1 can be drawn as G_2, while G_3 can be drawn as G_4. In fact, G_1 (and G_2) is the graph of the tetrahedron. For each graph, its order n, its size m and the number r of regions are shown as well.

Observe that $n - m + r = 2$ for each graph of Figure 6.5. Of course, this is not surprising for G_2 since this is the graph of a polyhedron (the tetrahedron) and $n = V$, $m = E$ and $r = F$. In fact, this identity holds for every connected plane graph.

Figure 6.5: Planar graphs

Theorem 6.1 (**The Euler Identity**) *For every connected plane graph of order n, size m and having r regions,*

$$n - m + r = 2.$$

Proof. We proceed by induction on the size m of a connected plane graph. There is only one connected graph of size 0, namely K_1. In this case, $n = 1$, $m = 0$ and $r = 1$. Since $n - m + r = 2$, the base case of the induction holds.

Assume for a positive integer m that if H is a connected plane graph of order n' and size m', where $m' < m$ such that there are r' regions, then $n' - m' + r' = 2$. Let G be a connected plane graph of order n and size m with r regions. We consider two cases.

Case 1. *G is a tree.* In this case, $m = n - 1$ and $r = 1$. Thus $n - m + r = n - (n - 1) + 1 = 2$, producing the desired result.

Case 2. *G is not a tree.* Since G is connected and is not a tree, it follows by Theorem 2.10 that G contains an edge e that is not a bridge. In G, the edge e is on the boundaries of two regions. So in $G - e$ these two regions merge into a single region. Since $G - e$ has order n, size $m - 1$ and $r - 1$ regions and $m - 1 < m$, it follows by the induction hypothesis that $n - (m - 1) + (r - 1) = 2$ and so $n - m + r = 2$. ■

The Euler Polyhedron Formula is therefore a special case of Theorem 6.1. While Euler struggled with the verification of $V - E + F = 2$, he did not have the luxury of a developed graph theory at his disposal.

From Theorem 6.1, it follows that every two planar embeddings of a connected planar graph result in plane graphs having the same number of regions; thus one can speak of the number of regions of a connected planar graph. For planar graphs in general, we have the following result. (See Exercise 2.) Recall that $k(G)$ denotes the number of components of a graph G.

Corollary 6.2 *If G is a plane graph with n vertices, m edges and r regions, then*

$$n - m + r = 1 + k(G).$$

An Upper Bound for the Size of Planar Graphs

If G is a connected plane graph of order 4 or more, then the boundary of every region of G must contain at least three edges. This observation is helpful in showing that the size of a planar graph cannot be too large in terms of its order.

Theorem 6.3 *If G is a planar graph of order $n \geq 3$ and size m, then*

$$m \leq 3n - 6.$$

Proof. Since the size of every graph of order 3 cannot exceed 3, the inequality holds for $n = 3$. So we may assume that $n \geq 4$. Futhermore, we may assume that the planar graphs under consideration are connected, for otherwise edges can be added to produce a connected graph. Suppose that G is a connected planar graph of order $n \geq 4$ and size m and that there is a given planar embedding of G, resulting in r regions. By the Euler Identity, $n - m + r = 2$. Let $R_1, R_2, \ldots, R_r$ be the regions of G and suppose that we denote the number of edges on the boundary of R_i $(1 \leq i \leq r)$ by m_i. Then $m_i \geq 3$. Since each edge of G is on the boundary of at most two regions of G, it follows that

$$3r \leq \sum_{i=1}^{r} m_i \leq 2m.$$

Hence

$$6 = 3n - 3m + 3r \leq 3n - 3m + 2m = 3n - m$$

and so $m \leq 3n - 6$. ■

By expressing Theorem 6.3 in its contrapositive form, we obtain the following reformulation of the theorem.

Theorem 6.4 *If G is a graph of order $n \geq 5$ and size m such that $m > 3n - 6$, then G is nonplanar.*

The Graphs K_5 and $K_{3,3}$

Theorem 6.4 provides us with a large class of nonplanar graphs.

Corollary 6.5 *The graph K_5 is nonplanar.*

Proof. The graph K_5 has order $n = 5$ and size $m = 10$. Since $m = 10 > 9 = 3n - 6$, it follows by Theorem 6.4 that K_5 is nonplanar. ■

Since it is evident that any graph containing a nonplanar subgraph is itself nonplanar, it follows that once we know that K_5 is nonplanar, we can conclude that K_n is nonplanar for every integer $n \geq 5$. Of course, K_n is planar for $1 \leq n \leq 4$. Another corollary of Theorem 6.4 provides us with a useful property of planar graphs.

Corollary 6.6 *Every planar graph contains a vertex of degree* 5 *or less.*

Proof. The result is obvious for planar graphs of order 6 or less. Let G be a graph of order n and size m all of whose vertices have degree 6 or more. Then $n \geq 7$ and

$$2m = \sum_{v \in V(G)} \deg v \geq 6n$$

and so $m \geq 3n$. By Theorem 6.4, G is nonplanar. ■

We will soon see that K_5 is an especially important nonplanar graph. Another important nonplanar graph is $K_{3,3}$. Since $K_{3,3}$ has order $n = 6$ and size $m = 9$ but $m < 3n - 6$, Theorem 6.4 cannot be used to establish the nonplanarity of $K_{3,3}$, however. On the other hand, we can use the fact that $K_{3,3}$ is bipartite to establish this property.

Corollary 6.7 *The graph $K_{3,3}$ is nonplanar.*

Proof. Suppose that $K_{3,3}$ is planar. Let there be given a planar embedding of $K_{3,3}$, resulting in r regions. Thus by the Euler Identity, $n-m+r = 6-9+r = 2$ and so $r = 5$. Let $R_1, R_2, \ldots, R_5$ be the five regions and let m_i be the number of edges on the boundary of R_i $(1 \leq i \leq 5)$. Since $K_{3,3}$ is bipartite, $K_{3,3}$ contains no triangles and so $m_i \geq 4$ for $1 \leq i \leq 5$. Since every edge of $K_{3,3}$ lies on the boundary of a cycle, every edge of $K_{3,3}$ belongs to the boundary of two regions. Thus

$$20 = 4r \leq \sum_{i=1}^{5} m_i = 2m = 18,$$

which is impossible. ■

Maximal Planar Graphs

A planar graph G is **maximal planar** if the addition to G of any edge joining two nonadjacent vertices of G results in a nonplanar graph. Necessarily then, if a maximal planar graph G of order $n \geq 3$ and size m is embedded in the plane, then the boundary of every region of G is a triangle and so $3r = 2m$. It then follows by the proof of Theorem 6.3 that $m = 3n - 6$. All of the graphs shown in Figure 6.6 are maximal planar.

Figure 6.6: Maximal planar graphs

A graph G is **nearly maximal planar** if there exists a planar embedding of G such that the boundary of every region of G is a cycle, at most one of which is not a triangle. Thus every maximal planar graph is nearly maximal planar (see Figure 6.7(a)). Also, the wheels $W_n = C_n \vee K_1$ $(n \geq 3)$ are nearly maximal planar (see Figure 6.7(b)). The graphs in Figures 6.7(c) and 6.7(d) (where the graph in Figure 6.7(d) is redrawn in Figure 6.7(e)) are nearly maximal planar.

Figure 6.7: Nearly maximal planar graphs

We now derive some results concerning the degrees of the vertices of a maximal planar graph.

Theorem 6.8 *If G is a maximal planar graph of order 4 or more, then the degree of every vertex of G is at least 3.*

Proof. Let G be a maximal planar graph of order $n \geq 4$ and size m and let v be a vertex of G. Since $m = 3n - 6$, it follows that $G - v$ has order $n - 1$ and

size $m - \deg v$. Since $G - v$ is planar and $n - 1 \geq 3$,

$$m - \deg v \leq 3(n-1) - 6$$

and so $m - \deg v = 3n - 6 - \deg v \leq 3n - 9$. Thus $\deg v \geq 3$. ■

Not only is the minimum degree of every maximal planar graph G of order 4 or more at least 3, the graph G is 3-connected (see Exercise 6).

In the early 20th century, Paul August Ludwig Wernicke studied under the supervision of Hermann Minkowski. (Dénes König, who, as we saw, wrote the first book on graph theory, also studied under Minkowski. Thus Wernicke and König were "academic brothers.") By Corollary 6.6 and Theorem 6.8, every maximal planar graph of order 4 or more contains a vertex of degree 3, 4 or 5. In the very same year that he received his Ph.D., Wernicke [236] proved that every planar graph that didn't have a vertex of a degree less than 5 must contain a vertex of degree 5 that is adjacent either to a vertex of degree 5 or to a vertex of degree 6. In the case of maximal planar graphs, Wernicke's result states the following.

Theorem 6.9 *If G is a maximal planar graph of order 4 or more, then G contains at least one of the following:* (1) *a vertex of degree* 3, (2) *a vertex of degree* 4, (3) *two adjacent vertices of degree* 5, (4) *two adjacent vertices, one of which has degree* 5 *and the other has degree* 6.

Proof. Assume, to the contrary, that there exists a maximal planar graph G of order $n \geq 4$ and size m containing none of (1)–(4). By Corollary 6.6, $\delta(G) = 5$. Let there be given a planar embedding of G, resulting in r regions. Then

$$n - m + r = 2.$$

Suppose that G has n_i vertices of degree i for $5 \leq i \leq \Delta(G) = \Delta$. Then

$$\sum_{i=5}^{\Delta} n_i = n \quad \text{and} \quad \sum_{i=5}^{\Delta} i n_i = 2m = 3r.$$

We now compute the number of regions whose boundary contains either a vertex of degree 5 or a vertex of degree 6. Since the boundary of every region is a triangle, it follows, by assumption, that no region has two vertices of degree 5 on its boundary or a vertex of degree 5 and a vertex of degree 6 on its boundary. On the other hand, the boundary of a region could contain two or perhaps three vertices of degree 6. Each vertex of degree 5 lies on the boundaries of five regions and every vertex of degree 6 lies on the boundaries of six regions. Furthermore, every region containing a vertex of degree 6 on its boundary can contain as many as three vertices of degree 6. Therefore, G has $5n_5$ regions whose boundary contains a vertex of degree 5 and at least $6n_6/3 = 2n_6$ regions

whose boundary contains at least one vertex of degree 6. Thus

$$\begin{aligned} r &\geq 5n_5 + 2n_6 \geq 5n_5 + 2n_6 - n_7 - 4n_8 - \cdots - (20 - 3\Delta)n_\Delta \\ &= \sum_{i=5}^{\Delta}(20 - 3i)n_i = 20n - 3\sum_{i=5}^{\Delta} in_i = 20(m - r + 2) - 3(2m) \\ &= (20m - 20r + 40) - 9r = (30r - 20r + 40) - 9r \\ &= r + 40, \end{aligned}$$

which is a contradiction. ■

The following result gives a relationship among the degrees of the vertices in a maximal planar graph of order at least 4.

Theorem 6.10 *Let G be a maximal planar graph of order $n \geq 4$ and size m containing n_i vertices of degree i for $3 \leq i \leq \Delta = \Delta(G)$. Then*

$$3n_3 + 2n_4 + n_5 = 12 + n_7 + 2n_8 + \cdots + (\Delta - 6)n_\Delta.$$

Proof. Since $m = 3n - 6$, it follows that $2m = 6n - 12$. Therefore,

$$\sum_{i=3}^{\Delta} in_i = \sum_{i=3}^{\Delta} 6n_i - 12$$

and so

$$\sum_{i=3}^{\Delta}(6 - i)n_i = 12. \tag{6.1}$$

Hence $3n_3 + 2n_4 + n_5 = 12 + n_7 + 2n_8 + \cdots + (\Delta - 6)n_\Delta$. ■

Charges and Discharging

Heinrich Heesch introduced the idea of assigning what is called a "charge" to each vertex of a planar graph as well as **discharging rules** which indicate how charges are to be redistributed among the vertices. In a maximal planar graph G, every vertex v of G is assigned a **charge** of $6 - \deg v$. In particular, every vertex of degree 5 receives a charge of $+1$, every vertex of degree 6 receives a charge of 0, and every vertex of degree 7 or more receives a negative charge. By appropriately redistributing positive charges, some useful results can often be obtained. According to equation (6.1) in the proof of Theorem 6.10, the sum of the charges of the vertices of a maximal planar graph of order 4 or more is 12. This is restated below.

Theorem 6.11 *If G is a maximal planar graph of order $n \geq 4$, size m and maximum degree $\Delta(G) = \Delta$ such that G has n_i vertices of degree i for $3 \leq i \leq \Delta$, then*

$$\sum_{i=3}^{\Delta}(6 - i)n_i = 12.$$

According to Theorem 6.8, every vertex of a maximal planar graph of order 4 or more has degree 3 or more. Consequently, no vertex can contribute more than 3 to the sum in Theorem 6.11. Because this sum is 12, we have the following corollary, which is an extension of Corollary 6.6.

Corollary 6.12 *If G is a maximal planar graph of order at least 4, then G contains at least four vertices whose degrees are at most 5.*

We now use the discharging method to give an alternative proof of Theorem 6.9.

Theorem 6.13 *If G is a maximal planar graph of order 4 or more, then G contains at least one of the following:* (1) *a vertex of degree 3,* (2) *a vertex of degree 4,* (3) *two adjacent vertices of degree 5,* (4) *two adjacent vertices, one of which has degree 5 and the other has degree 6.*

Proof. Assume, to the contrary, that there exists a maximal planar graph G of order $n \geq 4$, where there are n_i vertices of degree i for $3 \leq i \leq \Delta = \Delta(G)$ such that G contains none of (1)–(4). Thus $\delta(G) = 5$. To each vertex v of G assign the charge $6 - \deg v$. Hence each vertex of degree 5 receives a charge of $+1$, each vertex of degree 6 receives no charge, and each vertex of degree 7 or more receives a negative charge. By Theorem 6.11, the sum of the charges of the vertices of G is

$$\sum_{i=3}^{\Delta}(6-i)n_i = 12.$$

Let there be given a planar embedding of G. For each vertex v of degree 5 in G, redistribute its charge of $+1$ by moving a charge of $\frac{1}{5}$ to each of its five neighbors, resulting in v now having a charge of 0. Hence the sum of the charges of the vertices of G remains 12. Since G contains neither (3) nor (4), no vertex of degree 5 or 6 will have its charges increased. Consider a vertex u with $\deg u = k \geq 7$. Thus u received an initial charge of $6 - k$. Because no consecutive neighbors of u in the embedding can have degree 5, the vertex u can receive an added charge of $+\frac{1}{5}$ from at most $k/2$ of its neighbors. After the redistribution of charges, the new charge of u is at most

$$6 - k + \frac{k}{2} \cdot \frac{1}{5} = 6 - \frac{9k}{10} < 0.$$

Hence no vertex of G now has a positive charge. This is impossible since the sum of the charges of the vertices of G is 12. ■

Another result concerning maximal planar graphs that can be proved with the aid of the discharging method (see Exercise 15) is due to Philip Franklin [85].

Theorem 6.14 *If G is a maximal planar graph of order 4 or more, then G contains at least one of the following:* (1) *a vertex of degree* 3, (2) *a vertex of degree* 4, (3) *a vertex of degree* 5 *that is adjacent to two vertices, each of which has degree* 5 *or* 6.

The Five Regular Polyhedra

By the Euler Polyhedron Formula, if V, E and F are the number of vertices, edges and faces of a polyhedron, then

$$V - E + F = 2.$$

When dealing with a polyhedron P (as well as the graph of the polyhedron P), it is customary to represent the number of vertices of degree k by V_k and number of faces (regions) bounded by a k-cycle by F_k. It follows then that

$$2E = \sum_{k\geq 3} kV_k = \sum_{k\geq 3} kF_k. \tag{6.2}$$

By Corollary 6.6, every polyhedron has at least one vertex of degree 3, 4 or 5. As an analogue to this result, we have the following.

Theorem 6.15 *At least one face of every polyhedron is bounded by a k-cycle for some $k = 3, 4, 5$.*

Proof. Assume, to the contrary, that $F_3 = F_4 = F_5 = 0$. By equation (6.2),

$$2E = \sum_{k\geq 6} kF_k \geq \sum_{k\geq 6} 6F_k = 6\sum_{k\geq 6} F_k = 6F.$$

Hence $E \geq 3F$. Also,

$$2E = \sum_{k\geq 3} kV_k \geq \sum_{k\geq 3} 3V_k = 3\sum_{k\geq 3} V_k = 3V.$$

By Theorem 6.1, $V - E + F = 2$ and so $3V - 3E + 3F = 6$. Hence $6 = 3V - 3E + 3F \leq 2E - 3E + E = 0$, which is a contradiction. ■

A **regular polyhedron** is a polyhedron whose faces are bounded by congruent regular polygons and whose polyhedral angles are congruent. In particular, for a regular polyhedron, $F = F_s$ for some s and $V = V_t$ for some t. For example, a cube is a regular polyhedron with $V = V_3$ and $F = F_4$. There are only four other regular polyhedra. These five regular polyhedra are the Platonic solids we saw in Figure 6.1. The Greeks were aware, over two thousand years ago, that there are only five such polyhedra.

Theorem 6.16 *There are exactly five regular polyhedra.*

Proof. Let P be a regular polyhedron and let G be an associated planar graph. Then $V - E + F = 2$, where V, E and F denote the number of vertices, edges and faces of P and G. Therefore,

$$\begin{aligned} -8 &= 4E - 4V - 4F \\ &= 2E + 2E - 4V - 4F \\ &= \sum_{k\geq 3} kF_k + \sum_{k\geq 3} kV_k - 4\sum_{k\geq 3} V_k - 4\sum_{k\geq 3} F_k \\ &= \sum_{k\geq 3}(k-4)F_k + \sum_{k\geq 3}(k-4)V_k. \end{aligned} \tag{6.3}$$

Since P is regular, there exist integers s (≥ 3) and t (≥ 3) such that $F = F_s$ and $V = V_t$. Hence

$$-8 = (s-4)F_s + (t-4)V_t.$$

Moreover, $3 \leq s, t \leq 5$ and $sF_s = 2E = tV_t$. If $s, t \geq 4$, then (6.3) yields $-8 = (s-4)F_s + (t-4)F_t \geq 0$, which is impossible. Hence either $s = 3$ or $t = 3$. This results in five possibilities for the pairs s, t.

Case 1. $s = 3$ *and* $t = 3$. Here we have

$$-8 = -F_3 - V_3 \quad \text{and} \quad 3F_3 = 3V_3,$$

so $F_3 = V_3 = 4$. Thus P is the *tetrahedron*. (That the tetrahedron is the only regular polyhedron with $V_3 = F_3 = 4$ follows from geometric considerations.)

Case 2. $s = 3$ *and* $t = 4$. Therefore

$$-8 = -F_3 \quad \text{and} \quad 3F_3 = 4V_4.$$

Hence $F_3 = 8$ and $V_4 = 6$, implying that P is the *octahedron*.

Case 3. $s = 3$ *and* $t = 5$. In this case,

$$-8 = -F_3 + V_5 \quad \text{and} \quad 3F_3 = 5V_5,$$

so $F_3 = 20$, $V_5 = 12$ and P is the *icosahedron*.

Case 4. $s = 4$ *and* $t = 3$. We find here that

$$-8 = -V_3 \quad \text{and} \quad 4F_4 = 3V_3.$$

Thus $V_3 = 8$, $F_4 = 6$ and P is the *cube*.

Case 5. $s = 5$ *and* $t = 3$. For these values,

$$-8 = F_5 - V_3 \quad \text{and} \quad 5F_5 = 3V_3.$$

Solving for F_5 and V_3, we find that $F_5 = 12$ and $V_3 = 20$, so P is the *dodecahedron.* ■

The graphs of the five regular polyhedra are shown in Figure 6.3.

Exercises for Section 6.1

1. Prove that a graph is planar if and only if each of its blocks is planar.
2. Prove Corollary 6.2: *If G is a plane graph with n vertices, m edges and r regions, then $n - m + r = 1 + k(G)$.*
3. Give an example of two non-isomorphic maximal planar graphs of the same order.
4. Prove that there exists only one 4-regular maximal planar graph.
5. Prove that a planar graph of order $n \geq 3$ and size m is maximal planar if and only if $m = 3n - 6$.
6. Prove that every maximal planar graph of order 4 or more is 3-connected.
7. Determine all connected regular planar graphs G such that the number of regions in a planar embedding of G equals its order.
8. Determine all maximal planar graphs G of order 3 or more such that the number of regions in a planar embedding of G equals its order.
9. Determine whether the graph G shown in Figure 6.8 is nearly maximal planar.

G :

Figure 6.8: The graph G in Exercise 9

10. A nontrivial tree T of order n has the property that $\overline{T}$ is a maximal planar graph.
    (a) What is n?
    (b) Give an example of a tree T with this property.
11. Show that every graph G of order $n \geq 6$ that contains three spanning trees T_1, T_2 and T_3 such that every edge of G belongs to exactly one of these three trees is nonplanar.

12. (a) Show that if G is a planar graph containing no vertex of degree less than 5, then G contains at least 12 vertices of degree 5.

    (b) Give an example of a planar graph that contains no vertex of degree less than 5.

13. Let $k \geq 3$ be an integer and let G be a plane graph of order n ($\geq k$) and size m.

    (a) If the length of every cycle is at least k, then determine an upper bound for m in terms of n and k.

    (b) Show that the upper bound B obtained in (a) is sharp by determining, for arbitrary $k \geq 3$, a plane graph G of order n and size B, every cycle of which has length at least k.

14. If the boundary of every interior region of a plane graph G of order n and size m is a triangle and the boundary of the exterior region is a k-cycle ($k \geq 3$), express m in terms of n and k.

15. Use a discharging method to prove Theorem 6.14: *If G is a maximal planar graph of order 4 or more, then G contains at least one of the following:* (1) *a vertex of degree* 3, (2) *a vertex of degree* 4, (3) *a vertex of degree* 5 *that is adjacent to two vertices, each of which has degree* 5 *or* 6.

6.2 Planarity versus Nonplanarity

In the preceding section, we discussed several characteristics of planar graphs. However, a fundamental question remains. For a given graph G, how does one determine whether G is planar or nonplanar? Of course, if G can be drawn in the plane without any of its edges crossing, then G is planar. On the other hand, if G cannot be drawn in the plane without edges crossing, then G is nonplanar. Nevertheless, it may very well be difficult to see how to draw a graph G in the plane without edges crossing or to know that such a drawing is impossible. We saw from Theorem 6.4 that if G has order $n \geq 3$ and size m where $m > 3n-6$, then G is nonplanar. Also, as a consequence of Theorem 6.4, we saw in Corollary 6.6 that if G contains no vertex of degree less than 6, then G is nonplanar. Of course, in that case, $m \geq 3n$.

Any graph that is a subgraph of a planar graph must surely be planar. Equivalently, every graph containing a nonplanar subgraph must itself be nonplanar. Thus to show that a disconnected graph G is planar it suffices to show that each component of G is planar. Hence when considering planarity, we may restrict our attention to connected graphs. Since a connected graph G is planar if and only if each block of G is planar (see Exercise 1 in Section 6.1), it is sufficient to concentrate on 2-connected graphs only.

According to Corollaries 6.5 and 6.7, the graphs K_5 and $K_{3,3}$ are nonplanar. Hence if a graph G should contain K_5 or $K_{3,3}$ as a subgraph, then G is

nonplanar. For the maximal planar graph G of order 5 and size 9 shown in Figure 6.9 (that is, G is obtained by deleting one edge from K_5), we consider the graph $F = G \square K_3$, shown in Figure 6.9. Thus F consists of three copies of G, denoted by G_1, G_2 and G_3, where $u_1u_2 \notin E(G_1)$, $v_1v_2 \notin E(G_2)$ and $w_1w_2 \notin E(G_3)$. To make it easier to draw G, the nine edges of each graph G_i $(1 \leq i \leq 3)$ are not drawn. The graph F has order 15 and size $m = 42$. Since $m = 42 > 39 = 3n - 6$, it follows that F is nonplanar. Furthermore, it can be shown that no subgraph of F is isomorphic to either K_5 or $K_{3,3}$. Thus, despite the fact that F contains neither K_5 nor $K_{3,3}$ as a subgraph, the graph F is nonplanar. Consequently, there must exist some other explanation as to why this graph is nonplanar.

Figure 6.9: The graph $F = G \square K_3$

Kuratowski's Theorem

A graph H is a **subdivision** of a graph G if either $H \cong G$ or H can be obtained from G by inserting vertices of degree 2 into the edges of G. Thus for the graph G of Figure 6.10, all of the graphs H_1, H_2 and H_3 are subdivisions of G. Indeed, H_3 is also a subdivision of H_2.

Certainly, a subdivision H of a graph G is planar if and only if G is planar. Therefore, K_5 and $K_{3,3}$ are nonplanar as is any subdivision of K_5 or $K_{3,3}$. This provides a necessary condition for a graph to be planar.

Theorem 6.17 *A graph G is planar only if G contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.*

The remarkable feature about this necessary condition for a graph to be

Figure 6.10: Subdivisions of a graph

planar is that the condition is also sufficient. The first published proof of this fact occurred in 1930. This theorem is due to the Polish topologist Kazimierz Kuratowski (1896–1980), who first announced this theorem in 1929. The title of Kuratowski's paper is "Sur le problème des courbes gauches en topologie" (On the problem of skew curves in topology), which suggests, and rightly so, that the setting of his theorem was in topology – not graph theory. Nonplanar graphs were sometimes called *skew graphs* during that period. The publication date of Kuratowski's paper was critical to having the theorem credited to him, for, as it turned out, later in 1930 the two American mathematicians Orrin Frink and Paul Althaus Smith submitted a paper containing a proof of this theorem as well but withdrew it after they became aware that Kuratowski's proof had preceded theirs, although just barely. They did publish a one-sentence announcement [86] of what they had accomplished in the *Bulletin of the American Mathematical Society* and, as the title "Irreducible non-planar graphs" of their note indicates, the setting for their proof was graph theoretical in nature.

It is believed by some that a proof of this theorem may have been discovered somewhat earlier by the Russian topologist Lev Semenovich Pontryagin, who was blind his entire adult life. Because the first proof of this theorem may have occurred in Pontryagin's unpublished notes, this result is sometimes referred to as the Pontryagin-Kuratowski theorem in Russia and elsewhere. However, since the possible proof of this theorem by Pontryagin did not satisfy the established practice of appearing in print in an accepted refereed journal, the theorem is generally credited to Kuratowski [142] and to Kuratowski alone.

Theorem 6.18 (**Kuratowski's Theorem**) *A graph G is planar if and only if G contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.*

Proof. We have already noted the necessity of this condition for a graph to be planar. Hence it remains to verify its sufficiency, namely that every graph containing no subgraph which is a subdivision of K_5 or $K_{3,3}$ is planar. Suppose that this statement is false. Then there is a nonplanar 2-connected graph G of minimum size containing no subgraph that is a subdivision of K_5 or $K_{3,3}$.

We claim in fact that G is 3-connected. Suppose that this is not the case. Then G contains a minimum vertex-cut consisting of two vertices x and y.

Since G has no cut-vertices, it follows that each of x and y is adjacent to one or more vertices in each component of $G - \{x, y\}$. Let F_1 be one component of $G - \{x, y\}$ and let F_2 be the union of the remaining components of $G - \{x, y\}$. Furthermore, let

$$G_i = G[V(F_i) \cup \{x, y\}] \text{ for } i = 1, 2.$$

We consider two cases, depending on whether x and y are adjacent or not. Suppose first that x and y are adjacent. We claim that in this case at least one of G_1 and G_2 is nonplanar. If both G_1 and G_2 are planar, then there exist planar embeddings of these two graphs in which xy is on the boundary of the exterior region in each embedding. This, however, implies that G itself is planar, which is impossible. Thus G_1, say, is nonplanar. Since G_1 is a subgraph of G, it follows that G_1 contains no subgraph that is a subdivision of K_5 or $K_{3,3}$. However, the size of G_1 is less than the size of G, which contradicts the defining property of G. Hence x and y must be nonadjacent.

Let f be the edge obtained by joining x and y, and let $H_i = G_i + f$ for $i = 1, 2$. If H_1 and H_2 are both planar, then, as above, there is a planar embedding of $G + f$ and of G as well. Since this is impossible, at least one of H_1 and H_2 is nonplanar, say H_1. Because the size of H_1 is less than the size of G, the graph H_1 contains a subgraph F that is a subdivision of K_5 or $K_{3,3}$. Since G_1 contains no such subgraph, it follows that $f \in E(F)$. Let P be an $x - y$ path in F_2. By replacing f in F by P, we obtain a subgraph of G that is a subdivision of K_5 or $K_{3,3}$. This produces a contradiction. Hence, as claimed, G is 3-connected.

To summarize then, G is a nonplanar graph of minimum size containing no subgraph that is a subdivision of K_5 or $K_{3,3}$ and, as we just saw, G is 3-connected. Let $e = uv$ be an edge of G. Then $H = G - e$ is planar. Let there be given a planar embedding of H. Since G is 3-connected, H is 2-connected. By Theorem 2.4, there are cycles in H containing both u and v. Among all such cycles, let

$$C = (u = v_0, v_1, \ldots, v_\ell = v, \ldots, v_k = u)$$

be one for which the number of regions interior to C is maximum.

It is convenient to define two subgraphs of H. The *exterior subgraph* of H is the subgraph induced by those edges lying exterior to C and the *interior subgraph* of H is the subgraph induced by those edges lying interior to C. Both subgraphs exist, for otherwise the edge e could be added either to the exterior or interior subgraph of H so that the resulting graph (namely G) is planar.

No two distinct vertices of $\{v_0, v_1, \ldots, v_\ell\}$ or of $\{v_\ell, v_{\ell+1}, \ldots, v_k\}$ are connected by a path in the exterior subgraph of H, for otherwise there is a cycle in H containing u and v and having more regions interior to it than C has. Since G is nonplanar, there must be a $v_s - v_t$ path P in the exterior subgraph of H, where $0 < s < \ell < t < k$, such that v_s and v_t are the only vertices of P that belong to C. (See Figure 6.11.) Necessarily, no vertex of P different from v_s

Figure 6.11: A step in the proof of Theorem 6.18

and v_t is adjacent to a vertex of C or is even connected to a vertex of C by a path, all of whose edges belong to the exterior subgraph of H.

Let S be the set of vertices on C different from v_s and v_t, that is,

$$S = V(C) - \{v_s, v_t\},$$

and let H_1 be the component of $H - S$ that contains P. By the defining property of C, the subgraph H_1 cannot be moved to the interior of C in a plane manner. This fact together with the fact that $G = H + e$ is nonplanar implies that the interior subgraph of H must contain one of the following:

(1) A $v_a - v_b$ path with $0 < a < s$ and $\ell < b < t$ such that only v_a and v_b belong to C. (See Figure 6.12(a).)

(2) A vertex w not on C that is connected to C by three internally disjoint paths such that the terminal vertex of one such path P' is one of v_0, v_s, v_ℓ and v_t. If, for example, the terminal vertex of P' is v_0, then the terminal vertices of the other two paths are v_a and v_b, where $s \leq a < \ell$ and $\ell < b \leq t$ where not both $a = s$ and $b = t$ occur. (See Figure 6.12(b).) If the terminal vertex of P' is one of v_s, v_ℓ and v_t, then there are corresponding bounds for a and b for the terminal vertices of the other two paths.

(3) A vertex w not on C that is connected to C by three internally disjoint paths P_1, P_2 and P_3 such that the terminal vertices of these paths are three of the four vertices v_0, v_s, v_ℓ and v_t, say v_0, v_ℓ and v_s, respectively, together with a $v_c - v_t$ path P_4 ($v_c \neq v_0, v_\ell, w$), where v_c is on P_1 or P_2 and P_4 is disjoint from P_1, P_2 and C except for v_c and v_t. (See Figure 6.12(c).) The remaining choices for P_1, P_2 and P_3 produce three analogous cases.

(4) A vertex w not on C that is connected to v_0, v_s, v_ℓ and v_t by four internally disjoint paths. (See Figure 6.12(d).)

Figure 6.12: Situations (1)-(4) in the proof of Theorem 6.18

In the first three cases, there is a subgraph of G that is a subdivision of $K_{3,3}$; while in the fourth case, there is a subgraph of G that is a subdivision of K_5. This is a contradiction. ■

While it is rarely easy to use Kuratowski's theorem to test a graph for planarity, a number of efficient algorithms have been developed that determine whether a graph is planar, including linear-time algorithms, the first of which was obtained by John Hopcropt and Robert Tarjan [121].

As a consequence of Kuratowski's theorem, the 4-regular graph G shown in Figure 6.13(a) is nonplanar since G contains the subgraph H in Figure 6.13(b) that is a subdivision of $K_{3,3}$.

Minors of Graphs

There is another characterization of planar graphs closely related to that given in Kuratowski's theorem. Before presenting this theorem, it is useful to introduce some additional terminology. If two adjacent vertices u and v in a

Figure 6.13: A nonplanar graph

graph G are identified, then we say that we have **contracted** the edge uv. For the graph G of Figure 6.14, the graph G' is obtained by contracting the edge uv in G and where G'' is obtained by contracting the edge wy in G'.

Figure 6.14: Contracting an edge

When dealing with edge contractions, it is often the case that we begin with a graph G, contract an edge in G to obtain a graph G', contract some edge in G' to obtain another graph G'', and so on, until finally arriving at a graph H. Any such graph H can be obtained in a different and perhaps simpler manner. In particular, H can be obtained from G by a succession of edge contractions if and only if the vertex set of H is the set of elements in a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$ where each induced subgraph $G[V_i]$ is connected and V_i is adjacent to V_j $(i \neq j)$ if some vertex in V_i is adjacent to some vertex in V_j in G. For example, in the graph G of Figure 6.14, if we were to let

$$V_1 = \{t\},\ V_2 = \{u, v\},\ V_3 = \{x\} \text{ and } V_4 = \{w, y\},$$

then the resulting graph H is shown in Figure 6.15. This is the graph G'' of Figure 6.14 obtained by successively contracting the edge uv in G and then the edge wy in G'.

A graph H is called a **minor** of a graph G if either $H \cong G$ or a graph isomorphic to H can be obtained from G by a succession of edge contractions, edge deletions and vertex deletions (in any order). Equivalently, H is a minor of G if $H \cong G$ or H can be obtained from a subgraph of G by a succession of

Figure 6.15: Edge contractions

edge contractions. Consequently, a graph G is a minor of itself. If H is a minor of G such that $H \ncong G$, then H is called a **proper minor** of G.

The planar graph H of Figure 6.15 is a minor of the planar graph G of that figure. In fact, every minor of a planar graph is planar.

Consider next the graph G_1 of Figure 6.16, where

$$V_1 = \{t_1, t_2\},\ V_2 = \{u_1, u_2, u_3, u_4\},$$
$$V_3 = \{v_1\},\ V_4 = \{w_1, w_2, w_3\},$$
$$V_5 = \{x_1, x_2\},\ V_6 = \{y_1\}, \text{ and } V_7 = \{z_1\}.$$

Then the graph H_1 of Figure 6.16 can be obtained from G_1 by successive edge contractions. Thus H_1 is a minor of G_1. By deleting the edge V_2V_6 and the vertices V_6 and V_7 from H_1 (or equivalently, deleting V_6 and V_7 from H_1), we see that K_5 is also a minor of G_1.

Figure 6.16: Minors of graphs

The example in Figure 6.16 serves to illustrate the following observation.

Theorem 6.19 *If a graph G is a subdivision of a graph H, then H is a minor of G.*

The converse of Theorem 6.19 is not true, however (see Exercise 10). The following is an immediate consequence of Theorem 6.19.

Theorem 6.20 *If G is a nonplanar graph, then K_5 or $K_{3,3}$ is a minor of G.*

Wagner's Theorem

The German mathematician Klaus Wagner (1910–2000) showed that the converse of Theorem 6.20 is true [234] in 1937, only a year after obtaining his Ph.D. from Universität zu Köln (University of Cologne), thereby giving another characterization of planar graphs.

Theorem 6.21 (**Wagner's Theorem**) *A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.*

Proof. We have already mentioned (in Theorem 6.20) that if a graph G is nonplanar, then either K_5 or $K_{3,3}$ is a minor of G. For the converse, suppose that G is a graph having K_5 or $K_{3,3}$ as a minor. If G were planar, then every minor of G is planar, contradicting the assumption that K_5 or $K_{3,3}$ is a minor of G. Thus G is nonplanar. ■

More specifically, Theorem 6.21 can be proved with the aid of the following result.

Theorem 6.22 *Let G be a graph.*

(*a*) *If G has $K_{3,3}$ as a minor, then G contains a subdivision of $K_{3,3}$.*

(*b*) *If G has K_5 as a minor, then G contains either a subdivision of K_5 or a subdivision of $K_{3,3}$.*

Proof. Suppose first that $H = K_{3,3}$ is a minor of G. The graph H can be obtained by first deleting edges and vertices of G (if necessary), obtaining a connected graph G', and then by a succession of edge contractions in G'. We show, in this case, that G' contains a subgraph that is a subdivision of $K_{3,3}$ and G', and G as well, is nonplanar.

Denote the vertices of H by U_i and W_i $(1 \le i \le 3)$, where $\{U_1, U_2, U_3\}$ and $\{W_1, W_2, W_3\}$ are the partite sets of H. Since H is obtained from G' by a succession of edge contractions, the subgraphs

$$F_i = G'[U_i] \text{ and } H_i = G'[W_i]$$

are connected. Since $U_iW_j \in E(H)$ for $1 \le i, j \le 3$, there is a vertex $u_{i,j} \in U_i$ that is adjacent in H to a vertex $w_{i,j} \in W_j$. Among the vertices $u_{i,1}, u_{i,2}, u_{i,3}$ in U_i, two or possibly all three may represent the same vertex. If $u_{i,1} = u_{i,2} = u_{i,3}$, then set $u_{i,j} = u_i$; if two of $u_{i,1}, u_{i,2}, u_{i,3}$ are the same, say $u_{i,1} = u_{i,2}$, then

set $u_{i,1} = u_i$; if $u_{i,1}, u_{i,2}$ and $u_{i,3}$ are distinct, then set u_i to be a vertex in U_i that is connected to $u_{i,1}, u_{i,2}$ and $u_{i,3}$ by internally disjoint paths in F_i. (Possibly $u_i = u_{i,j}$ for some j.) We proceed in the same manner to obtain vertices $w_i \in W_i$ for $1 \leq i \leq 3$. The subgraph of G induced by the nine edges $u_i w_j$ together with the edge sets of all of the previously mentioned paths in F_i and H_j $(1 \leq i, j \leq 3)$ is a subdivision of $K_{3,3}$.

Next, suppose that $H = K_5$ is a minor of G. Then H can be obtained by first deleting edges and vertices of G (if necessary), obtaining a connected graph G', and then by a succession of edge contradictions in G'. We show in this case that either G' contains a subgraph that is a subdivision of K_5 or G' contains a subgraph that is a subdivision of $K_{3,3}$.

We may denote the vertices of H by V_i $(1 \leq i \leq 5)$, where $G_i = G'[V_i]$ is a connected subgraph of G' and each subgraph G_i contains a vertex that is adjacent to G_j for each pair i, j of distinct integers where $1 \leq i, j \leq 5$. For $1 \leq i \leq 5$, let $v_{i,j}$ be a vertex of G_i that is adjacent to a vertex of G_j, where $1 \leq j \leq 5$ and $j \neq i$.

For a fixed integer i with $1 \leq i \leq 5$, if the vertices $v_{i,j}$ $(i \neq j)$ represent the same vertex, then denote this vertex by v_i. If three of the four vertices $v_{i,j}$ are the same, then we also denote this vertex by v_i. If two of the vertices $v_{i,j}$ are the same, the other two are distinct, and there exist internally disjoint paths from the coinciding vertices to the other two vertices, then we denote the two coinciding vertices by v_i. If the vertices $v_{i,j}$ are distinct and G_i contains a vertex from which there are four internally disjoint paths (one of which may be trivial) to the vertices $v_{i,j}$, then denote this vertex by v_i. Hence there are several instances in which we have defined a vertex v_i. Should v_i be defined for all i $(1 \leq i \leq 5)$, then G' (and therefore G as well) contains a subgraph that is a subdivision of K_5.

We may assume then that for one or more integers i $(1 \leq i \leq 5)$, the vertex v_i has not been defined. For each such i, there exist distinct vertices u_i and w_i, each of which is connected to two of the vertices $v_{i,j}$ by internally disjoint (possibly trivial) paths, while u_i and w_i are connected by a path none of whose internal vertices are the vertices $v_{i,j}$ and where every two of the five paths have only u_i or w_i in common. If two of the vertices $v_{i,j}$ coincide, then we denote this vertex by u_i. If the remaining two vertices $v_{i,j}$ should also coincide, then we denote this vertex by w_i. We may assume that $i = 1$, that u_1 is connected to $v_{1,2}$ and $v_{1,3}$ and that w_1 is connected to $v_{1,4}$ and $v_{1,5}$, as described above. Denote the edge set of these paths by E_1.

We now consider G_2. If $v_{2,1} = v_{2,4} = v_{2,5}$, then let w_2 be this vertex and set $E_2 = \emptyset$; otherwise, there is a vertex w_2 of G_2 (which may coincide with $v_{2,1}$, $v_{2,4}$ or $v_{2,5}$) connected by internally disjoint (possibly trivial) paths to the distinct vertices in $\{v_{2,1}, v_{2,4}, v_{2,5}\}$. We then let E_2 denote the edge set of these paths. Similarly, the vertices w_3, u_2, and u_3 and the sets E_3, E_4, and E_5 are defined with the aid of the sets $\{v_{3,1}, v_{3,4}, v_{3,5}\}$, $\{v_{4,1}, v_{4,4}, v_{4,5}\}$, and $\{v_{5,1}, v_{5,2}, v_{5,3}\}$, respectively. The subgraph of G' induced by the union of the sets E_i and the edges $v_{i,j}v_{j,i}$ contains a subdivision of $K_{3,3}$ with partite sets $\{u_1, u_2, u_3\}$ and

$\{w_1, w_2, w_3\}$. ■

In the proof of Theorem 6.22, it was shown that if K_5 is a minor of a graph G, then G contains a subdivision of K_5 or a subdivision of $K_{3,3}$. In other words, we were unable to show that G necessarily contains a subdivision of K_5. There is good reason for this, which is illustrated in the next example.

The Petersen graph P has order $n = 10$ and size $m = 15$. Since $m < 3n - 6$, no conclusion can be drawn from Theorem 6.3 regarding the planarity or nonplanarity of P. Nevertheless, the Petersen graph is, in fact, nonplanar. Theorems 6.18 and 6.21 give two ways to establish this fact. Figures 6.17(a) and 6.17(b) show P drawn in two ways. Since $P - x$ (shown in Figure 6.17(c)) is a subdivision of $K_{3,3}$, the Petersen graph is nonplanar. The partition $\{V_1, V_2, \ldots, V_5\}$ of $V(P)$ shown in Figure 6.17(d), where $V_i = \{u_i, v_i\}$, $1 \le i \le 5$, shows that K_5 in Figure 6.17(d) is a minor of P and is therefore nonplanar. Since P is a cubic graph, there is no subgraph of P that is a subdivision of K_5, however.

Figure 6.17: Showing that the Petersen graph is nonplanar

Outerplanar Graphs

We now turn our attention to a special class of planar graphs. A graph G is **outerplanar** if there exists a planar embedding of G so that every vertex of G lies on the boundary of the exterior region. Actually, if there is a planar embedding of G so that every vertex of G lies on the boundary of the same region of G, then G is outerplanar. The following two results provide characterizations of outerplanar graphs.

Theorem 6.23 *A graph G is outerplanar if and only if the join $G \vee K_1$ is planar.*

Proof. Let G be an outerplanar graph and suppose that G is embedded in the plane such that every vertex of G lies on the boundary of the exterior region. Then a vertex v can be placed in the exterior region of G and joined to all vertices of G in such a way that a planar embedding of $G \vee K_1$ results. Thus $G \vee K_1$ is planar.

For the converse, assume that G is a graph such that $G \vee K_1$ is planar. Hence $G \vee K_1$ contains a vertex v that is adjacent to every vertex of G. Let there be a planar embedding of $G \vee K_1$. Upon deleting the vertex v, we arrive at a planar embedding of G in which all vertices of G lie on the boundary of the same region. Thus G is outerplanar. ■

The following characterization of outerplanar graphs is analogous to the characterization of planar graphs stated in Kuratowski's theorem.

Theorem 6.24 *A graph G is outerplanar if and only if G contains no subgraph that is a subdivision of K_4 or $K_{2,3}$.*

Proof. Suppose first that there exists some outerplanar graph G that contains a subgraph H that is a subdivision of K_4 or $K_{2,3}$. By Theorem 6.23, $G \vee K_1$ is planar. Since $H \vee K_1$ is a subdivision of K_5 or a subdivision of $K_{3,3}$, it follows that $G \vee K_1$ contains a subgraph that is a subdivision of K_5 or a subdivision of $K_{3,3}$ and so is nonplanar. This produces a contradiction.

For the converse, assume, to the contrary, that there exists a graph G that is not outerplanar but contains no subgraph that is a subdivision of K_4 or $K_{2,3}$. By Theorem 6.23, $G \vee K_1$ is not planar but contains a subgraph that is a subdivision of K_5 or $K_{3,3}$. This contradicts Theorem 6.18. ■

An outerplanar graph G is **maximal outerplanar** if the addition to G of any edge joining two nonadjacent vertices of G results in a graph that is not outerplanar. Necessarily then, there is a planar embedding of a maximal outerplanar graph G of order at least 3, the boundary of whose exterior region is a Hamiltonian cycle of G. In fact, every maximal outerplanar graph of order 3 or more is nearly maximal planar. We now describe some other facts about outerplanar graphs.

Theorem 6.25 *Every nontrivial outerplanar graph contains at least two vertices of degree* 2 *or less.*

Proof. Let G be a nontrivial outerplanar graph. The result is obvious if the order of G is 4 or less, so we may assume that the order of G is at least 5. Add edges to G, if necessary, to obtain a maximal outerplanar graph. Thus the boundary of the exterior region of G is a Hamiltonian cycle of G. Among the chords of C, let uv be one such that uv and a $u - v$ path on C produce a cycle containing a minimum number of interior regions. Necessarily, this minimum is 1. Then the degree of the remaining vertex y on the boundary of this region is 2. There is such a chord wx of C on the other $u - v$ path of C, producing another vertex z of degree 2. In G, the degrees of y and z are therefore 2 or less. ■

Theorem 6.26 *The size of every outerplanar graph of order $n \geq 2$ is at most* $2n - 3$.

Proof. We proceed by induction on n. The result clearly holds for $n = 2$. Assume that the size of every outerplanar graph of order k, where $k \geq 2$, is at most $2k - 3$ and let G be a outerplanar graph of order $k + 1$. We show that the size of G is at most $2(k + 1) - 3 = 2k - 1$. By Theorem 6.25, G contains a vertex v of degree at most 2. Then $G - v$ is an outerplanar graph of order k. By the induction hypothesis, the size of $G - v$ is at most $2k - 3$. Hence the size of G is at most

$$2k - 3 + \deg v \leq 2k - 3 + 2 = 2k - 1,$$

as desired. ■

In view of Theorem 6.26, an outerplanar graph of order $n \geq 2$ is maximal outerplanar if and only if its size is $2n - 3$.

Exercises for Section 6.2

1. Let U be a subset of the vertex set of a graph G, and suppose that $H = G[U]$ is an induced subgraph of G of order k. Let H' be the graph obtained by inserting a vertex of degree 2 into every edge of H. Let G' be the graph obtained by inserting a vertex of degree 2 into every edge of G.

   (a) Prove or disprove: If G is planar, then H' is planar.

   (b) What is $G'[U]$?

2. Let T be a tree of order at least 4, and let $e_1, e_2, e_3 \in E(\overline{T})$. Prove that $T + e_1 + e_2 + e_3$ is planar.

3. Let T be a tree of order at least 5, and let $e_1, e_2, \cdots, e_5 \in E(\overline{T})$. Let $G = T + \{e_1, e_2, \cdots, e_5\}$. Prove that if G does not contain a subdivision of $K_{3,3}$, then G is planar.

4. (a) Determine the order n, the size m and the number $3n - 6$ for the graph $K_4 \,\square\, K_2$.
   (b) What does the information in (a) say about the planarity of $K_4 \,\square\, K_2$?
   (c) Is $K_4 \,\square\, K_2$ planar or nonplanar?

5. Determine all graphs G of order $n \geq 5$ and size $m = 3n - 5$ such that for each edge e of G, the graph $G - e$ is planar.

6. Let $S_{a,b}$ denote the double star in which the degrees of the two vertices that are not end-vertices are a and b. Determine all pairs a, b of integers such that $\overline{S}_{a,b}$ is planar.

7. A nonplanar graph G of order 7 has the property that $G - v$ is planar for every vertex v of G.
   (a) Show that G does not contain $K_{3,3}$ as a subgraph.
   (b) Give an example of a graph with this property.

8. Determine all integers $n \geq 3$ such that $\overline{C}_n$ is planar.

9. Determine all integers $n \geq 3$ such that C_n^2 is nonplanar.

10. Show that the converse of Theorem 6.19 is not, in general, true.

11. It has been observed that if a graph H is a minor of a planar graph, then H is planar. Prove or disprove: If a minor H of a graph G is planar, then G is planar.

12. Let G be the graph shown in Figure 6.18.
   (a) Show that G contains $K_{3,3}$ as a subgraph.
   (b) Show that G does not contain a subdivision of K_5 as a subgraph.
   (c) Show that K_5 is a minor of G.

G :

Figure 6.18: The graph G in Exercise 12

13. Let G be a 4-regular graph of order 10 and size m.

(a) What can be deduced about the planarity of G by comparing the numbers m and $3n - 6$?

(b) Prove or disprove: There is a planar 2-connected 4-regular graph of order 10.

(c) Show that the 2-connected 4-regular graph H of order 10 shown in Figure 6.19 does not contain K_5 as a subgraph but does contain K_5 as a minor. What can you conclude from this?

H :

Figure 6.19: A 2-connected 4-regular graph of order 10

(d) Prove that if G is a graph of order $n \geq 5$ and size $m \geq 3n - 5$, then G need not contain K_5 as a subgraph but must contain a subgraph with minimum degree 4.

14. (a) What is the minimum possible order of a graph G containing only vertices of degree 3 and degree 4 and an equal number of each such that G contains a subdivision of K_5?

(b) Does the graph H of Figure 6.20 contain a subdivision of K_5 or a subdivision of $K_{3,3}$?

(c) Does the graph H of Figure 6.20 contain K_5 or $K_{3,3}$ as a minor?

(d) Is the graph H of Figure 6.20 planar or nonplanar?

H :

Figure 6.20: The graph H in Exercise 14

15. Determine all connected graphs G of order $n \geq 4$ such that $G \vee K_1$ is outerplanar.

16. Use Theorem 6.23 to prove Theorem 6.26: *The size of every outerplanar graph of order $n \geq 2$ is at most $2n - 3$.*

6.3 The Crossing Number of a Graph

While every graph is either planar or nonplanar, there are several ways of measuring just how close to or far from being planar a nonplanar graph may be. In this section, we discuss one of these measures. In the next chapter, we discuss the best known measure of nonplanarity.

Nonplanar graphs cannot, of course, be embedded in the plane. Hence, whenever a nonplanar graph is drawn in the plane, some of its edges must cross. This rather simple observation leads to the concept we discuss in this section.

The **crossing number** $\mathrm{cr}(G)$ of a graph G is the minimum number of crossings (of its edges) among the drawings of G in the plane. Before proceeding further, we comment on the assumptions we are making regarding all drawings under consideration. In particular, we assume that

- adjacent edges never cross
- two nonadjacent edges cross at most once
- no edge crosses itself
- no more than two edges cross at a point of the plane and
- the (open) arc in the plane corresponding to an edge of the graph contains no vertex of the graph.

A few observations will prove useful. If $G \subseteq H$, then $\mathrm{cr}(G) \leq \mathrm{cr}(H)$; while if H is a subdivision of G, then $\mathrm{cr}(G) = \mathrm{cr}(H)$. A graph G is planar if and only if $\mathrm{cr}(G) = 0$. In particular, if G is a maximal planar graph of order $n \geq 3$ and size m, then $m = 3n - 6$ and $\mathrm{cr}(G) = 0 = m - 3n + 6$. If $m > 3n - 6$ and so $m - 3n + 6 > 0$, then G is nonplanar and so $\mathrm{cr}(G) \geq 1$. In fact, the number $m - 3n + 6$ provides a lower bound for the crossing number of a graph of order $n \geq 3$ and size m.

Theorem 6.27 *If G is a graph of order $n \geq 3$ and size m, then*

$$\mathrm{cr}(G) \geq m - 3n + 6.$$

Proof. Let there be given a drawing of G in the plane with $\mathrm{cr}(G) = c$ crossings. At each crossing, a new vertex is introduced, producing a new plane graph

H of order $n+c$ and size $m+2c$. Since H is planar, it follows by Theorem 6.3 that

$$m + 2c \leq 3(n+c) - 6.$$

Thus $\mathrm{cr}(G) = c \geq m - 3n + 6$. ■

Crossing Numbers of Complete Graphs

One class of graphs whose crossing number has been a subject of study is complete graphs. By Theorem 6.27, it follows for $n \geq 3$ that

$$\mathrm{cr}(K_n) \geq \binom{n}{2} - 3n + 6 = \frac{(n-3)(n-4)}{2}. \tag{6.4}$$

Richard K. Guy [102] discovered an even better lower bound for $cr(K_n)$.

Theorem 6.28 *For $n \geq 5$, $\mathrm{cr}(K_n) \geq \frac{1}{5}\binom{n}{4}$.*

Proof. We proceed by induction on n. For $n = 5$, we have $\frac{1}{5}\binom{n}{4} = \frac{1}{5}\binom{5}{4} = 1$. Since K_5 is nonplanar, $\mathrm{cr}(K_5) \geq 1$. In fact, the drawing of K_5 in Figure 6.21 with one crossing shows that $\mathrm{cr}(K_5) = 1$.

Figure 6.21: A drawing of K_5 with one crossing

Assume that $\mathrm{cr}(K_{n-1}) \geq \frac{1}{5}\binom{n-1}{4}$ for an integer $n \geq 6$. Let there be a drawing of K_n in the plane with $\mathrm{cr}(K_n)$ crossings. When a vertex of K_n is deleted, a drawing of K_{n-1} is obtained, where the number of crossings in this copy of K_{n-1} is at least $\mathrm{cr}(K_{n-1})$. Hence the n vertex-deleted subgraphs of K_n produce n graphs K_{n-1} having a total of at least $n\,\mathrm{cr}(K_{n-1})$ crossings.

A crossing in K_n involves two nonadjacent edges, say uv and xy. This crossing occurs in every vertex-deleted subgraph K_{n-1} of K_n except when u, v, x or y is deleted, that is, this crossing occurs in $n-4$ subgraphs K_{n-1} of K_n and so

$$(n-4)\,\mathrm{cr}(K_n) \geq n\,\mathrm{cr}(K_{n-1}) \geq \frac{n}{5}\binom{n-1}{4}.$$

Hence

$$\mathrm{cr}(K_n) \geq \frac{n}{5(n-4)}\binom{n-1}{4} = \frac{1}{5}\binom{n}{4},$$

as desired. ■

The lower bounds for $\mathrm{cr}(K_n)$ given in (6.4) and in Theorem 6.28 are the same when $n = 5$ and $n = 6$, while

$$\frac{1}{5}\binom{n}{4} > \frac{(n-3)(n-4)}{2}$$

when $n \geq 7$. Thus the lower bound $\mathrm{cr}(K_n) \geq \frac{1}{5}\binom{n}{4}$ is an improvement over that given in (6.4). When $n = 6$, these bounds state that $\mathrm{cr}(K_6) \geq 3$. The drawing of K_6 with three crossings in Figure 6.22 shows that $\mathrm{cr}(K_6) = 3$.

Figure 6.22: A drawing of K_6 with three crossings

It has been shown by Jaroslav Blažek and Milan Koman [22] and Richard K. Guy [101], among others, that for complete graphs,

$$\mathrm{cr}(K_n) \leq \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor. \tag{6.5}$$

Guy conjectured, in fact, that the upper bound in (6.5) is, in fact, the crossing number of K_n for all positive integers n. As far as known results are concerned, the best obtained is the following.

Theorem 6.29 *For $1 \leq n \leq 12$,*

$$\mathrm{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor. \tag{6.6}$$

Guy [102] established Theorem 6.29 for $1 \leq n \leq 10$, while Shengjun Pan and R. Bruce Richter [169] verified Theorem 6.29 for $n = 11$ and $n = 12$.

We now consider the class P_n^k of the kth powers of paths for various values of k and n. Of course, $P_n^1 = P_n$. For $n \geq 2$, the square P_n^2 of P_n is a maximal outerplanar graph and for $n \geq 3$, the cube P_n^3 of P_n is a maximal planar graph (see Exercise 2). In the case of 4th powers of paths, the following result obtained independently by Gary Chartrand, Futaba Okamoto and Ping Zhang [40] and Wenping Zheng, Xiaohui Lin, Yuansheng Yang and Gui Yang [248] gives the crossing number. The proof we present is that given in [40].

Theorem 6.30 *For $n \geq 4$, the crossing number of P_n^4 is*

$$\mathrm{cr}(P_n^4) = n - 4.$$

Proof. Since $P_4^4 = K_4$ and $P_5^4 = K_5$, the result is true for $n = 4, 5$. Hence we may assume that $n \geq 6$. Since the order of P_n^4 is n and the size of P_n^4 is $m = 4n - 10$, it follows by Theorem 6.27 that

$$\mathrm{cr}(P_n^4) \geq m - 3n + 6 = (4n - 10) - 3n + 6 = n - 4.$$

It remains to show that $\mathrm{cr}(P_n^4) \leq n - 4$. It has already been observed that P_n^3 is maximal planar. Let $P_n = (u_1, u_2, \ldots, u_n)$. We claim in fact that there is a a planar embedding of P_n^3 such that

(1) for each integer k with $6 \leq k \leq n$, the edge $u_{k-3}u_{k-1}$ lies on the boundaries of the two regions R_k and R'_k whose boundaries have vertex sets $\{u_{k-4}, u_{k-3}, u_{k-1}\}$ and $\{u_{k-3}, u_{k-1}, u_k\}$, respectively, and

(2) the edge $u_{n-2}u_n$ lies on the boundaries of the two regions R_{n+1} and R^*_{n+1} whose boundaries have vertex sets $\{u_{n-3}, u_{n-2}, u_n\}$ and $\{u_{n-2}, u_{n-1}, u_n\}$, respectively.

We verify this claim by induction on n. As the drawing P_6^3 in Figure 6.23 shows, the claim is true for $n = 6$.

Figure 6.23: A planar embedding of P_6^3

Assume that the claim is true for some integer $n \geq 6$. Hence there is a planar embedding of P_n^3 with the properties (1) and (2) so that the boundary of the region R^*_{n+1} contains the three vertices u_{n-2}, u_{n-1} and u_n. We now place a new vertex u_{n+1} into the region R^*_{n+1} and join u_{n+1} to u_{n-2}, u_{n-1} and u_n. Thus the edge $u_{n-1}u_{n+1}$ lies on the boundaries of two regions R_{n+2} and R^*_{n+2} with the boundary vertex sets

$$\{u_{n-2}, u_{n-1}, u_{n+1}\} \text{ and } \{u_{n-1}, u_n, u_{n+1}\},$$

respectively. This verifies (2) for P_{n+1}^3. Furthermore, the edge $u_{n-2}u_n$ lies on the boundaries of the two regions R_{n+1} and R'_{n+1} with boundary vertex sets

$$\{u_{n-3}, u_{n-2}, u_n\} \text{ and } \{u_{n-2}, u_n, u_{n+1}\},$$

respectively. By the induction hypothesis, (1) is satisfied for P_{n+1}^3 and the claim is verified for every integer $n \geq 6$.

From the claim, it now follows that there is a planar embedding of the maximal planar graph P_n^3 for $n \geq 6$ such that $2n-8$ of the $2n-4$ regions can be partitioned into $n-4$ pairs R_k, R'_k of regions ($6 \leq k \leq n$) for which the edge $u_{k-3}u_{k-1}$ lies on the boundaries of these two regions and the distance between the remaining two vertices u_{k-4} and u_k on the boundaries of these regions is 4 in P_n. Hence for $6 \leq k \leq n$, the edge $u_{k-4}u_k$ can be drawn in R_k and R'_k crossing only the edge $u_{k-3}u_{k-1}$, thereby producing $n-4$ crossings such that these are the only crossings in P_n^4 and so $\mathrm{cr}(P_n^4) \leq n-4$. Therefore, $\mathrm{cr}(P_n^4) = n-4$. ■

The Turán Brick-Factory Problem

We now turn to the crossing number of complete bipartite graphs. The problem of determining $\mathrm{cr}(K_{s,t})$ has a rather curious history. It is sometimes referred to as the **Turán Brick-Factory Problem**, named for the Hungarian mathematician Paul Turán (1910–1976).

Born in Budapest, Hungary, Paul Turán displayed remarkable mathematical ability at a very early age. Turán was one of many young students in Budapest who studied graph theory under Dénes König. Turán met Paul Erdős in September 1930 and became and remained friends with him. Turán received a Ph.D. in 1935 from the University of Budapest, with a dissertation in number theory. By the end of 1935, Turán had seven papers in print. Despite his outstanding record at such an early age, Turán had great difficulty securing a faculty position because of his Jewish heritage. He could only make a living as a private mathematics tutor although he continued his research.

While Turán finally secured a position in 1938 (as a school teacher), his personal situation had grown worse. After the German invasion of Poland which began World War II, Hungary was not involved in the war at first but was nevertheless greatly influenced by Nazi policies. In 1940 Turán was sent to a labor camp. Indeed, Turán was in and out of several labor camps during the war. Turán himself [220] wrote:

> *We worked near Budapest, in a brick factory. There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all the storage yards. The bricks were carried on small wheeled trucks to the storage yards. All we had to do was to put the bricks on the trucks at the kilns, push the trucks to the storage yards, and unload them there. We had a reasonable piece rate for the trucks, and the work itself was not difficult; the trouble was only at the crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short this caused a lot of trouble and loss of time which was precious to all of us. We were all sweating and cursing at such occasions, I too; but nolens volens the idea occurred to me that this loss of time could have been minimized if the number of crossings of*

the rails had been minimized. But what is the minimum number of crossings? I realized after several days that the actual situation could have been improved, but the exact solution of the general problem with s kilns and t storage yards seemed to be very difficult ... the problem occurred to me again ... at my first visit to Poland where I met Zarankiewicz. I mentioned to him my 'brick-factory'–problem ... and Zarankiewicz thought to have solved (it). But Ringel found a gap in his published proof, which nobody has been able to fill so far – in spite of much effort. This problem has also become a notoriously difficult unsolved problem ....

Crossing Numbers of Complete Bipartite Graphs

Kazimierz Zarankiewicz [247] thus thought that he had proved

$$\mathrm{cr}(K_{s,t}) = \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{t-1}{2} \right\rfloor \tag{6.7}$$

but, in actuality, he had only verified that the right hand expression of (6.7) is an upper bound for $\mathrm{cr}(K_{s,t})$. As it turned out, both Paul C. Kainen and Gerhard Ringel found flaws in Zarankiewicz's argument. Hence, (6.7) remains only a conjecture. The best general result on crossing numbers of complete bipartite graphs is the following, due to the combined work of Daniel Kleitman [135] and Douglas Woodall [243].

Theorem 6.31 *If s and t are positive integers with $s \le t$ and either* (1) $s \le 6$ *or* (2) $s = 7$ *and* $t \le 10$, *then*

$$\mathrm{cr}(K_{s,t}) = \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{t-1}{2} \right\rfloor .$$

It follows, therefore, from Theorem 6.31 that

$$\mathrm{cr}(K_{3,t}) = \left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{t-1}{2} \right\rfloor, \qquad \mathrm{cr}(K_{4,t}) = 2 \left\lfloor \tfrac{t}{2} \right\rfloor \left\lfloor \tfrac{t-1}{2} \right\rfloor,$$

$$\mathrm{cr}(K_{5,t}) = 4 \left\lfloor \frac{t}{2} \right\rfloor \left\lfloor \frac{t-1}{2} \right\rfloor \quad \text{and} \quad \mathrm{cr}(K_{6,t}) = 6 \left\lfloor \tfrac{t}{2} \right\rfloor \left\lfloor \tfrac{t-1}{2} \right\rfloor$$

for all t. For example, $\mathrm{cr}(K_{3,3}) = 1$, $\mathrm{cr}(K_{4,4}) = 4$, $\mathrm{cr}(K_{5,5}) = 16$, $\mathrm{cr}(K_{6,6}) = 36$ and $\mathrm{cr}(K_{7,7}) = 81$. A drawing of $K_{4,4}$ with four crossings is shown in Figure 6.24.

As would be expected, the situation regarding crossing numbers of complete k-partite graphs, $k \ge 3$, is even more complicated. For the most part, only bounds and highly specific results have been obtained in these cases. On the other hand, some of the proof techniques employed have been enlightening. As an example, the following result of Arthur T. White [237, p. 67] establishes the crossing number of $K_{2,2,3}$.

Figure 6.24: A drawing of $K_{4,4}$ with four crossings

Theorem 6.32 *The crossing number of $K_{2,2,3}$ is 2.*

Proof. The graph $K_{2,2,3}$ has order 7 and size 16. Suppose that $\mathrm{cr}(K_{2,2,3}) = c$. Since $K_{3,3}$ is nonplanar and $K_{3,3} \subseteq K_{2,2,3}$, it follows that $K_{2,2,3}$ is nonplanar and so $c \geq 1$. Let there be given a drawing of $K_{2,2,3}$ in the plane with c crossings. At each crossing we introduce a new vertex, producing a connected plane graph G of order $n = 7 + c$ and size $m = 16 + 2c$. By Theorem 6.3, $m \leq 3n - 6$.

Let u_1u_2 and v_1v_2 be two (nonadjacent) edges of $K_{2,2,3}$ that cross in the given drawing, giving rise to a new vertex. If G is a maximal planar graph, then $C = (u_1, v_1, u_2, v_2, u_1)$ is a cycle of G, implying that the subgraph induced by $\{u_1, u_2, v_1, v_2\}$ in $K_{2,2,3}$ is K_4. However, $K_{2,2,3}$ contains no such subgraph; thus, G is not a maximal planar graph and so $m < 3n - 6$. Therefore,

$$16 + 2c < 3(7 + c) - 6,$$

from which it follows that $c \geq 2$. The inequality $c \leq 2$ follows from the fact that there exists a drawing of $K_{2,2,3}$ with two crossings (see Figure 6.25). ■

Figure 6.25: A drawing of $K_{2,2,3}$ with two crossings

Crossing Numbers of Cartesian Products of Graphs

Other graphs whose crossing numbers have been investigated with little success are the n-cubes Q_n. Since Q_n is planar for $n = 1, 2, 3$, $\mathrm{cr}(Q_n) = 0$ for

each such n. Roger Eggleton and Richard Guy [67] have shown that $\mathrm{cr}(Q_4) = 8$ but $\mathrm{cr}(Q_n)$ is unknown for $n \geq 5$. One might observe that

$$Q_4 = K_2 \,\square\, K_2 \,\square\, K_2 \,\square\, K_2 = C_4 \,\square\, C_4$$

so that $\mathrm{cr}(C_4 \,\square\, C_4) = 8$. This raises the question of determining $\mathrm{cr}(C_s \,\square\, C_t)$ for $s, t \geq 3$. For the case $s = t = 3$, Frank Harary, Paul Kainen and Allen Schwenk [110] showed that $\mathrm{cr}(C_3 \,\square\, C_3) = 3$. Their proof consisted of the following three steps:

Step 1. Exhibiting a drawing of $C_3 \,\square\, C_3$ with three crossings so that $\mathrm{cr}(C_3 \,\square\, C_3) \leq 3$.

Step 2. Showing that $C_3 \,\square\, C_3 - e$ is nonplanar for every edge e of $C_3 \,\square\, C_3$ so that $\mathrm{cr}(C_3 \,\square\, C_3) \geq 2$.

Step 3. Showing, by case exhaustion, that it is impossible to have a drawing of $C_3 \,\square\, C_3$ with exactly two crossings so that $\mathrm{cr}(C_3 \,\square\, C_3) \geq 3$ (see Exercise 8).

Richard D. Ringeisen and Lowell W. Beineke [185] then significantly extended this result by determining $\mathrm{cr}(C_3 \,\square\, C_t)$ for all integers $t \geq 3$.

Theorem 6.33 *For each integer $t \geq 3$,*

$$\mathrm{cr}(C_3 \,\square\, C_t) = t.$$

Proof. We label the vertices of $C_3 \,\square\, C_t$ by the $3t$ ordered pairs $(0, j), (1, j)$ and $(2, j)$, where $j = 0, 1, \ldots, t-1$, and, for convenience, we let

$$u_j = (0, j),\ v_j = (1, j), \text{ and } w_j = (2, j).$$

First, we note that $\mathrm{cr}(C_3 \,\square\, C_t) \leq t$. This observation follows from the fact that there exists a drawing of $C_3 \,\square\, C_t$ with t crossings. A drawing of $C_3 \,\square\, C_4$ with four crossings is shown in Figure 6.26. Drawings of $C_3 \,\square\, C_t$ with t crossings for other values of t can be given similarly.

To complete the proof, we show that $\mathrm{cr}(C_3 \,\square\, C_t) \geq t$. We verify this by induction on $t \geq 3$. For $t = 3$, we recall the previously mentioned result $\mathrm{cr}(C_3 \,\square\, C_3) = 3$.

Assume that $\mathrm{cr}(C_3 \,\square\, C_k) \geq k$ for some integer $k \geq 3$, and consider the graph $C_3 \,\square\, C_{k+1}$. We show that $\mathrm{cr}(C_3 \,\square\, C_{k+1}) \geq k+1$. Let there be given a drawing of $C_3 \,\square\, C_{k+1}$ with $\mathrm{cr}(C_3 \,\square\, C_{k+1})$ crossings. We consider two cases.

Case 1. *No edge of any triangle $T_j = G[\{u_j, v_j, w_j\}]$, $j = 0, 1, \ldots, k$, is crossed.* For $j = 0, 1, \ldots, k$, define

$$H_j = G[\{u_j, v_j, w_j, u_{j+1}, v_{j+1}, w_{j+1}\}],$$

where the subscripts are expressed modulo $k+1$. We show that for each $j = 0, 1, \ldots, k$, the number of times edges of H_j are crossed totals at least 2. Since,

Figure 6.26: A drawing of $C_3 \square C_4$ with four crossings

by assumption, no triangle T_j has an edge crossed and since every edge not in any triangle T_j belongs to exactly one subgraph H_j, it will follow that there are at least $k+1$ crossings in the drawing because then every crossing of an edge in H_j involves either two edges of H_j or an edge of H_j and an edge of H_i for some $i \neq j$.

If two of the edges $u_j u_{j+1}$, $v_j v_{j+1}$ and $w_j w_{j+1}$ cross each other, then two edges of H_j are crossed. Assume then that no two edges of H_j cross each other. Thus, H_j is a plane subgraph in the drawing of $C_3 \square C_{k+1}$ (see Figure 6.27). The triangle T_{j+2} must lie within some region of H_j. If T_{j+2} lies in a region of H_j bounded by a triangle, say T_j, then at least one edge of the cycle $(u_0, u_1, \ldots, u_k, u_0)$, for example, must cross an edge of T_j, contradicting our assumption. Thus, T_{j+2} must lie in a region of H_j bounded by a 4-cycle, say $(u_j, u_{j+1}, w_{j+1}, w_j, u_j)$, without loss of generality. However, then edges of the cycle $(v_0, v_1, \ldots, v_k, v_0)$ must cross edges of the cycle $(u_j, u_{j+1}, w_{j+1}, w_j, u_j)$ at least twice and hence edges of H_j at least twice, as asserted.

Figure 6.27: The subgraph H_j in the proof of Theorem 6.33

Case 2. *Some triangle, say T_0, has at least one of its edges crossed.* Suppose that $\mathrm{cr}(C_3 \square C_{k+1}) < k+1$. Then the graph $C_3 \square C_{k+1} - E(T_0)$, which is a subdivision of $C_3 \square C_k$, is drawn with fewer than k crossings, contradicting the inductive hypothesis. ■

The only other result giving the crossing number of graphs $C_s \square C_t$ is the following formula by Beineke and Ringeisen [12].

Theorem 6.34 *For all $t \geq 4$,*

$$\mathrm{cr}(C_4 \square C_t) = 2t.$$

Beineke and Ringeisen [12] have also found a formula for $\mathrm{cr}(K_4 \square C_t)$.

Theorem 6.35 *For all $t \geq 4$,*

$$\mathrm{cr}(K_4 \square C_t) = 3t.$$

Fáry's Theorem

In a planar embedding of a graph G, an edge of G can be any curve, including a straight-line segment. The **rectilinear crossing number** $\overline{\mathrm{cr}}(G)$ of a graph G is the minimum number of crossings among all those drawings of G in the plane in which each edge is a straight-line segment. Since the crossing number $\mathrm{cr}(G)$ considers all drawings of G in the plane (not just those for which edges are straight-line segments), we have the obvious inequality

$$\mathrm{cr}(G) \leq \overline{\mathrm{cr}}(G). \tag{6.8}$$

Clearly, $\overline{\mathrm{cr}}(G) \geq 0$ for every planar graph G. However, an interesting feature of planar graphs is that they can be embedded in the plane so that every edge is a straight-line segment. Such an embedding is referred to as a **straight-line embedding**. This result is known as Fáry's Theorem but was proved independently by István Fáry [79], Sherman K. Stein [207] and Klaus Wagner [233].

Theorem 6.36 (**Fáry's Theorem**) *If G is a planar graph, then*

$$\overline{\mathrm{cr}}(G) = 0.$$

Proof. If the rectilinear crossing number of every maximal planar graph is 0, then the rectilinear crossing number of every planar graph is 0. Hence it suffices to prove the theorem for maximal planar graphs. This result is obvious for K_1 and K_2.

We prove by induction on $n \geq 3$ that for every maximal plane graph G of order n, the boundary of whose exterior region contains the vertices u, v and w, there exists a straight-line embedding of G each region of which has the same boundary as the given planar embedding of G and whose exterior region has boundary vertices u, v and w. The result is certainly true for $n = 3$ and $n = 4$. Assume that the statement is true for all maximal plane graphs of order k for

some integer $k \geq 4$. Let G be a maximal plane graph of order $k+1$ whose exterior region has boundary vertices u, v and w.

By Corollary 6.12, G contains a vertex $x \notin \{u, v, w\}$ such that $\deg x = r$ and $3 \leq r \leq 5$. Let $N_G(x) = \{x_1, x_2, \ldots, x_r\}$. Remove x from G and let R be the region of $G - x$ whose boundary vertices are $N_G(x)$. Add $r-3$ edges to the region R of $G - x$ so that a maximal plane graph G' results. Since G' is a maximal plane graph of order k, it follows by the induction hypothesis that there is a straight-line embedding of G' resulting in a graph G'' each region of which has the same boundary and whose exterior region has boundary vertices u, v and w.

Now remove the $r-3$ edges that were added to $G-x$ to produce a straight-line embedding G^* of $G - x$ such that the boundary of the region R^* with boundary vertices $N_G(x)$ is an r-gon, where $3 \leq r \leq 5$. If the r-gon is convex, then the vertex x can be added anywhere in R^* and joined to the vertices of $N_G(x)$ by straight-line segments, producing a straight-line embedding of G.

Suppose that the r-gon P is not convex and so $r = 4$ or $r = 5$. We then triangulate P by adding $r-3$ straight-line segments. If $r = 4$, then one straight-line segment is added. Place a new vertex x on this segment and remove this segment. Straight-line segments can then be drawn from x to each of the vertices x_i $(1 \leq i \leq 4)$, producing a straight-line embedding of G. See Figure 6.28.

Figure 6.28: A step ($r = 4$) in the proof of Theorem 6.36

If P is a pentagon, then two straight-line segments can be added to triangulate P, producing three triangles. One of the five vertices of P, say x_1, lies on each of these three triangles. One of these triangles, say T, is a neighbor of the other two triangles. Since a triangle is convex, a straight-line segment can be drawn in T and so in P to each of x_2, x_3, x_4 and x_5 (see Figure 6.29), again producing a straight-line embedding of G. ■

By Fáry's Theorem, there is a straight-line embedding of every planar graph. In the proof of this theorem, we used the fact that within the interior of each triangle, quadrilateral and pentagon, a point can be placed that can be joined by straight-line segments to the vertices of the polygon P so that they lie within the interior of P. Actually this is a special case of a result from geometry that is commonly called "The Art Gallery Problem."

Figure 6.29: A step ($r = 5$) in the proof of Theorem 6.36

The Art Gallery Problem

> *Suppose that a certain art gallery consists of a single large room with n walls on which paintings are hung. What is the minimum number of security guards that must be hired and stationed in the gallery to guarantee that for every painting hung on a wall there is a guard who has straight line vision of the artwork?*

This problem was posed in 1973 by the geometer Victor Klee after a discussion with Vašek Chvátal. It was shown by Chvátal [46], with a simpler proof by Steve Fisk [81], that no more than $\lfloor n/3 \rfloor$ guards are needed and that examples exist where $\lfloor n/3 \rfloor$ guards are required. In the case of an art gallery with three, four or five walls, only one guard need be hired, which is the fact used in the proof of Theorem 6.36.

It has been conjectured that $\mathrm{cr}(K_{s,t}) = \overline{\mathrm{cr}}(K_{s,t})$ for all positive integers s and t. In the case of complete graphs, not only is $\overline{\mathrm{cr}}(K_n) = \mathrm{cr}(K_n)$ for $1 \le n \le 4$ (when K_n is planar), it is also known that $\overline{\mathrm{cr}}(K_n) = \mathrm{cr}(K_n)$ for $5 \le n \le 7$ and $n = 9$. However, $\mathrm{cr}(K_8) = 18$ and $\overline{\mathrm{cr}}(K_8) = 19$ (see Guy [102]), so strict inequality in (6.8) is indeed a possibility. Furthermore, Alex Brodsky, Stephane Durocher and Ellen Gethner [29] showed that $\overline{\mathrm{cr}}(K_{10}) = 62$, while $\mathrm{cr}(K_{10}) = 60$. The members Oswin Aichholzer, Franz Aurenhammer and

Hannes Krasser of the *Rectilinear Crossing Number Project* determined $\overline{\mathrm{cr}}(K_n)$ for all n with $11 \leq n \leq 21$ except $n = 20$. In particular

- $\mathrm{cr}(K_{11}) = 100$ and $\overline{\mathrm{cr}}(K_{11}) = 102$ and
- $\mathrm{cr}(K_{12}) = 150$ and $\overline{\mathrm{cr}}(K_{12}) = 153$.

Exercises for Section 6.3

1. Draw K_7 in the plane with nine crossings.

2. (a) Let k and n be integers with $1 \leq k \leq n-1$. Determine the size of P_n^k.

   (b) For each integer $n \geq 2$, prove that the graph P_n^2 is maximal outerplanar.

   (c) For each integer $n \geq 3$, prove that the graph P_n^3 is maximal planar.

   (d) For each pair k, n of integers with $4 \leq k \leq n-1$, prove that

$$\mathrm{cr}(P_n^k) \geq (k-3)n + 6 - \tbinom{k+1}{2}.$$

3. Prove that $5 \leq \mathrm{cr}(P_7^5) \leq 6$.

4. Determine $\mathrm{cr}(K_{3,3})$ without using Theorem 6.31.

5. Show that $\mathrm{cr}(K_{7,7}) \leq 81$.

6. Determine $\mathrm{cr}(K_{2,2,2})$.

7. Determine $\mathrm{cr}(K_{1,2,3})$.

8. Show that $2 \leq \mathrm{cr}(C_3 \,\square\, C_3) \leq 3$ without using Theorem 6.33.

9. (a) It is known that $\mathrm{cr}(W_4 \,\square\, K_2) = 2$, where W_4 is the wheel $C_4 \vee K_1$ of order 5. Draw $W_4 \,\square\, K_2$ in the plane with two crossings.

   (b) Prove or disprove: If G is a nonplanar graph containing an edge e such that $G - e$ is planar, then $\mathrm{cr}(G) = 1$.

10. Prove that $\overline{\mathrm{cr}}(C_3 \,\square\, C_t) = t$ for $t \geq 3$.

11. Give an example of a straight-line embedding of a maximal planar graph of order exceeding 4 containing exactly four vertices of degree 5 or less.

6.4 Hamiltonian Planar Graphs

In Section 6.1, we saw necessary conditions for a connected graph to be planar and necessary conditions for a graph of order at least 4 to be maximal planar. In this section, we will be introduced to a single result, namely a necessary condition for a planar graph to be Hamiltonian.

Grinberg's Theorem

Let G be a Hamiltonian planar graph of order n with Hamiltonian cycle C and let there be given a planar embedding of G. An edge of G not lying on C is a **chord** of G. Every chord and every region of G then lies interior to C or exterior to C. For $i = 3, 4, \ldots, n$, let r_i denote the number of regions interior to C whose boundary contains exactly i edges and let r'_i denote the number of regions exterior to C whose boundary contains exactly i edges.

The plane graph G of Figure 6.30 of order 12 is Hamiltonian. The edges of the Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{12}, v_1)$ are drawn with bold lines. With respect to C, we have

$$r_3 = r'_3 = 1,\ r_4 = 3,\ r'_4 = 2,\ r_5 = r'_7 = 1,$$

while $r_i = 0$ for $6 \leq i \leq 12$ and $r'_i = 0$ for $i = 5, 6$ and $8 \leq i \leq 12$.

Figure 6.30: A Hamiltonian planar graph

In 1968 a necessary condition for a planar graph to be Hamiltonian was discovered by the Latvian mathematician Emanuels Ja. Grinberg [99].

Theorem 6.37 (**Grinberg's Theorem**) *For a plane graph G of order n with Hamiltonian cycle C,*

$$\sum_{i=3}^{n} (i-2)(r_i - r'_i) = 0.$$

Proof. Suppose that c chords of G lie interior to C. Then $c+1$ regions of G lie interior to C. Therefore,

$$\sum_{i=3}^{n} r_i = c+1 \text{ and so } c = \sum_{i=3}^{n} r_i - 1.$$

Let N denote the result obtained by summing over all regions interior to C the number of edges on the boundary of each such region. Then each edge on C is counted once and each chord interior to C is counted twice, that is,

$$N = \sum_{i=3}^{n} ir_i = n + 2c.$$

Therefore,

$$\sum_{i=3}^{n} ir_i = n + 2c = n + 2\sum_{i=3}^{n} r_i - 2$$

and so

$$\sum_{i=3}^{n} (i-2)r_i = n - 2.$$

Similarly,

$$\sum_{i=3}^{n} (i-2)r_i' = n - 2,$$

giving the desired result $\sum_{i=3}^{n}(i-2)(r_i - r_i') = 0$. ■

The following observations often prove quite useful in applying Theorem 6.37. Let G be a plane graph with Hamiltonian cycle C. Furthermore, suppose that the edge e of G is on the boundary of two regions R_1 and R_2 of G. If e is an edge of C, then one of R_1 and R_2 is in the interior of C and the other is in the exterior of C. If, on the other hand, e is not an edge of C, then R_1 and R_2 are either both in the interior of C or both in the exterior of C.

The Tutte Graph

In 1880, the English mathematician Peter Guthrie Tait conjectured that every 3-connected cubic planar graph is Hamiltonian. This conjecture was disproved in 1946 by William T. Tutte [221], who produced the graph G in Figure 6.31 as a counterexample. In addition to disproving Tait's conjecture, Tutte [225] proved that every 4-connected planar graph is Hamiltonian.

Since Theorem 6.37 gives a necessary condition for a planar graph to be Hamiltonian, this theorem also provides a sufficient condition for a planar graph to be non-Hamiltonian. We see how Grinberg's theorem can be used to show that the plane graph of Figure 6.31 is not Hamiltonian. This graph is called

Figure 6.31: The Tutte graph

the **Tutte graph** and has a great deal of historical interest. We will encounter this graph again in Chapter 11.

Suppose that the Tutte graph G is Hamiltonian. Then G has a Hamiltonian cycle C. Necessarily, C must contain exactly two of the three edges e, f_1 and f_2, say f_1 and either e of f_2. Similarly, C must contain exactly two edges of the three edges e', f_2 and f_3. Since we may assume that C contains f_2, we may further assume that e is not on C. Consequently, R_1 and R_2 lie interior to C.

Let G_1 denote the component of $G - \{e, f_1, f_2\}$ containing w. Thus G_1 contains a Hamiltonian $v_1 - v_2$ path P'. Therefore, $G_2 = G_1 + v_1v_2$ is Hamiltonian and contains a Hamiltonian cycle C' consisting of P' and v_1v_2. Applying Grinberg's theorem to G_2 with respect to C', we obtain

$$1(r_3 - r_3') + 2(r_4 - r_4') + 3(r_5 - r_5') + 6(r_8 - r_8') = 0. \tag{6.9}$$

Since v_1v_2 is on C' and the exterior region of G_2 lies exterior to C', it follows that

$$r_3 - r_3' = 1 - 0 = 1 \quad \text{and} \quad r_8 - r_8' = 0 - 1 = -1.$$

Therefore, from (6.9), we have

$$2(r_4 - r_4') + 3(r_5 - r_5') = 5.$$

Necessarily, both ww_1 and ww_2 are edges of C' and so $r_4 \geq 1$, implying that either

$$r_4 - r_4' = 1 - 1 = 0 \text{ or } r_4 - r_4' = 2 - 0 = 2.$$

If $r_4 - r_4' = 0$, then $3(r_5 - r_5') = 5$, which is impossible. On the other hand, if $r_4 - r_4' = 2$, then $3(r_5 - r_5') = 1$, which is also impossible. Hence G is not Hamiltonian.

For many years, Tutte's graph was the only known example of a 3-connected cubic planar graph that was not Hamiltonian. Much later, however, other

such graphs have been found; for example, Grinberg himself found another counterexample to Tait's conjecture (see Exercise 3).

Exercises for Section 6.4

1. Show, by applying Theorem 6.37, that $K_{2,3}$ is not Hamiltonian.

2. Show, by applying Theorem 6.37, that each of the graphs in Figure 6.32 is not Hamiltonian.

Figure 6.32: Graphs in Exercise 2

3. Show, by applying Theorem 6.37, that the **Grinberg graph** in Figure 6.33 is not Hamiltonian.

Figure 6.33: The Grinberg graph in Exercise 3

4. Show, by applying Theorem 6.37, that the **Herschel graph** in Figure 6.34 is not Hamiltonian.

Figure 6.34: The Herschel graph in Exercise 4

5. Show, by applying Theorem 6.37, that no Hamiltonian cycle in the graph of Figure 6.35 contains both the edges e and f.

Figure 6.35: A Hamiltonian planar graph in Exercise 5

Chapter 7

Graph Embeddings

While only planar graphs can be embedded in the plane, there is a host of common and increasingly complex surfaces on which a nonplanar graph G might possibly be embedded. Determining the simplest of these surfaces on which G can be embedded gives an indication of how close G is to being planar. We will see that the embeddings of interest are those called 2-cell embeddings and discuss the possible surfaces on which a given graph has such an embedding.

7.1 The Genus of a Graph

We have seen that a graph G is planar if G can be drawn in the plane in such a way that no two edges cross and that such a drawing is called an embedding of G in the plane or a planar embedding. Furthermore, a graph G can be embedded in the plane if and only if G can be embedded on (the surface of) a sphere.

Of course, not all graphs are planar. Indeed, Kuratowski's theorem (Theorem 6.18) and Wagner's theorem (Theorem 6.21) describe conditions (involving the two nonplanar graphs K_5 and $K_{3,3}$) under which G can be embedded in the plane. Graphs that are not embeddable in the plane (or on a sphere) may be embeddable on other surfaces, however.

Embeddings on a Torus

Another common surface on which a graph may be embedded is the **torus**, a doughnut-shaped surface (see Figure 7.1(a)). Two different embeddings of the (planar) graph K_4 on a torus are shown in Figures 7.1(b) and 7.1(c).

Figure 7.1: Embedding K_4 on a torus

While it is easy to see that every planar graph can be embedded on a torus, some nonplanar graphs can be embedded on a torus as well. For example, embeddings of K_5 and $K_{3,3}$ on a torus are shown in Figures 7.2(a) and 7.2(b).

Figure 7.2: Embedding K_5 and $K_{3,3}$ on a torus

Another way to represent a torus and to visualize an embedding of a graph on a torus is to begin with a rectangular piece of (flexible) material as in Figure 7.3 and first make a cylinder from it by identifying sides a and c, which are the same after the identification occurs. Sides b and d are then circles. These circles are then identified to produce a torus.

Figure 7.3: Constructing a torus

After seeing how a torus can be constructed from a rectangle, it follows that the points labeled A in the rectangle in Figure 7.4(a) represent the same point

on the torus. This is also true of the points labeled B and the points labeled C. Figures 7.4(b) and 7.4(c) show embeddings of K_5 and $K_{3,3}$ on the torus. There are five regions in the embedding of K_5 on the torus shown in Figure 7.4(b) as R is a single region in this embedding. Moreover, there are three regions in the embedding of $K_{3,3}$ on the torus shown in Figure 7.4(c) as R' is a single region.

Figure 7.4: Embedding K_5 and $K_{3,3}$ on a torus

Another way to represent a torus and an embedding of a graph on a torus is to begin with a sphere, insert two holes in its surface (as in Figure 7.5(a)) and attach a handle on the sphere, where the ends of the handle are placed over the two holes (as in Figure 7.5(b)). An embedding of K_5 on the torus constructed in this manner is shown in Figure 7.5(c).

