Riphah Society For Excellence

FCAS Digital Study Circle Chapter

SOLUTION MANUAL OF COMPLEX VARIABLE

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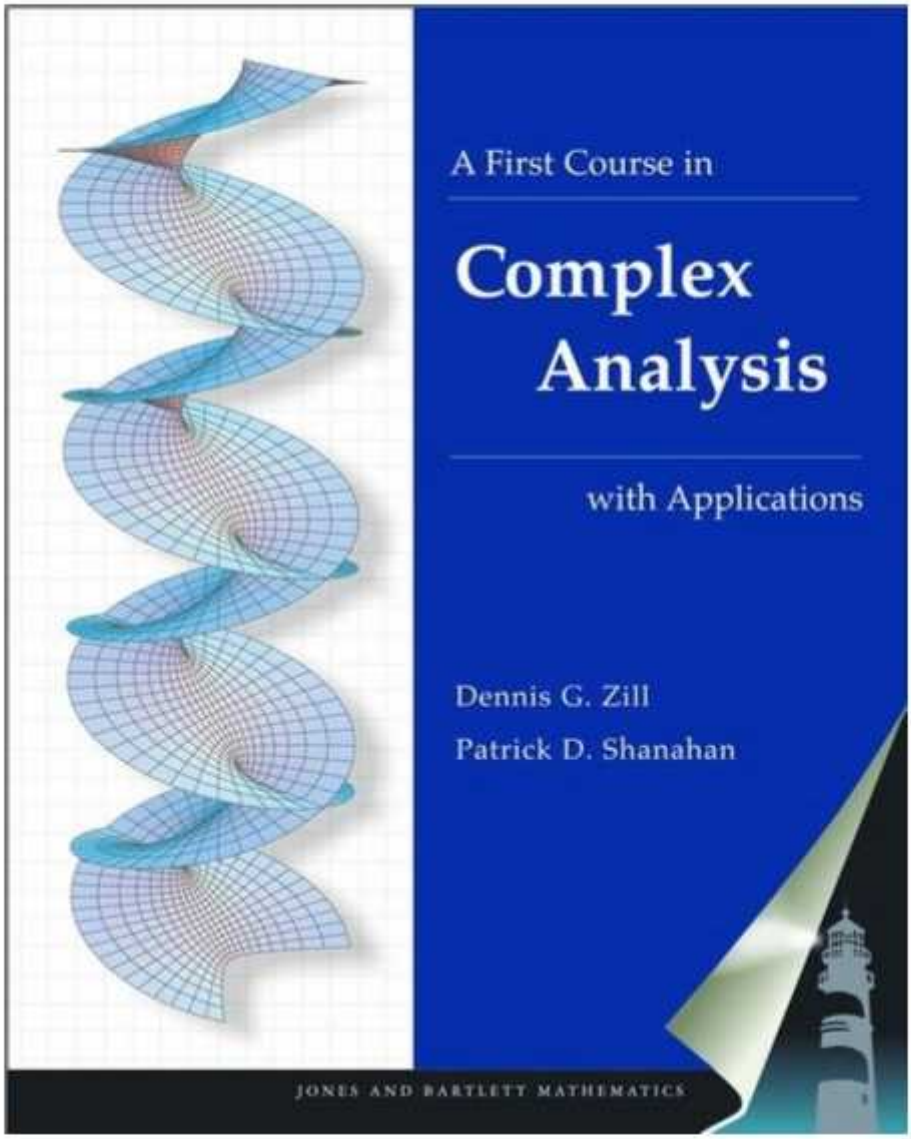
A First Course in
Complex Analysis
with Applications
Dennis G. Zill
Patrick D. Shanahan
JONES AND BARTLETT MATHEMATICS