Figure 7.5: Embedding K_5 on a torus

While a torus is a sphere with one handle, a sphere with k handles, $k \geq 0$, is called a **surface of genus** k and is denoted by S_k. Thus, S_0 is a sphere and S_1 is a torus. The surfaces S_k are the **orientable surfaces**.

Let G be a nonplanar graph. When drawing G on a sphere, some edges of G will cross. The graph G can always be drawn so that only two edges cross at any point of intersection. At each such point of intersection, a handle can be suitably placed on the sphere so that one of these two edges passes over the handle and the intersection of the two edges has been avoided. Consequently, every graph can be embedded on some orientable surface. The smallest nonnegative integer k such that a graph G can be embedded on S_k is called the **genus** of G and is denoted by $\gamma(G)$. Therefore, $\gamma(G) = 0$ if and only if G is planar; while $\gamma(G) = 1$ if and only if G is nonplanar but G can be embedded on the torus. An embedding of a graph G on the torus is called a **toroidal embedding** of G. In particular,

$$\gamma(K_5) = 1 \text{ and } \gamma(K_{3,3}) = 1.$$

Figure 7.6(a) shows an embedding of a disconnected graph H on a sphere. In this case, $n = 8$, $m = 9$ and $r = 4$. Thus $n - m + r = 8 - 9 + 4 = 3$. That $n - m + r \neq 2$ is not particularly surprising as the Euler Identity (Theorem 6.1) requires that H be a *connected* plane graph. Although this is a major reason why we will restrict our attention to connected graphs here, it is not the only reason. There is a desirable property possessed by every embedding of a connected planar graph on a sphere that is possessed by no embedding of a disconnected planar graph on a sphere.

2-Cell Embeddings of Graphs

Suppose that G is a graph embedded on a surface S_k, $k \geq 0$. A region of this embedding is a **2-cell** if every closed curve in that region can be continuously deformed in that region to a single point. (Topologically, a region is a 2-cell if it is homeomorphic to a disk.) While the closed curve C in R in the embedding of the graph on a sphere shown in Figure 7.6(b) can in fact be continuously deformed in R to a single point, the curve C' cannot. Hence R is not a 2-cell in this embedding.

Figure 7.6: An embedding on a sphere that is not a 2-cell embedding

An embedding of a graph G on some surface is a **2-cell embedding** if every region in the embedding is a 2-cell. Consequently, the embedding of the graph shown in Figure 7.6(a) is not a 2-cell embedding. It turns out, however, that every embedding of a connected graph on a sphere is necessarily a 2-cell embedding. Of course, such a graph is necessarily planar. If a connected graph is embedded on a surface S_k where $k > 0$, then the embedding may or may not be a 2-cell embedding, however. For example, the embedding of K_4 in Figure 7.1(b) is not a 2-cell embedding. The curves C and C' shown in Figures 7.7(a) and 7.7(b) cannot be continuously deformed to a single point in the region in which these curves are drawn. On the other hand, the embed-

ding of K_4 shown in Figure 7.1(c) and shown again in Figure 7.7(c) is a 2-cell embedding.

(a) (b) (c)

Figure 7.7: Non-2-cell and 2-cell embeddings of K_4 on the torus

The Generalized Euler Identity

The embeddings of K_4, K_5 and $K_{3,3}$ on a torus given in Figures 7.1(c), 7.2(a) and 7.2(b), respectively, are all 2-cell embeddings. Furthermore, in each case, $n - m - r = 0$. As it turns out, if G is a connected graph of order n and size m that is 2-cell embedded on a torus resulting in r regions, then it is always the case that $n - m - r = 0$. This fact together with the Euler Identity (Theorem 6.1) are special cases of a more general result. The mathematician Simon Antoine Jean Lhuilier spent much of his life working on problems related to the Euler Identity. Lhuilier, like Euler, was from Switzerland and was taught mathematics by one of Euler's former students (Louis Bertrand). Lhuilier saw that the Euler Identity did *not* hold for graphs embedded on spheres containing handles. In fact, he proved a more general form of this identity [146].

Theorem 7.1 (**The Generalized Euler Identity**) *If G is a connected graph of order n and size m that is 2-cell embedded on a surface of genus $k \geq 0$, resulting in r regions, then*

$$n - m + r = 2 - 2k.$$

Proof. We proceed by induction on k. If G is a connected graph of order n and size m that is 2-cell embedded on a surface of genus 0, then G is a plane graph. By the Euler Identity, $n - m + r = 2 = 2 - 2 \cdot 0$. Thus the basis step of the induction holds.

Assume, for every connected graph G' of order n' and size m' that is 2-cell embedded on a surface S_k for some nonnegative integer k, resulting in r' regions, that

$$n' - m' + r' = 2 - 2k.$$

Let G be a connected graph of order n and size m that is 2-cell embedded on S_{k+1}, resulting in r regions. We may assume, without loss of generality, that no vertex of G lies on any handle of S_{k+1} and that the edges of G are drawn

on the handles so that a closed curve can be drawn around each handle that intersects no edge of G more than once.

Let H be one of the $k+1$ handles of S_{k+1}. There are necessarily edges of G on H; for otherwise, the handle belongs to a region R in which case any closed curve around H cannot be continuously deformed in R to a single point, contradicting the assumption that R is a 2-cell. We now draw a closed curve C around H, which intersects some edges of G on H but intersects no edge more than once. Suppose that there are $t \geq 1$ points of intersection of C and the edges on H. Let the points of intersection be vertices, where each of the t edges becomes two edges. Also, the segments of C between vertices become edges. We add two vertices of degree 2 along C to produce two additional edges. (This guarantees that the resulting structure will be a graph, not a multigraph.)

Let G_1 be the graph just constructed, where G_1 has order n_1, size m_1 and r_1 regions. Then

$$n_1 = n + t + 2 \text{ and } m_1 = m + 2t + 2.$$

Since each portion of C that became an edge of G_1 is in a region of G, the addition of such an edge divides that region into two regions, each of which is a 2-cell. Since there are t such edges,

$$r_1 = r + t.$$

We now cut the handle H along C and "patch" the two resulting holes, producing two duplicate copies of the vertices and edges along C (see Figure 7.8). Denote the resulting graph by G_2, which is now 2-cell embedded on S_k.

Figure 7.8: Converting a 2-cell embedding of G_1 on S_{k+1} into a 2-cell embedding of G_2 on S_k

Let G_2 have order n_2, size m_2 and r_2 regions, all of which are 2-cells. Then

$$n_2 = n_1 + t + 2,\ m_2 = m_1 + t + 2 \text{ and } r_2 = r_1 + 2.$$

Furthermore, $n_2 = n + 2t + 4$, $m_2 = m + 3t + 4$ and $r_2 = r + t + 2$. By the induction hypothesis, $n_2 - m_2 + r_2 = 2 - 2k$. Therefore,

$$\begin{aligned} n_2 - m_2 + r_2 &= (n + 2t + 4) - (m + 3t + 4) + (r + t + 2) \\ &= n - m + r + 2 = 2 - 2k. \end{aligned}$$

Therefore, $n - m + r = 2 - 2(k+1)$. ■

The following result [245] was obtained by J. W. T. (Ted) Youngs.

Theorem 7.2 *Every embedding of a connected graph G of genus k on S_k, where k is a nonnegative integer, is a 2-cell embedding.*

Lower Bounds for the Genus of a Graph

With the aid of Theorems 7.1 and 7.2, we have the following.

Corollary 7.3 *If G is a connected graph of order n and size m that is embedded on a surface of genus $\gamma(G)$, resulting in r regions, then*

$$n - m + r = 2 - 2\gamma(G).$$

The following result is a consequence of Corollary 7.3.

Theorem 7.4 *If G is a connected graph of order $n \geq 3$ and size m, then*

$$\gamma(G) \geq \frac{m}{6} - \frac{n}{2} + 1.$$

Proof. Suppose that G is embedded on a surface of genus $\gamma(G)$, resulting in r regions. By Corollary 7.3, $n - m + r = 2 - 2\gamma(G)$. Let $R_1, R_2, \ldots, R_r$ be the regions of G and let m_i be the number of edges on the boundary of R_i $(1 \leq i \leq r)$. Thus $m_i \geq 3$. Since every edge is on the boundary of one or two regions, it follows that

$$3r \leq \sum_{i=1}^{r} m_i \leq 2m$$

and so $3r \leq 2m$. Therefore,

$$6 - 6\gamma(G) = 3n - 3m + 3r \leq 3n - 3m + 2m = 3n - m \tag{7.1}$$

Solving (7.1) for $\gamma(G)$, we have $\gamma(G) \geq \frac{m}{6} - \frac{n}{2} + 1$. ■

Theorem 7.4 is a generalization of Theorem 6.3, for when G is planar (and so $\gamma(G) = 0$) Theorem 7.4 becomes Theorem 6.3. The lower bound for $\gamma(G)$ presented in Theorem 7.4 can be improved when more information on cycle lengths in G is available (see Exercise 3).

Theorem 7.5 *If G is a connected graph of order n and size m whose smallest cycle has length k, then*

$$\gamma(G) \geq \frac{m}{2}\left(1 - \frac{2}{k}\right) - \frac{n}{2} + 1.$$

The following result is a consequence of Theorem 7.5 that includes bipartite graphs as a special case. Recall that a graph is called triangle-free if it contains no triangles.

Corollary 7.6 *If G is a connected, triangle-free graph of order $n \geq 3$ and size m, then*

$$\gamma(G) \geq \frac{m}{4} - \frac{n}{2} + 1.$$

While there is no general formula for the genus of an arbitrary graph, the following result by Joseph Battle, Frank Harary and Yukihiro Kodama [6] implies that, as far as genus formulas are concerned, only 2-connected graphs need be investigated.

Theorem 7.7 *If G is a graph having blocks B_1, B_2, ..., B_k, then*

$$\gamma(G) = \sum_{i=1}^{k} \gamma(B_i).$$

The following corollary is a consequence of the preceding result (see Exercise 4).

Corollary 7.8 *If G is a graph with components $G_1, G_2, \ldots, G_k$, then*

$$\gamma(G) = \sum_{i=1}^{k} \gamma(G_i).$$

The Genus of Special Graphs

As is often the case when no general formula exists for the value of a parameter for an arbitrary graph, formulas (or partial formulas) are established for certain families of graphs. Ordinarily the first classes to be considered are the complete graphs, the complete bipartite graphs and the n-cubes. The genus offers no exception to this rule.

According to Theorem 7.4,

$$\gamma(K_5) \geq \tfrac{1}{6},\ \gamma(K_6) \geq \tfrac{1}{2} \ \text{ and } \ \gamma(K_7) \geq 1.$$

This says that all three graphs K_5, K_6 and K_7 are nonplanar. Of course, we already knew this by Corollary 6.5, for since K_5 is nonplanar so are K_6 and K_7. We have seen that $\gamma(K_5) = 1$. Actually, $\gamma(K_7) = 1$ as well. Figure 7.9 shows an embedding of K_7 on the torus where the vertex set of K_7 is $\{v_1, v_2, \ldots, v_7\}$. Because K_6 is nonplanar and K_6 is a subgraph of a graph that can be embedded on a torus, $\gamma(K_6) = 1$.

Applying Theorem 7.4 to the complete graph K_n, $n \geq 3$, we have

$$\gamma(K_n) \geq \frac{\binom{n}{2}}{6} - \frac{n}{2} + 1 = \frac{(n-3)(n-4)}{12}.$$

Figure 7.9: An embedding of K_7 on the torus

Since $\gamma(K_n)$ is an integer,

$$\gamma(K_n) \geq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil. \tag{7.2}$$

Born in India, Ted Youngs (1910–1970) received his Ph.D. in 1934. In 1964 he was appointed one of the first faculty members of the University of California Santa Cruz. Born in Austria, Gerhard Ringel (1919–2008) was one of the pioneers of graph theory. He received his Ph.D. from the University of Bonn in Germany in 1951. In addition to his mathematical skills, Ringel was well known for his interest in collecting and breeding butterflies. Ringel and Youngs [189] worked together to complete a proof of a famous theorem in graph theory, which will be discussed in Chapter 9. However, the truth of this theorem was based on showing that the lower bound for $\gamma(K_n)$ in (7.2) is, in fact, the value of $\gamma(K_n)$. Ringel left Germany in 1970 to continue his career at the University of California Santa Cruz.

Theorem 7.9 *For every integer $n \geq 3$,*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

Ringel [188] also discovered a formula for the genus of every complete bipartite graph.

Theorem 7.10 *For every two integers $s, t \geq 2$,*

$$\gamma(K_{s,t}) = \left\lceil \frac{(s-2)(t-2)}{4} \right\rceil.$$

In particular, Theorem 7.10 implies that a complete bipartite graph G can be embedded on a torus if and only if G is planar or is a subgraph of $K_{4,4}$ or $K_{3,6}$.

A formula for the genus of the n-cube was found by Ringel [186] and by Beineke and Harary [11]. We prove this result to illustrate some of the techniques involved.

Theorem 7.11 *For $n \geq 2$, the genus of the n-cube is given by*

$$\gamma(Q_n) = (n-4) \cdot 2^{n-3} + 1.$$

Proof. Since the n-cube is a triangle-free graph of order $2n$ and size $n \cdot 2^{n-1}$, it follows by Corollary 7.6 that

$$\gamma(Q_n) \geq (n-4) \cdot 2^{n-3} + 1.$$

To verify the reverse inequality, we employ induction on n. In fact, we show for every integer $n \geq 2$, that there is an embedding of Q_n on the surface of genus $(n-4) \cdot 2^{n-3} + 1$ such that the boundary of every region is a 4-cycle and such that there exist 2^{n-2} regions with pairwise disjoint boundaries. Since Q_2 and Q_3 are planar and $(n-4) \cdot 2^{n-3} + 1 = 0$ for $n = 2$ and $n = 3$, this is certainly true for $n = 2$ and $n = 3$.

Assume for an integer $k \geq 4$ that there is an embedding of Q_{k-1} on the surface S of genus $(k-5) \cdot 2^{k-4} + 1$ such that the boundary of every region is a 4-cycle and such that there exist 2^{k-3} regions with pairwise disjoint boundaries. Since the order of Q_{k-1} is 2^{k-1}, each vertex of Q_{k-1} belongs to the boundary of precisely one of the aforementioned 2^{k-3} regions. Furthermore, let Q_{k-1} be embedded on another copy S' of the surface of genus $(k-5) \cdot 2^{k-4} + 1$ such that the embedding of Q_{k-1} on S' is a "mirror image" of the embedding of Q_{k-1} on S (that is, if v_1, v_2, v_3, v_4 are the vertices of the boundary of a region of Q_{k-1} on S, where the vertices are listed clockwise about the 4-cycle, then there is a region on S', with the vertices v_1, v_2, v_3, v_4 on its boundary listed counterclockwise).

We now consider the 2^{k-3} distinguished regions of S together with the corresponding regions of S' and join each pair of associated regions by a handle. The addition of the first handle produces the surface of genus $2[(k-5) \cdot 2^{k-4} + 1]$ while the addition of each of the other $2^{k-3} - 1$ handles results in an increase of 1 to the genus. Thus, the surface just constructed has genus $(k-4) \cdot 2^{k-3} + 1$. Now each set of four vertices on the boundary of a distinguished region can be joined to the corresponding four vertices on the boundary of the associated region so that the four edges are embedded on the handle joining the regions. It is now immediate that the resulting graph is isomorphic to Q_k and that every region is bounded by a 4-cycle. Furthermore, each added handle gives rise to four regions, "opposite" ones of which have disjoint boundaries, so there exist 2^{k-2} regions of Q_k that are pairwise disjoint. ■

The Möbius Strip and Projective Plane

There are other surfaces on which graphs can be embedded. The **Möbius strip** (or **Möbius band**) is a one-sided surface that can be constructed from a

rectangular piece of material by giving one side of the rectangle a half-twist (or a rotation through 180°) and then identifying opposite sides of the rectangle (see Figure 7.10). Thus A represents the same point on the Möbius strip. The Möbius strip is named for the German mathematician August Ferdinand Möbius who discovered it in 1858 (even though the mathematician Johann Benedict Listing discovered it shortly before Möbius).

Figure 7.10: The Möbius strip

Certainly every planar graph can be embedded on the Möbius strip. Figure 7.11 shows that $K_{3,3}$ can also be embedded on the Möbius strip.

Figure 7.11: An embedding of $K_{3,3}$ on the Möbius strip

Of more interest are the nonorientable surfaces (or the nonorientable 2-dimensional manifolds), the simplest example of which is the projective plane. The **projective plane** can be represented by identifying opposite sides of a rectangle in the manner shown in Figure 7.12(a). Note that A represents the same point in the projective plane, as does B. Figure 7.12(b) shows an embedding of K_5 on the projective plane.

The projective plane can also be represented by a circle where antipodal pairs of points on the circumference are the same point. Using this representation, we can give an embedding of K_6 on the projective plane shown in Figure 7.13.

For the embedding of K_5 on the projective plane shown in Figure 7.12(b), $n = 5$, $m = 10$ and $r = 6$; while for the embedding of K_6 shown in Figure 7.13, $n = 6$, $m = 15$ and $r = 10$. In both cases, $n - m + r = 1$. In fact, for any

Figure 7.12: An embedding of K_5 on the projective plane

Figure 7.13: An embedding of K_6 on the projective plane

connected graph of order n and size m that is 2-cell embedded on the projective plane, resulting in r regions,

$$n - m + r = 1.$$

Exercises for Section 7.1

1. Determine $\gamma = \gamma(K_{4,4})$ without using Theorem 7.10 and label the regions in a 2-cell embedding of $K_{4,4}$ on the surface of genus γ.

2. (a) Show that $\gamma(G) \leq \mathrm{cr}(G)$ for every graph G.

   (b) Prove that for every positive integer k, there exists a graph G such that $\gamma(G) = 1$ and $\mathrm{cr}(G) = k$.

3. Prove Theorem 7.5: *If G is a connected graph of order n and size m whose smallest cycle has length k, then* $\gamma(G) \geq \frac{m}{2}\left(1 - \frac{2}{k}\right) - \frac{n}{2} + 1$.

4. Use Theorem 7.7 to prove Corollary 7.8: *If G is a graph with components $G_1, G_2, \ldots, G_k$, then $\gamma(G) = \sum_{i=1}^{k} \gamma(G_i)$.*

5. Show for every two integers $s, t \geq 2$ that
$$\gamma(K_{s,t}) \geq \left\lceil \frac{(s-2)(t-2)}{4} \right\rceil.$$

6. (a) Find a lower bound for $\gamma(K_{3,3} \vee \overline{K}_n)$.
   (b) Determine $\gamma(K_{3,3} \vee \overline{K}_n)$ exactly for $n = 1, 2, 3$.

7. Determine $\gamma(K_2 \,\square\, C_4 \,\square\, C_6)$.

8. Prove, for every positive integer γ, that there exists a connected graph G of genus γ.

9. Prove, for each positive integer k, that there exists a connected planar graph G such that $\gamma(G \,\square\, K_2) \geq k$.

10. Show, in a manner similar to the embedding of $K_{3,3}$ shown in Figure 7.11, that K_5 can be embedded on the Möbius strip.

11. By Theorem 7.9, $\gamma(K_7) = 1$. Let there be an embedding of K_7 on the torus and let R_1 and R_2 be two neighboring regions. Let G be the graph obtained by adding a new vertex v in R_1 and joining v to the vertices on the boundaries of both R_1 and R_2. What is $\gamma(G)$?

12. The graph H is a certain 6-regular graph of order 12. It is known that $G = H \,\square\, K_2$ can be embedded on S_3. What is $\gamma(G)$?

7.2 2-Cell Embeddings of Graphs

In the preceding section we saw that every graph G has a genus; that is, there exists a surface (a compact orientable 2-manifold) of minimum genus on which G can be embedded. Indeed, by Theorem 7.2 if G is a connected graph that is embedded on the surface of genus $\gamma(G)$, then the embedding is necessarily a 2-cell embedding. On the other hand, if G is disconnected, then no embedding of G on any surface is a 2-cell embedding.

Our primary interest lies with embeddings of (connected) graphs that are 2-cell embeddings. In this section, we investigate graphs and the surfaces on which they can be 2-cell embedded. Recall that S_k denotes the surface of genus k. Thus, S_0 represents the sphere (or plane), S_1 represents the torus and S_2 represents the **double torus** (or sphere with two handles).

We have already mentioned that the torus can be represented as a square with opposite sides identified. More generally, the surface S_k $(k > 0)$ can be

represented as a regular $4k$-gon whose $4k$ sides can be listed in clockwise order as

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_k b_k a_k^{-1} b_k^{-1}, \tag{7.3}$$

where, for example, a_1 is a side directed clockwise and a_1^{-1} is a side also labeled a_1 but directed counterclockwise. These two sides are then identified in a manner consistent with their directions. Thus, the double torus can be represented by a regular octagon, as shown in Figure 7.14. The "two" points labeled X are actually the same point on S_2 while the "eight" points labeled Y are, in fact, a single point.

Figure 7.14: A representation of the double torus

Although it is probably obvious that there exist numerous graphs that can be embedded on the surface S_k of a given nonnegative integer k, it may not be entirely obvious that there always exist graphs for which a 2-cell embedding on S_k exists.

Theorem 7.12 *For every nonnegative integer k, there exists a connected graph that has a 2-cell embedding on S_k.*

Proof. For $k = 0$, every connected planar graph has the desired property; thus, we assume that $k > 0$.

We represent S_k as a regular $4k$-gon whose $4k$ sides are described and identified as in (7.3). First, we define a pseudograph H as follows. At each vertex of the $4k$-gon, let there be a vertex of H. Actually, the identification process associated with the $4k$-gon implies that there is only one vertex of H. Let each side of the $4k$-gon represent an edge of H. The identification produces $2k$ distinct edges, each of which is a loop. This completes the construction of H. Hence, the pseudograph H has order 1 and size $2k$. Furthermore, there is only one region, namely the interior of the $4k$-gon; this region is clearly a 2-cell. Therefore, there exists a 2-cell embedding of H on S_k.

To convert the pseudograph H into a graph, we subdivide each loop twice, producing a graph G having order $4k + 1$, size $6k$ and again a single 2-cell region. ■

Figure 7.15 illustrates the construction given in the proof of Theorem 7.12 in the case of the torus S_1. The graph G so constructed is shown in Figure 7.15(a). In Figure 7.15(b)-(e) we see a variety of ways of visualizing the embedding. In Figure 7.15(b), a 3-dimensional embedding is described. In Figure 7.15(c) and (d), the torus is represented as a rectangle with opposite sides identified. (Figure 7.15(c) is the actual drawing described in the proof of the theorem.) In Figure 7.15(e), a portion of G is drawn in the plane. Then two circular holes are made in the plane and a tube (or handle) is placed over the plane joining the two holes. The edge uv is then drawn over the handle, completing the 2-cell embedding.

Figure 7.15: A graph 2-cell embedded on the torus

The graphs G constructed in the proof of Theorem 7.12 are planar. Hence, for every nonnegative integer k, there exist planar graphs that can be 2-cell embedded on S_k. It is also true that for every planar graph G and every positive integer k, there exists an embedding of G on S_k that is *not* a 2-cell embedding. In general, for a given graph G and *positive* integer k with $k > \gamma(G)$, there always exists an embedding of G on S_k that is not a 2-cell embedding, which can be obtained from an embedding of G on $S_{\gamma(G)}$ by adding $k - \gamma(G)$ handles to the interior of some region of G. If $k = \gamma(G)$ and G is connected, then by Theorem 7.2 every embedding of G on S_k is a 2-cell embedding. Of course, if $k < \gamma(G)$, there is no embedding whatsoever of G on S_k.

The Rotational Embedding Scheme

Thus far, whenever we have described a 2-cell embedding (or, in fact, any embedding) of a graph G on a surface S_k, we have resorted to a geometric description, such as the ones shown in Figure 7.15. There is another method, algebraic in nature, which we now discuss.

Consider the 2-cell embedding of K_5 on S_1 shown in Figure 7.16, with the vertices of K_5 labeled as indicated. Observe that in this embedding the edges incident with v_1 are arranged cyclically counterclockwise about v_1 in the order $v_1v_2, v_1v_3, v_1v_4, v_1v_5$ or, equivalently, $v_1v_3, v_1v_4, v_1v_5, v_1v_2$ and so on, which is indicated by the counterclockwise arrow drawn outside the rectangle. This embedding induces a cyclic permutation π_1 of the subscripts of the vertices adjacent with v_1, namely $\pi_1 = (2\ 3\ 4\ 5)$, expressed as a permutation cycle. Similarly, this embedding induces a cyclic permutation π_2 of the subscripts of the vertices adjacent with v_2; in particular, $\pi_2 = (1\ 5\ 4\ 3)$. In fact, one can associate a cyclic permutation π_i with each vertex v_i $(1 \leq i \leq 5)$. In this case, we have

$$\begin{aligned}
\pi_1 &= (2\ 3\ 4\ 5)\\
\pi_2 &= (1\ 5\ 4\ 3)\\
\pi_3 &= (1\ 2\ 5\ 4)\\
\pi_4 &= (1\ 3\ 2\ 5)\\
\pi_5 &= (1\ 4\ 3\ 2).
\end{aligned}$$

Figure 7.16: A 2-cell embedding of K_5 on the torus

In the 2-cell embedding of K_5 on S_1 shown in Figure 7.16, there are five regions, labeled $R_1, R_2, \ldots, R_5$. Each region R_i $(1 \leq i \leq 5)$ is, of course, a 2-cell. The boundary of the region R_1 consists of the vertices v_1, v_2 and v_5 and the edges v_1v_2, v_2v_5 and v_5v_1. If we trace out the edges of the boundary of R_1 in a clockwise direction, that is, keeping the boundary at our left and the region to our right (Figure 7.17), beginning with the edge v_1v_2, we have v_1v_2, followed by v_2v_5, and finally v_5v_1. This information can also be obtained from the cyclic permutations $\pi_1, \pi_2, \ldots, \pi_5$; indeed, the edge following $v_1v_2 = v_2v_1$ as we trace

the boundary edges of R_1 in a clockwise direction is precisely the edge incident with v_2 that follows v_2v_1 if one proceeds counterclockwise about v_2; that is, the edge following v_1v_2 in the boundary of R_1 is $v_2v_{\pi_2(1)} = v_2v_5$. Similarly, the edge following $v_2v_5 = v_5v_2$ as we trace out the edges of the boundary of R_1 in a clockwise direction is $v_5v_{\pi_5(2)} = v_5v_1$. Hence with the aid of the cyclic permutations $\pi_1, \pi_2, \ldots, \pi_5$, we can trace out the edges of the boundary of R_1. In a like manner, the boundary of every region of the embedding can be so described.

Figure 7.17: Tracing out a region

Since the direction (namely, clockwise) in which the edges of the boundary of a region are traced in the above description is of utmost importance, it is convenient to regard each edge of K_5 as a symmetric pair of arcs and, thus, to interpret K_5 itself as a digraph D. With this interpretation, the boundary of the region R_1 and thus R_1 itself can be described, starting at v_1, as

$$(v_1, v_2), (v_2, v_{\pi_2(1)}), (v_5, v_{\pi_5(2)})$$

or

$$(v_1, v_2), (v_2, v_5), (v_5, v_1). \tag{7.4}$$

We now define a mapping $\pi : E(D) \to E(D)$ as follows. Let $a \in E(D)$, where $a = (v_i, v_j)$. Then

$$\pi(a) = \pi((v_i, v_j)) = \pi(v_i, v_j) = (v_j, v_{\pi_j(i)}).$$

The mapping π is one-to-one and is therefore a permutation of $E(D)$. Thus, π can be expressed as a product of disjoint permutation cycles. In this context, each permutation cycle of π is referred to as an "orbit" of π. Hence (7.4) corresponds to an orbit of π and is often denoted more compactly as $v_1 - v_2 - v_5 - v_1$. (Although this orbit corresponds to a cycle in the graph, this is not always the case for an arbitrary orbit in a graph that is 2-cell embedded.) For the embedding of K_5 on S_1 shown in Figure 7.16, the list of all five orbits (one for each region) is given below:

R_1 : $\quad v_1 - v_2 - v_5 - v_1$

R_2 : $\quad v_1 - v_3 - v_2 - v_1$

$$\begin{aligned} R_3: &\quad v_1 - v_4 - v_3 - v_1 \\ R_4: &\quad v_1 - v_5 - v_4 - v_1 \\ R_5: &\quad v_2 - v_3 - v_5 - v_2 - v_4 - v_5 - v_3 - v_4 - v_2. \end{aligned}$$

The orbits of π form a partition of $E(D)$ and, as such, each arc of D appears in exactly one orbit of π. Since D is the digraph obtained by replacing each edge of K_5 by a symmetric pair of arcs, each edge of K_5 appears twice among the orbits of π, once for each of the two possible directions that are assigned to the edge.

The 2-cell embedding of K_5 on S_1 shown in Figure 7.16 uniquely determines the collection $\{\pi_1, \pi_2, \ldots, \pi_5\}$ of permutations of the subscripts of the vertices adjacent to the vertices of K_5. This set of permutations, in turn, completely describes the embedding of K_5 on S_1 shown in Figure 7.16.

This method of describing an embedding is referred to as the **Rotational Embedding Scheme**. Such a scheme was observed and used by Walther von Dyck [232] in 1888 and by Lothar Heffter [117] in 1891. It was formalized by Jack Edmonds [64] in 1960 and discussed in more detail by Ted Youngs [245] in 1963.

We now describe the Rotational Embedding Scheme in a more general setting. Let G be a nontrivial connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let

$$V(i) = \{j : \ v_j \in N(v_i)\}.$$

For each i $(1 \le i \le n)$, let $\pi_i : V(i) \to V(i)$ be a cyclic permutation (or rotation) of $V(i)$. Thus, each permutation π_i can be represented by a (permutation) cycle of length $|V(i)| = |N(v_i)| = \deg v_i$. The Rotational Embedding Scheme states that there is a one-to-one correspondence between the 2-cell embeddings of G (on all possible surfaces) and the n-tuples $(\pi_1, \pi_2, \ldots, \pi_n)$ of cyclic permutations.

Theorem 7.13 (**The Rotational Embedding Scheme**) *Let G be a nontrivial connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. For each 2-cell embedding of G on a surface, there exists a unique n-tuple $(\pi_1, \pi_2, \ldots, \pi_n)$, where for $i = 1, 2, \ldots, n$, $\pi_i : V(i) \to V(i)$ is a cyclic permutation that describes the subscripts of the vertices adjacent to v_i in counterclockwise order about v_i. Conversely, for each such n-tuple $(\pi_1, \pi_2, \ldots, \pi_n)$, there exists a 2-cell embedding of G on some surface such that for $i = 1, 2, \ldots, n$, the subscripts of the vertices adjacent to v_i and in counterclockwise order about v_i are given by π_i.*

Proof. Let there be given a 2-cell embedding of G on some surface. For each vertex v_i of G, define $\pi_i : V(i) \to V(i)$ as follows: If $v_iv_j \in E(G)$ and v_iv_t (possibly $t = j$) is the next edge encountered after v_iv_j as we proceed counterclockwise about v_i, then we define $\pi_i(j) = t$. Each function π_i so defined is a cyclic permutation.

Conversely, suppose that we are given an n-tuple $(\pi_1, \pi_2, \ldots, \pi_n)$ such that for each i $(1 \le i \le n)$, $\pi_i : V(i) \to V(i)$ is a cyclic permutation. We show that

this determines a 2-cell embedding of G on some surface. (By necessity, this proof requires the use of properties of compact orientable 2-manifolds.)

Let D denote the digraph obtained from G by replacing each edge of G by a symmetric pair of arcs. We define a mapping $\pi : E(D) \to E(D)$ by

$$\pi((v_i, v_j)) = \pi(v_i, v_j) = (v_j, v_{\pi_j(i)}).$$

The mapping π is one-to-one and, thus, is a permutation of $E(D)$. Hence, π can be expressed as a product of disjoint permutation cycles. These permutation cycles are thus the orbits of π and produce a partition of $E(D)$. Assume that

$$R: \ ((v_i, v_j)(v_j, v_t) \cdots (v_\ell, v_i))$$

is an orbit of π, which we also write as

$$R: \ v_i - v_j - v_t - \cdots - v_\ell - v_i.$$

Hence, this implies that in the desired embedding, if we begin at v_i and proceed along (v_i, v_j) to v_j, then the next arc we must encounter after (v_i, v_j) in a counterclockwise direction about v_j is

$$(v_j, v_{\pi_j(i)}) = (v_j, v_t).$$

Continuing in this manner, we must finally arrive at the arc (v_ℓ, v_i) and return to v_i, in the process describing the boundary of a (2-cell) region (considered as a subset of the plane) corresponding to the orbit R. Therefore, each orbit of π gives rise to a 2-cell region in the desired embedding.

To obtain the surface S on which G is 2-cell embedded, pairs of regions, with their boundaries, are "pasted" along certain arcs; in particular, if (v_i, v_j) is an arc on the boundary of R_1 and (v_j, v_i) is an arc on the boundary of R_2, then (v_i, v_j) is identified with (v_j, v_i) as shown in Figure 7.18. The properties of compact orientable 2-manifolds imply that S is indeed an appropriate surface.

Figure 7.18: A step in the proof of Theorem 7.13

In order to determine the genus of S, one needs only to observe that the number r of regions equals the number of orbits. Thus, if G has order n and size m, then by Theorem 7.1, $S = S_k$ where k is the nonnegative integer satisfying the equation $n - m + r = 2 - 2k$. ■

As an illustration of the Rotational Embedding Scheme, we once again consider the complete graph K_5, with $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Let there be given the 5-tuple $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$ of cyclic permutations

$$\begin{aligned}
\pi_1 &= (2\ 3\ 4\ 5) \\
\pi_2 &= (1\ 3\ 4\ 5) \\
\pi_3 &= (1\ 2\ 4\ 5) \\
\pi_4 &= (1\ 2\ 3\ 5) \\
\pi_5 &= (1\ 2\ 3\ 4).
\end{aligned}$$

Thus, by Theorem 7.13, this 5-tuple describes a 2-cell embedding of K_5 on some surface S_k. To evaluate k, we consider the digraph D obtained by replacing each edge of K_5 by a symmetric pair of arcs and determine the orbits of the permutation $\pi : E(D) \to E(D)$ defined in the proof of Theorem 7.13. The orbits are

$$\begin{aligned}
R_1 &: \quad v_1 - v_2 - v_3 - v_4 - v_5 - v_1 \\
R_2 &: \quad v_1 - v_3 - v_2 - v_4 - v_3 - v_5 - v_4 - v_1 - v_5 - v_2 - v_1 \\
R_3 &: \quad v_1 - v_4 - v_2 - v_5 - v_3 - v_1
\end{aligned}$$

and each orbit corresponds to a 2-cell region. Thus, the number of regions in the embedding is $r = 3$. Since K_5 has order $n = 5$ and size $m = 10$, and since

$$n - m + r = -2 = 2 - 2k,$$

it follows that $k = 2$, so that the given 5-tuple describes an embedding of K_5 on S_2.

Given an n-tuple of cyclic permutations as we have described, it is not necessarily an easy problem to present a geometric description of the embedding, particularly on surfaces of high genus. For the example just presented, however, we give two geometric descriptions in Figure 7.19. In Figure 7.19(a), a portion of K_5 is drawn in the plane. Two handles are then inserted over the plane, as indicated, and the remainder of K_5 is drawn along these handles. The edge $e_1 = v_2v_5$ is drawn along the handle H_1, the edge $e_2 = v_3v_5$ is drawn along H_2 while $e_3 = v_1v_3$ is drawn along both H_1 and H_2. The three 2-cell regions produced are denoted by R_1, R_2 and R_3. In Figure 7.19(b), this 2-cell embedding of K_5 on S_2 is shown on the regular octagon. The labeling of the eight sides (as in (7.3)) indicates the identification used in producing S_2.

(a)

(b)

Figure 7.19: A 2-cell embedding of K_5 on the double torus

As a more general illustration of Theorem 7.13 we determine the genus of the complete bipartite graph $K_{2a,2b}$, where $1 \leq a \leq b$. According to Theorem 7.10,

$$\gamma(K_{2a,2b}) = (a-1)(b-1).$$

That $(a-1)(b-1)$ is a lower bound for $\gamma(K_{2a,2b})$ follows from Corollary 7.6 (see also Exercise 5 in Section 7.1). We use Theorem 7.13 to show that $K_{2a,2b}$ is 2-cell embeddable on $S_{(a-1)(b-1)}$, thereby proving that $\gamma(K_{2a,2b}) \leq (a-1)(b-1)$ and completing the argument.

Denote the partite sets of $K_{2a,2b}$ by U and W, where $|U| = 2a$ and $|W| = 2b$. Furthermore, label the vertices so that

$$U = \{v_1, v_3, v_5, \ldots, v_{4a-1}\} \text{ and } W = \{v_2, v_4, v_6, \ldots, v_{4a}, v_{4a+2}, \ldots, v_{4b}\}.$$

Let there be given the $(2a+2b)$-tuple

$$(\pi_1, \pi_2, \ldots, \pi_{4a-1}, \pi_{4a}, \pi_{4a+2}, \pi_{4a+4}, \ldots, \pi_{4b}),$$

where

$$\begin{aligned}
\pi_1 &= \pi_5 = \cdots = \pi_{4a-3} = (2\ 4\ 6\ \ldots\ 4b),\\
\pi_3 &= \pi_7 = \cdots = \pi_{4a-1} = (4b\ \ldots\ 6\ 4\ 2),\\
\pi_2 &= \pi_6 = \cdots = \pi_{4b-2} = (1\ 3\ 5\ \ldots\ 4a-1),\\
\pi_4 &= \pi_8 = \cdots = \pi_{4b} = (4a-1\ \ldots\ 5\ 3\ 1).
\end{aligned}$$

By Theorem 7.13, then, this $(2a+2b)$-tuple describes a 2-cell embedding of $K_{2a,2b}$ on some surface S_h. In order to evaluate h, let D denote the digraph obtained by replacing each edge of $K_{2a,2b}$ by a symmetric pair of arcs and determine the orbits of the permutation $\pi : E(D) \to E(D)$ defined in the proof of Theorem 7.13.

Every orbit of π contains an arc of the type (v_s, v_t), where $v_s \in U$ and $v_t \in W$. If $s \equiv 1 \pmod 4$ and $t \equiv 2 \pmod 4$, then the resulting orbit R containing (v_s, v_t) is

$$R:\ v_s - v_t - v_{s+2} - v_{t-2} - v_s,$$

with $s+2$ expressed modulo $4a$ and $t-2$ expressed modulo $4b$. Note that R also contains the arc (v_{s+2}, v_{t-2}), where, then, $s+2 \equiv 3 \pmod 4$ and $t-2 \equiv 0 \pmod 4$. If $s \equiv 1 \pmod 4$ and $t \equiv 0 \pmod 4$, then the orbit R' containing (v_s, v_t) is

$$R':\ v_s - v_t - v_{s-2} - v_{t-2} - v_s,$$

where, again, $s-2$ is expressed modulo $4a$ and $t-2$ is expressed modulo $4b$. The orbit R' also contains the arc (v_{s-2}, v_{t-2}), where, $s-2 \equiv 3 \pmod 4$ and $t-2 \equiv 2 \pmod 4$. Thus every orbit of π is either of the type R (where $s \equiv 1 \pmod 4$ and $t \equiv 2 \pmod 4$) or of the type R' (where $s \equiv 1 \pmod 4$ and $t \equiv 0 \pmod 4$). Since there are a choices for s and b choices for t in each case, the total number of orbits is $2ab$; therefore, the number of regions in this embedding is $r = 2ab$.

Since $K_{2a,2b}$ has order $n = 2a+2b$ and size $m = 4ab$ and because $n-m+r = 2-2h$, we have

$$(2a+2b) - 4ab + 2ab = 2 - 2h,$$

so that $h = (a-1)(b-1)$. Hence there is a 2-cell embedding of $K_{2a,2b}$ on $S_{(a-1)(b-1)}$, as we wished to show.

As a theoretical application of Theorem 7.13, we present a result that is referred to as the Ringeisen-White Edge-Adding Lemma (see [184]), named for Richard D. Ringeisen and Arthur T. White.

Theorem 7.14 (**The Ringeisen-White Edge-Adding Lemma**) *Let G be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that v_i and v_j $(i \neq j)$ are*

nonadjacent vertices. Suppose that there exists a 2*-cell embedding of* G *on some surface* S_h *with* r *regions such that* v_i *is on the boundary of region* R_i *and* v_j *is on the boundary of region* R_j*. Let* $H = G + v_iv_j$*. Then*

(*i*) *if* $R_i \neq R_j$*, then there exists a* 2*-cell embedding of* H *on* S_{h+1} *with* $r - 1$ *regions in which* v_i *and* v_j *are on the boundary of the same region;*

(*ii*) *if* $R_i = R_j$*, then there exists a* 2*-cell embedding of* H *on* S_h *with* $r + 1$ *regions in which each of* v_i *and* v_j *belongs to the boundaries of* (*the same*) *two distinct regions.*

Proof. By hypothesis, there exists a 2-cell embedding of the graph G of order n and size m on S_h with r regions such that v_i is on the boundary of region R_i and v_j is on the boundary of region R_j. By the Rotational Embedding Scheme (Theorem 7.13), an n-tuple $(\pi_1, \pi_2, \ldots, \pi_n)$ of cyclic permutations corresponds to this embedding, namely for $t = 1, 2, \ldots, n$, $\pi_t : V(t) \to V(t)$ is a cyclic permutation of the subscripts of the vertices of $N(v_t)$ in counterclockwise order about v_t.

Let D denote the symmetric digraph obtained from G by replacing each edge by a symmetric pair of arcs and let $\pi : E(D) \to E(D)$ be defined by $\pi(v_\alpha, v_\beta) = (v_\beta, \pi_{\beta(\alpha)})$. Since the given embedding has r regions, π has r orbits. Denote each region and its corresponding orbit by the same symbol; in particular, R_i and R_j are orbits of π.

Suppose that $R_i \neq R_j$. We can therefore represent the orbits R_i and R_j as

$$R_i : \ v_i - v_k - \cdots - v_{k'} - v_i$$

and

$$R_j : \ v_j - v_\ell - \cdots - v_{\ell'} - v_j.$$

It therefore follows that

$$\pi_i(k') = k \text{ and } \pi_j(\ell') = \ell.$$

We now consider the graph $H = G + v_iv_j$ and define

$$V'(t) = \{s : \ v_sv_t \in E(H)\}$$

for $t = 1, 2, \ldots, n$. Thus $V'(t) = V(t)$ for $t \neq i, j$, and $V'(i) = V(i) \cup \{j\}$ while $V'(j) = V(j) \cup \{i\}$. For the graph H, we define an n-tuple $(\pi'_1, \pi'_2, \ldots, \pi'_n)$ of cyclic permutations, where $\pi'_i : V'(t) \to V'(t)$ for $t = 1, 2, \ldots, n$ such that $\pi'_t = \pi_t$ for $t \neq i, j$. Furthermore,

$$\pi'_i(a) = \begin{cases} \pi_i(a) & \text{if } a \neq k' \\ j & \text{if } a = k' \\ k & \text{if } a = j \end{cases}$$

and

$$\pi'_j(a) = \begin{cases} \pi_j(a) & \text{if } a \neq \ell' \\ i & \text{if } a = \ell' \\ \ell & \text{if } a = i. \end{cases}$$

Let D' be the digraph obtained from H by replacing each edge of H by a symmetric pair of arcs. Define the permutation $\pi' : E(D') \to E(D')$ by $\pi'(v_\alpha, v_\beta) = (v_\beta, v_{\pi'_\beta(\alpha)})$. The orbits of π' then consist of all orbits of π different from R_i and R_j together with the orbit

$$R: \ v_i - v_j - v_\ell - \cdots - v_{\ell'} - v_j - v_i - v_k - \cdots - v_{k'} - v_i.$$

Thus, π' has $r-1$ orbits and the corresponding 2-cell embedding of H has $r-1$ regions. Moreover, v_i and v_j lie on the boundary of R. Since $n-m+r = 2-2h$, it follows that $n-(m+1)+(r-1) = 2-2(h+1)$ and H is 2-cell embedded on S_{h+1}. This completes the proof of (i).

Suppose next that $R_i = R_j$. We can represent the orbit R_i $(= R_j)$ as

$$R_i: \ v_i - v_k - \cdots - v_{\ell'} - v_j - v_\ell - \cdots - v_{k'} - v_i.$$

(Note that v_i and v_j cannot be consecutive in R_i since $v_iv_j \notin E(G)$.) It follows that

$$\pi_i(k') = k \text{ and } \pi_j(\ell') = \ell.$$

We again consider the graph $H = G + v_iv_j$ and once more define

$$V'(t) = \{s: \ v_sv_t \in E(H)\}$$

for $t = 1, 2, \ldots, n$. We define an n-tuple $(\pi'_1, \pi'_2, \ldots, \pi'_n)$ of cyclic permutations, where $\pi'_t : V'(t) \to V'(t)$ for $t = 1, 2, \ldots, n$ such that $\pi'_t = \pi_t$ for $t \neq i, j$. Also,

$$\pi'_i(a) = \begin{cases} \pi_i(a) & \text{if } a \neq k' \\ j & \text{if } a = k' \\ k & \text{if } a = j \end{cases}$$

and

$$\pi'_j(a) = \begin{cases} \pi_j(a) & \text{if } a \neq \ell' \\ i & \text{if } a = \ell' \\ \ell & \text{if } a = i. \end{cases}$$

Again we denote by D' the digraph obtained from H by replacing each edge of H by a symmetric pair of arcs and define the permutation $\pi' : E(D') \to E(D')$ by $\pi'(v_\alpha, v_\beta) = (v_\beta, v_{\pi'_\beta(\alpha)})$. The orbits of π' consist of all orbits of π different from R_i together with the orbits

$$R': \ v_i - v_j - v_\ell - \cdots - v_{k'} - v_i$$

and

$$R'' : v_j - v_i - v_k - \cdots - v_{\ell'} - v_j.$$

Therefore, π' has $r+1$ orbits and the resulting 2-cell embedding of H has $r+1$ regions. Furthermore, each of v_i and v_j belongs to the boundaries of both R' and R''. Here $n - m + r = 2 - 2h$ implies that $n - (m+1) + (r+1) = 2 - 2h$, and H is 2-cell embedded on S_h, which verifies (ii). ■

A consequence of Theorem 7.14 will prove to be useful.

Corollary 7.15 *Let e and f be adjacent edges of a connected graph G. If there exists a 2-cell embedding of $G' = G - e - f$ with one region, then there exists a 2-cell embedding of G with one region.*

Proof. Let $e = uv$ and $f = vw$, where then $u \neq w$. Let there be given a 2-cell embedding of G' with one region R. Thus all vertices of G' belong to the boundary of R, including u and v. By Theorem 7.14(ii), there exists a 2-cell embedding of $G' + e$ with two regions where u and v lie on the boundary of both regions. Therefore, v is on the boundary of one region and w is on the boundary of the other region in the 2-cell embedding of $G' + e$. Applying Theorem 7.14(i), we conclude that there exists a 2-cell embedding of $G' + e + f = G$ with one region. ■

Exercises for Section 7.2

1. (a) For the 2-cell embedding of $K_{3,3}$ shown in Figure 7.20(a), determine the 6-tuple of cyclic permutations π_i associated with this embedding. Determine the orbits of the resulting permutation π.

Figure 7.20: The embeddings of $K_{3,3}$ and K_7 in Exercise 1

(b) For the 2-cell embedding of K_7 on S_1 shown in Figure 7.20(b), determine the 7-tuple of cyclic permutations π_i $(0 \leq i \leq 6)$ associated with this embedding. Determine the orbits of the resulting permutation π.

2. Let $G = K_4 \,\square\, K_2$. Show that $\gamma(G) = 1$ by finding an 8-tuple of cyclic permutations that describes a 2-cell embedding of G on S_1. Determine the orbits of the resulting permutation π.

3. Let G be a graph with $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and let (π_1, π_2, π_3, π_4, π_5, π_6) describe a 2-cell embedding of G on the surface S_k, where

$$\pi_1 = (2\ 5\ 6\ 3),\ \pi_2 = (3\ 6\ 1\ 4),\ \pi_3 = (4\ 1\ 2\ 5),$$
$$\pi_4 = (5\ 2\ 3\ 6),\ \pi_5 = (6\ 3\ 4\ 1),\ \pi_6 = (1\ 4\ 5\ 2).$$

   (a) What is this familiar graph G?
   (b) What is k?
   (c) Is $k = \gamma(G)$?

4. A graph G is obtained from $K_{4,4}$ by removing four pairwise nonadjacent edges. Let $V(G) = \{v_1, v_2, \ldots, v_8\}$. The 8-tuple $(\pi_1, \pi_2, \ldots, \pi_8)$ of cyclic permutations defined by

$$\pi_1 = (2\ 8\ 6)\ \ \pi_2 = (1\ 5\ 3)\ \ \pi_3 = (2\ 48\ )\ \ \pi_4 = (3\ 5\ 7)$$
$$\pi_5 = (2\ 6\ 4)\ \ \pi_6 = (7\ 5\ 1)\ \ \pi_7 = (4\ 6\ 8)\ \ \pi_8 = (1\ 7\ 3)$$

   describes a 2-cell embedding of G on some surface S_k.

   (a) What is k?
   (b) Is $k = \gamma(G)$?

5. How many of the 2-cell embeddings of K_4 are embeddings in the plane? On the torus? On the double torus?

6. (a) Describe an embedding of $K_{3,3}$ on S_2 by means of a 6-tuple of cyclic permutations.
   (b) Show that there exists no 2-cell embedding of $K_{3,3}$ on S_3.

7.3 The Maximum Genus of a Graph

We now turn our attention to the following problem:

For a given connected graph G, determine those surfaces S_k on which there exists a 2-cell embedding of G.

Duke's Theorem

As a major step toward solving this problem, we present an "interpolation theorem" due to Richard A. Duke [63].

Theorem 7.16 (**Duke's Theorem**) *If there exist 2-cell embeddings of a connected graph G on the surfaces S_p and S_q, where $p \leq q$, and k is any integer such that $p \leq k \leq q$, then there exists a 2-cell embedding of G on the surface S_k.*

Proof. Observe that there exist 2-cell embeddings of K_1 only on the sphere; thus, we assume that G is nontrivial. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 2$.

Assume that there exists a 2-cell embedding of G on some surface S_ℓ. By the Rotational Embedding Scheme (Theorem 7.13), there exists an n-tuple $(\pi_1, \pi_2, \ldots, \pi_n)$ of cyclic permutations associated with this embedding such that for $i = 1, 2, \ldots, n$, $\pi_i : V(i) \to V(i)$ is a cyclic permutation of the subscripts of the vertices of $N(v_i)$ in counterclockwise order about v_i.

Let D be the symmetric digraph obtained from G by replacing each edge of G by a symmetric pair of arcs. Let $\pi : E(D) \to E(D)$ be the permutation defined by $\pi(v_i, v_j) = (v_j, v_{\pi_j(i)})$. Denote the number of orbits in π by r; that is, assume that there are r 2-cell regions in the given embedding of G on S_ℓ.

Assume that there exists some vertex of G, say v_1, such that $\deg v_1 \geq 3$. Then $\pi_1 = (a\ b\ c\ \cdots)$, where a, b and c are distinct. Let v_x be any vertex adjacent to v_1 other then v_a and v_b, and suppose that $\pi_1(x) = y$. Thus

$$\pi_1 = (a\ b\ c\ \cdots\ x\ y\ \cdots),$$

where, possibly, $x = c$ or $y = a$. Let E_1 be the subset of $E(D)$ consisting of the three pairs of arcs

$$(v_a, v_1), (v_1, v_b);\ (v_b, v_1), (v_1, v_c);\ (v_x, v_1), (v_1, v_y). \tag{7.5}$$

Note that the six arcs listed in (7.5) are all distinct. By the definition of the permutation π, we have

$$\pi(v_a, v_1) = (v_1, v_b),\ \ \pi(v_b, v_1) = (v_1, v_c)\ \text{ and }\ \pi(v_x, v_1) = (v_1, v_y).$$

This implies that the arc (v_a, v_1) is followed by the arc (v_1, v_b) in some orbit of π, and that the edge $v_a v_1$ of G is followed by the edge $v_1 v_b$ as we proceed clockwise around the boundary of the corresponding region in the given embedding of G in S_ℓ. Also, (v_b, v_1) is followed by (v_1, v_c) in some orbit of π and (v_x, v_1) is followed by (v_1, v_y) in some orbit.

We now define a new permutation $\pi' : E(D) \to E(D)$ with the aid of the n-tuple $(\pi'_1, \pi'_2, \ldots, \pi'_n)$, where for $i = 1, 2, \ldots, n$, $\pi'_i : V(i) \to V(i)$ is a cyclic permutation defined by

$$\pi'_i = \begin{cases} (a\ c\ \cdots x\ b\ y\ \cdots) & \text{if } i = 1 \\ \pi_i & \text{if } 2 \leq i \leq n. \end{cases}$$

We then define $\pi'(v_i, v_j) = (v_j, v_{\pi'_j(i)})$. By the Rotational Embedding Scheme (Theorem 7.13), the n-tuple $(\pi'_1, \pi'_2, \ldots, \pi'_n)$ determines a 2-cell embedding of G on some surface, where for $i = 1, 2, \ldots, n$, π'_i is a cyclic permutation of the subscripts of the vertices adjacent to v_i in counterclockwise order about v_i.

Three cases are now considered, depending on the possible distribution of the pairs (7.5) of arcs in E_1 among the orbits of π.

Case 1. *All arcs of E_1 belong to a single orbit R of π.* Suppose, first, that the orbit R has the form

$$R: \ v_1 - v_y - \cdots - v_b - v_1 - v_c - \cdots - v_a - v_1 - v_b - \cdots - v_x - v_1.$$

Here the orbits of π' are the orbits of π except that the orbit R is replaced by the three orbits

$$\begin{aligned} R'_1 &: \ v_1 - v_y - \cdots - v_b - v_1, \\ R'_2 &: \ v_1 - v_c - \cdots - v_a - v_1, \\ R'_3 &: \ v_1 - v_b - \cdots - v_x - v_1. \end{aligned}$$

Hence, π' describes a 2-cell embedding of G with $r + 2$ regions on a surface S'. Necessarily, then, $S' = S_{\ell-1}$.

The other possible form that the orbit R may take is

$$R: \ v_1 - v_y - \cdots - v_a - v_1 - v_b - \cdots - v_b - v_1 - v_c - \cdots - v_x - v_1.$$

In this situation, the orbits of π' are the orbits of π, except for R, which is replaced by the orbit

$$R': \ v_1 - v_y - \cdots - v_a - v_1 - v_c - \cdots - v_x - v_1 - v_b - \cdots - v_b - v_1.$$

Hence, π' has r orbits.

Case 2. *The permutation π has two orbits, say R_1 and R_2, where R_1 contains two of the pairs of arcs in E_1 and R_2 contains the remaining pair of arcs.* In this case, the orbits of π' are those of π, except for R_1 and R_2, which are replaced by two orbits R'_1 and R'_2, where one of R'_1 and R'_2 contains two arcs of E_1 and the other contains the remaining four arcs of E_1. In this case, π' has r orbits.

Case 3. *The permutation π has three orbits R_1, R_2 and R_3 such that (v_a, v_1) is followed by (v_1, v_b) in R_1, (v_b, v_1) is followed by (v_1, v_c) in R_2 and (v_x, v_1) is followed by (v_1, v_y) in R_3.* In this case, the orbits of π' are the orbits of π, except for R_1, R_2 and R_3, which are replaced by a single orbit R' of the form

$$R': \ v_1 - v_y - \cdots - v_x - v_1 - v_b - \cdots - v_a - v_1 - v_c - \cdots - v_b - v_1.$$

In this case, π' has $r - 2$ orbits so that π' describes a 2-cell embedding of G on $S_{\ell+1}$.

Thus, we can now conclude that the shifting of a single term in π_1 (producing π_1') changes the genus of the resulting surface on which G is 2-cell embedded by at most 1. Having made this observation, we can now complete the proof.

Let $(\mu_1, \mu_2, \ldots, \mu_n)$ be the n-tuple of cyclic permutations associated with a 2-cell embedding of G on S_p and let $(\nu_1, \nu_2, \ldots, \nu_n)$ be the n-tuple of cyclic permutations associated with a 2-cell embedding of G on S_q. If the degree of v_i is 1 or 2 for each i, $1 \leq i \leq n$, then $\mu_i = \nu_i$ so that $p = q$ and the desired result follows. Hence, we may assume that for some i, $1 \leq i \leq n$, $\deg v_i \geq 3$. For each such i, μ_i can be transformed into ν_i by a finite number of single term shifts, as described above. Each such single term shift describes an embedding of G on a surface whose genus differs by at most 1 from the genus of the surface on which G is embedded prior to the shift. Therefore, by performing sequences of single term shifts on those μ_i for which $\deg v_i \geq 3$, the n-tuple $(\mu_1, \mu_2, \ldots, \mu_n)$ can be transformed into $(\nu_1, \nu_2, \ldots, \nu_n)$. Since $p \leq k \leq q$, there must be at least one term $(\pi_1, \pi_2, \ldots, \pi_n)$ in the aforementioned sequence beginning with $(\mu_1, \mu_2, \ldots, \mu_n)$ and ending with $(\nu_1, \nu_2, \ldots, \nu_n)$ that describes a 2-cell embedding of G on S_k. ■

Suppose that G is a connected graph with $\gamma(G) = p$ and q is the largest positive integer such that G is 2-cell embeddable on S_q. It then follows from Theorem 7.16 that G can be 2-cell embedded on S_k if and only if $p \leq k \leq q$. This leads us to the following concept.

Let G be a connected graph. The **maximum genus** $\gamma_M(G)$ of G is the maximum integer k for which G can be 2-cell embedded on S_k. At the outset, it may not even be clear that every graph has a maximum genus since, perhaps, some graphs are 2-cell embeddable on infinitely many surfaces. However, there are no graphs that can be 2-cell embedded on infinitely many surfaces, for suppose that G is a nontrivial connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. By the Rotational Embedding Scheme (Theorem 7.13), there exists a one-to-one correspondence between the set of all 2-cell embeddings of G and the n-tuples $(\pi_1, \pi_2, \ldots, \pi_n)$, where for $i = 1, 2, \ldots, n$, $\pi_i : V(i) \to V(i)$ is a cyclic permutation. Since the number of such n-tuples is finite, and in fact is equal to

$$\prod_{i=1}^{n} (\deg v_i - 1)!,$$

it follows that there are only finitely many 2-cell embeddings of G and therefore that there exists a surface of maximum genus on which G can be 2-cell embedded. We can now state an immediate consequence of Theorem 7.16.

Corollary 7.17 *A connected graph G has a 2-cell embedding on the surface S_k if and only if*

$$\gamma(G) \leq k \leq \gamma_M(G).$$

We now present an upper bound for the maximum genus of any connected graph. This bound employs a new but useful concept.

The **Betti number** $\mathcal{B}(G)$ of graph G of order n and size m having k components is defined as

$$\mathcal{B}(G) = m - n + k.$$

Thus, if G is a connected graph, then

$$\mathcal{B}(G) = m - n + 1,$$

which is the same as the cycle rank of G. The following result is due to Edward A. Nordhaus, Bonnie M. Stewart and Arthur T. White [165].

Theorem 7.18 *If G is a connected graph, then*

$$\gamma_M(G) \leq \left\lfloor \frac{\mathcal{B}(G)}{2} \right\rfloor.$$

Furthermore, equality holds if and only if there exists a 2-cell embedding of G on the surface of genus $\gamma_M(G)$ with exactly one or exactly two regions according to whether $\mathcal{B}(G)$ is even or odd, respectively.

Proof. Let G be a connected graph of order n and size m that is 2-cell embedded on the surface of genus $\gamma_M(G)$, producing r (2-cell) regions. By Theorem 7.1

$$n - m + r = 2 - 2\gamma_M(G).$$

Thus,

$$\mathcal{B}(G) = m - n + 1 = 2\gamma_M(G) + r - 1,$$

so that

$$\gamma_M(G) = \frac{\mathcal{B}(G) + 1 - r}{2} \leq \frac{\mathcal{B}(G)}{2},$$

producing the desired bound.

Moreover, we have

$$\gamma_M(G) = \frac{\mathcal{B}(G) + 1 - r}{2} = \left\lfloor \frac{\mathcal{B}(G)}{2} \right\rfloor,$$

if and only if $r = 1$ (which can only occur when $\mathcal{B}(G)$ is even) or $r = 2$ (which is only possible when $\mathcal{B}(G)$ is odd). ■

Upper Embeddable Graphs

A (connected) graph G is called **upper embeddable** if the maximum genus of G attains the upper bound given in Theorem 7.18, that is, if $\gamma_M(G) = \lfloor \mathcal{B}(G)/2 \rfloor$. The graph G is said to be **upper embeddable on a surface** S if $S = S_{\gamma_M(G)}$. We can now state an immediate consequence of Theorem 7.18.

Corollary 7.19 *Let G be a connected graph with even (odd) Betti number. Then G is upper embeddable on a surface S if and only if there exists a 2-cell embedding of G on S with one region (two regions).*

In order to present a characterization of upper embeddable graphs, we introduce a special kind of spanning tree of a connected graph.

A spanning tree T of a connected graph G is a **splitting tree** of G if at most one component of $G - E(T)$ has odd size. It follows therefore that if $G - E(T)$ is connected, then T is a splitting tree. For the graph G of Figure 7.21, the tree T_1 is a splitting tree. On the other hand, T_2 is not a splitting tree of G.

Figure 7.21: A splitting tree and a non-splitting tree

The following observation that relates splitting trees and Betti numbers is elementary, but useful.

Theorem 7.20 *Let T be a splitting tree of a graph G of order n and size m. Then every component of $G - E(T)$ has even size if and only if $\mathcal{B}(G)$ is even.*

Proof. Suppose that every component of $G - E(T)$ has even size. Then $G - E(T)$ has even size. Since every tree of order n has size $n - 1$, the graph $G - E(T)$ has size $m - (n - 1) = m - n + 1$. Therefore, $\mathcal{B}(G) = m - n + 1$ is even.

Conversely, suppose that $\mathcal{B}(G)$ is even. The graph $G - E(T)$ has size $m - n + 1 = \mathcal{B}(G)$. Since T is a splitting tree of G, at most one component of $G - E(T)$ has odd size. Since the sum of the sizes of the components of $G - E(T)$ is even, it is impossible for exactly one such component to have odd size, producing the desired result. ■

We now state a characterization of upper embeddable graphs, which was discovered independently by Mark Jungerman [128] and Nguyen Huy Xuong [244].

Theorem 7.21 *A graph G is upper embeddable if and only if G has a splitting tree.*

Proof. Suppose that G is a graph of order n and size m having a splitting tree T. We show that G is upper embeddable by considering two cases, depending on the parity of the Betti number of G.

Case 1. $\mathcal{B}(G) = m - n + 1$ *is even.* By Theorem 7.20, each component of $G - E(T)$ has even size. By Theorems 3.1 and 3.3, every component F of $G - E(T)$ is trivial or Eulerian, or the edge set of F can be partitioned into subsets each of which induces an open trail of even length. Thus the edge set of each nontrivial component of $G - E(T)$ can be partitioned into adjacent pairs of edges, implying that $E(G) - E(T)$ can be partitioned into $(m-n+1)/2 = \mathcal{B}(G)/2$ subsets, each inducing a subgraph F_i, $1 \leq i \leq \mathcal{B}(G)/2$, isomorphic to P_3.

The tree T is 2-cell embeddable on the sphere with a single region. By repeated application of Corollary 7.15, it follows that the graph

$$T + \bigcup_{i=1}^{k} E(F_i)$$

is 2-cell embeddable for $k = 1, 2, \ldots, \mathcal{B}(G)/2$ on some surface with a single region. Since

$$G = T + \bigcup_{i=1}^{\mathcal{B}(G)/2} E(F_i),$$

we conclude that G is 2-cell embeddable on a surface with one region, which, by Corollary 7.19, implies that G is upper embeddable.

Case 2. $\mathcal{B}(G)$ *is odd.* Necessarily, $G - E(T)$ has exactly one component of odd size, which we denote by H. If H is a tree, let e be a pendant edge of H; otherwise, let e be an edge lying on a cycle of H. Since every bridge of G belongs to T, it follows that $G - e$ is connected. Furthermore, since $H - e$ has only components of even size, every component of $G - E(T) - e = G - e - E(T)$ has even size and T is a splitting tree of $G - e$. Since $\mathcal{B}(G - e)$ is even, we conclude from Case 1 that $G - e$ is upper embeddable and so is 2-cell embeddable on a surface with one region. Applying Theorem 7.14(ii), we see that there is a 2-cell embedding of $(G - e) + e = G$ on a surface with two regions which, by Corollary 7.19, implies that G is upper embeddable.

For the converse, suppose that G is an upper embeddable graph of order n and size m (and consequently a connected graph) with vertex set $\{v_1, v_2, \ldots, v_n\}$. To show that G has a splitting tree, we again consider two cases according to the parity of the Betti number of G.

Case 1. $\mathcal{B}(G) = m - n + 1$ *is even.* We proceed by induction on the size m of G. If $m = 0$, then $G = K_1$ and G is a splitting tree of itself. Assume that all upper embeddable graphs with even Betti number and having size less than m contain splitting trees.

Since G is upper embeddable, it follows from Corollary 7.19 that G is 2-cell embeddable on the surface of genus $\gamma_M(G)$ with one region, say R. By Theorem 7.13, corresponding to this 2-cell embedding, there exists an n-tuple $(\pi_1, \pi_2, \ldots, \pi_n)$, where for $i = 1, 2, \ldots, n$, $\pi_i : V(i) \to V(i)$ is a cyclic permutation that describes the subscripts of the vertices adjacent to v_i in counterclockwise order about v_i. Denote by D the digraph obtained from G by replacing each edge of G by a symmetric pair of arcs. Since the embedding has one region, the mapping $\pi : V(D) \to V(D)$ defined by

$$\pi(v_i, v_j) = (v_j, v_{\pi_j(i)})$$

has a single orbit, also denoted by R. Thus in R all arcs of D appear in a fixed (cyclic) order. Among the edges of G, let $e = v_\ell v_q$ be one with the property that in the orbit of R, the resulting arcs $a_1 = (v_\ell, v_q)$ and $a_2 = (v_q, v_\ell)$ have the minimum number k of arcs between them. Without loss of generality, we assume that in R the number of arcs following a_1 but preceding a_2 is k. Thus the number of arcs following a_2 and preceding a_1 is $2m - 2 - k$, where then $k \leq 2m - 2 - k$.

If $k = 0$, then a_1 and a_2 appear consecutively in the orbit R, in the order a_1, a_2. This further implies that $\deg_G v_q = 1$. The graph $G - v_q$ is clearly 2-cell embeddable on the same surface on which G is 2-cell embedded and also with a single region. Thus $G - v_q$ is upper embeddable. Since $G - v_q$ has size $m - 1$ and since $\mathcal{B}(G - v_q)$ is even, we may apply the induction hypothesis and conclude that $G - v_q$ has a splitting tree T_1. Thus the tree T produced by adding v_q and the edge e to T_1 is a splitting tree of G since the components of $G - E(T)$ are the components of $G - v_q - E(T_1)$ together with an additional trivial component, which, of course, has even size.

Assume, then that $k > 0$. Thus, by assumption, in the orbit R there are k arcs following a_1 but preceding a_2. Suppose that b_1 is the arc immediately following a_1 in R, so that $b_1 = (v_q, v_p)$, where $p \neq \ell$. Further, let $f = v_q v_p$ denote the corresponding edge of G and let $b_2 = (v_p, v_q)$ denote the other related arc of D. Necessarily the orbit R is of the form

$$R : (a_1\ b_1\ \cdots\ a_2\ \cdots\ b_2\ \cdots);$$

that is, b_2 must follow a_2 and precede a_1. Otherwise, if b_2 follows b_1 but precedes a_2, then the number of arcs in R between b_1 and b_2 is less than k, contradicting the defining property of the edge e. The single orbit R of π can also be described as follows:

$$\begin{aligned} R :\ & v_\ell - v_q - v_p - v_r - \cdots - v_s - v_q - v_\ell - v_t - \cdots \\ & -v_x - v_p - v_q - v_y - \cdots - v_z - v_\ell. \end{aligned}$$

This implies that $G - e - f$ contains the closed spanning walk

$$(v_\ell, v_t, \ldots, v_x, v_p, v_r, \ldots, v_s, v_q, v_y, \ldots, v_z, v_\ell);$$

thus $G - e - f$ is connected. The description of the orbit R also implies, for example, that

$$\pi_\ell(q) = t, \;\; \pi_q(p) = y \;\text{ and }\; \pi_p(q) = r.$$

We next show that $G - e - f$ is 2-cell embeddable on a surface with a single region. Define

$$V'(i) = \{j : \; v_i v_j \in E(G - e - f)\}$$

for $i = 1, 2, \ldots, n$. Therefore, $V(i) = V'(i)$ for $i \neq \ell, q, p$, while $V(\ell) = V'(\ell) \cup \{q\}$, $V(q) = V'(q) \cup \{\ell, p\}$ and $V(p) = V'(p) \cup \{q\}$. For the graph $G - e - f$, we define an n-tuple $(\pi'_1, \pi'_2, \ldots, \pi'_n)$ of cyclic permutations, where $\pi'_i : V'(i) \to V'(i)$ for $i = 1, 2, \ldots, n$ such that $\pi'_i = \pi_i$ for $i \neq \ell, q, p$. Moreover, $\pi'_\ell(j) = \pi_\ell(j)$ for $j \neq z$ while $\pi'_\ell(z) = \pi'_\ell(q) = t$. Also $\pi'_q(j) = \pi_q(j)$ if $j \neq s$ and $\pi'_q(s) = \pi'_q(p) = y$. Finally, $\pi'_p(j) = \pi_p(j)$ for $j \neq x$, while $\pi'_p(x) = \pi_p(q) = r$.

Let D' be the digraph obtained from $G - e - f$ by replacing each edge of $G - e - f$ by a symmetric pair of arcs. Define the permutation $\pi' : E(D') \to E(D')$ by $\pi'(v_i, v_j) = (v_j, v_{\pi'_j(i)})$. There is a single orbit of π', namely

$$R' : \; v_\ell - v_t - \cdots - v_x - v_p - v_r - \cdots - v_s - v_q - v_y - \cdots - v_z - v_\ell.$$

Hence, corresponding to the n-tuple $(\pi'_1, \pi'_2, \ldots, \pi'_n)$ is a 2-cell embedding of $G - e - f$ with one region. Therefore, $G - e - f$ is upper embeddable. Since $\mathcal{B}(G - e - f)$ is even, $G - e - f$ has a splitting tree T by the induction hypothesis.

By Theorem 7.20, every component of $G - e - f - E(T)$ has even size. Returning the two adjacent edges e and f to $G - e - f - E(T)$ produces either the same number of components (if e and f are incident only to vertices in the same component of $G - e - f - E(T)$ or fewer components (if e and f are incident with vertices in two or three components of $G - e - f - E(T)$). In either case, every component of $G - E(T)$ has even size and T is a splitting tree of G.

Case 2. *$\mathcal{B}(G)$ is odd.* Since G is upper embeddable, it follows by Corollary 7.19 that there exists a 2-cell embedding of G with two regions. Let e be an edge of G that is on the boundary of both regions. (Necessarily, e is not a bridge of G.) Deleting e produces a 2-cell embedding of $G - e$ with one region. Therefore, $G - e$ is upper embeddable and since $\mathcal{B}(G - e)$ is even, it follows by the preceding case that $G - e$ has a splitting tree T. Furthermore, by Theorem 7.20 every component of $G - e - E(T)$ has even size. Returning the edge e to $G - e - E(T)$ produces a graph, namely $G - E(T)$, having exactly one component of odd size. Therefore, T is also a splitting tree of G. ■

Returning to the graph G of Figure 7.21, we now see that G is upper embeddable since G contains T_1 as a splitting tree. On the other hand, neither the graph G_1 nor the graph G_2 of Figure 7.22 has a single splitting tree, so, by Theorem 7.21, neither of these graphs is upper embeddable.

Figure 7.22: Graphs that are not upper embeddable

A Formula for the Maximum Genus

We mentioned earlier that no formula is known for the genus of an arbitrary graph. However, such is not the case with maximum genus. With the aid of Theorem 7.21, Xuong [244] developed a formula for the maximum genus of every connected graph.

For a graph H we denote by $\xi_0(H)$ the number of components of H of odd size. For a connected graph G, we define the number $\xi(G)$ as follows:

$$\xi(G) = \min\left(\xi_0(G - E(T))\right),$$

where the minimum is taken over all spanning trees T of G.

Theorem 7.22 *The maximum genus of a connected graph G is given by*

$$\gamma_M(G) = \frac{1}{2}\left(\mathcal{B}(G) - \xi(G)\right).$$

Proof. Let G have order n and size m and suppose that G is 2-cell embedded on the surface of genus $\gamma_M(G)$ such that r regions are produced. First we show that $r = 1 + \xi(G)$. Note that if $\xi(G) = 0$, then G contains a splitting tree and $\mathcal{B}(G)$ is even. Thus G is upper embeddable on $S_{\gamma_M(G)}$ with one region; that is, $r = 1 + \xi(G)$ if $\xi(G) = 0$. Similarly, if $r = 1$, then $\mathcal{B}(G)$ is even and G is upper embeddable. Thus G has a splitting tree and $\xi(G) = 0$, so that $r = 1 + \xi(G)$ if $r = 1$. Therefore, we may assume that $\xi(G) > 0$ and $r \geq 2$.

Let T_1 be a spanning tree of G such that

$$\xi_0(G - E(T_1)) = \xi(G).$$

Let G_i, $i = 1, 2, \ldots, \xi(G)$, be the components of odd size in $G - E(T_1)$. For $i = 1, 2, \ldots, \xi(G)$, let e_i be a pendant edge of G_i if G_i is a tree and let e_i be an edge belonging to a cycle of G_i if G_i is not a tree. Define

$$H = G - \left\{e_1, e_2, \ldots, e_{\xi(G)}\right\}.$$

Since T_1 is a spanning tree of H, the graph H is connected. Since every component of $H - E(T_1)$ has even size, T_1 is a splitting tree of H. Therefore, by Theorem 7.21, H is upper embeddable. Also, by Theorem 7.20, $\mathcal{B}(G)$ is even. Hence by Corollary 7.19, H can be 2-cell embedded on $S_{\gamma_M(G)}$ with one region. Adding the edges $e_1, e_2, \ldots, e_{\xi(G)}$ to H produces the graph G. By the Ringeisen-White Edge-Adding Lemma (Theorem 7.14), there exists a 2-cell embedding of $H + e_1$ on some surface (namely on $S_{\gamma_M(H)}$ in this case) with two regions.

By $\xi(G)$ applications of Theorem 7.14, it follows that there exists a 2-cell embedding of $G = H + e_1 + e_2 + \cdots + e_{\xi(G)}$ on some surface S with at most $1 + \xi(G)$ regions. Therefore, if G is 2-cell embedded on S with s regions, then necessarily, $s \leq 1 + \xi(G)$. Since the minimum number of regions of any 2-cell embedding of G occurs when G is 2-cell embedded on $S_{\gamma_M(G)}$ and since such an embedding produces r regions, by assumption, we conclude that $r \leq s$ so that $r \leq 1 + \xi(G)$.

To verify that $r \geq 1 + \xi(G)$, we again assume that G is 2-cell embedded on the surface of genus $\gamma_M(G)$ with r (≥ 2) regions. Let f_1 be an edge belonging to the boundary of two regions of G. (Necessarily, f_1 is not a bridge of G.) Then $G - f_1$ is 2-cell embedded on the surface of genus $\gamma_M(G)$ with $r - 1$ regions. Furthermore, if $r > 2$, then for $k = 2, 3, \ldots, r - 1$, let f_k be an edge belonging to the boundary of two regions of $G - \{f_1, f_2, \ldots, f_{k-1}\}$. Then for $k = 1, 2, \ldots, r - 1$, the graph $G - \{f_1, f_2, \ldots, f_k\}$ is 2-cell embedded on the surface of genus $\gamma_M(G)$ with $r - k$ regions; in particular, the graph

$$G' = G - \{f_1, f_2, \cdots, f_{r-1}\}$$

is 2-cell embeddable on the surface of genus $\gamma_M(G)$ with one region. Therefore, $\mathcal{B}(G')$ is even and, by Corollary 7.19, the graph G' is upper embeddable on the surface of genus $\gamma_M(G)$. By Theorem 7.21, G' contains a splitting tree T' and all components of $G' - E(T')$ have even size. Thus $\xi_0(G - E(T')) \leq r - 1$. Consequently,

$$\xi(G) \leq \xi_0(G - E(T')) \leq r - 1$$

so that $r \geq 1 + \xi(G)$. Therefore, $r = 1 + \xi(G)$.

By Theorem 7.1,

$$n - m + r = 2 - 2\gamma_M(G).$$

Since $r = 1 + \xi(G)$, it follows that

$$2\gamma_M(G) = m - n + 1 - \xi(G)$$

or

$$\gamma_M(G) = \frac{1}{2}\left(\mathcal{B}(G) - \xi(G)\right),$$

as desired. ■

Returning to the graph G_1 of Figure 7.22, we see that $\mathcal{B}(G_1) = 6$ and that $\xi_0(G_1 - E(T)) = 6$ for every spanning tree T. Therefore, $\xi(G_1) = 6$ so that

$$\gamma_M(G_1) = \frac{1}{2}(\mathcal{B}(G_1) - \xi(G_1)) = 0$$

and G_1 is 2-cell embeddable only on the sphere.

With the aid of Theorem 7.22 (or Theorem 7.21), it is possible to show that a wide variety of graphs are upper embeddable. The following result is due to Hudson V. Kronk, Richard D. Ringeisen and Arthur T. White [140] (see Exercise 6).

Corollary 7.23 *Every complete k-partite graph, $k \geq 2$, is upper embeddable.*

From Corollary 7.23, it follows at once that every complete graph is upper embeddable, a result due to Nordhaus, Stewart and White [165]. We present a proof using Theorem 7.21.

Corollary 7.24 *The maximum genus of K_n is given by*

$$\gamma_M(K_n) = \left\lfloor \frac{(n-1)(n-2)}{4} \right\rfloor.$$

Proof. If T is a spanning path of K_n, then $K_n - E(T)$ contains at most one nontrivial component. Therefore, T is a splitting tree of K_n and, by Theorem 7.21, K_n is upper embeddable. Since $\mathcal{B}(K_n) = (n-1)(n-2)/2$, the result follows. ■

A formula for the maximum genus of complete bipartite graphs was discovered by Ringeisen [184] (see Exercise 7).

Corollary 7.25 *The maximum genus of $K_{s,t}$ is given by*

$$\gamma_M(K_{s,t}) = \left\lfloor \frac{(s-1)(t-1)}{2} \right\rfloor.$$

Joseph Zaks [246] discovered a formula for the maximum genus of the n-cube (see Exercise 8).

Corollary 7.26 *The maximum genus of Q_n, $n \geq 2$, is given by*

$$\gamma_M(Q_n) = (n-2)2^{n-2}.$$

Exercises for Section 7.3

1. Describe an embedding of K_5 on $S_{\gamma_M(K_5)}$ by means of a 5-tuple of cyclic permutations.

2. Determine the maximum genus of the graph G_2 of Figure 7.22.

3. Determine the maximum genus of the Petersen graph.

4. (a) Let G be a connected graph with blocks $B_1, B_2, \ldots, B_k$. Prove that

$$\gamma_M(G) \geq \sum_{i=1}^{k} \gamma_M(B_i).$$

   (b) Show that the inequality in (a) may be strict.

5. Theorem 7.21 states that a graph G is upper embeddable if and only if G has a splitting tree. Prove Theorem 7.21 as a corollary to Theorem 7.22.

6. Prove Corollary 7.23: *Every complete k-partite graph, $k \geq 2$, is upper embeddable.*

7. Prove Corollary 7.25: *The maximum genus of $K_{s,t}$ is given by $\gamma_M(K_{s,t}) = \left\lfloor \frac{(s-1)(t-1)}{2} \right\rfloor$.*

8. Prove Corollary 7.26: *The maximum genus of Q_n, $n \geq 2$, is given by $\gamma_M(Q_n) = (n-2)2^{n-2}$.*

9. Prove or disprove: For every positive integer k, there exists a connected graph G_k such that $\lfloor \mathcal{B}(G)/2 \rfloor - \gamma_M(G_k) = k$.

10. Prove or disprove: If H is a connected spanning subgraph of an upper embeddable graph G, then H is upper embeddable.

11. For $G = C_s \,\square\, C_t$ $(s, t \geq 3)$, determine $\gamma(G)$ and $\gamma_M(G)$.

12. Prove that if each vertex of a connected graph G lies on at most one cycle, then G is only 2-cell embeddable on the sphere.

13. Prove, for positive integers p and q with $p \leq q$, that there exists a graph G of genus p that can be 2-cell embedded on S_q.

14. Prove that if G is upper embeddable, then $G \,\square\, K_2$ is upper embeddable.

15. Let $G = K_{3,3} - e$, where $K_{3,3}$ has partite sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, and let $e = v_5v_6$.

   (a) Describe a 2-cell embedding of G on a surface S_k, where $k \neq \gamma(G)$, by giving an appropriate 6-tuple of cyclic permutations. What is k in your example?

   (b) On which surfaces is G 2-cell embeddable?

16. Let $G = K_{2,2,2}$ where $V(G) = \{v_1, v_2, \cdots, v_6\}$. The 6-tuple $(\pi_1, \pi_2, \cdots, \pi_6)$ of cyclic permutations defined by

$$\pi_1 = (3\ 4\ 5\ 6), \quad \pi_2 = (3\ 4\ 6\ 5), \quad \pi_3 = (1\ 2\ 5\ 6),$$
$$\pi_4 = (2\ 1\ 6\ 5), \quad \pi_5 = (2\ 1\ 3\ 4), \quad \pi_6 = (1\ 2\ 4\ 3)$$

describes a 2-cell embedding of G on S_k.

(a) What is k?

(b) Is k the genus of G?

(c) Is k the maximum genus of G?

(d) Determine all surfaces on which G can be 2-cell embedded.

7.4 The Graph Minor Theorem

We have seen by Wagner's theorem (Theorem 6.21) that a graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G. That is, Wagner's theorem is a **forbidden minor** characterization of planar graphs – in this case two forbidden minors: K_5 and $K_{3,3}$. A natural question to ask is whether a forbidden minor characterization may exist for graphs embedded on other surfaces. It was shown by Daniel Archdeacon and Philip Huneke [3] that there are exactly 35 forbidden minors for graphs that can be embedded on the projective plane.

In recent years, much more general results involving minors have been obtained. Klaus Wagner conjectured that in every infinite collection of graphs, there are always two graphs where one is isomorphic to a minor of the other. In what must be considered one of the major theorems of graph theory, Neil Robertson and Paul Seymour [191] verified this conjecture. Its lengthy proof is a consequence of a sequence of several papers that required years to complete.

Theorem 7.27 (**The Graph Minor Theorem**) *In every infinite set of graphs, there are two graphs where one is* (*isomorphic to*) *a minor of the other.*

One consequence of Theorem 7.27 is another major theorem in graph theory, also due to Robertson and Seymour [191]. A set S of graphs is said to be **minor-closed** if for every graph G in S, every minor of G also belongs to S. For example, the set S of planar graphs is minor-closed because every minor of a planar graph is planar; that is, if $G \in S$ and H is a minor of G, then $H \in S$ as well.

Theorem 7.28 *Let S be a minor-closed set of graphs. Then there exists a finite set M of graphs such that $G \in S$ if and only if no graph in M is a minor of G.*

Proof. Let $\overline{S}$ be the set of graphs not belonging to S and let M be the set of all graphs F in $\overline{S}$ such that every proper minor of F belongs to S. We claim that this set M has the required properties. First, we show that $G \in S$ if and only if no graph in M is a minor of G.

First, suppose that there is some graph $G \in S$ such that there is a graph F that is a minor of G and for which $F \in M$. Since $G \in S$ and S is minor-closed, it follows that $F \in S$. However, since $F \in M$, we have $F \in \overline{S}$, which is a contradiction.

For the converse, assume to the contrary that there is a graph $G \in \overline{S}$ such that no graph in M is a minor of G. We consider two cases.

Case 1. *All of the proper minors of G are in S.* Then by the defining property of M, it follows that $G \in M$. Since $G \in M$ and G is a minor of itself, this contradicts our assumption that no graph in M is a minor of G.

Case 2. *Some proper minor of G, say G', is not in S.* Thus $G' \in \overline{S}$. Then G' either satisfies the condition of Case 1 or of Case 2. We proceed in this manner as long as we remain in Case 2, producing a sequence $G = G^{(0)}, G^{(1)}, G^{(2)}, \ldots$ of proper minors.

If this process terminates, we have a finite sequence

$$G = G^{(0)}, G' = G^{(1)}, \ldots, G^{(p)},$$

where each graph in the sequence is a proper minor of all those graphs that precede it. Then $G^{(p)} \in M$, which returns us to Case 1. Otherwise, the sequence $G = G^{(0)}, G' = G^{(1)}, G^{(2)}, \ldots$ is infinite and where each graph $G^{(i)}$, $i \geq 1$, is a proper minor of all those graphs that precede it. This, however, is impossible since for each $i \geq 0$, either the order of $G^{(i+1)}$ is less than that of $G^{(i)}$ or the orders are the same and the size of $G^{(i+1)}$ is less than that of $G^{(i)}$.

It therefore remains only to show that M is finite. Assume, to the contrary, that M is infinite. By the Graph Minor Theorem, M contains two graphs, H_1 and H_2 say, such that one is a minor of the other. Suppose that H_2 is a minor (necessarily a proper minor) of H_1. Since every proper minor of H_1 belongs to S, it follows that $H_2 \in S$. However, since $H_2 \in M$, it follows that $H_2 \in \overline{S}$, producing a contradiction. ■

We now return to the question concerning the existence of a forbidden minor characterization for graphs embeddable on a surface S_k of genus $k \geq 0$. Certainly, if G is a sufficiently small graph (in terms of order and/or size), then G can be embedded on S_k. Hence if we begin with a graph F that cannot be embedded on S_k and perform successive edge contractions, edge deletions and vertex deletions, then eventually we arrive at a graph F' that also cannot be embedded on S_k but such that any additional edge contraction, edge deletion or vertex deletion of F' produces a graph that *can* be embedded on S_k. Such a graph F' is said to be **minimally nonembeddable on** S_k. Consequently, a graph F' is minimally nonembeddable on S_k if F' cannot be embedded on S_k but every proper minor F' can be embedded on S_k. Thus the set of graphs embeddable on S_k is minor-closed. As a consequence of the Graph Minor Theorem, we have the following.

Theorem 7.29 *For each integer $k \geq 0$, the set of minimally nonembeddable graphs on S_k is finite.*

Consequently, for each nonnegative integer k, there is a finite set $\mathcal{M}_k$ of graphs such that a graph G is embeddable on S_k if and only if no graph in $\mathcal{M}_k$ is a minor of G. Of course, the set of minimally nonembeddable graphs on the sphere is $\mathcal{M}_0 = \{K_5, K_{3,3}\}$. Although the number of minimally nonembeddable graphs on the torus is finite, it is known that this number exceeds 800 and so $|\mathcal{M}_1| > 800$.

Exercises for Section 7.4

1. (a) Show that the set $\mathcal{F}$ of forests is a minor-closed family of graphs.
   (b) What are the forbidden minors of $\mathcal{F}$?

2. Is the set B of bipartite graphs a minor-closed family of graphs? If so, what are the forbidden minors of B?

3. We have seen that the set of planar graphs is a minor-closed family of graphs. Is the set O of outerplanar graphs a minor-closed family of graphs? If so, what are the forbidden minors of O?

4. Use the Graph Minor Theorem (Theorem 7.27) to show that for any infinite set $S = \{G_1, G_2, G_3, \ldots\}$ of graphs, there exist infinitely many pairwise disjoint 2-element sets $\{i, j\}$ of integers such that one of G_i and G_j is a minor of the other.

Chapter 8

Vertex Colorings

This and the next chapter as well as Chapter 11 will be devoted to perhaps the best known and most studied area of graph theory: graph colorings. In 1993 Noga Alon of Tel Aviv University in Israel, well known for his work in combinatorics and theoretical computer science, wrote:

Graph coloring is arguably the most popular subject in graph theory.

Graph colorings has become a subject of great interest, largely because of its intriguing history (which will be discussed in Chapter 9), its diverse theoretical results, its unsolved problems and its numerous applications. The problems on graph colorings that have received the most attention involve coloring the vertices of a graph. Furthermore, the problems on vertex colorings that have been studied most often are those concerning proper vertex colorings. It is these colorings that will be discussed in this chapter.

8.1 The Chromatic Number of a Graph

By a **vertex coloring** of a graph G is meant an assignment of colors to the vertices of G, one color to each vertex. Vertex colorings in which adjacent vertices are colored differently are **proper vertex colorings**. Since the only vertex colorings we will consider here are proper vertex colorings, we ordinarily refer to these simply as **colorings** of G. While the colors used can be elements of any set, we typically use positive integers, say $1, 2, \ldots, k$ for some positive integer k. Thus a (proper) coloring can be considered as a function $c : V(G) \to \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers) such that $c(u) \neq c(v)$ if u and v are adjacent in G. If each color used is one of k given colors, then we refer to the coloring as a **k-coloring**. In a k-coloring, we may then assume that it is the colors $1, 2, \ldots, k$ that are being used.