Aaron Morrisett

1.3) $(5-9i)+(2-4i) = 7-13i$

1.6) $i(4-i) + 4i(1+2i) = 4i + 1 + 4i + 8\overset{-1}{i^2} = -7+8i$

1.9) $3i + \frac{1}{2-i} = 3i + \frac{2+i}{2^2+1} = 3i + \frac{2}{5} + \frac{1}{5}i = \frac{2}{5} + \frac{16}{5}i$

1.12) $\frac{10-5i}{6+2i} = \frac{(10-5i)(6-2i)}{6^2+2^2} = \frac{1}{40}[60-20i-30i+10i^2] = \frac{1}{40}[50-50i] = \frac{5}{4} - \frac{5}{4}i$

1.15) $\frac{(5-4i)-(3+7i)}{(4+2i)+(2-3i)} = \frac{2-11i}{6-i} = \frac{1}{37}[(2-11i)(6+i)] = \frac{1}{37}[(12-66i+2i+11)]$

$= \frac{23}{37} - \frac{64}{37}i$

1.18) $(1+i)^2(1-i)^3 = (1+i)(1+i)(1-i)(1-i)(1-i) = 2i(1-i)(1-i)(1-i)$

$= (2+2i)(1-i)(1-i) = (4)(1-i) = 4-4i$

1.21) using Pascal's triangle

$(2+3i)^2$ Using the second row of the Pascal's triangle. 1 2 1

$(1)2^2 + (2)(2)(3i) + (3i)^2 = 4-9 + 12i = -5+12i$

1.24) $(1+i)^8$ Using Pascels Triangle

1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1 Eighth Row

$(1)^8 + (8)(1)^7(i)^1 + (28)(1)^6(i)^2 + (56)(1)^5(i)^3 + (70)(1)^4(i)^4 + (56)(1)^3(i)^5$

$+(28)(1)^2(i)^6 + (8)(1)^1(i)^7 + 1(i)^8 = 1 + 8i - 28 - 56i + 70 + 56i - 28 - 8i + 1$

$= 16$

1.27) $Re(\frac{1}{z})$ $\frac{1}{x+yi} = \frac{x-yi}{x^2+y^2}$ $Re(\frac{1}{z}) = \frac{x}{x^2+y^2}$

1.30) $Im(\bar{z}^2 + z^2) = (x-yi)(x-yi) + (x+yi)(x+yi)$

$= x^2+y^2-2xyi + x^2-y^2 + 2xyi$ Imaginary Part $= 0$

1.33) $Im((1+i)z) \Rightarrow x+yi+xi-y$ Imaginary Part $x+y$

1.36) $z^4 = -4$; $z_1 = 1+i$, $z_2 = -1+i$

$(1+i)(-1+i)z_3z_4 = -4$ $z_3 = \sqrt{2}$ $z_4 = -\sqrt{2}$

or $z_3 = -\sqrt{2}$ $z_4 = \sqrt{2}$

1.39) $z^2 = i$ $(a+bi) = z$

Definition 1.1.2 says $z_1 = z_2$ when $a_1 = a_2$ and $b_1 = b_2$

$a+bi = \sqrt{i}$

answer

$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

42) $\frac{z}{1+z} = 3+4i$

$\frac{a+bi}{(1+a)+bi} = 3+4i$

$\frac{1}{(1+a)^2+b^2}[(a+bi)(1+a-bi)] = 3+4i$

I am not sure about 1.39 or 1.42

1.45 if $z=\bar{z}$ then $Im(z)=0$

if $(z)^2=(\bar{z})^2$ then $Im(z)=0$ or $Re(z)$ can be zero.