Suppose that c is a k-coloring of a graph G, where each color is one of the integers $1, 2, \ldots, k$ as mentioned above. If V_i $(1 \leq i \leq k)$ is the set of vertices in G colored i (where one or more of these sets may be empty), then

each nonempty set V_i is called a **color class** and the nonempty elements of $\{V_1, V_2, \ldots, V_k\}$ produce a partition of $V(G)$. Because no two adjacent vertices of G are assigned the same color by c, each nonempty color class V_i $(1 \leq i \leq k)$ is an independent set of vertices of G.

A graph G is **k-colorable** if there exists a coloring of G from a set of k colors. In other words, G is k-colorable if there exists a k-coloring of G. The minimum positive integer k for which G is k-colorable is the **chromatic number** of G and is denoted by $\chi(G)$. (The symbol χ is the Greek letter *chi*.) The chromatic number of a graph G is therefore the minimum number of independent sets into which $V(G)$ can be partitioned. A graph G with chromatic number k is a **k-chromatic graph**. Therefore, if $\chi(G) = k$, then there exists a k-coloring of G but not a $(k-1)$-coloring. In fact, a graph G is k-colorable if and only if $\chi(G) \leq k$. Certainly, every graph of order n is n-colorable. Necessarily, if a k-coloring of a k-chromatic graph G is given, then all k colors must be used.

Two different colorings of a graph H are shown in Figure 8.1. The coloring in Figure 8.1(a) is a 5-coloring and the coloring in Figure 8.1(b) is a 4-coloring. Because the order of H is 10, the graph H is k-colorable for every integer k with $4 \leq k \leq 10$. Since H is 4-colorable, $\chi(H) \leq 4$. There is, however, no 3-coloring of H as we now explain. In any coloring c of H, the vertices u_1, u_2 and v_2 must be colored differently as they induce a triangle. Hence we may assume that u_1 is colored 1, u_2 is colored 2 and v_2 is colored 3 as indicated in Figure 8.1(b). If we attempt to color H with only the three colors $1, 2$ and 3, then we are forced to color the vertices of H as follows:

$$c(u_1) = 1,\ c(u_2) = 2,\ c(v_2) = 3,\ c(v_3) = 1,\ c(u_3) = 3,$$
$$c(v_4) = 2,\ c(u_4) = 1,\ c(v_5) = 3,\ c(u_5) = 2,\ c(v_1) = 2.$$

However, the adjacent vertices v_1 and u_5 are now both colored 2, which is impossible. So there is no 3-coloring of H. Therefore, $\chi(H) \geq 4$ and so $\chi(H) = 4$.

Figure 8.1: Colorings of a graph H

The argument used to verify that the graph H of Figure 8.1 has chromatic number 4 is a common one. In general, to show that some graph G has chro-

matic number k, say, we need to show that there exists a k-coloring of G (and so $\chi(G) \leq k$) and to show that every coloring of G requires at least k colors (and so $\chi(G) \geq k$).

There is no general formula for the chromatic number of a graph. Indeed, the problem of showing that a graph is k-colorable for some $k \geq 3$ is **NP**-complete. Consequently, we will often be concerned with and must be content with (1) determining the chromatic number of some specific graphs of interest or of graphs belonging to some classes of interest and (2) determining upper and/or lower bounds for the chromatic number of a graph. Certainly, for every graph G of order n,

$$1 \leq \chi(G) \leq n.$$

Clearly, $\chi(K_n) = n$. Furthermore, if G is a graph of order n that is not complete, then assigning the color 1 to two nonadjacent vertices of G and distinct colors to the remaining $n-2$ vertices of G produces an $(n-1)$-coloring of G. Therefore:

A graph G of order n has chromatic number n if and only if $G = K_n$.

Since distinct colors are needed to color any two adjacent vertices of a graph, we also have:

A graph G of order n has chromatic number 1 if and only if $G = \overline{K}_n$.

Thus for a graph G to have chromatic number at least 2, G must have at least one edge. Also, for a nonempty graph G to have chromatic number 2, there must be some way to partition $V(G)$ into two independent subsets V_1 (the vertices of G colored 1) and V_2 (the vertices of G colored 2). Since every edge of G must join a vertex of V_1 and a vertex of V_2, the graph G is bipartite. That is:

A nonempty graph G has chromatic number 2 if and only if G is bipartite.

From these observations, we have the following.

Theorem 8.1 *A nontrivial graph G is 2-colorable if and only if G is bipartite.*

As a consequence of Theorem 1.18, we also have the following:

The chromatic number of a graph G is at least 3 if and only if G has an odd cycle.

A rather obvious, but often useful, lower bound for the chromatic number of a graph involves the chromatic numbers of its subgraphs.

Theorem 8.2 *If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.*

Proof. Suppose that $\chi(G) = k$. Then there exists a k-coloring c of G. Since c assigns distinct colors to every two adjacent vertices of G, the coloring c also assigns distinct colors to every two adjacent vertices of H. Therefore, H is k-colorable and so $\chi(H) \leq k = \chi(G)$. ■

As a result of Theorem 8.2, if a graph G contains K_k as a subgraph, then $\chi(G) \geq k$. The **clique number** $\omega(G)$ of a graph G is the order of the largest **clique** (complete subgraph) of G. (The symbol ω is the Greek letter *omega.*) A clique of order k is called a k**-clique**. The following result is then an immediate consequence of Theorem 8.2.

Corollary 8.3 *For every graph G, $\chi(G) \geq \omega(G)$.*

The lower bound for the chromatic number of a graph in Corollary 8.3 is related to a much studied class of graphs called "perfect graphs," which will be visited in Section 8.4.

If a graph G is the union or the join of graphs $G_1, G_2, \ldots, G_k$, then the chromatic number of G can be easily expressed in terms of the chromatic numbers of these k graphs (see Exercise 13).

Theorem 8.4 *For graphs $G_1, G_2, \ldots, G_k$ and $G = G_1 + G_2 + \cdots + G_k$,*

$$\chi(G) = \max\{\chi(G_i) : 1 \leq i \leq k\}.$$

The following is then an immediate consequence of Theorem 8.4.

Corollary 8.5 *If G is a graph with components $G_1, G_2, \ldots, G_k$, then*

$$\chi(G) = \max\{\chi(G_i) : 1 \leq i \leq k\}.$$

There is a result analogous to Corollary 8.5 that expresses the chromatic number of a graph in terms of the chromatic numbers of its blocks (see Exercise 14).

Theorem 8.6 *If G is a graph with blocks $B_1, B_2, \ldots, B_k$, then*

$$\chi(G) = \max\{\chi(B_i) : 1 \leq i \leq k\}.$$

Corollary 8.5 and Theorem 8.6 tell us that we can restrict our attention to 2-connected graphs when studying the chromatic number of graphs. In the case of joins, we have the following (see Exercise 15).

Theorem 8.7 *For graphs $G_1, G_2, \ldots, G_k$ and $G = G_1 \vee G_2 \vee \cdots \vee G_k$,*

$$\chi(G) = \sum_{i=1}^{k} \chi(G_i).$$

A special case of Theorem 8.7 is the following:

Every complete k-partite graph has chromatic number k.

Certainly every even cycle is 2-chromatic since these are bipartite graphs and the chromatic number of every odd cycle is at least 3. The coloring c defined on the vertices of an odd cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$ by

$$c(v_i) = \begin{cases} 1 & \text{if } i \text{ is odd and } 1 \leq i < n \\ 2 & \text{if } i \text{ is even} \\ 3 & \text{if } i = n \end{cases}$$

is a 3-coloring. Thus we have the following.

Theorem 8.8 *For every integer $n \geq 3$,*

$$\chi(C_n) = \begin{cases} 2 & \textit{if } n \textit{ is even} \\ 3 & \textit{if } n \textit{ is odd.} \end{cases}$$

Many bounds (both upper and lower bounds) have been developed for the chromatic number of a graph. Two of the most elementary bounds for the chromatic number of a graph G involve the independence number $\alpha(G)$, which, recall, is the maximum cardinality of an independent set of vertices of G. The lower bound is especially useful.

Theorem 8.9 *If G is a graph of order n, then*

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n - \alpha(G) + 1.$$

Proof. Suppose that $\chi(G) = k$ and let there be given a k-coloring of G with resulting color classes V_1, V_2, …, V_k. Since

$$n = |V(G)| = \sum_{i=1}^{k} |V_i| \leq k\alpha(G),$$

it follows that

$$\frac{n}{\alpha(G)} \leq \chi(G).$$

Next, let U be a maximum independent set of vertices of G and assign the color 1 to each vertex of U. Assigning distinct colors different from 1 to each vertex of $V(G) - U$ produces a proper coloring of G. Hence

$$\chi(G) \leq |V(G) - U| + 1 = n - \alpha(G) + 1$$

as desired. ■

According to Theorem 8.9, for the complete 3-partite graph $G = K_{1,2,3}$, which has order $n = 6$ and independence number $\alpha(G) = 3$, we have

$$n/\alpha(G) = 2 \leq \chi(G) \leq 4 = n - \alpha(G) + 1.$$

Since G is a complete 3-partite graph, it follows that $\chi(G) = 3$ and so neither bound in Theorem 8.9 is attained in this case. On the other hand, $\omega(G) = 3$ and so $\chi(G) = \omega(G)$.

For the graph G of order 10 shown in Figure 8.2(a), we have $\alpha(G) = 2$ and $\omega(G) = 4$. By Corollary 8.3, $\chi(G) \geq 4$; while, according to Theorem 8.9, $5 \leq \chi(G) \leq 9$. However, the 5-coloring of G in Figure 8.2(b) shows that $\chi(G) \leq 5$ and so $\chi(G) = 5$.

Figure 8.2: A 5-chromatic graph G with $\alpha(G) = 2$ and $\omega(G) = 4$

Applications of Vertex Colorings

In Example 1.1 of Section 1.1 a situation was described that could be modeled by a graph. We repeat this example below.

Example 8.10

The service department of an automobile dealer has advertised maintenance specials for a particular afternoon. One of these is a 30-minute maintenance special and the other is a 60-minute maintenance special. Eight customers, denoted by $c_1, c_2, \ldots, c_8$, have made appointments, listed below:

c_1 : 1 : 30 – 2 : 00 pm	c_2 : 1 : 40 – 2 : 40 pm
c_3 : 1 : 40 – 2 : 10 pm	c_4 : 1 : 50 – 2 : 50 pm
c_5 : 2 : 30 – 3 : 00 pm	c_6 : 2 : 15 – 3 : 15 pm
c_7 : 2 : 55 – 3 : 55 pm	c_8 : 3 : 10 – 3 : 40 pm

This situation can be modeled by the graph G shown in Figure 8.3(a). The obvious question to ask here is: What is the minimum number of mechanics needed to provide the necessary maintenance?

Solution. Certainly a mechanic cannot work on two cars whose owners have overlapping appointments. In other words, we seek the chromatic number of

G. The clique number of G is 4. Indeed, the vertices c_1, c_2, c_3 and c_4 induce a complete graph of order 4. Hence $\chi(G) \geq \omega(G) = 4$. Color the vertices c_i $(1 \leq i \leq 4)$ by color i and color c_5, c_6, c_7 and c_8 with colors $1, 3, 2$ and 4, respectively as shown in Figure 8.3(b). Thus $\chi(G) = 4$. Since c_1 and c_5 are colored 1, mechanic #1 can service the cars of the customers c_1 and c_5, and so on. In general, we have the following

mechanic #1: c_1, c_5 mechanic #2: c_2, c_7
mechanic #3: c_3, c_6 mechanic #4: c_4, c_4.

♦

Figure 8.3: The graph G in Example 8.10 and a 5-coloring of G

Example 8.11

Eight faculty members, denoted by $f_1, f_2, \ldots, f_8$, have been invited to a workshop to discuss seven topics of interest. It is decided that it would be more efficient to divide these faculty members into seven committees of three each, namely

$$S_1 = \{f_1, f_2, f_3\},\ S_2 = \{f_2, f_3, f_4\},\ S_3 = \{f_4, f_5, f_6\},\ S_4 = \{f_5, f_6, f_7\},$$
$$S_5 = \{f_1, f_7, f_8\},\ S_6 = \{f_1, f_5, f_7\},\ S_7 = \{f_2, f_5, f_8\}.$$

If each committee must meet during one of the time periods

9 - 10 am, 10 - 11 am, 11am - 12 noon, 2 - 3 pm, 3 - 4 pm, 4 - 5 pm,

then what is the minimum number of time periods needed for all seven committees to meet?

Solution. No two committees can meet during the same time period if some faculty member belongs to both committees. Define a graph G by $V(G) = \{S_1, S_2, \ldots, S_7\}$, where two vertices S_i and S_j are adjacent if $S_i \cap S_j \neq \emptyset$ (and so S_i and S_j must meet at different times). The graph G is shown in Figure 8.4(a).

The answer to this question is $\chi(G)$. Here $\omega(G) = 4$ and so $\chi(G) \geq 4$. The 5-coloring in Figure 8.4(b) shows that $\chi(G) \leq 5$. Since no 4-coloring of G is possible, $\chi(G) = 5$. Possible meeting times for these committees are therefore

$$\text{9 - 10 am: } S_1 \quad \text{10 - 11 am: } S_6 \quad \text{11am - 12 noon: } S_7$$

$$\text{2 - 3 pm: } S_2, S_4 \quad \text{3 - 4 pm: } S_3,\ S_5$$

and so all committee meetings are over by 4 pm. ♦

Figure 8.4: The graph G in Example 8.11 and a 5-coloring of G

Example 8.12

Figure 8.5 shows six traffic lanes $L_1, L_1, \ldots, L_6$ at the intersection of two streets. A traffic light is located at the intersection. During each phase of the traffic light, those cars in lanes for which the light is green may proceed safely through the intersection into certain permitted lanes. What is the minimum number of phases needed for the traffic light so that all cars may proceed safely through the intersection?

Solution. A graph G is constructed with vertex set $V(G) = \{L_1, L_2, \ldots, L_6\}$, where L_i is adjacent to L_j $(i \neq j)$ if cars in lanes L_i and L_j cannot proceed safely through the intersection at the same time. (See Figure 8.6(a).) The minimum number of phases needed for the traffic light so that all cars may proceed, in time, through the intersection is $\chi(G)$. Since $\{L_1, L_2, L_4\}$ induces a triangle $\chi(G) \geq 3$. Since there is a 3-coloring of G (see Figure 8.6(b)), it follows that $\chi(G) = 3$. For example, since L_1, L_5, and L_6 belong to the same color class, cars in those three lanes may proceed safely through the intersection at the same time. Three possible phases for all cars to proceed safely through the intersection are therefore

$$\text{Phase \#1: } L_1,\ L_5,\ L_6, \quad \text{Phase \#2: } L_3,\ L_4, \quad \text{Phase \#3: } L_2.$$

♦

Figure 8.5: Traffic lanes at street intersections in Example 8.12

Figure 8.6: The graph G of Example 8.12 and a 3-coloring of G

Exercises for Section 8.1

1. The vertices of a graph G are colored with three colors in such a way that each vertex is adjacent to vertices colored with only one of the three colors. Show that $\chi(G) = 2$.

2. Prove that the size of every k-chromatic graph is at least $\binom{k}{2}$.

3. Let G be a graph with $\chi(G) = k$ and let a k-coloring of G be given. Let H be the graph with $V(H) = V(G)$ such that $uv \in E(H)$ if u and v are assigned different colors in G. Determine $\chi(H)$.

4. Prove that if S is a color class resulting from a k-coloring of a k-chromatic graph G, where $k \geq 2$, then there is a component H of $G - S$ such that $\chi(H) = k - 1$.

5. Show, for every connected graph G of order n and diameter d, that

$$\chi(G) \leq n - d + 1.$$

6. For each integer $k \geq 3$, give an example of a regular k-chromatic graph G such that $G \neq K_k$.

7. Let G be a k-chromatic graph, where $k \geq 2$, and let r be a positive integer such that $r \geq \Delta(G)$. Prove that there exists an r-regular k-chromatic graph H such that G is an induced subgraph of H.

8. Determine the largest positive integer k such that $\chi(H) = \chi(G) = k$, where H is obtained from a nonempty graph G by subdividing each edge of G exactly once.

9. Let G be a nonempty graph and let H be a graph obtained from G by subdividing a single edge of G exactly once.

   (a) Show that $|\chi(G) - \chi(H)| \leq 1$.

   (b) Show that each of the following is possible:

      (i) $\chi(H) = \chi(G)$

      (ii) $\chi(H) = \chi(G) - 1$

      (iii) $\chi(H) = \chi(G) + 1$.

10. A **balanced coloring** of a graph G is an assignment of colors to the vertices of G such that (i) every two adjacent vertices are assigned different colors and (ii) the numbers of vertices assigned any two different colors differ by at most one. The smallest number of colors used in a balanced coloring of G is the **balanced chromatic number** $\chi_b(G)$ of G.

    (a) Prove that the balanced chromatic number is defined for every graph G.

    (b) Determine $\chi_b(G)$ for the graph G in Figure 8.7.

Figure 8.7: The graph in Exercise 10(b)

11. Let G be an r-regular graph, $r \geq 3$, such that $3 \leq \chi(G) \leq r + 1$. Prove or disprove the following.

    (a) For every edge e of G, there exists an odd cycle containing e.

(b) For every edge e of G, there exists an odd cycle not containing e.

12. (a) Show for every graph G that

$$\chi(G) = \max\{\chi(H) : H \text{ is a subgraph of } G\}.$$

(b) The problem in (a) should suggest a question to you. Ask and answer this question.

13. Prove Theorem 8.4: *For graphs* $G_1, G_2, \ldots, G_k$ *and* $G = G_1 + G_2 + \cdots + G_k$, $\chi(G) = \max\{\chi(G_i) : 1 \le i \le k\}$.

14. Prove Theorem 8.6: *If* G *is a graph with blocks* B_1, B_2, ..., B_k, *then* $\chi(G) = \max\{\chi(B_i) : 1 \le i \le k\}$.

15. Prove Theorem 8.7: *For graphs* $G_1, G_2, \ldots, G_k$ *and* $G = G_1 \vee G_2 \vee \cdots \vee G_k$, $\chi(G) = \sum_{i=1}^{k} \chi(G_i)$.

16. Let G be a graph of order n with independence number $\alpha(G) = 2$.

(a) Prove that if G is disconnected, then G contains $K_{\lceil \frac{n}{2} \rceil}$ as a subgraph.

(b) Prove that if G is connected, then G contains a path (u, v, w) such that $uw \notin E(G)$ and every vertex in $G - \{u, v, w\}$ is adjacent to either u or w (or both).

17. By Theorem 8.9, it follows that for every graph G of order n,

$$\frac{n}{\alpha(G)} \le \chi(G) \le n - \alpha(G) + 1.$$

Prove or disprove: The chromatic number of a graph G can never be closer to $n - \alpha(G) + 1$ than to $\frac{n}{\alpha(G)}$.

18. For a given positive integer n, determine all graphs of order n for which the two bounds in Theorem 8.9 are equal.

19. Let $k \ge 2$ be an integer. For each integer i with $1 \le i \le 2k + 1$, let G_i be a copy of K_k. The graph G of order $2k^2 + k$ is obtained from the graphs $G_1, G_2, \ldots, G_{2k+1}, G_{2k+2} = G_1$ by joining each vertex in G_i to every vertex in G_{i+1} $(1 \le i \le 2k + 1)$. Compute the bounds $n/\alpha(G)$, $n + 1 - \alpha(G)$ and $\omega(G)$ for $\chi(G)$ for an arbitrary $k \ge 2$. Determine $\chi(G)$.

20. Give an example of a graph G of order n for which $n/\alpha(G)$ is an integer and for which $\chi(G)$ is none of the numbers $n/\alpha(G)$, $n - \alpha(G) + 1$ and $\omega(G)$.

21. Give an example of a graph G with $\chi(G) = \alpha(G)$ in which no $\chi(G)$-coloring of G results in a color class containing $\alpha(G)$ vertices.

22. Does there exist a k-chromatic graph G in which no color class of a k-coloring of G contains at least $\alpha(G) - 2$ vertices?

23. Let $k \geq 2$ be an integer. Prove that if G is a k-colorable graph of order n such that $\delta(G) > \left(\frac{k-2}{k-1}\right) n$, then G is k-chromatic. Is the bound sharp?

24. Prove that a connected graph G has chromatic number at least 3 if and only if for every vertex v of G, there exist two adjacent vertices u and w in G such that $d(u, v) = d(v, w)$.

25. An independent set of vertices in a graph G is **maximal** if it is not properly contained in any other independent set of G.

    (a) Let v be a vertex of a k-chromatic graph G and let $U_1, U_2, \ldots, U_\ell$ be the maximal independent sets of G containing v. Prove that for some k-coloring of G, one of the resulting k color classes is U_i for some i with $1 \leq i \leq k$.

    (b) Let G be a k-chromatic graph, $k \geq 2$, and S an independent set of vertices in G. Prove that $\chi(G[V(G) - S]) = k - 1$ if and only if S is a color class in some k-coloring of G.

26. For a nonempty graph G, let v be a vertex of G and let $U_1, U_2, \ldots, U_\ell$ be the maximal independent sets of G containing v (see the definition in Exercise 25).

    (a) Prove that $\chi(G) = 1 + \min_{1 \leq i \leq \ell} \chi(G[V(G) - U_i])$.

    (b) What is the relationship between $\chi(G)$ and

    $$1 + \max_{1 \leq i \leq \ell} \chi(G[V(G) - U_i])?$$

27. Prove that every Hamiltonian-connected graph of order at least 3 has chromatic number at least 3.

28. Determine (and prove) a necessary and sufficient condition for a connected graph to have a 2-colorable line graph.

29. Two dentists are having new offices designed for themselves. In the common waiting room for their patients, they have decided to have an aquatic area containing fish tanks. Because some fish require a coldwater environment while others are more tropical and because some fish are aggressive with other types of fish, not all fish can be placed in a single tank. It is decided to have ten exotic fish, denoted by $F_1, F_2, \ldots, F_{10}$, where the fish that cannot be placed in the same tank as F_i $(1 \leq i \leq 10)$ are listed to the right of F_i.

F_1: F_2, F_3, F_4, F_5, F_{10}	F_2: F_1, F_3, F_6	F_3: F_1, F_2, F_4
F_4: F_1, F_3, F_5	F_5: F_1, F_4, F_9	F_6: F_2, F_7, F_{10}
F_7: F_6, F_8, F_{10}	F_8: F_7, F_9, F_{10}	F_9: F_5, F_8, F_{10}
F_{10}: F_1, F_6, F_7, F_8, F_9.		

What is the minimum number of tanks required?

30. Figure 8.8 shows traffic lanes $L_1, L_1, \ldots, L_7$ at the intersection of two streets. A traffic light is located at the intersection. During a certain phase of the traffic light, those cars in lanes for which the light is green may proceed safely through the intersection in permissible directions. What is the minimum number of phases needed for the traffic light so that (eventually) all cars may proceed through the intersection?

Figure 8.8: Traffic lanes at street intersections in Exercise 30

8.2 Color-Critical Graphs

For every k-chromatic graph G with $k \geq 2$ and every vertex v of G, either $\chi(G - v) = k$ or $\chi(G - v) = k - 1$. Furthermore, for every edge e of G, either $\chi(G - e) = k$ or $\chi(G - e) = k - 1$. In fact, if $\chi(G - e) = k - 1$ and u is a vertex incident with e, then $\chi(G - u) = k - 1$. Graphs that are k-chromatic (but just barely) are often of great interest. A graph G is called **color-critical** if $\chi(H) < \chi(G)$ for every proper subgraph H of G. If G is a color-critical k-chromatic graph, then G is called **critically k-chromatic** or simply **k-critical**. The graph K_2 is the only 2-critical graph. In fact, K_n is n-critical for every integer $n \geq 2$. The odd cycles are the only 3-critical graphs. There is no characterization of k-critical graphs for any integer $k \geq 4$.

Let G be a k-chromatic graph, where $k \geq 2$, and suppose that H is a k-chromatic subgraph of minimum size in G having no isolated vertices. Then for every proper subgraph F of H, $\chi(F) < \chi(H)$, that is, H is a k-critical subgraph of G. From this observation, it follows that every k-chromatic graph, $k \geq 2$, contains a k-critical subgraph. By Corollary 8.5, every k-critical graph, $k \geq 2$,

must be connected; while by Theorem 8.6, every k-critical graph, $k \geq 3$, must be 2-connected and therefore 2-edge-connected. The following theorem tells us even more.

Theorem 8.13 *Every k-critical graph, $k \geq 2$, is $(k-1)$-edge-connected.*

Proof. The only 2-critical graph is K_2, which is 1-edge-connected; while the only 3-critical graphs are odd cycles, each of which is 2-edge-connected. Since the theorem holds for $k = 2, 3$, we may assume that $k \geq 4$.

Suppose that there is a k-critical graph G, $k \geq 4$, that is not $(k-1)$-edge-connected. This implies by Theorem 2.33 that there exists a partition $\{V_1, V_2\}$ of $V(G)$ such that the number of edges joining the vertices of V_1 and the vertices of V_2 is at most $k-2$. Since G is k-critical, the two induced subgraphs

$$G_1 = G[V_1] \text{ and } G_2 = G[V_2]$$

are $(k-1)$-colorable. Let there be given colorings of G_1 and G_2 from the same set of $k-1$ colors and suppose that E' is the set of edges of G that join the vertices in V_1 and the vertices in V_2. It cannot occur that every edge in E' joins vertices of different colors, for otherwise, G itself is $(k-1)$-colorable. Hence there are some edges in E' joining vertices that are assigned the same color. We now show that there exists a permutation of the colors assigned to the vertices of V_1 that results in a proper coloring of G in which every edge of E' joins vertices of different colors, which again shows that G is $(k-1)$-colorable, producing a contradiction.

Let $U_1, U_2, \ldots, U_t$ denote the color classes of G_1 for which there is some vertex in U_i $(1 \leq i \leq k-2)$ adjacent to a vertex of V_2. Suppose that there are k_i edges joining the vertices of U_i and the vertices of V_2. Then each $k_i \geq 1$ and

$$\sum_{i=1}^{t} k_i \leq k-2.$$

If, for every vertex $u_1 \in U_1$, the neighbors of u_1 are assigned a color different from that assigned to u_1, then the color of the vertices in U_1 is not altered. If, on the other hand, some vertex $u_1 \in U_1$ is adjacent to a vertex in V_2 that is colored the same as u_1, then the $k-1$ colors used to color the vertices of G_1 may be permuted so that no vertex in U_1 is adjacent to a vertex of V_2 having the same color. This is possible since there are at most k_1 colors to avoid when coloring the vertices of U_1 but there are at least $k-1-k_1 \geq 1$ colors available for this purpose. If, upon giving this new coloring to the vertices of G_1, each vertex $u_2 \in U_2$ is adjacent only to the vertices in V_2 assigned a color different from that of u_2, then no (additional) permutation of the colors of V_1 is performed. Suppose, however, that there is some vertex $u_2 \in U_2$ that is assigned the same color as one of its neighbors in V_2. In this case, we may once again permute the $k-1$ colors used to color the vertices of V_1, where the color assigned to the vertices in U_1 is not changed. This too is possible since there are at most

$k_2 + 1$ colors to avoid when coloring the vertices of U_2 but the number of colors available for U_2 is at least

$$(k-1) - (k_2+1) \geq (k-1) - (k_2+k_1) \geq 1.$$

This procedure is continued until a $(k-1)$-coloring of G is produced, which, as we noted, is impossible. ■

As a consequence of Theorem 8.13, $\chi(G) \leq 1 + \lambda(G)$ for every color-critical graph G. A related theorem of David W. Matula [152] provides an upper bound for the chromatic number of an arbitrary graph in terms of the edge connectivity of its subgraphs.

Theorem 8.14 *For every graph G,*

$$\chi(G) \leq 1 + \max\{\lambda(H)\},$$

where the maximum is taken over all subgraphs H of G.

Proof. Suppose that F is a color-critical subgraph of G with $\chi(G) = \chi(F)$. By Theorem 8.13,

$$\chi(G) = \chi(F) \leq 1 + \lambda(F) \leq 1 + \max\{\lambda(H)\},$$

where the maximum is taken over all subgraphs H of G. ■

Since the edge connectivity of a graph never exceeds its minimum degree (by Theorem 2.29), we have the following corollary of Theorem 8.13.

Corollary 8.15 *If G is a color-critical graph, then*

$$\chi(G) \leq 1 + \delta(G).$$

We have noted that every k-critical graph, $k \geq 3$, is 2-connected. While every 4-critical graph must be 3-edge-connected (by Theorem 8.13), a 4-critical graph need not be 3-connected. The 4-critical graph G of Figure 8.9 has connectivity 2. In fact, $S = \{u, v\}$ is a minimum vertex-cut of G.

Figure 8.9: A critically 4-chromatic graph

Observe that for the vertex-cut $S = \{u, v\}$ of the 4-critical graph G of Figure 8.9, the induced subgraph $G[S]$ is disconnected and is certainly not complete. As it turns out, this observation is not surprising. Recall, for a vertex-cut S of a graph G and a component H of $G - S$, that the subgraph $G[V(H) \cup S]$ is referred to as a branch of G at S or an S-branch.

Theorem 8.16 *If S is a vertex-cut of a (noncomplete) k-critical graph G, $k \geq 3$, then the subgraph $G[S]$ is not a clique.*

Proof. Since no color-critical graph contains a cut-vertex, $|S| \geq 2$. Let H_1, H_2, $\cdots$, H_s be the components of $G - S$ and let

$$G_i = G[V(H_i) \cup S] \ (1 \leq i \leq s)$$

be the resulting S-branches of G. Since each S-branch is a proper subgraph of G, it follows that $\chi(G_i) \leq k - 1$ for each i $(1 \leq i \leq s)$.

If there exists a $(k-1)$-coloring of each S-branch G_i in which no two vertices of S are assigned the same color, then by permuting the colors assigned to the vertices of S in each S-branch, if necessary, a $(k-1)$-coloring of G can be produced. Since this is impossible, there is some S-branch in which no $(k-1)$-coloring assigns distinct colors to the vertices of S. This, in turn, implies that $G[S]$ contains two nonadjacent vertices and so $G[S]$ is not a clique. ■

The following result is a consequence of Theorem 8.16 and its proof.

Corollary 8.17 *Let G be a k-critical (noncomplete) graph, $k \geq 3$, where $\kappa(G) = 2$. If $S = \{u, v\}$ is a vertex-cut of G, then $uv \notin E(G)$ and there exists some S-branch G' of G such that $\chi(G' + uv) = k$.*

Proof. If $S = \{u, v\}$, then $uv \notin E(G)$ by Theorem 8.16. As we saw in the proof of Theorem 8.16, there is an S-branch G' of G in which no $(k-1)$-coloring of G' assigns distinct colors to u and v. Therefore, $G' + uv$ is not $(k-1)$-colorable and so $\chi(G' + uv) = k$. ■

Exercises for Section 8.2

1. Let G be a k-chromatic graph where $k \geq 2$.
   (a) Prove that for every vertex v of G, either $\chi(G - v) = k$ or $\chi(G - v) = k - 1$.
   (b) Prove that for every edge e of G, either $\chi(G - e) = k$ or $\chi(G - e) = k - 1$.
2. Let G be a k-chromatic graph such that $\chi(G - e) = k - 1$ for some edge $e = uv$ of G. Prove that $\chi(G - u) = \chi(G - v) = k - 1$.
3. Show that the odd cycles are the only 3-critical graphs.

4. Determine all k-critical graphs with $k \geq 3$ such that $G-v$ is $(k-1)$-critical for every vertex v of G.

5. Prove or disprove: If G and H are color-critical graphs, then the join $G \vee H$ of G and H is color-critical.

6. It has been mentioned that every k-critical graph, $k \geq 3$, is 2-connected. Show that there exists a k-critical graph having connectivity 2 for every integer $k \geq 3$.

7. Prove or disprove the following.

   (a) There exists no graph G with $\chi(G) = 3$ without isolated vertices such that $\chi(G - v) = 2$ for exactly 75% of the vertices v of G.

   (b) There exists no graph G with $\chi(G) = 3$ without isolated vertices and containing a 3-critical component such that $\chi(G-v) = 2$ for exactly 75% of the vertices v of G.

8. Define a graph G to be **k-degenerate**, $k \geq 0$, if for every induced subgraph H of G, $\delta(H) \leq k$. Then the 0-degenerate graphs are the empty graphs, and by Exercise 6 in Section 2.2, the 1-degenerate graphs are precisely the forests. By Corollary 6.6, every planar graph is 5-degenerate. A k-degenerate graph is **maximal k-degenerate** if, for every two nonadjacent vertices u and v of G, the graph $G + uv$ is not k-degenerate.

   For $k \geq 0$, let $\mathcal{P}_k$ denote the family of k-degenerate graphs. Then $\chi_{\mathcal{P}_k}(G)$ is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a k-degenerate subgraph of G. A graph is said to be **ℓ-critical with respect to** $\chi_{\mathcal{P}_k}$, $\ell \geq 2$, if $\chi_{\mathcal{P}_k}(G) = \ell$ and $\chi_{\mathcal{P}_k}(G - v) = \ell - 1$ for every $v \in V(G)$.

   (a) Prove that if G is a maximal k-degenerate graph of order n, where $n \geq k + 1$, then $\delta(G) = k$.

   (b) Determine $\chi_{\mathcal{P}_k}(K_n)$.

   (c) Prove that if G is a graph that is ℓ-critical with respect to $\chi_{\mathcal{P}_k}$, then $\delta(G) \geq (k + 1)(\ell - 1)$.

8.3 Bounds for the Chromatic Number

Since determining the chromatic number of a graph is known to be a very challenging problem in general, it is not surprising that much of the research emphasis on coloring has centered on finding bounds (both lower bounds and upper bounds) for the chromatic number of a graph. For a graph G of order n with clique number $\omega(G)$ and independence number $\alpha(G)$, we have already seen that $\omega(G)$ and $n/\alpha(G)$ are lower bounds for $\chi(G)$ while $n - \alpha(G) + 1$ is

an upper bound for $\chi(G)$. Of course, n is also an upper bound for $\chi(G)$. In particular,

$$\omega(G) \leq \chi(G) \leq n.$$

Bruce Reed [182] showed that $\chi(G)$ can never be closer to n than to $\omega(G)$.

Theorem 8.18 *For every graph G of order n,*

$$\chi(G) \leq \left\lfloor \frac{n+\omega(G)}{2} \right\rfloor.$$

Proof. We proceed by induction on the nonnegative integer $|V(G)| - \omega(G)$. If G is a graph such that $|V(G)| - \omega(G) = 0$, then $G = K_n$ for some positive integer n and so $\chi(G) = \omega(G) = n$. Thus $\chi(G) = \left\lfloor \frac{n+\omega(G)}{2} \right\rfloor$. This verifies the basis step of the induction.

Assume for a positive integer k and every graph H (having order n') such that $n' - \omega(H) < k$ that $\chi(H) \leq \left\lfloor \frac{n'+\omega(H)}{2} \right\rfloor$. Let G be a graph of order n such that $n - \omega(G) = k \geq 1$. Since G is not complete, G contains two nonadjacent vertices u and v. Let $H = G - u - v$. Then H has order $n-2$ and either $\omega(H) = \omega(G)$ or $\omega(H) = \omega(G) - 1$. In either case, $0 \leq (n-2) - \omega(H) < k$ and so

$$\chi(H) \leq \left\lfloor \frac{n-2+\omega(H)}{2} \right\rfloor$$

by the induction hypothesis. Hence there exists a $\left\lfloor \frac{n-2+\omega(H)}{2} \right\rfloor$-coloring of H. Assigning u and v the same new color implies that

$$\chi(G) \leq \left\lfloor \frac{n-2+\omega(H)}{2} \right\rfloor + 1 = \left\lfloor \frac{n+\omega(H)}{2} \right\rfloor \leq \left\lfloor \frac{n+\omega(G)}{2} \right\rfloor,$$

giving the desired result. ■

The Greedy Coloring Algorithm

Suppose that we would like to assign colors to the vertices of some graph G. Ideally of course, we would like to provide a coloring of G using as few colors as possible, namely $\chi(G)$ colors. Since we have mentioned that this is an extraordinarily difficult problem in general, this suggests consideration of the problem of providing a coloring of G that does not use an excessive number of colors (if this is possible). One possible approach is to use a greedy method, which provides a step-by-step strategy for coloring the vertices of a graph such that at each step, a potentially optimal choice for a color of a vertex is made. While this method may not result in coloring the vertices of G using the minimum number of colors, it does provide an upper bound (in fact, several upper bounds) for the chromatic number of G.

Let G be a graph of order n whose n vertices are listed in some specified order. In a greedy coloring of G, the vertices are successively colored with positive integers according to an algorithm that assigns to the vertex under consideration the smallest available color. Hence, if the vertices of G are listed in the order $v_1, v_2, \ldots, v_n$, then the resulting greedy coloring c assigns the color 1 to v_1, that is, $c(v_1) = 1$. If v_2 is not adjacent to v_1, then also define $c(v_2) = 1$; while if v_2 is adjacent to v_1, then define $c(v_2) = 2$. In general, suppose that the first j vertices $v_1, v_2, \ldots, v_j$ in the sequence have been colored, where $1 \leq j < n$, and t is the smallest positive integer not used in coloring any neighbor of v_{j+1} from among $v_1, v_2, \ldots, v_j$. We then define $c(v_{j+1}) = t$. When the algorithm ends, the vertices of G have been assigned colors from the set $\{1, 2, \ldots, k\}$ for some positive integer k. Thus $\chi(G) \leq k$ and so k is an upper bound for the chromatic number of G.

While there is always an ordering of the vertices of a graph G so that this algorithm gives a vertex coloring of G with $\chi(G)$ colors (see Exercise 1(a)), it is also possible that this algorithm gives a vertex coloring of G using k colors such that k is considerably larger than $\chi(G)$ (see Exercise 1(b)). This algorithm is now stated more formally.

The Greedy Coloring Algorithm *Suppose that the vertices of a graph G are listed in the order $v_1, v_2, \ldots, v_n$.*

1. *The vertex v_1 is assigned the color* 1.
2. *Once the vertices $v_1, v_2, \ldots, v_j$ have been assigned colors, where $1 \leq j < n$, the vertex v_{j+1} is assigned the smallest color that is not assigned to any neighbor of v_{j+1} belonging to the set $\{v_1, v_2, \ldots, v_j\}$.*

As an illustration of the greedy coloring algorithm, suppose that we consider the graph C_6 of Figure 8.10. If we list the vertices of C_6 in the order u, w, v, y, z, x, then the greedy coloring algorithm yields the coloring c of G defined by

$$c(u) = 1,\ c(w) = 1,\ c(v) = 2,\ c(y) = 1,\ c(z) = 2,\ c(x) = 2.$$

This gives $\chi(C_6) \leq 2$. Of course, $\chi(C_6) = 2$ and so with this ordering of the vertices of C_6, a $\chi(C_6)$-coloring is produced. On the other hand, if the vertices of G are listed in the order u, x, v, w, z, y, then the greedy coloring algorithm yields the coloring c' of G defined by

$$c'(u) = 1,\ c'(x) = 1,\ c'(v) = 2,\ c'(w) = 3,\ c'(z) = 2,\ c'(y) = 3.$$

This gives a 3-coloring of C_6, which, of course, is not the chromatic number of C_6.

From the second listing of the vertices of C_6, we see that $\chi(C_6) \leq 3$. As the proof of the following theorem shows, no greater upper bound for $\chi(C_6)$ is possible using the greedy coloring algorithm.

Figure 8.10: The graph C_6

Theorem 8.19 *For every graph G,*

$$\chi(G) \leq 1 + \Delta(G).$$

Proof. Suppose that the vertices of G are listed in the order $v_1, v_2, \ldots, v_n$ and the greedy coloring algorithm is applied. Then v_1 is assigned the color 1 and for $2 \leq i < n$, the vertex v_i is either assigned the color 1 or is assigned the color $k+1$, where k is the largest integer such that all of the colors $1, 2, \ldots, k$ are used to color the neighbors of v_i in the set $S = \{v_1, v_2, \ldots, v_{i-1}\}$. Since at most $\deg v_i$ neighbors of v_i belong to S, the largest value of k is $\deg v_i$. Hence the color assigned to v_i is at most $1 + \deg v_i$. Thus

$$\chi(G) \leq \max_{1 \leq i \leq n} \{1 + \deg v_i\} = 1 + \Delta(G),$$

as desired. ■

The following theorem of George Szekeres and Herbert S. Wilf [212] gives an upper bound for the chromatic number of a graph that is an improvement over that stated in Theorem 8.19 but which is more difficult to compute.

Theorem 8.20 *For every graph G,*

$$\chi(G) \leq 1 + \max\{\delta(H)\},$$

where the maximum is taken over all subgraphs H of G.

Proof. Let $\chi(G) = k$ and let F be a k-critical subgraph of G. By Corollary 8.15, $\delta(F) \geq k - 1$. Thus

$$k - 1 \leq \delta(F) \leq \max\{\delta(H)\},$$

where the maximum is taken over all subgraphs H of G. Therefore, $\chi(G) \leq 1 + \max\{\delta(H)\}$. ■

Since $\delta(H') \leq \delta(H)$ for each spanning subgraph H' of an induced subgraph H of G, it follows that in Theorem 8.20 we may restrict the subgraphs H of G only to those that are induced.

For an r-regular graph G, both Theorems 8.19 and 8.20 give the same upper bound for $\chi(G)$, namely $1+r$. On the other hand, if T is a tree of order at least 3, then Theorem 8.19 gives $1+\Delta(T) \geq 3$ for an upper bound for $\chi(T)$; while Theorem 8.20 gives the improved and best possible upper bound $\chi(T) \leq 2$ since every subgraph of a tree contains a vertex of degree 1 or less.

When applying the greedy coloring algorithm to a graph G, there are, in general, fewer colors to avoid when coloring a vertex if the vertices of higher degree are listed early in the ordering of the vertices of G. The following result is due to Dominic J. A. Welsh and Martin B. Powell [235].

Theorem 8.21 *Let G be a graph of order n whose vertices are listed in the order $v_1, v_2, \ldots, v_n$ so that $\deg v_1 \geq \deg v_2 \geq \cdots \geq \deg v_n$. Then*

$$\chi(G) \leq 1 + \min_{1 \leq i \leq n} \{\max\{i-1, \deg v_i\}\} = \min_{1 \leq i \leq n} \{\max\{i, 1+\deg v_i\}\}.$$

Proof. Suppose that $\chi(G) = k$ and let H be a k-critical subgraph of G. Hence the order of H is at least k and $\delta(H) \geq k-1$ by Corollary 8.15. Therefore, for $1 \leq i \leq k$,

$$\max\{i, 1+\deg v_i\} \geq k;$$

while for $k+1 \leq i \leq n$,

$$\max\{i, 1+\deg v_i\} \geq k+1.$$

Consequently, $\chi(G) = k \leq \min_{1 \leq i \leq n} \{\max\{i, 1+\deg v_i\}\}$. ■

Brooks' Theorem

While as we saw in Theorem 8.19 that $1+\Delta(G)$ is an upper bound for the chromatic number of a connected graph G, Rowland Leonard Brooks (1916–1993) showed [30] that $\chi(G) = 1+\Delta(G)$ occurs only in very special cases.

Theorem 8.22 (**Brooks' Theorem**) *For every connected graph G that is not an odd cycle or a complete graph,*

$$\chi(G) \leq \Delta(G).$$

Proof. Let $\chi(G) = k \geq 2$ and let H be a k-critical subgraph of G. Thus H is 2-connected and $\Delta(H) \leq \Delta(G)$. Suppose first that either $H = K_k$ or H is an odd cycle. Then $G \neq H$ since G is neither an odd cycle nor a complete graph. Since G is connected, $\Delta(G) \geq k$ if $H = K_k$; while $\Delta(G) \geq 3$ if H is an odd cycle. If $H = K_k$, then $k = \chi(H) = \chi(G) \leq \Delta(G)$; while if H is an odd cycle, then $3 = \chi(H) = \chi(G) \leq \Delta(G)$. Therefore, in both cases, $\chi(G) \leq \Delta(G)$, as desired. Hence we may assume that H is a k-critical subgraph that is neither an odd cycle nor a complete graph. This implies that $k \geq 4$.

Suppose that H has order n. Since $\chi(G) = k \geq 4$ and H is not complete, $n > k$ and so $n \geq 5$. Since H is 2-connected, either H is 3-connected or H has connectivity 2. We consider these two cases.

Case 1. *H is 3-connected.* Since H is not complete, there are two vertices u and w of H such that $d_H(u, w) = 2$. Let (u, v, w) be a $u - w$ geodesic in H. Since H is 3-connected, $H - u - w$ is connected. Let $v = u_1, u_2, \ldots, u_{n-2}$ be the vertices of $H - u - w$, so listed that each vertex u_i $(2 \leq i \leq n-2)$ is adjacent to some vertex preceding it. Let $u_{n-1} = u$ and $u_n = w$. Consequently, for each set

$$U_j = \{u_1, u_2, \ldots, u_j\},\ 1 \leq j \leq n,$$

the induced subgraph $H[U_j]$ is connected.

We now apply a greedy coloring to H with respect to the reverse ordering

$$w = u_n, u = u_{n-1}, u_{n-2}, \ldots, u_2, u_1 = v \tag{8.1}$$

of the vertices of H. Since w and u are not adjacent, each is assigned the color 1. Furthermore, each vertex u_i $(2 \leq i \leq n-2)$ is assigned the smallest color in the set $\{1, 2, \ldots, \Delta(H)\}$ that was not used to color a neighbor of u_i that preceded it in the sequence (8.1). Since each vertex u_i has at least one neighbor following it in the sequence (8.1), u_i has at most $\Delta(H) - 1$ neighbors preceding it in the sequence and so a color is available for u_i. Moreover, the vertex $u_1 = v$ is adjacent to two vertices colored 1 (namely $w = u_n$ and $u = u_{n-1}$) and so at most $\Delta(H) - 1$ colors are assigned to the neighbors of v, leaving a color for v. Hence

$$\chi(G) = \chi(H) \leq \Delta(H) \leq \Delta(G). \tag{8.2}$$

Case 2. $\kappa(H) = 2$. We claim that H contains a vertex x such that $2 < \deg_H x < n - 1$. Suppose that this is not the case. Then the degree of every vertex of H is either 2 or $n - 1$. Because $\chi(H) \geq 4$, it follows that H cannot contain only vertices of degree 2; and because H is not complete, H cannot contain only vertices of degree $n - 1$. If H contains vertices of both degrees 2 and $n - 1$ and no others, then either

$$H = K_1 \vee \left(\frac{n-1}{2}\right) K_2 \quad \text{or} \quad H = K_{1,1,n-2}.$$

In both cases, $\chi(H) = 3$ and H is not critical, which is impossible. Thus, as claimed, H contains a vertex x such that $2 < \deg_H x < n - 1$.

Since $\kappa(H) = 2$, either $\kappa(H - x) = 2$ or $\kappa(H - x) = 1$. If $\kappa(H - x) = 2$, then x belongs to no minimum vertex-cut of H, which implies that H contains a vertex y such that $d_H(x, y) = 2$. Proceeding as in Case 1 with $u = x$ and $w = y$, we see that there is a coloring of H with at most $\Delta(H)$ colors and so once again we have (8.2), that is, $\chi(G) \leq \Delta(G)$.

Finally, we may assume that $\kappa(H - x) = 1$. Thus $H - x$ has end-blocks B_1 and B_2, containing cut-vertices x_1 and x_2, respectively, of $H - x$. Since H is 2-connected, there exist vertices $y_1 \in V(B_1) - \{x_1\}$ and $y_2 \in V(B_2) - \{x_2\}$ such that x is adjacent to both y_1 and y_2. Proceeding as in Case 1 with $u = y_1$ and $w = y_2$, we obtain a coloring of H with at most $\Delta(H)$ colors, once again giving us (8.2) and so $\chi(G) \leq \Delta(G)$. ■

We have now seen a number of upper bounds for the chromatic number of a graph G of order n and two lower bounds, namely, the clique number $\omega(G)$ and another $n/\alpha(G)$ in terms of its order and independence number. Dennis Paul Geller [92] gave a lower bound for the chromatic number of a graph in terms of its order and size.

Theorem 8.23 *If G is a graph of order n and size m, then*

$$\chi(G) \geq \frac{n^2}{n^2 - 2m}.$$

Proof. Suppose that $\chi(G) = k$. Let c be a k-coloring of G resulting in color classes $V_1, V_2, \ldots, V_k$ with $|V_i| = n_i$ for $1 \leq i \leq k$. Then the largest possible size of G occurs when G is a complete k-partite graph with partite sets $V_1, V_2, \ldots, V_k$ and the cardinalities of these partite sets are as equal as possible (or each $|V_i|$ is as close to $\frac{n}{k}$ as possible for $1 \leq i \leq k$). This implies that

$$m \leq \binom{k}{2} \frac{n^2}{k^2}$$

and so

$$2m \leq \frac{(k-1)n^2}{k}.$$

Thus

$$\frac{n^2}{n^2 - 2m} \leq \frac{n^2}{n^2 - \frac{(k-1)n^2}{k}} = k = \chi(G),$$

giving the desired result. ■

If G is a graph of order n and size $n^2/4$, then certainly $\chi(G) \geq 2$. Actually, this cannot be improved since $K_{\frac{n}{2},\frac{n}{2}}$ is a bipartite graph of order n and size $n^2/4$. On the other hand, if G is a graph of order n whose size is greater than $n^2/4$, then $\chi(G) \geq 3$ since by Theorem 1.9, G has a triangle.

The Nordhaus-Gaddum Theorem

We have seen that $\chi(K_n) = n$ and $\chi(\overline{K}_n) = 1$. So $\chi(K_n) + \chi(\overline{K}_n) = n + 1$. By Theorem 8.1, if G is a graph such that $\overline{G}$ is a nonempty bipartite graph, then $\chi(\overline{G}) = 2$, while $\chi(G) = \omega(G) \leq n - 1$. Thus $\chi(G) + \chi(\overline{G}) \leq n + 1$. Edward A.

Nordhaus and Jerry W. Gaddum [164] showed that this inequality holds for all graphs G of order n when they established two pairs of inequalities involving the sum and product of the chromatic numbers of a graph and its complement. Such inequalities established for any parameter have become known as **Nordhaus-Gaddum inequalities**. The following proof is due to Hudson V. Kronk (see [39]).

Theorem 8.24 (**The Nordhaus-Gaddum Theorem**) *If G is a graph of order n, then*

(*i*) $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n+1$

(*ii*) $n \leq \chi(G) \cdot \chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2$.

Proof. Suppose that $\chi(G) = k$ and $\chi(\overline{G}) = \ell$. Let a k-coloring c of G and an ℓ-coloring $\overline{c}$ of $\overline{G}$ be given. Using these colorings, we obtain a coloring of K_n. With each vertex v of G (and of $\overline{G}$), we associate the ordered pair $(c(v), \overline{c}(v))$. Since every two vertices of K_n are either adjacent in G or in $\overline{G}$, they are assigned different colors in that subgraph of K_n. Thus this is a coloring of K_n using at most $k\ell$ colors. Therefore,

$$n = \chi(K_n) \leq k\ell = \chi(G) \cdot \chi(\overline{G}).$$

This establishes the lower bound in (ii). Since the geometric mean of two positive real numbers never exceeds their arithmetic mean, it follows that

$$\sqrt{n} \leq \sqrt{\chi(G) \cdot \chi(\overline{G})} \leq \frac{\chi(G) + \chi(\overline{G})}{2}. \tag{8.3}$$

Consequently,

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}),$$

which verifies the lower bound in (i).

To verify the upper bound in (i), let $p = \max\{\delta(H)\}$, where the maximum is taken over all subgraphs H of G. Hence the minimum degree of every subgraph of G is at most p. By Theorem 8.20, $\chi(G) \leq 1 + p$.

We claim that the minimum degree of every subgraph of $\overline{G}$ is at most $n-p-1$. Assume, to the contrary, that there is a subgraph H of G such that $\delta(\overline{H}) \geq n-p$ for the subgraph $\overline{H}$ in $\overline{G}$. Thus every vertex of H has degree $p-1$ or less in G. Let F be a subgraph of G such that $\delta(F) = p$. So every vertex of F has degree p or more. This implies that no vertex of F belongs to H. Since the order of F is at least $p+1$, the order of H is at most $n-(p+1) = n-p-1$. This, however, contradicts the fact that $\delta(\overline{H}) \geq n-p$. Thus, as claimed, the minimum degree of every subgraph of $\overline{G}$ is at most $n-p-1$. By Theorem 8.20, $\chi(\overline{G}) \leq 1 + (n-p-1) = n-p$ and so

$$\chi(G) + \chi(\overline{G}) \leq (1+p) + (n-p) = n+1.$$

This verifies the upper bound in (i). By (8.3),

$$\chi(G) \cdot \chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2,$$

verifying the final inequality. ■

As the proof of Theorem 8.24 indicates, the key inequality of the four listed is $\chi(G) + \chi(\overline{G}) \leq n+1$. Bonnie M. Stewart [208] and Hans-Joachim Finck [80] showed that no improvement in Theorem 8.24 is possible.

Theorem 8.25 *Let n be a positive integer. For every two positive integers a and b such that*

$$2\sqrt{n} \leq a+b \leq n+1 \text{ and } n \leq ab \leq \left(\tfrac{n+1}{2}\right)^2$$

there is a graph G of order n such that $\chi(G) = a$ and $\chi(\overline{G}) = b$.

Proof. Let $n_1, n_2, \ldots, n_a$ be a positive integers such that $\sum_{i=1}^{a} n_i = n$ and $n_1 \leq n_2 \leq \ldots \leq n_a = b$. Since $a+b-1 \leq n \leq ab$, such integers n_i $(1 \leq i \leq a)$ exist. The graph $G = K_{n_1,n_2,\ldots,n_a}$ has order n and $\overline{G} = K_{n_1} + K_{n_2} + \cdots + K_{n_a}$. Hence $\chi(G) = a$ and $\chi(\overline{G}) = b$. ■

Of the several bounds for the chromatic number of a graph G, the clique number $\omega(G)$ is the best known and simplest lower bound for $\chi(G)$, while $1 + \Delta(G)$ is the best known and simplest upper bound for $\chi(G)$. If $\Delta(G) \geq 3$ and G is not complete, then $\Delta(G)$ is an improved upper bound for $\chi(G)$ by Brooks' theorem (Theorem 8.22). Bruce Reed [182] conjectured that $\chi(G)$ is always at least as close to $\omega(G)$ as to $1 + \Delta(G)$.

Reed's Conjecture *For every graph G,*

$$\chi(G) \leq \frac{\omega(G) + 1 + \Delta(G)}{2}.$$

As we saw in Theorem 8.9, for a graph G of order n, the number $n - \alpha(G) + 1$ is also an upper bound for $\chi(G)$. By Theorem 8.18, $(n + \omega(G))/2$ is also an upper bound for $\chi(G)$. The following improved upper bound for $\chi(G)$ is due to Robert C. Brigham and Ronald D. Dutton [28].

Theorem 8.26 *For every graph G of order n,*

$$\chi(G) \leq \frac{\omega(G) + n + 1 - \alpha(G)}{2}.$$

Proof. We proceed by induction on n. When $n = 1$, $G = K_1$ and $\chi(G) = \omega(G) = \alpha(G) = 1$ and so

$$\chi(G) = \frac{\omega(G) + n + 1 - \alpha(G)}{2}.$$

Thus the basis step holds for the induction.

Assume that the inequality holds for all graphs of order less than n where $n \geq 2$ and let G be a graph of order n. If $G = \overline{K}_n$, then $\chi(G) = \omega(G) = 1$ and $\alpha(G) = n$; so

$$\chi(G) = \frac{\omega(G) + n + 1 - \alpha(G)}{2}.$$

Hence we may assume that $G \neq \overline{K}_n$. Thus $1 \leq \alpha(G) \leq n - 1$. Let V_0 be a maximum independent set of vertices in G. Therefore, $|V_0| = \alpha(G)$. Let $G_1 = G - V_0$, where $\alpha(G_1) = \alpha_1$ and $\omega(G_1) = \omega_1$. Furthermore, let $V(G_1) = V_1$, where, then, $|V_1| = n - \alpha(G)$. We consider two cases.

Case 1. *G_1 is a complete graph.* Thus $V(G)$ can be partitioned into V_0 and V_1, where $G[V_0] = \overline{K}_{\alpha(G)}$ and $G_1 = G[V_1] = K_{n-\alpha(G)}$. Therefore, either $\chi(G) = \omega(G) = n - \alpha(G)$ or $\chi(G) = \omega(G) = n - \alpha(G) + 1$. We now consider these two subcases.

Subcase 1.1. $\chi(G) = \omega(G) = n - \alpha(G)$. So

$$\begin{aligned} \chi(G) &= n - \alpha(G) = \frac{(n - \alpha(G)) + (n - \alpha(G))}{2} \\ &= \frac{\omega(G) + n - \alpha(G)}{2} < \frac{\omega(G) + n + 1 - \alpha(G)}{2}. \end{aligned}$$

Subcase 1.2. $\chi(G) = \omega(G) = n - \alpha(G) + 1$. Here

$$\begin{aligned} \chi(G) &= n - \alpha(G) + 1 = \frac{(n - \alpha(G) + 1) + (n - \alpha(G) + 1)}{2} \\ &= \frac{\omega(G) + (n - \alpha(G) + 1)}{2}. \end{aligned}$$

Case 2. *G_1 is not a complete graph.* In this case, $\alpha_1 \geq 2$. Since $\chi(G) \leq \chi(G_1) + 1$, it follows by the induction hypothesis that

$$\begin{aligned} \chi(G) &\leq \chi(G_1) + 1 \leq \frac{\omega_1 + (n - \alpha(G)) + 1 - \alpha_1}{2} + 1 \\ &\leq \frac{\omega(G) + (n - \alpha(G)) + 1 - \alpha_1}{2} + 1 \leq \frac{\omega(G) + (n - \alpha(G) + 1)}{2}, \end{aligned}$$

completing the proof. ■

Applying Theorem 8.26 to both a graph G of order n and its complement $\overline{G}$ (where then $\omega(\overline{G}) = \alpha(G)$ and $\alpha(\overline{G}) = \omega(G)$), we have

$$\chi(G) \leq \frac{\omega(G) + n + 1 - \alpha(G)}{2}$$

and

$$\chi(\overline{G}) \leq \frac{\omega(\overline{G}) + n + 1 - \alpha(\overline{G})}{2}.$$

Adding these inequalities gives us an alternative proof of the major inequality stated in the Nordhaus-Gaddum theorem (Theorem 8.24): For every graph G of order n,

$$\chi(G) + \chi(\overline{G}) \leq n + 1.$$

The Gallai-Roy-Vitaver Theorem

Bounds for the chromatic number of a graph G can also be given in terms of the length $\ell(D)$ of a longest (directed) path in an orientation D of G. Suppose that G is a k-chromatic graph, where a k-coloring of G is given using the colors $1, 2, \ldots, k$. An orientation D of G can be constructed by directing each edge uv of G from u to v if the color assigned to u is smaller than the color assigned to v. Thus the length of every directed path in D is at most $k-1$. In particular, $\ell(D) \leq \chi(G) - 1$. So there exists an orientation D of G such that $\chi(G) \geq 1 + \ell(D)$. On the other hand, Tibor Gallai [90], Bernard Roy [197] and L. M. Vitaver [227] independently discovered the following result.

Theorem 8.27 (**The Gallai-Roy-Vitaver Theorem**) *For every orientation D of a graph G,*

$$\chi(G) \leq 1 + \ell(D).$$

Proof. Let D be an orientation of G and let D' be a spanning acyclic subdigraph of D of maximum size. A coloring c is defined on G by assigning to each vertex v of G the color 1 plus the length of a longest path in D' whose terminal vertex is v. Then, as we proceed along the vertices in any directed path in D', the colors are strictly increasing.

Let (u, v) be an arc of D. If (u, v) belongs D', then $c(u) < c(v)$. On the other hand, if (u, v) is not in D', then adding (u, v) to D' creates a directed cycle, which implies that $c(v) < c(u)$. Consequently, $c(u) \neq c(v)$ for every two adjacent vertices u and v of G and so

$$c : V(G) \to \{1, 2, \ldots, 1 + \ell(D)\}$$

is a proper coloring of G. Therefore, $\chi(G) \leq 1 + \ell(D)$. ■

The following result is therefore a consequence of Theorem 8.27 and the observation that precedes it.

Corollary 8.28 *Let G be a graph and let $\ell = \min\{\ell(D)\}$, where the minimum is taken over all orientations D of G. Then*

$$\chi(G) = 1 + \ell.$$

There is also a corollary to Corollary 8.28.

Corollary 8.29 *Every orientation of a graph G contains a directed path with at least $\chi(G)$ vertices.*

Exercises for Section 8.3

1. (a) Show, for every k-chromatic graph G, that there exists an ordering of the vertices of G such that the greedy coloring algorithm gives a k-coloring of G.

   (b) Show, for every positive integer p, that there exists a graph G and an ordering of the vertices of G such that the greedy coloring algorithm gives a k-coloring of G, where $k = p + \chi(G)$.

2. Since every nontrivial tree T is bipartite, $\chi(T) = 2$. Prove, for every integer $k \geq 2$, that there exists a tree T_k with $\Delta(T_k) = k$ and an ordering s of the vertices of T_k that produces a greedy coloring of T_k using $k + 1$ colors.

3. Let T be the tree of Figure 8.11.

   (a) What is the greedy coloring c produced by the ordering s: v_1, v_2, ..., v_8 of the vertices of T?

   (b) Does there exist a different ordering of the vertices of T giving a greedy coloring that uses fewer colors?

   (c) Does there exist a different ordering of the vertices of T giving a greedy coloring that uses more colors than that in (a)?

Figure 8.11: The tree T in Exercise 3

4. (a) What upper bound for $\chi(G)$ is given by Theorem 8.20 for the graph G in Figure 8.12?

   (b) What is $\chi(G)$ for this graph G?

5. For the double star T containing two vertices of degree 4, what upper bound for $\chi(T)$ is given by Theorems 8.19, 8.20 and 8.21?

6. What bound is given for $\chi(G)$ by Theorem 8.20 in the case that G is (a) a tree? (b) an outerplanar graph?

7. Among all 5-regular graphs, let s be the smallest chromatic number and let t be the largest chromatic number. For each integer k with $s \leq k \leq t$, show that there exists a 5-regular graph G_k such that $\chi(G_k) = k$.

Figure 8.12: The graph G in Exercise 4

8. Prove a result analogous to Theorem 8.22 for disconnected graphs.

9. Let G be a connected cubic graph of order $n > 4$ having girth 3. Determine $\chi(G)$.

10. Show that there is no graph of order 6 and size 13 that has chromatic number 3.

11. (a) If G is a graph of order n and size $n^2/4$, what bound does Theorem 8.23 give about $\chi(G)$? How sharp is this bound?

    (b) If G is a graph of order n and size m where $m > n^2/4$, what bound does Theorem 8.23 give about $\chi(G)$? How sharp is this bound?

12. What does the bound in Theorem 8.23 say for a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ of order n where $n_1 = n_2 = \cdots = n_k$?

13. Use Corollary 8.28 to determine the chromatic number of every nonempty bipartite graph.

14. By a proper k-coloring of a graph G is meant a function $c : V(G) \to \mathbb{N}_k = \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ if $uv \in E(G)$. This can also be defined as a function $c : V(G) \to \mathcal{P}(\mathbb{N}_k)$, the power set of $\mathbb{N}_k$, such that $|c(x)| = 1$ for each $x \in V(G)$ and $c(u) \cap c(v) = \emptyset$ if $d(u, v) = 1$, where no condition is placed on pairs u, v of vertices if $d(u, v) \geq 2$. A **proper 2-set k-coloring** of G is a function $c : V(G) \to \mathcal{P}(\mathbb{N}_k)$ such that $|c(x)| = 2$ for each $x \in V(G)$ and $c(u) \cap c(v) = \emptyset$ if $d(u, v) = 1$ and $|c(u) \cap c(v)| \leq 1$ if $d(u, v) = 2$, where there is no condition placed on $c(u) \cap c(v)$ if $d(u, v) \geq 3$. The **2-set chromatic number** $\chi_2(G)$ of G is the minimum k for which G has a proper 2-set k-coloring.

    (a) Prove that if the maximum degree of a graph G is Δ, then

$$\chi_2(G) \geq \left\lceil \frac{5 + \sqrt{8\Delta + 1}}{2} \right\rceil.$$

(b) Compute $\chi_2(C_7)$ and $\chi_2(C_8)$, thereby showing that there are graphs for which the bound in (a) is attained and graphs for which the bound in (a) is not attained.

8.4 Perfect Graphs

In Corollary 8.3 we saw that the clique number $\omega(G)$ of a graph G is a lower bound for $\chi(G)$. While there are many examples of graphs G for which $\chi(G) = \omega(G)$, such as complete graphs and bipartite graphs, there are also many graphs whose chromatic number exceeds its clique number such as the Petersen graph (see Figure 8.13) and the odd cycles of length 5 or more. In these two instances, the graphs have chromatic number 3 and clique number 2. As we are about to see, among the graphs with clique number 2, the chromatic number of a graph can be arbitrarily large.

Figure 8.13: The Petersen graph: A 3-chromatic graph with clique number 2

Triangle-Free Graphs With Large Chromatic Number

For a given graph F, recall that a graph G is called F-free if no induced subgraph of G is isomorphic to F. Recall also that a K_3-free graph is commonly called a triangle-free graph. Consequently, every nonempty triangle-free graph has clique number 2. The graph G of Figure 8.14 is triangle-free (and so $\omega(G) = 2$) but $\chi(G) = 4$. Hence $\chi(G)$ exceeds $\omega(G)$ by 2 in this case. This graph is called the **Grötzsch graph**, named for the German mathematician Herbert Grötzsch. It is known to be the unique smallest graph (in terms of order) that is both 4-chromatic and triangle-free. The fact that a graph can be triangle-free and yet have a large chromatic number has been established by a number of mathematicians, including Blanche Descartes [55], John B. Kelly and Leroy M. Kelly [132] and Alexander Zykov [249]. The proof of this fact that we present here, however, is due to Jan Mycielski [162].

Figure 8.14: The Grötzsch graph: A 4-chromatic triangle-free graph

Theorem 8.30 *For every positive integer k, there exists a triangle-free k-chromatic graph.*

Proof. Since no graph with chromatic number 1 or 2 contains a triangle, the theorem is obviously true for $k = 1$ and $k = 2$. To verify the theorem for $k \geq 3$, we proceed by induction on k. Since $\chi(C_5) = 3$ and C_5 is triangle-free, the statement is true for $k = 3$.

Assume that there exists a triangle-free graph with chromatic number k, where $k \geq 3$. We show that there exists a triangle-free $(k+1)$-chromatic graph. Let H be a triangle-free graph with $\chi(H) = k$, where $V(H) = \{v_1, v_2, \ldots, v_n\}$. We construct a graph G from H by adding $n+1$ new vertices $u, u_1, u_2, \ldots, u_n$, joining u to each vertex u_i $(1 \leq i \leq n)$ and joining u_i to each neighbor of v_i in H. (See Figure 8.15 for the case where $k = 3$ and $H = C_5$. Here the resulting graph G is the Grötzsch graph of Figure 8.15.)

Figure 8.15: The Mycielski construction

We claim that G is a triangle-free $(k+1)$-chromatic graph. First, we show that G is triangle-free. Since $S = \{u_1, u_2, \ldots, u_n\}$ is an independent set of vertices of G and u is adjacent to no vertex of H, it follows that u belongs to no triangle in G. Hence if there is a triangle T in G, then two of the three vertices of T must belong to H and the third vertex must belong to S, say

$V(T) = \{u_i, v_j, v_k\}$. Since u_i is adjacent to v_j and v_k, it follows that v_i is adjacent to v_j and v_k. Since v_j and v_k are adjacent, H contains a triangle, which is a contradiction. Thus, as claimed, G is triangle-free.

Next, we show that $\chi(G) = k+1$. Since H is a subgraph of G and $\chi(H) = k$, it follows that $\chi(G) \geq k$. Let a k-coloring of H be given and assign to u_i the same color that is assigned to v_i for $1 \leq i \leq n$. Assigning the color $k+1$ to u produces a $(k+1)$-coloring of G and so $\chi(G) \leq k+1$. Hence either $\chi(G) = k$ or $\chi(G) = k+1$. Suppose that $\chi(G) = k$. Then there is a k-coloring of G with colors $1, 2, \ldots, k$, where u is assigned the color k, say. Necessarily, none of the vertices $u_1, u_2, \ldots, u_n$ is assigned the color k; that is, each vertex of S is assigned one of the colors $1, 2, \ldots, k-1$. Since $\chi(H) = k$, one or more vertices of H are assigned the color k. For each vertex v_i of H colored k, recolor it with the color assigned to u_i. This produces a $(k-1)$-coloring of H, which is impossible. Thus $\chi(G) = k+1$. ■

If the Mycielski construction (described in the proof of Theorem 8.30) is applied to the Grötzsch graph (Figures 8.14 and 8.15), then a 5-chromatic triangle-free graph of order 23 is produced. Using a computer search, Tommy Jensen and Gordon F. Royle [127] showed that the smallest order of a 5-chromatic triangle-free graph is actually 22. Applying the Mycielski construction to this graph produces a 6-chromatic triangle-free graph of order 45. Whether 45 is the smallest order of such a graph is unknown.

With the aid of Theorem 8.30, it can be seen that for every two integers ℓ and k with $2 \leq \ell \leq k$, there exists a graph G with $\omega(G) = \ell$ and $\chi(G) = k$ (see Exercise 2). A symmetric graph G that is not triangle-free but for which $\chi(G) > \omega(G)$ is shown in Figure 8.16. This graph has order 15 and consists of five mutually vertex-disjoint triangles T_i $(1 \leq i \leq 5)$ where every vertex of T_i is adjacent to every vertex of T_j if either $|i-j| = 1$ or if $\{i, j\} = \{1, 5\}$. All of the edges between these pairs of triangles are indicated in the figure by bold lines. (We will visit this graph again in Section 9.3.) This graph G has clique number 6 and independence number 2. By Theorem 8.9, $\chi(G) \geq n/\alpha(G) = 15/2$ and so $\chi(G) \geq 8$. If we color the vertices of T_i $(1 \leq i \leq 5)$ as indicated in Figure 8.16, then it follows that G is 8-colorable and so $\chi(G) \leq 8$. Therefore, $\chi(G) = 8$.

While much interest has been shown in graphs G for which $\chi(G) > \omega(G)$, even more interest has been shown in graphs G for which not only $\chi(G) = \omega(G)$ but $\chi(H) = \omega(H)$ for *every* induced subgraph H of G. A graph G is called **perfect** if $\chi(H) = \omega(H)$ for every induced subgraph H of G. This definition, introduced in [17], is due to Claude Berge. As is often the case when a new concept is introduced, it is not initially known whether the concept will lead to interesting results and whether any intriguing characterizations will be forthcoming. For this particular class of graphs, however, all of this occurred.

Certainly, if $G = K_n$, then $\chi(G) = \omega(G) = n$. Furthermore, every induced subgraph H of K_n is also a complete graph and so $\chi(H) = \omega(H)$. Thus every complete graph is perfect. On the other hand, if $G = \overline{K}_n$ and H is any induced subgraph of G, then $\chi(H) = \omega(H) = 1$. So every empty graph is also perfect.

Figure 8.16: A graph G with $\chi(G) = 8$

The bipartite graphs constitute a more interesting class of perfect graphs.

Theorem 8.31 *Every bipartite graph is perfect.*

Proof. Let G be a bipartite graph and let H be an induced subgraph of G. If H is nonempty, then $\chi(H) = \omega(H) = 2$; while if H is empty, then $\chi(H) = \omega(H) = 1$. In either case, $\chi(H) = \omega(H)$ and so G is perfect. ■

The next theorem, which is a consequence of a result due to Tibor Gallai [88], describes a related class of perfect graphs.

Theorem 8.32 *Every graph whose complement is bipartite is perfect.*

Proof. Let G be a graph of order n such that $\overline{G}$ is bipartite. Since the complement of every (nontrivial) induced subgraph of G is also bipartite, to verify that G is perfect, it suffices to show that $\chi(G) = \omega(G)$. Suppose that $\chi(G) = k$ and $\omega(G) = \ell$. Then $k \geq \ell$. Let there be given a k-coloring of G. Then each color class of G consists either of one or two vertices; for if G contains a color class with three or more vertices, then this would imply that $\overline{G}$ has a triangle, which is impossible.

Of the k color classes, suppose that p of these classes consist of a single vertex and that each of the remaining q classes consists of two vertices. Hence $p + q = k$ and $p + 2q = n$. Let W be the set of vertices of G belonging to a singleton color class. Since every two vertices of W are necessarily adjacent, $G[W] = K_p$ and so $\overline{G}[W] = \overline{K}_p$.

Since no k-coloring of G results in more than q color classes having two vertices, it follows that $\overline{G}$ has a set M of q pairwise nonadjacent edges but no more than q such edges. We claim that for each edge $uv \in M$, either u is adjacent to no vertex of W or v is adjacent to no vertex of W. Suppose that

this is not the case. Then we may assume that u is adjacent to some vertex $w_1 \in W$ and v is adjacent to some vertex $w_2 \in W$. Since $\overline{G}$ is triangle-free, $w_1 \neq w_2$. However then, $(M - \{uv\}) \cup \{uw_1, vw_2\}$ is a set of $q+1$ pairwise nonadjacent edges in $\overline{G}$. This, however, is impossible and so, as claimed, for each edge uv in M either u is adjacent to no vertex of W or v is adjacent to no vertex of W.

Therefore, $\overline{G}$ contains an independent set of at least $p+q=k$ vertices and so $\omega(G) = \ell \geq k$. Hence $\chi(G) = \omega(G)$. ■

From what we've seen, if G is a graph that is either complete or bipartite, then both G and $\overline{G}$ are perfect. Indeed, in 1961 Claude Berge made the following conjecture.

The Perfect Graph Conjecture *A graph is perfect if and only if its complement is perfect.*

In 1970, at the age of 22, László Lovász received his Ph.D. from the Hungarian Academy of Sciences. His advisor was Tibor Gallai. Only two years later Lovász [147] verified the Perfect Graph Conjecture.

Theorem 8.33 (**The Perfect Graph Theorem**) *A graph is perfect if and only if its complement is perfect.*

Interval Graphs

For a collection $S = \{I_1, I_2, \ldots, I_n\}$ of finite closed intervals of real numbers, the **interval graph** G representing S has vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ where $v_iv_j \in E(G)$ if $I_i \cap I_j \neq \emptyset$. If G is an interval graph, then every induced subgraph of G is also an interval graph (see Exercise 11). In the definition of "interval graph," we arrive at the same concept if open intervals are used rather than closed intervals. Indeed, we can use half-open (half-closed) intervals or a mixture of any of these (see Exercises 13–15).

For example, the graph G in Figure 8.17 is an interval graph as can be seen by considering the five intervals $I_1 = [0,2]$, $I_2 = [1,5]$, $I_3 = [3,6]$, $I_4 = [4,8]$, $I_5 = [7,9]$, where v_i and v_j are adjacent in G, $1 \leq i,j \leq 5$, if $I_i \cap I_j \neq \emptyset$. Observe that $\chi(G) = \omega(G) = 3$ for this graph G. This graph is also a perfect graph. Indeed, every interval graph is a perfect graph.

Theorem 8.34 *Every interval graph is a perfect graph.*

Proof. Let G be an interval graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Since every induced subgraph of an interval graph is also an interval graph, it suffices to show that $\chi(G) = \omega(G)$. Because G is an interval graph, there exist n closed intervals $I_i = [a_i, b_i]$, $1 \leq i \leq n$, such that v_i is adjacent to v_j $(i \neq j)$ if $I_i \cap I_j \neq \emptyset$. We may assume that the intervals (and consequently, the vertices of G) have been labeled so that $a_1 \leq a_2 \leq \cdots \leq a_n$.

Figure 8.17: An interval graph

We now define a vertex coloring of G by applying the greedy coloring algorithm to the vertices of G listed in the order $v_1, v_2, \ldots, v_n$. We thus assign v_1 the color 1. If v_1 and v_2 are not adjacent (that is, if I_1 and I_2 are disjoint), then assign v_2 the color 1 as well; otherwise, assign v_2 the color 2. Suppose then that we have assigned colors to $v_1, v_2, \ldots, v_r$ for some integer r with $2 \leq r < n$. We now assign v_{r+1} the smallest color that has not been assigned to any neighbor of v_{r+1} in the set $\{v_1, v_2, \ldots, v_r\}$. Thus if v_{r+1} is adjacent to no vertex in $\{v_1, v_2, \ldots, v_r\}$, then v_{r+1} is assigned the color 1. This results in a k-coloring of G for some positive integer k and so $\chi(G) \leq k$. If $k = 1$, then $G = \overline{K}_n$ and $\chi(G) = \omega(G) = 1$. Hence we may assume that $k \geq 2$.

Suppose that the vertex v_t has been assigned the color k. Since it was not possible to assign v_t any of the colors $1, 2, \ldots, k-1$, this means that the interval $I_t = [a_t, b_t]$ must have a nonempty intersection with $k-1$ intervals $I_{j_1}, I_{j_2}, \ldots, I_{j_{k-1}}$, where say $1 \leq j_1 < j_2 < \cdots < j_{k-1} < t$. Thus $a_{j_1} \leq a_{j_2} \leq \ldots \leq a_{j_{k-1}} \leq a_t$. Since $I_{j_i} \cap I_t \neq \emptyset$ for $1 \leq i \leq k-1$, it follows that

$$a_t \in I_{j_1} \cap I_{j_2} \cap \cdots \cap I_{j_{k-1}} \cap I_t.$$

Thus for $U = \{v_{j_1}, v_{j_2}, \ldots, v_{j_{k-1}}, v_t\}$,

$$G[U] = K_k$$

and so $\chi(G) \leq k \leq \omega(G)$. Since $\chi(G) \geq \omega(G)$, we have $\chi(G) = \omega(G)$, as desired. ■

Chordal Graphs

We now consider a more general class of graphs. Recall that a chord of a cycle C in a graph is an edge that joins two nonconsecutive vertices of C. For example, wz and xz are chords in the cycle $C = (u, v, w, x, y, z, u)$ in the graph G of Figure 8.18; while in the cycle $C' = (w, x, y, z, w)$ in G, the edge xz is a chord and wz is not. The cycle $C'' = (u, v, w, z, u)$ has no chords. Obviously no triangle contains a chord.

A graph G is a **chordal graph** if every cycle of length 4 or more in G has a chord. Since the cycle $C'' = (u, v, w, z, u)$ in the graph G of Figure 8.18 contains no chords, the graph G is not a chordal graph.

While every complete graph is a chordal graph, no complete bipartite graph $K_{s,t}$, where $s, t \geq 2$, is chordal, for if u_1 and v_1 belong to one partite set and u_2

Figure 8.18: Chords in cycles

and v_2 belong to the other partite set, then the cycle $(u_1, u_2, v_1, v_2, u_1)$ contains no chord. Indeed, no graph having girth 4 or more is chordal. The graphs G_1 and G_2 of Figure 8.19 are chordal graphs. For the subset $S_1 = \{u_1, v_1, x_1\}$ of $V(G_1)$ and the subset $S_2 = \{u_2, w_2, x_2\}$ of $V(G_2)$, let the graph G_3 be obtained by identifying the vertices in the complete subgraph $G_1[S_1]$ with the vertices in the complete subgraph $G_2[S_2]$, where, say, u_1 and u_2 are identified, v_1 and x_2 are identified and x_1 and w_2 are identified. The graph G_3 shown in Figure 8.19 is also a chordal graph.

Figure 8.19: Chordal graphs

More generally, suppose that G_1 and G_2 are two graphs containing complete subgraphs H_1 and H_2, respectively, of the same order and G_3 is the graph obtained by identifying the vertices of H_1 with the vertices of H_2 (in a one-to-one manner). If G_3 contains a cycle of length 4 or more having no chord, then C must belong to G_1 or G_2. That is, if G_1 and G_2 are chordal, then G_3 is chordal. Furthermore, if G_3 is chordal, then both G_1 and G_2 are chordal.

Theorem 8.35 *Let G be a graph obtained by identifying two complete subgraphs of the same order in two graphs G_1 and G_2. Then G is chordal if and only if G_1 and G_2 are chordal.*

Proof. We have already noted that if G_1 and G_2 are two chordal graphs containing complete subgraphs H_1 and H_2, respectively, of the same order, then the graph G obtained by identifying the vertices of H_1 with the vertices of H_2 is also chordal. On the other hand, if G_1, say, were not chordal, then it would contain a cycle C of length 4 or more having no chords. However, then C would also be a cycle in G having no chords. ■

We have now observed that every graph obtained by identifying two complete subgraphs of the same order in two chordal graphs is also chordal. These are not only sufficient conditions for a graph to be chordal, they are necessary conditions as well. The following characterization of chordal graphs is due to Andras Hajnal and János Surányi [105] and Gabriel Dirac [62].

Theorem 8.36 *A graph G is chordal if and only if G can be obtained by identifying two complete subgraphs of the same order in two chordal graphs.*

Proof. As a consequence of Theorem 8.35, we need only show that every chordal graph can be obtained from two chordal graphs by identifying two complete subgraphs of the same order in these two graphs. If G is complete, say $G = K_n$, then G is chordal and can trivially be obtained by identifying the vertices of $G_1 = K_n$ and the vertices of $G_2 = K_n$ in any one-to-one manner. Hence we may assume that G is a connected chordal graph that is not complete.

Let S be a minimum vertex-cut of G. Now let V_1 be the vertex set of one component of $G - S$ and let $V_2 = V(G) - (V_1 \cup S)$. Consider the two subgraphs

$$G_1 = G[V_1 \cup S] \text{ and } G_2 = G[V_2 \cup S]$$

of G. Consequently, G is obtained by identifying the vertices of S in G_1 and in G_2. We now show that $G[S]$ is complete. Since this is certainly true if $|S| = 1$, we may assume that $|S| \geq 2$.

Each vertex v in S is adjacent to at least one vertex in each component of $G - S$, for otherwise $S - \{v\}$ is a vertex-cut of G, which is impossible. Let $u, w \in S$. Hence there are $u - w$ paths in G_1, where every vertex except u and w belongs to V_1. Among all such paths, let $P = (u, x_1, x_2, \ldots, x_s, w)$ be one of minimum length. Similarly, let $P' = (u, y_1, y_2, \ldots, y_t, w)$ be a $u - w$ path of minimum length where every vertex except u and w belongs to V_2. Hence

$$C = (u, x_1, x_2, \ldots, x_s, w, y_t, y_{t-1}, \ldots, y_1, u)$$

is a cycle of length 4 or more in G. Since G is chordal, C contains a chord. No vertex x_i $(1 \leq i \leq s)$ can be adjacent to a vertex y_j $(1 \leq j \leq t)$ since S is a vertex-cut of G. Furthermore, no nonconsecutive vertices of P or of P' can be adjacent due to the manner in which P and P' are defined. Thus $uw \in E(G)$, implying that $G[S]$ is complete. By Theorem 8.35, G_1 and G_2 are chordal. ■

With the aid of Theorem 8.36, we now have an even larger class of perfect graphs.

Corollary 8.37 *Every chordal graph is perfect.*

Proof. Since every induced subgraph of a chordal graph is also a chordal graph, it suffices to show that if G is a connected chordal graph, then $\chi(G) = \omega(G)$. We proceed by induction on the order n of G. If $n = 1$, then $G = K_1$ and $\chi(G) = \omega(G) = 1$. Assume therefore that $\chi(H) = \omega(H)$ for every chordal graph H of order less than n, where $n \geq 2$ and let G be a chordal graph of order $n \geq 2$.

If G is a complete graph, then $\chi(G) = \omega(G) = n$. Hence we may assume that G is not complete. By Theorem 8.36, G can be obtained from two chordal graphs G_1 and G_2 by identifying two complete subgraphs of the same order in G_1 and G_2. Observe that

$$\chi(G) \leq \max\{\chi(G_1), \chi(G_2)\} = k.$$

By the induction hypothesis, $\chi(G_1) = \omega(G_1)$ and $\chi(G_2) = \omega(G_2)$. Thus

$$\chi(G) \leq \max\{\omega(G_1), \omega(G_2)\} = k.$$

On the other hand, let S denote the set of vertices in G that belong to G_1 and G_2. Thus $G[S]$ is complete and no vertex in $V(G_1) - S$ is adjacent to a vertex in $V(G_2) - S$. Hence

$$\omega(G) = \max\{\omega(G_1), \omega(G_2)\} = k.$$

Thus $\chi(G) \geq k$. Therefore, $\chi(G) = k = \omega(G)$. ■

The graph G of Figure 8.20 has clique number 3. Thus $\chi(G) \geq 3$. In this case, however, $\chi(G) \neq \omega(G)$. Indeed, $\chi(G) = 4$. A 4-coloring of G is shown in Figure 8.20. Thus G is not perfect. By Corollary 8.37, G is not chordal. In fact, $C = (u, x, v, y, u)$ is a 4-cycle containing no chord. Since G is not perfect, it follows by the Perfect Graph Theorem (Theorem 8.33) that $\overline{G}$ is not perfect either. Indeed, $\overline{G} = C_7$ and so $\chi(\overline{G}) = 3$ and $\omega(\overline{G}) = 2$. In fact, the complement of every odd cycle of length 5 or more is not perfect since its chromatic number exceeds its clique number (see Exercise 20).

The graph F of Figure 8.20 is also not chordal, but yet $\chi(H) = \omega(H)$ for every induced subgraph H of F. Hence the converse of Corollary 8.37 is not true.

We now consider a class of perfect graphs that can be obtained from a given perfect graph. Let G be a graph where $v \in V(G)$. Then the **replication graph** $R_v(G)$ of G (with respect to v) is that graph obtained from G by adding a new vertex v' to G and joining v' to the vertices in the closed neighborhood $N[v]$ of v.

The wheel $G = W_4$ of Figure 8.21 is a perfect graph. Since $\omega(G) = 3$, it follows that $\chi(G) = 3$. For the central vertex v of G, the replication graph

Figure 8.20: Non-chordal graphs

$R_v(G)$ of G is also shown in Figure 8.21. The graph $R_v(G)$ is also perfect and since its clique number is 4, it follows that $\chi(R_v(G)) = 4$ as well. Indeed, in 1972 Lovász [148] obtained the following result, which was used to prove the Perfect Graph Theorem (Theorem 8.33).

Figure 8.21: The replication graph $R_v(G)$
with respect to a vertex v of a graph G

Theorem 8.38 (**The Replication Lemma**) *Let G be a graph where $v \in V(G)$. If G is perfect, then $R_v(G)$ is perfect.*

Proof. Let $G' = R_v(G)$. First, we show that $\chi(G') = \omega(G')$. We consider two cases, depending on whether v belongs to a maximum clique of G.

Case 1. *v belongs to a maximum clique of G.* Then $\omega(G') = \omega(G)+1$. Since

$$\chi(G') \leq \chi(G) + 1 = \omega(G) + 1 = \omega(G'),$$

it follows that $\chi(G') = \omega(G')$.

Case 2. *v does not belong to any maximum clique of G.* Suppose that $\chi(G) = \omega(G) = k$. Let there be given a k-coloring of G using the colors $1, 2, \ldots, k$. We may assume that v is assigned the color 1. Let V_1 be the color class consisting of the vertices of G that are colored 1. Thus $v \in V_1$. Since $\omega(G) = k$, every maximum clique of G must contain a vertex of each color. Since v does not belong to a maximum clique, it follows that $|V_1| \geq 2$. Let

$U_1 = V_1 - \{v\}$. Because every maximum clique of G contains a vertex of U_1, it follows that $\omega(G-U_1) = \omega(G)-1 = k-1$. Since G is perfect, $\chi(G-U_1) = k-1$. Let a $(k-1)$-coloring of $G - U_1$ be given, using the colors $1, 2, \ldots, k-1$. Since V_1 is an independent set of vertices, so is $U_1 \cup \{v'\}$. Assigning the vertices of $U_1 \cup \{v'\}$ the color k produces a k-coloring of G'. Therefore,

$$k = \omega(G) \leq \omega(G') \leq \chi(G') \leq k$$

and so $\chi(G') = \omega(G')$.

It remains to show that $\chi(H) = \omega(H)$ for every induced subgraph H of G'. This is certainly the case if H is a subgraph of G. If H contains v' but not v, then $H \cong G[(V(H) - \{v'\}) \cup \{v\}]$ and so $\chi(H) = \omega(H)$. If H contains both v and v' but $H \ncong G'$, then H is the replication graph of $G[V(H) - \{v'\}]$ and the argument used to show that $\chi(G') = \omega(G')$ can be applied to show that $\chi(H) = \omega(H)$. ■

The Strong Perfect Graph Theorem

As we have mentioned, if G is an odd cycle of length 5 or more or is the complement of such an odd cycle, then $\chi(G) \neq \omega(G)$. In 1961 Claude Berge conjectured that this fact is critical in determining whether a graph is perfect.

The Strong Perfect Graph Conjecture A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length 5 or more.

After an intensive 28-month assault on this conjecture, its truth was established in 2002 by Maria Chudnovsky, Neil Robertson, Paul Seymour and Robin Thomas [43].

Theorem 8.39 (**The Strong Perfect Graph Theorem**) *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length 5 or more.*

Exercises for Section 8.4

1. Determine G if, in the proof of Theorem 8.30,

   (a) $H = K_2$; (b) $H = C_5$.

2. Show, for every two integers ℓ and k with $2 \leq \ell \leq k$, that there exists a graph G with $\omega(G) = \ell$ and $\chi(G) = k$.

3. Prove or disprove: For every integer $k \geq 3$, there exists a triangle-free, k-critical graph.

4. Give an example of a regular triangle-free 4-chromatic graph.

5. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The **shadow graph** $Shad(G)$ of G is that graph with vertex set $V(G) \cup \{u_1, u_2, \ldots, u_n\}$, where u_i is called the **shadow vertex** of v_i and u_i is adjacent to u_j if v_i is adjacent to v_j and u_i is adjacent to v_j if v_i is adjacent to v_j for $1 \le i, j \le n$. How are $\omega(G)$ and $\omega(Shad(G))$ related and how are $\chi(G)$ and $\chi(Shad(G))$ related?

6. Let $n \ge 2$ be an integer. Prove that every $(n-1)$-chromatic graph of order n has clique number $n-1$.

7. Prove or disprove the following.

   (a) If G is a graph of order $n \ge 3$ with $\chi(G) = n-2$, then $\omega(G) = n-2$.

   (b) If G is a graph of sufficiently large order n with $\chi(G) = n-2$, then $\omega(G) = n-2$.

8. Prove that a critically k-chromatic graph G is perfect if and only if $G = K_k$.

9. Let A be a set and let S be a collection of nonempty subsets of A. The **intersection graph** of S is that graph whose vertices are the elements of S and where two vertices are adjacent if the subsets have a nonempty intersection.

   (a) Prove that every graph is an intersection graph.

   (b) Let G be a nonempty graph. Show that a set $\mathcal{F}$ can be associated with G so that the intersection graph of $\mathcal{F}$ is the line graph of G.

10. For the graph $G = K_4 - e$, determine the smallest positive integer k for which there is a collection S of subsets of $A = \{1, 2, \ldots, k\}$ such that the intersection graph (see the definition in Exercise 9) of S is isomorphic to G.

11. Show that every induced subgraph of an interval graph is an interval graph.

12. Determine which connected graphs of order 4 are interval graphs.

13. (a) For the four open intervals $I_1 = (0, 1)$, $I_2 = (1, 2)$, $I_3 = (2, 3)$, $I_4 = (3, 4)$ of length 1 and the five open intervals

$$J_1 = \left(0, \tfrac{4}{5}\right),\ J_2 = \left(\tfrac{4}{5}, 1\tfrac{3}{5}\right),\ J_3 = \left(1\tfrac{3}{5}, 2\tfrac{2}{5}\right),$$
$$J_4 = \left(2\tfrac{2}{5}, 3\tfrac{1}{5}\right),\ J_5 = \left(3\tfrac{1}{5}, 4\right)$$

   of length $\frac{4}{5}$, let $S = \{I_i :\ 1 \le i \le 4\} \cup \{J_j :\ 1 \le j \le 5\}$. Draw the interval graph G with $V(G) = S$ and determine the size of G.

   (b) Repeat the problem in part (a) for the open intervals $I_1 = (0, 1)$, $I_2 = (1, 2)$, $I_3 = (2, 3)$ and the open intervals $J_j = \left(\frac{3}{5}(j-1), \frac{3}{5}j\right)$ for $j = 1, 2, \ldots, 5$.

(c) Repeat the problem in part (a) for the open intervals $I_1 = (0, 1)$, $I_2 = (1, 2)$, $I_3 = (2, 3)$, $I_4 = (3, 4)$ and the open intervals $J_j = \left(\frac{4}{6}(j-1), \frac{4}{6}j\right)$ for $j = 1, 2, \ldots, 6$.

14. Let a and b be integers with $2 \leq a < b$. For $i = 1, 2, \ldots, a$, let $I_i = (i-1, i)$ and for $j = 1, 2, \ldots, b$, let $J_j = \left(\frac{a}{b}(j-1), \frac{a}{b}j\right)$. Let G be the interval graph with $V(G) = S$.

    (a) Show that G is acyclic.

    (b) Determine the size of G.

15. (a) Draw the interval graph G with vertex set $S = \{I_1, I_2, \ldots, I_5\}$ consisting of the five closed intervals $I_1 = [1, 7], I_2 = [6, 8], I_3 = [6, 9], I_4 = [7, 20]$ and $I_5 = [20, 22]$.

    (b) Show that the same graph G can be obtained from a set $T = \{J_1, J_2, \ldots, J_5)$ of five open intervals where $I_i \subseteq J_i$ for $i = 1, 2, \ldots, 5$.

16. Show, for every integer $k \geq 3$, that there exists a k-chromatic graph that is not chordal.

17. Prove that every interval graph is a chordal graph.

18. Give an example of a chordal graph that is not an interval graph.

19. For each integer $n \geq 7$, give an example of a graph G_n of order n such that no induced subgraph of G_n is an odd cycle of length at least 5 but G_n is not perfect.

20. Let $G = C_{2k+1}$ for some integer $k \geq 2$. Show that $\overline{G}$ is not perfect by determining $\chi(\overline{G})$ and $\omega(\overline{G})$.

21. Determine whether $\overline{C}_8$ is perfect.

22. Let $\mathcal{G}$ be the set of all connected graphs of order 25, say $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$, and let $G = G_1 + G_2 + \cdots + G_k$ be the union of these graphs. Prove or disprove the following.

    (a) For the graph G, $\chi(G) = \omega(G)$.

    (b) The graph G is perfect.

23. Let G be a graph where $v \in V(G)$. According to Theorem 8.38, if G is perfect, then the replication graph $R_v(G)$ is perfect. Is the converse true?

8.5 List Colorings

In recent decades there has been increased interest in colorings of graphs in which the color of each vertex must be chosen from a specified list of allowable colors. Let G be a graph for which there is an associated set $L(v)$ of permissible colors for each vertex v of G. The set $L(v)$ is commonly called a **color list** for v. A **list coloring** of G is then a proper coloring c of G such that $c(v) \in L(v)$ for each vertex v of G. A list coloring is also referred to as a **choice function**. If

$$\mathfrak{L} = \{L(v) : \ v \in V(G)\}$$

is a set of color lists for the vertices of G and there exists a list coloring for this set $\mathfrak{L}$ of color lists, then G is said to be **$\mathfrak{L}$-choosable** or **$\mathfrak{L}$-list-colorable**. A graph G is **k-choosable** or **k-list-colorable** if G is $\mathfrak{L}$-choosable for every collection $\mathfrak{L}$ of lists $L(v)$ for the vertices v of G such that $|L(v)| \geq k$ for each vertex v. The **list chromatic number** $\chi_\ell(G)$ of G is the minimum positive integer k such that G is k-choosable. Then $\chi_\ell(G) \geq \chi(G)$. The concept of list colorings was introduced by Vadim Vizing [230] in 1976 and, independently, by Paul Erdős, Arthur L. Rubin and Herbert Taylor [72] in 1979.

Suppose that G is a graph with $\Delta(G) = \Delta$. By Theorem 8.19 if we let

$$L(v) = \{1, 2, \ldots, \Delta, 1 + \Delta\}$$

for each vertex v of G, then for these color lists there is a list coloring of G. Indeed, if $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $\mathfrak{L} = \{L(v_i) : \ 1 \leq i \leq n\}$ is a collection of color lists for G where each set $L(v_i)$ consists of any $1 + \Delta$ colors, then a greedy coloring of G produces a proper coloring and so G is $\mathfrak{L}$-choosable. Therefore, $\chi_\ell(G) \leq 1 + \Delta(G)$. (See Exercise 1.) Summarizing these observations, we have

$$\chi(G) \leq \chi_\ell(G) \leq 1 + \Delta(G)$$

for every graph G.

We now consider some examples. First, $\chi(C_4) = 2$ and so $\chi_\ell(C_4) \geq 2$. Consider the cycle C_4 of Figure 8.22 and suppose that we are given any four color lists $L(v_i)$, $1 \leq i \leq 4$, with $|L(v_i)| = 2$. Let $L(v_1) = \{a, b\}$. We consider three cases.

Case 1. $a \in L(v_2) \cap L(v_4)$. In this case, assign v_1 the color b and v_2 and v_4 the color a. Then there is at least one color in $L(v_3)$ that is not a. Assigning v_3 that color gives C_4 a list coloring for this collection of lists.

Case 2. *The color a belongs to exactly one of $L(v_2)$ and $L(v_4)$, say $a \in L(v_2) - L(v_4)$.* If there is some color $x \in L(v_2) \cap L(v_4)$, then assign v_2 and v_4 the color x and v_1 the color a. There is at least one color in $L(v_3)$ different from x. Assign v_3 that color. Hence there is a list coloring of C_4 for this collection of lists. Next, suppose that there is no color belonging to both $L(v_2)$ and $L(v_4)$. If $a \in L(v_3)$, then assign a to both v_1 and v_3. There is a color available for

both v_2 and v_4. If $a \notin L(v_3)$, then assign v_1 the color a, assign v_2 the color y in $L(v_2)$ different from a, assign v_3 any color z in $L(v_3)$ different from y, and assign v_4 any color in $L(v_4)$ different from z. This is a list coloring for C_4.

Case 3. $a \notin L(v_2) \cup L(v_4)$. Then assign v_1 the color a and v_3 any color from $L(v_3)$. Hence there is an available color from $L(v_2)$ and $L(v_4)$ to assign to v_2 and v_4, respectively. Therefore, there is a list coloring of C_4 for this collection of lists.

C_4 :

$L(v_1) = \{a, b\}$

Figure 8.22: The graph C_4 is 2-list colorable

Actually, $\chi_\ell(C_n) = 2$ for every even integer $n \geq 4$. Before showing this, however, it is useful to show that $\chi_\ell(T) = 2$ for every nontrivial tree T.

Theorem 8.40 *Every tree is* 2*-choosable. Furthermore, for every tree* T*, for a vertex* u *of* T *and for a collection* $\mathfrak{L} = \{L(v) : v \in V(T)\}$ *of color lists of size* 2*, where* $a \in L(u)$*, there exists an* $\mathfrak{L}$*-list-coloring of* T *in which* u *is assigned color* a*.*

Proof. We proceed by induction on the order of the tree. The result is obvious for a tree of order 1 or 2. Assume that the statement is true for all trees of order k, where $k \geq 2$. Let T be a tree of order $k+1$ and let

$$\mathfrak{L} = \{L(v) : v \in V(T)\}$$

be a collection of color lists of size 2. Let $u \in V(T)$ and suppose that $a \in L(u)$. Let x be an end-vertex of T such that $x \neq u$ and let

$$\mathfrak{L}' = \{L(v) : v \in V(T - x)\}.$$

Let y be the neighbor of x in T. By the induction hypothesis, there exists an $\mathfrak{L}'$-list-coloring c' of $T - x$ in which u is colored a. Now let $b \in L(x)$ such that $b \neq c'(y)$. Then the coloring c defined by

$$c(v) = \begin{cases} b & \text{if } v = x \\ c'(v) & \text{if } v \neq x \end{cases}$$

is an $\mathfrak{L}$-list-coloring of T in which u is colored a. ■

Theorem 8.41 *Every even cycle is* 2*-choosable.*

Proof. We already know that C_4 is 2-choosable. Let C_n be an n-cycle, where $n \geq 6$ is even. Suppose that $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Let there be given a collection $\mathfrak{L} = \{L(v_i) : 1 \leq i \leq n\}$ of color lists of size 2 for the vertices of C_n. We show that C_n is $\mathfrak{L}$-list-colorable. We consider two cases.

Case 1. *All of the color lists are the same, say* $L(v_i) = \{1, 2\}$ *for* $1 \leq i \leq n$. If we assign the color 1 to v_i for odd i and the color 2 to v_i for even i, then C_n is $\mathfrak{L}$-list-colorable.

Case 2. *The color lists in* $\mathfrak{L}$ *are not all the same.* Then there are adjacent vertices v_i and v_{i+1} in G such that $L(v_i) \neq L(v_{i+1})$. Thus there exists a color $a \in L(v_{i+1}) - L(v_i)$. The graph $C_n - v_i$ is a path of order $n - 1$. Let $\mathfrak{L}' = \{L(v) : v \in V(C_n - v_i)\}$. By Theorem 8.40, there exists an $\mathfrak{L}'$-list-coloring c' of $C_n - v_i$ in which $c'(v_{i+1}) = a$. Let $b \in L(v_i)$ such that $b \neq c'(v_{i-1})$. Then the coloring c defined by

$$c(v) = \begin{cases} b & \text{if } v = v_i \\ c'(v) & \text{if } v \neq v_i \end{cases}$$

is an $\mathfrak{L}$-list-coloring of G. ■

Since the chromatic number of every odd cycle is 3, the list chromatic number of every odd cycle is at least 3. Indeed, every odd cycle is 3-choosable (see Exercise 2).

We have seen that all trees and even cycles are 2-choosable. Of course, these are both classes of bipartite graphs. Not every bipartite graph is 2-choosable, however. To illustrate this, we consider $\chi_\ell(K_{3,3})$, where $K_{3,3}$ is shown in Figures 8.23(a). While $\chi_\ell(K_{3,3}) \leq 1 + \Delta(K_{3,3}) = 4$, we can show that $\chi_\ell(K_{3,3}) \leq 3$. Let there be given lists $L(v_i)$, $1 \leq i \leq 6$, where $|L(v_i)| = 3$. We consider two cases.

Case 1. *Some color occurs in two or more of the lists* $L(v_1)$, $L(v_2)$, $L(v_3)$ *or in two or more of the lists* $L(v_4)$, $L(v_5)$, $L(v_6)$, *say color* a *occurs in* $L(v_1)$ *and* $L(v_2)$. Then assign v_1 and v_2 the color a and assign v_3 any color in $L(v_3)$. Then there is an available color in $L(v_i)$ for v_i $(i = 4, 5, 6)$.

Case 2. *The sets* $L(v_1)$, $L(v_2)$, $L(v_3)$ *are pairwise disjoint as are the sets* $L(v_4)$, $L(v_5)$, $L(v_6)$. Let $a_1 \in L(v_1)$ and $a_2 \in L(v_2)$. If none of the sets $L(v_4)$, $L(v_5)$, $L(v_6)$ contain both a_1 and a_2, then let a_3 be any color in $L(v_3)$. Then there is an available color for each of v_4, v_5, v_6 to construct a proper coloring of $K_{3,3}$. If exactly one of the sets $L(v_4)$, $L(v_5)$, $L(v_6)$ contains both a_1 and a_2, then select a color $a_3 \in L(v_3)$ so that none of $L(v_4)$, $L(v_5)$, $L(v_6)$ contains all of a_1, a_2, a_3. By assigning v_3 the color a_3, we see that there is an available color for each of v_4, v_5, and v_6.

Hence, as claimed, $\chi_\ell(K_{3,3}) \leq 3$. We show in fact that $\chi_\ell(K_{3,3}) = 3$. Consider the sets $L(v_i)$, $1 \leq i \leq 6$, shown in Figures 8.23(b). Assume, without

loss of generality, that v_1 is colored 1. Then v_4 must be colored 2 and v_5 must be colored 3. Whichever color is chosen for v_3 is the same color as that of either v_4 or v_5. This produces a contradiction. Hence $K_{3,3}$ is not 2-choosable and so $\chi_\ell(K_{3,3}) = 3$.

Figure 8.23: The graph $K_{3,3}$ is 3-choosable

The graph $G = K_{3,3}$ shows that it is possible for $\chi_\ell(G) > \chi(G)$. In fact, $\chi_\ell(G)$ can be considerably larger than $\chi(G)$.

Theorem 8.42 *If r and k are positive integers such that $r \geq \binom{2k-1}{k}$, then*

$$\chi_\ell(K_{r,r}) \geq k + 1.$$

Proof. Assume, to the contrary, that $\chi_\ell(K_{r,r}) \leq k$. Then there exists a k-list-coloring of $K_{r,r}$. Let U and W be the partite sets of $K_{r,r}$, where

$$U = \{u_1, u_2, \ldots, u_r\} \text{ and } W = \{w_1, w_2, \ldots, w_r\}.$$

Let $S = \{1, 2, \ldots, 2k-1\}$. There are $\binom{2k-1}{k}$ distinct k-element subsets of S. Assign these color lists to $\binom{2k-1}{k}$ vertices of U and to $\binom{2k-1}{k}$ vertices of W. Any remaining vertices of U and W are assigned any of the k-element subsets of S. For $i = 1, 2, \ldots, r$, choose a color $a_i \in L(u_i)$ and let $T = \{a_i : 1 \leq i \leq r\}$. We consider two cases.

Case 1. $|T| \leq k - 1$. Then there exists a k-element subset S' of S that is disjoint from T. However, $L(u_j) = S'$ for some j with $1 \leq j \leq r$. This is a contradiction.

Case 2. $|T| \geq k$. Hence there exists a k-element subset T' of T. Thus $L(w_j) = T'$ for some j with $1 \leq j \leq r$. Whichever color from $L(w_j)$ is assigned for w_j, this color has been assigned to some vertex u_i. Thus u_i and w_j have been assigned the same color and $u_i w_j$ is an edge of $K_{r,r}$. This is a contradiction. ■

Exercises for Section 8.5

1. Use the greedy coloring algorithm to show that $\chi_\ell(G) \leq 1 + \Delta(G)$ for every graph G.

2. Prove that every odd cycle is 3-choosable.

3. Prove that $\chi_\ell(K_{2,3}) = 2$.

4. Prove that $\chi_\ell(K_{2,4}) = 3$.

5. Prove that $\chi_\ell(K_{3,27}) > 3$.

6. We have seen that $\chi_\ell(K_{3,3}) > 2$; in fact, $\chi_\ell(K_{3,3}) = 3$.

   (a) Show that $\chi_\ell(K_{10,10}) > 3$.

   (b) Show for each integer $k \geq 4$, that there exists a positive integer r such that $\chi_\ell(K_{r,r}) > k$.

7. Prove that the list-chromatic number of $P_n \,\square\, K_2$ is 3 for every integer $n \geq 4$.

8. Suppose for $G = K_{3,3}$ that a set $\mathfrak{L} = \{L(v) : v \in V(G)\}$ of color lists is given for the vertices v in G, where $|L(u)| = 3$ for one vertex u in G and $|L(w)| = 2$ for all other vertices w in G. Does there exist a list coloring of G for these lists?

Chapter 9

Map Colorings

For well over a century the most famous unsolved problem in graph theory and one of the most famous unsolved problems in mathematics was a map coloring problem called the Four Color Problem. This problem, dealing with determining whether a conjecture called the Four Color Conjecture was true or false, had a major impact on the development of graph theory. Indeed, in 1967 William T. Tutte wrote:

> *It would not be hard to present the history of graph theory as an account of the struggle to prove the four color conjecture, or at least to find out why the problem is difficult.*

This problem will be discussed in the current chapter as will related coloring problems dealing with graphs that are embeddable on certain surfaces.

9.1 The Four Color Problem

Vertex colorings (introduced in the preceding chapter), edge colorings (to be introduced in Chapter 11) and map colorings (to be introduced in the current chapter) can all be traced to a single problem.

If the countries in a map of South America (see Figure 9.1) were to be colored in such a way that every two countries with a common boundary are colored differently, then this map can be colored using only four colors. Is this true of every map?

This map coloring problem would eventually acquire a name that would become known throughout the world of mathematics.

The Four Color Problem *Can the countries of every map be colored with four or fewer colors so that every two countries with a common boundary are colored differently?*

Figure 9.1: Map of South America

It is not only the subject of graph colorings that owes its origin to the Four Color Problem. Much of the early development of graph theory can be traced to the study of this problem.

The Guthrie Brothers

The origin of the Four Color Problem is rather well documented. On October 23, 1852, Frederick Guthrie (1833–1886), a student at University College London, visited his mathematics professor, the famous Augustus De Morgan (1806–1871), to describe an apparent mathematical discovery of his older brother Francis. While coloring the counties of a map of England, Francis observed that he could color them with four colors, which led him to conjecture that no more than four colors would be needed to color the regions of any map.

The Four Color Conjecture *The regions of every map can be colored with four or fewer colors in such a way that every two regions sharing a common boundary are colored differently.*

Francis Guthrie (1831–1899) attempted to prove this Four Color Conjecture and although he thought he may have been successful, he was not completely satisfied with his proof. Francis discussed his discovery with Frederick. With Francis' approval, Frederick mentioned this apparent theorem to Professor De Morgan, who expressed pleasure with it and believed it to be a new result. Evidently Frederick asked Professor De Morgan if he was aware of an argument that would establish the truth of the theorem.

This led De Morgan to write a letter to his friend, the famous Irish mathematician Sir William Rowan Hamilton. These two mathematical giants had corresponded for years, although apparently had met only once. De Morgan wrote (in part):

> *My dear Hamilton:*
>
> *A student of mine asked me to day to give him a reason for a fact which I did not know was a fact – and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary lines are differently coloured – four colours may be wanted but not more – the following is his case in which four are wanted.*

A B C D are names of colours

> *Query cannot a necessity for five or more be invented . . .*
>
> *My pupil says he guessed it colouring a map of England . . . . The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did . . .*

In De Morgan's letter to Hamilton, De Morgan attempted to explain why the problem appeared to be difficult. He followed this explanation by writing:

> *But it is tricky work and I am not sure of all convolutions – What do you say? And has it, if true been noticed?*

Since De Morgan had shown an interest in Hamilton's research on quaternions as well as other subjects Hamilton had studied, it is likely that De Morgan

expected an enthusiastic reply to his letter to Hamilton. Such was not the case, however. Indeed, on October 26, 1852, Hamilton gave an unexpected response:

> *I am not likely to attempt your "quaternion" of colours very soon.*

Hamilton's response did nothing however to diminish De Morgan's interest in the Four Color Problem.

Since De Morgan's letter to Hamilton did not mention Frederick Guthrie by name, there may be reason to question whether Frederick was in fact the student to whom De Morgan was referring and that it was Frederick's older brother Francis who was the originator of the Four Color Problem. It was during 1880 (only six years before Frederick died) that Frederick wrote:

> *Some thirty years ago, when I was attending Professor De Morgan's class, my brother, Francis Guthrie, who had recently ceased to attend them (and who is now professor of mathematics at the South African University, Cape Town), showed me the fact that the greatest necessary number of colours to be used in colouring a map so as to avoid identity colour in lineally contiguous districts is four. I should not be justified, after this lapse of time, in trying to give his proof, but the critical diagram was as in the margin.*

> *With my brother's permission I submitted the theorem to Professor De Morgan, who expressed himself very pleased with it; accepted it as new; and, as I am informed by those who subsequently attended his classes, was in the habit of acknowledging where he had got his information.*
>
> *If I remember rightly, the proof which my brother gave did not seem altogether satisfactory to himself; but I must refer to him those interested in the subject. ....*

The first statement in print of the Four Color Problem evidently occurred in an anonymous review written in the April 14, 1860, issue of the literary journal *Athenaeum*. Although the author of the review was not identified, De Morgan was quite clearly the writer. This review led to the Four Color Problem becoming known in the United States.

De Morgan had made little progress towards a solution of the Four Color Problem at the time of his death in 1871 and overall interest in this problem

had faded. During a meeting of the London Mathematical Society on June 13, 1878, Arthur Cayley (1821–1895) raised a question about the Four Color Problem that brought renewed attention to the problem:

> *Has a solution been given of the statement that in colouring a map of a country, divided into counties, only four distinct colours are required, so that no two adjacent counties should be painted in the same colour?*

This question appeared in the Proceedings of the Society's meeting. In the April 1879 issue of the *Proceedings of the Royal Geographical Society*, Cayley reported:

> *I have not succeeded in obtaining a general proof; and it is worth while to explain wherein the difficulty consists.*

Cayley observed that if a map with a certain number of regions has been colored with four colors and a new map is obtained by adding a new region, then there is no guarantee that the new map can be colored with four colors – without first recoloring the original map. This showed that any attempted proof of the Four Color Conjecture by mathematical induction would not be straightforward.

Alfred Bray Kempe

Among those who studied under Arthur Cayley was Alfred Bray Kempe (1849–1922). Despite his great enthusiasm for mathematics, Kempe took up a career in the legal profession. Kempe was present at the meeting of the London Mathematical Society in which Cayley had inquired about the status of the Four Color Problem. Kempe worked on the problem and obtained a "solution" in 1879. Indeed, on July 17, 1879, a statement of Kempe's accomplishment appeared in the British journal *Nature*, with the complete proof published in Volume 2 of the *American Journal of Mathematics* (edited by James Joseph Sylvester, a good friend of Cayley's).

Kempe's approach for solving the Four Color Problem involved locating a region R in a map M such that R is surrounded by five or fewer neighboring regions. The existence of such a region R is a consequence of Corollary 6.6. Once R was found, Kempe then showed that for every coloring of M (minus the region R) with four colors, there is a coloring of the entire map M with four colors. Such an argument would show that M could not be a minimum counterexample.

For example, no minimum counterexample M could possibly contain a region R surrounded by three regions R_1, R_2 and R_3 as shown in Figure 9.2(a). In this case, we could shrink the region R to a point, producing a new map M' with one less region. The map M' can then be colored with four colors, only

three of which are used to color R_1, R_2 and R_3 as in Figure 9.2(b). Returning to the original map M, we see that there is now an available color for R as shown in Figure 9.2(c), implying that M could be colored with four colors after all, thereby producing a contradiction. Certainly, if the map M contains a region surrounded by fewer than three regions, a contradiction can be obtained in a similar manner.

(a) in M (b) in M' (c) in M

Figure 9.2: A region surrounded by three regions in a map

Kempe Chains

Suppose, however, that the map M contained no region surrounded by three or fewer regions but did contain a region R surrounded by four regions, say R_1, R_2, R_3 and R_4, as shown in Figure 9.3(a). If, once again, we shrink the region R to a point, producing a map M' with one less region, then we know that M' can be colored with four colors. If two or three colors are used to color R_1, R_2, R_3 and R_4, then we can return to M and there is a color available for R. However, this technique does not work if the regions R_1, R_2, R_3 and R_4 are colored with four distinct colors, as shown in Figure 9.3(b).

(a) in M (b) in M'

Figure 9.3: A region surrounded by four regions in a map

What we can do in this case, however, is to determine whether the map M' has a chain of regions, beginning at R_1 and ending at R_3, all of which

are colored red or green. If no such chain exists, then the two colors of every red-green chain of regions beginning at R_1 can be interchanged. We can then return to the map M, where the color red is now available for R. That is, the map M can be colored with four colors, producing a contradiction. But what if the map contains a red-green chain of regions beginning at R_1 and ending at R_3? (See Figure 9.4, where r, b, g, y denote the colors red, blue, green, yellow.) Then interchanging the colors red and green offers no benefit to us. However, in this case, there can be no blue-yellow chain of regions, beginning at R_2 and ending at R_4. Then the colors of every blue-yellow chain of regions beginning at R_2 can be interchanged. Returning to M, we see that the color blue is now available for R, which once again says that M can be colored with four colors and produces a contradiction.

Figure 9.4: A red-green chain of regions from R_1 to R_3

So if R were surrounded by four or fewer neighboring regions, Kempe's proof involved the use of chains of regions whose colors alternate between two colors and then interchanging these colors, if appropriate, to arrive at a coloring of the regions of M (minus R) with four colors so that the neighboring regions of R used at most three of these colors and thereby leaving a color available for R. In fact, these chains of regions became known as **Kempe chains**.

There was one case, however, that still needed to be resolved, namely the case where no region in the map is surrounded by four or fewer neighboring regions. As we noted, the map must then contain some region R surrounded by exactly five neighboring regions. At least three of the four colors must be used to color the five neighboring regions of R. If only three colors are used to color these five regions, then a color is available for R. Hence we are left with the single situation in which all four colors are used to color the five neighboring regions surrounding R (see Figure 9.5), where once again r, b, g, y indicate the colors red, blue, green, yellow. In this case, two neighboring regions of R are colored blue.

Figure 9.5: The final case in Kempe's solution of the Four Color Problem

Kempe proceeded as follows with this final case. Among the regions adjacent to R, only the region R_1 is colored yellow. Consider all the regions of the map M that are colored either yellow or red and that, beginning at R_1, can be reached by an alternating sequence of neighboring yellow and red regions, that is, by a yellow-red Kempe chain. If the region R_3 (which is the neighboring region of R that is colored red) cannot be reached by a yellow-red Kempe chain, then the colors yellow and red can be interchanged for all regions in M that can be reached by a yellow-red Kempe chain beginning at R_1. This results in a coloring of all regions in M (except R) in which neighboring regions are colored differently and such that each neighboring region of R is colored red, blue or green. We can then color R yellow to arrive at a 4-coloring of the entire map M. From this, we may assume that the region R_3 can be reached by a yellow-red Kempe chain beginning at R_1. (See Figure 9.6.)

Figure 9.6: A yellow-red Kempe chain in the map M

Consider the region R_5, which is colored green. We consider all regions of M

colored green or red and that, beginning at R_5, can be reached by a green-red Kempe chain. If the region R_3 cannot be reached by a green-red Kempe chain that begins at R_5, then the colors green and red can be interchanged for all regions in M that can be reached by a green-red Kempe chain beginning at R_5. Upon doing this, a 4-coloring of all regions in M (except R) is obtained, in which each neighboring region of R is colored red, blue, or yellow. We can then color R green to produce a 4-coloring of the entire map M. We may therefore assume that R_3 can be reached by a green-red Kempe chain that begins at R_5. (See Figure 9.7.)

Figure 9.7: Yellow-red and green-red Kempe chains in the map M

Because there is a ring of regions consisting of R and a green-red Kempe chain, there cannot be a blue-yellow Kempe chain in M beginning at R_4 and ending at R_1. In addition, because there is a ring of regions consisting of R and a yellow-red Kempe chain, there is no blue-green Kempe chain in M beginning at R_2 and ending at R_5. Hence we interchange the colors blue and yellow for all regions in M that can be reached by a blue-yellow Kempe chain beginning at R_4 and interchange the colors blue and green for all regions in M that can be reached by a blue-green Kempe chain beginning at R_2. Once these two color interchanges have been performed, each of the five neighboring regions of R is colored red, yellow or green. Then R can be colored blue and a 4-coloring of the map M has been obtained, completing the proof.

As it turned out, this proof given by Kempe contained a fatal flaw, but one that would go unnoticed for a decade. Despite the fact that Kempe's attempted proof of the Four Color Problem was erroneous, he made a number of interesting observations in his article. He noticed that if a piece of tracing paper was placed over a map and a point was marked on the tracing paper over each region of the map and two points were joined by a line segment whenever the corresponding

regions had a common boundary, then a diagram of a "linkage" was produced. Furthermore, the problem of determining whether the regions of the map can be colored with four colors so that neighboring regions are colored differently is the same problem as determining whether the points in the linkage can be colored with four colors so that every two points joined by a line segment are colored differently. (See Figure 9.8.)

Figure 9.8: A map and corresponding plane graph

In 1878 Sylvester referred to a linkage as a graph and it is this terminology that became standard. Later it became commonplace to refer to the points and lines of a linkage as the vertices and edges of the graph. Since the graphs constructed from maps in this manner (referred to as the **planar dual** or the **dual graph** of the map) can themselves be drawn in the plane without two edges (line segments) intersecting, the graphs produced were plane graphs. In terms of graphs, the Four Color Conjecture can then be restated.

The Four Color Conjecture *Every planar graph is* 4*-colorable.*

In 1951 Gabriel Dirac (1925–1984) wrote:

> *The colouring of abstract graphs is a generalization of the colouring of maps, and the study of the colouring of abstract graphs . . . opens a new chapter in the combinatorial part of mathematics.*

Kempe's proof of the theorem, which then had become known as the Four Color Theorem, was accepted both within the United States and England. In particular, Arthur Cayley had accepted Kempe's argument as a valid proof. This led to Kempe to be elected a Fellow of the Royal Society in 1881.

Mistakes concerning the Four Color Problem were not limited to mathematical errors however. Prior to establishing Francis Guthrie as the true and sole originator of the Four Color Problem, it was often stated in print that cartographers were aware that the regions of every map could be colored with four or less colors so that adjacent regions are colored differently. The well-known mathematical historian Kenneth O. May investigated this claim and found no justification to it. He conducted a study of atlases in the Library of Congress

and found no evidence of attempts to minimize the number of colors used in maps. Most maps used more than four colors and even when four colors were used, often fewer colors could have been used. There was never a mention of a "four color theorem."

Percy John Heawood

The next important figure in the history of the Four Color Problem was Percy John Heawood (1861–1955), who spent the period 1887–1939 as a lecturer, professor and vice-chancellor at Durham College in England. When Heawood was a student at Oxford University in 1880, one of his teachers was Professor Henry Smith, who spoke often of the Four Color Problem. Heawood read Kempe's paper and it was he who discovered the serious error in the proof. In 1889 Heawood [115] wrote a paper of his own, published in 1890, in which he presented the map shown in Figure 9.9.

Figure 9.9: The Heawood map: A counterexample to Kempe's proof

In the **Heawood map**, two of the five neighboring regions surrounding the uncolored region R are colored red; while for each of the colors blue, yellow and green, there is exactly one neighboring region of R with that color. According to Kempe's argument, since blue is the color of the region that shares a boundary with R as well as with the two neighboring regions of R colored red, we are concerned with whether this map contains a blue-yellow Kempe chain between two neighboring regions of R as well as a blue-green Kempe chain between

two neighboring regions of R. It does. These Kempe chains are shown in Figures 9.10(a) and 9.10(b), respectively.

Figure 9.10: Blue-yellow and blue-green Kempe chains in the Heawood map

Because the Heawood map contains these two Kempe chains, it follows by Kempe's proof that this map does not contain a red-yellow Kempe chain between the two neighboring regions of R that are colored red and yellow *and* does not contain a red-green Kempe chain between the two neighboring regions of R that are colored red and green. This is, in fact, the case. Figure 9.11(a) indicates all regions that can be reached by a red-yellow Kempe chain beginning at the red region that borders R and that is not adjacent to the yellow region bordering R. Furthermore, Figure 9.11(b) indicates all regions that can be reached by a red-green Kempe chain beginning at the red region that borders R and that is not adjacent to the green region bordering R.

In the final step of Kempe's proof, the two colors within each Kempe chain are interchanged, resulting in a coloring of the Heawood map with four colors. This double interchange of colors is shown in Figure 9.11(c). However, as Figure 9.11(c) shows, this results in neighboring regions with the same color, namely red. Consequently, Kempe's proof is unsuccessful when applied to the Heawood map, as colored in Figure 9.9. What Heawood had shown was that Kempe's method of proof was incorrect. That is, Heawood had discovered a counterexample to Kempe's technique, not to the Four Color Conjecture itself. Indeed, it is not particularly difficult to give a 4-coloring of the regions of the Heawood map so that every two neighboring regions are colored differently.

Figure 9.11: Steps in illustrating Kempe's technique

Unavoidable Sets of Reducible Configurations

At the beginning of the 20th century, the Four Color Problem was still unsolved. Although possibly seen initially as a rather frivolous problem, not worthy of a serious mathematician's attention, it would become clear that the Four Color Problem was a very challenging mathematics problem. Many mathematicians, using a variety of approaches, would attack this problem during the 20th century. It was known that if the Four Color Conjecture could be verified for **cubic maps** (maps where three boundary lines meet at each meeting point), then the Four Color Conjecture would be true for all maps. In fact, every map M that has no region completely surrounded by another region can be converted into a cubic map M' by drawing a circle about each meeting point in M and creating new meeting points and one new region (see Figure 9.12). If the map M' can be colored with four colors, then so can M.

Every cubic map must contain a region surrounded by two, three, four or five neighboring regions. These four kinds of **configurations** (arrangements of regions) were called **unavoidable** because every cubic map had to contain at least one of them. Thus the arrangements of regions shown in Figure 9.13 make up an unavoidable set of configurations.

Figure 9.12: Converting a map into a cubic map

Figure 9.13: An unavoidable set of configurations in a cubic map

A **reducible configuration** is any configuration of regions that cannot occur in a minimum counterexample of the Four Color Conjecture. Many mathematicians who attempted to solve the Four Color Problem attempted to do so by trying to find an unavoidable set S of reducible configurations. Since S is unavoidable, this means that every cubic map must contain at least one configuration in S. Because each configuration in S is reducible, this means that it cannot occur in a minimum counterexample. Essentially then, a proof of the Four Color Conjecture by this approach would be a proof by minimum counterexample resulting in a number of cases (one case for each configuration in the unavoidable set S) where each case then leads to a contradiction (that is, each configuration is shown to be reducible).

Since the only configuration in the unavoidable set shown in Figure 9.13 that could not be shown to be reducible was the pentagon, this suggested searching for configurations different from the pentagon that must also be part of an unavoidable set with the hope that these more complicated configurations could somehow be shown to be reducible. A region surrounded by k neighboring regions is called a **k-gon**. In 1903 Paul Wernicke proved that every cubic map containing no k-gon where $k < 5$ must either contain two adjacent pentagons or two adjacent regions, one of which is a pentagon and the other a hexagon (see Theorem 6.13). That is, the troublesome case of a cubic map containing a pentagon could be eliminated and replaced by two different cases.

Finding new, large unavoidable sets of configurations was not a problem. Finding reducible configurations was. In 1913 the distinguished mathematician George David Birkhoff (1884–1944) published a paper called "The reducibility

of maps" in which he considered rings of regions for which there were regions interior to as well as exterior to the ring. Since the map was a minimum counterexample, the ring together with the interior regions and the ring together with the exterior regions could both be colored with four colors. If two 4-colorings could be chosen so that they match along the ring, then there is a 4-coloring of the entire map. Since this can always be done if the ring consists of three regions, rings of three regions can never appear in a minimum counterexample. Birkhoff proved that rings of four regions also cannot appear in a minimum counterexample. In addition, he was successful in proving that rings of five regions cannot appear in a minimum counterexample either – unless the interior of the ring consisted of a single region. This generalized Kempe's approach. While Kempe's approach to solving the Four Color Problem involved the removal of a single region from a map, Birkhoff's method allowed the removal of regions inside or outside some ring of regions. For example, a configuration that Birkhoff was able to prove was reducible consisted of a ring of six pentagons enclosing four pentagons. This became known as the **Birkhoff diamond** (see Figure 9.14).

Figure 9.14: The Birkhoff diamond (a reducible configuration)

Philip Franklin wrote his doctoral dissertation in 1921 titled *The Four Color Problem* under the direction of Oswald Veblen. Veblen later became the first professor at the Institute for Advanced Study at Princeton University. He was well known for his work in geometry and topology (called analysis situs at the time) as well as for his lucid writing. In his thesis, Franklin showed that if a cubic map does not contain a k-gon, where $k < 5$, then it must contain a pentagon adjacent to two other regions, each of which is a pentagon or a hexagon (see Theorem 6.14). This resulted in a larger unavoidable set of configurations.

The German mathematician Heinrich Heesch was an assistant to Hermann Weyl, a gifted mathematician who was a colleague of Albert Einstein and a student of the famous mathematician David Hilbert. Through one of Heesch's friends, Ernst Witt, Heesch was introduced to and became interested in the Four Color Problem. As Heesch studied this famous problem, he had become increasingly convinced that the problem could be solved by finding an unavoidable set of reducible configurations, even though such a set may very well be extremely large. He began lecturing on his ideas in the 1940s at the Universities of Hamburg and Kiel. A 1948 lecture at the University of Kiel was attended by the student Wolfgang Haken (born in 1928), who recalls Heesch saying that an unavoidable set of reducible configurations may contain as many as ten thou-

sand members. Heesch discovered a method for creating many unavoidable sets of configurations. Since the method had an electrical flavor to it, electrical terms were chosen for the resulting terminology.

Heesch's plan consisted of establishing rules, called discharging rules (see Chapter 6), for moving a positive charge from one vertex to others in a manner that did not change the sum of the charges, which was 12 (see Theorem 6.11). The goal was to use these rules to create an unavoidable set of configurations by showing that if a minimum counterexample to the Four Color Conjecture contained none of these configurations, then the sum of the charges of its vertices was not 12.

Since Heesch's discharging method was successful in finding unavoidable sets, much of the early work in the 20th century on the Four Color Problem was focused on showing that certain configurations were reducible. Often showing that even one configuration was reducible became a monumental task. In the 1960s Heesch had streamlined Birkhoff's approach of establishing the reducibility of certain configurations. One of these techniques was sufficiently algorithmic in nature to allow this technique to be executed on a computer.

Even though Haken had attended Heesch's talk at the University of Kiel on the Four Color Problem, he had become interested in a long-standing unsolved problem in knot theory. He attempted to solve this problem and was successful. Haken's solution of the problem in knot theory led to his being invited to the University of Illinois as a visiting professor. After leaving the University of Illinois to spend some time at the Institute for Advanced Study in Princeton, Haken then returned to the University of Illinois, this time to take on a permanent position.

Heesch remained interested in the Four Color Problem and felt that a computer-aided solution of it was possible. As time went on, however, Haken was losing faith in a computer-aided solution of the Four Color Problem despite the fact that he had a doctoral student at the University of Illinois whose research was related to the problem. One of the members of this student's thesis committee was Kenneth Appel. After completing his undergraduate degree at Queens College with a special interest in actuarial mathematics, Appel worked at an insurance company and shortly afterwards was drafted and began a period of military service. He then went to the University of Michigan for his graduate studies in mathematics. During the spring of 1956, the University of Michigan acquired an IBM 650 and Appel audited the very first programming course offered at the university.

Kenneth Appel's area of research was mathematical logic. In fact, Appel asked Haken to give a talk at the logic seminar in the Department of Mathematics so he could better understand the thesis. In his talk, Haken included a discussion of the computer difficulties that had been encountered in his approach to solve the Four Color Problem and explained that he was finished with the problem for the present. With his knowledge of computer programming, however, Appel convinced Haken that the two of them should "take a shot at it."

Solution of Four Color Problem

Together, Appel and Haken took a somewhat different approach. They devised an algorithm that tested for "reduction obstacles." The work of Appel and Haken was greatly aided by Appel's doctoral student John Koch who wrote an efficient program that tested certain kinds of configurations for reducibility. Much of Appel and Haken's work involved refining Heesch's method for finding an unavoidable set of reducible configurations.

The partnership in the developing proof concerned the active involvement of a team of three, namely Appel, Haken and a computer. As their work progressed, Appel and Haken needed ever-increasing amounts of time on a computer. Because of Appel's political skills, he was able to get time on the IBM 370-168 located in the university's administration building. Eventually, everything paid off. In June of 1976, Appel and Haken had constructed an unavoidable set of 1936 reducible configurations, which was later reduced to 1482. The proof was finally announced at the 1976 Summer Meeting of the American Mathematical Society and the Mathematical Association of America at the University of Toronto. Shortly afterwards, the University of Illinois employed the postmark

FOUR COLORS SUFFICE

on its outgoing mail.

In 1977 Frank Harary, editor-in-chief of the newly founded *Journal of Graph Theory*, asked William Tutte if he would contribute something to the first volume of the journal in connection with this announcement. Tutte responded that although he wasn't able to comply with this request, he just happened to be visited by Blanche Descartes, who wrote, under the title "Some Recent Progress in Combinatorics":

> *Wolfgang Haken*
> *Smote the Kraken*
> *One! Two! Three! Four!*
> *Quoth he: "The monster is no more".*

In the poem, Descartes likened solving the Four Color Problem to slaying the legendary sea monster known as a kraken.

A second proof of the Four Color Theorem, using the same overall approach, was announced and described by Frank Allaire of Lakehead University, Canada in 1977, although the complete details were never published. Due in part to the history of incorrect proofs of the Four Color Conjecture and the controversy surrounding a proof that made use of computers so substantially, in 1996 Neil Robertson, Daniel P. Sanders, Paul Seymour and Robin Thomas constructed their own proof of the Four Color Theorem, but this too was heavily computer-aided.

9.2 Colorings of Planar Graphs

While Heawood [115] discovered an error in Kempe's proposed solution of the Four Color Problem and neither he nor Kempe was able to correct the error, Heawood was able to use Kempe's technique to prove that every map could be colored with five or fewer colors. This led to the following theorem.

Theorem 9.1 (**The Five Color Theorem**) *Every planar graph is* 5*-colorable.*

Proof. We proceed by induction on the order n of the graph. Clearly, the result is true if $1 \leq n \leq 5$. Assume that every planar graph of order $n-1$ is 5-colorable, where $n \geq 6$, and let G be planar graph of order n. We show that G is 5-colorable.

By Corollary 6.6, G contains a vertex v with $\deg v \leq 5$. Since $G - v$ is a planar graph of order $n-1$, it follows by the induction hypothesis that $G-v$ is 5-colorable. Let there be given a 5-coloring of $G-v$, where the colors used are denoted by 1, 2, 3, 4, 5. If one of these colors is not used to color the neighbors of v, then this color can be assigned to v, producing a 5-coloring of G. Hence we may assume that $\deg v = 5$ and that all five colors are used to color the neighbors of v.

Let there be a planar embedding of G and suppose that v_1, v_2, v_3, v_4, v_5 are the neighbors of v arranged cyclically about v. We may assume that v_i has been assigned the color i for $1 \leq i \leq 5$. Let H be the subgraph of $G-v$ induced by the set of vertices colored 1 or 3. Thus $v_1, v_3 \in V(H)$. If v_1 and v_3 should belong to different components of H, then by interchanging the colors of the vertices belonging to the component H_1 of H containing v_1, a 5-coloring of G can be produced by assigning the color 1 to v.

Suppose then that v_1 and v_3 belong to the same component of H. This implies that $G-v$ contains a $v_1 - v_3$ path P, every vertex of of which is colored 1 or 3. The path P and the path (v_1, v, v_3) in G produce a cycle, enclosing either v_2 or both v_4 and v_5. In particular, this implies that there is no $v_2 - v_4$ path in $G - v$, every vertex of which is colored 2 or 4.

Let F be the subgraph of $G-v$ induced by the set of vertices colored 2 or 4, and let F_2 be the component of F containing v_2. Necessarily, $v_4 \notin V(F_2)$. By interchanging the colors of the vertices of F_2, a 5-coloring of G can be produced by assigning the color 2 to v. ■

Thus after the publication of Heawood's famous paper in 1890, it was known that every planar graph could be colored with five or fewer colors but it was not known if any planar graph actually required five colors.

As we saw, some 86 years after the Five Color Theorem was proved, Kenneth Appel and Wolfgang Haken [2] proved the Four Color Theorem.

Theorem 9.2 (**The Four Color Theorem**) *Every planar graph is* 4*-colorable.*

Of course, every maximal planar graph is also 4-colorable. Since every maximal planar graph of order 3 or more contains triangles, the chromatic

number of each such graph is at least 3. That is, if G is a maximal planar graph of order 3 or more, then $\chi(G) = 3$ or $\chi(G) = 4$. It is possible for a maximal planar graph of order 4 or more to have chromatic number 3. Both maximal planar graphs in Figure 9.15 not only have chromatic number 3, they also have something else in common. The following was first observed by Heawood [116] in 1898.

Figure 9.15: Two 3-colorable maximal planar graphs

Theorem 9.3 *A maximal planar graph G of order 3 or more has chromatic number 3 if and only if G is Eulerian.*

Proof. Let there be given a planar embedding of G. Suppose first that G is not Eulerian. Then G contains a vertex v of odd degree $k \geq 3$. Since G is maximal planar, G contains an odd cycle $C = (v_1, v_2, \ldots, v_k, v_1)$ consisting of the neighbors of v in G. Because v is adjacent to every vertex of C, it follows that $\chi(G) = 4$.

We verify the converse by induction on the order of maximal planar Eulerian graphs. If the order of G is 3, then $G = K_3$ and $\chi(G) = 3$. Assume that every maximal planar Eulerian graph of order k has chromatic number 3 for an integer $k \geq 3$ and let G be a maximal planar Eulerian graph of order $k + 1$. Let there be given a planar embedding of G and let uw be an edge of G. Then uw is on the boundary of two (triangular) regions of G. Let x be the third vertex on the boundary of one of these regions and y the third vertex on the boundary of the other region. Suppose that

$$N(x) = \{u = x_1, x_2, \ldots, x_s = w\} \text{ and } N(y) = \{u = y_1, y_2, \ldots, y_t = w\},$$

where s and t are even. Then

$$C = (x_1, x_2, \ldots, x_s, x_1) \text{ and } C' = (y_1, y_2, \ldots, y_t, y_1)$$

are even cycles. Let G' be the graph obtained from G by (1) deleting x, y and uw from G and (2) adding a new vertex z and joining z to every vertex of C and C'. Thus z is adjacent to every vertex of the $(s + t - 2)$-cycle

$$C'' = (u = x_1, x_2, \ldots, x_s = y_t, y_{t-1}, \cdots, y_1 = u).$$

The graph G' is thus a maximal planar Eulerian graph of order k. By the induction hypothesis, $\chi(G') = 3$. Let a 3-coloring of G' be given using the colors 1, 2 and 3. Since z is adjacent to every vertex of C and C', we may assume that z is colored 1 and that the vertices of C and C' alternate in the colors 2 and 3. Necessarily, one of u and w is colored 2 in G' and the other is colored 3. We may then remove z from G', add x and y back in together with uw and all edges of G incident with x or y. Assigning x and y the color 1 produces a 3-coloring of G. ■

Thomassen's Theorem

In 1976 Vadim Vizing [230] and in 1980 Paul Erdős, Arthur L. Rubin and Herbert Taylor [72] conjectured that the maximum list chromatic number of a planar graph is 5. In 1993, Margit Voigt [231] gave an example of a planar graph of order 238 that is not 4-choosable and in 1994 Carsten Thomassen [216] completed the verification of this conjecture by showing that every planar graph is 5-choosable. To show that every planar graph is 5-choosable, it suffices to verify this result for maximal planar graphs. In fact, it suffices to verify this result for a slightly more general class of graphs.

Recall that a planar graph G is nearly maximal planar if there exists a planar embedding of G such that the boundary of every region is a cycle, at most one of which is not a triangle. If G is a nearly maximal planar graph, then we may assume that there is a planar embedding of G such that the boundary of every interior region is a triangle, while the boundary of the exterior region is a cycle of length 3 or more. Therefore, every maximal planar graph is nearly maximal planar.

Theorem 9.4 (**Thomassen's Theorem**) *Every planar graph is* 5*-choosable.*

Proof. It suffices to verify the theorem for nearly maximal planar graphs. In fact, we verify the following somewhat stronger statement by induction on the order of nearly maximal planar graphs:

> *Let G be a nearly maximal planar graph of order $n \geq 3$ such that the boundary of its exterior region is a cycle C (of length 3 or more) and such that $\mathfrak{L} = \{L(v) : v \in V(G)\}$ is a collection of prescribed color lists for G with $|L(v)| \geq 3$ for each $v \in V(C)$ and $|L(v)| \geq 5$ for each $v \in V(G) - V(C)$. If x and y are any two consecutive vertices on C with $a \in L(x)$ and $b \in L(y)$ where $a \neq b$, then there exists an $\mathfrak{L}$-list-coloring of G with x and y colored a and b, respectively.*

The statement is certainly true for $n = 3$. Assume for an integer $n \geq 4$ that the statement is true for all nearly maximal planar graphs of order less than n satisfying the conditions in the statement and let G be a nearly maximal planar graph of order n the boundary of whose exterior region is the cycle C and such that

$$\mathfrak{L} = \{L(v) : \ v \in V(G)\}$$

is a collection of color lists for G for which $|L(v)| \geq 3$ for each $v \in V(C)$ and $|L(v)| \geq 5$ for each $v \in V(G) - V(C)$. Let x and y be any two consecutive vertices on C and suppose that $a \in L(x)$ and $b \in L(y)$ where $a \neq b$. We show that there exists an $\mathfrak{L}$-list-coloring c of G in which x and y are colored a and b, respectively. We consider two cases, according to whether C has a chord.

Case 1. *The cycle C has a chord uw.* The cycle C contains two $u-w$ paths P' and P'', exactly one of which, say P', contains both x and y. Let C' be the cycle determined by P' and uw and let G' be the nearly maximal planar subgraph of G induced by those vertices lying on or interior to C'. Let

$$\mathfrak{L}' = \{L(v) : \ v \in V(G')\}.$$

By the induction hypothesis, there is an $\mathfrak{L}'$-list coloring c' of G' in which x and y are colored a and b, respectively. Suppose that $c'(u) = a'$ and $c'(w) = b'$.

Let C'' be the cycle determined by P'' and uw and let G'' be the nearly maximal planar subgraph of G induced by those vertices lying on or interior to C''. Furthermore, let

$$\mathfrak{L}'' = \{L(v) : \ v \in V(G'')\}.$$

Again, by the induction hypothesis, there is an $\mathfrak{L}''$-list coloring c'' of G'' such that $c''(u) = c'(u) = a'$ and $c''(w) = c'(w) = b'$. Now the coloring c of G defined by

$$c(v) = \begin{cases} c'(v) & \text{if } v \in V(G') \\ c''(v) & \text{if } v \in V(G'') \end{cases}$$

is an $\mathfrak{L}$-list-coloring of G.

Case 2. *The cycle C has no chord.* Let v_0 be the vertex on C that is adjacent to x such that $v_0 \neq y$ and let

$$N(v_0) = \{x, v_1, v_2, \ldots, v_k, z\},$$

where z is on C. Since G is nearly maximal planar, we may assume that $xv_1, v_kz \in E(G)$ and $v_iv_{i+1} \in E(G)$ for $i = 1, 2, \ldots, k-1$ (see Figure 9.16).

Figure 9.16: A step in the proof of Theorem 9.4

Let P be the $x - z$ path on C that does not contain v_0 and let

$$P^* = (x, v_1, v_2, \ldots, v_k, z).$$

Furthermore, let C^* be the cycle determined by P and P^*. Then $G - v_0$ is a nearly maximal planar graph of order $n-1$ in which C^* is the boundary of the exterior region. Since $|L(v_0)| \geq 3$, there are (at least) two colors a^* and b^* in $L(v_0)$ different from a. We now define a collection $\mathfrak{L}^*$ of color lists $L^*(v)$ for the vertices v of $G - v_0$ by

$$L^*(v) = L(v) \text{ if } v \neq v_i \ (1 \leq i \leq k)$$

and

$$L^*(v_i) = L(v_i) - \{a^*, b^*\} \ (1 \leq i \leq k)$$

and let

$$\mathfrak{L}^* = \{L^*(v) : \ v \in V(G - v_0)\}.$$

Hence $|L^*(v)| \geq 3$ for $v \in V(C^*)$ and $|L^*(v)| \geq 5$ for $v \in V(G^*) - V(C^*)$. By the induction hypothesis, there is an $\mathfrak{L}^*$-list coloring of $G - v_0$ with x and y colored a and b, respectively. Since at least one of the colors a^* and b^* has not been assigned to z, one of these colors is available for v_0, producing an $\mathfrak{L}$-list coloring of G. Thus G is $\mathfrak{L}$-choosable and so is 5-choosable. ■

The Mirzakhani Graph

As we mentioned, in 1993 Margit Voigt gave an example of a planar graph of order 238 that is not 4-choosable. In 1996 Maryam Mirzakhani [159] gave an example of a planar graph of order 63 that is not 4-choosable. We now describe the Mirzakhani graph and verify that it is, in fact, not 4-choosable.

First, let H be the planar graph of order 17 shown in Figure 9.17(a). For each vertex u of H, a list $L(u)$ of three or four colors is given in Figure 9.17(b), where $L(u) \subseteq \{1, 2, 3, 4\}$. In fact, if $\deg_H u = 4$, then $L(u) = \{1, 2, 3, 4\}$, while if $\deg_H v \neq 4$, then $|L(u)| = 3$. Let $\mathfrak{L} = \{L(u) : \ u \in V(H)\}$. We claim that H is not $\mathfrak{L}$-choosable.

Theorem 9.5 *The planar graph H of Figure 9.17(a) with the set $\mathfrak{L}$ of color lists in Figure 9.17(b) is not $\mathfrak{L}$-choosable.*

Proof. Assume, to the contrary, that H is $\mathfrak{L}$-choosable. Then there exists a 4-coloring c of H such that $c(u) \in L(u)$ for each vertex u in H. Since each vertex of degree 4 in H is adjacent to vertices assigned either two or three distinct colors, it follows that each vertex of degree 4 in H is adjacent to two (nonadjacent) vertices assigned the same color. We claim that $c(x) = 1$ or $c(w) = 2$. Suppose that $c(x) \neq 1$ and $c(w) \neq 2$. Then there are two possibilities. Suppose first that $c(x) = 3$ and $c(w) = 4$. Then either $c(s_1) = 3$ or $c(s_2) = 4$. This is impossible, however, since $3 \notin L(s_1)$ and $4 \notin L(s_2)$. Next, suppose that $c(x) = 4$ and

Figure 9.17: A planar graph of order 17

$c(w) = 3$. Then either $c(v) = 4$ or $c(y) = 3$. This is impossible as well since $4 \notin L(v)$ and $3 \notin L(y)$. Hence, as claimed, $c(x) = 1$ or $c(w) = 2$. We consider these two cases.

Case 1. $c(x) = 1$. Since none of the vertices t_1, t_2 and y can be assigned the color 1, it follows that $c(t_1) = c(y) = 2$. Since none of the vertices u_1, u_2 and v can be assigned the color 2, it follows that $c(u_1) = c(v) = 3$. Therefore, none of the vertices r_1, r_2 and w can be assigned the color 3. Thus $c(r_1) = c(w) = 4$. Hence $c(x) = 1$, $c(y) = 2$, $c(v) = 3$ and $c(w) = 4$, which is impossible.

Case 2. $c(w) = 2$. Proceeding as in Case 1, we first see that $c(r_2) = c(v) = 1$. From this, it follows that $c(u_2) = c(y) = 4$. Next, we find that $c(t_2) = c(x) = 3$. Hence $c(v) = 1$, $c(w) = 2$, $c(x) = 3$ and $c(y) = 4$, again an impossibility.

Therefore, as claimed, the graph H is not $\mathfrak{L}$-choosable for the set $\mathfrak{L}$ of lists described in Figure 9.17(b). ■

Continuing with the description of the Mirzakhani graph, let H_1, H_2, H_3 and H_4 be four copies of the graph H in Figure 9.17. For $i = 1, 2, 3, 4$, the color i in the color list of every vertex of H_i is replaced by 5 and the color i is then added to the color list of each vertex not having degree 4. The graphs H_i $(i = 1, 2, 3, 4)$ and the color lists of their vertices are shown in Figure 9.18.

The **Mirzakhani graph** G (a planar graph of order 63) is now constructed from the graphs H_i $(i = 1, 2, 3, 4)$ of Figure 9.18 by identifying the two vertices labeled x_i and the two vertices labeled y_i for $i = 2, 3, 4$ and adding a new vertex p with $L'(p) = \{1, 2, 3, 4\}$ and joining p to each vertex of each copy H_i of H whose degree is not 4. The Mirzakhani graph is shown in Figure 9.19 along with the resulting color lists for each vertex. Let $\mathfrak{L}' = \{L'(u) : u \in V(G)\}$. We show that G is not $\mathfrak{L}'$-choosable.

Theorem 9.6 *The Mirzakhani graph is not* 4*-choosable.*

Figure 9.18: The graphs H_i $(i = 1, 2, 3, 4)$

Proof. Let $L'(u)$ be the color list for each vertex u in G shown in Figure 9.19 and let $\mathfrak{L}' = \{L'(u) : u \in V(G)\}$. We claim that G is not $\mathfrak{L}'$-choosable. Suppose, to the contrary, that G is $\mathfrak{L}'$-choosable. Then there is a coloring c' such that $c'(u) \in L'(u)$ for each $u \in V(G)$. Since the graph H of Figure 9.17(a) is not $\mathfrak{L}$-choosable for the set $\mathfrak{L}$ of lists in Figure 9.17(b), the only way for G to be $\mathfrak{L}'$-choosable is that $c'(v_i) = i$ for some $v_i \in V(H_i)$ for $i = 1, 2, 3, 4$. However then, regardless of the value of $c'(p)$, the vertex p is adjacent to a vertex in G having the same color as p, producing a contradiction. Thus, as claimed, G is not $\mathfrak{L}'$-choosable. Because $|L'(u)| = 4$ for each $u \in V(G)$, it follows that G is not 4-choosable. ■

Since the Mirzakhani graph has chromatic number 3, it follows that a 3-colorable planar graph need not be 4-choosable.

Figure 9.19: The Mirzakhani graph:
A non-4-choosable planar graph of order 63

Exercises for Section 9.2

1. Show that the countries in a map of South America (see Figure 9.1) cannot be colored with three or fewer colors so that every two neighboring countries are colored differently.

2. It was once thought that the regions of every map can be colored with four or fewer colors because no map contains five mutually adjacent regions. Show that there exist maps that do not contain four mutually adjacent regions. Does this mean that every such map can be colored with three or fewer colors?

3. Prove that if G is a maximal outerplanar graph of order 3 or more, then $\chi(G) = 3$.

4. Use the result in Exercise 3 to prove the Art Gallery Theorem: The walls of an art gallery form the n sides of an n-gon. It is possible to position $\lfloor n/3 \rfloor$ security guards in the art gallery so that every location in the gallery has a straight line view by one of the guards.

5. It is known that the minimum degree of every induced subgraph of an outerplanar graph is at most 2. Use this fact to prove that every outerplanar graph is 3-choosable.

6. Use the fact that every planar graph contains a vertex of degree 5 or less to prove that every planar graph is 6-choosable.

9.3 The Conjectures of Hajós and Hadwiger

We have often mentioned that $\chi(G) \geq \omega(G)$ for every graph G and that this inequality can be strict. Indeed, Theorem 8.30 states that for every integer $k \geq 3$, there is a graph G such that $\chi(G) = k$ and $\omega(G) = 2$. Even though K_k need not be present in a k-chromatic graph G, it has been thought (and conjectured over the years) that K_k may be indirectly present in G. Of course, K_k is present in a k-chromatic graph for $k = 1$ and $k = 2$. This is not true for $k = 3$, however. Indeed, every odd cycle of order at least 5 is 3-chromatic but none of these graphs contains K_3 as a subgraph. All of these do contain a subdivision of K_3, however. Since every 3-chromatic graph contains an odd cycle, it follows that if G is a graph with $\chi(G) \geq 3$, then G must contain a subdivision of K_3 as a subgraph. In 1952 Gabriel Dirac [60] showed that the corresponding result is also true for graphs having chromatic number 4.

Theorem 9.7 *Every 4-chromatic graph contains a subdivision of K_4.*

Proof. We proceed by induction on the order $n \geq 4$ of a 4-chromatic graph G. The basis step of the induction follows since K_4 is the only graph of order 4 having chromatic number 4. For an integer $n \geq 5$, assume that every graph of

order n' with $4 \leq n' < n$ having chromatic number 4 contains a subdivision of K_4. Let G be a graph of order n such that $\chi(G) \geq 4$. We show that G contains a subdivision of K_4.

Let H be a critically 4-chromatic subgraph of G. If the order of H is less than n, then it follows by the induction hypothesis that H (and G as well) contains a subdivision of K_4. Hence we may assume that H has order n. Therefore, H is 2-connected.

Suppose first that $\kappa(H) = 2$ and $S = \{u, v\}$ is a vertex-cut of H. By Corollary 8.17, $uv \notin E(H)$ and H contains an S-branch H' such that $\chi(H' + uv) = 4$. Because the order of $H' + uv$ is less than n, it follows by the induction hypothesis that $H' + uv$ contains a subdivision F of K_4. If F does not contain the edge uv, then H' and therefore G contains F. Hence we may assume that F contains the edge uv. In this case, let H'' be an S-branch of H distinct from H'. Because S is a minimum vertex-cut, both u and v are adjacent to vertices in each component of $H - S$. Hence H'' contains a $u - v$ path P. Replacing the edge uv in F by P produces a subdivision of K_4 in H.

We may now assume that H is 3-connected. Let $w \in V(H)$. Then $H - w$ is 2-connected and so contains a cycle C. Let w_1, w_2, and w_3 be three vertices belonging to C. With the aid of Corollary 2.37, H contains $w - w_i$ paths P_i $(1 \leq i \leq 3)$ such that every two of these paths have only w in common and w_i is the only vertex of P_i on C. Then C and the paths P_i $(1 \leq i \leq 3)$ produce a subdivision of K_4 in H.

Since H contains a subdivision of K_4, the graph G does as well. ■

Hajós' Conjecture

Consequently, for $2 \leq k \leq 4$, every k-chromatic graph contains a subdivision of K_k. In 1961 Győrgy Hajós [106] conjectured that this is true for every integer $k \geq 2$.

Hajós' Conjecture If G is a k-chromatic graph, where $k \geq 2$, then G contains a subdivision of K_k.

In 1979 Paul Catlin [33] constructed a family of graphs that showed that Hajós' Conjecture is false for every integer $k \geq 7$. For example, recall the graph G of Figure 8.16 of order 15 (shown again in Figure 9.20), consisting of five mutually vertex-disjoint triangles T_i $(1 \leq i \leq 5)$ where every vertex of T_i is adjacent to every vertex of T_j if either $|i - j| = 1$ or if $\{i, j\} = \{1, 5\}$. It was shown that $\omega(G) = 6$ and $\chi(G) = 8$. Since $\omega(G) = 6$, the graph G does not contain K_8 as a subgraph (or even K_7). We claim, in fact, that G does not contain a subdivision of K_8 either, for suppose that H is such a subgraph of G. Then H contains eight vertices of degree 7 and all other vertices of H have degree 2.

Figure 9.20: A counterexample to Hajós' Conjecture for $k = 8$

First, we show that no triangle T_i $(1 \leq i \leq 5)$ can contain exactly one vertex of degree 7 in H. Suppose that the triangle T_1, say, contains exactly one such vertex v. Then v is the initial vertex of seven paths to the remaining seven vertices of degree 7 in H, where every two of these paths have only v in common. Since v is adjacent only to six vertices outside of T_1, this is impossible.

Next, we show that no triangle can contain exactly two vertices of degree 7 in H. Suppose that the triangle T_1, say, contains exactly two such vertices, namely u and v. Then u is the initial vertex of six paths to six vertices outside of T_1. Necessarily, these six paths must contain all six vertices in T_2 and T_5. However, this is true of the vertex v as well. This implies that the six vertices of T_2 and T_5 are the remaining vertices of degree 7 in H. Therefore, there are two vertices u_5 and v_5 in T_5 and two vertices u_2 and v_2 in T_2 such that the interior vertices of some $u_5 - u_2$ path, $u_5 - v_2$ path, $v_5 - u_2$ path and $v_5 - v_2$ path contain only vertices of T_3 and T_4. Since these four paths must be internally disjoint and since T_3 (and T_4) contains only three vertices, this is impossible.

Therefore, no triangle T_i $(1 \leq i \leq 5)$ can contain exactly one or exactly two vertices of degree 7 in H. This, however, says that no triangle T_i $(1 \leq i \leq 5)$ can contain exactly three vertices of degree 7 in H either. Thus, as claimed, G does not contain a subdivision of K_8. Hence the graph G is a counterexample to Hajós' Conjecture for $k = 8$.

Let k be an integer such that $k \geq 9$ and consider the graph $F = G \vee K_{k-8}$. Then $\chi(F) = k$. We claim that F does not contain a subdivision of K_k, for suppose that H is such a subgraph in F. Delete all vertices of K_{k-8} that belong to H, arriving at a subgraph H' of G. This says that H' contains a subdivision of K_8, which is impossible. Hence Hajós' Conjecture is false for all integers $k \geq 8$. As we noted, Catlin showed that this conjecture is false for every integer $k \geq 7$ (see Exercise 3).

Recall that a graph G is perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G. Furthermore, for a graph G and a vertex v of G, the replication graph

$R_v(G)$ of G is that graph obtained from G by adding a new vertex v' to G and joining v' to every vertex in the closed neighborhood $N[v]$ of v. We saw in Theorem 8.38 that if G is perfect, then $R_v(G)$ is perfect for every $v \in V(G)$. Carsten Thomassen [217] showed that there is a connection between perfect graphs and Hajós' Conjecture.

Theorem 9.8 *A graph G is perfect if and only if every replication graph of G satisfies Hajós' Conjecture.*

Hadwiger's Conjecture

Recall also that a graph H is a minor of a graph G if H can be obtained from G by a succession of contractions, edge deletions and vertex deletions (in any order). Furthermore, from Theorem 6.19, if a graph G contains a subdivision of a graph H, then H is a minor of G. In particular, if a k-chromatic graph G contains a subdivision of K_k, then K_k is a minor of G. Of course, we have seen that for $k \geq 7$, a k-chromatic graph need not contain a subdivision of K_k. This does not imply, however, that a k-chromatic graph need not contain K_k as a minor. Indeed, years before Hajós' Conjecture, on December 15, 1942, Hugo Hadwiger made the following conjecture during a lecture he gave at the University of Zürich in Switzerland.

Hadwiger's Conjecture Every k-chromatic graph contains K_k as a minor.

A published version [103] of Hadwiger's lecture appeared in 1943. Hadwiger's paper not only contained this conjecture, it contained three theorems of interest:

(1) Hadwiger's Conjecture is true for $1 \leq k \leq 4$.

(2) If G is a graph with $\delta(G) \geq k-1$ where $1 \leq k \leq 4$, then G contains K_k as a minor.

(3) If G is a connected graph of order n and size m that has K_k as a minor, then $m \geq n + \binom{k}{2} - k$ (see Exercise 4).

In 1937 Klaus Wagner [234] proved that every planar graph is 4-colorable if and only if every 5-chromatic graph contains K_5 as a minor. That is, Wagner had shown the equivalence of the Four Color Conjecture and Hadwiger's Conjecture for $k = 5$ six years before Hadwiger stated his conjecture. In his 1943 paper [103], Hadwiger mentioned that the truth of his conjecture for $k = 5$ implies the truth of the Four Color Conjecture and referenced Wagner's paper but he did not refer to the equivalence. Hadwiger's Conjecture can therefore be considered as a generalization of the Four Color Conjecture.

Using the Four Color Theorem, Neil Robertson, Paul Seymour and Robin Thomas [192] verified Hadwiger's Conjecture for $k = 6$. While Hadwiger's Conjecture is true for $k \leq 6$, it is open for every integer $k > 6$.

Exercises for Section 9.3

1. Show that $K_{4,4}$ contains no subdivision of K_5.
2. Show that $K_{4,5}$ contains a subdivision of K_5.
3. Show that there exists a 7-chromatic graph that does not contain a subdivision of K_7.
4. Prove that if G is a connected graph of order n and size m that has K_k as a minor, then $m \geq n + \binom{k}{2} - k$.

9.4 Chromatic Polynomials

During the period that the Four Color Problem was unsolved, which spanned more than a century, many approaches were introduced with the hopes that they would lead to a solution of this famous problem. In 1912 George David Birkhoff [20] defined a function $P(M, \lambda)$ that gives the number of proper λ-colorings of a map M for a positive integer λ. As we will see, $P(M, \lambda)$ is a polynomial in λ for every map M and is called the chromatic polynomial of M. Consequently, if it could be verified that $P(M, 4) > 0$ for every map M, then this would have established the truth of the Four Color Conjecture.

In 1932 Hassler Whitney [239] expanded the study of chromatic polynomials from maps to graphs. While Whitney and others obtained a number of results on chromatic polynomials of graphs, this did not contribute to a proof of the Four Color Conjecture. Renewed interest in chromatic polynomials of graphs occurred in 1968, however, when Ronald C. Read [180] wrote a survey paper on chromatic polynomials.

For a graph G and a positive integer λ, the number of different proper λ-colorings of G is denoted by $P(G, \lambda)$ and is called the **chromatic polynomial** of G. Two λ-colorings c and c' of G from the same set $\{1, 2, \ldots, \lambda\}$ of λ colors are considered different if $c(v) \neq c'(v)$ for some vertex v of G. Obviously, if $\lambda < \chi(G)$, then $P(G, \lambda) = 0$. By convention, $P(G, 0) = 0$. Indeed, we have the following.

> *Let G be a graph. Then $\chi(G) = k$ if and only if k is the smallest positive integer for which $P(G, k) > 0$.*

As an example, we determine the number of ways that the vertices of the graph G of Figure 9.21 can be colored from the set $\{1, 2, 3, 4, 5\}$. The vertex v can be assigned any of these 5 colors, while w can be assigned any color other than the color assigned to v. That is, w can be assigned any of the 4 remaining colors. Both u and t can be assigned any of the 3 colors not used to color v and w. Therefore, the number $P(G, 5)$ of 5-colorings of G is $5 \cdot 4 \cdot 3^2 = 180$. More generally, $P(G, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$ for every integer λ.

There are some classes of graphs G for which $P(G, \lambda)$ can be easily computed.

Figure 9.21: A graph G with $P(G, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$

Theorem 9.9 *For every positive integer λ,*

(a) $P(K_n, \lambda) = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1) = \lambda^{(n)}$,

(b) $P(\overline{K}_n, \lambda) = \lambda^n$.

In particular, if $\lambda \geq n$ in Theorem 9.9(a), then

$$P(K_n, \lambda) = \lambda^{(n)} = \frac{\lambda!}{(\lambda - n)!}.$$

A Technique for Constructing Chromatic Polynomials

We now determine the chromatic polynomial of C_4 in Figure 9.22. There are λ choices for the color of v_1. The vertices v_2 and v_4 must be assigned colors different from that assigned to v_1. The vertices v_2 and v_4 may be assigned the same color or may be assigned different colors. If v_2 and v_4 are assigned the same color, then there are $\lambda - 1$ choices for that color. The vertex v_3 can then be assigned any color except the color assigned to v_2 and v_4. Hence the number of distinct λ-colorings of C_4 in which v_2 and v_4 are colored the same is $\lambda(\lambda-1)^2$.

Figure 9.22: Computing the chromatic polynomial of C_4

If, on the other hand, v_2 and v_4 are colored differently, then there are $\lambda - 1$ choices for v_2 and $\lambda - 2$ choices for v_4. Since v_3 can be assigned any color except the two colors assigned to v_2 and v_4, the number of λ-colorings of C_4 in

which v_2 and v_4 are colored differently is $\lambda(\lambda-1)(\lambda-2)^2$. Hence the number of distinct λ-colorings of C_4 is

$$\begin{aligned} P(C_4,\lambda) &= \lambda(\lambda-1)^2+\lambda(\lambda-1)(\lambda-2)^2 \\ &= \lambda(\lambda-1)(\lambda^2-3\lambda+3) \\ &= \lambda^4-4\lambda^3+6\lambda^2-3\lambda \\ &= (\lambda-1)^4+(\lambda-1). \end{aligned}$$

The preceding example illustrates an important observation. Suppose that u and v are nonadjacent vertices in a graph G. The number of λ-colorings of G equals the number of λ-colorings of G in which u and v are colored differently plus the number of λ-colorings of G in which u and v are colored the same. Since the number of λ-colorings of G in which u and v are colored differently is the number of λ-colorings of $G+uv$ while the number of λ-colorings of G in which u and v are colored the same is the number of λ-colorings of the graph H obtained by identifying u and v, it follows that

$$P(G,\lambda)=P(G+uv,\lambda)+P(H,\lambda).$$

This observation is summarized below:

Theorem 9.10 *Let G be a graph containing nonadjacent vertices u and v and let H be the graph obtained from G by identifying u and v. Then*

$$P(G,\lambda)=P(G+uv,\lambda)+P(H,\lambda).$$

Note that if G is a graph of order $n\geq 2$ and size $m\geq 1$, then $G+uv$ has order n and size $m+1$ while H has order $n-1$ and size at most m.

Of course, the equation stated in Theorem 9.10 can also be expressed as

$$P(G+uv,\lambda)=P(G,\lambda)-P(H,\lambda).$$

In this context, Theorem 9.10 can be rephrased in terms of an edge deletion and a contraction.

Corollary 9.11 *Let G be a graph containing adjacent vertices u and v and let F be the graph obtained from G by identifying u and v. Then*

$$P(G,\lambda)=P(G-uv,\lambda)-P(F,\lambda).$$

By systematically applying Theorem 9.10 to pairs of nonadjacent vertices in a graph G, we eventually arrive at a collection of complete graphs whose chromatic polynomials are known. We now illustrate this. Suppose that we wish to compute the chromatic polynomial of the graph G of Figure 9.23. For the nonadjacent vertices u and v of G and the graph H obtained by identifying u and v, it follows that by Theorem 9.10 that the chromatic polynomial of G is the sum of the chromatic polynomials of $G+uv$ and H.

Figure 9.23: $P(G, \lambda) = P(G + uv, \lambda) + P(H, \lambda)$

At this point it is useful to adopt a convention introduced by Alexander Zykov [249] and utilized later by Read [180]. Rather than repeatedly writing the equation that appears in the statement of Theorem 9.10, we represent the chromatic polynomial of a graph by a drawing of the graph and indicate on the drawing which pair u, v of nonadjacent vertices will be separately joined by an edge and identified. So, for the graph G of Figure 9.23, we have the "equation" also shown in Figure 9.23.

Continuing in this manner, as shown in Figure 9.24, we obtain

$$P(G, \lambda) = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda.$$

Using this approach, we see that the chromatic polynomial of every graph is the sum of chromatic polynomials of complete graphs. A consequence of this observation is the following.

Theorem 9.12 *The chromatic polynomial $P(G, \lambda)$ of a graph G is a polynomial in λ.*

Properties of Chromatic Polynomials

There are some interesting properties possessed by the chromatic polynomial of every graph. In fact, if G is a graph of order n and size m, then the chromatic polynomial $P(G, \lambda)$ of G can be expressed as

$$P(G, \lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n,$$

where $c_0 = 1$ (and so $P(G, \lambda)$ is a polynomial of degree n with leading coefficient 1), $c_1 = -m$, $c_i \geq 0$ if i is even with $0 \leq i \leq n$, and $c_i \leq 0$ if i is odd with $1 \leq i \leq n$. Since $P(G, 0) = 0$, it follows that $c_n = 0$.

The following theorem is due to Whitney [239].

$$= \lambda^{(5)} + 4\lambda^{(4)} + 3\lambda^{(3)} = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda$$

Figure 9.24: $P(G, \lambda) = P(G + uv, \lambda) + P(H, \lambda)$

Theorem 9.13 *Let G be a graph of order n and size m. Then $P(G, \lambda)$ is a polynomial of degree n with leading coefficient 1 such that the coefficient of λ^{n-1} is $-m$, and whose coefficients alternate in sign.*

Proof. We proceed by induction on m. If $m = 0$, then $G = \overline{K}_n$ and $P(G, \lambda) = \lambda^n$, as we have seen. Then $P(\overline{K}_n, \lambda) = \lambda^n$ has the desired properties.

Assume that the result holds for all graphs whose size is less than m, where $m \geq 1$. Let G be a graph of size m and let $e = uv$ be an edge of G. By Corollary 9.11,

$$P(G, \lambda) = P(G - e, \lambda) - P(F, \lambda),$$

where F is the graph obtained from G by identifying u and v. Since $G - e$ has order n and size $m - 1$, it follows by the induction hypothesis that

$$P(G - e, \lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n,$$

where $a_0 = 1$, $a_1 = -(m - 1)$, $a_i \geq 0$ if i is even with $0 \leq i \leq n$, and $a_i \leq 0$ if i is odd with $1 \leq i \leq n$. Furthermore, since F has order $n - 1$ and size m',

where $m' \leq m-1$, it follows that

$$P(F,\lambda) = b_0\lambda^{n-1} + b_1\lambda^{n-2} + b_2\lambda^{n-3} + \cdots + b_{n-2}\lambda + b_{n-1},$$

where $b_0 = 1$, $b_1 = -m'$, $b_i \geq 0$ if i is even with $0 \leq i \leq n-1$, and $b_i \leq 0$ if i is odd with $1 \leq i \leq n-1$. By Corollary 9.11,

$$\begin{aligned} P(G,\lambda) &= P(G-e,\lambda) - P(F,\lambda) \\ &= (a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n) - \\ &\quad (b_0\lambda^{n-1} + b_1\lambda^{n-2} + b_2\lambda^{n-3} + \cdots + b_{n-2}\lambda + b_{n-1}) \\ &= a_0\lambda^n + (a_1 - b_0)\lambda^{n-1} + (a_2 - b_1)\lambda^{n-2} + \cdots \\ &\qquad +(a_{n-1} - b_{n-2})\lambda + (a_n - b_{n-1}). \end{aligned}$$

Since $a_0 = 1$, $a_1 - b_0 = -(m-1) - 1 = -m$, $a_i - b_{i-1} \geq 0$ if i is even with $2 \leq i \leq n$, and $a_i - b_{i-1} \leq 0$ if i is odd with $1 \leq i \leq n$, $P(G,\lambda)$ has the desired properties and the theorem follows by mathematical induction. ■

Suppose that a graph G contains an end-vertex v whose only neighbor is u. Then, of course, $P(G-v,\lambda)$ is the number of λ-colorings of $G-v$. The vertex v can then be assigned any of the λ colors except the color assigned to u. This observation gives the following.

Theorem 9.14 *If G is a graph containing an end-vertex v, then*

$$P(G,\lambda) = (\lambda-1)P(G-v,\lambda).$$

One consequence of Theorem 9.14 is the following result, the converse of which is true as well (see Exercise 12).

Corollary 9.15 *If T is a tree of order $n \geq 1$, then*

$$P(T,\lambda) = \lambda(\lambda-1)^{n-1}.$$

Proof. We proceed by induction on n. For $n = 1$, $T = K_1$ and certainly $P(K_1,\lambda) = \lambda$. Thus the basis step of the induction is true. Suppose that $P(T',\lambda) = \lambda(\lambda-1)^{n-2}$ for every tree T' of order $n-1 \geq 1$ and let T be a tree of order n. Let v be an end-vertex of T. Thus $T-v$ is a tree of order $n-1$. By Theorem 9.14 and the induction hypothesis,

$$P(T,\lambda) = (\lambda-1)P(T-v,\lambda) = (\lambda-1)\left[\lambda(\lambda-1)^{n-2}\right] = \lambda(\lambda-1)^{n-1},$$

as desired. ■

Two graphs are **chromatically equivalent** if they have the same chromatic polynomial. By Theorem 9.13, two chromatically equivalent graphs must have the same order, the same size, and the same chromatic number. By Corollary 9.15, every two trees of the same order are chromatically equivalent. It

is not known under what conditions two graphs are chromatically equivalent in general. A graph G is **chromatically unique** if $P(H, \lambda) = P(G, \lambda)$ implies that $H \cong G$. Here too it is not known under what conditions a graph is chromatically unique.

Exercises for Section 9.4

1. Prove that the chromatic polynomial of every graph can be expressed as the sum and difference of the chromatic polynomials of empty graphs.

2. (a) Determine $P(C_6, \lambda)$ by repeated application of Theorem 9.10.
   (b) Use the polynomial obtained in (a) to determine $P(C_6, 2)$. Explain why this answer is not surprising.

3. (a) Determine $P(K_{2,2,2}, \lambda)$ by repeated application of Theorem 9.10.
   (b) Use the polynomial obtained in (a) to determine $P(K_{2,2,2}, 3)$. Explain why this answer is not surprising.

4. Prove that $P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$ each integer $n \geq 3$.

5. We know that every two trees of the same order are chromatically equivalent.
   (a) Which unicyclic graphs of the same order are chromatically equivalent?
   (b) How many distinct chromatic polynomials are there for unicyclic graphs of order $n \geq 3$?

6. Prove that if G is a graph with components $G_1, G_2, \ldots, G_k$, then
$$P(G, \lambda) = \prod_{i=1}^{k} P(G_i, \lambda).$$

7. (a) Prove that if G is a nontrivial connected graph, then $P(G, \lambda) = \lambda g(\lambda)$, where $g(0) \neq 0$.
   (b) Prove that a graph G has exactly k components if and only if
$$P(G, \lambda) = \lambda^k f(\lambda),$$
   where $f(\lambda)$ is a polynomial with $f(0) \neq 0$.

8. Show that if F is a forest of order n with k components, then
$$P(F, \lambda) = \lambda^k (\lambda - 1)^{n-k}.$$

9. Prove that if G is a connected graph with blocks $B_1, B_2, \ldots, B_r$, then

$$P(G, \lambda) = \frac{\prod_{i=1}^{r} P(B_i, \lambda)}{\lambda^{r-1}}.$$

10. It has been stated that if G and H are two chromatically equivalent graphs, then G and H have the same order, the same size and the same chromatic number. Show that the converse of this statement is false.

11. Prove for each integer $r \geq 2$ that $K_{r,r}$ is chromatically unique.

12. Let G be a graph. Prove that if $P(G, \lambda) = \lambda(\lambda - 1)^{n-1}$, then G is a tree of order n.

13. Prove that if G is a connected graph of order n, then

$$P(G, \lambda) \leq \lambda(\lambda - 1)^{n-1}.$$

14. Prove or disprove: The polynomial $\lambda^4 - 3\lambda^3 + 3\lambda^2$ is the chromatic polynomial of some graph.

15. Prove or disprove: The graph G in Figure 9.25 is chromatically unique.

G :

Figure 9.25: The graph G in Exercise 15

9.5 The Heawood Map-Coloring Problem

In addition to the counterexample to Kempe's proof and a proof of the Five Color Theorem, Heawood's paper [115] contained several other interesting results, observations and comments. For example, Heawood also considered the problem of coloring maps that can be drawn on other surfaces. As we have observed, maps that can be drawn in the plane are precisely those maps that can be drawn on the surface of a sphere. Heawood proved that the regions of every map drawn on the surface of a torus can be colored with seven or fewer colors and that there is, in fact, a map on the torus that requires seven colors. The map M shown in Figure 9.26 is drawn on the torus. It has seven regions (colored with $1, 2, \ldots, 7$) and, in fact, requires seven colors. The **toroidal dual** of this map is the graph K_7 drawn on the torus (see Figure 7.9).

More generally, Heawood showed that the regions of every map drawn on a surface of genus k ($k > 0$) can be colored with $\left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor$ colors. In addition,

Figure 9.26: A map on the torus

he stated that such maps requiring this number of colors exist. He never proved this latter statement, however. In fact, it would take another 78 years to verify this statement.

Heawood became interested in the largest chromatic number of a graph that could be embedded on certain surfaces. The **chromatic number of a surface** S is defined by

$$\chi(S) = \max\{\chi(G)\}$$

where the maximum is taken over all graphs G that can be embedded on S. That $\chi(S_0) = 4$ is the Four Color Theorem. Heawood was successful in determining the chromatic number of the torus.

Theorem 9.16 $\chi(S_1) = 7$.

Proof. In Figure 7.9, we saw that the complete graph K_7 can be embedded on the torus. Since $\chi(K_7) = 7$, it follows that $\chi(S_1) \geq 7$.

Now let G be a graph that can be embedded on the torus. Among the subgraphs of G, let H be one having the largest minimum degree. We show that $\delta(H) \leq 6$. Suppose that H has order n and size m. If $n \leq 7$, then certainly $\delta(H) \leq 6$. Hence we may assume that $n > 7$.

Since G is embeddable on the torus, so is H. Thus $\gamma(H) \leq 1$. It therefore follows by Theorem 7.4 that

$$1 \geq \gamma(H) \geq \frac{m}{6} - \frac{n}{2} + 1$$

and so $m \leq 3n$. Hence

$$n\delta(H) \leq \sum_{v \in V(H)} \deg_H v = 2m \leq 6n$$

and so $\delta(H) \leq 6$. Therefore, in any case, $\delta(H) \leq 6$. By Theorem 8.20,

$$\chi(G) \leq 1 + \delta(H) \leq 7.$$

Hence $\chi(S_1) = 7$. ■

In his important paper, Heawood [115] obtained an upper bound for the chromatic number of S_k for every positive integer k.

Theorem 9.17 *For every nonnegative integer k,*

$$\chi(S_k) \leq \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor.$$

Proof. Let G be a graph that is embeddable on the surface S_k and let

$$h = \frac{7 + \sqrt{1 + 48k}}{2}.$$

Hence $1 + 48k = (2h - 7)^2$. Solving for $h - 1$, we have

$$h - 1 = 6 + \frac{12(k-1)}{h}. \tag{9.1}$$

Among the subgraphs of G, let H be one having the largest minimum degree. We show that $\delta(H) \leq h - 1$. Suppose that H has order n and size m. If $n \leq h$, then $\delta(H) \leq h - 1$. Hence we may assume that $n > h$. Since G is embeddable on S_k, so is H. Therefore, $\gamma(H) \leq k$. By Theorem 7.4,

$$k \geq \gamma(H) \geq \frac{m}{6} - \frac{n}{2} + 1.$$

Thus $m \leq 3n + 6(k-1)$. We therefore have

$$n\delta(H) \leq \sum_{v \in V(H)} \deg_H v = 2m \leq 6n + 12(k-1)$$

and so, by (9.1),

$$\delta(H) \leq 6 + \frac{12(k-1)}{n} \leq 6 + \frac{12(k-1)}{h} = h - 1.$$

Hence $\delta(H) \leq h - 1$ in any case. By Theorem 8.20,

$$\chi(G) \leq 1 + \delta(H) \leq h = \frac{7 + \sqrt{1 + 48k}}{2},$$

giving the desired result. ■

In fact, Heawood was under the impression that he had shown that

$$\chi(S_k) = \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor \tag{9.2}$$

for every positive integer k, but, in fact, all he had established was the bound given in Theorem 9.17. During the year following the publication of Heawood's paper, Lothar Heffter [117] wrote a paper in which he drew attention to the incomplete nature of Heawood's argument. Heffter was in fact able to show that equality held in (9.2) not only for $k = 1$ but for $1 \leq k \leq 6$ and some other values of k as well. To verify equality in (9.2) for every positive integer k, it would be necessary to show, for every positive integer k, that there is a graph G_k that is embeddable on S_k for which

$$\chi(G_k) = \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor.$$

The problem concerning whether equality held in (9.2) for every positive integer k would become known as:

The Heawood Map-Coloring Problem For every positive integer k, is it true that

$$\chi(S_k) = \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor?$$

There was a great deal of confusion surrounding this famous problem and the origin of this confusion is also unknown. For example, in their well-known book *What is Mathematics*?, Richard Courant and Herbert E. Robbins [52] reported that

$$\chi(S_k) = \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor$$

for every positive integer k. Whether it was the belief that this formula is true which led Courant and Robbins to include this premature statement in their book or whether writing this statement in their book led to mathematical folklore is not known. Indeed, this was not even known to Courant and Robbins. There were reports that the Heawood Map-Coloring Problem had been solved as early as the early 1930s in Göttingen in Germany.