1.48) $\frac{5+6i}{1+i} = a+bi$

$5+6i = (a+bi)(1+i) = a+ai+bi-b$

$5+6i = a-b+i(a+b)$

$a-b=5 \Rightarrow a=5+b \qquad \Rightarrow a = 11/2$

$a+b=6 \Rightarrow 5+b+b=6 \Rightarrow 2b=1 \Rightarrow b=1/2$

1.2

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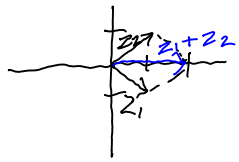
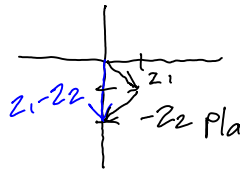
3) $z_1 = 1 - i \quad z_2 = 1 + i$

a) $z_1 + z_2$



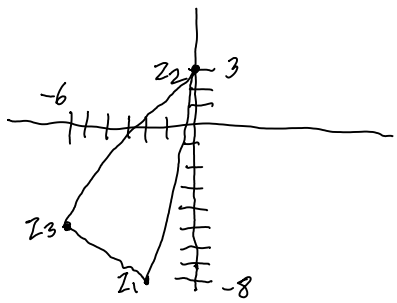
b) $z_1 - z_2$

$-2i$



$-z_2$ placed at end of z_1

6) a) Plot the Points $z_1 = -2 - 8i$, $z_2 = 3i$, $z_3 = -6 - 5i$



b) the side $z_2 z_1 = z_2 - z_3$
the side $z_2 z_3 = z_2 - z_1$
the side $z_3 z_1 = z_3 - z_1$

9) find the modulus of $(1-i)^2$

$(1-i)(1-i) = 1 - 2i - 1 = -2i$

Modulus $= \sqrt{(-2)^2} = 2$

12) find the modulus of $\frac{1-2i}{1+i} + \frac{2-i}{1-i}$

$\frac{1}{2}(1-2i)(1-i) + \frac{1}{2}(2-i)(1+i) = \frac{1}{2}[1 - 3i - \cancel{2} + \cancel{2} + i + 1]$

$= \frac{1}{2}[2 + 2i] = 1 + i \qquad$ Modulus $= \sqrt{1^2 + 1^2} = \sqrt{2}$

15) $z_1 = 10 + 8i \qquad z_2 = 11 - 6i$

a) which is closer to the origin?

z_1 is $\sqrt{10^2 + 8^2} = \sqrt{164} \approx 12.806$ units from the origin

z_2 is $\sqrt{11^2 + 6^2} \approx 13.6$ units from the origin

thus $10 + 8i$ is closer to the origin

b) which is closer to the point $1 + i$? ☆ using the distance formula.

$d_1 = \sqrt{(1-10)^2 + (1-8)^2} = \sqrt{(-9)^2 + (-7)^2} = 11.401$

$d_2 = \sqrt{(1-11)^2 + (1+6)^2} = \sqrt{(-10)^2 + (7)^2} = 12.206$

$10 + 8i$ is closer to the point $1 + i$

18) $[\mathrm{Im}(iz)]^2 = 2$

$[\mathrm{Im}(xi - y)]^2 = 2$

$x^2 = 2 \Rightarrow x = \sqrt{2}$ the line in the Imaginary Axis where the Real value is $\sqrt{2}$ satisfies this equation.