Solving the Heawood Map-Coloring Problem required the work of many mathematicians over a period of 78 years. However, primarily through the efforts of Gerhard Ringel and J. W. T. (Ted) Youngs [189], this problem was finally solved when a formula for the genus of the complete graph was determined (see Theorem 7.9).

Theorem 9.18 (**The Heawood Map-Coloring Theorem**) *For every positive integer k,*

$$\chi(S_k) = \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor.$$

Proof. By Theorem 9.17,

$$\chi(S_k) \leq \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor.$$

Hence it remains only to verify the reverse inequlaity. Define

$$n = \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor.$$

Thus $n \leq (7+\sqrt{1+48k})/2$. Solving this inequlaity for k, we have

$$k \geq (n-3)(n-4)/12$$

and so by Theorem 7.9,

$$k \geq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil = \gamma(K_n).$$

Therefore, $\gamma(K_n) \leq k$, which implies that

$$\chi(S_{\gamma(K_n)}) \leq \chi(S_k).$$

Since K_n is clearly embeddable on $S_{\gamma(K_n)}$, it follows that $\chi(S_{\gamma(K_n)}) \geq n$ and so

$$\chi(S_k) \geq n = \left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor,$$

giving the desired result. ■

Exercises for Section 9.5

1. It is known that the Petersen graph P is not planar. Thus P cannot be embedded on the sphere.

   (a) Show that P can be embedded on the torus however.

   (b) How many regions does P have when it's embedded on the torus? Explain.

   (c) What is the minimum number of colors that can be assigned to these regions so that every two adjacent regions are colored differently?

2. Prove or disprove: If G is a graph such that $\chi(G) \leq \chi(S_k)$ for some positive integer k, then G can be embedded on S_k.

3. Give an example of a graph G with genus 2 and $\chi(G) = \chi(S_2)$. Verify that your example has these properties.

4. Use the result given in Theorem 8.20 to establish an upper bound for the chromatic numbers of graphs embeddable on the torus. Discuss the sharpness of your bound.

Chapter 10

Matchings, Factorization and Domination

Of the numerous problems concerning sets of edges or sets of vertices in graphs, many of these deal with independent sets. We saw the importance of these sets of vertices when vertex colorings were discussed in Chapter 8 and again in Chapter 9. We will discuss independent sets of edges in the current chapter and see their connections to edge colorings in the following chapter. A number of important related concepts will also be discussed in the current chapter.

10.1 Matchings and Independence in Graphs

A set of edges in a graph G is **independent** if no two edges in the set are adjacent in G. The edges in an independent set of edges of G are called a **matching** in G. A matching of maximum cardinality in G is a **maximum matching** in G. The **edge independence number** $\alpha'(G)$ of G is the number of edges in a maximum matching of G. In fact, $\alpha'(G)$ is sometimes referred to as the **matching number** of G. In the graph G of Figure 10.1, the set $M_1 = \{e_1, e_4\}$ is a matching that is not a maximum matching, while $M_2 = \{e_1, e_3, e_5\}$ and $M_3 = \{e_1, e_3, e_6\}$ are maximum matchings in G. For the graph G of Figure 10.1 then, $\alpha'(G) = 3$.

Figure 10.1: Matchings and maximum matchings

If M is a matching in a graph G with the property that every vertex of G is incident with an edge of M, then M is a **perfect matching** in G. Clearly, if G has a perfect matching M, then G has even order and the edge-induced subgraph $G[M]$ is a 1-regular spanning subgraph of G. Thus, the graph G of Figure 10.1 cannot have a perfect matching.

If M is a matching in a graph G, then every vertex of G is incident with at most one edge of M. A vertex that is incident with no edges of M is called an **M-unmatched vertex** or simply an **unmatched vertex** if the matching M is clear. The following theorem will prove to be useful.

Theorem 10.1 *Let M_1 and M_2 be matchings in a graph G. Then each component of the spanning subgraph H of G with $E(H) = (M_1 - M_2) \cup (M_2 - M_1)$ is one of the following types:*

(*i*) *an isolated vertex,*

(*ii*) *an even cycle whose edges are alternately in M_1 and in M_2,*

(*iii*) *a nontrivial path whose edges are alternately in M_1 and in M_2 and such that each end-vertex of the path is either M_1-unmatched or M_2-unmatched but not both.*

Proof. First we note that $\Delta(H) \leq 2$, for if H contains a vertex v such that $\deg_H v \geq 3$, then v is incident with at least two edges in the same matching. Since $\Delta(H) \leq 2$, every component of H is a path (possibly trivial) or a cycle. Since no two edges in a matching are adjacent, the edges of each cycle and path in H are alternately in M_1 and in M_2. Thus each cycle in H is even.

Suppose that $e = uv$ is an edge of H and u is the end-vertex of a path P that is a component of H. The proof will be complete once we have shown that u is M_1-unmatched or M_2-unmatched but not both. Since $e \in E(H)$, it follows that $e \in M_1 - M_2$ or $e \in M_2 - M_1$. If $e \in M_1 - M_2$, then u is not M_1-unmatched. We show that u is M_2-unmatched. If this is not the case, then there is an edge f in M_2 (thus $f \neq e$) such that f is incident to u. Since e and f are adjacent, $f \notin M_1$. Thus, $f \in M_2 - M_1 \subseteq E(H)$. This, however, is impossible since u is the end-vertex of P. Therefore, u is M_2-unmatched. Similarly, if $e \in M_2 - M_1$, then u is M_1-unmatched. ■

Augmenting Paths and Maximum Matchings

In order to present a characterization of matchings in a graph G that are maximum matchings, we introduce two new terms. Let M be a matching in a graph G. An **M-alternating path** of G is a path whose edges are alternately in M and not in M. An **M-augmenting path** is an M-alternating path P both of whose end-vertices are unmatched, that is, the first and last edges of P do not belong to M. Necessarily, every M-augmenting path has odd length.

Figure 10.2 shows a graph G of order 13 and a matching

$$M = \{v_2v_4, v_3v_6, v_5v_8, v_7v_{10}, v_{11}v_{12}\}$$

of size 5, whose edges are indicated by bold lines. The path $P =(v_1,\ v_2,\ v_4,\ v_7,\ v_{10},\ v_9)$ is then an M-augmenting $v_1 - v_9$ path in G.

Figure 10.2: A matching M in a graph G and an M-augmenting path $P = (v_1, v_2, v_4, v_7, v_{10}, v_9)$

The following result is due to Claude Berge [14].

Theorem 10.2 *A matching M in a graph G is a maximum matching if and only if G contains no M-augmenting path.*

Proof. First, assume, to the contrary, that a graph G contains an M-augmenting path P, where M is a maximum matching in G. Let M' denote the edges of P belonging to M, and let $M'' = E(P) - M'$. Since $|M''| = |M'| + 1$, the set $(M - M') \cup M''$ is a matching having cardinality exceeding that of M, producing a contradiction.

For the converse, suppose that M_1 is a matching of a graph G that is not a maximum matching in G. Let M_2 be a maximum matching in G and let H be the spanning subgraph of G with

$$E(H) = (M_1 - M_2) \cup (M_2 \cup M_1).$$

Since $|M_2| > |M_1|$, it follows by Theorem 10.1 that there is a component P of H that is a path whose edges are alternately in M_1 and in M_2 such that P contains more edges belonging to M_2 than to M_1. Necessarily, each end-vertex of P is M_1-unmatched, which implies that P is an M_1-augmenting path. ■

Since the path $P = (v_1, v_2, v_4, v_7, v_{10}, v_9)$ in the graph G of Figure 10.2 is an M-augmenting path for the matching

$$M = \{v_2v_4, v_3v_6, v_5v_8, v_7v_{10}, v_{11}v_{12}\},$$

it follows by Theorem 10.2 that M is not a maximum matching in G. Following the proof of Theorem 10.2, we let $M' = E(P) \cap M$ and $M'' = E(P) - M'$.

Figure 10.3: A maximum matching in a graph G

Then $(M - M') \cup M''$ is a matching whose size exceeds that of M. In fact, $(M - M') \cup M''$ is a maximum matching in the graph G of Figure 10.2 (see Figure 10.3).

Although matchings are of interest in all graphs, they are of particular interest in bipartite graphs. Let G be a bipartite graph with partite sets U and W, where $|U| \leq |W|$. A matching in G is therefore a set $M = \{e_1, e_2, \ldots, e_k\}$ of edges, where $e_i = u_i w_i$ for $1 \leq i \leq k$ such that $u_1, u_2, \ldots, u_k$ are k distinct vertices of U and $w_1, w_2, \ldots, w_k$ are k distinct vertices of W. In this case, M **matches** the set $\{u_1, u_2, \ldots, u_k\}$ to the set $\{w_1, w_2, \ldots, w_k\}$. Necessarily, for any matching of k edges, $k \leq |U|$. If $|U| = k$, then U is said to be **matched** to a subset of W.

Hall's Theorem

For a bipartite graph G with partite sets U and W and for $S \subseteq U$, let $N(S)$ be the set of all vertices in W having a neighbor in S. The condition that

$$|N(S)| \geq |S| \text{ for all } S \subseteq U$$

is referred to as **Hall's condition**. This condition is named for Philip Hall (1904–1982). The following 1935 theorem of Hall [108] shows that this condition is necessary and sufficient for one partite set of a bipartite graph to be matched to a subset of the other.

Theorem 10.3 (**Hall's Theorem**) *Let G be a bipartite graph with partite sets U and W. Then U can be matched to a subset of W if and only if Hall's condition is satisfied.*

Proof. Suppose the U can be matched to a subset of W under a matching M^*. Then every nonempty subset S of U can be matched under M^* to some subset of W, implying that $|N(S)| \geq |S|$; so U satisfies Hall's condition.

To verify the converse, suppose that there exists a bipartite graph G for which Hall's condition is satisfied but U cannot be matched to a subset of W. Let M be a maximum matching in G. By assumption, there is a vertex u in

U that is unmatched. Let S be the set of all vertices of G that are connected to u by an M-alternating path. Since M is a maximum matching, it follows by Theorem 10.2 that u is the only unmatched vertex in S.

Let $U_1 = S \cap U$ and let $W_1 = S \cap W$. Since no vertex of $S - \{u\}$ is unmatched, it follows that $U_1 - \{u\}$ is matched under M to W_1. Therefore, $|W_1| = |U_1| - 1$ and $W_1 \subseteq N(U_1)$. Furthermore, for every $w \in N(U_1)$, the graph G contains an M-alternating $u - w$ path so that $N(U_1) \subseteq W_1$. Thus, $N(U_1) = W_1$ and

$$|N(U_1)| = |W_1| = |U_1| - 1 < |U_1|.$$

This, however, contradicts the fact that Hall's condition is satisfied. ■

If G is a bipartite graph with partite sets U and W where $|U| \leq |W|$ and Hall's condition is satisfied, then by Theorem 10.3, G contains a matching of size $|U|$, which is a maximum matching. If $|U| = |W|$, then such a matching is a perfect matching in G. The following result is another consequence of Theorem 10.3.

Theorem 10.4 *Every r-regular bipartite graph ($r \geq 1$) has a perfect matching.*

Proof. Let G be an r-regular bipartite graph with partite sets U and W. Then $|U| = |W|$. Let S be a nonempty subset of U. Suppose that a total of k edges join the vertices of S and the vertices of $N(S)$. Thus $k = r|S|$. Since there are $r|N(S)|$ edges incident with the vertices of $N(S)$, it follows that $k \leq r|N(S)|$ or $r|S| \leq r|N(S)|$. Therefore, $|S| \leq |N(S)|$ and Hall's condition is satisfied in G. By Theorem 10.3, G has a perfect matching. ■

A collection $S_1, S_2, \ldots, S_n$, $n \geq 1$, of finite nonempty sets is said to have a **system of distinct representatives** if there exists a set $\{s_1, s_2, \ldots, s_n\}$ of distinct elements such that $s_i \in S_i$ for $1 \leq i \leq n$.

Theorem 10.5 *A collection $\{S_1, S_2, \ldots, S_n\}$ of finite nonempty sets has a system of distinct representatives if and only if for each integer k with $1 \leq k \leq n$, the union of any k of these sets contains at least k elements.*

Proof. Assume first that $\{S_1, S_2, \ldots, S_n\}$ has a system of distinct representatives. Then, for each integer k with $1 \leq k \leq n$, the union of any k of these sets contains at least k elements.

For the converse, suppose that $\{S_1, S_2, \ldots, S_n\}$ is a collection of n sets such that for each integer k with $1 \leq k \leq n$, the union of any k of these sets contains at least k elements. We now consider the bipartite graph G with partite sets

$$U = \{S_1, S_2, \ldots, S_n\} \text{ and } W = S_1 \cup S_2 \cup \cdots \cup S_n$$

such that a vertex S_i ($1 \leq i \leq n$) in U is adjacent to a vertex w in W if $w \in S_i$. Let X be any subset of U with $|X| = k$, where $1 \leq k \leq n$. Since the union of any k sets in U contains at least k elements, it follows that $|N(X)| \geq |X|$. Thus G satisfies Hall's condition. By Theorem 10.3, G contains a matching from U

to a subset of W. This matching pairs off the sets $S_1, S_2, \ldots, S_n$ with n distinct elements in $S_1 \cup S_2 \cup \ldots \cup S_n$, producing a system of distinct representatives for $\{S_1, S_2, \ldots, S_n\}$. ■

Theorem 10.5 is also due to Philip Hall. Indeed, it was through systems of distinct representative that Hall proved the theorem that bears his name. Born in England, Hall's interest in mathematics was greatly influenced by the mathematics teachers he had. In 1927 Hall obtained an important result in group theory, generalizing the Sylow theorems for finite solvable groups, which is now often called Hall's theorem. He then published a paper in 1932 on groups of prime power order, perhaps his best known work. In 1933 he was then appointed a Lecturer at Cambridge. In 1935 his theorem on matchings was published, although it was not stated in terms of graph theory. Except for a period during World War II when he worked for a Foreign Office at Bletchley Park, he remained at Cambridge from 1933 to 1967. Hall spent much of his life making important contributions to algebra and is considered one of the great mathematicians of the 20th century.

The preceding discussion is directly related to a well-known combinatorial problem called the **Marriage Problem**:

> *Given a set of boys and a set of girls where each girl knows some of the boys, under what conditions can all girls get married, each to a boy she knows?*

In this context, Theorem 10.5 may be reformulated to produce what is often referred to as **Hall's Marriage Theorem**:

> *If there are n girls, then the Marriage Problem has a solution if and only if every subset of k girls $(1 \leq k \leq n)$ collectively know at least k boys.*

We have already noted that if M is a perfect matching in a graph G, then $G[M]$ is a 1-regular spanning subgraph of G. Any spanning subgraph of a graph G is referred to as a **factor** of G. A k-regular factor is called a **k-factor**. Thus F is a 1-factor of a graph G if and only if $E(F)$ is a perfect matching in G. Therefore, the theorems stated earlier concerning perfect matchings can be restated in terms of 1-factors. For example, we restate Theorem 10.4 in terms of 1-factors.

Theorem 10.6 *Every r-regular bipartite graph $(r \geq 1)$ has a 1-factor.*

The determination of whether a given graph contains a 1-factor is a problem that has received much attention in the literature. Of course, if a graph G has a 1-factor, then G has even order. The graph G of Figure 10.4 has even order but does not contain a 1-factor. Let $S = \{s_1, s_2, s_3\}$. Since each of the components $G_1, G_2, \ldots, G_5$ of $G - S$ is odd, if G has a 1-factor F, then some edge of F must join a vertex of G_i to S for each i $(1 \leq i \leq 5)$. However, since there are more odd components in $G - S$ than vertices of S, this is impossible.

Figure 10.4: A graph that contains no 1-factor

Tutte's Theorem on 1-Factors

William Tutte characterized graphs containing a 1-factor while he was a graduate student. The proof of Tutte's theorem [224] we present is due to Ian Anderson [1]. A component of a graph is **odd** or **even** according to whether its order is odd or even. The number of odd components in a graph G is denoted by $k_o(G)$.

Theorem 10.7 (**Tutte's Theorem**) *A nontrivial graph G contains a 1-factor if and only if $k_o(G-S) \leq |S|$ for every proper subset S of $V(G)$.*

Proof. First, suppose that G contains a 1-factor F. Let S be a proper subset of $V(G)$. If $G-S$ has no odd components, then $k_o(G-S) = 0$ and $k_o(G-S) \leq |S|$. Thus we may assume that $k_o(G-S) = k \geq 1$. Let $G_1, G_2, \ldots, G_k$ be the odd components of $G-S$. (There may be some even components of $G-S$ as well.) For each odd component G_i of $G-S$, there is at least one edge of F joining a vertex of G_i and a vertex of S. Thus $k_o(G-S) \leq |S|$.

We now verify the converse. Let G be a graph such that $k_o(G-S) \leq |S|$ for every proper subset S of $V(G)$. In particular, $k_o(G-\emptyset) \leq |\emptyset| = 0$, implying that every component of G is even and so G itself has even order. We now show that G has a 1-factor by employing induction on the (even) order of G. Since K_2 is the only graph of order 2 having no odd components and K_2 has a 1-factor, the base case of the induction is verified.

For a given even integer $n \geq 4$, assume that all graphs H of even order less than n and satisfying $k_o(H-S) \leq |S|$ for every proper subset S of $V(H)$ contain a 1-factor. Now let G be a graph of order n satisfying $k_o(G-S) \leq |S|$ for every proper subset S of $V(G)$. As we saw above, every component of G has even order. We show that G has a 1-factor.

For a vertex v of G that is not a cut-vertex and $R = \{v\}$, it follows that $k_o(G - R) = |R| = 1$. Hence there are nonempty proper subsets T of $V(G)$ for which $k_o(G - T) = |T|$. Among all such sets T, let S be one of maximum cardinality. Suppose that $k_o(G - S) = |S| = k \geq 1$ and let $G_1, G_2, \ldots, G_k$ be the odd components of $G - S$.

We claim that $k(G-S) = k$, that is, $G_1, G_2, \ldots, G_k$ are the *only* components of $G - S$. Assume, to the contrary, that $G - S$ has an even component G_0. Let v_0 be a vertex of G_0 that is not a cut-vertex of G_0. Let $S_0 = S \cup \{v_0\}$. Since G_0 has even order,

$$k_o(G - S_0) \geq k_o(G - S) + 1 = k + 1.$$

On the other hand, $k_o(G - S_0) \leq |S_0| = k + 1$. Therefore,

$$k_o(G - S_0) = |S_0| = k + 1,$$

which is impossible. Thus, as claimed, $G_1, G_2, \ldots, G_k$ are the only components of $G - S$.

For each integer i $(1 \leq i \leq k)$, let S_i denote the set of those vertices in S adjacent to at least one vertex of G_i. Since G has only even components, each set S_i is nonempty.

We claim, for each integer ℓ with $1 \leq \ell \leq k$, that the union of any ℓ of the sets $S_1, S_2, \ldots, S_k$ contains at least ℓ vertices. Assume, to the contrary, that this is not the case. Then there is an integer j such that the union S' of j of the sets $S_1, S_2, \ldots, S_k$ has fewer than j elements. Suppose that $S_1, S_2, \ldots, S_j$ have this property. Thus

$$S' = S_1 \cup S_2 \cup \cdots \cup S_j \text{ and } |S'| < j.$$

Then $G_1, G_2, \ldots, G_j$ are at least some of the components of $G - S'$ and so

$$k_o(G - S') \geq j > |S'|,$$

which contradicts the hypothesis. Thus, as claimed, for each integer ℓ with $1 \leq \ell \leq k$, the union of any ℓ of the sets $S_1, S_2, \ldots, S_k$ contains at least ℓ vertices.

By Theorem 10.5, there is a set $\{v_1, v_2, \ldots, v_k\}$ of k distinct vertices of S such that $v_i \in S_i$ for $1 \leq i \leq k$. Since every component G_i of $G - S$ contains a vertex u_i such that $u_i v_i$ is an edge of G, it follows that $\{u_i v_i : 1 \leq i \leq k\}$ is a matching of G.

We now show that for each nontrivial component G_i of $G - S$ $(1 \leq i \leq k)$, the graph $G_i - u_i$ contains a 1-factor. Let W be a proper subset of $V(G_i - u_i)$. We claim that

$$k_o(G_i - u_i - W) \leq |W|.$$

Assume, to the contrary, that $k_o(G_i - u_i - W) > |W|$. Since G_i has odd order, $G_i - u_i$ has even order and so $k_o(G_i - u_i - W)$ and $|W|$ are both even or both odd. Hence

$$k_o(G_i - u_i - W) \geq |W| + 2.$$

Let $X = S \cup W \cup \{u_i\}$. Then

$$\begin{aligned} |X| &= |S| + |W| + 1 = |S| + (|W| + 2) - 1 \\ &\leq k_o(G - S) + k_o(G_i - u_i - W) - 1 \\ &= k_o(G - X) \leq |X|, \end{aligned}$$

which implies that $k_o(G - X) = |X|$ and contradicts the defining property of S. Thus, as claimed, $k_o(G_i - u_i - W) \leq |W|$.

Therefore, by the induction hypothesis, for each nontrivial component G_i of $G - S$ $(1 \leq i \leq k)$, the graph $G_i - u_i$ has a 1-factor. The collection of 1-factors of $G_i - u_i$ for all nontrivial graphs G_i of $G - S$ together with the edges in $\{u_i v_i : 1 \leq i \leq k\}$ produce a 1-factor of G. ■

The Tutte-Berge Formula

By Tutte's theorem, it follows, of course, that if G is a graph of order $n \geq 2$ such that for each proper subset S of $V(G)$, the number of odd components of $G - S$ does not exceed $|S|$, then n is even and $\alpha'(G) = n/2$. Berge [16] observed that Tutte's theorem (Theorem 10.7) gives rise to a formula for the size of a maximum matching in a graph G, that is, a formula for the edge independence number $\alpha'(G)$.

Theorem 10.8 (**The Tutte-Berge Formula**) *If G is a graph of order n, then*

$$\alpha'(G) = \min_{S \subset V(G)} \left\{ \frac{n + |S| - k_o(G - S)}{2} \right\} \tag{10.1}$$

where the minimum is taken over all proper subsets S of $V(G)$.

Proof. Let S be a proper subset of $V(G)$ and let M be a maximum matching in G. Furthermore, let M_1 be the subset of M consisting of those edges incident with at least one vertex of S and let $M_2 = M - M_1$. Then

$$\begin{aligned} \alpha'(G) &= |M| = |M_1| + |M_2| \leq |S| + \alpha'(G - S) \\ &\leq |S| + \frac{(n - |S|) - k_o(G - S)}{2} = \frac{n + |S| - k_o(G - S)}{2}. \end{aligned}$$

Thus

$$\alpha'(G) \leq \min_{S \subset V(G)} \left\{ \frac{n + |S| - k_o(G - S)}{2} \right\}.$$

Next, let

$$p = \max_{S \subset V(G)} \{k_o(G - S) - |S|\}, \tag{10.2}$$

where the maximum is taken over all proper subsets S of $V(G)$. If $S = \emptyset$, then $k_o(G - S) - |S| = k_o(G) \geq 0$ and so $p \geq 0$. If $p = 0$, then $k_o(G - S) \leq |S|$ for

every proper subset S of $V(G)$ and so by Tutte's theorem (Theorem 10.7), G has a 1-factor. Therefore, the formula (10.1) holds. Hence we may assume that $p \geq 1$. Let $G' = G \vee K_p$. Then the order of G' is $n' = n + p$. Observe that G' is connected and that n and $k_o(G - S) - |S|$ are of the same parity for each proper subset S of $V(G)$. Thus n' is even.

We now show that $k_o(G' - S') \leq |S'|$ for every proper subset S' of $V(G')$. First, the inequality holds for $S' = \emptyset$. So we may assume that $S' \neq \emptyset$. Suppose first that S' does not contain all p vertices of K_p. Then $G' - S'$ is connected and so $k_o(G' - S') \leq |S'|$. Hence we may assume that S' contains all p vertices of K_p. Let $S = S' - V(K_p)$. So $|S'| = |S| + p$ and $G' - S' = G - S$. Consequently, by (10.2),

$$k_o(G' - S') = k_o(G - S) \leq |S| + p = |S'|.$$

Therefore, by Tutte's theorem, G' has a 1-factor and thus a perfect matching. Deleting the p vertices of K_p from G' produces a matching in G of size at least

$$\begin{aligned} \frac{n'}{2} - p &= \frac{n+p}{2} - p = \frac{n-p}{2} \\ &= \frac{1}{2}\left(n - \max_{S \subset V(G)} \{k_o(G - S) - |S|\}\right) \\ &= \min_{S \subset V(G)} \left\{\frac{n + |S| - k_o(G - S)}{2}\right\} \end{aligned}$$

and so

$$\alpha'(G) \geq \min_{S \subset V(G)} \left\{\frac{n + |S| - k_o(G - S)}{2}\right\}.$$

Therefore, (10.1) holds. ■

Petersen's Theorem

Clearly, every 1-regular graph contains a 1-factor, while the only 2-regular graphs containing a 1-factor are those that are the union of even cycles. Determining which cubic graphs contain a 1-factor is considerably more complex. One of the best known theorems in this connection is due to Julius Petersen [171].

Peter Christian Julius Petersen (1839–1910) was born in Sorø, Denmark. Petersen received his Ph.D. in 1871 from Copenhagen University. On the occasion of receiving his doctorate, Petersen wrote (translated from Danish):

> *Mathematics had, from the time I started to learn it, taken my complete interest, and most of my work consisted of solving problems of my own and of my friends, and in seeking the trisection of the angle, a problem that has had a great influence on my whole development.*

Although Petersen worked in and made contributions to many areas of mathematics, it is only graph theory for which he is known. Indeed in his day he enjoyed an international reputation. Petersen's contributions to graph theory were primarily contained within a single paper he wrote, published in 1891 and titled "Die Theorie der regulären Graphen." Prior to 1891, the important results on graph theory (including Leonhard Euler's work on Eulerian graphs and Gustav Kirchhoff's work on spanning trees) were not results in graph *theory* as there really was no graph theory at that time. In the case of Petersen's paper, however, an argument could be made that for the first time a paper had been written containing fundamental results on the theory of graphs. Among the important results occurring in this paper were two theorems we will see shortly, namely, Theorem 10.9, referred to as Petersen's theorem (*Every* 3-*regular bridgeless graph contains a* 1-*factor.*) and Theorem 10.17 (*A graph* G *is* 2-*factorable if and only if* G *is* r-*regular for some positive even integer* r.) although these theorems appeared in reverse order in Petersen's paper.

Even though Petersen's major contribution to graph theory was his 1891 paper, that is not what he is known for. His primary claim to fame lies not for a single paper he wrote but for a single graph that appeared in one of his papers: the Petersen graph. We have encountered this graph several times already and we will continue to encounter it. Petersen first mentioned this graph, not in his 28-page classic 1891 paper, but in his 3-page 1898 note [172] "Sur le théorème de Tait" in which he presented this graph as a counterexample to a statement we will encounter in the next section. His graph did not appear in the aesthetic way in which it is commonly drawn, as shown in Figure 10.5(a), but in the less appealing way shown in Figure 10.5(b).

Figure 10.5: The Petersen graph

Theorem 10.9 (**Petersen's Theorem**) *Every bridgeless cubic graph contains a* 1-*factor.*

Proof. Let G be a bridgeless cubic graph and let S be a proper subset of $V(G)$ with $|S| = k$. We show that $k_o(G - S) \leq |S|$. This is true if $G - S$ has no odd components; so we assume that $G - S$ has $\ell \geq 1$ odd components, say $G_1, G_2, \ldots, G_\ell$.

Let E_i $(1 \leq i \leq \ell)$ denote the set of edges joining the vertices of G_i and the vertices of S. Since G is cubic, every vertex of G_i has degree 3 in G. Because

the sum of the degrees in G of the vertices of G_i is odd and the sum of the degrees in G_i of the vertices of G_i is even, it follows that $|E_i|$ is odd. Because G is bridgeless, $|E_i| \neq 1$ and so $|E_i| \geq 3$ for $1 \leq i \leq \ell$. This implies that there are at least 3ℓ edges joining the vertices of $G - S$ and the vertices of S. Since $|S| = k$, at most $3k$ edges join the vertices of $G - S$ and the vertices of S. Thus

$$3k_o(G - S) = 3\ell \leq 3k = 3|S|$$

and so $k_o(G-S) \leq |S|$. By Tutte's theorem (Theorem 10.7), G has a 1-factor. ■

Indeed, Petersen showed that a cubic graph with at most two bridges contains a 1-factor (see Exercise 8). This result cannot be extended further, however, since the graph G of Figure 10.6 is cubic and contains three bridges but no 1-factor since $k_o(G - v) = 3 > 1 = |\{v\}|$.

Figure 10.6: A cubic graph with no 1-factor

Errera's Theorem

If a connected cubic graph G contains two bridges e and f, then e and f lie on a single path of G. In fact, if all the bridges of a cubic graph, regardless of how many there might be, lie on a single path, then Alfred Errera [75] showed that G must have a 1-factor.

Theorem 10.10 (**Errera's Theorem**) *If all the bridges of a connected cubic graph G lie on a single path of G, then G has a 1-factor.*

Proof. We proceed by induction on the number of bridges in a connected cubic graph, all of whose bridges lie on a single path. We have already mentioned that if any such a graph G contains one bridge or two bridges, then G has a 1-factor. Assume that every connected cubic graph with k (≥ 2) bridges all of which lie on a single path contains a 1-factor.

Let G be a connected cubic graph with $k + 1$ bridges $e_1, e_2, \ldots, e_{k+1}$, all of which lie on a single path of G. Thus G contains $k + 2$ blocks $B_1, B_2, \ldots, B_{k+2}$ such that $e_i = u_i w_i$ ($1 \leq i \leq k + 1$), where $u_i \in V(B_i)$ and $w_i \in V(B_{i+1})$. (See

Figure 10.7.) Thus B_i has odd order when $i = 1$ or $i = k + 2$; while B_i has even order if $2 \leq i \leq k + 1$. We now construct two connected graphs H and H', each containing k bridges, all of which lie on a single path. The graph H is obtained from G by deleting $V(B_2)$ and joining u_1 and w_2 by the edge e. The graph H' is obtained from G by deleting $V(B_{k+1})$ and joining u_k and w_{k+1} by the edge e'.

Figure 10.7: The graphs G, H and H' in the proof of Theorem 10.10

By the induction hypothesis, H contains a 1-factor F and H' contains a 1-factor F'. Let $M_1 = E(F) \cap E(B_1)$. Then u_1 is the only M_1-unmatched vertex in B_1. Since F is a 1-factor in H, it follows that $e \in E(F)$. This in turn implies that neither w_2 nor u_3 is incident with an edge of F lying in B_3. Thus $u_3w_3 \in E(F)$. Continuing in this manner, we see that every bridge of H belongs to F. In a similar manner, every bridge of H' belongs to F'. This says that in G, it follows that $B_1 - u_1$ and $B_{k+2} - w_{k+1}$ contain 1-factors and that $B_i - w_{i-1} - u_i$ contain 1-factors for $2 \leq i \leq k + 1$. These 1-factors together with the $k + 1$ bridges of G produce a 1-factor in G. ■

The three bridges of the cubic graph G of Figure 10.6 do not lie on a single path of G, of course, and G does not have a 1-factor. It is a direct consequence of Errera's theorem that if the bridges of a cubic graph G of order n lie on a single path, then $\alpha'(G) = n/2$.

Vertex Covers and Edge Covers

A vertex and an edge are said to **cover each other** in a graph G if they are incident in G. A **vertex cover** in G is a set of vertices that covers all the edges of G. An **edge cover** in a graph G without isolated vertices is a set of edges that covers all the vertices of G. The minimum cardinality of a vertex cover in a graph G is called the **vertex covering number** of G and is denoted by $\beta(G)$. As expected, the **edge covering number** $\beta'(G)$ of a graph G (without isolated vertices) is the minimum cardinality of an edge cover in G.

For $s \leq t$, we have $\beta(K_{s,t}) = s$ and $\beta'(K_{s,t}) = t$. Recall that the (vertex) independence number $\alpha(G)$ of G is the maximum cardinality among the independent sets of vertices in G. We have already seen that $\alpha(K_{s,t}) = t$. Furthermore, $\alpha'(K_{s,t}) = s$. As another illustration of these four parameters, we note that for $n \geq 2$, $\alpha(K_n) = 1$, $\alpha'(K_n) = \lfloor n/2 \rfloor$, $\beta(K_n) = n - 1$ and $\beta'(K_n) = \lceil n/2 \rceil$. The independence and covering concepts are summarized below.

$\alpha(G)$	vertex independence number	maximum number of vertices, no two of which are adjacent
$\alpha'(G)$	edge independence number	maximum number of edges, no two of which are adjacent
$\beta(G)$	vertex covering number	minimum number of vertices that cover the edges of G
$\beta'(G)$	edge covering number	minimum number of edges that cover the vertices of G

Gallai's Theorem

Observe that for the two graphs G of order n considered above, namely $K_{s,t}$ with $n = s + t$ and K_n, we have

$$\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G) = n.$$

These two examples illustrate the next theorem [89], due to the Hungarian mathematician Tibor Gallai (1912–1992).

Gallai was a winner of national mathematics competitions, along with Paul Erdős and Paul Turán, and became lifelong friends of both. As a consequence of his accomplishments, Gallai was admitted to Pázmány University in Budapest. He was one of a group of enthusiastic students in the 1930s in Budapest that included Erdős, Turán, George Szekeres and Esther Klein. Some of these students attended the graph theory course given by Dénes König, who was a professor at the Technical University of Budapest. This was to have a profound effect on Gallai's mathematical interests. Gallai, who was König's only doctoral student,

helped König with his graph theory book and König mentioned some of Gallai's results in the book and used other ideas of Gallai. Many of Gallai's contributions to graph theory were to prove fundamental to the subject and aided in the rapid development of graph theory and combinatorics. For example, he was among the first to recognize the importance of so-called min-max theorems (such as Menger's theorem).

Gallai was an exceptionally modest individual and rarely made public appearances or attended conferences. In fact, much of his work became known only because of the efforts of his students. While Gallai was quick to praise the work of others, he often underestimated the merits of his own contributions, even though he had important results in many areas of graph theory. Consequently, he was notoriously slow to publish his own results. Several of his results went unpublished, later to be independently rediscovered (and published) by others.

Theorem 10.11 (**Gallai's Theorem**) *If G is a graph of order n having no isolated vertices, then*

$$\alpha(G) + \beta(G) = n \tag{10.3}$$

and

$$\alpha'(G) + \beta'(G) = n. \tag{10.4}$$

Proof. We begin with (10.3). Let U be an independent set of vertices of G with $|U| = \alpha(G)$. Then the set $V(G) - U$ is a vertex cover in G. Therefore, $\beta(G) \leq n - \alpha(G)$. If, however, W is a set of $\beta(G)$ vertices that covers all edges of G, then $V(G) - W$ is independent; thus $\alpha(G) \geq n - \beta(G)$. This proves (10.3).

To verify (10.4), let E_1 be an independent set of edges of G with $|E_1| = \alpha'(G)$. Then E_1 covers $2\alpha'(G)$ vertices of G. For each vertex of G not covered by E_1, select an incident edge and define E_2 to be the union of this set of edges and E_1. Necessarily, E_2 is an edge cover in G so that $|E_2| \geq \beta'(G)$. Also we note that $2|E_1| + |E_2| = n$; hence $\alpha'(G) + \beta'(G) \leq n$.

Next, suppose that E' is an edge cover in G with $|E'| = \beta'(G)$. The minimality of E' implies that each component of $G[E']$ is a tree. Select from each component of $G[E']$ one edge, denoting the resulting set of edges by E''. Then $|E''| \leq \alpha'(G)$. If $G[E']$ is a forest with k components, then by Corollary 2.16 the size of $G[E']$ is $n - k$. Thus

$$\alpha'(G) + \beta'(G) \geq |E''| + |E'| = k + (n - k) = n,$$

completing the proof of (10.4) and the theorem. ■

An elementary relationship involving the independence and covering numbers is stated next.

Theorem 10.12 *If G is a graph of order n having no isolated vertices, then*

$$\beta(G) \geq \alpha'(G) \quad \textit{and} \quad \beta'(G) \geq \alpha(G).$$

Proof. Let S be a vertex cover in G and let X be an independent set of edges with $|X| = \alpha'(G)$. For each edge e of X there is a vertex v_e in S that is incident with e. Furthermore, for every two distinct edges e and f of X, the vertices v_e and v_f are distinct. Thus $|S| \geq |X| = \alpha'(G)$, which implies that $\beta(G) \geq \alpha'(G)$.

By Gallai's theorem (Theorem 10.11), it follows that

$$n - \alpha(G) = \beta(G) \geq \alpha'(G) = n - \beta'(G)$$

and so $\beta'(G) \geq \alpha(G)$. ■

The König-Egerváry Theorem

While $\beta(G) \geq \alpha'(G)$ for a graph G, equality does not hold in general. If, however, G is bipartite, then we do have $\beta(G) = \alpha'(G)$. This result was discovered independently by König [138] and Eugene Egerváry [66].

Theorem 10.13 (**The König-Egerváry Theorem**) *If G is a bipartite graph, then*

$$\beta(G) = \alpha'(G).$$

Proof. Since $\beta(G) \geq \alpha'(G)$, it suffices to show that $\beta(G) \leq \alpha'(G)$. Let U and W be the partite sets of G and let M be a maximum matching in G. Then $\alpha'(G) = |M|$. Denote by A the set of all M-unmatched vertices in U. (If $A = \emptyset$, then the proof is complete.) Observe that $|M| = |U| - |A|$. Let S be the set of all vertices of G that are connected to some vertex in A by an M-alternating path. Define $U' = S \cap U$ and $W' = S \cap W$.

As in the proof of Hall's theorem (Theorem 10.3), we have that $U' - A$ is matched to W' and that $N(U') = W'$. Since $U' - A$ is matched to W', it follows that $|U'| - |W'| = |A|$.

Observe that $C = (U - U') \cup W'$ is a vertex cover in G; for otherwise, there is an edge vw in G such that $v \in U'$ and $w \notin W'$. Furthermore,

$$|C| = |U| - |U'| + |W'| = |U| - |A| = |M|.$$

Therefore, $\beta(G) \leq |C| = |M| = \alpha'(G)$ and the proof is complete. ■

As a consequence of the König-Egerváry theorem (Theorem 10.13) and Gallai's theorem (Theorem 10.11), we have the following result of König and Richard Rado [176].

Theorem 10.14 (**The König-Rado Theorem**) *If G is a bipartite graph without isolated vertices, then*

$$\alpha(G) = \beta'(G).$$

Proof. By Theorems 10.11 and 10.13,

$$\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G) \text{ and } \beta(G) = \alpha'(G).$$

Thus $\alpha(G) = \beta'(G)$. ■

Exercises for Section 10.1

1. Show that every tree has at most one perfect matching.

2. Determine the maximum size m of a graph of order n having a maximum matching of k edges if (a) $n = 2k$, (b) $n = 2k + 2$.

3. Use Menger's theorem (Theorem 2.36) to prove the following implication in Hall's theorem (Theorem 10.3): *Let G be a bipartite graph with partite sets U and W. If Hall's condition is satisfied, then U can be matched to a subset of W.*

4. Let G be a bipartite graph with partite sets U and W such that $|U| = |W| = r \geq 1$. Suppose that $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_r\}$. Two vertices u_i and w_j are adjacent in G if and only if $i + j \geq r + 1$. Use Hall's theorem (Theorem 10.3) to show that G has a perfect matching.

5. Let G be a bipartite graph of size m with partite sets U and W where $|U| = |W| = k$.
   (a) Prove that if $m \geq k^2 - k + 1$, then G has a perfect matching.
   (b) Show that for $m = k^2 - k$, there exists a bipartite graph H with partite sets U and W such that $|U| = |W| = k$ but H does not have a perfect matching.

6. For which connected graphs G does the subdivision graph $S(G)$ contain a perfect matching?

7. Let G be a bipartite graph with partite sets U and W where $|U| \leq |W|$. The **deficiency** $\text{def}(S)$ of a set $S \subseteq U$ is defined as $\max\{|A| - |N(A)|\}$, where the maximum is taken over all nonempty subsets A of S. Show that
$$\alpha'(G) = \min\{|U|, |U| - \text{def}(U)\}.$$

8. Prove that every cubic graph with at most two bridges contains a 1-factor.

9. (a) Use Tutte's theorem (Theorem 10.7) to show that the graph G shown in Figure 10.8 does not have a 1-factor.
   (b) Petersen's theorem (Theorem 10.9) states that if G is a bridgeless, cubic graph, then G has a 1-factor. Show that Petersen's theorem can be extended somewhat by proving that if G is a bridgeless graph having exactly one vertex of degree 7 and all others of degree 3, then G has a 1-factor.

(c) Show that the result in (b) cannot be extended any further by giving an example of a bridgeless graph G having exactly two vertices of degree 7 and all others of degree 3 but G has no 1-factor.

Figure 10.8: The graph G in Exercise 9

10. (a) Prove that if a connected cubic graph has a 1-factor and a bridge, then the 1-factor must contain the bridge.

    (b) Suppose that a connected cubic graph G has a minimum edge-cut consisting of two edges e and f. What can we conclude about any 1-factor of G? May it contain neither e nor f? May it contain both e and f? May it contain exactly one of e and f?

11. (a) Let G be a graph every vertex of which has odd degree and let $\{V_1, V_2\}$ be a partition of $V(G)$, where $[V_1, V_2]$ is the set of edges joining V_1 and V_2. Prove that $|V_1|$ and $|[V_1, V_2]|$ are of the same parity.

    (b) Prove that every $(2k+1)$-regular, $2k$-edge-connected graph, $k \geq 1$, contains a 1-factor.

12. Prove that if G is an r-regular, $(r-2)$-edge-connected graph ($r \geq 3$) of even order containing at most $r-1$ distinct edge-cuts of cardinality $r-2$, then G has a 1-factor.

13. Give an example of a connected graph G with bridges having a 1-factor such that the bridges of G do not lie on a single path.

14. A graph G is **factor-critical** if $G - v$ contains a 1-factor for every vertex v of G. Prove that a graph G of order n is factor-critical if and only if n is odd and $k_o(G - S) \leq |S|$ for every nonempty proper subset S of $V(G)$.

15. Show that a graph G is bipartite if and only if $\alpha(H) \geq \frac{1}{2}|V(H)|$ for every subgraph H of G.

16. Let G be a connected graph of order n. Prove that

$$\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \alpha'(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$$

and show that these bounds are sharp.

17. Let G be a graph of order n without isolated vertices. Prove that

$$\left\lceil \frac{n}{2} \right\rceil \leq \beta'(G) \leq \left\lfloor \frac{n \cdot \Delta(G)}{1+\Delta(G)} \right\rfloor$$

and show that these bounds are sharp.

18. An independent set of vertices in a graph G is **maximal independent** if S is not a proper subset of any other independent set of vertices in G. Characterize those nonempty graphs with the property that every two distinct maximal independent sets of vertices are disjoint.

19. The **matching graph** $M(G)$ of a nonempty graph G has the maximum matchings of G as its vertices, and two vertices M_1 and M_2 of $M(G)$ are adjacent if M_1 and M_2 differ in only one edge. Show that each cycle C_n, $n = 3, 4, 5, 6$, is the matching graph of some graph.

20. Prove or disprove: A graph G without isolated vertices has a perfect matching if and only if $\alpha'(G) = \beta'(G)$.

21. Let M be a perfect matching in the graph $G = K_{16}$. Can the graph $G - M$ be embedded on the torus?

22. (a) Show that there is only one regular maximal planar graph G whose order $n \in \{5, 6, \ldots, 11\}$.

    (b) For the graph G in (a), show that $\overline{G}$ has a perfect matching M. Determine the genus of the graph $G + M$.

    (c) Prove that if G is a maximal planar graph G of order $n > 4$ whose complement contains a perfect matching M, then the genus

$$\gamma(G + M) \geq \frac{n}{12}.$$

10.2 Factorization

A graph G is said to be **factorable** into the factors $G_1, G_2, \ldots, G_t$ if these factors are pairwise edge-disjoint and $\cup_{i=1}^{t} E(G_i) = E(G)$. If G is factored into $G_1, G_2, \ldots, G_t$, then $\mathcal{F} = \{G_1, G_2, \ldots, G_t\}$ is called a **factorization** of G.

If there exists a factorization $\mathcal{F}$ of a graph G such that each factor in $\mathcal{F}$ is a k-factor (for a fixed positive integer k), then G is **k-factorable**. If G is a

k-factorable graph, then necessarily G is r-regular for some integer r that is a multiple of k.

If a graph G is factorable into $G_1, G_2, \ldots, G_t$, where each $G_i \cong H$ for some graph H, then we say that G is **H-factorable** and that G has an **isomorphic factorization** into the factor H. Certainly, if a graph G is H-factorable, then the size of H divides the size of G. A graph G of order $n = 2k$ is 1-factorable if and only if G is kK_2-factorable.

The problem in this area that has received the most attention is the determination of which graphs are 1-factorable. Of course, only regular graphs of even order can be 1-factorable. Trivially, every 1-regular graph is 1-factorable. Since a 2-regular graph contains a 1-factor if and only if every component is an even cycle, it is precisely these 2-regular graphs that are 1-factorable. The situation for r-regular graphs, $r \geq 3$, in general, or even only 3-regular graphs in particular, is considerably more complicated. By Petersen's theorem, every bridgeless cubic graph contains a 1-factor. Consequently, every bridgeless cubic graph can be factored into a 1-factor and a 2-factor. In 1884 Peter Guthrie Tait wrote that every cubic graph is 1-factorable but this result was *not true without limitation*. Petersen interpreted this rather vague statement to mean that every bridgeless cubic graph is 1-factorable. Not every bridgeless cubic graph is 1-factorable, however. Indeed, as Petersen himself observed [172], the Petersen graph shown in Figure 10.9 is not 1-factorable (see Exercise 1).

Figure 10.9: The Petersen graph: a bridgeless cubic graph that is not 1-factorable

We now describe two classes of 1-factorable graphs. The first of these is due to König [137].

Theorem 10.15 *Every r-regular bipartite graph, $r \geq 1$, is 1-factorable.*

Proof. We proceed by induction on r. The result is obvious if $r = 1$. Assume that every $(r-1)$-regular bipartite graph is 1-factorable where $r - 1 \geq 1$ and let G be an r-regular bipartite graph. By Theorem 10.6, G contains a 1-factor F_1. Then $G - E(F_1)$ is an $(r-1)$-regular bipartite graph. By the induction hypothesis, $G - E(F_1)$ can be factored into $r - 1$ 1-factors, say $F_2, F_3, \ldots, F_r$. Then $\{F_1, F_2, \ldots, F_r\}$ is a 1-factorization of G. ■

As a consequence of Theorem 10.15 then, the 4-regular bipartite graph $K_{4,4}$ can be factored into four 1-factors. If $U = \{A, K, Q, J\}$ and $W = \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ are the two partite sets of $K_{4,4}$, then one possible 1-factorization of $K_{4,4}$ is shown in Figure 10.10.

Figure 10.10: A 1-factorization of $K_{4,4}$

The 1-factorization of $K_{4,4}$ shown in Figure 10.10 then gives rise to a partition of the edge set of $K_{4,4}$ into four perfect matchings F_1, F_2, F_3, F_4, whose edges can be listed as follows:

F_1	A ♠	K ♡	Q ♢	J ♣
F_2	J ♢	Q ♣	K ♠	A ♡
F_3	K ♣	A ♢	J ♡	Q ♠
F_4	Q ♡	J ♠	A ♣	K ♢

This, of course, is very suggestive of the 16 playing cards of the four denominations ace, king, queen, jack and the four suits spades, hearts, diamonds, clubs. In the square table above, each denomination and each suit appears exactly once in each row and in each column. This card problem dates back to the year 1723. Thus the elements of the set U as well as the elements of W can be arranged in a 4×4 table so that every element appears exactly once in each row and column. More generally, when the elements of an n-element set are arranged in an $n \times n$ table so that the every element appears exactly once in each row and column, a **Latin square** results. So the table above gives rise to a pair of Latin squares. The topic of Latin squares has a long history and is an area of much interest in the subject of combinatorics.

1-Factorable Complete Graphs

Theorem 10.16 *For each positive integer k, the complete graph K_{2k} is 1-factorable.*

Proof. The result is obvious for $k = 1$, so we assume that $k \geq 2$. Denote the vertex set of K_{2k} by $\{v_0, v_1, \ldots, v_{2k-1}\}$ and arrange the vertices $v_1, v_2, \ldots, v_{2k-1}$ in a regular $(2k-1)$-gon, placing v_0 in the center. Now join every two vertices by a straight line segment, producing K_{2k}. For $i = 1, 2, \ldots, 2k-1$, define the 1-factor F_i to consist of the edge v_0v_i together with all those edges perpendicular to v_0v_i. Then $\{F_1, F_2, \cdots, F_{2k-1}\}$ is a 1-factorization of K_{2k}. ■

The construction described in the proof of Theorem 10.16 is illustrated in Figure 10.11 for the graph K_6.

Figure 10.11: A 1-factorization of K_6

In the 1-factorization of K_{2k} described in the proof of Theorem 10.16, the 1-factor F_1 consists of the edge v_0v_1 and all edges perpendicular to v_0v_1, namely, $v_2v_{2k-1}, v_3v_{2k-2}, \ldots, v_kv_{k+1}$. If the k edges of F_1 are rotated clockwise through an angle of $2\pi/(2k-1)$ radians, then the 1-factor F_2 is obtained. In general, if the edges of F_1 are rotated clockwise through an angle of $2\pi j/(2k-1)$ radians, where $0 \leq j \leq 2k-2$, then the 1-factor F_{j+1} is produced. A factorization of a graph obtained in this manner is referred to as a **cyclic factorization**.

Such a factorization can be viewed in another way. Let K_{2k} be drawn as described in the proof of Theorem 10.16. We now label each edge of K_{2k} with one of the integers $0, 1, \ldots, k-1$. Indeed, we will assign $2k-1$ edges of K_{2k} the label i for $i = 0, 1, \ldots, k-1$. Every edge of the type v_0v_i $(1 \leq i \leq 2k-1)$ is labeled 0. Now let C denote the cycle $(v_1, v_2, \ldots, v_{2k-1}, v_1)$. For $1 \leq s < t \leq 2k-1$, the edge v_sv_t is assigned the distance label $d_C(v_s, v_t)$. Observe that $1 \leq d_C(v_s, v_t) \leq k-1$. Thus, the $2k-1$ edges of C are labeled 1; in general,

then, $2k-1$ edges of K_{2k} are labeled the integer i for $0 \le i \le k-1$. Observe, further, that F_1 contains k edges, one of which is labeled i for $0 \le i \le k-1$. Moreover, when an edge of F_1 labeled i $(0 \le i \le k-1)$ is rotated clockwise through an angle of $2\pi j/(2k-1)$ radians, $0 \le j \le 2k-2$, an edge of F_{j+1} also labeled i is obtained. Hence a 1-factorization of K_{2k} is produced.

The 1-Factorization Conjecture

We saw in Dirac's theorem (Corollary 3.7) that if G is a graph of order $n \ge 3$ such that $\deg v \ge n/2$ for every vertex v of G, then G is Hamiltonian. Consequently, if G is an r-regular graph of even order $n \ge 4$ such that $r \ge n/2$, then G contains a Hamiltonian cycle C. Since C is an even cycle, C can be factored into two 1-factors. If there exists a 1-factorization of $G - E(C)$, then clearly G is 1-factorable. This is certainly the case if $r = 3$. For which values of r and n this can be done is not known; however, there is a conjecture in this connection.

The 1-Factorization Conjecture If G is an r-regular graph of even order n such that $r \ge n/2$, then G is 1-factorable.

The origin of this conjecture is unclear. While its first mention in print appears to be in a 1986 paper of Amanda G. Chetwynd and Anthony J. W. Hilton [42], it is possible that it was known to Dirac as early as the 1950s. If, in fact, the 1-Factorization Conjecture is true, then it cannot be improved. To see this, let H_1 and H_2 be two graphs with $H_i = K_k$, where $k \ge 5$ and k is odd. Let $u_1v_1 \in E(H_1)$ and $u_2v_2 \in E(H_2)$. Furthermore, let

$$H = H_1 + H_2 \text{ and } G = H - u_1v_1 - u_2v_2 + u_1u_2 + v_1v_2.$$

Then G is a $(k-1)$-regular graph of order $2k$. We claim that G is not 1-factorable, for suppose that it is. Let $\mathcal{F}$ be a 1-factorization into $k-1$ (≥ 4) 1-factors. Then there exists a 1-factor $F \in \mathcal{F}$ containing neither u_1u_2 nor v_1v_2. Since H_1 and H_2 have odd orders, some edge of F must join a vertex of H_1 and a vertex of H_2. This is impossible.

Thus it is not true that if G is an r-regular graph of even order n such that $r \ge \frac{n}{2} - 1$, then G is 1-factorable. Hence if the 1-Factorization Conjecture is true, then the resulting theorem is sharp.

2-Factorable Graphs

An obvious necessary condition for a graph G to be 2-factorable is that G is $2k$-regular for some positive integer k. Julius Petersen [171] showed that this condition is sufficient as well as necessary.

Theorem 10.17 *A graph G is 2-factorable if and only if G is $2k$-regular for some positive integer k.*

Proof. We have already noted that if G is a 2-factorable graph, then G is regular of positive even degree. Conversely, suppose that G is $2k$-regular for some integer $k \geq 1$. Assume, without loss of generality, that G is connected. Hence, G is Eulerian and so contains an Eulerian circuit C.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. We define a bipartite graph H with partite sets $U = \{u_1, u_2, \ldots, u_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$, where

$$E(H) = \{u_i w_j : \ v_j \ \text{ immediately follows } v_i \text{ on } C\}.$$

The graph H is k-regular and so, by Theorem 10.15, is 1-factorable. Hence, $\{F_1, F_2, \cdots, F_k\}$ is a 1-factorization of H.

Corresponding to each 1-factor F_ℓ of H is a permutation α_ℓ on the set $\{1, 2, \ldots, n\}$, defined by $\alpha_\ell(i) = j$ if $u_i w_j \in E(F_\ell)$. Let α_ℓ be expressed as a product of disjoint permutation cycles. There is no permutation cycle of length 1 in this product; for if (i) were a permutation cycle, then this would imply that $\alpha_\ell(i) = i$. However, this further implies that $u_i w_i \in E(F_\ell)$ and that $v_i v_i \in E(C)$, which is impossible. Also there is no permutation cycle of length 2 in this product; for if $(i\ j)$ were a permutation cycle, then $\alpha_\ell(i) = j$ and $\alpha_\ell(j) = i$. This would indicate that $u_i w_j, u_j w_i \in E(F_\ell)$ and that v_j both immediately follows and precedes v_i on C, contradicting the fact that no edge is repeated on a circuit. Thus, the length of every permutation cycle in α_ℓ is at least 3.

Each permutation cycle in α_ℓ therefore gives rise to a cycle in G, and the product of the disjoint permutation cycles in α_ℓ produces a collection of mutually disjoint cycles in G containing all vertices of G, that is, a 2-factor in G. Since the 1-factors in H are mutually edge-disjoint, the resulting 2-factors in G are mutually edge-disjoint. Hence, G is 2-factorable. ■

Hamiltonian Factorable Graphs

By Theorem 10.17 then, there exists a factorization of every regular graph G of positive even degree in which every factor is a union of cycles. We next consider the problem of whether there exists a factorization of G such that every factor is a single cycle. A **Hamiltonian factorization** of a graph G is a factorization of G such that every factor is a Hamiltonian cycle of G and a graph possessing such a factorization is **Hamiltonian factorable**. Certainly, every Hamiltonian factorable graph is a 2-connected regular graph of positive even degree. The converse of this statement is not true, however, as the graph of Figure 10.12 shows.

Complete graphs of odd order are not only 2-factorable, they are Hamiltonian factorable.

Theorem 10.18 *For every positive integer k, the graph K_{2k+1} is Hamiltonian factorable.*

Figure 10.12: A 2-factorable graph that is not Hamiltonian factorable

Proof. Since the result is clear for $k = 1$, we may assume that $k \geq 2$. Let $V(K_{2k+1}) = \{v_0, v_1, \ldots, v_{2k}\}$. Arrange the vertices $v_1, v_2, \ldots, v_{2k}$ in a regular $2k$-gon and place v_0 in some convenient position. Join every two vertices by a straight line segment, thereby producing K_{2k+1}. We define the edge set of F_1 to consist of v_0v_1, v_0v_{k+1}, all edges parallel to v_1v_2 and all edges parallel to $v_{2k}v_2$ (see F_1 in Figure 10.13 for the case $k = 3$). In general, for i $= 1, 2, \ldots, k$, we define the edge set of the factor F_i to consist of v_0v_i, v_0v_{k+i}, all edges parallel to v_iv_{i+1} and all edges parallel to $v_{i-1}v_{i+1}$, where the subscripts are expressed modulo $2k$. Since

$$(v_0, v_i, v_{i+1}, v_{i-1}, v_{i+2}, v_{i-2}, \cdots, v_{k+i-1}, v_{k+i+1}, v_{k+i}, v_0)$$

is a Hamiltonian cycle for each i $(1 \leq i \leq k)$, it follows that $\{F_1, F_2, \ldots, F_k\}$ is a Hamiltonian factorization of K_{2k+1}. ■

The Hamiltonian factorization described in the proof of Theorem 10.18 is illustrated in Figure 10.13 for the complete graph K_7.

Figure 10.13: A Hamiltonian factorization of K_7

The factorization described in the proof of Theorem 10.18 is also a cyclic factorization. If we place the vertex v_0 in the center of the regular $2k$-gon and rotate the edges of the Hamiltonian cycle F_1 clockwise through an angle of $2\pi/2k = \pi/k$ radians, then the Hamiltonian cycle F_2 is produced. Indeed, if we rotate the edges of F_1 clockwise through an angle of $\pi j/k$ radians for any integer j with $1 \leq j \leq k-1$, then the Hamiltonian cycle F_{j+1} is produced and the desired Hamiltonian factorization of K_{2k+1} is obtained (see Exercise 5).

Another factorization result now follows readily from Theorem 10.18 (see Exercise 8).

Corollary 10.19 *For each positive integer k, the complete graph K_{2k} can be factored into k Hamiltonian paths.*

Using the construction employed in the proof of Theorem 10.18, we can obtain another factorization result (see Exercise 10).

Theorem 10.20 *For each positive integer k, the graph K_{2k} can be factored into $k-1$ Hamiltonian cycles and a 1-factor.*

Exercises for Section 10.2

1. Show that the Petersen graph is not 1-factorable.

2. Give an example of a connected graph G of composite size having the property that whenever F is a factor of G and the size of F divides the size of G, then G is F-factorable.

3. (a) Prove that Q_n is 1-factorable for all $n \geq 1$.

   (b) Prove that Q_n is k-factorable if and only if $k \mid n$.

4. Use the proof of Theorem 10.17 to give a 2-factorization of the graph of the octahedron (namely $K_{2,2,2}$).

5. Use the proof of Theorem 10.18 to produce a Hamiltonian factorization of K_9.

6. Use Theorems 10.16 and 10.18 to prove the following: Let r and n be integers with $0 \leq r \leq n-1$. Then there exists an r-regular graph of order n if and only if r and n are not both odd.

7. Let k be a nonnegative even integer and $n \geq 5$ an odd integer with $k \leq n-3$. Prove that there exists a graph G of order n, all of whose vertices have degree k or $k+2$.

8. Prove Corollary 10.19: *For each positive integer k, the complete graph K_{2k} can be factored into k Hamiltonian paths.*

9. Show that K_{2k+1} cannot be factored into Hamiltonian paths.

10. Give a constructive proof of Theorem 10.20: *For each positive integer k, the graph K_{2k} can be factored into $k-1$ Hamiltonian cycles and a 1-factor.*

10.3 Decomposition and Graceful Graphs

Very similar to the concept of factorization is decomposition. A **decomposition** $\mathcal{D}$ of a graph G is a collection $\{H_1, H_2, \ldots, H_t\}$ of nonempty subgraphs such that $H_i = G[E_i]$ for some (nonempty) subset E_i of $E(G)$, where $\{E_1, E_2, \ldots, E_t\}$ is a partition of $E(G)$. Thus no subgraph H_i in a decomposition of G contains isolated vertices. If $\mathcal{D}$ is a decomposition of G, then we say G is decomposed into the subgraphs $H_1, H_2, \ldots, H_t$. Indeed, if $\mathcal{D}$ is a decomposition of a graph G of order n and we define $F_i = H_i + (n - |V(H_i)|)\, K_1$ for $1 \leq i \leq t$, then $\{F_1, F_2, \ldots, F_t\}$ is a factorization of G. On the other hand, every factorization of a nonempty graph G also gives rise to a decomposition of G. Suppose that $\{F_1, F_2, \ldots, F_s\}$ is a factorization of a nonempty graph G, so written that $F_1, F_2, \ldots, F_t$, are nonempty ($t \leq s$). Let $H_i = G[E(F_i)]$ for $i = 1, 2, \ldots, t$. Then $\{H_1, H_2, \ldots, H_t\}$ is a decomposition of G.

If $\mathcal{D} = \{H_1, H_2, \ldots, H_t\}$ is a decomposition of a graph G such that $H_i \cong H$ for some graph H for each i ($1 \leq i \leq t$), then $\mathcal{D}$ is an **H-decomposition** of G. If there exists an H-decomposition of a graph G, then G is said to be **H-decomposable**. The graph $G = K_{2,2,2}$ (the graph of the octahedron) is H-decomposable for the graph H shown in Figure 10.14. An H-decomposition of G is also shown in Figure 10.14.

Figure 10.14: An H-decomposable graph

The decomposition shown in Figure 10.14 is a *cyclic* decomposition. In general, a **cyclic decomposition** of a graph G into k copies of a subgraph H is obtained by

(a) drawing G in an appropriate manner,

(b) selecting a suitable subgraph H_1 of G that is isomorphic to H and

(c) rotating the vertices and edges of H_1 through an appropriate angle $k-1$ times to produce the k copies of H in the decomposition.

If G is an H-decomposable graph for some graph H, then certainly H is a subgraph of G and the size of H divides the size of G. Although this last condition is necessary, it is not sufficient. For example, the graph $K_{1,4}$ is a subgraph of the graph G of Figure 10.14 and the size 4 of $K_{1,4}$ divides the size 12 of G, but yet G is not $K_{1,4}$-decomposable (see also Exercise 2).

The basic problem in this context is whether a given graph G is H-decomposable for some subgraph H of G such that the size of H divides the size of G. Of course, every nonempty graph is K_2-decomposable. For a connected graph G to be P_3-decomposable, the size of G must be even. It turns out that this condition is also sufficient for a graph to be P_3-decomposable.

Theorem 10.21 *A nontrivial connected graph G is P_3-decomposable if and only if G has even size.*

Proof. We have already noted that if G is P_3-decomposable, then G has even size. For the converse, assume that G has even size. Suppose first that G is Eulerian, where the edges of G are encountered in the order $e_1, e_2, \ldots, e_m$. Then each of the sets $\{e_1, e_2\}$, $\{e_3, e_4\}$, $\ldots$,$\{e_{m-1}, e_m\}$ induces a copy of P_3; so G is P_3-decomposable. Otherwise, G has $2k$ odd vertices for some $k \geq 1$. By Theorem 3.3, $E(G)$ can be partitioned into subsets $E_1, E_2, \ldots, E_k$, where for each i, $G[E_i]$ is an open trail T_i of even length connecting odd vertices of G. Then, as with the Eulerian circuit above, the edges of each trail T_i can be paired off so that each pair of edges induce a copy of P_3. Thus G is P_3-decomposable. ■

Steiner Triple Systems

The vast majority of factorization and decomposition results deal with factoring or decomposing complete graphs into a specific graph or graphs. A problem of particular interest concerns determining those complete graphs that are K_3-decomposable.

A **Steiner triple system of order** n consists of a set S of n elements and a collection T of 3-element subsets of S, called **triples**, such that every pair of elements of S belongs to exactly one triple in T. For example

$$S = \{0, 1, \ldots, 6\} \text{ and } T = \{013, 124, 235, 346, 450, 561, 602\}$$

is a Steiner triple system of order 7. A Steiner triple system of order n corresponds to a K_3-decomposition of K_n. In particular, the Steiner triple system of order 7 mentioned above results in the cyclic K_3-decomposition of K_7 shown in Figure 10.15.

Figure 10.15: A K_3-decomposition of K_7

Steiner triple systems are named for the Swiss mathematician Jakob Steiner (1796–1863). Although Steiner asked for which integers n such a triple system of order n exists, he was not the first to ask this question. The problem of determining those integers n for which there exists a Steiner triple system of order n was initially posed by the Reverend Wesley S. B. Woolhouse as Prize question 1733 in the *Lady's and Gentlemen's Diary* of 1844. This problem was solved three years later by the Reverend Thomas Penygton Kirkman [134].

Theorem 10.22 *There exists a Steiner triple system of order $n \geq 3$ if and only if $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$.*

Kirkman's Schoolgirl Problem

While the necessity for the existence of a Steiner triple system of order n stated in Theorem 10.22 is quite straightforward (see Exercise 3), the sufficiency requires the construction of appropriate Steiner triple systems. Despite the fact that Kirkman's work preceded Steiner's, these systems are named for Steiner, not Kirkman. Since $7 \equiv 1 \pmod 6$, there is a Steiner triple system of order 7 as we saw above. Furthermore, $9 \equiv 3 \pmod 6$ and so there is a Steiner triple system of order 9. Likewise, there is a Steiner triple system of order 13 and a Steiner triple system of order 15. A Steiner triple system of order 15 must then consist of 35 triples. Also, in the *Lady's and Gentlemen's Diary* of 1850, Kirkman posed an even more challenging question which led to a famous problem.

Kirkman's Schoolgirl Problem : *A school mistress has fifteen schoolgirls whom she wishes to take on a daily walk. The girls are to walk in five rows of three girls each. It is required that every two girls should walk in the same row exactly once a week. Can this be done?*

If we think about this a bit, we see that the question can be rephrased as follows: Is there a $5K_3$-factorization of K_{15}? If we label the vertices of K_{15} by the schoolgirls, numbered $1, 2, \ldots, 15$ say, then we see that a solution is given below.

Sunday	{1, 2, 3}, {4, 8, 12}, {5, 10, 14}, {6, 9, 15}, {7, 11, 13}
Monday	{1, 4, 5}, {2, 9, 11}, {3, 13, 15}, {6, 8, 14}, {7, 10, 12}
Tuesday	{1, 6, 7}, {2, 8, 10}, {3, 12, 14}, {4, 9, 13}, {5, 11, 15}
Wednesday	{1, 8, 9}, {2, 5, 7}, {3, 13, 14}, {4, 10, 15}, {6, 11, 12}
Thursday	{1, 10, 11}, {2, 12, 15}, {3, 4, 6}, {5, 8, 13}, {7, 9, 14}
Friday	{1, 12, 13}, {2, 5, 6}, {3, 9, 10}, {4, 11, 14}, {7, 8, 15}
Saturday	{1, 14, 15}, {2, 4, 7}, {3, 8, 11}, {5, 9, 12}, {6, 10, 13}

Although there is a $5K_3$-factorization of K_{15}, it turns out that there is no cyclic $5K_3$-factorization of K_{15} (which makes it more difficult to construct such a factorization). Suppose that a graph G of order n and size m has a 2-factorization in which every component of each 2-factor is a triangle. Then G is tK_3-factorable for some positive integer t. Therefore, $n = 3t$. Since the degree of every vertex of G is 2 in each 2-factor of G, every vertex has even degree in G. Therefore, $n-1$ is even and so $n = 3t$ is odd, which implies that t is odd. Therefore, $t = 2k+1$ for some nonnegative integer k and so $n = 6k+3$. Hence G is $(2k+1)K_3$-factorable.

A **Kirkman triple system** of order n is an n-element set S, a collection T of triples of S and a partition $\mathcal{P}$ of T such that

(1) every two distinct elements of S belong to a unique triple in T and

(2) every element of S belongs to a unique triple in each element of $\mathcal{P}$.

Consequently, if there is a Kirkman triple system of order n, then $n = 6k+3$ for some nonnegative integer k. In fact, there is a Kirkman triple system of order $6k+3$ if and only if there is a $(2k+1)K_3$-factorization of K_{6k+3}. In 1971 Dijen Ray-Chaudhuri and Richard Wilson [179] established the existence of a Kirkman triple system of order $n = 6k+3$ for every nonnegative integer k.

Theorem 10.23 *There exists a Kirkman triple system of order $n \geq 3$ if and only if $n \equiv 3 \pmod 6$.*

Richard Wilson [241] proved that for every graph H without isolated vertices, there exist infinitely many positive integers n such that K_n is H-decomposable.

Theorem 10.24 *For every graph H without isolated vertices and having size m, there exists a positive integer N such that if (i) $n \geq N$, (ii) $m \mid \binom{n}{2}$ and (iii) $d \mid (n-1)$, where*

$$d = \gcd\{\deg v : \ v \in V(H)\},$$

then K_n is H-decomposable.

Graceful Graphs

A **graph labeling** is an assignment of labels (typically positive integers or nonnegative integers) to the vertices or edges or both of a graph, often satisfying some condition. Many labelings can be traced to a 1967 paper titled "On certain valuations of the vertices of a graph" authored by Alexander Rosa [196]. Rosa received his Ph.D. in 1968 from the Slovak Academy of Science under Anton Kotzig, whom we will encounter shortly. The labeling that has drawn the most attention is Rosa's β-valuation, which was called a graceful labeling by Solomon W. Golomb.

A graph G of size m is **graceful** if it is possible to label the vertices of G with distinct elements from the set $\{0, 1, \ldots, m\}$ in such a way that the induced edge labeling, which prescribes the integer $|i - j|$ to the edge joining vertices labeled i and j, assigns the labels $1, 2, \ldots, m$ to the edges of G. Such a labeling is called a **graceful labeling**. Thus, a graceful graph is a graph that admits a graceful labeling.

All of the graphs K_3, K_4, $K_4 - e$ and C_4 are graceful, as is illustrated in Figure 10.16. Here the vertex labels are placed within the vertices and the induced edge labels are placed near the relevant edges.

Figure 10.16: Graceful graphs

All connected graphs of order at most 4 are graceful. There are exactly three connected graphs of order 5 that are not graceful, namely K_5, C_5 and $K_1 \vee 2K_2$. These graphs are shown in Figure 10.17.

Figure 10.17: The three connected graphs of order 5 that are not graceful

For any graceful graph H of size m, the complete graph K_{2m+1} is H-decomposable. In fact, there is a cyclic decomposition of K_{2m+1} into H. This observation is due to Rosa [196].

Theorem 10.25 *If H is a graceful graph of size m, then the complete graph K_{2m+1} is H-decomposable. In fact, K_{2m+1} can be cyclically decomposed into copies of H.*

Proof. Since H is graceful, there is a graceful labeling of H, that is, the vertices of H can be labeled from a subset of $\{0, 1, \ldots, m\}$ so that the induced edge labels are $1, 2, \ldots, m$. Let $V(K_{2m+1}) = \{v_0, v_1, \ldots, v_{2m}\}$ where the vertices of K_{2m+1} are arranged cyclically in a regular $(2m+1)$-gon, denoting the resulting $(2m+1)$-cycle by C. A vertex labeled i $(0 \le i \le m)$ in H is placed at v_i in K_{2m+1} and this is done for each vertex of H. Every edge of H is drawn as a straight line segment in K_{2m+1}, denoting the resulting copy of H in K_{2m+1} as H_1. Hence $V(H_1) \subseteq \{v_0, v_1, \ldots, v_m\}$.

Each edge $v_s v_t$ of K_{2m+1} $(0 \le s, t \le 2m)$ is labeled $d_C(v_s, v_t)$, where then $1 \le d_C(v_s, v_t) \le m$. Consequently, K_{2m+1} contains exactly $2m+1$ edges labeled i for each i $(1 \le i \le m)$ and H_1 contains exactly one edge labeled i $(1 \le i \le m)$. Whenever an edge of H_1 is rotated through an angle (clockwise, say) of $2\pi k/(2m+1)$ radians, where $1 \le k \le 2m$, an edge of the same label is obtained. Denote the subgraph obtained by rotating H_1 through a clockwise angle of $2\pi k/(2m+1)$ radians by H_{k+1}. Then $H_{k+1} = H$ and a cyclic decomposition of K_{2m+1} into $2m+1$ copies of H results. ■

As an illustration of Theorem 10.25, we consider the graceful graph $H = P_3$. A graceful labeling of H is shown in Figure 10.18 as well as the resulting cyclic H-decomposition of K_5.

Figure 10.18: A cyclic decomposition of K_5 into the graceful graph P_3

Although K_{2m+1} has a cyclic decomposition into every graceful graph H of size m, it is not necessary for H to be graceful in order for K_{2m+1} to have

a cyclic H-decomposition. For example, we have seen that C_5 is not graceful; yet K_{11} has a cyclic C_5-decomposition. Such a decomposition is depicted in Figure 10.19.

Figure 10.19: A cyclic decomposition of K_{11} into the nongraceful graph C_5

The Graceful Tree Conjecture

It has been conjectured by Anton Kotzig that every nontrivial tree is graceful.

The Graceful Tree Conjecture *Every nontrivial tree is graceful.*

If the Graceful Tree Conjecture is true and T is a tree of size m, then by Theorem 10.25, K_{2m+1} is T-decomposable. Regardless of whether every tree is graceful, Gerhard Ringel [187] made the following conjecture concerning decompositions of complete graphs into trees.

Ringel's Conjecture *For every tree T of size m, the complete graph K_{2m+1} is T-decomposable.*

Not only does the truth of Kotzig's conjecture imply the truth of Ringel's conjecture, the truth of Kotzig's conjecture also implies the truth of the following conjecture, due jointly to Ringel and Kotzig.

The Ringel-Kotzig Conjecture *For every tree T of size m, K_{2m+1} can be cyclically decomposed into T.*

Although there are no general sufficient conditions for a graph to be graceful, there are necessary conditions.

Theorem 10.26 *If G is a graceful graph of size m, then there exists a partition of $V(G)$ into two subsets V_e and V_o such that the number of edges joining V_e and V_o is $\lceil m/2 \rceil$.*

Proof. Let a graceful labeling of G be given. Denote the set of vertices labeled with an even integer by V_e and the set of vertices labeled with an odd integer by V_o. All edges labeled with an odd integer must then join a vertex of V_e and a vertex of V_o. Since there are $\lceil m/2 \rceil$ such edges, the result follows. ■

A necessary condition for an Eulerian graph to be graceful was discovered by Alexander Rosa [196].

Theorem 10.27 *If G is a graceful Eulerian graph of size m, then*

$$m \equiv 0 \pmod 4 \text{ or } m \equiv 3 \pmod 4.$$

Proof. Let $C = (v_0, v_1, \ldots, v_{m-1}, v_m = v_0)$ be an Eulerian circuit of G, and let a graceful labeling of G be given that assigns the integer a_i $(0 \le a_i \le m)$ to v_i for $0 \le i \le m$, where, of course, $a_i = a_j$ if $v_i = v_j$. Thus the label of the edge $v_{i-1}v_i$ is $|a_i - a_{i-1}|$. Observe that

$$|a_i - a_{i-1}| \equiv (a_i - a_{i-1}) \pmod 2$$

for $1 \le i \le m$. Thus the sum of the labels of the edges of G is

$$\sum_{i=1}^{m} |a_i - a_{i-1}| \equiv \sum_{i=1}^{m} (a_i - a_{i-1}) \equiv 0 \pmod 2,$$

that is, the sum of the edge labels of G is even. However, the sum of the edge labels is

$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2};$$

so $m(m+1)/2$ is even. Consequently, $4 \mid m(m+1)$, which implies that $4 \mid m$ or $4 \mid (m+1)$ so that $m \equiv 0 \pmod 4$ or $m \equiv 3 \pmod 4$. ■

Classes of Graceful Graphs

We now determine which graphs belonging to some well-known classes of graphs are graceful. Rosa [196] determined the graceful cycles.

Theorem 10.28 *The cycle C_n is graceful if and only if*

$$n \equiv 0 \pmod 4 \text{ or } n \equiv 3 \pmod 4.$$

Proof. Since C_n is an Eulerian graph, it follows by Theorem 10.27 that if $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$, then C_n is not graceful; so it remains only to show that if $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$, then C_n is graceful. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Assume first that $n \equiv 0 \pmod 4$. We assign v_i the label a_i, where

$$a_i = \begin{cases} \frac{i-1}{2} & \text{if } i \text{ is odd} \\ n+1-\frac{i}{2} & \text{if } i \text{ is even and } i \le n/2 \\ n-\frac{i}{2} & \text{if } i \text{ is even and } i > n/2. \end{cases}$$

It remains to observe that this labeling is graceful. This labeling is illustrated in Figure 10.20(a) for $n = 12$.

Next, assume that $n \equiv 3 \pmod 4$. In this case we assign v_i the label b_i, where

$$b_i = \begin{cases} n+1-\frac{i}{2} & \text{if } i \text{ is even} \\ \frac{i-1}{2} & \text{if } i \text{ is odd and } i < (n-1)/2 \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } i > (n-1)/2. \end{cases}$$

This is a graceful labeling of C_n. An illustration is given in Figure 10.20(b) for $n = 11$. ■

Figure 10.20: Graceful labelings of C_{12} and C_{11}

If G is a graceful graph of order n and size m, then, of course, the vertices of G can be labeled with the elements of a set $\{a_1, a_2, \ldots, a_n\} \subseteq \{0, 1, \ldots, m\}$ so that the induced edge labels are precisely $1, 2, \ldots, m$. This means one vertex in some pairs of adjacent vertices is labeled 0 and the other vertex in the pair is labeled m. Also, if we were to replace each vertex label a_i by $m - a_i$, then we have a new graceful labeling, called the **complementary labeling** of G.

We saw in Figure 10.16 that the complete graphs K_3 and K_4 are graceful. It is very easy to show that K_2 is graceful. The following result of Golomb [96] shows that there are no other graceful complete graphs.

Theorem 10.29 *The complete graph K_n $(n \geq 2)$ is graceful if and only if $n \leq 4$.*

Proof. We have already observed that K_n is graceful if $2 \leq n \leq 4$. Assume then that $n \geq 5$ and suppose, to the contrary, that K_n is graceful. Hence there exists a graceful labeling of the vertices of K_n from an n-element subset of $\{0, 1, \ldots, m\}$, where $m = \binom{n}{2}$.

We have already seen that every graceful labeling of a graph of size m requires 0 and m to be vertex labels. Since some edge of K_n must be labeled $m-1$, some vertex of K_n must be labeled 1 or $m-1$. We may assume, without loss of generality, that a vertex of K_n is labeled 1; otherwise, we may use the complementary labeling.

To produce an edge labeled $m-2$, we must have adjacent vertices labeled 0, $m-2$ or 1, $m-1$ or 2, m. If a vertex is labeled 2 or $m-1$, then we have two edges labeled 1, which is impossible. Thus, some vertex of K_n must be labeled $m-2$.

Since we now have vertices labeled $0, 1, m-2$ and m, we have edges labeled $1, 2, m-3, m-2, m-1$ and m. To have an edge labeled $m-4$, we must have a vertex labeled 4 because all other choices result in two edges with the same label.

Now we have vertices labeled $0, 1, 4, m-2$ and m, which results in edges labeled $1, 2, 3, 4, m-6, m-4, m-2, m-1$ and m. However, it is quickly seen that there is no vertex label that will produce the edge label $m-5$ without also producing a duplicate edge label. Hence no graceful labeling of K_n exists. ■

Unlike the classes of graphs we have considered, every complete bipartite graph is graceful.

Theorem 10.30 *Every complete bipartite graph is graceful.*

Proof. Let $K_{s,t}$ have partite sets U and W, where $|U| = s$ and $|W| = t$. Label the vertices of U with $0, 1, \ldots, s-1$ and label the vertices of W by $s, 2s, \ldots, (t-1)s, ts$ (see Figure 10.21). This is a graceful labeling. ■

Exercises for Section 10.3

1. Show that every planar graph can be decomposed into two bipartite graphs.

2. Show that the graph of the octahedron is not $K_{1,4}$-decomposable.

Figure 10.21: A graceful labeling of $K_{s,t}$

3. Prove that if there exists a Steiner triple system of order $n \geq 3$, then either $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$.

4. Prove that nine schoolgirls can take four daily walks in three rows of three girls each so that no two girls walk in the same row twice.

5. (a) Use the fact that K_3 is graceful to find a K_3-decomposition of K_7.
   (b) Find a noncomplete regular connected K_3-decomposable graph.

6. Find an F-decomposition of K_{12} where $F = 2P_2 + 2P_3$.

7. Find a P_6-decomposition of K_{10}.

8. For each integer $k \geq 1$, show that
   (a) K_{2k+1} is $K_{1,k}$-decomposable.
   (b) K_{2k} is $K_{1,k}$-decomposable.

9. Let m be an even integer and let G be a graceful graph of size m. Show that K_{3m+1} can be decomposed into $3m+1$ copies of G and an m-regular graph of order $3m+1$.

10. (a) Use the drawing of the Petersen graph shown in Figure 10.22 to find cyclic F_i-decompositions for $i = 1, 2, 3$.

Figure 10.22: The Petersen graph and graphs F_1, F_2 and F_3 in Exercise 10

   (b) Does there exist a decomposition $\{F_1, F_2, F_3\}$ of the Petersen graph?

11. Find all graphs F of size 3 that are subgraphs of the Petersen graph P for which P is F-decomposable. (Hint: Use the drawing of the Petersen graph shown in Exercise 10.)

12. Determine graceful labelings of C_{15} and C_{16}.

13. By Theorem 10.28, the cycle C_6 is not graceful. Show, however, that there is a cyclic C_6-decomposition of K_{13}.

14. (a) By Theorem 10.28, the graph $G = C_9$ is not graceful. Thus there is no labeling of the vertices of G with distinct elements from the set $\{0, 1, \ldots, 9\}$ in such a way that the induced edge labeling (which prescribes the integer $|i-j|$ to the edge joining vertices labeled i and j) assigns distinct labels from the set $\{1, 2, \ldots, 9\}$ to the edges of G. Show, however, that there is a labeling of the vertices of G with distinct elements of the set $\{0, 1, \ldots, 18\}$ such that the set of edge labels of G produced by the induced edge labeling has a nonempty intersection with each of the sets $S_1 = \{1, 18\}$, $S_2 = \{2, 17\}$, $\ldots$, $S_9 = \{9, 10\}$.

    (b) What information about G can be obtained from (a)?

    (c) From (a) and (b), what more general information can be obtained?

15. Determine graceful labelings of P_6, P_7, P_9 and P_{10}.

16. Show that the following classes of trees are graceful:

    (a) double stars, (b) caterpillars.

17. (a) Show that no disconnected forest is graceful.

    (b) Give an example of a disconnected graph without isolated vertices having order n and size m with $m \geq n - 1$ that is not graceful.

18. Prove that for every integer $k \geq 2$, there exists a graceful graph having chromatic number k.

19. Let G be a graceful graph of order n and size m where there is a graceful labeling of G that assigns the labels $a_1, a_2, \ldots, a_n$ to the vertices of G. Show that the complementary labeling that replaces the vertex label a_i by $m - a_i$ is also a graceful labeling of G.

10.4 Domination

An area of graph theory that has received increased attention during recent decades is that of domination in graphs. A vertex v in a graph G is said to **dominate** itself and each of its neighbors, that is, v dominates the vertices in its closed neighborhood $N[v]$. A set S of vertices of G is a **dominating set** of G if every vertex of G is dominated by at least one vertex of S. Equivalently, a set S

of vertices of G is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality among the dominating sets of G is called the **domination number** of G and is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is then referred to as a **minimum dominating set**. Although $\gamma(G)$ is the same notation that is used for the genus of a graph G, the notation in both instances is common and we will never use the terms domination number and the genus of a graph in the same discussion.

The sets $S_1 = \{v_1, v_2, y_1, y_2\}$ and $S_2 = \{w_1, w_2, x\}$ are both dominating sets in the graph G of Figure 10.23, indicated by solid circles. Since S_2 is a dominating set of minimum cardinality, $\gamma(G) = 3$.

Figure 10.23: Dominating sets

Dominating sets appear to have their origins in the game of chess, where the goal is to cover or dominate various squares of a chessboard by certain chess pieces. In 1862 Carl Friedrich de Jaenisch [54] considered the problem of determining the minimum number of queens (which can move either horizontally, vertically or diagonally over any number of unoccupied squares) that can be placed on a chessboard such that every square is either occupied by a queen or can be occupied by one of the queens in a single move. The minimum number of such queens is 5 and one possible placement of five such queens is shown in Figure 10.24.

Two queens on a chessboard are **attacking** if the square occupied by one of the queens can be reached by the other queen in a single move; otherwise, they are **nonattacking queens**. Clearly, every pair of queens on the chessboard of Figure 10.24 are attacking. The minimum number of nonattacking queens such that every square of the chessboard can be reached by one of the queens is also 5. A possible placement of five nonattacking queens is shown in Figure 10.25.

The connection between the chessboard problem described above and dominating sets in graphs is immediate. The 64 squares of a chessboard are the vertices of a graph G and two vertices (squares) are adjacent in G if each of the two squares can be reached by a queen on the other square in a single move. The graph G is referred to as the **queen's graph**. Then the minimum number of queens that dominate all the squares of a chessboard is $\gamma(G)$. The minimum number of nonattacking queens that dominate all the squares of a chessboard is the minimum cardinality of an independent dominating set in G.

Q							
		Q					
				Q			
					Q		
						Q	

Figure 10.24: The minimum number of queens that dominate the squares of a chessboard

Domination as a theoretical area in graph theory was formalized by Claude Berge in 1958 [15, p. 40] and Oystein Ore [167, Chapter 13] in 1962. Since 1977, when Ernest Cockayne and Stephen Hedetniemi [50] presented a survey of domination results, domination theory has received considerable attention.

Minimal Dominating Sets

A **minimal dominating set** in a graph G is a dominating set that contains no dominating set as a proper subset. A minimal dominating set of minimum cardinality is, of course, a minimum dominating set and consists of $\gamma(G)$ vertices. For the graph G of Figure 10.23, the set $S_1 = \{v_1, v_2, y_1, y_2\}$ is a minimal dominating set that is not a minimum dominating set. Minimal dominating sets were characterized by Ore [167, p. 206].

Theorem 10.31 *A dominating set S of a graph G is a minimal dominating set of G if and only if every vertex v in S satisfies at least one of the following two properties:*

(i) there exists a vertex w in $V(G) - S$ such that $N(w) \cap S = \{v\}$;

(ii) v is adjacent to no vertex of S.

Proof. First, observe that if each vertex v in S has at least one of the properties (i) and (ii), then $S - \{v\}$ is not a dominating set of G. Consequently, S is a minimal dominating set of G.

Conversely, assume that S is a minimal dominating set of G. Then certainly for each $v \in S$, the set $S - \{v\}$ is not a dominating set of G. Hence there is a vertex w in $V(G) - (S - \{v\})$ that is adjacent to no vertex of $S - \{v\}$. If $w = v$, then v is adjacent to no vertex of S. Suppose then that $w \neq v$. Since

Figure 10.25: The minimum number of nonattacking queens that dominate the squares of a chessboard

S is a dominating set of G and $w \notin S$, the vertex w is adjacent to at least one vertex of S. However, w is adjacent to no vertex of $S - \{v\}$. Consequently, $N(w) \cap S = \{v\}$. ■

Theorem 10.31 can be reworded as follows: A dominating set S of a graph G is a minimal dominating set of G if and only if for every vertex v in S either (1) v dominates some vertex of $V(G) - S$ that no other vertex of S dominates or (2) no other vertex of S dominates v.

The following result of Ore [167, p. 207] gives a property of the complementary set of a minimal dominating set in a graph without isolated vertices.

Theorem 10.32 *If S is a minimal dominating set of a graph G without isolated vertices, then $V(G) - S$ is a dominating set of G.*

Proof. Let $v \in S$. Then v has at least one of the two properties (i) and (ii) described in the statement of Theorem 10.31. Suppose first that there exists a vertex w in $V(G) - S$ such that $N(w) \cap S = \{v\}$. Hence v is adjacent to some vertex in $V(G) - S$. Suppose next that v is adjacent to no vertex in S. Then v is an isolated vertex of the subgraph $G[S]$. Since v is not isolated in G, the vertex v is adjacent to some vertex of $V(G) - S$. Thus $V(G) - S$ is a dominating set of G. ■

For graphs G without isolated vertices, we now have an upper bound for $\gamma(G)$ in terms of the order of G.

Corollary 10.33 *If G is a graph of order n without isolated vertices, then $\gamma(G) \leq n/2$.*

Proof. Let S be a minimal dominating set of G. By Theorem 10.32, $V(G) - S$ is a dominating set of G. Thus

$$\gamma(G) \leq \min\{|S|, |V(G) - S|\} \leq n/2,$$

as desired. ■

Nearly all connected graphs attaining the bound in Corollary 10.33 can be produced with the aid of the following operation. The **corona** $cor(H)$ of a graph H is that graph obtained from H by adding a pendant edge to each vertex of H. Let $G = cor(H)$, where G has order n. Then G has no isolated vertices and $\gamma(G) = n/2$. Indeed, Charles Payan and Nguyen Huy Xuong [170] showed that every component of a graph G of order n without isolated vertices having $\gamma(G) = n/2$ is either C_4 or the corona of some (connected) graph.