21) $\mathrm{Im}(z^2) = 2$

$\mathrm{Im}(x^2 - y^2 + 2xyi) = 2 \Rightarrow 2xy = 2 \Rightarrow xy = 1$

the hyperbola $xy = 1$

24) $|z - i| = 2|z - 1|$

$|x + yi - i| = 2|x + yi - 1|$

$\sqrt{x^2 + (y-1)^2} = 2\sqrt{(x-1)^2 + y^2}$

$x^2 + (y-1)^2 = 4[(x-1)^2 + y^2]$

$x^2 + y^2 - 2y + 1 = 4[x^2 - 2x + 1 + y^2]$

$x^2 + y^2 - 2y + 1 = 4x^2 - 8x + 4 + 4y^2$

$-3y^2 - 2y - 3x^2 + 8x = 3$

$-3(y + \frac{1}{3})^2 - 3(x - 4/3) = 3 - 3(\frac{1}{9} + \frac{16}{9})$

$(y + \frac{1}{3})^2 + (x - 4/3)^2 = -1 + \frac{1}{9} + \frac{16}{9} = 8/9$

$y^2 + \frac{2}{3}y + \frac{1}{9} = (y + \frac{1}{3})(y + \frac{1}{3})$

$x^2 - \frac{8}{3}x + \frac{16}{9} = (x - 4/3)(x - 4/3)$

a circle satisfies this equation with center $(4/3, -1/3)$ and radius $8/9$

27) establish the given simultaneous inequality.
if $|z| = 2$, then $8 \le |z + 6 + 8i| \le 12$

$8 \le |z| + |6 + 8i| \le 12$

$8 \le 2 + |6 + 8i| \le 12$

$8 \le 2 + \sqrt{6^2 + 8^2} \le 12$

at $|z| = 2$

$8 \le 12 \le 12$

30) find the upper and lower bound for the reciprocal of the Modulus of $z^4 - 5z^2 + 6$ if $|z| = 2$

$z^4 - 5z^2 + 6 = (z^2 - 3)(z^2 - 2)$

$N \le \frac{1}{|(z^2-3)(z^2-2)|} \le M$

Using $|z_1 z_2| = |z_1||z_2|$ and $||z_1| - |z_2|| \le |z_1 - z_2|$

$|(z^2 - 3)(z^2 - 2)| = |z^2 - 3| \cdot |z^2 - 2| \ge ||z^2| - 3| \cdot ||z^2| - 2|$

$\ge |4 - 3||4 - 2|$

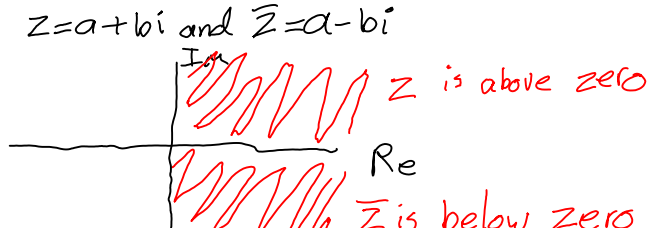
≥ 2

$M = \frac{1}{2}$

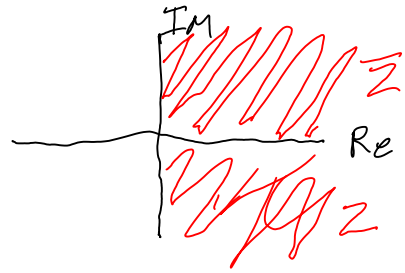
I dont Know how to find the lower bounds.

33) a) draw the Points $z = a + bi$ and $\bar{z} = a - bi$

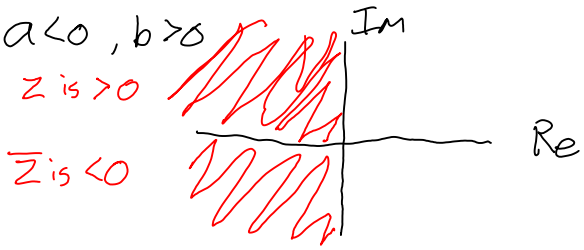
if $a > 0$ and $b > 0$



$a>0,\ b<0$



$a<0,\ b>0$



$a<0,\ b<0$

b) they are points Reflected about the Real Axis

c) they are points Reflected about the Imaginary Axis

36) $0+0i$ is the only complex number with modulus 0

39) Euler's Formula : $e^{i\theta} = \cos\theta + i\sin\theta$

thus the set of points follow $e^{i\theta}$ with θ in Radians.

42) Using complex notation find the Parametric equations of the line segment between any two complex numbers z_1 and z_2.