Béla Bollobás and Ernest Cockayne [24] showed that every graph without isolated vertices contains a minimum dominating set in which every vertex satisfies property (i) described in the statement of Theorem 10.31.

Theorem 10.34 *Every graph G without isolated vertices contains a minimum dominating set S such that for every vertex v of S, there exists a vertex w of $G - S$ such that $N(w) \cap S = \{v\}$.*

Proof. Among all minimum dominating sets of G, let S be one such that $G[S]$ has maximum size. Suppose, to the contrary, that S contains a vertex v that does not have the desired property. Then by Theorem 10.31, v is an isolated vertex in $G[S]$. Moreover, every vertex of $V(G) - S$ that is adjacent to v is adjacent to some other vertex of S as well. Since G contains no isolated vertices, v is adjacent to a vertex w in $V(G) - S$. Consequently, $(S - \{v\}) \cup \{w\}$ is a minimum dominating set of G whose induced subgraph contains at least one edge incident with w and hence has a greater size than $G[S]$. This produces a contradiction. ■

The domination number of a graph without isolated vertices is also bounded above by all of the independence and covering numbers.

Theorem 10.35 *If G is a graph without isolated vertices, then*

$$\gamma(G) \leq \min\{\alpha(G), \alpha'(G), \beta(G), \beta'(G)\}.$$

Proof. Since every vertex cover of a graph without isolated vertices is a dominating set, as is every maximum independent set of vertices, $\gamma(G) \leq \beta(G)$ and $\gamma(G) \leq \alpha(G)$. Let X be an edge cover of cardinality $\beta'(G)$. Then every vertex of G is incident with at least one edge in X.

Let S be a set of vertices, obtained by selecting an incident vertex with each edge in X. Then S is a dominating set of vertices and $\gamma(G) \leq |S| \leq |X| = \beta'(G)$.

Next, let M be a maximum matching in G. We construct a set S of vertices consisting of one vertex incident with an edge of M for each edge of M. Let $uv \in M$. The vertices u and v cannot be adjacent to distinct unmatched vertices x and y, respectively; for otherwise, (x, u, v, y) is an M-augmenting path in G, contradicting Theorem 10.2. If u is adjacent to a unmatched vertex, place u

in S; otherwise, place v in S. This is done for each edge of M. Thus, S is a dominating set of G, and $\gamma(G) \leq |S| = |M| = \alpha'(G)$. ■

Vadim Vizing conjectured that the domination number of the Cartesian product of two graphs is always at least as large as the product of the domination numbers of these two graphs.

Vizing's Conjecture *For every two graphs G and H,*

$$\gamma(G \,\square\, H) \geq \gamma(G) \cdot \gamma(H).$$

Independent Dominating Sets

Recall that an independent set S of vertices of a graph G is a maximal independent set if S is not a proper subset of any independent set in G. Thus every maximum independent set is maximal. The minimum cardinality of a maximal independent set of vertices of G is denoted by $i(G)$. Thus, for $K_{s,t}$, where $s < t$, there are only two maximal independent sets of vertices, namely, the partite sets of $K_{s,t}$. Hence, $\alpha(K_{s,t}) = t$ while $i(K_{s,t}) = s$.

Since every maximal independent set of vertices in a graph G is a dominating set of G, it follows that $\gamma(G) \leq i(G)$. Not every dominating set is independent, however. Indeed, not every minimum dominating set is independent. For example in the graph G of Figure 10.26, the set $S_1 = \{u_1, u_2, v_1, v_2, w_1, w_2\}$ is a maximal independent set (and consequently a dominating set) of G; while $S_2 = \{x, y, z\}$ is a minimum dominating set of G and certainly S_2 is not independent. (These sets are indicated in Figure 10.26 by solid circles.) However, G does contain a minimum dominating set of G that is independent, namely, $S_3 = \{u, v, w\}$.

A set S of vertices in a graph G is an **independent dominating set** of G if S is both an independent and a dominating set of G. Thus the sets S_1 and S_3 in Figure 10.26 are independent dominating sets while S_2 is not. The **independent domination number** $i(G)$ of G is the minimum cardinality among all independent dominating sets of G. That this is precisely the notation used for the minimum cardinality of a maximal independent set of vertices in a graph is justified by the following observation of Berge [18].

Theorem 10.36 *A set S of vertices in a graph is an independent dominating set if and only if S is maximal independent.*

Proof. We have already noted that every maximal independent set of vertices is a dominating set. Conversely, suppose that S is an independent dominating set. Then S is independent and every vertex not in S is adjacent to a vertex of S, that is, S is maximal independent. ■

Another observation now follows.

Figure 10.26: Dominating sets and maximal independent sets

Corollary 10.37 *Every maximal independent set of vertices in a graph is a minimal dominating set.*

Proof. Let S be a maximal independent set of vertices in a graph G. By Theorem 10.36, S is a dominating set. Since S is independent, certainly every vertex of S is adjacent to no vertex of S. Thus, every vertex of S satisfies property (ii) of Theorem 10.31. So, by Theorem 10.31, S is a minimal dominating set. ■

We have seen that $\gamma(G) \leq i(G)$ for every graph G. Indeed, equality can hold as $\gamma(K_{1,t}) = i(K_{1,t}) = 1$ for every positive integer t. For $1 \leq s < t$, let H be the graph obtained from $K_{s,t}$ by adding a pendant edge to each vertex of the partite set of cardinality s. Then $\gamma(H) = i(H) = s$. For the queen's graph G, we also have $\gamma(G) = i(G) = 5$, as Figures 10.24 and 10.25 show. That the difference between the independent domination number and domination number of a graph can be arbitrarily large can be seen in the double star T containing two vertices of degree $k \geq 2$, where $i(T) = k$ and $\gamma(T) = 2$.

Exercises for Section 10.4

1. Determine the domination numbers of the 3-cube Q_3 and the 4-cube Q_4.

2. (a) Determine and verify a formula for $\gamma(C_n)$.

(b) Determine and verify a formula for $\gamma(P_n)$.

3. State and prove a characterization of those graphs G with $\gamma(G) = 1$.

4. Prove that if G is a graph of order n, then $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$.

5. Prove that if the diameter of a connected graph G of order n is at least 3, then $\gamma(\overline{G}) = 2$.

6. (a) Does there exist a graph G without isolated vertices such that $\gamma(G) = \beta(G)$ but $\gamma(G)$ is strictly less than each of the numbers $\alpha(G)$, $\alpha'(G)$ and $\beta'(G)$?

   (b) The question in (a) suggests three other questions. State and answer these questions.

7. Use Theorem 10.35 to give an alternative proof of Corollary 10.33: *If G is a graph of order n without isolated vertices, then $\gamma(G) \leq n/2$.*

8. Show that if G is a graph of order $n \geq 2$, then $3 \leq \gamma(G) + \gamma(\overline{G}) \leq n + 1$.

9. (a) Prove or disprove: If a graph G contains a minimum dominating set that is also independent, then $\gamma(G) = i(G)$.

   (b) Show that a graph need not have any minimum dominating set that is independent.

10. (a) Prove that if G is a connected bipartite graph of order n, then $i(G) \leq n/2$.

    (b) Show that the bound given in (a) is sharp.

11. For each integer $k \geq 3$, show that there exists a graph G such that $i(G) = k$ and $\gamma(G) = 3$.

12. (a) Prove that if G is a graph of order n without isolated vertices, then $\gamma(G) + i(G) \leq n$.

    (b) Investigate the sharpness of the bound given in (a).

13. A set S of vertices of a graph G is a **total dominating set** of G if for every vertex v of G, there is a vertex $u \in S$ such that $v \in N(u)$. The **total domination number** $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G.

    (a) Prove for every nontrivial connected graph G, $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$.

    (b) Show that the bounds in (a) are sharp by showing that there are two infinite classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of graphs such that (1) for each $H \in \mathcal{C}_1$, $\gamma(H) = \gamma_t(H)$ and (2) for each $F \in \mathcal{C}_2$, $\gamma_t(F) = 2\gamma(F)$.

Chapter 11

Edge Colorings

The graph colorings that we have considered are vertex colorings in Chapter 8 and map colorings (region colorings of plane graphs) in Chapter 9. With the aid of matchings and 1-factors in Chapter 10, we now consider a third coloring, namely edge colorings. As with vertex colorings where the primary emphasis has been on proper vertex colorings, the customary requirement for edge colorings is that adjacent edges be colored differently, resulting in proper edge colorings. This too will be our focus in the current chapter.

11.1 Chromatic Index and Vizing's Theorem

An **edge coloring** of a graph G is an assignment of colors to the edges of G, one color to each edge. If adjacent edges are assigned distinct colors, then the edge coloring is a **proper edge coloring**. Proper edge colorings are the most studied edge coloring and it is these colorings that we investigate in this chapter. When we refer to an edge coloring of a graph, we will mean a proper edge coloring unless stated otherwise.

Since a proper edge coloring of a nonempty graph G is a proper vertex coloring of its line graph $L(G)$, edge colorings of graphs is the same subject as vertex colorings of line graphs. Because investigating vertex colorings of line graphs provides no apparent advantage to investigating edge colorings of graphs, we will study this subject strictly in terms of edge colorings.

An edge coloring that uses colors from a set of k colors is a **k-edge coloring**. Thus a k-edge coloring of a graph G can be described as a function $c : E(G) \to \{1, 2, \ldots, k\}$ such that $c(e) \neq c(f)$ for every two adjacent edges e and f in G. A graph G is **k-edge colorable** if there exists a k-edge coloring of G. In Figure 11.1(a), a 5-edge coloring of a graph H is shown; while in Figures 11.1(b) and 11.1(c), a 4-edge coloring and a 3-edge coloring of H are shown.

As with vertex colorings, we are often interested in edge colorings using a minimum number of colors. The **chromatic index** (or **edge chromatic**

Figure 11.1: Edge colorings of a graph

number) $\chi'(G)$ of a graph G is the minimum positive integer k for which G is k-edge colorable. Furthermore, $\chi'(G) = \chi(L(G))$ for every nonempty graph G.

If a graph G is k-edge colorable for some positive integer k, then $\chi'(G) \leq k$. In particular, since the graph H of Figure 11.1 is 3-edge colorable, $\chi'(H) \leq 3$. On the other hand, since H contains three mutually adjacent edges (indeed, several such sets of three edges), at least three distinct colors are required in any edge coloring of H and so $\chi'(H) \geq 3$. Therefore, $\chi'(H) = 3$.

Let there be given a k-edge coloring of a nonempty graph G using the colors $1, 2, \ldots, k$ and let E_i $(1 \leq i \leq k)$ be the set of edges of G assigned the color i. Then the nonempty sets among $E_1, E_2, \ldots, E_k$ of $E(G)$ are the **edge color classes** of G for the given k-edge coloring. Thus the nonempty sets in $\{E_1, E_2, \ldots, E_k\}$ produce a partition of $E(G)$ into edge color classes. Since no two adjacent edges of G are assigned the same color in a (proper) edge coloring of G, every nonempty edge color class consists of an independent set of edges of G. Indeed, the chromatic index of G is the minimum number of independent sets of edges into which $E(G)$ can be partitioned.

Recall that the edge independence number $\alpha'(G)$ of a nonempty graph G is the maximum number of edges in an independent set of edges of G, that is, the size of a maximum matching in G. Furthermore, if the order of G is n, then $\alpha'(G) \leq n/2$. The following gives a simple yet useful lower bound for the chromatic index of a graph and is an analogue to the lower bound for the chromatic number of a graph presented in Theorem 8.9.

Theorem 11.1 *If G is a graph of size $m \geq 1$, then*

$$\chi'(G) \geq \frac{m}{\alpha'(G)}.$$

Proof. Suppose that $\chi'(G) = k$ and that $E_1, E_2, \ldots, E_k$ are the edge color classes in a k-edge coloring of G. Thus $|E_i| \leq \alpha'(G)$ for each i $(1 \leq i \leq k)$. Hence

$$m = |E(G)| = \sum_{i=1}^{k} |E_i| \leq k\alpha'(G)$$

and so $\chi'(G) = k \geq \frac{m}{\alpha'(G)}$. ■

Since every edge coloring of a graph G must assign distinct colors to adjacent edges, for each vertex v of G it follows that $\deg v$ colors must be used to color the edges incident with v in G. Therefore,

$$\chi'(G) \geq \Delta(G) \tag{11.1}$$

for every nonempty graph G.

In the graph G of order $n = 7$ and size $m = 10$ of Figure 11.2, $\Delta(G) = 3$. Hence by (11.1), $\chi'(G) \geq 3$. On the other hand, $X = \{uz, vx, wy\}$ is an independent set of three edges of G and so $\alpha'(G) \geq 3$. Because, $\alpha'(G) \leq n/2 = 7/2$, it follows that $\alpha'(G) = 3$. By Theorem 11.1, $\chi'(G) \geq m/\alpha'(G) = 10/3$ and so $\chi'(G) \geq 4$. The 4-edge coloring of G in Figure 11.2 shows that $\chi'(G) \leq 4$ and so $\chi'(G) = 4$.

Figure 11.2: A graph with chromatic index 4

Vizing's Theorem

While $\Delta(G)$ is a rather obvious lower bound for the chromatic index of a nonempty graph G, the graph G of Figure 11.2 shows that it's possible for $\chi'(G) > \Delta(G)$. The Russian graph theorist Vadim G. Vizing [228] established a remarkable upper bound for the chromatic index of a graph. Vizing's theorem, published in 1964, must be considered the major theorem in the area of edge colorings. Vizing's theorem was rediscovered in 1966 by Ram Prakash Gupta [100].

Theorem 11.2 (**Vizing's Theorem**) *For every nonempty graph G,*

$$\chi'(G) \leq 1 + \Delta(G).$$

Proof. Suppose that the theorem is false. Then among all those graphs H for which $\chi'(H) \geq 2 + \Delta(H)$, let G be one of minimum size and let $\Delta = \Delta(G)$.

Thus G is not $(1+\Delta)$-edge colorable. On the other hand, if $e = uv$ is an edge of G, then $G - e$ is $(1 + \Delta(G - e))$-edge colorable. Since $\Delta(G - e) \leq \Delta(G)$, the graph $G - e$ is $(1 + \Delta)$-edge colorable.

Let there be given a $(1+\Delta)$-edge coloring of $G-e$. Hence, with the exception of e, every edge of G is assigned one of $1 + \Delta$ colors such that adjacent edges are colored differently. For each edge $e' = uv'$ of G incident with u (including the edge e), we define the **dual color** of e' as any of the $1+\Delta$ colors that is not used to color the edges incident with v'. (See Figure 11.3.) Since $\deg v' \leq \Delta$, there is always at least one available color for the dual color of the edge uv'. It may occur that distinct edges have the same dual color.

Figure 11.3: A step in the proof of Theorem 11.2

Denote the edge e by $e_0 = uv_0$ as well (where then $v_0 = v$) and suppose that e_0 has dual color α_1. (Thus α_1 is not the color of any edge incident with v.) Necessarily, some edge $e_1 = uv_1$ incident with u is colored α_1, for otherwise the color α_1 could be assigned to e, producing a $(1 + \Delta)$-edge coloring of G.

Let α_2 be the dual color of e_1. (Thus no edge incident with v_1 is colored α_2.) If there should be some edge incident with u that is colored α_2, then denote this edge by $e_2 = uv_2$ and let its dual color be denoted by α_3. (See Figure 11.4.) Proceeding in this manner, we then construct a sequence $e_0, e_1, \ldots, e_k$ $(k \geq 1)$ containing a maximum number of distinct edges, where $e_i = uv_i$ for $0 \leq i \leq k$. Consequently, the final edge e_k of this sequence is colored α_k and has dual color α_{k+1}. Therefore, each edge e_i $(0 \leq i \leq k)$ is colored α_i and has dual color α_{i+1}.

We claim that there is some edge incident with u that is colored α_{k+1}. Suppose that this is not the case. Then each of the edges $e_0, e_1, \ldots, e_k$ can be assigned its dual color, producing a $(1 + \Delta)$-edge coloring of G. This, however, is impossible. Thus, as claimed, there is an edge e_{k+1} incident with u that is colored α_{k+1}.

Since the sequence $e_0, e_1, \ldots, e_k$ contains the maximum number of distinct edges, it follows that $e_{k+1} = e_j$ for some j with $1 \leq j \leq k$ and so $\alpha_{k+1} = \alpha_j$. Since the color assigned to e_k is not the same as its dual color, it follows that $\alpha_{k+1} \neq \alpha_k$. Therefore, $1 \leq j < k$. Let $j = t + 1$ for $0 \leq t < k - 1$. Hence $\alpha_{k+1} = \alpha_{t+1}$ and so e_k and e_t have the same dual color.

There must be a color β used to color an edge incident with v in $G - e$ that is not used to color any edge incident with u. If this were not the case, then there would be $\deg_G u - 1 \leq \Delta - 1$ colors used to color the edges incident with u

Figure 11.4: A step in the proof of Theorem 11.2

or v, leaving two or more colors available for e. Assigning e one of these colors produces a $(1+\Delta)$-edge coloring of G, resulting in a contradiction.

The color β must also be assigned to some edge incident with v_i for each i with $1 \le i \le k$. If this were not the case, then there would exist a vertex v_r with $1 \le r \le k$ such that no edge incident with v_r is colored β. However, we could then change the color of e_r to β and color each edge e_i $(0 \le i < r)$ with its dual color to obtain a $(1+\Delta)$-edge coloring of G, which is impossible.

Let P be a path of maximum length with initial vertex v_k whose edges are alternately colored β and α_{k+1}, and let Q be a path of maximum length with initial vertex v_t whose edges are alternately colored β and $\alpha_{t+1} = \alpha_{k+1}$. Suppose that P is a v_k–x path and Q is a v_t–y path. We now consider four cases depending on whether the vertices x and y belong to the set $\{v_0, v_1, \ldots, v_{k-1}, u\}$.

Case 1. *$x = v_r$ for some integer r with* $0 \le r \le k-1$. Since α_{k+1} is the dual color of e_k, no edge incident with v_k is colored α_{k+1} and so the initial edge of P must be colored β. We have seen that for every integer i with $0 \le i \le k$, there is an edge incident with v_i that is colored β. Because of the defining property of P, the color of the terminal edge of P cannot be α_{k+1}. This implies that no edge incident with v_r is colored α_{k+1} and so both the initial and terminal edges of P are colored β. Unless $v_r = v_t$, the vertex v_t is not on P as no edge incident with v_t is colored α_{k+1}.

We now interchange the colors β and α_{k+1} of the edges of P. If $r = 0$, then e can be colored β; otherwise, $r > 0$ and no edge incident with v_r is colored β and the dual color of e_i with $0 \le i < r$ is not changed. Then the edge e_r can be colored β and each edge e_i with $0 \le i < r$ can be colored with its dual color. This, however, results in a $(1+\Delta)$-edge coloring of G, which is impossible.

Case 2. *$y = v_r$ for some integer r with $0 \le r \le k$ where $r \ne t$.* As in Case 1, the initial and terminal edges of Q must also be colored β and no edge incident with v_r is colored α_{k+1}. Furthermore, Q does not contain the vertex v_k unless $v_r = v_k$. We now interchange the colors β and α_{k+1} of the edges of Q. If $r < t$, then we proceed as in Case 1. On the other hand, if $r > t$, we change the color of e to β if $t = 0$; while if $t > 0$, we change the color of e_t to β and color each edge e_i $(0 \le i < t)$ with its dual color. This implies that G is $(1 + \Delta)$-edge colorable, producing a contradiction.

Case 3. *Either* (1) *$x \ne v_r$ for $0 \le r \le k - 1$ and $x \ne u$ or* (2) *$y \ne v_r$ for $r \ne t$ and $y \ne u$.* Since (1) and (2) are similar, we consider (1) only. Upon interchanging the colors β and α_{k+1} of the edges of P, the edge incident with v_k is colored β. Furthermore, the dual color of each edge e_i $(0 \le i < k)$ has not been altered. Thus e_k is colored β and each edge e_i $(0 \le i < k)$ is colored with its dual color, producing a contradiction.

Case 4. *$x = y = u$.* Necessarily, the initial edges of P and Q are colored β and the terminal edges of P and Q are colored α_{k+1}. Since no edge incident with u is colored β, the paths P and Q cannot be edge-disjoint for this would imply that u is incident with two distinct edges having the same color (namely α_{k+1}), which is impossible. Thus P and Q have the same terminal edge and so there is a first edge f that P and Q have in common. Since f is adjacent to another edge of P and another edge of Q, there are three mutually adjacent edges of G belonging to P or Q and so there are adjacent edges of $G - e$ that are colored the same. Since this is impossible, this case cannot occur. ■

We now look at a few well-known graphs and classes of graphs and determine their chromatic index. We begin with the cycles. Since the cycle C_n $(n \ge 3)$ is 2-regular, $\chi'(C_n) = 2$ or $\chi'(C_n) = 3$. If n is even, then the edges may be alternately colored 1 and 2, producing a 2-edge coloring of C_n. If n is odd, then $\alpha'(C_n) = (n-1)/2$. Since the size of C_n is n, it follows by Theorem 11.1 that $\chi'(C_n) \ge n/\alpha'(C_n) = 2n/(n-1) > 2$ and so $\chi'(C_n) = 3$. Therefore,

$$\chi'(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

We now turn to complete graphs. Since K_n is $(n-1)$-regular, either $\chi'(K_n) = n - 1$ or $\chi'(K_n) = n$. If n is even, then it follows by Theorem 10.16 that K_n is 1-factorable, that is, K_n can be factored into $n - 1$ 1-factors $F_1, F_2, \ldots, F_{n-1}$. By assigning each edge of F_i $(1 \le i \le n-1)$ the color i, an $(n-1)$-edge coloring of K_n is produced. If n is odd, then $\alpha'(K_n) = (n-1)/2$. Since the size m of K_n is $n(n-1)/2$, it follows by Theorem 11.1 that $\chi'(K_n) \ge m/\alpha'(K_n) = n$. Thus $\chi'(K_n) = n$. In summary,

$$\chi'(K_n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Consequently, the chromatic index of every nonempty complete graph is an odd integer.

We now consider the complete bipartite graphs $K_{s,t}$ where $1 \leq s \leq t$. Since $\Delta(K_{s,t}) = t$, it follows that $\chi'(K_{s,t}) = t$ or $\chi'(K_{s,t}) = t + 1$. Suppose that the partite sets of $K_{s,t}$ are $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$. For $1 \leq i \leq s$ and $1 \leq j \leq t$, assign the color $j - i + 1$ to the edge $u_i w_j$, where $j - i + 1$ is expressed modulo t (see Figure 11.5 for $s = 3$ and $t = 4$). Since this is a proper t-edge coloring of $K_{s,t}$, it follows that $\chi'(K_{s,t}) = t$.

Figure 11.5: A 4-edge coloring of $K_{3,4}$

While the chromatic number of a multigraph is the same as the chromatic number of its underlying graph, quite obviously this is not true for the chromatic index. Theorem 11.1 holds for multigraphs as well as graphs, however. That is, if G is a multigraph of size $m \geq 1$, then

$$\chi'(G) \geq \frac{m}{\alpha'(G)}. \tag{11.2}$$

For a multigraph G, we write $\mu(G)$ for the **maximum multiplicity** or the **strength** of G, which is the maximum number of parallel edges joining the same pair of vertices of G. Vizing [228] and, independently, Gupta [100] found an upper bound for $\chi'(G)$ in terms of $\Delta(G)$ and $\mu(G)$.

Theorem 11.3 *For every nonempty multigraph G,*

$$\chi'(G) \leq \Delta(G) + \mu(G).$$

Shannon's Theorem

For a graph G, Theorem 11.3 reduces to $\chi'(G) \leq \Delta(G)+1$, which is Vizing's theorem (Theorem 11.2). Claude Elwood Shannon found an upper bound for the chromatic index of a multigraph G in terms of $\Delta(G)$ alone [205].

Theorem 11.4 (**Shannon's Theorem**) *If G is a multigraph, then*

$$\chi'(G) \leq \frac{3\Delta(G)}{2}.$$

Proof. Suppose that the theorem is false. Among all multigraphs H with $\chi'(H) > 3\Delta(H)/2$, let G be one of minimum size. Let $\Delta(G) = \Delta$ and $\mu(G) = \mu$. Suppose that $\chi'(G) = k$. Hence $\chi(G - f) = k - 1$ for every edge f of G. By Theorem 11.3, $k \leq \Delta + \mu$ and, by assumption, $k > 3\Delta/2$.

Suppose that μ edges join the vertices u and v. Let e be one of these edges. Since $\chi'(G - e) = k - 1$, there exists a $(k-1)$-edge coloring of $G - e$. The number of colors not used in coloring the edges incident with u is at least $(k-1) - (\Delta - 1) = k - \Delta$. Similarly, the number of colors not used in coloring the edges incident with v is at least $k - \Delta$ as well.

Each of these $k - \Delta$ or more colors not used to color an edge incident with u must be used to color an edge incident with v, for otherwise there is a color available for e, contradicting our assumption that $\chi'(G) = k$. Similarly, each of the $k - \Delta$ or more colors not used to color an edge incident with v must be used to color an edge incident with u. Hence the total number of colors used to color the edges incident with u or v is at least $2(k - \Delta) + \mu - 1$ and so

$$2(k - \Delta) + \mu - 1 \leq k - 1.$$

Since $3\Delta/2 < k \leq \Delta + \mu$, it follows that $\mu > \Delta/2$ and so

$$2(k - \Delta) + (\Delta/2) - 1 < 2(k - \Delta) + \mu - 1 \leq k - 1.$$

Therefore,

$$2k - (3\Delta/2) - 1 < k - 1,$$

implying that $k < 3\Delta/2$, which is a contradiction. ■

There are occasions when the upper bound for the chromatic index of a multigraph given by Shannon's theorem (Theorem 11.4) is an improvement over that provided by Theorem 11.3. For the multigraph G of Figure 11.6(a), $\Delta(G) = 7$ and $\mu(G) = 4$. Thus $7 \leq \chi(G) \leq 11$ by Theorem 11.3. By Shannon's theorem, $\chi'(G) \leq (3 \cdot 7)/2$ and so $\chi'(G) \leq 10$. Since the size of G is 16 and at most two edges of G can be assigned the same color, it follows by Theorem 11.1 for multigraphs (11.2) that $\chi'(G) \geq 16/2 = 8$. In fact, $\chi'(G) = 8$ as the 8-edge coloring of G in Figure 11.6(b) shows.

Exercises for Section 11.1

1. Determine the chromatic index of the graph shown in Figure 11.7.

2. What can be said about the chromatic index of a graph obtained by adding a pendant edge to a vertex of maximum degree in a graph?

3. Prove that $\chi'(G \,\square\, K_2) = \Delta(G \,\square\, K_2)$ for every graph G.

4. (a) Show that $\chi'(G \vee K_1) \geq \chi'(G)$ for every nonempty graph G.

Figure 11.6: A multigraph G with $\Delta(G) = 7$, $\mu(G) = 4$ and $\chi'(G) = 8$

Figure 11.7: The graph in Exercise 1

(b) Show that there are infinitely many graphs G such that $\chi'(G \vee K_1) = \chi'(G)$.

5. For $i = 1, 2$, let G_i be a graph of order n_i and chromatic index k_i and let

$$B = \max\{n_1, n_2\} + \max\{k_1, k_2\}.$$

(a) Show that $\chi'(G_1 \vee G_2) \leq B$.

(b) Give an example of graphs G_1 and G_2 such that $\chi'(G_1 \vee G_2) < B$.

6. (a) Determine the upper bounds for $\chi'(G)$ given by Theorems 11.3 and 11.4 for the multigraph G shown in Figure 11.8.

(b) Determine $\chi'(G)$ for the multigraph G shown in Figure 11.8.

Figure 11.8: The multigraph G in Exercise 6

11.2 Class One and Class Two Graphs

By Vizing's theorem (Theorem 11.2), it follows that for every nonempty graph G, either $\chi'(G) = \Delta(G)$ or $\chi'(G) = 1 + \Delta(G)$. A graph G is said to belong to or is of **Class one** if $\chi'(G) = \Delta(G)$ and is of **Class two** if $\chi'(G) = 1 + \Delta(G)$. Consequently, a major question in the area of edge colorings is that of determining to which of these two classes does a given graph belong. This is far from being an easy question to answer in general, however, as Ian Holyer [120] showed that the problem of determining the chromatic index of an arbitrary graph is **NP**-complete. In fact, the problem is **NP**-complete even for cubic graphs.

We have therefore seen in the preceding section that every complete bipartite graph is of Class one, while a cycle and complete graph are of Class one if their orders are even and are of Class two if their orders are odd. Of course, both the cycles and complete graphs are regular graphs. For an r-regular graph G, either $\chi'(G) = r$ or $\chi'(G) = r + 1$. If $\chi'(G) = r$, then there is an r-edge coloring of G, resulting in r color classes $E_1, E_2, \ldots, E_r$. Since every vertex v of G has degree r, the vertex v is incident with exactly one edge in each set E_i $(1 \leq i \leq r)$. Therefore, each color class E_i is a perfect matching and G is 1-factorable. Conversely, if G is 1-factorable, then $\chi'(G) = r$.

Theorem 11.5 *A regular graph G is of Class one if and only if G is 1-factorable.*

We saw in Exercise 1 of Section 10.2 that the Petersen graph P is not 1-factorable and so it is of Class two, that is, $\chi'(P) = 4$. The formulas mentioned above for the chromatic index of cycles and complete graphs are immediate consequences of Theorem 11.5, as is the following.

Corollary 11.6 *Every regular graph of odd order is of Class two.*

König's Theorem

We have already observed that the even cycles are of Class one. The four graphs shown in Figure 11.9 are of Class one as well. The even cycles and the four graphs of Figure 11.9 are all bipartite. These graphs serve to illustrate a theorem of Denés König [137] (see Exercise 7).

Theorem 11.7 (**König's Theorem**) *If G is a nonempty bipartite graph, then*

$$\chi'(G) = \Delta(G).$$

Proof. Suppose that the theorem is false. Then among the counterexamples, let G be one of minimum size. Thus G is a bipartite graph such that $\chi'(G) =$

Figure 11.9: Some Class one graphs

$\Delta(G)+1$. Let $e \in E(G)$, where $e = uv$. Necessarily, u and v belong to different partite sets of G. Then $\chi'(G-e) = \Delta(G-e)$. Now $\Delta(G-e) = \Delta(G)$, for otherwise G is $\Delta(G)$-edge colorable.

Let there be given a $\Delta(G)$-edge coloring of $G-e$. Each of the $\Delta(G)$ colors must be assigned to an edge incident either with u or with v in $G-e$, for otherwise this color could be assigned to e producing a $\Delta(G)$-edge coloring of G. Because $\deg_{G-e} u < \Delta(G)$ and $\deg_{G-e} v < \Delta(G)$, there is a color α of the $\Delta(G)$ colors not used in coloring the edges of $G-e$ incident with u and a color β of the $\Delta(G)$ colors not used in coloring the edges of $G-e$ incident with v. Then $\alpha \neq \beta$ and, furthermore, some edge incident with v is colored α and some edge incident with u is colored β.

Let P be a path of maximum length having initial vertex v whose edges are alternately colored α and β. The path P cannot contain u, for otherwise P has odd length, implying that the initial and terminal edges of P are both colored α. This is impossible, however, since u is incident with no edge colored α. Interchanging the colors α and β of the edges of P produces a new $\Delta(G)$-edge coloring of $G-e$ in which neither u nor v is incident with an edge colored α. Assigning e the color α produces a $\Delta(G)$-edge coloring of G, which is a contradiction. ■

An immediate consequence of Theorem 11.7 is that for integers s and t with $1 \leq s \leq t$, $\chi'(K_{s,t}) = t$, as we observed in the preceding section. We have seen that if G is a graph of size m, then any partition of $E(G)$ into independent sets must contain at least $\frac{m}{\alpha'(G)}$ sets. If v is a vertex with $\deg v = \Delta(G)$, then each of the $\Delta(G)$ edges incident with v must belong to distinct independent sets. Thus $\frac{m}{\alpha'(G)} \geq \Delta(G)$ and so $m \geq \alpha'(G) \cdot \Delta(G)$. If $m > \alpha'(G) \cdot \Delta(G)$, then we can say more.

Theorem 11.8 *If G is a graph of size m such that*

$$m > \alpha'(G) \cdot \Delta(G),$$

then G is of Class two.

Proof. By Theorem 11.1, $\chi'(G) \geq \frac{m}{\alpha'(G)}$. Thus

$$\chi'(G) \geq \frac{m}{\alpha'(G)} > \frac{\alpha'(G) \cdot \Delta(G)}{\alpha'(G)} = \Delta(G),$$

which implies that $\chi'(G) = 1 + \Delta(G)$ and so G is of Class two. ■

If G is a graph of order n, then $\alpha'(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$. Therefore, the largest possible value of $\alpha'(G) \cdot \Delta(G)$ is $\left\lfloor \frac{n}{2} \right\rfloor \cdot \Delta(G)$. A graph G of order n and size m is called **overfull** if $m > \left\lfloor \frac{n}{2} \right\rfloor \cdot \Delta(G)$. If n is even, then $\lfloor n/2 \rfloor = n/2$ and

$$2m = \sum_{v \in V(G)} \deg v \leq n\Delta(G).$$

Therefore, $m \leq (n/2) \cdot \Delta(G) = \left\lfloor \frac{n}{2} \right\rfloor \cdot \Delta(G)$ and so G is not overfull. Thus no graph of even order is overfull.

Since $\alpha'(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$ for every graph G of order n, Theorem 11.8 has an immediate corollary (see Exercise 9).

Corollary 11.9 *Every overfull graph is of Class two.*

Applications of Edge Colorings

We now look at two problems whose solutions involve edge colorings.

Example 11.10 *Seven countries, namely Argentina (A), Brazil (B), Chile (C), France (F), Germany (G), Italy (I) and Spain (S), have been invited to send their soccer teams to participate in a tournament. The schedule calls for Argentina to play all other teams except Brazil and Chile. France will play all other teams except Germany and Italy, while Spain plays all other teams except Italy. In general, all pairs of teams play each other except for the five pairs of exceptions noted above. If no team is to play two matches on the same day, what is the minimum number of days needed to schedule this tournament?*

Solution. We construct a graph H with $V(H) = \{A, B, C, F, G, I, S\}$ whose vertices are the seven soccer teams. Two vertices x and y are adjacent in H if they play a soccer match against each other. The graph H is shown in Figure 11.10. The answer to the question is the chromatic index of H. The order of H is $n = 7$ and the degrees of its vertices are $5, 5, 5, 5, 4, 4, 4$. Thus $\Delta(H) = 5$ and the size of H is $m = 16$. Since

$$16 = m > \left\lfloor \frac{n}{2} \right\rfloor \cdot \Delta(H) = 5 \cdot 3 = 15,$$

the graph H is overfull. By Corollary 11.9, H is of Class two and so $\chi(H) = 1 + \Delta(H) = 6$. A 6-edge coloring of H is also shown in Figure 11.10. This provides us with a schedule for the soccer tournament taking place over a minimum of six days. ♦

Day 1: AG, BC, FS
Day 2: AI, BG, CS
Day 3: GI, SB
Day 4: AF, CG
Day 5: AS, BF, CI
Day 6: BI, CF, GS

Figure 11.10: The graph H in Example 11.10 and a 6-edge coloring of H

Example 11.11 *One year it is decided to have a charity tennis tournament consisting entirely of doubles matches. Five tennis players (denoted by A, B, C, D, E) have agreed to participate. Each pair $\{W, X\}$ of tennis players will play a match against every other pair $\{Y, Z\}$ of tennis players, where then $\{W, X\} \cap \{Y, Z\} = \emptyset$, but no 2-person team is to play two matches on the same day. What is the minimum number of days needed to schedule such a tournament? Give an example of such a tournament using a minimum number of days.*

Solution. We construct a graph G whose vertex set is the set of 2-element subsets of $\{A, B, C, D, E\}$. Thus the order of G is $\binom{5}{2} = 10$. Two vertices $\{W, X\}$ and $\{Y, Z\}$ are adjacent if these sets are disjoint. The graph G is shown in Figure 11.11. Thus G is the Petersen graph. To answer the question, we determine the chromatic index of G, which we have seen is 4. A 4-edge coloring of G is given in Figure 11.11 together with a possible schedule of tennis matches over a period of four days. ♦

Overfull Subgraphs

As we have seen, the size of a graph G of Class one and having order n cannot exceed $\left\lfloor \frac{n}{2} \right\rfloor \cdot \Delta(G)$. Since the size of any overfull graph of order n exceeds this number, every overfull graph is therefore of Class two. There are related subgraphs such that if a graph G should contain one of these, then G must also be of Class two.

A subgraph H of odd order n' and size m' of a graph G is an **overfull subgraph** of G if

$$m' > \left\lfloor \frac{n'}{2} \right\rfloor \cdot \Delta(G) = \frac{n'-1}{2} \cdot \Delta(G).$$

Day 1: AB-CE, AC-BD, AE-BC, AD-BE
Day 2: AB-CD, AC-BE, AE-BD, BC-DE
Day 3: AB-DE, AD-BC, AE-CD, BD-CE
Day 4: AC-DE, AD-CE, BE-CD

Figure 11.11: The Petersen graph G in Example 11.11 and a 4-edge coloring of G

Actually, if H is an overfull subgraph of G, then $\Delta(H) = \Delta(G)$ (see Exercise 10). This says that H is itself of Class two. Not only is an overfull subgraph of a graph G of Class two, G itself is of Class two.

Theorem 11.12 *Every graph having an overfull subgraph is of Class two.*

Proof. Let H be an overfull subgraph of a graph G. As we noted, $\Delta(H) = \Delta(G)$ and H is of Class two; so $\chi'(H) = 1 + \Delta(H)$. Thus

$$\chi'(G) \geq \chi'(H) = 1 + \Delta(H) = 1 + \Delta(G)$$

and so $\chi'(G) = 1 + \Delta(G)$. ■

The following result provides a useful property of overfull subgraphs of a graph.

Theorem 11.13 *Let G be an r-regular graph of even order $n = 2k$, where $\{V_1, V_2\}$ is a partition of $V(G)$ such that $|V_1| = n_1$ and $|V_2| = n_2$ are odd. Suppose that $G_1 = G[V_1]$ is an overfull subgraph of G. Then $G_2 = G[V_2]$ is also an overfull subgraph of G. Furthermore, if k is odd, then $r < k$; while if k is even, then $r < k - 1$.*

Proof. Let the size of G_i be m_i for $i = 1, 2$ and let m' denote the number of edges in the set $[V_1, V_2]$. Thus

$$rn_1 = 2m_1 + m' \text{ and } rn_2 = 2m_2 + m'.$$

Since G_1 is overfull, $m_1 > \left(\frac{n_1-1}{2}\right) r$. We show that $m_2 > \left(\frac{n_2-1}{2}\right) r$. Now

$$m_2 = \frac{rn_2 - m'}{2} = \frac{r(n-n_1) - (rn_1 - 2m_1)}{2} = \frac{rn}{2} - rn_1 + m_1.$$

Since $2m_1 > rn_1 - r$, it follows that

$$\begin{aligned} 2m_2 &= rn - 2rn_1 + 2m_1 > rn - 2rn_1 + rn_1 - r \\ &= r(n - n_1 - 1) = r(n_2 - 1). \end{aligned}$$

Hence $m_2 > \left(\frac{n_2-1}{2}\right) r$ and G_2 is overfull. Therefore, both G_1 and G_2 are overfull.

Now, either $n_1 \le k$ or $n_2 \le k$, say the former. If k is even, then $n_1 \le k-1$. Since G_1 is overfull,

$$m_1 > \left(\frac{n_1-1}{2}\right) r \text{ and so } 2m_1 > (n_1 - 1)r.$$

Since the size m_1 of G_1 is at most $\binom{n_1}{2}$, it follows that

$$2\binom{n_1}{2} \ge 2m_1 > (n_1 - 1)r \text{ and so } n_1(n_1 - 1) > (n_1 - 1)r.$$

Consequently, $r < n_1 \le k - 1$ and so $r < k - 1$. If k is odd, then $r < k$. ■

In the definition of an overfull subgraph H of order n' and size m' in a graph G, we have $m' > \frac{n'-1}{2} \cdot \Delta(G)$. As we saw in Theorem 11.12, this implies that G is of Class two. On the other hand, if G should contain a subgraph H of order n' and size m' such that $m' > \frac{n'-1}{2} \cdot \Delta(H)$, where $\Delta(H) < \Delta(G)$, then H is an overfull graph and so H is of Class two. This need not imply that G is of Class two, however. For example, $C_5 \times K_2$ is of Class one but C_5 is of Class two.

While every graph containing an overfull subgraph must be of Class two, a graph can be of Class two without containing any overfull subgraph. The Petersen graph P (which is 3-regular of order 10) contains no overfull subgraph; yet we saw that P is of Class two.

The Overfull Conjecture

The following conjecture is due to Amanda G. Chetwynd and Anthony J. W. Hilton [42].

The Overfull Conjecture Let G be a graph of order n such that $\Delta(G) > n/3$. Then G is of Class two if and only if G contains an overfull subgraph.

Let v be a vertex of the Petersen graph P. Then $P - v$ has order $n = 9$ and $\Delta(P - v) = \frac{n}{3} = 3$. Even though $P - v$ is of Class two, it has no overfull subgraph. Hence if the Overfull Conjecture is true, the resulting theorem cannot be improved in general.

We encountered the following conjecture in Chapter 10.

The 1-Factorization Conjecture If G is an r-regular graph of even order n such that $r \geq n/2$, then G is 1-factorable.

The Overfull Conjecture implies the 1-Factorization Conjecture.

Theorem 11.14 *If the Overfull Conjecture is true, then so too is the 1-Factorization Conjecture.*

Proof. Assume, to the contrary, that the Overfull Conjecture is true but the 1-Factorization Conjecture is false. Then there exists an r-regular graph G of even order n with $r \geq n/2$ such that G is not 1-factorable. Thus G is of Class two. By the Overfull Conjecture, G contains an overfull subgraph G_1.

Let G_2 be the subgraph of G induced by $V(G) - V(G_1)$. By Theorem 11.13, G_2 is also overfull. At least one of G_1 and G_2 has order at most $n/2$. Suppose that G_1 has order at most $n/2$. Again, by Theorem 11.13, if $n/2$ is odd, then $r < n/2$; while if $n/2$ is even, then $r < (n/2) - 1$. Since $r \geq n/2$, a contradiction is produced. ■

Exercises for Section 11.2

1. Let H be a nonempty regular graph of odd order, and let G be a graph obtained from H by deleting $\frac{1}{2}(\Delta(H) - 1)$ or fewer edges. Show that G is of Class two.

2. Prove or disprove: If G_1 and G_2 are Class one graphs and H is a graph with $G_1 \subseteq H \subseteq G_2$, then H is of Class one.

3. Prove that every Hamiltonian cubic graph is of Class one.

4. Show that graphs G_1 and G_2 in Figure 11.12 are of Class two.

Figure 11.12: Graphs in Exercise 4

5. Determine the classes of the graphs of the five regular polyhedra.

6. Prove that every cubic graph having connectivity 1 is of Class two.

7. Use the fact that every r-regular bipartite graph is 1-factorable to give an alternative proof of Theorem 11.7: *If G is a nonempty bipartite graph, then $\chi'(G) = \Delta(G)$.*

8. Show that Corollary 11.6 (*Every regular graph of odd order is of Class two.*) is also a corollary of Theorem 11.8 (*If G is a graph of size m such that $m > \alpha'(G)\Delta(G)$, then G is of Class two.*).

9. Prove Corollary 11.9: *Every overfull graph is of Class two.*

10. Show that if H is an overfull subgraph of a graph G, then $\Delta(H) = \Delta(G)$.

11. Show that the condition that G_1 is an overfull subgraph of G is needed in Theorem 11.13.

12. A nonempty graph G is of Type one if $\chi'(L(G)) = \omega(L(G))$ and of Type two if $\chi'(L(G)) = 1+\omega(L(G))$. Prove or disprove: Every nonempty graph is either of Type one or of Type two.

13. The **total deficiency** of a graph G of order n and size m is the number $n \cdot \Delta(G) - 2m$. (Thus every graph that is not regular has positive total deficiency.) Prove that if G is a graph of odd order whose total deficiency is less than $\Delta(G)$, then G is of Class two.

14. Show that for every integer k with $2 \leq k \leq 5$, there is a planar graph of Class one and a planar graph of Class two, both having maximum degree k.

15. For a positive integer k, let H be a $2k$-regular graph of order $4k+1$. Let G be obtained from H by removing a set of $k-1$ independent edges from H. Prove that G is of Class two.

16. For a positive integer k, let G be a $(2k+2)$-regular graph of order $4k+1$.

    (a) Show that G is Hamiltonian

    (b) Let $C = (v_1, v_2, \ldots, v_{4k+1}, v_{4k+2} = v_1)$ be a Hamiltonian cycle of G and let $X = \{v_{2i-1}v_{2i} : 1 \leq i \leq 2k+1\}$. Prove that $H = G - X$ is of Class two.

17. In Figure 11.10, a solution of Example 11.10 is given by providing a 6-edge coloring of the graph H shown in that figure. Give a characteristic of the resulting soccer schedule which might not be considered ideal. Correct this deficiency by giving a different 6-edge coloring of H.

18. In Figure 11.11, a solution of Example 11.11 is given by providing a 4-edge coloring of the graph G shown in that figure. Give a characteristic of the resulting tennis schedule which might not be considered ideal. Correct this deficiency by giving a different 4-edge coloring of G.

11.3 Tait Colorings

The Scottish physicist Peter Guthrie Tait (1831–1901) was one of many individuals who played a role in the story of the Four Color Problem (see Chapter 9). In addition to his interest in physics, Tait was also interested in mathematics and golf. His son Frederick (better known as Freddie Tait) shared his father's interest in golf and became the finest amateur golfer of his time.

Peter Tait became acquainted with the Four Color Problem through Arthur Cayley and became interested in Alfred Bray Kempe's solution of the problem. In fact, Tait felt that Kempe's solution was too lengthy and came up with several solutions of his own. Unfortunately, as in the case of Kempe's solution, none of Tait's solutions proved to be correct. Despite this, one of his attempted proofs contained an interesting and useful idea. Tait's idea was to consider coloring the boundary lines of cubic maps. In fact, he stated as a lemma that:

> *The boundary lines of every cubic map can always be colored with three colors so that the three lines at each meeting point are colored differently.*

Tait also mentioned that this lemma could be easily proved and showed how the lemma could be used to prove the Four Color Theorem. Although Tait was correct that this lemma could be used to to prove the Four Color Theorem, he was incorrect when he said that the lemma could be easily proved. Indeed, as it turned out, this lemma is equivalent to the Four Color Theorem [214] and, of course, was equally difficult to prove.

Tait's Theorem

In the proof of the following theorem, we use the so-called Klein four-group which is the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2$ of two copies of the cyclic group of order 2.

Theorem 11.15 (**Tait's Theorem**) *A bridgeless cubic plane graph G is 4-region colorable if and only if G is 3-edge colorable.*

Proof. Suppose first that G is 4-region colorable. Let a 4-region coloring of G be given, where the colors are the four elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus each color can be expressed as (a, b) or ab, where $a, b \in \{0, 1\}$. The four colors used are then $c_0 = 00$, $c_1 = 01$, $c_2 = 10$ and $c_3 = 11$. Addition of colors is defined as coordinate-wise addition in $\mathbb{Z}_2$. For example, $c_1 + c_3 = 01 + 11 = 10 = c_2$. Since every element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is self-inverse, the sum of two distinct elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is never $c_0 = 00$.

We now define an edge coloring of G. Since G is bridgeless, each edge e lies on the boundary of two distinct regions. Define the color of e as the sum of the colors of the two regions having e on their boundary. Thus no edge of G is assigned the color c_0 and a 3-edge coloring of G is produced. It remains to

show that this 3-edge coloring is proper. Let e_1 and e_2 be two adjacent edges of G and let v be the vertex incident with e_1 and e_2. Then v is incident with a third edge e_3 as well. For $1 \leq i < j \leq 3$, suppose that e_i and e_j are on the boundary of the region R_{ij}. Since the colors of R_{13} and R_{23} are different, the sum of the colors of R_{13} and R_{23} and the sum of the colors of R_{12} and R_{23} are different and so this 3-edge coloring is proper.

We now turn to the converse. Suppose that G is 3-edge colorable. Let a 3-edge coloring of G be given using the colors $c_1 = 01$, $c_2 = 10$ and $c_3 = 11$, as described above. This produces a partition of $E(G)$ into three perfect matchings E_1, E_2 and E_3, where E_i $(1 \leq i \leq 3)$ is the set of edges colored c_i.

Let G_1 be the spanning subgraph of G with edge set $E(G_1) = E_1 \cup E_2$ and let G_2 be the spanning subgraph of G with edge set $E(G_2) = E_2 \cup E_3$. Thus both G_1 and G_2 are 2-regular spanning subgraphs of G and so each of G_1 and G_2 is the disjoint union of even cycles. For $i = 1, 2$, every region of G_i is the union of regions of G. For each cycle C in the graph G_i $(i = 1, 2)$, every region of G either lies interior or exterior to C. Furthermore, since G is bridgeless, each edge of G_i $(i = 1, 2)$ belongs to a cycle C' of G_i and is on the boundary of two distinct regions of G_i, one of which lies interior to C' and the other exterior to C'.

We now define a 4-region coloring of G. We assign to a region R of G the color a_1a_2 with $a_i \in \{0, 1\}$ for $i = 1, 2$, where $a_i = 0$ if R lies interior to an even number of cycles in G_i and $a_i = 1$ otherwise. Figure 11.13(a) shows a 3-edge coloring of a cubic graph G, Figure 11.13(b) shows the cycles of G_1, Figure 11.13(c) shows the cycles of G_2 and Figure 11.13(d) shows the resulting 4-region coloring of G as defined above.

It remains to show that this 4-region coloring of G is proper, that is, that every two adjacent regions of G are assigned different colors. Let R_1 and R_2 be two adjacent regions of G. Thus there is an edge e that lies on the boundary of both R_1 and R_2. If e is colored c_1 or c_2, then e lies on a cycle in G_1; while if e is colored c_2 or c_3, then e lies on a cycle in G_2. Thus e lies on a cycle in G_1, on a cycle in G_2 or on a cycle in both G_1 and G_2. Let C be a cycle in G_1, say, containing the edge e. Exactly one of R_1 and R_2 lies interior to C, while for every other cycle C' of G_1, either R_1 and R_2 are both interior to C' or both exterior to C'. Hence the first coordinate of the colors of R_1 and R_2 are different. Therefore, the colors of every two adjacent regions of G differ in the first coordinate or the second coordinate or both. Hence this 4-region coloring of G is proper. ■

Eventually proper 3-edge colorings of cubic graphs became known as **Tait colorings**. Certainly, a cubic graph G has a Tait coloring if and only if G is 1-factorable. Every Hamiltonian cubic graph necessarily has a Tait coloring, for the edges of a Hamiltonian cycle can be alternately colored 1 and 2, with the remaining edges (constituting a perfect matching) colored 3. Tait believed that every 3-connected cubic planar graph is Hamiltonian. If Tait was correct, then this would mean that every 3-connected cubic planar graph is 3-edge colorable.

Figure 11.13: A step in the proof of Theorem 11.15

However, as we are about to see, this implies that every 2-connected cubic planar graph is 3-edge colorable. But the 2-connected cubic graphs are precisely the connected bridgeless cubic graphs and so by Tait's theorem (Theorem 11.15), the Four Color Conjecture would be true.

Theorem 11.16 *If every* 3*-connected cubic planar graph is* 3*-edge colorable, then every* 2*-connected cubic planar graph is* 3*-edge colorable.*

Proof. Suppose that the statement is false. Then all 3-connected cubic planar graphs are 3-edge colorable, but there exist cubic planar graphs having connectivity 2 that are not 3-edge colorable. Among the cubic planar graphs having connectivity 2 that are not 3-edge colorable, let G be one of minimum order n. Certainly n is even and since there is no such graph of order 4, it follows that $n \geq 6$. As we saw in Theorem 2.31, $\kappa(G) = \lambda(G) = 2$. This implies that every minimum edge-cut of G consists of two nonadjacent edges of G.

Let $\{u_1v_1, x_1y_1\}$ be a minimum edge-cut of G. Thus the vertices u_1, v_1, x_1, and y_1 are distinct and G has the appearance shown in Figure 11.14, where F_1 and H_1 are the two components of $G - u_1v_1 - x_1y_1$.

Suppose that $u_1x_1, v_1y_1 \notin E(G)$. Then $F_1 + u_1x_1$ and $H_1 + v_1y_1$ are 2-connected cubic planar graphs of order less than n and so are 3-colorable. Let 3-edge colorings of $F_1 + u_1x_1$ and $H_1 + v_1y_1$ be given using the colors 1, 2 and 3. Now permute the colors of the edges in both $F_1 + u_1x_1$ and $H_1 + v_1y_1$, if necessary, so that both u_1x_1 and v_1y_1 are assigned the color 1. Deleting the

Figure 11.14: A step in the proof of Theorem 11.16

edges u_1x_1 and v_1y_1, adding the edges u_1v_1 and x_1y_1 and assigning both u_1v_1 and x_1y_1 the color 1 results in a 3-edge coloring of G, which is impossible.

Thus we may assume that at least one of u_1x_1 and v_1y_1 is an edge of G, say $u_1x_1 \in E(G)$. Then u_1 is adjacent to a vertex u_2 in F_1 that is different from x_1, and x_1 is adjacent to a vertex x_2 in F_1 that is different from u_1. Since $\kappa(G) = 2$, it follows that $u_2 \neq x_2$. We then have the situation shown in Figure 11.15, where $\{u_2u_1, x_2x_1\}$ is a minimum edge-cut of G, and F_2 and H_1 are the two components of $G - u_1 - x_1$.

Figure 11.15: A step in the proof of Theorem 11.16

Suppose that $u_2x_2, v_1y_1 \notin E(G)$. Then $F_2 + u_2x_2$ and $H_1 + v_1y_1$ are 2-connected cubic planar graphs of order less than n and so are 3-edge colorable. Let 3-edge colorings of $F_2 + u_2x_2$ and $H_1 + v_1y_1$ be given using the colors 1, 2 and 3. Now permute the colors of the edges in both $F_2 + u_2v_2$ and $H_1 + v_1y_1$, if necessary, so that u_2x_2 is colored 2 and v_1y_1 is colored 1. Deleting the edges u_2x_2 and v_1y_1, assigning u_1u_2 and x_1x_2 the color 2, assigning u_1v_1 and x_1y_1 the color 1 and assigning u_1x_1 the color 3 produces a 3-edge coloring of G, which is a contradiction.

Thus we may assume that at least one of u_2x_2 and v_1y_1 is an edge of G. Continuing in this manner, we have a sequence

$$\{u_1, x_1\},\ \{u_2, x_2\},\ \ldots,\ \{u_k, x_k\}\ (k \geq 1)$$

of pairs of vertices of F_1 such that $u_kx_k \notin E(G)$ and $u_ix_i \in E(G)$ for $1 \leq i < k$ and a sequence

$$\{v_1, y_1\},\ \{v_2, y_2\},\ \ldots,\ \{v_\ell, y_\ell\}\ (\ell \geq 1)$$

of pairs of vertices of H_1 such that $v_\ell y_\ell \notin E(G)$ and $v_i y_i \in E(G)$ for $1 \le i < \ell$, as shown in Figure 11.16, where F_k and H_ℓ are the two components of

$$G-(\{u_1, u_2, \ldots, u_{k-1}\} \cup \{x_1, x_2, \ldots, x_{k-1}\} \cup \{v_1, v_2, \ldots, v_{\ell-1}\} \cup \{y_1, y_2, \ldots, y_{\ell-1}\}).$$

Figure 11.16: A step in the proof of Theorem 11.16

Since $F_k + u_k x_k$ and $H_\ell + v_\ell y_\ell$ are 2-connected cubic planar graphs of order less than n, each is 3-edge colorable. Let 3-edge colorings of $F_k + u_k x_k$ and $H_\ell + v_\ell y_\ell$ be given using the colors 1, 2 and 3. Permute the colors 1, 2 and 3 of the edges in both $F_k + u_k x_k$ and $H_\ell + v_\ell y_\ell$, if necessary, so that both $u_k x_k$ and $v_\ell y_\ell$ are colored 1 if k and ℓ are of the same parity and $u_k x_k$ is colored 2 and $v_\ell y_\ell$ is colored 1 if k and ℓ are of the opposite parity. Deleting the edges $u_k x_k$ and $v_\ell y_\ell$, alternating the colors 1 and 2 along the paths

$$(v_\ell, v_{\ell-1}, \ldots, v_1, u_1, u_2, \ldots, u_k) \text{ and } (y_\ell, y_{\ell-1}, \ldots, y_1, x_1, x_2, \ldots, x_k),$$

and assigning the color 3 to the edges

$$u_1x_1,\ u_2x_2,\ \ldots,\ u_kx_k,\ v_1y_1,\ v_2y_2,\ \ldots,\ v_\ell y_\ell$$

produces a 3-edge coloring of G, again a contradiction. ■

Class One Planar Graphs

As a consequence of Theorems 11.15 and 11.16, every planar graph is 4-colorable – *provided* Tait was correct that every 3-connected cubic planar graph is Hamiltonian. As we mentioned in Section 6.4, the Tutte graph, shown again in Figure 11.17, is a 3-connected cubic graph that is not Hamiltonian.

Despite the fact that Tait was wrong and that 3-edge colorability of bridgeless cubic planar graphs was never used to prove the Four Color Theorem, the eventual verification of the Four Color Theorem did instead show that every bridgeless cubic planar graph is 3-edge colorable.

Corollary 11.17 *Every bridgeless cubic planar graph is of Class one.*

An immediate consequence of Corollary 11.17 is that there are planar graphs G of Class one with $\Delta(G) = 3$. In fact, there are planar graphs G of Class two

Figure 11.17: The Tutte graph

with $\Delta(G) = 3$. Indeed, one can say more. For every integer k with $2 \leq k \leq 5$, there is a planar graph of Class one and a planar graph of Class two, both having maximum degree k (see Exercise 14 in Section 11.2). This story does not go much further, however, for in 1965 Vadim Vizing [229] proved the following.

Theorem 11.18 *If G is a planar graph with $\Delta(G) \geq 8$, then G is of Class one.*

In 2001 Daniel Sanders and Yue Zhao [200] resolved one of the two missing cases.

Theorem 11.19 *If G is a planar graph with $\Delta(G) = 7$, then G is of Class one.*

Thus only one case remains. Vizing conjectured that this remaining case also results in a planar graph of Class one.

Vizing's Planar Graph Conjecture Every planar graph with maximum degree 6 is of Class one.

Snarks

While every bridgeless cubic planar graph is of Class one, there are many cubic graphs that are of Class two. First, every cubic graph containing a bridge is of Class two (see Exercise 6 in Section 11.2). Furthermore, every bridgeless cubic graph of Class two is necessarily non-Hamiltonian and nonplanar. The Petersen graph is an example of a bridgeless cubic graph of Class two.

Recall that the girth of a graph G that is not a forest is the length of a smallest cycle in G and that the girth of the Petersen graph is 5. The **cyclic edge-connectivity** of a graph is the smallest number of edges whose removal results in a disconnected graph, each component of which contains a cycle. The cyclic edge-connectivity of the Petersen graph is 5.

There is a class of bridgeless cubic graphs of Class two that are of special interest. A **snark** is a cubic graph of Class two that has girth at least 5 and cyclic edge-connectivity at least 4. The Petersen graph is therefore a snark. The girth and cyclic edge-connectivity requirements are present in the definition to rule out trivial examples. The term "snark" was coined for these graphs in 1976 by Martin Gardner, a longtime popular writer for the magazine *Scientific American*. Gardner borrowed this name from Lewis Carroll (the pen-name of the mathematician Charles Lutwidge Dodgson), well known for his book *Alice's Adventures in Wonderland*. One century earlier, in 1876, Carroll wrote a nonsensical poem titled *The Hunting of the Snark* in which a group of adventurers are in pursuit of a legendary and elusive beast: the snark. Gardner chose to call these graphs "snarks" because just as Carroll's snarks were difficult to find, so too were these graphs difficult to find (at least for the first several years).

The oldest snark is the Petersen graph, discovered in 1891. In 1946 Danilo Blanuša [21] discovered two more snarks (the **Blanuša snarks**), both of order 18 (see Figure 11.18(a)).

(a) The Blanuša snarks

(b) A flower snark

(c) The double-star snark

Figure 11.18: Snarks

The **Descartes snark** (discovered by Tutte [56] in 1948) has order 210. George Szekeres [211] discovered the **Szekeres snark** of order 50 in 1973. Until 1973 these were the only known snarks. In 1975, however, Rufus Isaacs [122] described two infinite families of snarks, one of which essentially contained all previously known snarks while the second family was completely new. This second family contained the so-called **flower snarks**, one example of which is shown in Figure 11.18(b). In addition, Isaacs found a snark that belonged to neither family. This **double-star snark** is shown in Figure 11.18(c).

All of the snarks shown in Figure 11.18 appear to have a certain resemblance to the Petersen graph. In fact, William Tutte conjectured that every snark has the Petersen graph as a minor. Neil Robertson, Daniel Sanders, Paul Seymour and Robin Thomas announced in 2001 that they had verified this conjecture.

Theorem 11.20 *Every snark has the Petersen graph as a minor.*

Exercises for Section 11.3

1. Let G be the graph shown in Figure 11.19. We have seen that every Hamiltonian cubic graph has a Tait coloring.

Figure 11.19: The graph in Exercise 1

   (a) Is the graph G Hamiltonian?

   (b) Does the graph G have a Tait coloring?

2. (a) Let G be a cubic graph and let H be the cubic graph obtained from G by replacing an edge uv of G by four new vertices w, x, y and z and the edges uw, vz, wx, wy, xy, xz and yz (see Figure 11.20(a)). Prove that G has a Tait coloring if and only if H has a Tait coloring.

   (b) Use the result in (a) to determine whether the graph F shown in Figure 11.20(b) has a Tait coloring.

Figure 11.20: The graph in Exercise 2

3. Let G be the cubic graph shown in Figure 11.21.

   (a) Is G planar?

   (b) Is G Hamiltonian?

   (c) Does G have a Tait coloring?

Figure 11.21: The graph in Exercise 3

4. Let G be a cubic graph and let H be the cubic graph obtained from G by replacing each edge uv of G by the structure shown in Figure 11.22. Prove or disprove: The graph G has a Tait coloring if and only if H has a Tait coloring.

5. (a) Determine the chromatic index of the graph G shown in Figure 11.23.

   (b) Is this graph a snark?

Figure 11.22: The graphs in Exercise 4

Figure 11.23: The graph in Exercise 5

11.4 Nowhere-Zero Flows

As we saw in the preceding section, Tait colorings are edge colorings that are intimately tied to colorings of the regions of bridgeless cubic plane graphs. There are labelings of the arcs of orientations of bridgeless cubic plane graphs that also have a connection to colorings of the regions of such graphs.

For an oriented graph D, a **flow** on D is a function $\phi : E(D) \to \mathbb{Z}$ such that for each vertex v of D,

$$\sigma^+(v;\phi) = \sum_{(v,w)\in E(D)} \phi(v,w) = \sum_{(w,v)\in E(D)} \phi(w,v) = \sigma^-(v;\phi). \tag{11.3}$$

That is, for each vertex v of D, the sum of the flow values of the arcs directed away from v equals the sum of the flow values of the arcs directed toward v. The property (11.3) of ϕ is called the **conservation property**. Since $\sigma^+(v;\phi) - \sigma^-(v;\phi) = 0$ for every vertex v of D, the sum of the flow values of the arcs incident with v is even. This, in turn, implies that the number of arcs incident with v having an odd flow value is even.

For an integer $k \geq 2$, if a flow ϕ on an oriented graph D has the property that $|\phi(e)| < k$ for every arc e of D, then ϕ is called a **k-flow** on D. Furthermore, if $0 < |\phi(e)| < k$ for every arc e of D (that is, $\phi(e)$ is never 0), then ϕ is called a **nowhere-zero k-flow** on D. Hence a nowhere-zero k-flow on D has the property that

$$\phi(e) \in \{\pm 1, \pm 2, \ldots, \pm(k-1)\}$$

for every arc e of D. It is the nowhere-zero k-flows for particular values of k that will be of special interest to us.

Those graphs for which some orientation has a nowhere-zero 2-flow can be described quite easily.

Theorem 11.21 *A nontrivial connected graph G has an orientation with a nowhere-zero 2-flow if and only if G is Eulerian.*

Proof. Let G be an Eulerian graph and let C be an Eulerian circuit of G. Direct the edges of C so that C becomes a directed Eulerian circuit in the resulting Eulerian digraph D, where then $\operatorname{od} v = \operatorname{id} v$ for every vertex v of D. Since the function ϕ defined by $\phi(e) = 1$ for each arc e of D satisfies the conservation property, ϕ is a nowhere-zero 2-flow on D.

Conversely, suppose that G is a nontrivial connected graph that is not Eulerian. Then G contains a vertex u of odd degree. Let D be any orientation of G. Then any function ϕ defined on $E(D)$ for which $\phi(e) \in \{-1, 1\}$ has an odd number of arcs incident with v having an odd flow value. Thus ϕ is not a nowhere-zero 2-flow. ■

The graph $G = C_4$ of Figure 11.24 is obviously Eulerian and therefore has an orientation D_1 with a nowhere-zero 2-flow. However, D_1 is not the only orientation of G with this property. The orientations D_2 and D_3 of G also have a nowhere-zero 2-flow.

Figure 11.24: Oriented graphs with a nowhere-zero 2-flow

The graph and digraphs shown in Figure 11.24 may suggest that the existence of a nowhere-zero k-flow for some integer $k \geq 2$ on all orientations of a graph G depends only on the existence of a nowhere-zero k-flow on a single orientation of G. This is, in fact, what happens.

Theorem 11.22 *Let G be a graph. If some orientation of G has a nowhere-zero k-flow, where $k \geq 2$, then every orientation of G has a nowhere-zero k-flow.*

Proof. Let D be an orientation of G having a nowhere-zero k-flow ϕ. Thus $\sigma^+(v; \phi) = \sigma^-(v; \phi)$ for each vertex v of D. Let D' be the orientation of G obtained by reversing the direction of some arc $f = (x, y)$ of D, resulting in the arc $f' = (y, x)$ of D'. We now define the function

$$\phi' : E(D') \to \{\pm 1, \pm 2, \ldots, \pm(k-1)\}$$

by

$$\phi'(e) = \begin{cases} \phi(e) & \text{if } e \neq f' \\ -\phi(f) & \text{if } e = f'. \end{cases}$$

For $v \in V(D) - \{x, y\}$, $\sigma^+(v; \phi') = \sigma^-(v; \phi')$. Also,

$$\begin{aligned} \sigma^+(x; \phi') &= \sigma^+(x; \phi) - \phi(f) = \sigma^-(x; \phi) + \phi'(f') = \sigma^-(x; \phi') \\ \sigma^+(y; \phi') &= \sigma^+(y; \phi) + \phi'(f') = \sigma^-(y; \phi) + (-\phi(f)) = \sigma^-(y; \phi'). \end{aligned}$$

Thus ϕ' is a nowhere-zero k-flow of D'.

Now, if D'' is any orientation of G, then D'' can be obtained from D by a sequence of arc reversals in D. Since a nowhere-zero k-flow can be defined on each orientation, as described above, at each step of the sequence, a nowhere-zero k-flow can be defined on D''. ■

According to Theorem 11.22, the property of an orientation (indeed of *all* orientations) of a graph G having a nowhere-zero k-flow is a characteristic of G rather than a characteristic of its orientations. We therefore say that a **graph G has a nowhere-zero k-flow** if every orientation of G has a nowhere-zero k-flow. As a consequence of the proof of Theorem 11.22, we also have the following.

Theorem 11.23 *If G is a graph having a nowhere-zero k-flow for some $k \geq 2$, then there is an orientation D of G and a nowhere-zero k-flow on D all of whose flow values are positive.*

By Theorem 11.21, the graph G of Figure 11.25 does not have a nowhere-zero 2-flow. Trivially, it has an orientation D possessing a 2-flow. This orientation does have a nowhere-zero 3-flow, however. By Theorem 11.22, G itself has a nowhere-zero 3-flow.

Figure 11.25: A graph with a nowhere-zero 3-flow

If D is an orientation of a graph G with a flow ϕ and a is an integer, then $a\phi$ is also a flow on D. In fact, if ϕ is a k-flow and $a \geq 1$, then $a\phi$ is an ak-flow. Indeed, if ϕ is a nowhere-zero k-flow, then $a\phi$ is a nowhere-zero ak-flow. More generally, we have the following (see Exercise 3).

Theorem 11.24 *If ϕ_1 and ϕ_2 are flows on an orientation D of a graph, then every linear combination of ϕ_1 and ϕ_2 is also a flow on D.*

Let G be a graph with a nowhere-zero k-flow for some integer $k \geq 2$ and let ϕ be a nowhere-zero k-flow defined on some orientation D of G. Then $\sigma^+(v;\phi) = \sigma^-(v;\phi)$ for every vertex v of D. Let $S = \{v_1, v_2, \ldots, v_t\}$ be a proper subset of $V(G)$ and let $T = V(G) - S$. Define

$$\sigma^+(S;\phi) = \sum_{(u,v)\in[S,T]} \phi(u,v)$$

and

$$\sigma^-(S;\phi) = \sum_{(v,u)\in[T,S]} \phi(v,u).$$

Since ϕ is a flow on D,

$$\sum_{i=1}^{t} \sigma^+(v_i;\phi) = \sum_{i=1}^{t} \sigma^-(v_i;\phi). \tag{11.4}$$

For every arc (v_a, v_b) with $v_a, v_b \in S$, the flow value $\phi(v_a, v_b)$ occurs in both the left and the right sums in (11.4). Cancelling all such terms in (11.4), we are left with $\sigma^+(S;\phi)$ on the left side of (11.4) and $\sigma^-(S;\phi)$ on the right side of (11.4). Thus $\sigma^+(S;\phi) = \sigma^-(S;\phi)$. In summary, we have the following.

Theorem 11.25 *Let ϕ be a nowhere-zero k-flow defined on some orientation D of a graph G and let S be a nonempty proper subset of $V(G)$. Then the sum of the flow values of the arcs directed from S to $V(G) - S$ equals the sum of the flow values of the arcs directed from $V(G) - S$ to S.*

A fundamental question concerns determining those nontrivial connected graphs having a nowhere-zero k-flow for some integer $k \geq 2$. Suppose that ϕ is a nowhere-zero k-flow defined on some orientation D of a nontrivial connected graph G. If G should contain a bridge $e = uv$, where S and $V(G) - S$ are the vertex sets of the components of $G - e$, then the conclusion of Theorem 11.25 cannot occur. As a consequence of this observation, we have the following.

Corollary 11.26 *No graph with a bridge has a nowhere-zero k-flow for any integer $k \geq 2$.*

The nowhere-zero k-flows of bridgeless planar graphs will be of special interest to us because of their connection with region colorings. Let D be an orientation of a bridgeless plane graph G. Suppose that c is a k-region coloring of G for some integer $k \geq 2$. Thus for each region R of G, we may assume that the color $c(R)$ is one of the colors $1, 2, \ldots, k$. For each edge uv of G belonging to the boundaries of two regions R_1 and R_2 of G and the arc $e = (u, v)$ of D, define

$$\phi(e) = c(R_1) - c(R_2),$$

where R_1 is the region that lies to the right of e and R_2 is the region that lies to the left of e (see Figure 11.26).

Figure 11.26: Defining $\phi(e)$ for $e = (u, v)$

A 4-region coloring of the plane graph of Figure 11.27(a) is given in that figure. A resulting nowhere-zero 4-flow of an orientation D of G is shown in Figure 11.27(b).

(a) a 4-region coloring of a plane graph

(b) a nowhere-zero 4-flow of an orientation D of G

Figure 11.27: Constructing a nowhere-zero flow from a region coloring of a plane graph

For bridgeless plane graphs, k-region colorability and the existence of nowhere-zero k-flows for an integer $k \geq 2$ are equivalent.

Theorem 11.27 *For an integer $k \geq 2$, a bridgeless plane graph G is k-region colorable if and only if G has a nowhere-zero k-flow.*

Proof. First, let there be given a k-region coloring c of G and let D be an orientation of G. For each arc $e = (u, v)$ of D, let R_1 be the region of G that lies to the right of e and R_2 the region of G that lies to the left of e. Define an integer-valued function ϕ of $E(D)$ by

$$\phi(e) = c(R_1) - c(R_2).$$

We show that ϕ is a nowhere-zero k-flow. Since uv is not a bridge of G, $c(R_1) \neq c(R_2)$ and since $1 \leq c(R) \leq k$ for each region R of G, it follows that

$$\phi(e) \in \{\pm 1, \pm 2, \ldots, \pm(k-1)\}.$$

It remains only to show that $\sigma^+(v;\phi) = \sigma^-(v;\phi)$ for each vertex v of D. Suppose that $\deg_G v = t$ and that $v_1, v_2, \ldots, v_t$ are the neighbors of v as we proceed cyclically about v in some direction. For $i = 1, 2, \ldots, t$, let R_i denote the region having vv_i and vv_{i+1} on its boundary where $v_{t+1} = v_1$. Thus in D each edge vv_i $(1 \leq i \leq t)$ is either the arc (v, v_i) or the arc (v_i, v). Let $c(R_i) = c_i$ for $i = 1, 2, \ldots, t$. Let $j \in \{1, 2, \ldots, t\}$ and consider $c(R_j) = c_j$. If $(v, v_j), (v, v_{j+1}) \in E(D)$, then c_j and $-c_j$ occur in $\sigma^+(v;\phi)$ and c_j does not occur in $\sigma^-(v;\phi)$. The situation is reversed if $(v_j, v), (v_{j+1}, v) \in E(D)$. If $(v, v_{j+1}), (v_j, v) \in E(D)$, then the term c_j occurs in both $\sigma^+(v;\phi)$ and $\sigma^-(v;\phi)$; while if $(v_{j+1}, v), (v, v_j) \in E(D)$, then the term $-c_j$ occurs in both $\sigma^+(v;\phi)$ and $\sigma^-(v;\phi)$. Thus

$$\sigma^+(v;\phi) = \sigma^-(v;\phi)$$

and ϕ is a nowhere-zero k-flow.

Next, suppose that G is a bridgeless plane graph having a nowhere-zero k-flow. This implies that for a given orientation D of G, there exists a nowhere-zero k-flow ϕ of D. By definition then, $\sigma^+(v;\phi) = \sigma^-(v;\phi)$ for every vertex v of D.

We now consider directed closed curves in the plane that do not pass through any vertex of D. Such closed curves may enclose none, one, or several vertices of G. For a directed closed curve C, we define the number $\sigma(C;\phi)$ to be the sum of terms $\phi(e)$ or $-\phi(e)$ for each occurrence of an arc e crossed by C. In particular, as we proceed along C in the direction of C and cross an arc e, we contribute $\phi(e)$ to $\sigma(C;\phi)$ if e is directed to the right of C and contribute $-\phi(e)$ to $\sigma(C;\phi)$ if e is directed to the left of C.

If C is a directed simple closed curve in the plane that encloses no vertex of D, then for each occurrence of an arc e crossed by C that is directed to the right of C, there is an occurrence of e crossed by C that is directed to the left of C. Hence in this case, $\sigma(C;\phi) = 0$. If C encloses a single vertex v, then because $\sigma^+(v;\phi) = \sigma^-(v;\phi)$, it follows that $\sigma(C;\phi) = 0$ here as well. (See Figure 11.28 for example.)

$$\sigma(C;\phi) = (\phi(e_1) + \phi(e_4)) - (\phi(e_2) + \phi(e_3) + \phi(e_5)) = 0$$

Figure 11.28: Computing $\sigma(C;\phi)$

Suppose now that C is a directed simple closed curve in the plane that encloses two or more vertices. Let $S = \{v_1, v_2, \cdots, v_s\}$, $s \geq 2$, be the set of vertices of D lying interior to C. Because, by Theorem 11.25, that the sum of the flow values of the arcs directed from S to $V(G) - S$ equals the sum of the flow values of the arcs directed from $V(G) - S$ to S, it follows that $\sigma(C; \phi) = 0$ as well.

If C is a directed closed curve that is not a simple closed curve, then C is a union of directed simple closed curves $C_1, C_2, \ldots, C_r$ and so

$$\sigma(C; \phi) = \sum_{i=1}^{s} \sigma(C_i; \phi) = 0.$$

Consequently, $\sigma(C; \phi) = 0$ for *every* directed closed curve C in the plane.

We now show that there is a proper coloring of the regions of G using the colors $1, 2, \ldots, k$. Assign the color k to the exterior region of G. Let R be some interior region in G. Choose a point A in the exterior region and a point B in R and let P be an open curve directed from A to B so that P passes through no vertices of D. The number $\sigma(P; \phi)$ is defined as the sum (addition performed modulo k) of the numbers $\phi(e)$ or $-\phi(e)$, for each occurrence of an arc e crossed by P, where either $\phi(e)$ or $-\phi(e)$ is contributed to the sum $\sigma(P; \phi)$ according to whether e is directed to the right or to the left of P, respectively. The least positive integer in the equivalence class containing $\sigma(P; \phi)$ in the ring of integers $\mathbb{Z}_k$ is the color $c(R)$ assigned to R. Thus $c(R) \in \{1, 2, \ldots, k\}$.

We now show that the color $c(R)$ assigned to R is well-defined. Suppose that the color assigned to R by the curve P above is $c(R) = a$. Let Q be another directed open curve from A to B and let $\sigma(Q; \phi) = b$. We claim that $a = b$. If $\tilde{Q}$ is the directed open curve from B to A obtained by reversing the direction of Q, then $\sigma(\tilde{Q}; \phi) = -b$. Now let C be the directed closed curve obtained by following P by $\tilde{Q}$. Then, as we saw, $\sigma(C; \phi) = 0$. But

$$\sigma(C; \phi) = \sigma(P; \phi) + \sigma(\tilde{Q}; \phi) = a - b.$$

So $a - b = 0$ and $a = b$. Thus the color $c(R)$ of R defined in this manner is, in fact, well-defined.

It remains to show that the region coloring c of G is proper. Let R' and R'' be two adjacent regions of D, where e' is an arc on the boundaries of both R' and R''. Let B' be a point in R' and let B'' be a point in R''. Furthermore, let P' be a directed open curve from A to B' that does not cross e' and suppose that P'' is a directed open curve from A to B'' that extends P' to B'' so that e' is the only additional arc crossed by P''. Therefore,

$$\sigma(P''; \phi) = \sigma(P'; \phi) + \phi(e')$$

or

$$\sigma(P''; \phi) = \sigma(P'; \phi) - \phi(e').$$

Since $\phi(e') \not\equiv 0 \pmod{k}$, it follows that the colors assigned to R' and R'' are distinct. Hence the region coloring c of G is proper. ■

Letting $k = 4$ in Theorem 11.27, we have the following corollary of the Four Color Theorem.

Corollary 11.28 *Every bridgeless planar graph has a nowhere-zero* 4*-flow.*

While every bridgeless planar graph has a nowhere-zero 4-flow, it is not the case that every bridgeless nonplanar graph has a nowhere-zero 4-flow.

Theorem 11.29 *The Petersen graph does not have a nowhere-zero* 4*-flow.*

Proof. Suppose that the Petersen graph P has a nowhere-zero 4-flow. Then there exists an orientation D of P and a nowhere-zero 4-flow ϕ on D such that $\phi(e) \in \{1, 2, 3\}$ for every arc e of D (by Theorem 11.23). Since $\sigma^+(v; \phi) = \sigma^-(v; \phi)$ for every vertex v of D, the only possible flow values of the three arcs incident with v are $1, 1, 2$ and $1, 2, 3$. In particular, this implies that every vertex of D is incident with exactly one arc having flow value 2. Thus the arcs of D with flow value 2 correspond to a 1-factor F of P. The remaining arcs of D then correspond to a 2-factor H of P. Because the Petersen graph is not Hamiltonian and has girth 5, the 2-factor H must consist of two disjoint 5-cycles. Every vertex v of H is incident with one or two arcs having flow value 1. Furthermore, if a vertex v of H is incident with two arcs having flow value 1, then these two arcs are either both directed toward v or both directed away from v. A vertex v of H is said to be of *type I* if there is an arc having flow value 1 directed toward v; while v is of *type II* if there is an arc having flow value 1 directed away from v. Consequently, each vertex v of H is either of type I or of type II, but not both. Moreover, the vertices in each of the 5-cycles of H must alternate between type I and type II, which is impossible for an odd cycle. ■

While the Petersen graph does not have a nowhere-zero 4-flow, it does have a nowhere-zero 5-flow, however (see Exercise 4). The Petersen graph plays an important role among cubic graphs as Neil Robertson, Daniel Sanders, Paul Seymour and Robin Thomas verified in a series of papers [193, 194, 195, 198, 199].

Theorem 11.30 *Every bridgeless cubic graph not having the Petersen graph as a minor has a nowhere-zero* 4*-flow.*

There is a question of determining the smallest positive integer k such that every bridgeless graph has a nowhere-zero k-flow. In this connection, William Tutte [223] conjectured the following.

Conjecture 11.31 *Every bridgeless graph has a nowhere-zero* 5*-flow.*

While this conjecture is still open, Paul Seymour [204] did establish the following.

Theorem 11.32 *Every bridgeless graph has a nowhere-zero* 6*-flow.*

Tait colorings deal exclusively with cubic graphs, of course, and it is this class of graphs that has received added attention when studying nowhere-zero flows. By Theorem 11.21, no cubic graph has a nowhere-zero 2-flow. In the case of nowhere-zero 3-flows, it is quite easy to determine which cubic graphs have these.

Theorem 11.33 *A cubic graph G has a nowhere-zero 3-flow if and only if G is bipartite.*

Proof. Let G be a cubic bipartite graph with partite sets U and W. Since G is regular, G contains a 1-factor F. Orient the edges of F from U to W and assign each arc the value 2. Orient all other edges of G from W to U and assign each of these arcs the value 1. This is a nowhere-zero 3-flow.

For the converse, let G be a cubic graph having a nowhere-zero 3-flow. By Theorem 11.23, there is an orientation D on which is defined a nowhere-zero 3-flow having only the values 1 and 2. In particular, the values must be 1, 1, 2 for the three arcs incident with each vertex of D. The arcs having the value 2 form a subdigraph H of D and the underlying graph of H is a 1-factor of G. Let U be the set of vertices of G where each vertex of U has outdegree 1 in H and let W be the remaining vertices of G. Every arc of D not in H must then have a value of 1 and be directed from a vertex of W to a vertex of U. Thus G is a bipartite graph with partite sets U and W. ■

Those cubic graphs having a nowhere-zero 4-flow depend only on their chromatic index.

Theorem 11.34 *A cubic graph G has a nowhere-zero 4-flow if and only if G is of Class one.*

Proof. Let G be a cubic graph that has a nowhere-zero 4-flow. By Theorem 11.23, there exists an orientation D of G and a nowhere-zero 4-flow ϕ on D such that $\phi(e) \in \{1, 2, 3\}$ for each arc e of D. Then each vertex is incident with arcs having either the values 1, 1, 2 or 1, 2, 3. In particular, each vertex of D is incident with exactly one arc having the value 2. The arcs having the value 2 produce a 1-factor in G and so the remaining arcs of D produce a 2-factor of G. If the 2-factor is a Hamiltonian cycle of G, then G is of Class one. Hence we may assume that this 2-factor consists of two or more mutually disjoint cycles. Let C be one of these cycles and let H be the orientation of C in D.

Suppose that C is an r-cycle and let $S = V(C)$. Since each arc of D joining a vertex of S and a vertex of $V(D) - S$ has flow value 2 and since $\sigma^+(S, \phi) = \sigma^-(S; \phi)$ by Theorem 11.25, it follows that there is an even number t of arcs joining the vertices of S and the vertices of $V(D) - S$. Since this set of arcs is independent, it follows that $G[S]$ contains t vertices of degree 2 and $r - t$ vertices of degree 3. Because $G[S]$ has an even number of odd vertices, $r - t$ is even. However, since t is also even, r is even as well and C is an even cycle. Thus the 2-factor of G is the union of even cycles and so G is of Class one.

We now verify the converse. Let G be a cubic graph of Class one. Therefore, G is 3-edge colorable and 1-factorable into factors F_1, F_2, F_3, where F_i is the 1-factor ($1 \leq i \leq 3$) whose edges are colored i. Every two of these three 1-factors produce a 2-factor consisting of a union of disjoint cycles. Since the edges of each cycle alternate in colors, the cycles are even and each 2-factor is bipartite. Let G_1 be the 2-factor obtained from F_1 and F_3 and G_2 the 2-factor obtained from F_2 and F_3. Now let D be an orientation of G and let D_i be the resulting orientation of G_i ($i = 1, 2$). Since each component of each graph G_i is Eulerian, it follows by Theorem 11.21 that there is a nowhere-zero 2-flow ϕ_i on D_i. For $e \in E(D) - E(D_i)$, define $\phi_i(e) = 0$ for $i = 1, 2$. Then ϕ_i is a 2-flow on D. By Theorem 11.24, the function ϕ defined by $\phi = \phi_1 + 2\phi_2$ is also a flow on D. Because $\phi(e) \in \{\pm 1, \pm 2, \pm 3\}$, it follows that ϕ is a nowhere-zero 4-flow on D. ■

Since the Petersen graph is a cubic graph that is of Class two, it follows by Theorem 11.34 that it has no nowhere-zero 4-flow (which we also saw in Theorem 11.29). We noted that the Petersen graph does have a nowhere-zero 5-flow, however, which has been conjectured to be true for every bridgeless graph.

Cycle Double Covers

There is another concept and conjecture related to bridgeless graphs which ultimately returns us to snarks.

If G is a connected bridgeless plane graph, then the boundary of every region is a cycle and every edge of G lies on the boundaries of two regions. Thus if S is the set of cycles of G that are the boundaries of the regions of G, then every edge of G belongs to exactly two elements of S. For which other graphs G is there a collection S of cycles of G such that every edge of G belongs to exactly two cycles of S?

A **cycle double cover** of a graph G is a set (actually a multiset) S of not necessarily distinct cycles of G such that every edge of G belongs to exactly two cycles of S. Certainly no cycle of G can appear more than twice in S. Also, if G contains a bridge e, then e belongs to no cycle and G can contains no cycle double cover. If G is Eulerian, then G contains an Eulerian circuit and therefore a set S' of cycles such that every edge of G belongs to exactly one cycle of S'. If S is the set of cycles of G that contains each cycle of S' twice, then S is a cycle double cover of G.

That every bridgeless graph has a cycle double cover was conjectured by Paul Seymour [203] in 1979. George Szekeres [211] conjectured this for cubic graphs even earlier – in 1973.

The Cycle Double Cover Conjecture Every nontrivial connected bridgeless graph has a cycle double cover.

As we have seen, this conjecture is true for all nontrivial connected bridgeless

planar graphs and for all Eulerian graphs. Initially, it may seem apparent that the Cycle Double Cover Conjecture is true, for if we were to replace each edge of a bridgeless graph G by two parallel edges then the resulting multigraph H is Eulerian. This implies that there is a set S of cycles of H such that each edge of H belongs to exactly one cycle in S. This, in turn, implies that each edge of G belongs to exactly two cycles in S, completing the proof. This argument is faulty, however, for one or more cycles of H in S may be 2-cycles, which do not correspond to cycles of G. Nevertheless, no counterexample to the Cycle Double Cover Conjecture is known. If the Cycle Double Cover Conjecture is false, then there exists a minimum counterexample, namely, a connected bridgeless graph of minimum size having no cycle double cover. Francois Jaeger (1947–1997) proved that a minimum counterexample to the Cycle Double Cover Conjecture must be a snark [125]. However, all known snarks possess a cycle double cover.

Exercises for Section 11.4

1. Let G be the Eulerian graph shown in Figure 11.29.

Figure 11.29: The graph G in Exercise 1

   (a) Give an example of an orientation D_1 of G and a nowhere-zero 2-flow ϕ_1 on D such that all flow values are -1.
   (b) Give an example of an orientation D_2 of G and a nowhere-zero 2-flow ϕ_2 on D such that exactly five flow values are $+1$.
   (c) Give an example of an orientation D_3 of G and a nowhere-zero 2-flow ϕ_3 on D such that half of the flow values are $+1$.

2. Let G be a graph having a nowhere-zero k-flow for some integer $k \geq 2$. Prove that for each partition $\{E_1, E_2\}$ of $E(G)$, there exists a nowhere-zero k-flow ϕ such that $\phi(e) > 0$ if and only if $e \in E_1$.

3. Prove Theorem 11.24: *If ϕ_1 and ϕ_2 are flows on an orientation D of a graph G, then every linear combination of ϕ_1 and ϕ_2 is also a flow on D.*

4. Show that the Petersen graph has a nowhere-zero 5-flow.

5. Prove that every bridgeless graph containing an Eulerian trail has a nowhere-zero 3-flow.

6. Can a graph have a cycle double cover consisting only of three Hamiltonian cycles?

7. Show that K_6 has a cycle double cover consisting only of Hamiltonian cycles.

11.5 List Edge Colorings and Total Colorings

In Section 8.5 we encountered the topic of list colorings. In a list coloring of a graph G, there is a list (or set) of available colors for each vertex of G, with the goal being to select a color from each list so that a proper vertex coloring of G results. One of the major problems concerns the determination of the smallest positive integer k such that if every list contains k or more colors, then a proper vertex coloring of G can be constructed. This smallest positive integer k is called the list chromatic number $\chi_\ell(G)$ of G. In this section we consider the edge analogue of this concept.