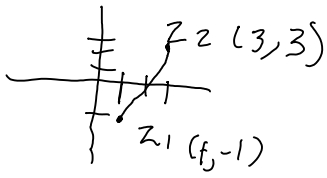
$z_1 = a_1 + b_1 i \qquad z_2 = a_2 + b_2 i$

the Line is defined by $z_2 - z_1$

$a(t) = a_2 - a_1$

$b(t) = b_2 - b_1$

$z(t) = a(t) + b(t)i$



45) $w = \bar{z}/z$

$|w| = 1$ always because the Modulus of a complex number and the Modulus of it's conjugate are always the same.

48) for any two complex numbers z_1 and z_2 show that

$$|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$$

$$|z_1+z_2||z_1+z_2|+|z_1-z_2||z_1-z_2|$$

$$|z_1^2+2z_1z_2+z_2^2|+|z_1^2-2z_1z_2+z_2^2|$$

$$2(|z_1|^2+|z_2|^2)=2(|z_1|^2+|z_2|^2)$$

Aaron Morrisett
Math 459
Homework # 2

3) $r=3 \qquad z = 3[\cos -\frac{\pi}{2} + i\sin -\frac{\pi}{2}]$

or $z = 3[\cos \frac{3\pi}{2} + i\sin \frac{3\pi}{2}]$

6) $z = 5-5i \qquad r = \sqrt{50}$

$\theta = \tan^{-1}(\frac{-5}{5}) = \tan^{-1}(-1) = -\pi/4$

$z = \sqrt{50}[\cos -\pi/4 + i\sin -\pi/4] = \sqrt{50}[\cos \frac{7\pi}{4} + i\sin \frac{7\pi}{4}]$

9) $z = \frac{3}{-1+i} = \frac{1}{2}[3(-1-i)] = \frac{1}{2}(-3-3i) = -\frac{3}{2} - \frac{3}{2}i$

$r = \sqrt{9/4 + 9/4} = \frac{3}{2}\sqrt{2} \qquad \theta = \tan^{-1}(1) = \pi/4 - \pi = \frac{-3\pi}{4}$

$z = \frac{3}{2}\sqrt{2}[\cos \frac{-3\pi}{4} + i\sin \frac{-3\pi}{4}] = \frac{3}{2}\sqrt{2}[\cos \frac{5\pi}{4} + i\sin \frac{5\pi}{4}]$

12) $z = -12-5i \qquad r = \sqrt{144+25} = 13 \qquad \tan^{-1}(\frac{5}{12}) = .39479 - \pi = -2.7468$ rads

$z = 13[\cos -2.748 + i\sin -2.748]$

$z = 13[\cos 3.536 + i\sin 3.536]$

15) $z = 5(\cos 7\pi/6 + i\sin 7\pi/6) \qquad a = 5\cos 7\pi/6 = -\frac{5\sqrt{3}}{2}$

$b = 5\sin 7\pi/6 = -5/2$

$z = -\frac{5\sqrt{2}}{3} - \frac{5}{2}i$

18) $z = 10(\cos \frac{\pi}{5} + i\sin \frac{\pi}{5}) \qquad a = 10\cos \pi/5 = \frac{5\sqrt{5}+5}{2} = 8.09 \qquad b = 10\sin \pi/5 = 5.877$

$8.09 + 5.877i$

21) $(3-3i)(5+5\sqrt{3}i) = 15 + 15\sqrt{3}i - 15i + 15\sqrt{3}$

$a = 15 + 15\sqrt{3} \qquad b = 15\sqrt{3} - 15$

$r = \sqrt{a^2+b^2} = 30\sqrt{2} \qquad \theta = \tan^{-1}(\frac{b}{a}) = \pi/12$

$z = 30\sqrt{2}[\cos \pi/12 + i\sin \pi/12]$

$a = 30\sqrt{2}\cos \pi/12 = 15\sqrt{3} + 15$

$b = 30\sqrt{2}\sin \pi/12 = 15\sqrt{3} - 15$

$z = 15\sqrt{3} + 15 + i(15\sqrt{3} - 15)$

24) $z = \frac{\sqrt{2}+\sqrt{6}i}{-1+\sqrt{3}i} = \frac{1}{4}[(\sqrt{2}+\sqrt{6}i)(-1-\sqrt{3}i)] = \frac{1}{4}[-\sqrt{2} - \sqrt{6}i - \sqrt{6}i + \sqrt{18}]$