Let G be a nonempty graph and for each edge e of G, let $L(e)$ be a list (or set) of colors. Furthermore, let $\mathfrak{L} = \{L(e) : e \in E(G)\}$. The graph G is **$\mathfrak{L}$-edge choosable** (or **$\mathfrak{L}$-list edge colorable**) if there exists a proper edge coloring c of G such that $c(e) \in L(e)$ for every edge e of G. For a positive integer k, a nonempty graph G is **k-edge choosable** (or **k-list edge colorable**) if for every set $\mathfrak{L} = \{L(e) : e \in E(G)\}$ where each $|L(e)| \geq k$, the graph G is $\mathfrak{L}$-edge choosable. The **list chromatic index** $\chi'_\ell(G)$ is the minimum positive integer k for which G is k-edge choosable. Necessarily then,

$$\chi'(G) \leq \chi'_\ell(G)$$

for every nonempty graph G. Applying a greedy edge coloring to a nonempty graph G (see Exercise 1) gives

$$\chi'_\ell(G) \leq 2\Delta(G) - 1.$$

Since the graph K_3 is of Class two, $\chi'(K_3) = 1 + \Delta(K_3) = 3$. Thus $\chi'_\ell(K_3) \geq 3$. However, if e_1, e_2 and e_3 are the three edges of K_3 and $\mathfrak{L}(e_1)$, $\mathfrak{L}(e_2)$ and $\mathfrak{L}(e_3)$ are three sets of three or more colors each, then three distinct colors $c(e_1), c(e_2)$ and $c(e_3)$ can be chosen such that $c(e_i) \in \mathfrak{L}(e_i)$ for $1 \leq i \leq 3$ and so K_3 is 3-edge choosable. Therefore, $\chi'_\ell(K_3) = 3$. Although it would be natural now to expect to see an example of a graph G with $\chi'(G) < \chi'_\ell(G)$, no graph is known to have this property.

The List Coloring Conjecture

While the following conjecture was made independently by Vadim Vizing, Ram Prakash Gupta, and Michael Albertson and Karen Collins (see [104]), it first appeared in print in a 1985 paper by Béla Bollobás and Andrew J. Harris [23].

The List Coloring Conjecture For every nonempty graph G,

$$\chi'(G) = \chi'_\ell(G).$$

The identical conjecture has been made for multigraphs as well. Since the list chromatic index of a graph equals the list chromatic number of its line graph, the List Coloring Conjecture can also be stated as $\chi_\ell(G) = \chi(G)$ for every line graph G.

In 1979 Jeffrey Howard Dinitz had already conjectured that $\chi'(G) = \chi'_\ell(G)$ when G is a regular complete bipartite graph. By Theorem 11.7, it is known that $\chi'(K_{r,r}) = r$.

Dinitz's Conjecture For every positive integer r, $\chi'_\ell(K_{r,r}) = r$.

Galvin's Theorem

Fred Galvin [91] not only verified the Dinitz Conjecture, he showed that the List Coloring Conjecture is true for all bipartite graphs (indeed, for all bipartite multigraphs). The proof of this result that we present is based on a proof of Tomaž Slivnik [206], which in turn is based on Galvin's proof. We begin with a theorem of Slivnik. First, we introduce some notation. Let G be a nonempty bipartite graph with partite sets U and W. For each edge e of G, let u_e denote the vertex of U incident with e and let w_e denote the vertex of W incident with e. For adjacent edges e and f, it therefore follows that $u_e = u_f$ or $w_e = w_f$, but not both.

Lemma 11.35 *Let G be a nonempty bipartite graph and let $c : E(G) \to \mathbb{N}$ be an edge coloring of G. For each edge e of G, let $\sigma_G(e)$ be the sum*

$$\begin{aligned}\sigma_G(e) \quad = \quad & 1 + |\{f \in E(G) : u_e = u_f \text{ and } c(f) > c(e)\}| \\ & + |\{f \in E(G) : w_e = w_f \text{ and } c(f) < c(e)\}|\end{aligned}$$

and let $L(e)$ be a set of $\sigma_G(e)$ colors. If

$$\mathfrak{L} = \{L(e) : e \in E(G)\},$$

then G is $\mathfrak{L}$-edge choosable.

Proof. We proceed by induction on the size m of G. Since the result holds if $m = 1$, the basis step for the induction is true. Assume that the statement of the theorem is true for all nonempty bipartite graphs of size less than m, where $m \geq 2$, and let G be a nonempty bipartite graph of size m on which is defined an edge coloring c and where the numbers $\sigma_G(e)$, the sets $L(e)$, and the set $\mathfrak{L}$ are defined in the statement of the theorem.

Let A be a set of edges of G. A matching $M \subseteq A$ is said to be **optimal** (in A) if the following is satisfied:

For every edge $e \in A - M$, there is an edge $f \in M$ such that either

(i) $u_e = u_f$ and $c(f) > c(e)$ or (ii) $w_e = w_f$ and $c(f) < c(e)$.

First, we show (by induction on the size of A) that for every $A \subseteq E(G)$, there is an optimal matching $M \subseteq A$.

Observe that if A itself is a matching, then $M = A$ is vacuously optimal. If $|A| = 1$, then $M = A$ is optimal and the basis step of this induction is satisfied. Assume for an integer k with $1 < k \leq m$ that for each set A' of edges of G with $|A'| = k - 1$, there is an optimal matching in A'. Let A be a set of edges of G with $|A| = k$. We show that A contains an optimal matching M.

An edge e belonging to A is **U-maximum** if there is no edge f in A for which $u_e = u_f$ and $c(f) > c(e)$, while e is **W-maximum** if there is no edge f in A for which $w_e = w_f$ and $c(f) > c(e)$. An edge $e \in A$ that is both U-maximum and W-maximum is called **c-maximum**. Consequently, an edge $e \in A$ is c-maximum if $c(f) < c(e)$ for every edge f adjacent to e. We consider two cases.

Case 1. *Every U-maximum edge in A is W-maximum.* Let

$$M = \{e \in A : \ e \text{ is } c\text{-maximum}\}.$$

We claim that M is optimal. Since no two edges of M can be adjacent, M is a matching. Let $e \in A - M$. Since e is not c-maximum, e is not U-maximum. So there exists an edge $f \in A$ for which $c(f)$ is maximum and $u_e = u_f$. This implies that f is U-maximum and is consequently c-maximum as well. Therefore, $f \in M$ and $c(f) > c(e)$. Hence M is optimal.

Case 2. *There exists an edge g in A that is U-maximum but not W-maximum.* Because g is not W-maximum, there exists an edge $h \in A$ such that $w_h = w_g$ and $c(h) > c(g)$. We consider the set $A - \{h\}$ which consists of $k - 1$ edges. By the induction hypothesis, there is an optimal matching M in $A - \{h\}$. Hence for every edge e in the set $(A - \{h\}) - M = A - (M \cup \{h\})$, there is an edge $f \in M$ for which either (i) $u_e = u_f$ and $c(f) > c(e)$ or (ii) $w_e = w_f$ and $c(f) < c(e)$.

We show that M is optimal in the set A. First, we establish the existence of an edge f in A such that either (i) $u_f = u_h$ and $c(f) > c(h)$ or (ii) $w_f = w_h$ and $c(f) < c(h)$. We consider two subcases.

Subcase 2.1. $g \notin M$. Then $g \in A - (M \cup \{h\})$. By the induction hypothesis, there is an edge $f \in M$ for which either (i) $u_g = u_f$ and $c(f) > c(g)$ or (ii) $w_g = w_f$ and $c(f) < c(g)$. Since g is U-maximum, (i) cannot occur and so (ii) must hold. Thus $c(f) < c(g) < c(h)$ and M is optimal.

Subcase 2.2. $g \in M$. Then g is the desired edge f and, again, M is optimal.

Therefore, M is optimal in either subcase and consequently, for every set A of edges of G, there exists an optimal matching $M \subseteq A$.

It now remains to show that there is an $\mathfrak{L}$-list edge coloring of G. Select a color $a \in \cup_{e \in E(G)} L(e)$ and let

$$A = \{e \in E(G) : a \in L(e)\}.$$

As verified above, there is an optimal matching M in A. Let $G' = G - M$. For each edge e in G', let $L'(e) = L(e) - \{a\}$. If $e \in E(G) - A$, then $a \notin L(e)$ and

$$|L'(e)| = |L(e)| = \sigma_G(e) \geq \sigma_{G'}(e).$$

On the other hand, if $e \in A - M$, then $a \in L(e)$. Since M is optimal in A, there is an edge $f \in M$ such that either (i) $u_e = u_f$ and $c(f) > c(e)$ or (ii) $w_e = w_f$ and $c(f) < c(e)$. Thus

$$|L'(e)| = |L(e)| - 1 = \sigma_G(e) - 1 \geq \sigma_{G'}(e).$$

Let $\mathfrak{L}' = \{L'(e) : e \in E(G')\}$.

Since the size of G' is less than that of G, it follows by the induction hypothesis that G' is $\mathfrak{L}'$-edge choosable. Thus there exists a proper edge coloring $c' : E(G') \to \mathbb{N}$ of G' such that $c'(e) \in L'(e)$ for every edge e of G'. Define $c : E(G) \to \mathbb{N}$ by

$$c(e) = \begin{cases} c'(e) & \text{if } e \in E(G') \\ a & \text{if } e \in M. \end{cases}$$

Then $c(e) \in L(e)$ for every edge e of G and $c(e) \neq c(f)$ for every two adjacent edges e and f of G. Hence c is a proper edge coloring and G is $\mathfrak{L}$-edge choosable. ■

From Lemma 11.35, we can now present a proof of Galvin's theorem.

Theorem 11.36 (**Galvin's Theorem**) *If G is a bipartite graph, then*

$$\chi'_\ell(G) = \chi'(G).$$

Proof. Since G is bipartite, it follows by König's theorem (Theorem 11.7) that $\chi'(G) = \Delta(G) = \Delta$. Thus there exists a proper edge coloring $c : E(G) \to \{1, 2, \ldots, \Delta\}$. For each edge e of G, let $L(e)$ be a list of colors such that

$$\begin{aligned} |L(e)| &= \sigma_G(e) = 1 + |\{f \in E(G) : u_e = u_f \text{ and } c(f) > c(e)\}| \\ &\qquad + |\{f \in E(G) : w_e = w_f \text{ and } c(f) < c(e)\}| \\ &\leq 1 + (\chi'(G) - c(e)) + (c(e) - 1) = \chi'(G). \end{aligned}$$

Let $\mathfrak{L} = \{L(e) : e \in E(G)\}$. By Lemma 11.35, G is $\mathfrak{L}$-edge choosable. Therefore, G is $\chi'(G)$-edge choosable and so $\chi'_\ell(G) \leq \chi'(G)$. Since $\chi'(G) \leq \chi'_\ell(G)$, we have the desired result. ■

Since every bipartite graph is of Class one, it follows that the list chromatic index of every bipartite graph G equals $\Delta(G)$.

The Total Coloring Conjecture

We now consider colorings that assign colors to both the vertices and the edges of a graph. A **total coloring** of a graph G is an assignment of colors to

the vertices and edges of G such that distinct colors are assigned to (1) every two adjacent vertices, (2) every two adjacent edges and (3) every incident vertex and edge. A **k-total coloring** of a graph G is a total coloring of G from a set of k colors. A graph G is **k-total colorable** if there is a k-total coloring of G. The **total chromatic number** $\chi''(G)$ of a graph G is the minimum positive integer k for which G is k-total colorable.

If c is a total coloring of a graph G and v is a vertex of G with $\deg v = \Delta(G)$, then c must assign distinct colors to the $\Delta(G)$ edges incident with v as well as to v itself. This implies that

$$\chi''(G) \geq 1 + \Delta(G) \textit{ for every graph } G.$$

Let $G = K_6 - e$ be the graph of order $n = 6$ and size $m = \binom{6}{2} - 1$ shown in Figure 11.30, where $e = xz$. Since $\Delta(G) = 5$, it follows that $\chi''(G) \geq 1 + \Delta(G) = 6$. The 7-total coloring of G indicated in the figure shows that $\chi''(G) \leq 7$. We show that $\chi''(G) = 7$. Assume, to the contrary, that $\chi''(G) = 6$. Thus there exists a 6-total coloring of G. No color can be assigned to more than two vertices. Suppose that some color is assigned to two vertices. Then these vertices must be x and z. This color can be assigned to at most two edges and so this color can be used for at most four vertices and edges. If some color is assigned to one vertex, then this color can be assigned to at most two edges; while if some color is assigned to no vertices, then this color can be assigned to at most three edges. Therefore, the total number of vertices and edges that can be colored with six colors cannot exceed $4 + 5 \cdot 3 = 19$. However, since $n + m = 6 + 14 = 20$, this is a contradiction. Thus, as claimed, $\chi''(G) = 7$.

Figure 11.30: A graph with total chromatic number 7

The total chromatic number of complete graphs shows that both $\chi''(G) = 1 + \Delta(G)$ and $\chi''(G) > 1 + \Delta(G)$ are possible (see Exercise 4).

Theorem 11.37 *For an integer $n \geq 2$,*

$$\chi''(K_n) = \begin{cases} n & \textit{if } n \textit{ is odd} \\ n+1 & \textit{if } n \textit{ is even.} \end{cases}$$

In the 1960s Mehdi Behzad [8] and Vadim Vizing [228] independently conjectured that, similar to the upper bound for the chromatic index established

by Vizing, the total chromatic number cannot exceed this lower bound by more than 1. This conjecture has become known as the Total Coloring Conjecture.

The Total Coloring Conjecture For every graph G,

$$\chi''(G) \leq 2 + \Delta(G).$$

The sum of the chromatic number and chromatic index of a graph is always an upper bound for the total chromatic number of the graph.

Theorem 11.38 *If G is a nonempty graph, then*

$$\chi''(G) \leq \chi(G) + \chi'(G).$$

Proof. Suppose that $\chi(G) = k$ and $\chi'(G) = \ell$. Then there is a k-vertex coloring of G using the colors $1, 2, \ldots, k$ and an ℓ-edge coloring of G using the colors $k+1, k+2, \ldots, k+\ell$. Since such a coloring of $V(G) \cup E(G)$ also assigns distinct colors to every incident vertex and edge, this produces a $(k+\ell)$-total coloring of G and so $\chi''(G) \leq k + \ell = \chi(G) + \chi'(G)$. ■

Since $3 = \chi''(K_2) = \chi(K_2) + \chi'(K_2) = 2 + 1$, equality is possible in Theorem 11.38. On the other hand, if G is a graph with $\chi(G) \geq 3$ and $\chi''(G) = \chi(G) + \chi'(G)$, then $\chi''(G) = \chi(G) + \chi'(G) \geq 3 + \Delta(G)$, which would provide a counterexample to the Total Coloring Conjecture. No such counterexample is possible, as the following result of Mehdi Behzad, Gary Chartrand and John K. Cooper, Jr. [9] shows.

Theorem 11.39 *Let G be a nonempty graph. If $\chi''(G) = \chi(G) + \chi'(G)$, then G is bipartite.*

Proof. Suppose that G is a graph that is not bipartite. Then G is a graph of order at least 3 such that $\chi(G) = k$ for some integer $k \geq 3$. Then $V(G)$ can be partitioned into k vertex color classes (independent sets) $V_1, V_2, \ldots, V_k$. Suppose that $\chi'(G) = \ell$. Then $E(G)$ can be partitioned into ℓ edge color classes $E_1, E_2, \ldots, E_\ell$. For each edge $e = uv \in E_\ell$, there is a set V_i $(1 \leq i \leq k)$ containing neither u nor v. We move e from E_ℓ to V_i. After moving each edge of E_ℓ to one of the sets $V_1, V_2, \ldots, V_k$ in this manner, k new sets $X_1, X_2, \ldots, X_k$ are produced such that $V_i \subseteq X_i$ for all i $(1 \leq i \leq k)$ and $E_\ell \subseteq \cup_{i=1}^{k} X_i$, we now assign the color i $(1 \leq i \leq k)$ to each element of X_i and assign the color $k + j$ to each edge of E_j $(1 \leq j \leq \ell - 1)$. This produces a $(k + \ell - 1)$-total coloring of G and so

$$\chi''(G) \leq k + \ell - 1 = \chi(G) + \chi'(G) - 1,$$

completing the proof. ■

We now present an infinite class of (necessarily bipartite) graphs G for which $\chi''(G) = \chi(G) + \chi'(G)$.

Theorem 11.40 *For every positive integer r,*

$$\chi''(K_{r,r}) = \chi(K_{r,r}) + \chi'(K_{r,r}).$$

Proof. Let $G = K_{r,r}$. Then $|V(G) \cup E(G)| = r^2 + 2r$. Suppose that $\chi''(G) = k$. Then $V(G) \cup E(G)$ can be partitioned into sets $X_1, X_2, \ldots, X_k$ where each set X_i $(1 \le i \le k)$ consists of an independent set of vertices and an independent set of edges, where no vertex of X_i is incident with an edge of X_i. Necessarily, $|X_i| \le r$ for each i $(1 \le i \le k)$. Hence $k \ge r + 2$. Since $\chi'(K_{r,r}) = r$ by König's theorem (Theorem 11.7) and $\chi(K_{r,r}) = 2$, it follows by Theorem 11.38 that $\chi''(G) \le r + 2$ and so

$$\chi''(G) = 2 + r = \chi(K_{r,r}) + \chi'(K_{r,r}),$$

as desired. ■

Even though it is not known if $2 + \Delta(G)$ is an upper bound for the total chromatic number of every graph, the number $2 + \chi'_\ell(G)$ *is* an upper bound.

Theorem 11.41 *For every graph G,*

$$\chi''(G) \le 2 + \chi'_\ell(G).$$

Proof. Suppose that $\chi'_\ell(G) = k$. Then

$$\begin{aligned} \chi(G) &\le 1 + \Delta(G) \le 1 + \chi'(G) \le 1 + \chi'_\ell(G) \\ &< 2 + \chi'_\ell(G) = 2 + k. \end{aligned}$$

Thus G is $(k+2)$-colorable. Let a $(k+2)$-coloring c of G be given, using the colors $1, 2, \ldots, k+2$, say. For each edge $e = uv$ of G, let $L(e) = \{1, 2, \ldots, k+2\}$ and let

$$L'(e) = L(e) - \{c(u), c(v)\}.$$

Since $|L'(e)| \ge k$ for each edge e of G and $\chi'_\ell(G) = k$, it follows that there is a proper edge coloring c' of G such that $c'(e) \in L'(e)$ and so $c'(e) \notin \{c(u), c(v)\}$. Hence the total coloring c'' of G defined by

$$c''(x) = \begin{cases} c(x) & \text{if } x \in V(G) \\ c'(x) & \text{if } x \in E(G) \end{cases}$$

is a $(k+2)$-total coloring of G and so

$$\chi''(G) \le 2 + k = 2 + \chi'_\ell(G),$$

as desired. ■

The List Coloring Conjecture states that $\chi'(G) = \chi'_\ell(G)$ for every nonempty graph G. If this conjecture is true, then $\chi'_\ell(G) \le 1 + \Delta(G)$ by Vizing's theorem (Theorem 11.2) and so $\chi''(G) \le 3 + \Delta(G)$ by Theorem 11.41. In 1998 Michael Molloy and Bruce Reed [160] established the existence of a constant c such that $c + \Delta(G)$ is an upper bound for $\chi''(G)$ for every graph G. In particular, they proved the following.

Theorem 11.42 *For every graph G,*

$$\chi''(G) \leq 10^{26} + \Delta(G).$$

Just as the chromatic index of a nonempty graph G equals the chromatic number of its line graph $L(G)$, the total chromatic number of G also equals the chromatic number of a related graph.

Recall that the total graph $T(G)$ of a graph G is that graph for which $V(T(G)) = V(G) \cup E(G)$ and such that two distinct vertices x and y of $T(G)$ are adjacent if x and y are adjacent vertices of G, adjacent edges of G or an incident vertex and edge. It therefore follows that

$$\chi''(G) = \chi(T(G)) \textit{ for every graph } G.$$

A graph G and its total graph are shown in Figure 11.31, together with a total coloring of G and a vertex coloring of $T(G)$. In this case, $\chi''(G) = \chi(T(G)) = 4$

Figure 11.31: Total graphs and total colorings

Exercises for Section 11.5

1. Let G be a nonempty graph. Show that $\chi'_\ell(G) \leq 2\Delta(G) - 1$ by applying a greedy edge coloring to G.

2. Determine $\chi(G)$, $\chi'(G)$ and $\chi''(G)$ for the 4-regular graph $G = C_7^2$ shown in Figure 11.32.

G :

Figure 11.32: The graph in Exercise 2

3. Determine $\chi''(G)$ for the graph G of Figure 11.33.

G :

Figure 11.33: The graph in Exercise 3

4. Verify Theorem 11.37. *For an integer $n \geq 2$,*

   (*a*) $\chi''(K_n) = n$ *if n is odd;*

   (*b*) $\chi''(K_n) = n + 1$ *if n is even.*

5. Show for integers s and t with $1 \leq s < t$ that $\chi''(K_{s,t}) = t + 1$.

6. Verify the Total Coloring Conjecture for graphs G with $\Delta(G) \leq 2$.

Chapter 12

Extremal Graph Theory

In this final chapter, we introduce an area of graph theory referred to as *extremal graph theory*. Of the many problems encountered in this area, a common one asks for the minimum size of a graph G of a given order which guarantees that G contains a certain subgraph or possesses a certain property. While a result of this type due to Paul Turán is often considered the beginning of extremal graph theory, there is a host of problems that may be considered as belonging to this area, including problems dealing with the existence of regular graphs of a given degree and a given girth as well as several that lie in a well-known area of graph theory called Ramsey theory.

12.1 Turán's Theorem

The first result in graph theory that might be considered belonging to extremal graph theory is a 1907 theorem due to Willem Mantel [149] who determined the minimum size of a graph of a given order which guarantees that the graph contains a triangle. This result was presented (and proved) in Chapter 1 as Theorem 1.9, which we state again below.

Theorem 12.1 *If the size of a graph G of order $n \geq 3$ is at least $\lfloor n^2/4 \rfloor + 1$, then G contains a triangle.*

A much more general result was obtained by Paul Turán in 1941 and it is this result that is often credited as beginning extremal graph theory. For an integer $n \geq 3$ and for each positive integer $k \leq n$, let $n_1, n_2, \ldots, n_k$ be k integers such that

$$n = n_1 + n_2 + \ldots + n_k,\ 1 \leq n_1 \leq n_2 \leq \ldots \leq n_k, \text{ and } n_k - n_1 \leq 1.$$

For every two integers k and n with $1 \leq k \leq n$, the integers $n_1, n_2, \ldots, n_k$ are unique. For example, if $n = 11$ and $k = 3$, then n_1, n_2, n_3 are $3, 4, 4$; while if $n = 14$ and $k = 6$, then $n_1, n_2, \ldots, n_6$ are $2, 2, 2, 2, 3, 3$. The complete k-partite

graph $K_{n_1,n_2,\ldots,n_k}$ is called the **Turán graph** $T_{n,k}$. Thus the Turán graph $T_{n,k}$ is the complete k-partite graph of order n, the cardinalities of whose partite sets differ by at most 1. The number of elements in each partite set of $T_{n,k}$ is either $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$. If n/k is an integer, then $\lfloor \frac{n}{k} \rfloor = \lceil \frac{n}{k} \rceil = \frac{n}{k}$; while if n/k is not an integer and r is the remainder when n is divided by k, then exactly r of the partite sets of $T_{n,k}$ have cardinality $\lceil \frac{n}{k} \rceil$.

Since the Turán graph $T_{n,k}$ is a complete k-partite graph, $T_{n,k}$ contains no $(k+1)$-clique (a clique of order $k+1$). The size of $T_{n,k}$ is denoted by $t_{n,k}$. Thus there exists a graph of order n and size $t_{n,k}$ containing no $(k+1)$-clique. Turán [219] showed that for positive integers n and k with $n \geq k$, the graph $T_{n,k}$ is the unique graph of order n and maximum size having no $(k+1)$-clique.

Theorem 12.2 (**Turán's Theorem**) *Let n and k be positive integers with $n \geq 3$ and $n \geq k$. The Turán graph $T_{n,k}$ is the unique graph of order n and maximum size having no $(k+1)$-clique.*

Proof. We proceed by induction on k. Since the only graph of order n containing no 2-clique is $\overline{K}_n = T_{n,1}$, the result holds for $k = 1$. Assume for an integer $k \geq 2$ and an integer n' with $n' \geq k-1$ that $T_{n',k-1}$ is the unique graph of order n' and maximum size containing no k-clique. Next, let $n \geq k$ and let G be a graph of order n and maximum size containing no $(k+1)$-clique. We show that $G = T_{n,k}$.

Let v be a vertex of G with $\deg v = \Delta(G) = \Delta$ and let $H = G[N_G(v)]$. Thus H is a graph of order Δ containing no k-clique. By the induction hypothesis, $H' = T_{\Delta,k-1}$ is the unique graph of order Δ and maximum size containing no k-clique. Thus $|E(H)| \leq |E(H')|$ and

$$|E(H)| = |E(H')| \text{ only if } H = H' = T_{\Delta,k-1}.$$

Define $G' = \overline{K}_{n-\Delta} \vee H'$. Hence G' has order n. Also, since H' is a $(k-1)$-partite graph, G' is a k-partite graph and so contains no $(k+1)$-clique. Furthermore,

$$|E(G')| = |E(H')| + \Delta(n-\Delta).$$

Since at most Δ edges of G do not belong to H for each $u \in V(G) - V(H)$, it follows that

$$|E(G) \leq |E(H)| + \Delta(n-\Delta).$$

Therefore,

$$|E(G)| \leq |E(H)| + \Delta(n-\Delta) \leq |E(H')| + \Delta(n-\Delta) = |E(G')|. \tag{12.1}$$

Since G is a graph of order n and maximum size containing no $(k+1)$-clique, it follows that $|E(G)| \geq |E(G')|$ and so $|E(G)| = |E(G')|$. Thus we have equality throughout (12.1). Because $|E(G)| = |E(H)| + \Delta(n-\Delta)$, it follows that $G = \overline{K}_{n-\Delta} \vee H$. Since $|E(H)| = |E(H')|$ and $H' = T_{\Delta,k-1}$ is the unique graph of order Δ and maximum size containing no k-clique, it follows that

$H = H' = T_{\Delta,k-1}$. Thus $G = \overline{K}_{n-\Delta} \vee T_{\Delta,k-1}$ and so G is a complete k-partite graph, which implies that $G = G'$.

Suppose that the partite sets of G are $V_1, V_2, \ldots, V_k$ where $|V_1| \leq |V_2| \leq \cdots \leq |V_k|$. If $G \neq T_{n,k}$, then V_k contains at least two more vertices than V_1. Let $x \in V_k$. Replacing V_1 by $V_1 \cup \{x\}$ and V_k by $V_k - \{x\}$ produces a complete k-partite graph F whose size exceeds that of G by $|V_k| - |V_1| \geq 1$, which is a contradiction. Thus $T_{n,k}$ is the unique graph of order n and maximum size containing no $(k+1)$-clique. ■

As a consequence of Turán's theorem, we have the following.

Corollary 12.3 *Let k and n be integers with $n \geq k+1 \geq 2$. The minimum size of a graph G of order n which guarantees that G contains a $(k+1)$-clique is $t_{n,k} + 1$.*

Graphs with Two Disjoint Cycles

We have already seen that the minimum size of a graph G of order $n \geq 3$ which guarantees that G contains a cycle is n. Although barely a teenager at the time, Lajos Pósa (see [70]) determined the minimum size of a graph G of order $n \geq 6$ which guarantees that G contains two disjoint cycles.

Theorem 12.4 *Every graph of order $n \geq 6$ and size at least $3n - 5$ contains two disjoint cycles.*

Proof. It suffices to show that every graph of order n and size $3n-5$ contains two disjoint cycles for $n \geq 6$. We employ induction on n. There are only two graphs of order 6 and size 13, one obtained by removing two nonadjacent edges from K_6 and the other obtained by removing two adjacent edges from K_6. In either case, the resulting graph has two disjoint triangles. Thus, the result is true for $n = 6$.

Assume for a given integer $n > 6$ that for all integers k with $6 \leq k < n$, every graph of order k and size $3k - 5$ contains two disjoint cycles. Let G be a graph of order n and size $3n - 5$. Since

$$\sum_{v \in V(G)} \deg v = 6n - 10,$$

there exists a vertex v_0 of G such that $\deg v_0 \leq 5$. Suppose first that $\deg v_0 = 5$, and $N(v_0) = \{v_1, v_2, \ldots, v_5\}$. If the subgraph $G[N[v_0]]$ induced by the closed neighborhood $N[v_0]$ contains 13 or more edges, then we have already noted that $G[N[v_0]]$ has two disjoint cycles, implying that G has two disjoint cycles. If, on the other hand, $G[N[v_0]]$ contains 12 or fewer edges, then, since $\deg v_0 = 5$, some neighbor of v_0, say v_1, is not adjacent to two other neighbors of v_0, say v_2 and v_3. Add to G the edges v_1v_2 and v_1v_3 and delete the vertex v_0, obtaining

the graph G'; that is, $G' = G + v_1v_2 + v_1v_3 - v_0$. The graph G' is a graph of order $n-1$ and size $3n-8$ and, by the inductive hypothesis, contains two disjoint cycles C_1 and C_2. At least one of these cycles, say C_1, does not contain the vertex v_1 and thus contains neither the edge v_1v_2 nor the edge v_1v_3. Hence C_1 is a cycle of G. If C_2 contains neither v_1v_2 nor v_1v_3, then C_1 and C_2 are disjoint cycles of G. If C_2 contains v_1v_2 but not v_1v_3, then by removing v_1v_2 and adding v_0, v_0v_1 and v_0v_2, we produce a cycle of G that is disjoint from C_1. The procedure is similar if C_2 contains v_1v_3 but not v_1v_2. If C_2 contains both v_1v_2 and v_1v_3, then by removing v_1 from C_2 and adding v_0, v_0v_2 and v_0v_3, a cycle of G disjoint from C_1 is produced.

Suppose next that $\deg v_0 = 4$, where $N(v_0) = \{v_1, v_2, v_3, v_4\}$. If $G[N[v_0]]$ is not complete, then some two vertices of $N(v_0)$ are not adjacent, say v_1 and v_2. By adding v_1v_2 to G and deleting v_0, we obtain a graph G' of order $n-1$ and size $3n-8$, which, by the inductive hypothesis, contains two disjoint cycles. We may proceed as before to show now that G has two disjoint cycles. Assume then that $G[N[v_0]]$ is a complete graph of order 5. If some vertex of $V(G) - N[v_0]$ is adjacent to two or more neighbors of v_0, then G contains two disjoint cycles. Hence we may assume that no vertex of $V(G) - N[v_0]$ is adjacent to more than one vertex of $N(v_0)$. Remove the vertices v_0, v_1 and v_2 from G, and note that the resulting graph G'' has order $n-3$ and contains at least $(3n-5)-(n-5)-9 = 2n-9$ edges. However, $n \geq 6$ implies that $2n-9 \geq n-3$; so G'' contains at least one cycle C. The cycle C and the cycle (v_0, v_1, v_2, v_0) are disjoint and belong to G.

Finally, we assume that $\deg v_0 \leq 3$. The graph $G - v_0$ is a graph of order $n-1$ and size m, where $m \geq 3n-8$. Hence by the inductive hypothesis, $G - v_0$ (and therefore G) contains two disjoint cycles. ■

To see that the bound $3n-5$ presented in Theorem 12.4 is sharp, observe that the complete 4-partite graph $K_{1,1,1,n-3}$ has order n and size $3n-6$, and that every cycle in $K_{1,1,1,n-3}$ contains at least two of the three vertices having degree $n-1$. Thus no two cycles of $K_{1,1,1,n-3}$ are disjoint.

Subgraphs with Given Minimum Degree

Another example of a theorem in extremal graph theory gives the minimum number of edges that a graph G of order n must have to guarantee that G contains a subgraph with a specified minimum degree.

Theorem 12.5 *Let k and n be integers with $1 \leq k < n$. Every graph of order n and size at least*

$$(k-1)n - \binom{k}{2} + 1$$

contains a subgraph with minimum degree k.

Proof. We proceed by induction on $n \geq k+1$. First, assume that $n = k+1$. Let G be a graph of order n and size at least

$$(k-1)n - \binom{k}{2} + 1 = (n-2)n - \binom{n-1}{2} + 1 = \binom{n}{2}.$$

Then $G = K_n = K_{k+1}$ and so G itself is a graph with minimum degree k.

Assume that every graph of order $n-1 \geq k+1$ and size at least

$$(k-1)(n-1) - \binom{k}{2} + 1$$

contains a subgraph with minimum degree k. Let G be a graph of order n and size m, where

$$m \geq (k-1)n - \binom{k}{2} + 1.$$

We show that G contains a subgraph with minimum degree k. If G itself is not such a graph, then G contains a vertex v with $\deg v \leq k-1$. Then the order of $G - v$ is $n-1$ and its size is at least

$$m - \deg v \geq (k-1)n - \binom{k}{2} + 1 - (k-1) = (k-1)(n-1) - \binom{k}{2} + 1.$$

By the induction hypothesis, $G - v$, and therefore G as well, contains a subgraph with minimum degree k. ■

The bound given in Theorem 12.5 cannot be improved. If $k = 1$, then the graph $\overline{K}_n$ contains no subgraph with minimum degree 1. More generally, for $2 \leq k < n$, the graph $\overline{K}_{n-k+1} \vee K_{k-1}$ has order n and size $(k-1)n - \binom{k}{2}$ but contains no subgraph with minimum degree k. For $n = 9$ and $k = 4$, the graph $\overline{K}_{n-k+1} \vee K_{k-1} = \overline{K}_6 \vee K_3$ (shown in Figure 12.1) has no subgraph with minimum degree 4.

Figure 12.1: A graph of order 9 and size 21
containing no subgraph with minimum degree 4

By Theorem 2.20, if G is a graph such that $\delta(G) \geq k$ for some positive integer k, then G contains every tree of size k as a subgraph. Combining this result with Theorem 12.5 gives us the following corollary.

Corollary 12.6 *Let k and n be integers with $1 \leq k < n$. If G is a graph of order n and size at least*

$$(k-1)n - \binom{k}{2} + 1,$$

then G contains every tree of size k as a subgraph.

Corollary 12.6 is not best possible, however. A 1963 conjecture due to Paul Erdös and Vera Sós states that if k and n are integers with $1 \leq k < n$ and G is a graph of order n and size greater than $n(k-1)/2$, then G contains every tree of size k as a subgraph.

Exercises for Section 12.1

1. For each ordered pair $(n, k) \in \{(7, 2), (7, 3), (7, 4), (8, 2), (8, 3), (8, 4)\}$, determine the Turán graph $T_{n,k}$ and its size $t_{n,k}$.

2. By Corollary 12.6, if G is a graph of order n and size at least $(k-1)n - \binom{k}{2} + 1$, where $1 \leq k < n$, then G contains every tree of size k as a subgraph. Show that this bound on the size of G is not best possible by finding two integers k and n with $1 \leq k < n$ such that every graph of order n and size m contains every tree of size k as a subgraph but $m \leq (k-1)n - \binom{k}{2}$.

3. Show that every graph of order $n \geq 3$ and size $\lfloor n/2 \rfloor + 1$ contains P_3 as a subgraph. For each integer $n \geq 3$, describe all graphs of order n and size $\lfloor n/2 \rfloor$ not containing P_3 as a subgraph.

4. Show that every graph of order $n \geq 4$ and size n contains $2K_2$ as a subgraph. For each integer $n \geq 4$, describe all graphs of order n and size $n-1$ not containing $2K_2$ as a subgraph.

5. For each integer $n \geq 4$, determine the smallest integer m such that every graph of order n and size m contains $K_{1,3}$ as a subgraph. For each integer $n \geq 4$, describe all graphs of order n and size $m-1$ not containing $K_{1,3}$ as a subgraph.

6. Prove that every graph of order $n \geq 4$ and size at least $2n-2$ contains a subdivision of K_4 as a subgraph.

7. (a) Let G be a graph of order $n \geq 4$. Prove that if $\deg v \geq \frac{2n+1}{3}$ for every vertex v of G, then every edge of G belongs to a complete subgraph of order 4.

   (b) Show that the result in (a) is best possible in general by showing that $\frac{2n+1}{3}$ cannot be replaced by $\frac{2n}{3}$.

12.2 Cages

An extremal problem which has received a great deal of attention for more than half a century is that of determining for fixed integers $r \geq 2$ and $g \geq 3$ the smallest order of an r-regular graph whose smallest cycle has length g. Recall that the length of a smallest cycle in a graph G is the girth of G, denoted by g(G). The study of this problem was initiated by William Tutte [222] in 1947. A related problem of determining the minimum order of an r-regular Hamiltonian graph of girth g for given integers r and g was described by Francesco Kárteszi [131] in 1960.

For integers $r \geq 2$ and $g \geq 3$, an (r, g)**-graph** is an r-regular graph having girth g. For $r, g \geq 3$, define the integer $M(r, g)$ by

$$M(r,g) = \begin{cases} 1 + r + r(r-1) + r(r-1)^2 + \cdots + r(r-1)^{(g-3)/2} & \text{if } g \text{ is odd} \\ 2\left[1 + (r-1) + (r-1)^2 + \cdots + (r-1)^{(g-2)/2}\right] & \text{if } g \text{ is even.} \end{cases}$$

The number $M(r, g)$ is, in fact, a lower bound for the order of every (r, g)-graph and is called the **Moore bound**, named for Edward F. Moore, whom we will encounter later in this section.

Theorem 12.7 *If G is an (r, g)-graph of order n, then $n \geq M(r, g)$.*

Proof. First, suppose that g is odd. Then $g = 2k+1$ for some positive integer k. Let $v \in V(G)$. For $1 \leq i \leq k$, the number of vertices at distance i from v is $r(r-1)^{i-1}$. Hence

$$\begin{aligned} n &\geq 1 + r + r(r-1) + r(r-1)^2 + \cdots + r(r-1)^{k-1} \\ &= 1 + r + r(r-1) + r(r-1)^2 + \cdots + r(r-1)^{(g-3)/2}. \end{aligned}$$

Next, suppose that g is even. Then $g = 2k$ where $k \geq 2$. Let $e = uv \in E(G)$. For $1 \leq i \leq k-1$, the number of vertices at distance i from u or v is $2(r-1)^i$. Thus

$$\begin{aligned} n &\geq 2 + 2(r-1) + 2(r-1)^2 + \cdots + 2(r-1)^{k-1} \\ &= 2\left[1 + (r-1) + (r-1)^2 + \cdots + (r-1)^{(g-2)/2}\right]. \end{aligned}$$

Thus $n \geq M(r, g)$. ■

For integers $r \geq 2$ and $g \geq 3$, let $n(r, g)$ denote the smallest order of an (r, g)-graph. The r-regular graphs of order $n(r, g)$ having girth g are called (r, g)**-cages**. The $(3, g)$-cages are commonly referred to simply as g**-cages**. Thus, among the (r, g)-graphs, an (r, g)-cage is one of minimum order.

Existence of Cages

It is clear that $n(r,g) \geq \max\{r+1, g\}$. Since C_g is a 2-regular graph of girth g, it follows that $n(2,g) = g$. Likewise, $n(r,3) = r+1$ since K_{r+1} is an r-regular graph having girth 3. In fact, the complete graph K_4 is the unique 3-cage. A fundamental question is whether an (r,g)-cage exists for every pair r, g of integers with $r \geq 2$ and $g \geq 3$. This, of course, is equivalent to the question of whether an (r,g)-graph exists for every pair r, g of integers with $r \geq 2$ and $g \geq 3$. The following result of Paul Erdös and Horst Sachs [73] not only establishes the existence of all cages but provides an upper bound on their orders.

Theorem 12.8 *For every pair r, g of integers with $r \geq 2$ and $g \geq 3$, the number $n(r,g)$ exists and*

$$n(r,g) \leq 3(r-1) + 2\left[(r-1)^2 + (r-1)^3 + \cdots + (r-1)^{g-2}\right] + (r-1)^{g-1}.$$

Proof. Since $n(2,g) = g$ for $g \geq 3$ and $n(r,3) = r+1$ for $r \geq 2$, we may assume that $r \geq 3$ and $g \geq 4$. Let

$$n = 3(r-1) + 2\left[(r-1)^2 + (r-1)^3 + \cdots + (r-1)^{g-2}\right] + (r-1)^{g-1}$$

and let S be the set of all graphs H of order n such that $g(H) = g$ and $\Delta(H) \leq r$. Thus $n \geq g$. Since $C_g + (n-g)K_1 \in S$, it follows that $S \neq \emptyset$. For each graph H in S, let

$$V_r(H) = \{v \in V(H) : \deg_H v < r\}.$$

Should it occur that $V_r(H) = \emptyset$ for some $H \in S$, then this graph H has the desired properties. We may therefore assume that $V_r(H) \neq \emptyset$ for all $H \in S$. For a graph H in S, define $d(H)$ to be the maximum distance between two vertices of $V_r(H)$ if every two vertices are connected in H and define $d(H) = +\infty$ otherwise. Observe that if $|V_r(H)| = 1$, then $d(H) = 0$.

Among the graphs belonging to S, let S_1 be the set of those graphs having maximum size. Now, among the graphs belonging to S_1, let S_2 be the set of all those graphs H for which $|V_r(H)|$ is maximum. Finally, among those graphs belonging to S_2, let G be one for which $d(G)$ is maximum.

Let $u, v \in V_r(G)$ such that $d_G(u,v) = d(G)$. Suppose first that $d(G) \geq g-1 \geq 2$. Then the graph $G' = G + uv$ has order n, girth g and $\Delta(G') \leq r$, which implies that $G' \in S$. However, since the size of G' exceeds the size of G and $G \in S$, this produces a contradiction. Therefore, $d(G) \leq g-2$ and so $d_G(u,v) \leq g-2$.

Denote by W the set of all vertices w in G such that either $d_G(u,w) \leq g-2$ or $d_G(v,w) \leq g-1$. Thus $u, v \in W$. The number of vertices distinct from u and whose distance from u is at most $g-2$ cannot exceed

$$(r-1) + (r-1)^2 + \cdots + (r-1)^{g-2}.$$

Moreover, the number of vertices distinct from v and whose distance from v is at most $g-1$ cannot exceed

$$(r-1)+(r-1)^2+\cdots+(r-1)^{g-1}.$$

Therefore,

$$\begin{aligned}|W| &\leq 2\left[(r-1)+(r-1)^2+\cdots+(r-1)^{g-2}\right]+(r-1)^{g-1}\\ &= 3(r-1)+2\left[(r-1)^2+(r-1)^3+\cdots+(r-1)^{g-2}\right]+\\ &\quad (r-1)^{g-1}-(r-1)\\ &= n-r+1<n.\end{aligned}$$

Since $|W|<n$, there is a vertex $w_1 \in V(G)-W$. Hence $d_G(u,w_1) \geq g-1$ and $d_G(v,w_1) \geq g$. Because $d_G(u,w_1) > d(G)$ and $u \in V_r(G)$, it follows that $w_1 \notin V_r(G)$ and so $\deg_G w_1 = r \geq 3$. Therefore, there exists an edge $e = w_1w_2$ such that $g(G-e) = g$. Since $d_G(v,w_1) \geq g$, it follows that $d(v,w_2) \geq g-1$. Since $d(G) \leq g-2$ and $v \in V_r(G)$, it follows that $w_2 \notin V_r(G)$ and $\deg_G w_2 = r$.

Let $G_1 = G - w_1w_2 + uw_1$. Observe that G_1 also belongs to S and, in fact, belongs to S_1. The set $V_r(G_1)$ contains every vertex in $V_r(G)$ except possibly u and, in addition, contains w_2. Because of the way G was chosen, it follows that $|V_r(G_1)| \leq |V_r(G)|$. This implies that

(i) $u \notin V_r(G_1)$,

(ii) $\deg_G u = r-1$ and

(iii) $|V_r(G_1)| = |V_r(G)|$.

Hence G_1 also belongs to S_2.

We claim that u is not the only vertex in $V_r(G)$, for suppose that it is. Since G contains $n-1$ vertices of degree r and one vertex of degree $r-1$, it follows that r and n are both odd. However, from the way n is defined, n is even when r is odd. Thus, as claimed, u is not the only vertex in $V_r(G)$ and so u and v are distinct vertices of $V_r(G)$.

Next, we claim that G_1 contains a $v-w_2$ path, for suppose that no such path exists. Then $d(G_1) = +\infty$. However then,

$$+\infty = d(G_1) \leq d(G) \leq g-2,$$

which is impossible. So, as claimed, G_1 contains a $v-w_2$ path. Let P be a $v-w_2$ geodesic in G_1. If P is also a $v-w_2$ path in G, then the length of P is at least $d_G(v,w_2)$. However,

$$d_G(v,w_2) \geq g-1 > d(G),$$

which is impossible since $v, w_2 \in V_r(G)$. Therefore, P is not a $v-w_2$ path in G, which implies that the edge uw_1 lies on P. Thus in G, the path P contains a

$u-v$ subpath of length $d_G(u,v)$. Since the length of P exceeds $d_G(u,v) = d(G)$, a contradiction is produced.

Hence there is in fact some graph $H \in S$ with $V_r(H) = \emptyset$ and so H is an r-regular graph of order n having girth g. ■

Some Specific Cages

We now determine not only the value of the number $n(r,4)$ but the $(r,4)$-cages.

Theorem 12.9 *For $r \geq 2$, $n(r,4) = 2r$. Furthermore, for each integer $r \geq 2$ there is only one $(r,4)$-cage, namely $K_{r,r}$.*

Proof. Since the Moore bound $M(r,4) = 2r$, it follows by Theorem 12.7 that $n(r,4) \geq 2r$. Obviously, the graph $K_{r,r}$ is r-regular, has girth 4 and has order $2r$, thus implying that $n(r,4) = 2r$. Furthermore, $K_{r,r}$ is an $(r,4)$-cage. It remains to show that $K_{r,r}$ is the *only* $(r,4)$-cage. Let G be an $(r,4)$-graph of order $2r$ and let $u_1 \in V(G)$. Denote by $v_1, v_2, \ldots, v_r$ the vertices of G adjacent to u_1. Since $g(G) = 4$, v_1 is adjacent to none of the vertices v_i, $2 \leq i \leq r$. Hence G contains $r-1$ additional vertices $u_2, u_3, \ldots, u_r$. Since every vertex has degree r and G contains no triangle, each vertex u_i $(1 \leq i \leq r)$ is adjacent to every vertex v_j $(1 \leq j \leq r)$ and so $G = K_{r,r}$. ■

We now know then that K_4 is the unique 3-cage and that $K_{3,3}$ is the unique 4-cage. Since the Petersen graph is a cubic graph of girth 5, it is a $(3,5)$-graph. Because the order of the Petersen graph is 10, it follows that $n(3,5) \leq 10$. On the other hand, $M(3,5) = 10$ and so $n(3,5) \geq 10$. Hence $n(3,5) = 10$ and so the Petersen graph is a 5-cage. In fact, it is the only 5-cage.

Theorem 12.10 *The Petersen graph is the unique 5-cage.*

Proof. Let G be a $(3,5)$-graph of order 10. We show that G is isomorphic to the Petersen graph. Let $v_1 \in V(G)$ and suppose that v_1 is adjacent to v_2, v_3 and v_4. Since $g(G) = 5$, each vertex v_i, $i = 2,3,4$, is adjacent to two new vertices of G. Suppose that v_2 is adjacent to v_5 and v_6, v_3 is adjacent to v_7 and v_8, and v_4 is adjacent to v_9 and v_{10}. Hence

$$V(G) = \{v_i : \ i = 1, 2, \ldots, 10\}.$$

The fact that the girth of G is 5 and that every vertex of G has degree 3 implies that v_5 is adjacent to one of v_7 and v_8 and to one of v_9 and v_{10}. Without loss of generality, we may assume that v_5 is adjacent to v_8 and v_9. Necessarily, v_6 is adjacent to v_7 and v_{10}. Therefore, the edges v_7v_9 and v_8v_{10} are also present in G but no other edges are present. Thus, G is isomorphic to the Petersen graph (see Figure 12.2). ■

Figure 12.2: The Petersen graph: the unique 5-cage

Few g-cages are known when $g \geq 6$. For example, there is only one 6-cage, referred to as the **Heawood graph**, and this is shown in Figure 12.3. The Heawood graph is the toroidal dual of the embedding of K_7 on the torus (see Figure 7.9). That is, this graph is constructed by inserting a vertex in each of the 14 regions and joining two vertices by an edge if the corresponding regions have a common boundary edge.

Figure 12.3: The Heawood graph: the unique 6-cage

There is also only one 7-cage, known as the **McGee graph** and named after William F. McGee [153]. Although initially discovered by Horst Sachs, it was first published by McGee in 1960. There is also only one 8-cage, known as the **Tutte-Coxeter graph** and first discovered by Tutte [222]. These two cages are shown in Figure 12.4.

There is no unique 9-cage, however. In fact, there are 18 different 9-cages, each of order 58. There are three different 10-cages, each of order 70. The first 11-cage of order 112 was constructed by Alexandru Balaban [4] in 1973. Twenty-five years later, the **Balaban cage** was shown to be the only 11-cage by Brendan McKay, Wendy Myrvold and Jacqueline Nadon [155]. There is a unique 12-cage of order 126, commonly called the **Benson graph**, named for Clark T. Benson [13], who first constructed it in 1966.

Figure 12.4: The 7-cage and 8-cage

Some Orders of (r, g)-Cages

The orders $n(r, g)$ of all known (r, g)-cages for $3 \leq r \leq 7$ and $3 \leq g \leq 12$ are shown in Figure 12.5. As we have seen, $n(r, 3) = r + 1$ and $n(r, 4) = 2r$ for all $r \geq 3$.

r \ g	3	4	5	6	7	8	9	10	11	12
3	4	6	10	14	24	30	58	70	112	126
4	5	8	19	26	67	80	275	384	–	728
5	6	10	30	42	152	170	–	–	–	2730
6	7	12	40	62	294	312	–	–	–	7812
7	8	14	50	90	–	–	–	–	–	–

Figure 12.5: The orders of certain (r, g)-cages

Moore Graphs

There is a class of cages known as the Moore graphs that has received considerable attention. First observe that for each vertex v in an r-regular graph G of diameter d, the number of vertices at distance i $(1 \leq i \leq d)$ from v is at most $r(r-1)^{i-1}$. Thus the order of G is at most

$$1 + r + r(r-1) + r(r-1)^2 + \cdots + r(r-1)^{d-1}. \tag{12.2}$$

An r-regular graph of diameter d whose order is the expression in (12.2) is called a **Moore graph**. These graphs were named by Alan J. Hoffman and Robert R. Singleton [119] for Edward F. Moore, who posed the problem of describing and classifying these graphs.

Let G be an r-regular Moore graph of diameter d. For each vertex v in G, let $A_i(v)$ be the set of vertices whose distance from v is i $(0 \leq i \leq d)$. For $1 \leq i \leq d-1$, $A_i(v)$ is an independent set of $r(r-1)^{i-1}$ vertices in G and

$$V(G) = \bigcup_{i=0}^{d} A_i(v).$$

The graph G necessarily has girth $g = 2d+1$ since there are not enough vertices to have a higher girth and a cycle of length less than $2d+1$ would cause there to be too few vertices among the first d distance levels from a given vertex v. Hence

$$M(r,g) = M(r,2d+1) = 1 + r + r(r-1) + r(r-1)^2 + \cdots + r(r-1)^{d-1}.$$

Thus $n(r,g) = M(r,g)$. Consequently, a Moore graph must be a cage. That is, if G is an r-regular Moore graph of diameter d, then G is an (r,g)-cage, where $g = 2d+1$.

Every odd cycle C_{2k+1} $(k \geq 1)$ is a Moore graph. Indeed, C_{2k+1} is a 2-regular Moore graph of diameter k and girth $2k+1$. In addition, K_{r+1} $(r \geq 2)$ is an r-regular Moore graph of diameter 1 and girth 3. There are only two other known Moore graphs, namely the Petersen graph, which is a 3-regular Moore graph of diameter 2 and girth 5, and a 7-regular graph of diameter 2, girth 5 and order 50, known as the **Hoffman-Singleton graph**. This graph consists of ten pairwise disjoint 5-cycles denoted by $F_1, F_2, \ldots, F_5$ and $H_1, H_2, \ldots, H_5$ whose vertices are labeled as indicated in Figure 12.6. The vertex i $(0 \leq i \leq 4)$ in F_j $(1 \leq j \leq 5)$ is adjacent to the vertices $i + jk \pmod 5$ in H_k $(1 \leq k \leq 5)$. In addition to the ten 5-cycles, only the five additional edges incident with the vertex labeled 1 in F_3 are shown in Figure 12.6.

A theorem by Hoffman and Singleton [119] states everything that is known about r-regular Moore graphs of girth 5.

Theorem 12.11 *If G is an r-regular Moore graph $(r \geq 3)$ of girth 5, then $r = 3, r = 7$ or possibly $r = 57$.*

As we already noted, the 2-regular Moore graph of girth 5 is C_5, the 3-regular Moore graph of girth 5 is the Petersen graph and the 7-regular Moore graph of girth 5 is the Hoffman-Singleton graph. It is not known whether there is a 57-regular Moore graph of girth 5 but if such a graph does exist, its order must be 3250.

The definition of Moore graph has been extended to permit a Moore graph to have even girth. For every two adjacent vertices u and v in an r-regular graph G of diameter d, the number of vertices w in G such that

Figure 12.6: Constructing the Hoffman-Singleton graph

$$\min\{d(u,w), d(v,w)\} = i \text{ for } 0 \le i \le d-1$$

is at most $2(r-1)^i$. Thus the order of G is at most

$$2 + 2(r-1) + 2(r-1)^2 + \cdots + 2(r-1)^{d-1}. \tag{12.3}$$

An r-regular graph G of diameter d whose order is the expression in (12.3) is also called a **Moore graph**. Such a Moore graph G necessarily has girth $g = 2d$. Hence

$$M(r,g) = M(r,2d) = 2\left[1 + (r-1) + (r-1)^2 + \cdots + (r-1)^{d-1}\right].$$

Thus $n(r,g) = M(r,g)$. Therefore, in this case as well, a Moore graph is a cage. That is, if G is an r-regular Moore graph of diameter d, then the girth of G is $g = 2d$ and G is an (r,g)-cage.

In the more general definition of Moore graphs that allows even girth, the graph $K_{r,r}$ is an r-regular Moore graph of girth 4, the Heawood graph is a 3-regular Moore graph of girth 6 and the Tutte-Coxeter graph is a 3-regular Moore graph of girth 8. More generally, Robert Mark Damerell [53] and Eiichi Bannai and Tatsuro Ito [5] verified that other than the Moore graphs already mentioned, a Moore graph must have girth 5, 6, 8 or 12. The following theorem summarizes the information concerning Moore graphs.

Theorem 12.12 *There exists an r-regular Moore graph of odd girth g when*

- *$g = 3$ and $r \ge 3$, in which case K_{r+1} is the unique Moore graph;*
- *$g = 5$ and $r = 3$, in which case the Petersen graph is the unique Moore graph;*

- *$g = 5$ and $r = 7$, in which case the Hoffman-Singleton graph is the unique Moore graph; and*
- *possibly when $g = 5$ and $r = 57$.*

There exists an r-regular Moore graph of even girth g when

- *$g = 4$ and $r \geq 4$, in which case $K_{r,r}$ is the unique Moore graph;*
- *$g = 6$ and for all r for which there exists a projective plane of order $r - 1$;*
- *$g = 8$ and for all r for which there exists a certain projective geometry; and*
- *$g = 12$ and for all r for which there exists a certain projective geometry.*

As an additional example of a Moore graph referred to above, there is a unique 4-regular Moore graph of girth 6 and order $M(4, 6) = 26$. This graph is shown in Figure 12.7.

Figure 12.7: The 4-regular Moore graph of girth 6

Exercises for Section 12.2

1. Let G be a connected graph with cycles. Show that $g(G) \leq 2\,\mathrm{diam}(G) + 1$.
2. (a) Prove that $n(3, 6) = 14$.

   (b) Prove that the Heawood graph is the only 6-cage.
3. Let G be an (r, g)-graph ($r \geq 2$, $g \geq 3$) of order $n(r, g)$; that is, let G be an (r, g)-cage. Prove that if $H = G \,\square\, K_2$ is an (s, g)-graph, then H cannot be an (s, g)-cage.

4. Construct a cubic graph of order 12 with the maximum possible girth.

5. For integers $r \geq 2$ and $g \geq 3$, the number $n(r, g)$ can be defined as the smallest order of an r-regular graph containing an induced connected 2-regular subgraph of order g but of no smaller order. For integers r, k and g with $r \geq k \geq 2$ and $g \geq 3$, let $n(r, k, g)$ be the smallest order of an r-regular graph containing an induced connected k-regular graph of order g but of no smaller order. Certainly $n(r, k, g) \geq r+1$ and $n(r, 2, g) = n(r, g)$.

   (a) Compute $n(4, 3, 4)$.

   (b) Compute $n(4, 3, 6)$.

12.3 Ramsey Theory

Frank Plumpton Ramsey (1903–1930) was a British mathematician who had many other interests, including English literature, philosophy, economics and politics. Ramsey [178] wrote a collection of 18 papers called *The Foundations of Mathematics, and Other Logical Essays*, in which he intended to improve upon *Principia Mathematica* by Bertrand Russell and Alfred North Whitehead. These papers ranged in dates from 1923 to 1929 and were edited after his death. The result for which he is best known occurred in a paper, presented to the London Mathematical Society and published in 1930 (the year he died). This paper was titled "On a problem of formal logic" [177] and his famous result occurred only as a minor lemma that he used to solve a problem in logic. One statement of Ramsey's theorem is given below.

Theorem 12.13 (**Ramsey's Theorem**) *For any $k+1 \geq 3$ positive integers t, n_1, n_2, ..., n_k, there exists a positive integer n such that if each of the t-element subsets of the set $\{1, 2, \ldots, n\}$ is colored with one of the k colors 1, 2, ..., k, then for some integer i with $1 \leq i \leq k$, there is a subset S of $\{1, 2, \ldots, n\}$ containing n_i elements such that every t-element subset of S is colored i.*

In order to see the connection that Ramsey's theorem has with graph theory, suppose that $\{1, 2, \ldots, n\}$ is the vertex set of the complete graph K_n. In the case of Theorem 12.13 where $t = 2$, each 2-element subset of the set $\{1, 2, \ldots, n\}$, that is, each edge of K_n, is assigned one of the colors $1, 2, \ldots, k$, thereby producing a k-edge coloring of K_n (but not necessarily a *proper* k-edge coloring). It is this special case of Ramsey's theorem in which we have a special interest.

Theorem 12.14 (**Ramsey's Theorem**) *For any $k \geq 2$ positive integers n_1, n_2, ..., n_k, there exists a positive integer n such that for every k-edge coloring of K_n (that is not necessarily a proper edge coloring), there is a complete subgraph K_{n_i} of K_n for some i $(1 \leq i \leq k)$ such that every edge of K_{n_i} is colored i.*

Our primary interest in Ramsey's theorem is in fact the case where $k = 2$. In a **red-blue edge coloring** (or simply a **red-blue coloring**) of a graph

G, every edge of G is colored red or blue. Adjacent edges may be colored the same. It is possible that all edges of G are colored red or that all edges of G are colored blue. For two graphs F and H, the **Ramsey number** $R(F, H)$ is the minimum order n of a complete graph such that for every red-blue coloring of K_n, there is either a subgraph isomorphic to F all of whose edges are colored red (a **red** F) or a subgraph isomorphic to H all of whose edges are colored blue (a **blue** H). Then

$$R(F, H) = R(H, F)$$

for every two graphs F and H. The Ramsey number $R(F, F)$ is thus the minimum order n of a complete graph such that if every edge of K_n is colored with one of two given colors, then there is a subgraph isomorphic to F all of whose edges are colored the same (a **monochromatic** F). The Ramsey number $R(F, F)$ is sometimes called the **Ramsey number of the graph** F.

The Ramsey Number $R(K_3, K_3)$

We begin with perhaps the best known Ramsey number, namely $R(K_3, K_3)$.

Theorem 12.15 $R(K_3, K_3) = 6$.

Proof. Let $V(K_5) = \{v_1, v_2, \ldots v_5\}$ and define a red-blue coloring of K_5 by coloring each edge of the 5-cycle $(v_1, v_2, \ldots, v_5, v_1)$ red and the remaining edges blue (see Figure 12.8), where solid edges indicate red edges and dashed edges denote blue edges. Since this red-blue coloring produces neither a red K_3 nor a blue K_3, it follows that $R(K_3, K_3) \geq 6$.

Figure 12.8: A red-blue coloring of K_5

To show that $R(K_3, K_3) \leq 6$, let there be given a red-blue coloring of K_6. Consider some vertex v_1 of K_6. Since v_1 is incident with five edges, it follows that at least three of these five edges are colored the same, say red. Suppose that v_1v_2, v_1v_3 and v_1v_4 are red edges. If any of the edges v_2v_3, v_2v_4 and v_3v_4 is colored red, then we have a red K_3; otherwise, all three of these edges are colored blue, producing a blue K_3. Hence $R(K_3, K_3) \leq 6$. Therefore, $R(K_3, K_3) = 6$. ■

Theorem 12.15 provides a solution to a problem that appeared on an exam that has become well known over the years. Except for a 3-year period during World War II, the William Lowell Putnam mathematical competition for undergraduates has taken place every year since 1938. This exam, administered by the Mathematical Association of America, consists of (since 1962) twelve challenging mathematical problems. This competition was designed to stimulate a healthy rivalry among colleges and universities throughout the United States and Canada. The 1953 exam contained the following problem.

Problem A2 *The complete graph with* 6 *points* (*vertices*) *and* 15 *edges has each edge colored red or blue. Show that we can find* 3 *points such that the* 3 *edges joining them are the same color.*

Theorem 12.15 also provides the answer to a popular question. What is the smallest number n of individuals in a gathering of people, every two of whom are either acquaintances or strangers, for which there must be either three mutual acquaintances or three mutual strangers? A gathering of n people can be modeled by the graph K_n and a red-blue coloring of K_n, where a red edge, say, indicates that two people are acquaintances and a blue edge indicates that two people are strangers. By Theorem 12.15, the smallest such integer n is 6.

Example of a Ramsey Number

As an added illustration of a Ramsey number, we determine $R(P_4, K_4)$.

Example 12.16 $R(P_4, K_4) = 10$.

Proof. To show that $R(P_4, K_4) \geq 10$, consider the red-blue coloring of K_9 in which the red subgraph is $3K_3$. Since the order of every component of $3K_3$ is 3, there is no red P_4 in this red-blue coloring of K_9. The blue subgraph in this case is $K_{3,3,3}$, which does not contain a 4-clique. Hence there is no blue K_4 in this red-blue coloring of K_9. Therefore, $R(P_4, K_4) \geq 10$.

We now show that $R(P_4, K_4) \leq 10$. Let there be given a red-blue coloring of $G = K_{10}$. We show that there is either a red P_4 or a blue K_4. Suppose first that some vertex v of G is incident with four or more red edges, say vv_1, vv_2, vv_3 and vv_4 are red. If any edge joining two vertices of $S = \{v_1, v_2, v_3, v_4\}$ is red, then a red P_4 results; otherwise, $G[S] = K_4$ is blue.

Hence we may assume that every vertex of G is incident with six or more blue edges. Let $u \in V(G)$ and suppose that $uu_1, uu_2, \ldots, uu_6$ are blue edges of G. Let $A = \{u_1, u_2, \ldots, u_6\}$ and let $H = G[A] = K_6$. Since $R(K_3, K_3) = 6$ by Theorem 12.15, it follows that H contains either a red K_3 or a blue K_3. Since u is joined to every vertex of H by a blue edge, if H contains a blue K_3, then G contains a blue K_4. Hence we may assume that H contains a red K_3, which we denote by F_1. If some vertex of F_1 is joined to a vertex of $V(G) - V(F_1)$ by a red edge, then a red P_4 results. Thus we may assume that G contains a

blue $K_{3,7}$ where $V(F_1)$ is the partite set of order 3. Since each vertex of F_1 is joined to six vertices of $V(G) - V(F_1)$ by blue edges, we may use the same reasoning as above to conclude that G contains a red K_3, which we denote by F_2, where then F_1 and F_2 are disjoint. We may assume then that every vertex of $V(G) - V(F_2)$ is joined to the vertices of F_2 by blue edges. Thus every vertex of $V(F_1) \cup V(F_2)$ in the blue subgraph G_B of G has degree 7 and the remaining four vertices of G_B have degree at least 6. Therefore, the size of G_B is at least $(6 \cdot 7 + 4 \cdot 6)/2 = 33$. By Turán's theorem (Theorem 12.2), either G_B contains a 4-clique (and so G contains a blue K_4) or G_B has size 33 and so is the Turán graph $T_{10,3} = K_{3,3,4}$, implying that G contains a red K_4 and a red P_4 as well. ■

The Ramsey number $R(F, H)$ of two graphs F and H can be defined without regard to edge colorings. The Ramsey number $R(F, H)$ is also the smallest positive integer n such that for every graph G of order n, either G contains a subgraph isomorphic to F or its complement $\overline{G}$ contains a subgraph isomorphic to H. Assigning the color red to each edge of G and the color blue to each edge of $\overline{G}$ returns us to our initial definition of $R(F, H)$.

Historically, it is the Ramsey numbers $R(K_s, K_t)$ that were the first to be studied. The numbers $R(K_s, K_t)$ are commonly expressed as $R(s, t)$ as well and are referred to as the **classical Ramsey numbers**. By Ramsey's theorem, $R(s, t)$ exists for every two positive integers s and t. The Ramsey number $R(s, t)$ is thus the smallest positive integer n such that for *every* graph G of order n, either G contains an s-clique or $\overline{G}$ contains a t-clique. Equivalently, $R(s, t)$ is the smallest positive integer n such that every graph of order n contains an s-clique or an independent set of t vertices.

Bounds for Ramsey Numbers

We begin with some observations. First, as observed above,

$$R(s, t) = R(t, s)$$

for every two positive integers s and t. Also,

$$R(1, t) = 1 \text{ and } R(2, t) = t \tag{12.4}$$

for every positive integer t (see Exercise 2) and by Theorem 12.15, $R(3, 3) = 6$.

Indeed, the Ramsey number $R(F, H)$ exists for every two graphs F and H. In fact, if F has order s and H has order t, then

$$R(F, H) \leq R(s, t).$$

The existence of the Ramsey numbers $R(s, t)$ was also established in 1935 by Paul Erdös and George Szekeres [74], where upper bounds for $R(s, t)$ were obtained as well.

Theorem 12.17 *For every two integers $s, t \geq 2$, the Ramsey number $R(s,t)$ exists and*

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

Furthermore, if $R(s-1,t)$ and $R(s,t-1)$ are both even, then

$$R(s,t) \leq R(s-1,t) + R(s,t-1) - 1.$$

Proof. To establish the existence of $R(s,t)$, we apply induction on $s+t \geq 4$. We have already seen that $R(s,t)$ exists if s or t is 1 or 2, so $R(2,2)$ exists. Assume that $R(s',t')$ exists if $s',t' \geq 2$ and $s'+t' = k-1$ for some integer $k \geq 5$ and let $s,t \geq 2$ such that $s+t = k$. In particular, $R(s-1,t)$ and $R(s,t-1)$ exist.

Let $n = R(s-1,t) + R(s,t-1)$ and let there be given a red-blue coloring of $G = K_n$. It suffices to show that G contains either a red K_s or a blue K_t. Let v be a vertex of G. Thus the degree of v in G is $n-1 = R(s-1,t)+R(s,t-1)-1$. Let F be the spanning red subgraph of G and H the spanning blue subgraph of G. We now consider two cases, depending on the degree of v in F.

Case 1. $\deg_F v \geq R(s-1,t)$. Since the order of $G[N_F(v)]$ is at least $R(s-1,t)$, it follows that either $G[N_F(v)]$ contains a red K_{s-1} or a blue K_t. If $G[N_F(v)]$ contains a blue K_t, then so does G. If $G[N_F(v)]$ contains a red K_{s-1}, then F contains K_s and so G contains a red K_s.

Case 2. $\deg_F v \leq R(s-1,t)-1$. Thus $\deg_H v \geq R(s,t-1)$. Since the order of $G[N_H(v)]$ is at least $R(s,t-1)$, it follows that either $G[N_H(v)]$ contains a red K_s or a blue K_{t-1}. If $G[N_H(v)]$ contains a red K_s, then so does G. If $G[N_H(v)]$ contains a blue K_{t-1}, then H contains K_t and so G contains a blue K_t.

Therefore, $R(s,t)$ exists and

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

Next, suppose that $R(s-1,t)$ and $R(s,t-1)$ are both even and once again let $n = R(s-1,t) + R(s,t-1)$. Let there be given a red-blue coloring of $G' = K_{n-1}$. Let F' be the spanning red subgraph of G' and H' the spanning blue subgraph of G'. Since F' has odd order, some vertex v of F' has even degree. If $\deg_{F'} v \geq R(s-1,t)$, then, proceeding as above, G' has either a red K_s or a blue K_t. On the other hand, if $\deg_{F'} v < R(s-1,t)$, then $\deg_{F'} v \leq R(s-1,t)-2$ and so $\deg_{H'} v \geq R(s,t-1)$. Once again, proceeding as above, G' has either a red K_s or a blue K_t. ■

An upper bound for $R(s,t)$ can also be given as a binomial coefficient. For positive integers k and n with $k \leq n$, a well-known combinatorial identity is

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}. \tag{12.5}$$

Corollary 12.18 *For every two positive integers s and t,*

$$R(s,t) \leq \binom{s+t-2}{s-1}.$$

Proof. We proceed by induction on $n = s+t$. Since $R(1,t) = 1$ and $R(2,t) = t$ for every positive integer t, it follows that $R(s,t) \leq \binom{s+t-2}{s-1}$ when $n = s+t \leq 5$. Thus we may assume that $s \geq 3$ and $t \geq 3$ and so $n \geq 6$. Suppose that

$$R(s',t') \leq \binom{s'+t'-2}{s'-1}$$

for $s' + t' = k - 1$ where $k \geq 6$. We show for positive integers s and t with $s, t \geq 3$ and $k = s + t$ that

$$R(s,t) \leq \binom{s+t-2}{s-1}.$$

By Theorem 12.17,

$$\begin{aligned} R(s,t) &\leq R(s-1,t) + R(s,t-1) \\ &\leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}, \end{aligned}$$

completing the proof. ■

As we have already noted, the bound given in Corollary 12.18 for $R(s,t)$ is exact if one of s and t is 1 or 2. The bound is also exact for $s = t = 3$. By Corollary 12.18,

$$R(3,t) \leq \frac{t^2+t}{2}.$$

An improved bound for $R(3,t)$ is now presented.

Theorem 12.19 *For every integer $t \geq 3$,*

$$R(3,t) \leq \frac{t^2+3}{2}. \tag{12.6}$$

Proof. We proceed by induction on t. For $t = 3$, $R(3,t) = 6 = (t^2+3)/2$, so that (12.6) holds if $t = 3$. Assume that

$$R(3,t-1) \leq \frac{(t-1)^2+3}{2},$$

for some integer $t \geq 4$, and consider $R(3,t)$. By Theorem 12.17,

$$R(3,t) \leq t + R(3,t-1). \tag{12.7}$$

Moreover, strict inequality holds if t and $R(3, t-1)$ are both even. Combining (12.7) and the inductive hypothesis, we have

$$R(3,t) \leq t + \frac{(t-1)^2+3}{2} = \frac{t^2+4}{2}. \tag{12.8}$$

To complete the proof, it suffices to show that the inequality given in (12.8) is strict.

If t is odd, then $R(3,t) < (t^2+4)/2$ since t^2+4 is odd. Thus we may assume that t is even. If $R(3,t-1) < ((t-1)^2+3)/2$, then clearly the inequality in (12.8) is strict. If, on the other hand,

$$R(3,t-1) = \frac{(t-1)^2+3}{2} = \frac{t^2}{2} - t + 2,$$

then $R(3,t-1)$ is even since t is even. Therefore the inequality in (12.8) is strict, which implies the desired result. ■

According to Theorem 12.19, $R(3,4) \leq 9$ and $R(3,5) \leq 14$. Actually, equality holds in both of these cases. The equality $R(3,4) = 9$ follows since there exists a graph of order 8 containing neither a triangle nor an independent set of four vertices, while the equality $R(3,5) = 14$ follows since there exists a graph of order 13 containing neither a triangle nor an independent set of five vertices. These graphs are shown in Figure 12.9. On the other hand, $R(3,6) \leq 19$ by Theorem 12.19, but the value of $R(3,6)$ is 18.

Figure 12.9: Graphs showing $R(3,4) \geq 9$ and $R(3,5) \geq 14$

Corollary 12.18 gives an upper bound for the Ramsey numbers $R(s,s)$, namely $R(s,s) \leq \binom{2s-2}{s-1}$. There are three ways in which lower bounds for $R(s,s)$ have been obtained: the constructive method, a counting method and the probabilistic method. In the constructive method, a lower bound for $R(s,s)$ is established by explicitly constructing a graph G of an appropriate order such

that neither G nor $\overline{G}$ contains K_s as a subgraph. Better lower bounds, however, have been obtained using a counting method, which we briefly describe here. Suppose that we wish to establish the existence of a graph G of order n having some given property P. If we can estimate the number of graphs of order n that do not have property P and we can show that this number is strictly less than the total number of graphs of order n, then there must exist a graph G of order n having property P. Of course, this procedure offers no method for constructing G. In 1947 Erdös [69] provided one of the first applications of a counting method. The following bound for the Ramsey number $R(t,t)$ is presented for the proof technique used rather than the usefulness of the lower bound.

Theorem 12.20 *For every integer $t \geq 3$,*

$$R(t,t) > \lfloor 2^{t/2} \rfloor.$$

Proof. Let $n = \lfloor 2^{t/2} \rfloor$. We demonstrate the existence of a graph G of order n such that neither G nor $\overline{G}$ contains K_t as a subgraph. Consider the set $\mathcal{G}$ of all graphs of order n having a fixed vertex set V, where then $|V| = n$. Two graphs G_1 and G_2 in $\mathcal{G}$ are then different if $E(G_1) \neq E(G_2)$. Since the set E of all edges with vertex set V consists of $\binom{n}{2}$ elements, $|\mathcal{G}|$ is the number of all subsets of E, which then is $2^{\binom{n}{2}}$. For each subset S of V with $|S| = t$, consider those graphs in $\mathcal{G}$ for which S induces a complete graph. Since the $\binom{t}{2}$ edges joining two vertices in S belong to each such graph H, the remaining edges of H must come from a set of $\binom{n}{2} - \binom{t}{2}$ edges. So the total number of such graphs is therefore $2^{\binom{n}{2}-\binom{t}{2}}$. Thus, if N denotes the number of graphs with vertex set V that contain a subgraph isomorphic to K_t, then since there are $\binom{n}{t}$ choices for the set S, it follows that

$$N \leq \binom{n}{t} 2^{\binom{n}{2}-\binom{t}{2}} = \left[\frac{n(n-1)\cdots(n-t+1)}{t!}\right] 2^{\binom{n}{2}-\binom{t}{2}} < \frac{n^t}{t!} 2^{\binom{n}{2}-\binom{t}{2}}. \quad (12.9)$$

Since $n = \lfloor 2^{t/2} \rfloor \leq 2^{t/2}$, it follows that

$$n^t \leq 2^{t^2/2}. \quad (12.10)$$

Next, we verify by induction on $t \geq 3$ that

$$2^{t^2/2} < \left(\frac{1}{2}\right) t! 2^{\binom{t}{2}}. \quad (12.11)$$

Since

$$2^{9/2} = \sqrt{2} \cdot 2^4 < \left(\frac{3}{2}\right) \cdot 16 = 3 \cdot 2^3 = \frac{1}{2}(3!)2^{\binom{3}{2}},$$

the inequality holds for $t = 3$. Suppose that

$$2^{k^2/2} < \frac{1}{2} \cdot k! \cdot 2^{\binom{k}{2}}$$

for some integer $k \geq 3$. We show that

$$2^{(k+1)^2/2} < \frac{1}{2}(k+1)!2^{\binom{k+1}{2}}.$$

Observe that

$$\begin{aligned} 2^{(k+1)^2/2} &= 2^{k^2/2} \cdot 2^{(2k+1)/2} = \sqrt{2} \cdot 2^{k^2/2} \cdot 2^k \\ &< \sqrt{2}\left[\frac{1}{2} \cdot k! \cdot 2^{\binom{k}{2}}\right] \cdot 2^k = \sqrt{2} \cdot \frac{1}{2} \cdot k! \cdot 2^{\binom{k}{2}+k} \\ &< \frac{1}{2}(k+1) \cdot k! \cdot 2^{\binom{k+1}{2}} = \frac{1}{2}(k+1)!2^{\binom{k+1}{2}}. \end{aligned}$$

Thus the inequality in (12.11) holds for every integer $t \geq 3$ and so by (12.10)

$$n^t < \left(\frac{1}{2}\right) t!2^{\binom{t}{2}}. \tag{12.12}$$

Combining (12.9) and (12.12), we conclude that

$$N < \left(\frac{1}{2}\right) 2^{\binom{n}{2}}.$$

If we list the N graphs with vertex set V that contain a subgraph isomorphic to K_t, together with their complements, then there are at most $2N < 2^{\binom{n}{2}}$ different graphs in the list. Since there are $2^{\binom{n}{2}}$ graphs with vertex set V, we conclude that there is a graph G with vertex set V such that neither G nor $\overline{G}$ appears in the aforementioned list, that is, neither G nor $\overline{G}$ contains a subgraph isomorphic to K_t. ■

Known Classical Ramsey Numbers

Relatively few classical Ramsey numbers $R(s,t)$ are known for $s,t \geq 3$. The table in Figure 12.10 gives the known values of $R(s,t)$ for integers s and t with $s,t \geq 3$. Although it is known that $43 \leq R(5,5) \leq 49$, the value of $R(5,5)$ remains unknown.

s	3	3	3	3	3	3	3	4	4
t	3	4	5	6	7	8	9	4	5
$R(s,t)$	6	9	14	18	23	28	36	18	25

Figure 12.10: Some classical Ramsey numbers

Tree-Complete Graph Ramsey Numbers

We saw in Example 12.16 that $R(P_4, K_4) = 10$. Vašek Chvátal [47] extended this considerably by finding the exact value of $R(F, H)$ whenever F is any tree and H is any complete graph.

Theorem 12.21 *Let T be a tree of order $p \geq 2$. For every integer $n \geq 2$,*

$$R(T, K_n) = (p-1)(n-1) + 1.$$

Proof. First, we show that $R(T, K_n) \geq (p-1)(n-1)+1$. Let there be given a red-blue coloring of the complete graph $K_{(p-1)(n-1)}$ such that the resulting red subgraph is $(n-1)K_{p-1}$; that is, the red subgraph consists of $n-1$ copies of K_{p-1}. Since each component of the red subgraph has order $p-1$, it contains no connected subgraph of order greater than $p-1$. In particular, there is no red tree of order p. The blue subgraph is then the complete $(n-1)$-partite graph $K_{p-1,p-1,\ldots,p-1}$, where every partite set contains exactly $p-1$ vertices. Hence there is no blue K_n either. Since this red-blue coloring of $K_{(p-1)(n-1)}$ produces neither a red tree T nor a blue K_n, it follows that $R(T, K_n) \geq (p-1)(n-1)+1$.

We now show that $R(T, K_n) \leq (p-1)(n-1)+1$ for an arbitrary but fixed tree T of order $p \geq 2$ and an integer $n \geq 2$. We verify this inequality by induction on n. For $n = 2$, we show that $R(T, K_2) \leq (p-1)(2-1)+1 = p$. Since $R(K_p, K_2) = R(p, 2) = p$, as noted in (12.4), every red-blue coloring of K_p produces a red K_p or a blue K_2. Thus $R(T, K_2) \leq p$. Therefore, the inequality $R(T, K_n) \leq (p-1)(n-1)+1$ holds when $n = 2$. Assume for an integer $k \geq 2$ that

$$R(T, K_k) \leq (p-1)(k-1)+1.$$

Consequently, every red-blue coloring of $K_{(p-1)(k-1)+1}$ contains either a red T or a blue K_k. We now show that

$$R(T, K_{k+1}) \leq (p-1)k+1.$$

Let there be given a red-blue coloring of $K_{(p-1)k+1}$. We show that there is either a red tree T or a blue K_{k+1}. We consider two cases.

Case 1. *There exists a vertex v in $K_{(p-1)k+1}$ that is incident with at least $(p-1)(k-1)+1$ blue edges.* Suppose that vv_i is a blue edge for $1 \leq i \leq (p-1)(k-1)+1$. Consider the subgraph H induced by the set $\{v_i : 1 \leq i \leq (p-1)(k-1)+1\}$. Thus $H = K_{(p-1)(k-1)+1}$. By the induction hypothesis, H contains either a red T or a blue K_k. If H contains a red T, so does $K_{(p-1)k+1}$. On the other hand, if H contains a blue K_k, then, since v is joined to every vertex of H by a blue edge, there is a blue K_{k+1} in $K_{(p-1)k+1}$.

Case 2. *Every vertex of $K_{(p-1)k+1}$ is incident with at most $(p-1)(k-1)$ blue edges.* So every vertex of $K_{(p-1)k+1}$ is incident with at least $p-1$ red

edges. Thus the red subgraph of $K_{(p-1)k+1}$ has minimum degree at least $p-1$. By Theorem 2.20, this red subgraph contains a red T. Therefore, $K_{(p-1)k+1}$ contains a red T as well. ■

Ramsey Numbers of Three or More Graphs

Ramsey's theorem suggests that the Ramsey number $R(F, H)$ of two graphs F and H can be extended to more than two graphs. For $k \geq 3$ graphs $G_1, G_2, \ldots, G_k$, the **Ramsey number** $R(G_1, G_2, \ldots, G_k)$ is defined as the smallest positive integer n such that if every edge of K_n is colored with one of k given colors, a monochromatic G_i results for some i $(1 \leq i \leq k)$. While the existence of this more general Ramsey number is also a consequence of Ramsey's theorem, the existence of $R(F, H)$ for every two graphs F and H can also be used to show that $R(G_1, G_2, \ldots, G_k)$ exists for every $k \geq 3$ graphs $G_1, G_2, \ldots, G_k$ (see Exercise 15).

Theorem 12.22 *For every $k \geq 3$ graphs $G_1, G_2, \ldots, G_k$, the Ramsey number $R(G_1, G_2, \ldots, G_k)$ exists.*

If $G_i = K_{n_i}$ for $1 \leq i \leq k$, then we write the number $R(G_1, G_2, \ldots, G_k)$ as $R(n_1, n_2, \ldots, n_k)$. When the graphs G_i $(1 \leq i \leq k)$ are all complete graphs of order at least 3 and $k \geq 3$, only the Ramsey number $R(3, 3, 3)$ is known. The following result is due to Robert E. Greenwood and Andrew M. Gleason [98].

Theorem 12.23 $R(3, 3, 3) = 17$.

Proof. Let there be given an edge coloring of K_{17} with the three colors red, blue and green and let v be a vertex of K_{17}. Since v has degree 16, it follows that v is incident with six edges that are colored the same, say vv_i $(1 \leq i \leq 6)$ are colored green. Let $H = K_6$ be the subgraph induced by $\{v_1, v_2, \ldots, v_6\}$. If any edge of H is colored green, then K_{17} has a green triangle. Thus we may assume that no edge of H is colored green. Hence every edge of H is colored red or blue. Since $H = K_6$ and $R(3, 3) = 6$ (by Theorem 12.15), it follows that H, and K_{17} as well, contains either a red triangle or a blue triangle. Therefore, K_{17} contains a monochromatic triangle and so $R(3, 3, 3) \leq 17$.

To show that $R(3, 3, 3) = 17$, it remains to show that there is a 3-edge coloring of K_{16} for which there is no monochromatic triangle. In fact, there is an isomorphic factorization of K_{16} into a triangle-free graph that is commonly called the **Clebsch graph** (named for Alfred Clebsch [49] who discovered it in 1868) or the **Greenwood-Gleason graph**. This graph can be constructed by beginning with the Petersen graph with vertices u_i and v_i $(1 \leq i \leq 5)$, as illustrated in Figure 12.11 by solid vertices and solid edges. We then add six new vertices, namely x and w_i $(1 \leq i \leq 5)$. The Clebsch graph CG (a 5-regular graph of order 16) is constructed as shown in Figure 12.11. This graph has the property that for every vertex v of CG, the subgraph $CG - N[v]$ is isomorphic to the Petersen graph. ■

Figure 12.11: The Clebsch graph

The next result gives the Ramsey number of the stars $K_{1,3}$ and $K_{1,5}$ and the 5-cycle.

Theorem 12.24 $R(K_{1,3}, K_{1,5}, C_5) = 15$.

Proof. First, consider the edge coloring of K_{14} in which every edge of $K_{7,7}$ is colored green. At this point every edge of $2K_7$ is uncolored. Since K_7 can be factored into three Hamiltonian cycles by Theorem 10.18, we can color the edges red in one Hamiltonian cycle in each copy of K_7 and the remaining edges blue. The blue graph is then 4-regular in $2K_7$. With this edge coloring, there is no red $K_{1,3}$, no blue $K_{1,5}$ and no green C_5. Thus $R(K_{1,3}, K_{1,5}, C_5) \geq 15$.

Next, let there be given an edge coloring of $G = K_{15}$ with the colors red, blue and green and suppose there is no red $K_{1,3}$ and no blue $K_{1,5}$. Then every vertex of G is incident with at least 8 green edges. Thus in the green subgraph H of G, $\deg_H v \geq 8$ for every vertex v of H. Thus $\deg_H u + \deg_H v \geq 16$ for every pair u, v of nonadjacent vertices of H. Since H has order 15, it follows by Theorem 3.25 that H is pancyclic and so G contains a green C_5. Therefore, $R(K_{1,3}, K_{1,5}, C_5) = 15$. ■

Exercises for Section 12.3

1. Show that if G is a graph of order $R(s,t)-1$, then
   (a) $K_{s-1} \subseteq G$ or $K_t \subseteq \overline{G}$,
   (b) $K_s \subseteq G$ or $K_{t-1} \subseteq \overline{G}$.
2. Show that $R(1,t)=1$ and $R(2,t)=t$ for every positive integer t.
3. If $2 \le s' \le s$ and $2 \le t' \le t$, then prove that $R(s',t') \le R(s,t)$. Furthermore, prove that equality holds if and only if $s'=s$ and $t'=t$.
4. Show that $R(3,4)=9$.
5. The graph shown in Figure 12.12 has order 17 and contains neither four mutually adjacent vertices nor an independent set of four vertices. Thus $R(4,4)>17$. Show that $R(4,4)=18$.

Figure 12.12: The graph in Exercise 5

6. The value of the Ramsey number $R(5,5)$ is unknown. Establish a lower bound A and an upper bound B (with explanations) for $R(5,5)$ such that $B-A<25$.
7. For a red-blue coloring c of K_6, let t_c denote the number of monochromatic triangles produced. By Theorem 12.15, $t_c \ge 1$ for every red-blue coloring c of K_6. Determine $\min\{t_c\}$ over all red-blue colorings c of K_6.
8. Determine $R(C_3, C_4)$.
9. Use the fact that the classical Ramsey number $R(3,4)=9$ to determine the value of $R(K_3, C_5)$.
10. For the graphs F and H shown in Figure 12.13, determine $R(F,H)$.
11. Let $n \ge 3$ be an integer. Prove that

$$R(K_{1,n}, K_{1,n}) = \begin{cases} 2n & \text{if } n \text{ is odd} \\ 2n-1 & \text{if } n \text{ is even.} \end{cases}$$

Figure 12.13: The graphs F and H in Exercise 10

12. Let F and H be two nontrivial graphs, where $x \in V(F)$ and $y \in V(H)$. Let $F' = F - x$ and $H' = H - y$. Prove that $R(F, H) \leq R(F', H) + R(F, H')$.

13. For integers s and t with $1 \leq s \leq t$, define the Ramsey chromatic number $R_\chi(s, t)$ to be the smallest positive integer n such that for every red-blue coloring of K_n, either the red subgraph has chromatic number at least s or the blue subgraph has chromatic number at least t. Trivially, $R_\chi(1, t) = 1$ and $R_\chi(2, t) = t$.

    (a) Does $R_\chi(s, t)$ exist for all integers s and t with $1 \leq s \leq t$?
    (b) Prove that $R_\chi(s, t) \leq R_\chi(s - 1, t) + R_\chi(s, t - 1)$.
    (c) Determine $R_\chi(3, 3)$.
    (d) For $3 \leq s \leq t$, determine a formula for $R_\chi(s, t)$ in terms of s and t.

14. An edge-colored graph G has a **rainbow** H if G contains a subgraph G' isomorphic to H in which no two edges of G' are colored the same. Recall that an edge-colored graph G has a monochromatic F if G contains a subgraph G' isomorphic to F in which all edges of G' are colored the same. For a nonempty forest F, the **rainbow Ramsey number** $RR(F)$ is the smallest positive integer n such that if each edge of the complete graph K_n is colored from any number of colors, then either a monochromatic F or a rainbow F is produced. Determine $RR(K_{1,k})$ for each integer $k \geq 3$.

15. Prove Theorem 12.22: *For every $k \geq 2$ graphs $G_1, G_2, \ldots, G_k$, the Ramsey number $R(G_1, G_2, \ldots, G_k)$ exists.*

16. Show for graphs $G_1, G_2, \ldots, G_k$ $(k \geq 2)$ that
$$R(G_1, G_2, \ldots, G_k, K_2) = R(G_1, G_2, \ldots, G_k).$$

17. Show for positive integers $n_1, n_2, \ldots, n_k$ $(k \geq 2)$ that
$$R(K_{n_1}, K_{n_2}, \ldots, K_{n_k}, T_s) = 1 + (r - 1)(s - 1)$$
where T_s is any tree of order $s \geq 1$ and $r = R(n_1, n_2, \ldots, n_k)$.

18. Determine $R(K_{1,5}, K_{1,6}, K_{1,7})$.

19. Determine $R(K_{1,5}, K_{1,6}, K_{1,8})$.

Hints and Solutions to Odd-Numbered Exercises

Chapter 1 Section 1.1

1. Let the vertices of the graph represent the boxes where two vertices are adjacent if the correspond boxes contain a wire segment of the same color.
3. The degree of each remaining vertex is 3.
5. (a) $G_1 \cong G_2$. (b) $H_1 \ncong H_2$.
7. Consider the average degree $2m/n$ of G.
9. (a) Apply the definition of isomorphism to $G[S]$ and $H[T]$.
   (b) Consider $r = 3$.
11. If G is not r-regular, then let G' be another copy of G and join corresponding vertices whose degrees are less than r.
13. Observe that if G has a partite set with three or more vertices, then $\overline{G}$ is not bipartite.
15. Let $C = (v_1, v_2, v_3, v_4, v_5, v_1)$, replace v_i by a copy H_i of C_5 and join every vertex of H_i to every vertex of H_j if v_iv_j is an edge of C.
17. Use induction and the hint given in Exercise 15.
19. Assume that $m \leq \frac{1}{2}\binom{n}{2}$ and $\overline{m} \leq \frac{1}{2}\binom{n}{2}$ and obtain a contradiction.
21. (a) The degree of u_1 in $G_1 + G_2$ is $\deg_{G_1} u_1$.
    (b) The degree of u_1 in $G_1 \vee G_2$ is $\deg_{G_1} u_1 + n_2$.
    (c) The degree of (u_1, u_2) in $G_1 \,\square\, G_2$ is $\deg_{G_1} u_1 + \deg_{G_2} u_2$.
23. (a) $rK_1 + (s/2)K_2$. (b) The minimum size is $s/2$.
    (c) The maximum size is $\binom{n}{2} - s/2$.

Section 1.2

1. $G - uw - xy + ux + wy$.

G :

3. $G = P_4$.
5. Assume that for each graph H with $V(H) = \{v_1, v_2, \ldots, v_n\}$ where $\deg v_i = d_i$ for $1 \leq i \leq n$, there is a vertex v_k $(1 \leq k \leq n)$ such that both (1) and (2) fail.
7. (b), (c) and (e) are graphical.
9. If a graph G of order 6 contains two vertices of degree 5, it has no vertex of degree 1.
11. Observe that $\sum_{i=1}^{3} d_i = 17$ and $3(3-1) + \sum_{i=1}^{7} \min\{3, d_i\} = 16$.
13. Let $S = \{a_1, a_2, \cdots, a_n\}$ and $k = \text{lcm } \{a_1 + 1, a_2 + 1, \cdots, a_n + 1\}$. For $S = \{2, 6, 7\}$, $k = 4$.
15. In one direction, assume that s_1 and s_2 are bigraphical sequences. Let $V_1 = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = \{w_1, w_2, \ldots, w_t\}$. Suppose that there is no bipartite graph H with partite sets V_1 and V_2 and a vertex u of H of degree a_1 in V_1 adjacent to vertices of degrees $b_1, b_2, \ldots, b_{a_1}$.

Section 1.3

1. Let $V(G) = \{u = v_1, v_2, \cdots, v_n = v\}$ and let P_i be a $v_i - v_{i+1}$ path for $i = 1, 2, \cdots, n-1$.
3. (a) The statement is true. (b) The statement is false.
5. For $i \neq j$, the (i, j)-entry of A^2 is the number of different $v_i - v_j$ paths of length 2 and the (i, j)-entry of A^3 is the number of different $v_i - v_j$ walks of length 3. The vertices v_1 and v_4 belong to one triangle while the vertices v_2 and v_3 belong to two triangles.
7. $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{v_1v_2, v_1v_3, v_2v_3, v_3v_4, v_4v_5\}$.
9. If G contains a $u - v$ path and a $v - w$ path, then G contains a $u - w$ path by Theorem 1.16.
11. $m = \binom{n-1}{2} + 1$.
13. Consider a longest path in G.
15. Suppose that the statement is false and consider a longest path in G.
17. (a) and (b). Consider the graphs of orders 7 and 9 that are union of cycles.
19. (a) Consider a path connecting two vertices of distinct degrees. (b) No.
21. Suppose that G has k or more components and consider a component of minimum order.
23. For every nonempty proper subset S of $V(G)$, there is a vertex in S adjacent to a vertex in $G - S$.
25. (a) $d_1(v) = \deg v$. (b) Let m_k be the size of $G^{(k)}$ having vertex set $V(G)$ and edge set $\{uv : d(u, v) = k\}$. Then $\sum_{v \in V(G)} d_k(v) = 2m_k$. (c) $n^2 - n$.
27. Let G be obtained by identifying an end-vertex of P_{b-a+1} with a vertex of C_{2a}.
29. $n = 2 + (r-1)\sum_{i=1}^{s-1}(r-2)^i$.
31. See the graph below.
33. Let $2k+1 \geq 3$ and let G be obtained by identifying the center of $K_{1,2k+1}$ and an end-vertex of P_{2k+1}.

35. If F can be transformed into G by a 2-switches and G can be transformed into H by b 2-switches, then F can be transformed into H by $a+b$ 2-switches.
37. Consider a $u-v$ geodesic for nonadjacent vertices u and v in G.

Section 1.4
1. Let $E(G) = \{e_1, e_2, \ldots, e_m\}$ and replace each edge e_i of G by 2^{i-1} parallel edges.
3. Each of the sequences in (a), (c), (e), (g) and (h) is the degree sequence of a multigraph.
5. (a) $m = 20$. (b) $9 \le m \le 11$.

(c) Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$ where $\deg u_1 \ge \deg u_2 \ge \cdots \ge \deg u_s$ and $\deg w_1 \ge \deg w_2 \ge \cdots \ge \deg w_t$. Then $m = \min\left\{\sum_{i=1}^{s} i \deg u_i, \quad \sum_{i=1}^{t} i \deg w_i\right\}$.
7. The statement is true.
9. The statement is false.
11. The outdegree and indegree of a vertex in a regular tournament of order n is $(n-1)/2$.
13. (a) The sum of the entries in row i is $\operatorname{od} v_i$ and the sum of the entries in column i is $\operatorname{id} v_i$.

(b) A is a zero-one matrix if and only if every entry on the main diagonal is 0.
15. Any orientation of an odd cycle produces a directed path of length 2.
17. Begin with the following.

19. For (a), see the figure below. (b), (c), (e) are true.

Chapter 2 Section 2.1

1. Apply Theorems 2.2 and 2.3.
3. Let H be the graph obtained by adding two vertices u and w to G and joining u to the two vertices of U and joining w to the two vertices of W.
5. Let $u, w \in V(\overline{G}) - \{v\}$. Show that there is a $u - w$ path in $\overline{G}$ not containing v.
7. In the case when G is a nonseparable graph and the two elements of G are the two edges $e = uv$ and $f = xy$, subdivide e and f, introducing two new vertices s and t and producing a new nonseparable graph H.
9. The statement is false.
11. The statement is false.

Section 2.2

1. Either $\deg u \geq 2$ or $\deg v \geq 2$, say the latter. Then v is adjacent to a vertex w distinct from u. Assume that v is not a cut-vertex and obtain a contradiction to Theorem 2.10.
3. (a) There is only one example. (b) $T = K_{1,3}$ or T is the tree in (a).
5. There are 20 forests of order 6.
7. A graph G is a forest if and only if every connected subgraph of G is an induced subgraph of G.
9. In one direction, assume that least one of e_1 and e_2 is not a bridge and observe that $G - e_1 - e_2$ has at most two components.
11. Apply Theorem 2.9.
13. (a) Observe that $b(v_i) \leq \deg v_i$ and apply Theorem 1.4.
(b) Observe that $\sum_{i=1}^{n} b(v_i) = 2m$ if and only if every edge of G is a bridge.
15. Apply Theorem 2.8.
17. Show that $\deg u = \deg v = m - (n - 2)$.
19. Apply Theorem 2.20.
21. The 4-cycle C_4 is the only graph with this property.

Section 2.3

1. The Prüfer codes of the first 8 trees are (2), (1), (3), (1,1), (2, 2), (4,4), (3, 3), (3,4).
3. The labeled tree with this Prüfer code is a caterpillar with three vertices of degree 3.
5. Observe that every vertex v of a tree T appears $\deg v - 1$ times in its Prüfer code.
7. If G is a unicyclic graph of order n, then G has size n and at least three vertices of G lie on a cycle.
9. Let v be a vertex of degree 2 in the unicyclic graph of order 5 containing a 4-cycle.
11. Let u and v be the two end-vertices in the unicyclic graph of order 5 containing a vertex of degree 4.
13. (a) There are six paths and three stars. (b) $(n - 1)^{n-2}$.

15. In one direction, suppose that e is an edge that does not belong to every spanning tree of G.
17. $n/2$.
19. Show that for each positive integer $k \neq 2$, there is a connected graph with exactly k spanning trees.
21. For $2 \leq r < t$, let G be the graph obtained by attaching $r-1$ pendant edges to a vertex of K_{t-r+2}.
23. For the graph G, there are two non-isomorphic spanning trees, namely $K_{1,3}$ and P_4. For the graph H, there are three non-isomorphic spanning trees.
25. A minimum spanning tree has weight 20.
27. First, assume that T is the unique minimum spanning tree of G. Obtain a contradiction if the weight of some edge e that is not in T is at most the weight of some edge f on the cycle $T+e$. For the converse, assume that T is not unique. Consider another minimum spanning tree T' of G and obtain a contradiction.

Section 2.4
1. If G is a complete k-partite graph of order n whose largest partite set contains n_k vertices, then $\kappa(G) = \lambda(G) = \delta(G) = n - n_k$.
3. (a) Let $H = G \vee K_1$. Show that if S is a set of vertices of H with $|S| \leq k$, then $H - S$ is connected.

(b) Let $H = G \vee K_1$. Show that if S is a set of edges of H with $|S| \leq k$, then $H - S$ is connected.
5. Observe that $k \leq \kappa(G) \leq \delta(G)$ and apply Theorem 2.20.
7. Assume, to the contrary, that G is not (ℓ, k)-connected. Consider a set S of $\kappa_k(G)$ vertices such that $G - S$ contains at least k components and produce a contradiction to the degree condition.
9. Such a graph G can be constructed by beginning with two copies of K_{c+1}.
11. For each $k \geq 3$, a graph G can be constructed by beginning with two copies of K_k.
13. Every k-connected graph of order n has minimum degree at least k and so has size at least $kn/2$. Show that there exists a k-connected graph G of (even) order n and size $kn/2$.
15. (a) Let W be a set of $k-2$ vertices of $G - e$. Show that $(G-e) - W$ is connected.

(b) Let X be a set of $k-2$ edges of $G - e$. Since $|X \cup \{e\}| = k-1$, the graph $G - e - X$ is connected.
17. $\overline{\kappa}(G) \geq \kappa(G)$, $\overline{\lambda}(G) \geq \lambda(G)$ and $\overline{\lambda}(G) \geq \overline{\kappa}(G)$.

Section 2.5
1. For $n \geq k \geq 3$, consider $G = C_n$.
3. For a set W of $k-1$ vertices of H, show that $H - W$ is connected.
5. In one direction, let G be a graph of order $n \geq 2k$ that is not k-connected. Then G contains a vertex-cut S with $|S| = k-1$. Consider a component G_1

of $G - S$ of minimum order and consider appropriate sets V_1 and V_2.
7. Let $u \in S_1$ and $v \in S_2$. Then there exist k internally disjoint $u - v$ paths in G. Consider appropriate subpaths of these paths.
9. Assume first that G is k-edge-connected. Consider a $u - v$ separating set of edges and apply Theorem 2.42. For the converse, suppose that G contains k pairwise edge-disjoint $u - v$ paths for each pair u, v of distinct vertices of G. Then apply Theorem 2.42.
11. The statement is true.
13. Consider the step where each edge of S is subdivided and then the k new vertices are identified.
15. Let u and v be two adjacent vertices of a 3-connected graph G. Then consider three internally disjoint $u - v$ paths. The graph $G = C_n$, $n \geq 4$, shows that a 2-connected graph need not contain a chorded cycle.
17. (a) By Whitney's theorem, $\kappa(G) = k$. Let S be a vertex-cut of G with $|S| = k$ and let u and v be vertices belonging to different components of $G - S$. Let $w \in V(G) - (S \cup \{u, v\})$ and consider $S \cup \{w\}$.

(b) Let u and v be two vertices of G with $d(u, v) = \operatorname{diam} G = k$ and consider k internally disjoint $v - u$ paths.

Chapter 3 Section 3.1

1. Yes.
3. Show that each vertex of G is even.
5. The graph $G \,\square\, H$ is Eulerian if and only if both G and H are Eulerian or every vertex of G and H is odd.
7. No.
9. For each vertex v in G, either $\deg v \equiv 0 \pmod 4$ or $\deg v \equiv 2 \pmod 4$. Consider the sum of the degrees of the vertices of G.
11. (a) and (b). Both statements are false.

Section 3.2

1. In the dodecahedron in Figure 3.5, follow the consonants in alphabetical order.
3. (a) Apply Dirac's theorem (Corollary 3.7).

(b) Apply Theorem 3.16.

(c) The graph G is Hamiltonian if and only if $n_k \leq \sum_{i=1}^{k-1} n_i$.

5. For $k \geq 1$, consider the graph $G = K_1 \vee 2K_k$.
7. First, observe that $G - v$ is connected for every vertex v of G. Next, consider the Petersen graph.
9. Consider $G = K_{k,k+1}$.
11. Assume, to the contrary, that the result is false. Among all counterexamples of order $2k$, let G be one of maximum size and consider a Hamiltonian path in G. The bound is sharp.
13. Let $H = G \vee K_1$ and apply Theorem 3.15.
15. (a) Let $G = P_5 = (v_1, v_2, v_3, v_4, v_5)$ and let $S = \{v_2, v_4\}$.

(b) **Theorem** If G is a graph with a Hamiltonian path, then $k(G-S) \leq |S|+1$ for every nonempty proper subset S of $V(G)$.

17. (a) Since $\delta(G) > n/2$, it follows by Dirac's theorem (Corollary 3.7) that G has a a Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{101}, v_1)$. Consider the vertex v_1 and the set $S_i = \{v_i, v_{i+25}, v_{i+50}, v_{i+75}\}$ for $2 \leq i \leq 26$.

(b) **Theorem** Let k be a positive integer. If G is a graph of order $4k+1$ such that $\delta(G) \geq 2k+1$, then every vertex of G lies on a cycle of length $k+2$.

19. Observe that every vertex-cut of G is also a vertex-cut of H.

21. Let $G = K_{r,s}$ with partite sets U and W such that $|U| = r$ and $|W| = s$ with $r \leq s$. Then $t(G) \leq \frac{|U|}{k(G-U)} = \frac{r}{s}$. Show that $t(G) \geq r/s$.

23. (a) Let $n = \sum_{i=1}^{k} n_i$. Then $t(K_{n_1,n_2,\ldots,n_k}) = \frac{n-n_k}{n_k}$.

(b) Let $r = a/b$, where a and b are positive integers. Consider the result in (a).

25. Apply Theorem 3.22.

27. Let $H = K_3 \,\square\, K_2$, where $\{u, v, w\}$ are the vertices in one copy of K_3 and $\{u', v', w'\}$ are the vertices in the other copy of K_3, where $uu', vv', ww' \in E(G)$. Let G be the graph obtained by adding a new vertex x to H where x is joined to w and w'.

29. Suppose that the statement is false. Then there exist two vertices u and v for which there is no Hamiltonian $u-v$ path. Consider the Hamiltonian graph $G-v$.

Section 3.3

1. The subdivision graph $S(G)$ of a graph G is Eulerian if and only if G is Eulerian.

3. The exceptions are the trees.

5. Let $u, v \in V(G)$ such that $d_G(u,v) = \text{diam}(G) = \ell$. Consider a $u-v$ geodesic $P = (u = v_0, v_1, \ldots, v_\ell = v)$.

7. Apply Exercise 2 in Section 3.2.

9. First observe that every self-complementary graph is connected and $\text{diam}(G) = 2$ or $\text{diam}(G) = 3$. If $\text{diam}(G) = 2$, then G^2 is complete. Hence it may be assumed that $\text{diam}(G) = 3$. If G is 2-connected, then apply Theorem 3.28. Hence we can assume that G has cut-vertices and contains vertices having eccentricity 3.

11. Observe that (a, b, c, a) is a triangle in $L(G)$ if and only if the subgraph induced by the three edges a, b, c is either K_3 or $K_{1,3}$.

13. (a) For a graph G of order n, $G \cong L(G)$ if and only if $G = C_n$.

(b) Apply (a) to $L^2(G)$.

15. First observe that $L^2(G)$ is connected by Exercise 10. Show that if every vertex of $L^3(G)$ has even degree, then every vertex of $L^2(G)$ has even degree.

17. (a) Assume, to the contrary, that there is a k-edge-connected graph G, $k \geq 2$, such that $L(G)$ is not k-connected. Let S be a vertex-cut in $L(G)$ with $|S| = \ell \leq k-1$.

(b) Let $\{U, W\}$ be a partition of $V(L(G))$ and apply Theorem 2.33.

19. (a) Observe that $T(G^2) \cong (S(G^2))^2$ and apply Theorem 3.27.
(b) Consider $H = (S(G))^2$ and observe that $T(G) \cong H$.

Chapter 4 Section 4.1

1. Follow the proof of Theorem 1.16.
3. (a) By Robbins' theorem (Theorem 4.3), G has a strong orientation D. Consider D and $\vec{D}$. (b) Consider $G = C_n$ where $n \geq 4$.
5. (a) Consider two cases: (1) G contains exactly one bridge and (2) G contains exactly two bridges. (b) Consider $G = K_{1,3}$.
7. Follow the proof of Theorem 3.1.
9. Add a new vertex w to D together with the arcs (v, w) and (w, u). Then apply Theorem 4.4.
11. Assume that $\operatorname{od} v_i \geq \operatorname{id} v_i$ for $1 \leq i \leq k$ and that $\operatorname{od} v_i < \operatorname{id} v_i$ for $k+1 \leq i \leq n$. Construct a digraph D' by adding t new vertices x_i $(1 \leq i \leq t)$ so that D' is Eulerian.
13. Label the vertices of D as $v_1, v_2, \ldots, v_n$ so that $r(v_i) = i$ for $1 \leq i \leq n$. Show that D contains the Hamiltonian path $(v_n, v_{n-1}, \ldots, v_2, v_1)$.
15. Let D_1 and D_2 be copies of the digraph K_k^* obtained by replacing each edge uv of K_k by the symmetric pair (u, v) and (v, u) of arcs. Let D be obtained by identifying a vertex of D_1 and a vertex of D_2.
17. Let D be the digraph of order 4 and size 5 consisting of the cycle $C = (u, v, w, x, u)$ together with the arc (u, w).
19. Let D' be the digraph obtained from D in Figure 17 by replacing (u, v) and (v, w) by (v, u) and (w, v).

Section 4.2

1. See the tournaments below.

T' :

T'' :

3. If D is an r-regular tournament of order n, then $n = 2r + 1$.
5. If $\widetilde{T}$ is not transitive, then $\widetilde{T}$ contains a triangle (S_1, S_2, S_3, S_1). Consider $v_i \in V(S_i)$ for $i = 1, 2, 3$.
7. (b) and (c) are score sequences of tournaments.
9. The tournament T is the transitive tournament of order n.
11. If T is regular, then every vertex of T has maximum score. Show that $e(v) = 2$ for every vertex v of T.
13. (a) Show that every tournament other than the transitive tournament has at least three kings.

(b) Observe that if every vertex of a tournament T is a king, then T is strong. Consider the unique strong tournament of order 4. Show that tournaments of order $3, 5$ and 6 exist for which every vertex is a king. Show that if there is a tournament T_{n-2} of order $n-2 \geq 5$ where every vertex is a king, then there is tournament T_n of order n with this property.
15. If T is strong, then T is Hamiltonian. If T is not strong, then T contains at least two strong components at least one of which is nontrivial.
17. (a) and (b). Both statements are false.
19. (a) One. (b) Suppose that there is a tournament that contains three vertices not having both positive outdegree and positive indegree. (c) None.
21. Consider a shortest $u-v$ path.
23. Consider the set of vertices to which v is adjacent.
25. Consider the sets of vertices to which u is adjacent and from which u is adjacent.
27. Assume that $(u, v) \in E(T)$ and then consider the set of vertices to which v is adjacent.

Section 4.3
1. No.
3. (a) $a = 1$, $b = 2$ and $c = 1$. (b) $\text{val}(f) = 4$.
5. Show that for the flow f', the net flow out of each vertex in N' equals the net flow out of this vertex for the flow f.
7. (a) Let $\mathcal{U}$ be the set of all maximal paths P in D whose initial vertex is u and such that no arc of P belongs to A. Consider the set $X = \bigcup_{P \in \mathcal{U}} V(P)$.
(b) Let D be a $u-v$ path of length 2.
9. (a) Show that $\text{val}(f) = \text{cap}(K)$ and apply Corollary 4.31.
(b) Use the proof of Theorem 4.34 to show that $f(\overline{X}, X) = 0$ and $c(X, \overline{X}) = f(X, \overline{X})$.
11. Apply Algorithm 4.35 three times to obtain a maximum flow f'' with $\text{val}(f'') = 6$ and a minimum cut $[X, \overline{X}]$ with $X = \{u, s\}$.
13. The maximum number of pairwise arc-disjoint $u-v$ paths in D is 1.

Chapter 5 **Section 5.2**
1. $\text{Aut}(G_1) = \{\epsilon, (t\ u), (y\ z), (t\ u)(y\ z)\}$.
$\text{Aut}(G_2) = \{\epsilon$, (1 2), (6 7), (1 2)(6 7), (3 5)(1 6)(2 7), (3 5)(2 6)(1 7), (3 5)(1 7 2 6), (3 5)(1 6 2 7)$\}$.
3. See the figure below.

5. Such a pair (k, n) is realizable except when $k = n$ and $2 \leq n \leq 5$.

7. The graph G_3 has two cut-vertices. In G_4, consider the subgraphs induced by the neighborhoods of its vertices.

Section 5.3

1. (a) Let $\Delta_1 = \{a\}$. The Cayley color graph $D_{\Delta_1}(\Gamma)$ is shown below.

(b) Let $\Delta_2 = \{a, a^2, a^3\}$.

3. The group of color-preserving automorphisms of the Cayley color graph $D_\Delta(\Gamma)$ consists of ϵ, $\alpha = (g_1\ g_2\ g_3\ g_4)(g_5\ g_6\ g_7\ g_8)$, $(g_1\ g_3)(g_2\ g_4)(g_5\ g_7)(g_6\ g_8)$, $(g_1\ g_4\ g_3\ g_2)(g_5\ g_8\ g_7\ g_6)$, $\beta = (g_1\ g_5)(g_2\ g_6)(g_3\ g_7)(g_4\ g_8)$, $(g_1\ g_6\ g_3\ g_8)(g_2\ g_7\ g_4\ g_5)$, $(g_1\ g_7)(g_2\ g_8)(g_3\ g_5)(g_4\ g_6)$, $(g_1\ g_8\ g_3\ g_6)(g_2\ g_5\ g_4\ g_7)$.

5. Let $\Gamma \cong \mathbb{Z}_n = \{e, a, a^2, \cdots, a^{n-1}\}$ and $\Delta = \{a\}$.

Section 5.4

1. See the graph below.

3. The graph $G = K_{3,4}$.

5. (a) Observe that $\overline{\overline{G} - v} = G - v$.

(b) Apply Theorem 5.10 and (a).

7. Apply the proof of Theorem 5.10 to obtain $G = 2C_4 + P_3$.

9. No.

11. The only solution is the graph obtained from $C_6 = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$ by adding the two edges v_2v_5 and v_3v_5.

Chapter 6 Section 6.1

1. To prove that a graph G is planar if each block of G is planar, employ induction on the number of blocks of G. Consider an end-block.

3. Consider $K_{2,2,2}$ and $P_4 \vee K_2$.

5. Suppose that there is a planar graph of order $n \geq 3$ and size $m = 3n - 6$ that is not maximal planar.

7. Apply the Euler Identity (Theorem 6.1).

9. The graph is nearly maximal planar.

11. Determine the size of G.

13. (a) $m \leq k(n-2)/(k-2)$. See the proof of Theorem 6.3. (b) Let $G = C_k$.

15. Suppose that the result is false. Assign to each vertex v of G a charge of $6 - \deg v$. For each vertex v of G having degree 5, distribute its charge of $+1$ equally to four neighbors of v not having degree 5 or 6.

Section 6.2

1. (a) The statement is true. (b) $G[U] = \overline{K}_k$.
3. If G is nonplanar, then G contains a subgraph H that is a subdivision of K_5. Compute the size of H.
5. The only such graph is $G = K_5$.
7. (a) Give a proof by contradiction.
   (b) Let G be the graph obtained from $K_{3,3}$ by subdividing one edge of $K_{3,3}$.
9. The graph C_n^2 is nonplanar if and only if $n \geq 5$ and n is odd.
11. The statement is false.
13. (a) No conclusion can be made.
   (b) The statement is true.
   (c) The graph H is nonplanar.
   (d) Give a proof by contradiction.
15. The graph $G = P_n$.

Section 6.3

1. See the figure below.

3. For the lower bound, see Exercise 2(d). For the upper bound, exhibit a drawing of P_7^5 with 6 crossings.

5. See the figure below.

7. $\mathrm{cr}(K_{1,2,3}) = 1$.
9. (a) See the figure below.

(b) The statement is false.
11. See the figure below.

Section 6.4

1. Consider a planar embedding of $K_{2,3}$ and suppose that G contains a Hamiltonian cycle C.
3. Suppose that the Grinberg graph G is Hamiltonian and observe that
$3(r_5 - r'_5) + 6(r_8 - r'_8) + 7(r_9 - r'_9) = 0$.

5. Suppose that this graph G has a Hamiltonian cycle C that contains both e and f. Then one of the regions having e on its boundary is in the interior of C and the other is in the exterior of C.

Chapter 7 Section 7.1

1. Since $K_{3,3} \subseteq K_{4,4}$, it follows that $\gamma(K_{4,4}) \geq 1$. Exhibit an embedding of $K_{4,4}$ on the torus to show that $\gamma(K_{4,4}) = 1$.
3. Since the boundary of every region contains at least k edges and every edge is on the boundary of at most two regions, $kr \leq 2m$. The result follows by applying Corollary 7.3.
5. Apply Corollary 7.6.
7. $\gamma(C_4 \,\square\, C_6 \,\square\, K_2) = 7$.
9. For $n \geq 6k + 6$, let G be a maximal planar graph of order n. Apply Theorem 7.4.
11. $\gamma(G) = 2$.

Section 7.2

1. (a) $\pi_1 = (2\ 6\ 3)$, $\pi_2 = (1\ 4\ 5)$, $\pi_3 = (1\ 5\ 4)$, $\pi_4 = (2\ 3\ 6)$, $\pi_5 = (2\ 6\ 3)$, $\pi_6 = (1\ 4\ 5)$. There are three orbits (regions).

(b) $\pi_0 = (1\ 5\ 4\ 6\ 2\ 3)$, $\pi_1 = (0\ 3\ 4\ 2\ 6\ 5)$, $\pi_2 = (0\ 6\ 1\ 4\ 5\ 3)$, $\pi_3 = (0\ 2\ 5\ 6\ 4\ 1)$, $\pi_4 = (0\ 5\ 2\ 1\ 3\ 6)$, $\pi_5 = (0\ 1\ 6\ 3\ 2\ 4)$, $\pi_6 = (0\ 4\ 3\ 5\ 1\ 2)$. There are fourteen orbits (regions).
3. (a) $G = K_{2,2,2}$. (b) $k = 1$. (c) No.
5. Use the Generalized Euler Identity (Theorem 7.1) to conclude that for a 2-cell embedding of K_4 on S_0 we have $r = 4$, for a 2-cell embedding of K_4 on S_1 we have $r = 2$ and for a 2-cell embedding of K_4 on S_2 we have $r = 0$. There are two 2-cell embeddings of K_4 on S_0, fourteen 2-cell embeddings of K_4 on S_1 and no 2-cell embeddings of K_4 on S_2.

Section 7.3

1. $\pi_1 = (2\ 5\ 3\ 4)$, $\pi_2 = (1\ 3\ 4\ 5)$, $\pi_3 = (1\ 2\ 4\ 5)$, $\pi_4 = (1\ 2\ 3\ 5)$, $\pi_5 = (4\ 3\ 2\ 1)$.
3. $\gamma_M(P) = 3$.
5. If G is upper embeddable, then $\xi(G) = 0$ or $\xi(G) = 1$. This implies that G has a splitting tree.
7. Apply Corollary 7.23.
9. The statement is true.
11. $\gamma(C_s \,\square\, C_t) = 1$ and $\gamma_M(C_s \,\square\, C_t) = \left\lfloor \frac{st+1}{2} \right\rfloor$.
13. If $p \leq q \leq 3p$, let G be constructed from p copies $F_1, F_2, \ldots, F_p$ of K_5 by joining a vertex of F_i to a vertex of F_{i+1} for $1 \leq i \leq p-1$. If $q > 3p$, similarly construct G from p copies of K_5 and $q - 3p$ copies of $K_{2,3}$.
15. (a) $\pi_1 = (2\ 4\ 6)$, $\pi_2 = (1\ 3\ 5)$, $\pi_3 = (2\ 6\ 4)$, $\pi_4 = (1\ 3\ 5)$, $\pi_5 = (2\ 4)$, $\pi_6 = (1\ 3)$. (b) S_0 and S_1.

Section 7.4

1. (a) Apply the definition of a minor-closed family of graphs. (b) $\{K_3\}$.
3. Yes; $\{K_4, K_{2,3}\}$.

Chapter 8 Section 8.1

1. Suppose that the vertices of G are colored 1, 2, 3. For each vertex v colored 3, say, consider the colors of the neighbors of v.
3. $\chi(H) = k$.
5. Consider a $v_0 - v_d$ geodesic $P = (v_0, v_1, v_2, \ldots, v_d)$ of length d in G.
7. If G is not r-regular, let G' be another copy of G and join corresponding vertices whose degrees are less than r. Permute the colors of the vertices of G'. Continue in this manner.
9. (a) Begin with a coloring of G (of H) and produce a coloring of H (of G).
   (b) (i) Let $G = P_2$. (ii) Let $G = C_3$. (iii) Let $G = C_4$.
11. (a) The statement is false. (b) The statement is true.
13. Assume that $\chi(G_1) = \max\{\chi(G_i) : 1 \leq i \leq k\}$ and for each i $(1 \leq i \leq k)$, provide a proper coloring of G_i using the colors $1, 2, \ldots, \chi(G_1)$. Then $\chi(G) \leq \chi(G_1)$.
15. Proceed by induction on $k \geq 2$.
17. The statement is false.
19. $n = 2k^2 + k$, $\alpha(G) = k$, $\omega(G) = 2k$ and $\chi(G) = 2k + 1$.
21. See the graph G below.

G :

23. Assume, to the contrary, that $\chi(G) \leq k-1$ and consider the color classes in a $\chi(G)$-coloring of G. This bound is sharp.
25. (a) Consider some k-coloring of G and the resulting color classes V_1, V_2, $\ldots$, V_k. (b) For one direction, suppose that $\chi(G[V(G)-S]) = k-1$ for some independent set S of vertices of G. Then color the vertices of $G[V(G) - S]$ with the $k - 1$ colors $1, 2, \ldots, k - 1$ and assign each vertex of S the color k.
27. If $\chi(G) = 2$, then G is a bipartite graph and is not Hamiltonian-connected.
29. A graph G is constructed with $V(G) = \{F_1, F_2, \ldots, F_{10}\}$, where F_i is adjacent to F_j $(i \neq j)$ if F_i and F_j cannot be placed in the same tank. Then the minimum number of tanks required is $\chi(G)$.

Section 8.2

1. (a) Assume, to the contrary, that $\chi(G - v) \leq k - 2$.
   (b) Assume, to the contrary, that $\chi(G - e) \leq k - 2$.
3. First, observe that any 3-critical graph of order $n \geq 3$ contains an odd cycle C.
5. The statement is true.
7. Both (a) and (b) are false statements.

Section 8.3

1. (a) Among the k-colorings of G using the colors $1, 2, \ldots, k$, let C_1 be the set of those colorings that color a maximum number of vertices 1. Among the colorings in C_1, let C_2 be the set of those colorings that color a maximum number of vertices 2. Continue in this manner.

(b) Let $H = K_{p+2,p+2}$ with partite sets $U = \{u_1, u_2, \ldots, u_{p+2}\}$ and $W = \{w_1, w_2, \ldots, w_{p+2}\}$. Let $G = H - \{u_i w_i : 1 \leq i \leq p+2\}$.

3. (a) See the graph below. (b) Yes. (c) No.

5. The upper bounds are 5, 2 and 3, respectively.

7. $s = 2$ and $t = 6$.

9. $\chi(G) = 3$.

11. (a) $\chi(G) \geq 2$. This bound is sharp. (b) $\chi(G) \geq 3$. This bound is sharp.

13. Show that there is an orientation D of G such that $\ell(D) = 1$.

Section 8.4

1. (a) $G = C_5$. (b) G is the Grötzsch graph.

3. The statement is true.

5. $\chi(Shad(G)) = \chi(G)$ and $\omega(Shad(G)) = \omega(G)$.

7. (a) The statement is false. (b) The statement is true.

9. (a) Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. Define an intersection graph H by $V(H) = \{S_1, S_2, \ldots, S_n\}$ such that S_i consists of the subscripts of the edges of G that are incident with v_i in G.

(b) Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $A = \{1, 2, \ldots, n\}$. Let $\mathcal{F}$ be the set of 2-element subsets of A such that $\{i, j\} \in \mathcal{F}$ if and only if $v_i v_j \in E(G)$.

11. Suppose that G is an interval graph defined by a set S of closed intervals. For an induced subgraph H of G, consider an appropriate subset of S.

13. (a) The interval graph G is the path $(J_1, I_1, J_2, I_2, J_3, I_3, J_4, I_4, J_5)$.

(b) The interval graph G is a tree obtained from the path $(J_1, I_1, J_2, I_2, J_4, I_3, J_5)$ by adding the vertex J_3 and the edge $I_2 J_3$.

(c) The interval graph G is the forest $2P_5$.

15. (a) Let u_i correspond to I_i for $1 \leq i \leq 5$. Then G is the graph with $V(G) = \{u_1, u_2, \ldots, u_5\}$ and $E(G) = \{u_1u_2, u_1u_3, u_2u_3, u_2u_4, u_3u_4, u_4u_5\}$.

(b) Let $J_1 = (0.9, 7.1)$, $J_2 = (5.9, 8.1)$, $J_3 = (5.9, 9.1)$, $J_4 = (6.9, 20.1)$ and $J_5 = (19.9, 22.1)$.

17. Assume, to the contrary, that there is an interval graph G that is not a chordal graph.
19. For $n = 7$, let $G_7 = \overline{C}_7$.
21. $\overline{C}_8$ is perfect.
23. The converse is true.

Section 8.5
1. Let there be given any collection $\mathfrak{L} = \{L(v) : v \in V(G)\}$ of color lists of size $1 + \Delta(G)$ for the vertices of G, where the colors are positive integers. Suppose that the vertices of G are listed in the order $v_1, v_2, \ldots, v_n$ and the greedy algorithm is applied. At each step in the greedy algorithm, there is a color available in $L(v)$ for the current vertex v under consideration since $\deg v < 1 + \Delta(G)$.
3. Let $G = K_{2,3}$. Clearly, $\chi_\ell(G) \geq 2$. Let the partite sets of G be $\{u_1, u_2\}$ and $\{v_1, v_2, v_3\}$ and let there be given an arbitrary collection $\mathfrak{L} = \{L(v) : v \in V(G)\}$ of color lists of size 2 for the vertices of G. Consider two cases (1) $L(u_1) \cap L(u_2) \neq \emptyset$ and (2) $L(u_1) \cap L(u_2) = \emptyset$.
5. Show that the graph $K_{3,27}$ is not 3-choosable by exhibiting an appropriate list of size 3 for each vertex of $K_{3,27}$ that does not produce a list coloring.
7. First, show that $P_n \,\square\, K_2$ is not 2-choosable for $n \geq 4$ by exhibiting an appropriate list of size 2 for each vertex of $P_n \,\square\, K_2$ that does not produce a list coloring. Show by induction on $n \geq 1$ that $P_n \,\square\, K_2$ is 3-choosable.

Chapter 9 Section 9.2
1. Consider Brazil, Argentina, Bolivia and Paraguay.
3. Use Theorem 8.20 and the fact that there is a planar embedding of every maximal outerplanar graph of order at least 3 such that the boundary of every interior region is a triangle.
5. Proceed by induction on the order of an outerplanar graph.

Section 9.3
1. Assume, to the contrary, that $K_{4,4}$ contains a subdivision F of K_5. Suppose that $V(F) = X \cup Y$ such that $\deg_F x = 4$ for each $x \in X$ and $\deg_F y = 2$ for each $y \in Y$.
3. Consider the graph G of order $n = 13$ consisting of three copies G_1, G_2, G_3 of K_3 and two copies G_4, G_5 of K_2 where each vertex of G_i is adjacent to each vertex of G_{i+1} for $i = 1, 2, \ldots, 5$, where $G_6 = G_1$.

Section 9.4
1. The chromatic polynomial of a graph G of order n can be expressed as $P(G, \lambda) = \lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + (-1)^{n+1}a_{n-1}\lambda$.

3. (a) See the figure below.

$$= (K_6 + K_5) + 2(K_5 + K_4) + (K_4 + K_3)$$
$$= K_6 + 3K_5 + 3K_4 + K_3$$

(b) $P(K_{2,2,2}, 3) = 6 = \chi(K_{2,2,2})$!

5. (a) Every two unicyclic graphs of the same order and containing a cycle of the same length are chromatically equivalent.

(b) $n - 2$.

7. (a) Assume that the statement is false. Then among all counterexamples, let S be the set of those of minimum order n and let G be a graph in S of minimum size m. Then $P(G, \lambda) = \lambda g(\lambda)$ where $g(0) = 0$. Then G is not a tree. Let $e = uv$ be an edge of G that is not a bridge and let H be the graph obtained by identifying u and v in $G - e$. Consider the coefficients of λ in $P(G, \lambda)$, $P(G - e, \lambda)$ and $P(H, \lambda)$.

(b) Apply Exercise 6 and (a).

9. Proceed by induction on the number of blocks in G and apply Exercise 6.

11. Since $K_{r,r}$ is the only bipartite graph of order $2r$ and size r^2, it is chromatically unique.

13. The vertices of G can be listed as $v_1, v_2, \ldots, v_n$ such that v_i $(2 \leq i \leq n)$ is adjacent to at least one vertex of $\{v_1, v_2, \ldots, v_{i-1}\}$ by Exercise 23 in Section 1.3. Consider the number of ways that the vertices of G can be colored using this ordering.

15. The statement is true.

Section 9.5

1. (a) See the figure below. (b) By the Generalized Euler Identity (Theorem 7.1), $r = 5$. (c) Five colors are needed.

3. Consider the graph K_8.

Chapter 10 Section 10.1

1. Suppose that there exists a tree with two distinct perfect matchings. Apply Theorem 10.1.

3. Consider the graph H obtained from G by adding two new vertices u and w where u is joined to every vertex of U and w is joined to every vertex of W.

5. (a) Proceed by contradiction and apply Hall's theorem (Theorem 10.3).
   (b) Consider $G = K_{k,k-1} + K_1$.

7. After observing that $\alpha'(G) \leq |U|$, consider the cases where $\text{def}(U) \leq 0$ and $\text{def}(U) > 0$.

9. (a) Let $S = \{u, v\}$.
   (b) Proceed by contradiction and apply Tutte's theorem (Theorem 10.7).
   (c) See the graph G in (a).

11. (a) Apply the First Theorem of Graph Theory (Theorem 1.4).
   (b) Proceed by contradiction and apply Tutte's theorem (Theorem 10.7).

13. See the figure below.

15. In one direction, let G be a graph that is not bipartite and let C be a smallest odd cycle of G.
17. By Exercise 16, $\frac{n}{1+\Delta(G)} \leq \alpha'(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$. Apply Gallai's theorem (Theorem 10.11).
19. $M(C_3) = C_3$, $M(2P_3) = C_4$, $M(C_5) = C_5$, $M(K_{2,3}) = C_6$.
21. No.

Section 10.2
1. Suppose that there is a 1-factorization $\{F_1, F_2, F_3\}$ of the Petersen graph and consider the spanning subgraph with edge set $E(F_1) \cup E(F_2)$.
3. (a) Apply Theorem 10.15.
(b) Observe that if Q_n is k-factorable, then $k \mid n$. For the converse, apply (a).
5. See the figure below.

7. Observe first that there is a k-regular graph of order $n-2$.
9. Proceed by contradiction and consider the size of each factor.

Section 10.3
1. Apply the Four Color Theorem.
3. If there exists a Steiner triple system of order $n \geq 3$, then the complete graph K_n is K_3-decomposable and so $3 \mid \binom{n}{2}$.

5. (a) Label the seven vertices of K_7 by $0, 1, 2, \ldots, 6$ and observe that the vertex labels $0, 1$ and 3 produce a graceful labeling of K_3.

(b) Consider the graph $G = K_3 \,\square\, K_3$.

7. See the figure below.

9. Let $m = 2k$. Then $K_{3m+1} = K_{6k+1}$. Arrange the vertices $v_1, v_2, \cdots, v_{6k+1}$ cyclically about a regular $(6k+1)$-gon and join every two vertices by a straight line segment, producing the complete graph K_{6k+1}. Let $C =(v_1, v_2, \cdots, v_{6k+1}, v_1)$. Assign each edge xy of K_{6k+1} the value $d_C(x, y)$. Use the fact that G is graceful to place G in K_{6k+1} so that the edge labels coincide.

11. The Petersen graph is cyclically F-decomposable when F is any of the graphs $P_4, 3K_2, P_3 + K_2$.

13. See the figure below.

15. See the figure below.

17. (a) If F is a forest of order n with $k \geq 2$ components, then the size of F is $m = n - k \leq n - 2$.

(b) The graph $P_3 + C_3$ is not graceful.

19. Apply the definition of a graceful labeling to f and $\overline{f}$.

Section 10.4

1. $\gamma(Q_3) = 2$ and $\gamma(Q_4) = 4$.

3. Let G be a graph of order n. Then $\gamma(G) = 1$ if and only if $\Delta(G) = n - 1$.

5. Let $d = \text{diam}(G) \geq 3$ and let v be a vertex of G such that $e(v) = d$. For a vertex w of G with $d(v, w) \geq 3$, consider $S = \{v, w\}$.

7. Use the fact that $\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G) = n$.

9. (a) The statement is true.

(b) Consider the double star where every vertex has degree 1 or 3.

11. For $k \geq 4$, let G be the graph obtained from $P_3 = (u, v, w)$ by adding $3(k-2)$ new vertices, where $k-2$ of these vertices are adjacent to u, another $k - 2$ vertices are adjacent to v and the final $k - 2$ vertices are adjacent to w.

13. (a) To show that $\gamma_t(G) \leq 2\gamma(G)$, let $S = \{v_1, v_2, \ldots, v_k\}$ be a minimum dominating set of G. For $u_i \in N(v_i)$ $(1 \leq i \leq k)$, consider $S \cup \{u_1, u_2, \ldots, u_k\}$.

(b) For each $n \geq 3$, let H_n be the corona of the cycle C_n and let F_n be the graph obtained by adding a pendant edge to each end-vertex of H_n.

Chapter 11 Section 11.1

1. The chromatic index of this graph is 4.

3. Since $\Delta(G \,\square\, K_2) = 1 + \Delta(G)$, it suffices to show that there exists a $(1 + \Delta(G))$-edge coloring of $G \,\square\, K_2$. Because $\chi'(G) \leq 1 + \Delta(G)$, there exists a $(1 + \Delta(G))$-edge coloring of G.

5. (a) Observe that G contains G_1, G_2 and the complete bipartite graph K_{n_1,n_2} as subgraphs.

(b) Consider $G_1 = K_2$ and $G_2 = C_5$.

Section 11.2

1. Show that G is overfull and apply Corollary 11.9.
3. Since G is 3-regular, it follows that G has even order. Consider a Hamiltonian cycle C of G.
5. The graph of each regular polyhedra is of Class one.
7. Show that there exists a $\Delta(G)$-regular bipartite graph containing G as an induced subgraph.
9. Apply Theorem 11.8.
11. Consider the graph G of order 12 in the figure below and let $V_1 = \{v_1, v_2, \cdots, v_5\}$ and $V_2 = \{v_6, v_7, \cdots, v_{12}\}$.

G :

13. Apply Theorem 11.8.
15. Apply Corollary 11.9.
17. France has two consecutive days off.

Section 11.3

1. (a) No. (b) Yes.
3. (a) No. (b) No. (c) Yes.
5. (a) $\chi'(G) = 4$. (b) No.

Section 11.4

1. See the figure below.

(a) D_1 (b) D_2 (c) D_3

3. Show that $\sigma^+(v; a\phi_1 + b\phi_2) = \sigma^-(v; a\phi_1 + b\phi_2)$ for integers a and b and for every vertex v of D.

5. Let G be a bridgeless graph containing an Eulerian trail and let u and v be the two odd vertices of G. Then G is 2-edge-connected and by Theorem 2.43, G contains two edge disjoint $u-v$ paths P and P'. Begin by assigning flow values to the appropriately directed edges of these paths.
7. Let $V(K_6) = \{1,2,3,4,5,6\}$. A cycle double cover of K_6 consisting of five Hamiltonian cycles is (1,2,3,4,5,6,1), (1,3,6,2,5,4,1), (1,4,2,6,5,3,1), (1,5,2,3,4,6,1), (1,2,4,6,3,5,1).

Section 11.5
1. Let $\Delta(G) = \Delta$ and for each edge e of G, let $L(e)$ be a list of $2\Delta - 1$ colors (positive integers). Let $e_1, e_2, \ldots, e_m$ be an ordering of the edges of G and give a greedy coloring of the edges.
3. $\chi''(G) = 4$.
5. Since $\Delta(K_{s,t}) = t$, it follows that $\chi''(K_{s,t}) \geq t+1$. Describe a $(t+1)$-total coloring of $K_{s,t}$.

Chapter 12 Section 12.1

1. $T_{7,2} = K_{3,4}$, $t_{7,2} = 12$, $T_{7,3} = K_{2,2,3}$, $t_{7,3} = 16$,
$T_{7,4} = K_{1,2,2,2}$, $t_{7,4} = 18$, $T_{8,2} = K_{4,4}$, $t_{8,2} = 16$,
$T_{8,3} = K_{2,3,3}$, $t_{8,3} = 21$, $T_{8,4} = K_{2,2,2,2}$, $t_{8,4} = 24$.
3. Show that G contains a vertex of degree 2 or more.
5. For $n \geq 4$, the smallest integer m such that every graph of order n and size m contains $K_{1,3}$ as a subgraph is $n+1$.
7. (a) Let $e = uv$ be an edge of G, let $A = N(u) - \{v\}$ and let $B = N(v) - \{u\}$. First show that $|A \cap B| \geq (n+2)/3$.
(b) For an integer $r \geq 3$, let $G = K_{r,r,r}$.

Section 12.2
1. Let $g = g(G)$. Consider the cases where g is even or g is odd.
3. Suppose that H is an (s,g)-graph and assume, to the contrary, that H is an (s,g)-cage. Since H contains 4-cycles, either $g = 3$ or $g = 4$.
5. (a) $n(4,3,4) = 5$. (b) $n(4,3,6) = 8$.

Section 12.3
1. (a) Consider $H = G \vee K_1$. (b) Consider $F = G + K_1$.
3. To show that $R(s',t') \leq R(s,t)$, let G be a graph of order $R(s,t)$. Then apply the definition of Ramsey number.
5. To show that $R(4,4) \leq 18$, consider a red-blue coloring of K_{18}. Each vertex of K_{18} is incident with nine edges of the same color.
7. $\min\{t_c\} = 2$.
9. Suppose that there is a red-blue coloring of the edges of K_9 in which there is no red K_3. Since $R(K_3, K_4) = 9$, there must be a blue $F = K_4$ in K_9. Consider the five vertices in K_9 not in F.
11. Consider the cases when n is even and n is odd. Apply Theorems 10.18 and 10.20.

13. (a) Yes. (b) Let $n = R_\chi(s-1,t) + R_\chi(s,t-1)$. Let there be given a red-blue coloring of the complete graph K_n. Show that K_n contains a red subgraph with chromatic number at least s or a blue subgraph with chromatic number at least t.

(c) $R_\chi(3,3) = 5$. (d) For $3 \le s \le t$, $R_\chi(s,t) = 1 + (s-1)(t-1)$.

15. Proceed by induction on $k \ge 2$.

17. Let $n = R(K_{n_1}, K_{n_2}, \ldots, K_{n_k}, T_s)$. To show that $n \le 1 + (r-1)(s-1)$, apply Theorem 12.21. To show that $n \ge 1 + (r-1)(s-1)$, exhibit an edge coloring of $K_{(r-1)(s-1)}$ with the colors $1, 2, \ldots, k+1$ such that for each i with $1 \le i \le k$ there is no copy of K_{n_i} all of whose edges are colored i and there is no copy of T_s all of whose edges are colored $k+1$.

19. $R(K_{1,5}, K_{1,6}, K_{1,8}) = 17$.

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Index

of Names

A
Aichholzer, Oswin 261
Albertson, Michael 484
Allaire, Frank 375
Alon, Noga 311
Anderson, Ian 407
Appel, Kenneth 374-376
Archdeacon, Daniel 307
Archimedes 222
Aurenhammer, Franz 261

B
Balaban, Alexandru 503
Bannai, Eiichi 506
Battle, Joseph 276
Bauer, Douglas 126
Behzad, Mehdi 488-489
Beineke, Lowell W. 141, 257, 259, 278
Benson, Clark T. 503
Berge, Claude 23, 93, 197, 342, 344, 350, 403, 409, 440, 443
Bertrand, Louis 273
Bieberbach, Ludwig 98
Bielak, Halina 40
Birkhoff, George David 372-374, 388
Black, Cornelius 198
Blanuša, Danilo 470
Blažek, Jaroslav 252
Bollobás, Béla 442, 484
Bondy, J. Adrian 121-122, 133
Borůvka, Otakar 81
Brigham, Robert C. 335
Brodsky, Alex 261
Broersma, Hajo 126
Brooks, Rowland Leonard 198, 331

C
Camion, Paul 167
Cantor, Georg 98
Carroll, Lewis 470
Catlin, Paul 385-386
Cayley, Arthur 63, 70, 195, 207, 363, 368, 464
Chartrand, Gary 94, 112, 139, 143, 252, 489
Chetwynd, Amanda C. 423, 461
Chudnovsky, Maria 350
Chvátal, Vašek 121-123, 129, 261, 517
Clebsch, Alfred 518
Cockayne, Ernest 440, 442
Collins, Karen 484
Cooper, John K. Jr. 489
Courant, Richard 151, 398
Coxeter, H.S.M. (Donald) 199
Czekanowski, Jan 81

D
Damerell, Robert Mark 506
de Jaenisch, Carl Friedrich 439
De Morgan, Augustus 360-362
Descartes, Blanche 199, 340, 375
Descartes, René 199, 222
Dinic, Efim A. 180
Dinitz, Jeffrey Howard 485
Dirac, Gabriel Andrew 102, 118-120, 347, 368, 384
Dirac, Paul Adrien Maurice 119
Dodgson, Charles Lutwidge 470

Duke, Richard A. 295
Durocher, Stephane 261
Dutton, Ronald D. 335

E
Edmonds, Jack 180, 286
Egerváry, Eugene 416
Eggleton, Roger 257
Einstein, Albert 373
Elias, Peter 179
Erdős, Paul 26, 123, 198, 200, 254, 353, 378, 414, 498, 500, 511, 515
Errera, Alfred 412
Euler, Leonhard 1, 6, 107-112, 195, 199, 221-222, 273, 411

F
Fáry, István 259
Feinstein, Amiel 179
Finck, Hans-Joachim 335
Fisk, Steve 261
Fleischner, Herbert 139
Ford, Lester Randolph Jr. 178-180
Franklin, Philip 231, 373
Frink, Orrin 237
Frucht, Roberto 210
Fulkerson, Delbert Ray 178-180

G
Gaddum, Jerry W. 334
Gallai, Tibor 26, 337, 343-344, 414-415
Galvin, Fred 485
Gardner, Martin 470
Geller, Dennis Paul 333
Georgakopoulos, Agelos 139
Gethner, Ellen 261
Ghouila-Houri, Alain 153
Gleason, Andrew W. 518
Goffman, Casper 200
Goldbach, Christian 221
Golomb, Solomon W. 431, 436
Graham, Ronald L. 81
Greenwood, Robert E. 518
Grinberg, Emanuels Ja. 263-264, 266
Grötzsch, Herbert 340
Gupta, Ram Prakash 449, 453, 484
Guthrie, Francis 360-362, 368
Guthrie, Frederick 360-362
Guy, Richard K. 251-252, 257, 261

H
Hadwiger, Hugo 387
Hahn, Hans 97
Hajnal, Andras 347
Hajós, Győrgy 385
Haken, Wolfgang 373-376
Hakimi, S. Louis 24
Hall, Philip 404, 406
Hamilton, William Rowan 115-118, 195, 361-362
Harary, Frank 59, 93-94, 141, 164, 168, 198-199, 257, 276, 278, 375
Harris, Andrew J. 484
Hausdorff, Felix 98
Havel, Václav 24
Heawood, Percy John 369-370, 376-377, 395-398
Hedetniemi, Stephen 39, 440
Heesch, Heinrich 230, 373-375
Heffter, Lothar 286, 398
Hell, Pavol 81
Hierholzer, Carl 112
Hilbert, David 373
Hilton, Anthony J. 423, 461
Hobbs, Arthur M. 139
Hoffman, Alan J. 505
Holyer, Ian 456
Hopcropt, John 240
Huneke, Philip 307

I
Isaacs, Rufus 471
Ito, Tatsuro 506
Izbicki, Herbert 210

J
Jackson, Bill 124
Jaeger, Francois 483

Jarník, Vojtěch 83
Jensen, Tommy 342
Jordan, Camille 98
Jung, Heinz 139
Jungerman, Mark 299

K
Kainen, Paul C. 255, 257
Kapoor, S. F. 139
Karaganis, Jerome 138
Karp, Richard M. 180
Kárteszi, Francesco 499
Kelly, John 340
Kelly, Leroy M. 340
Kelly, Paul J. 213
Kempe, Alfred Bray 363, 366-370, 373, 376, 395, 464
Kirchhoff, Gustav 63, 77, 411
Kirkman, Thomas Penyngton 118, 429
Klee, Victor 261
Klein, Esther 414
Kleitman, Daniel 255
Knuth, Donald E. 204
Koch, John 375
Kodama, Yukihiro 276
Koman, Milan 252
König, Dénes 26, 98, 100, 196-197, 209-210, 229, 254, 414-416, 420, 456
Kotzig, Anton 431, 433
Krasser, Hannes 262
Kronk, Hudson V. 305, 334
Kruskal, John Bernard 81
Kuratowski, Kazimierz 213, 237

L
Landau, Hyman Garshin 162, 165
Legendre, Adrien-Marie 222
Leibniz, Gottfried 1
Lesniak, Linda 95
Lhuilier, Simon Antoine Jean 273
Lin, Xiaohui 252
Listing, Johann Benedict 279
Lovász, László 204, 344, 349

M
Mantel, Willem 15, 493
Manvel, Bennet 216
Matthews, Manton 130
Matula, David W. 325
May, Kenneth O. 368
McGee, William F. 503
McKay, Brendan 213, 503
McKee, Terry A. 113
Menger, Karl 97-98, 100
Meyniel, Henri 152
Minkowski, Hermann 196, 229
Mirzakhani, Maryam 380
Möbius, August Ferdinand 279
Molloy, Michael 490
Moon, John W. 168
Moore, Edward F. 499, 505
Moser, Leo 164, 168
Mycielski, Jan 340
Myrvold, Wendy 503

N
Nadon, Jacqueline 503
Nash, John 197
Nash-Williams, Crispin 139, 141
Nijenhuis, Albert 213
Nordhaus, Edward A. 298, 305, 334
Norman, Robert Z. 59

O
Okamoto, Futaba 252
Ore, Oystein 1, 118, 131, 197, 440-441

P
Pan, Shengjun 252
Payan, Charles 442
Peano, Giuseppe 98
Pétard, Hector 198
Petersen, Julius 13, 410-411, 420, 423
Plesník, Jan 95
Plummer, Michael D. 139
Polimeni, Albert D. 112
Pontryagin, Lev Semenovich 237

Pósa, Lajos 495
Powell, Martin B. 331
Prim, Robert Clay 83
Prüfer, Heinz 70

R
Rado, Richard 416
Ramsey, Frank Plumpton 508
Ray-Chaudhuri, Dijen 430
Read, Ronald C. 388, 391
Rédei, László 166-167
Reed, Bruce 335, 490
Richter, R. Bruce 252
Řiha, Stanislav 139
Ringeisen, Richard D. 257, 259, 290, 305
Ringel, Gerhard 255, 277-278, 398, 433
Robbins, Herbert E. 150-151, 398
Robertson, Neil 307, 350, 375, 387, 471, 480
Rosa, Alexander 431, 434
Rota, Gian-Carlo 198
Roy, Bernard 337
Royle, Gordon F. 204, 342
Rubin, Arthur L. 353, 378
Russell, Bertrand 508

S
Sachs, Horst 500, 503
Sale, Tony 199
Sanders, Daniel 375, 469, 471, 480
Schur, Issai 210
Schuster, Seymour 75
Schwenk, Allen J. 257
Sekanina, Milan 138
Seymour, Paul 307, 350, 375, 387, 471, 480, 482
Shannon, Claude Elwood 179, 453
Singleton, Robert R. 505
Slivnik, Tomaž 485
Smith, Cedric Austen Bardell 198
Smith, Henry 369
Smith, Paul Althaus 237
Sós, Vera 498
Stanton, Ralph G. 199
Stein, Sherman K. 259
Steiner, Jakob 429
Stewart, Bonnie M. 298, 305, 335
Stewart, M. James 112
Stockmeyer, Paul K. 214
Stone, Arthur 198
Sumner, David 130
Surányi, János 347
Sylvester, James Joseph 3, 363, 368
Syslo, Maciej 40
Szekeres, George 330, 414, 471, 482, 511
Szele, Tibor 167

T
Tait, Frederick 464
Tait, Peter Guthrie 264, 266, 420, 464-465, 468
Tarjan, Robert E. 240
Taylor, Herbert 353, 378
Teller, Edward 213
Thomas, Robin 350. 375, 387, 471, 480
Thomassen, Carsten 119, 162, 168, 378, 387
Toida, Shunichi 113
Toft, Bjarne 197
Turán, Paul 254, 414, 493-494
Turing, Alan 199
Tutte, William T. 198-200, 264, 359, 375, 407, 471, 480, 499, 503

U
Ulam, Stanislaw M. 213

V
Veblen, Oswald 113, 196, 373
Veldman, Henk Jan 126
Vitaver, L. M. 337
Vizing, Vadim G. 353, 378, 443, 449, 453, 469, 484, 488-489
Voigt, Margit 378, 380
von Dyck, Walther 286
von Neumann, John 213

W
Wagner, Klaus 243, 259, 307, 387
Wall, Curtiss E. 143
Welsh, Dominic J. A. 331
Wernicke, Paul 229, 372
Weyl, Hermann 373
White, Arthur T. 255, 290, 298, 305
Whitehead, Alfred North 508
Whitney, Hassler 91, 100-101, 141, 151, 388, 391
Wigner, Eugene Paul 119
Wilf, Herbert S. 330
Williamson, James E. 132
Wilson, Richard 430
Witt, Ernst 373
Woodall, Douglas 152, 255
Woolhouse, Wesley S. B. 429

X
Xuong, Nguyen Huy 299, 303, 442

Y
Yang, Gui 252
Yang, Yuansheng 252
Youngs, J.W.T. (Ted) 274, 277, 286, 398

Z
Zaks, Joseph 305
Zarankiewicz, Kazimierz 255
Zhang, Ping 252
Zhao, Yue 469
Zheng, Wenping 252
Zykov, Alexander 198, 340, 391

Index

of Mathematical Terms

Each term in the index is followed by a number that gives the page where the term is defined or described. In some cases, this is followed by a range of numbers including page numbers where this is the topic of discussion.

A

acyclic digraph 157
acyclic graph 62
adjacency matrix
    of a digraph 53
    of a graph 32 (32-34)
adjacent
    from (a vertex in a digraph) 47
    to (a vertex in a digraph) 47
    to (arcs) 154
adjacent edges 4
adjacent vertices 4
algorithm
    linear-time 85
    polynomial-time 85
    quadratic-time 85
M-alternating path 402
antipodal vertices 37
arc (of a digraph) 47
Arc Form of Menger's Theorem 186
Art Gallery Problem 261
attacking queens 439
M-augmenting path 402
f-augmenting semipath 178
automorphism
    coloring-preserving 208
    of a digraph 204
    of a graph 201
automorphism group
    of a digraph 205
    of a graph 201 (201-206)
average degree 6

B

backward arc 177
Balaban cage 503
balanced chromatic number 320
balanced coloring 320
Bauer-Broersma-Veldman graph 126 (126-128)
Benson graph 503
Betti number 298
bicentral tree 68
bigraphical sequences 30
bipartite graph 14 (14-16)
    complete 14
Birkhoff diamond 373
n-bit string 18
Blanuša snarks 470
block 57 (57-60)
blue graph 509
boundary of a region 224
branch 59, 89
S-branch 89
Brick-Factory Problem
    Turán 254 (254-255)
bridge 61 (61-62)
Brooks' Theorem 331 (331-333)

C

cage (499-508)
  g- 499
  (r, g)- 499
capacity
  of an arc 172
  of a cut 174
capacity function 171
Cartesian product 18
caterpillar 64
Cayley color graph 207 (206-212)
Cayley graph 211
Cayley's Tree Formula 74
2-cell (region) 272
2-cell embedding 272 (272-273, 281-294)
center
  of a graph 38 (38-40)
  of a strong digraph 53
central tree 68
central vertex 37, 64
charge (of a vertex) 230
choice function 353
choosable
  k- 353
  $\mathfrak{L}$- 353
  k-edge 484
  $\mathfrak{L}$-edge 484
chord 263
chordal graph 345 (345-350)
chorded cycle 105
k-chromatic graph 312
  critically 323 (323-327)
chromatic index 447
  list 484
chromatic number 312 (311-323)
  balanced 320
  edge 447
  list 353
  2-set 339
  of a surface 396
  total 488
chromatic polynomial 388 (388-395)
chromatically equivalent 393
chromatically unique 394
circuit
  in a digraph 50
  in a graph 34
circular tournament 170
Class one graph 456
Class two graph 456
classical Ramsey number 511
claw 14
claw-free 130
Clebsch graph 518 (518-519)
clique 314
  k- 314
clique number 314
closed neighborhood 4
closed walk 31
closure 121 (121-123)
cofactor 78
collaboration graph 200
color class 312
  edge 448
color list 353
color-critical graph 323 (323-327)
color-preserving automorphism 208
colorable
  k- 312
  k-edge 447
  k-list- 353
  $\mathfrak{L}$-list- 353
  k-list edge 484
  $\mathfrak{L}$-list edge 484
  k-total 488
coloring 311 (311-357)
  k- 311
  balanced 320
  edge 447 (447-491)
  k-edge 447
  list 353 (353-357)
  proper edge 447
  proper 2-set 339
  red-blue 508
  red-blue edge 508
  proper vertex 311 (311-357)
  Tait 465
  total 487
  k-total 488
  vertex 311 (311-357)

complement (of a graph) 16 (16-17)
complementary labeling 435
complete bipartite graph 14
complete graph 5
complete multipartite graph 16
complete k-partite graph 16
complete symmetric digraph 49
complexity of an algorithm 85
component
  even 407
  of a graph 35
  odd 407
  strong 159
configuration 371
  reducible 372
  unavoidable 371
connected digraph 50
  strongly 50
  weakly 50
connected graph 35
  k- 90
  (ℓ, k)- 96
connected vertices 35
connection number 97
connectivity 89 (89-90)
  k- 96
conservation equation 173
conservation property 473
contracted edge 241
converse (of a digraph) 150
corona (of a graph) 442
cost (of an edge) 80
cover 413
  edge 414
  vertex 414
covering number
  edge 414
  vertex 414
Coxeter graph 204 (204-205)
  truncated 204 (204-205)
critical graph
  k- 323 (323-327)
  color- 323 (323-327)
  factor- 418
ℓ-critical (with respect to χ_{P_k}) 327
critically k-chromatic graph 323 (323-327)
crossing number 250 (250-262)
  rectilinear 259
cube (hexahedron) 221
n-cube 18
cube of a graph 138
cubic graph 13
cubic map 371
cut (in a network) 174
  minimum 176
cut-vertex 55 (55-57)
cycle 5
  3- 5
  k- 35
  n- 5
  even 35
  in a digraph 50
  in a graph 35
  odd 35
cycle double cover 482 (482-483)
Cycle Double Cover Conjecture 482
cycle rank 75
cyclic decomposition 427
cyclic edge-connectivity 469
cyclic factorization 422

D

decision problem 134
deck (of graphs) 217
  solution to 217
H-decomposable graph 427
decomposition 427 (427-433)
  H- 427
  cyclic 427
deficiency
  of a set 417
  total 462
k-degenerate graph 327
  maximal 327
degree (of a vertex)
  in a digraph 47
  in a graph 6 (6-7)
  in a multigraph 46
degree matrix 78

degree sequence (of a graph) 22 (22-30)
Descartes snark 471
detour 45
detour distance 45
diameter
of a graph 37
of a strong digraph 50
digraph 47 (47-51)
Dinitz's Conjecture 485
Dirac's Theorem 119
directed circuit 50
directed cycle 50
directed distance 50
directed edge 47
directed graph 47 (47-51)
directed loop 51
directed path 50
Directed Vertex Form of Menger's Theorem 187
directed walk 50
discharging rules 230
disconnected graph 35
distance
detour 45
between graphs 45
in a graph 36 (36-38)
in a digraph 50
between subgraphs 44, 112
total 44
in a tournament 165 (165-166)
distance-preserving (from a vertex) 75
distinct labelings (of a graph) 70
divisor graph 53
divisor pseudodigraph 53
dodecahedron 117, 221
dominate 438
dominating circuit 141
dominating set of vertices 438
independent 443 (443-444)
minimal 440 (440-443)
minimum 439
total 445
domination (438-445)
domination number 439
independent 443
total 445
double star 64
double-star snark 471 (470-471)
double torus 281 (281-282)
dual color 450
dual graph 368
Duke's Theorem 295 (295-298)

E

eccentric vertex 44
eccentricity of a vertex
in a graph 37
in a strong digraph 50
edge 3
edge choosable
k- 484
$\mathfrak{L}$- 484
edge chromatic number 447
edge color class 448
edge colorable
k- 447
k-list 484
$\mathfrak{L}$-list 484
edge coloring 447 (447-491)
k- 447
proper 447
red-blue 508
k-edge-connected 91
edge-connectivity 90 (90-91)
cyclic 469
edge cover 414
edge covering number 414
edge-cut 90
minimum 90
edge exchange 75
edge independence number 401
edge-induced subgraph 10
edge rotation 29
edge set 4
element (of a graph) 60
embedding
2-cell 272 (272-273, 281-294)
in the plane 223
planar 223
straight-line 259

toroidal 271 (269-272)
empty graph 5
end-block 57
end-vertex 6
equal walks 33
Erdős-Gallai Theorem 27 (26-27)
Erdős number 200
Errera's Theorem 412 (412 -413)
Euler Identity 225 (221-235)
Generalized 273
Euler Polyhedral Formula 222
Euler tour 109
Eulerian circuit
in a digraph 151
in a graph 109
Eulerian digraph 151 (151-152)
Eulerian graph 109 (107-115)
Eulerian trail
in a digraph 151
in a graph 109
even component 407
even cycle 35
even vertex 7
exterior region (of a plane graph) 224

F

face (of a polyhedron) 221
factor 406
k- 406
factorable 419
H- 420
k- 419
factor-critical graph 418
factorization 419 (419-426)
cyclic 422
Hamiltonian 424
isomorphic 420
1-Factorization Conjecture 423, 462
Fáry's Theorem 259 (259-262)
First Theorem of Digraph Theory 48
First Theorem of Graph Theory 6
Five Color Theorem 376
flow
k- 473
along an arc 173
maximum 174
in a network 173 (171-194)
nowhere-zero k- 473
in an oriented graph 473
flower snark 471 (470-471)
forbidden minor 307
forest 66
forward arc 177
Four Color Conjecture 361, 368
Four Color Problem 359 (359-375)
Four Color Theorem 376
F-free 130
Frucht's Theorem 210 (209-211)

G

Gallai-Roy-Vitaver Theorem 337
Gallai's Theorem 415 (414-416)
Galvin's Theorem 487 (485-487)
Generalized Euler Identity 273
generated by (elements of a group) 206
generating set 207
generators (of a group) 206
genus
maximum 297 (294-307)
of a graph 271 (269-281)
of a surface 271
geodesic 36
girth 35
k-gon 372
graceful
graph 431 (431-436)
labeling 431
Graceful Tree Conjecture 433
graph 3
(r, g)- 499
graph labeling 431
Graph Minor Theorem 307 (307-309)
graphical sequence 22
Greedy Coloring Algorithm 329 (328-331)
Greenwood-Gleason graph 518 (518-519)
Grinberg graph 266
Grinberg's Theorem 263 (263-266)
Grötzsch graph 340
group of a graph 201

H
Hadwiger's Conjecture 387
Hajós' Conjecture 385 (385-387)
Hall's condition 404
Hall's Marriage Theorem 406
Hall's Theorem 404 (404-406)
Hamiltonian cycle
    in a digraph 152
    in a graph 117
Hamiltonian digraph 152 (152-154)
Hamiltonian factorable graph 424 (424-426)
Hamiltonian factorization 424
Hamiltonian graph 118 (115-133)
    k-ordered 135
Hamiltonian path
    in a digraph 166
    in a graph 117
Hamiltonian tournament (166-169)
Hamiltonian-connected digraph 170
Hamiltonian-connected graph 131 (131-132)
Handshaking Lemma 6
Havel-Hakimi Theorem 24 (23-26)
Heawood graph 503
Heawood map 369
Heawood Map-Coloring Problem 398 (395-400)
Heawood Map Coloring Theorem 399
Herschel graph 266
hexahedron 221
Hoffman-Singleton graph 505
hypercube 18

I
icosahedron 221
Icosian Game (116-118)
incident (vertex and edge) 5
indegree (of a vertex) 47
independence number 123
    edge 401
    vertex 123
independent dominating set 443 (443-444)
independent domination number 443
independent set of edges 401
independent set of vertices 123
    maximal 322, 419
induced subgraph 10
    edge- 10
in-neighborhood 47
intermediate vertex 172
internally disjoint paths 57, 98
intersection graph 351
interval graph 344 (344-345)
irregular graph 27 (27-28)
    nearly 27
isolated vertex 6
isomorphic digraphs 48
isomorphic factorization 420
isomorphic graphs 7 (7-10)
isomorphism
    of two digraphs 48
    of two graphs 7
iterated line graph 143

J
join (of two graphs) 17
joining two vertices
    in a digraph 47
    in a graph 4

K
Kempe chains 365 (364-369)
king (in a tournament) 165
Kirkman triple system 430
Kirkman's Schoolgirl Problem 429 (429-430)
Kneser graph 21
König's Theorem 456
König-Egerváry Theorem 416
König-Rado Theorem 416
Königsberg Bridge Problem 107 (107-109)
Kruskal's Algorithm 81 (81-83)
Kuratowski's Theorem 237 (236-240)

L
labeling
complementary 435
graceful 431
graph 431
Latin square 421
leaf 6
length of a walk
in a digraph 50
in a graph 31
line 3
line digraph 154
line graph 140 (140-144)
iterated 143
linear-time algorithm 85
link 3
list chromatic index 484
list chromatic number 353
list-colorable
k- 353
$\mathfrak{L}$- 353
list coloring 353 (353-357)
List Coloring Conjecture 484 (484-485)
list edge colorable
k- 484
$\mathfrak{L}$- 484
list edge colorings (484-491)
loop 45
directed 51

M
major determinant 78
Marriage Problem 406
Marriage Theorem
Hall's 406
matched 404
matches 404
matching 401 (401-419)
maximum 401
optimal (in A) 485
perfect 402
matching graph 419
matching number 401
Matrix-Tree Theorem 78 (77-80)
Max-Flow Min-Cut Theorem 179 (179-185)
maximal k-degenerate graph 327
maximal independent set (of vertices) 322, 419
maximal outerplanar graph 246
maximal planar graph 227 (227-232)
nearly 228
maximum degree 6
maximum edge
c- 486
U- 486
W- 486
maximum flow 174
maximum genus 297 (294-307)
maximum matching 401
maximum multiplicity 453
MeGee graph 503
median (of a graph) 44
median vertex 44
Menger theorems for digraphs (185-190)
Menger's Theorem 98 (97-100)
Arc Form 186
Directed Vertex Form 187
metric 37
metric space 37
Meyniel's Theorem 152
minimal dominating set 440 (440-443)
minimally nonembeddable 308
minimum cut (in a network) 176
minimum degree 6
minimum dominating set 439
minimum edge-cut 90
minimum separating set 98
minimum spanning tree 81
Minimum Spanning Tree Problem 81 (80-84)
minimum vertex-cut 89
minor 241(240-243)
forbidden 307
proper 242
minor-closed 307
Mirzakhani graph 381 (380-383)
Möbius band 278
Möbius strip 278

monochromatic graph 509
Moore bound 499
Moore graph 505, 506 (504-507)
multidigraph 51
multigraph 45
multiplicity
    maximum 453

N
nearly irregular graph 27
nearly maximal planar graph 228
neighbor 4
neighborhood 4
    closed 4
    in- 47
    open 4
    out- 47
net flow
    into a set of vertices 175
    into a vertex 173
    out of a set of vertices 175
    out of a vertex 173
network 171
node 3
nonattacking queens 439
nonempty graph 5
non-isomorphic graphs 7
nonseparable graph 56 (55-60)
nontrivial graph 5
Nordhaus-Gaddum inequalities 334
Nordhaus-Gaddum Theorem 334 (333-337)
nowhere-zero flow (473-484)
nowhere-zero k-flow
    in a graph 475
    in an oriented graph 473
NP 134
NP-complete 134

O
octahedron 221
odd component 407
odd cycle 35
odd vertex 7
open neighborhood 4
open walk 31
operations on graphs (17-18)
optimal matching (in A) 485
orbit 203
order
    of a digraph 47
    of a graph 5
k-ordered Hamiltonian 135
Ore's Theorem 119
orientable surface 271
orientation of a graph 49
oriented graph 49
outdegree of a vertex 47
outerplanar graph 246 (246-247)
    maximal 246
out-neighborhood 47
Overfull Conjecture 461 (461-462)
overfull graph 458
overfull subgraph 459 (459-462)

P
P 134
P=**NP**-Problem 134
panconnected graph 132
pancyclic digraph 168
    vertex- 168
pancyclic graph 133
parallel arcs 51
parallel edges 45
k-partite graph 16
    complete 16
partite sets 14, 16
path
    in a digraph 50
    in a graph 5, 31, 35
pendant edge 6
perfect graph 342 (340-352)
Perfect Graph Conjecture 344
    Strong 350
Perfect Graph Theorem 344
    Strong 350
perfect matching 402
peripheral vertex 37
periphery 39 (39-41)

Petersen graph 13
truncated 204 (204-205)
Petersen's Theorem 411 (410-412)
planar dual 368
planar embedding 223
planar graph 223 (223-232)
maximal 227
nearly maximal 228
plane graph 223
Platonic solid 221
point (vertex) 3
polyhedron 221
regular 232
polynomial-time algorithm 85
kth power of a graph 138
Prim's Algorithm 83 (83-84)
projective plane 279
proper edge coloring 447
proper minor 242
proper 2-set k-coloring 339
proper subgraph 10
proper vertex coloring 311 (311-357)
Prüfer code 70 (70-74)
pseudodigraph 51
pseudograph 45

Q

quadratic-time algorithm 85
queen's graph 439

R

radius
of a graph 37
of a strong digraph 50
rainbow graph 521
rainbow Ramsey number 521
Ramsey number 509
classical 511
of a graph 509
of $k \geq 3$ graphs 518
rainbow 521
Ramsey Theory (508-521)
Ramsey's Theorem 508
reachable (from a vertex) 156
recognizable property 215 (214-218)
reconstructible graph 213 (213-214)
Reconstruction Conjecture 213
Reconstruction Problem 213 (212-220)
rectilinear crossing number 259
red graph 509
red-blue coloring 508
red-blue edge coloring 508
reducible configuration 372
Reed's Conjecture 335
region (of a plane graph) 224
regular digraph
of degree r 49
r- 49
regular graph 12 (12-14)
of degree r 12
r- 12
regular polyhedron 232
replication graph 348
Replication Lemma 349
Ringeisen-White Edge-Adding Lemma 290
Ringel's Conjecture 433
Ringel-Kotzig Conjecture 433
Robbins' Theorem 150 (150-151)
Rotational Embedding Scheme 286 (284-293)

S

saturated arc 173
score (of a vertex) 158
score sequence 158 (160-164)
self-centered graph 39
self-complementary graph 16
semipath 177
separating set of arcs 186
separating set of edges 102
separating set of vertices (in a digraph) 186
separating set of vertices (in a graph) 98
minimum 98
serf (in a tournament) 170
2-set chromatic number 339
2-set k-coloring

proper 339
shadow graph 351
shadow vertex 351
Shannon's Theorem 453 (453-454)
similar vertices 203
simple graph 46
sink (in a network) 171
size
of a digraph 47
of a graph 5
snark 470 (469-471)
Blanuša 470
Descartes 471
double-star 471 (470-471)
flower 471 (470-471)
Szekeres 471
solution to a deck 217
source (in a network) 171
spanning subdigraph 48
spanning subgraph 10
spanning tree 74 (69-89)
minimum 81
spine (of a caterpillar) 64
splitting tree 299
square of a graph 138
square root (of a graph) 146
stable set 123
star 14
Steiner triple system 428
straight-line embedding 259
strength (of a multigraph) 453
string
n-bit 18
strong component 159
strong digraph 50 (149-156)
Strong Perfect Graph Conjecture 350
Strong Perfect Graph Theorem 350
strongly connected digraph 50
subdigraph 48
spanning 48
subdivision (of a graph) 236
subdivision graph 137
subgraph 10 (10-12)
edge-induced 10
induced 10
overfull 459 (459-462)
proper 10
spanning 10
subgraph induced by a set of edges 10
subgraph induced by a set of vertices 10
supergraph 10
surface of genus k 271
2-switch 23
symmetric digraph 49
complete 49
symmetric property 37
system of distinct representatives 405
Szekeres snark 471

T

Tait coloring 465 (464-473)
Tait's Theorem 464 (464-468)
tetrahedron 221
Thomassen's Theorem 378 (378-380)
toroidal dual 395
toroidal embedding 271 (269-272)
torus 269
double 281 (281-282)
total chromatic number 488
k-total colorable graph 488
total coloring 487
k- 488
Total Coloring Conjecture 489 (487-491)
total deficiency 463
total distance of a vertex 44
total dominating set 445
total domination number 445
total graph 145
t-tough graph 125
toughness 129 (125-131)
tournament 49, 156 (156-171)
transitive 157 (157-160)
trail 31
transitive tournament 157 (157-160)
tree 62 (61-89)
spanning 74 (69-89)
splitting 299
triangle 5, 35

triangle inequality 37
triangle-free graph 130
triple 428
triple system
    Kirkman 430
    Steiner 428
trivial digraph 48
trivial graph 5
trivial walk 31
truncated Coxeter graph 204 (204-205)
truncated Petersen graph 204 (204-205)
Turán Brick-Factory Problem 254 (254-255)
Turán graph 494
Turán's Theorem 494 (493-495)
Tutte graph 265 (264-266)
Tutte-Berge Formula 409 (409-410)
Tutte-Coxeter graph 503
Tutte's Theorem 407 (407-409)

U
unavoidable configuration 371
underlying digraph of a network 171
underlying graph of a digraph 49
underlying graph of a multigraph 45
unicyclic graph 75
union (of two graphs) 17
unmatched vertex 402
    M- 402
unsaturated arc 173
f-unsaturated semipath 177
upper embeddable graph 298
upper embeddable on a surface 298

V
value of a flow 174
vertex 3
    even 7
    odd 7
vertex coloring 311 (311-357)
    proper 311 (311-357)
vertex cover 414
vertex covering number 414
vertex independence number 123
vertex set 4
vertex-connectivity 89
vertex-cut 89
    minimum 89
vertex-pancyclic digraph 168
vertex-transitive graph 204 (204-205)
vertices
    of a digraph 47
    of a graph 3
Vizing's Conjecture 443
Vizing's Planar Graph Conjecture 469
Vizing's Theorem 449 (449-453)

W
Wagner's Theorem 243 (243-245)
walk in a digraph 50
walk(s) in a graph 31
    closed 31
    equal 33
    length of 31
    open 31
    trivial 31
weakly connected digraph 50
weight
    of an edge 80
    of a subgraph 80
weighted graph 80
Whitney's Theorem 101 (100-102)

Z
zero flow 173

List of Symbols

Symbol	Meaning	Page
V, $V(G)$	vertex set of G	3-4
E, $E(G)$	edge set of G	3-4
$N(v)$, $N_G(v)$	(open) neighborhood of v (in G)	4
$N[v]$	closed neighborhood of v	4
K_n	complete graph of order n	5
P_n	path of order n	5
C_n	cycle of order n	5
$\deg v$, $\deg_G v$	degree of a vertex v (in G)	6
$\Delta(G)$	maximum degree of G	6
$\delta(G)$	minimum degree of G	6
$G \cong H$	graphs G and H are isomorphic	7
$G \ncong H$	graphs G and H are not isomorphic	7
$H \subseteq G$	H is a subgraph of G	10
$G[S]$	subgraph of G induced by set S of vertices	10
$G[X]$	subgraph of G induced by set X of edges	10
$G - v$	subgraph of G obtained by deleting vertex v	11
$G - e$	subgraph of G obtained by deleting edge e	11
$G - U$	subgraph of G obtained by deleting U	11
$G - X$	subgraph of G obtained by deleting X	11
$G + uv$	graph obtained by adding uv to G	11
$[X, Y]$	set of edges joining X and Y	14
$K_{s,t}$	complete bipartite graph	14
$K_{1,t}$	star	14
$K_{1,3}$	claw	14
$K_{n_1,n_2,\cdots,n_k}$	complete k-partite graph	16
$\overline{G}$	complement of G	16
$G_1 + G_2$	union of G_1 and G_2	17
$G = kH$	G is the union of k copies of H	17
$G_1 \vee G_2$	join of G_1 and G_2	17
$G_1 \square G_2$	Cartesian product of G_1 and G_2	18
Q_n	n-cube	18
$A(G)$	adjacency matrix of G	32

$g(G)$	girth of G	35
$k(G)$	number of components of G	35
$d(u,v)$, $d_G(u,v)$	distance between u and v (in G)	36
$e(v)$	eccentricity of v	37
$\mathrm{diam}(G)$	diameter of G	37
$\mathrm{rad}(G)$	radius of G	37
$\mathrm{Cen}(G)$	center of G	38
$\mathrm{Per}(G)$	periphery of G	39
$td(v)$	total distance of v	44
$\mathrm{Med}(G)$	median of G	44
$d(F,H)$	distance between subgraphs F and H	44
$d(G,H)$	distance from graph G to graph H	45
$D(u,v)$	detour distance between u and v	45
V, $V(D)$	vertex set of a digraph D	47
E, $E(D)$	arc set of a digraph D	47
$\mathrm{od}\, v$	outdegree of v	47
$\mathrm{id}\, v$	indegree of v	47
$N^+(v)$	out-neighborhood of v in D	47
$N^-(v)$	in-neighborhood of v in D	47
$\deg v$	degree of v in D	47
$D_1 \cong D_2$	digraphs D_1 and D_2 are isomorphic	48
$D_1 \subseteq D$	D_1 is a subdigraph of D	48
K_n^*	complete symmetric digraph of order n	49
$\vec{d}(u,v)$	(directed) distance from u to v in D	50
$e(u)$	eccentricity of u in D	50
$\mathrm{rad}(D)$	radius of D	50
$\mathrm{diam}(D)$	diameter of D	50
$A(D)$	adjacency matrix of D	53
$\mathrm{Cen}(D)$	center of D	53
$D(G)$	degree matrix of G	78
$w(e)$	weight of edge e	80
$w(H)$	weight of subgraph H	80
$\kappa(G)$	connectivity of G	89
$\lambda(G)$	edge-connectivity of G	90
$\kappa_k(G)$	k-connectivity of G	96
$c(G)$	connection number of G	97
$CL(G)$	closure of G	121
$\alpha(G)$	(vertex) independence number of G	123
$t(G)$	toughness of G	129
$S(G)$	subdivision graph of G	137
G^k	kth power of G	138
G^2, G^3	square, cube of G	138
$L(G)$	line graph of G	140
$L^k(G)$	kth iterated line graph of G	143
$T(G)$	total graph of G	145

$\vec{D}$	converse of D	150
$L(D)$	line digraph of D	154
$\tilde{T}$	transitive tournament associated with T	159
$c(a)$	capacity of arc a	172
$f(a)$	flow along arc a	173
$f^+(x) - f^-(x)$	net flow out of vertex x	173
$f^-(x) - f^+(x)$	net flow into vertex x	173
$\mathrm{val}(f)$	value of flow f	174
$\mathrm{cap}(K)$	capacity of cut K	174
$\mathrm{Aut}(G)$	automorphism group of graph G	201
$\mathrm{Aut}(D)$	automorphism group of digraph D	205
$D_\Delta(\Gamma)$	Cayley color graph of Γ	207
$\mathrm{cr}(G)$	crossing number of G	250
$\overline{\mathrm{cr}}(G)$	rectilinear crossing number of G	259
S_k	surface of genus k	271
$\gamma(G)$	genus of G	271
$\gamma_M(G)$	maximum genus of G	297
$\mathcal{B}(G)$	Betti number of G	298
$\xi_o(H)$	number of components of H of odd size	303
$\xi(G)$	$\min(\xi_o(G - E(T))$ over all spanning trees	303
$\chi(G)$	chromatic number of G	312
$\omega(G)$	clique number of G	314
$\chi_b(G)$	balanced chromatic number of G	320
$\chi_2(G)$	2-set chromatic number of G	339
$R_v(G)$	replication graph of G	348
$Shad(G)$	shadow graph of G	351
$L(v)$	color list of vertex v	353
$\chi_\ell(G)$	list chromatic number of G	353
$P(G, \lambda)$	chromatic polynomial of G	388
$\chi(S)$	chromatic number of a surface S	396
$\alpha'(G)$	edge independence number of G	401
$N(S)$	neighborhood of S	404
$k_o(G)$	number of odd components of G	407
$\beta(G)$	vertex covering number of G	414
$\beta'(G)$	edge covering number of G	414
$\mathrm{def}(S)$	deficiency of a set S	417
$M(G)$	matching graph of G	419
$\gamma(G)$	domination number of G	439
$i(G)$	independent domination number of G	443
$\gamma_t(G)$	total domination number of G	445
$\chi'(G)$	chromatic index of G	448
$\mu(G)$	maximum multiplicity of multigraph G	453
$L(e)$	color list of edge e	484
$\chi'_\ell(G)$	list chromatic index of G	484
$\chi''(G)$	total chromatic number of G	488

$T_{n,k}$	Turan graph	494
$t_{n,k}$	size of $T_{n,k}$	494
(r,g)-graph	r-regular graph of girth g	499
$M(r,g)$	Moore bound	499
(r,g)-cage	(r,g)-graph of minimum order	499
g-cage	$(3,g)$-cage	499
$R(F,H)$	Ramsey number of F and H	509
$R(s,t)$	classical Ramsey number of K_s and K_t	511
$R(G_1,G_2,\ldots,G_k)$	Ramsey number of $G_1,G_2,\ldots,G_k$	518
$RR(F)$	rainbow Ramsey number of F	521

Mathematics

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CRC Press
Taylor & Francis Group
an informa business
www.crcpress.com

6000 Broken Sound Parkway, NW
Suite 300, Boca Raton, FL 33487
270 Madison Avenue
New York, NY 10016
2 Park Square, Milton Park
Abingdon, Oxon OX14 4RN, UK

K11263
ISBN: 978-1-4398-2627-0
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9 781439 826270