$= \frac{1}{4}[\sqrt{18} - \sqrt{2} - 2\sqrt{6}i] \qquad a = \frac{1}{4}(\sqrt{18} - \sqrt{2}) \qquad b = -\frac{\sqrt{6}}{2}$

$r = \sqrt{(\frac{1}{4}(\sqrt{18}-\sqrt{2}))^2 + 6} = \sqrt{2} \qquad \tan^{-1}(\frac{b}{a}) = -\pi/3$

$z = \sqrt{2}[\cos -\pi/3 + i\sin -\pi/3]$

$z = \frac{1}{4}(\sqrt{18} - \sqrt{2}) - \frac{\sqrt{6}}{2}i$

27) $(\frac{1}{2} + \frac{1}{2}i)^{10} \qquad \theta = \tan^{-1}(1) = \pi/4 \qquad r = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{2}}{2}$

$z = (\frac{\sqrt{2}}{2})^{10}[\cos 10(\frac{\pi}{4}) + i\sin 10(\frac{\pi}{4})] = \frac{1}{32}[\cos \frac{\pi}{2} + i\sin \frac{\pi}{2}] = \frac{1}{32}i$

30) $[\cos \frac{2\pi}{3} + i\sin \frac{2\pi}{3}]^6 = 3[\cos \frac{12\pi}{3} + i\sin \frac{12\pi}{3}] = 3[\cos 4\pi + i\sin 4\pi]$

30) $\left[\sqrt{3}\left(\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}\right)\right]^6 = 3^3\left[\cos\frac{12\pi}{9}+i\sin\frac{12\pi}{9}\right] = 27\left[\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right]$

$= -\frac{27}{2} - \frac{27\sqrt{3}}{2}i$

33) $n=2$ $\cos 2\theta$ and $\sin 2\theta$

Using de Moivre's: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$z = \cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$

$= \cos^2\theta - \sin^2\theta + i\,2\sin\theta\cos\theta$

$Re(z) = \cos 2\theta = \cos^2\theta - \sin^2\theta$

$Im(z) = \sin 2\theta = 2\sin\theta\cos\theta$

36) find n for which $\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^n = 1$

$r = \sqrt{a^2+b^2} = 1$ $\theta = \tan^{-1}(-1) + \pi = 3\pi/4$

$z = 1^n\left[\cos n\frac{3\pi}{4} + i\sin n\frac{3\pi}{4}\right]$ when $n = 8/3$ $\theta \cdot n = 2\pi$

$z = 1^{8/3}\left[\cos 2\pi + i\sin 2\pi\right] = 1$

1.4

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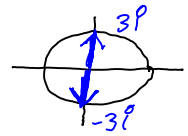
3) find the n'th root of $(-9)^{1/2}$ $\quad r=9 \quad n=2 \quad \theta = \tan^{-1}(0) = 0+\pi = \pi$

$w_0 = \sqrt{9}\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] = 3i$

$w_1 = \sqrt{9}\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right] = -3i$



6) $(-i)^{1/3}$ $\quad r=1 \quad \theta = \tan^{-1}\left(\frac{-1}{0}\right) = \tan^{-1}(\infty) = -\frac{\pi}{2}$

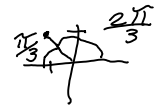
$n=3$

$w_0 = \sqrt[3]{1}\left[\cos -\frac{\pi}{6} + i\sin -\frac{\pi}{6}\right] = \frac{\sqrt{3}}{2} - \frac{i}{2}$

$w_1 = \sqrt[3]{1}\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] = i$

$w_2 = \sqrt[3]{1}\left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right] = -\frac{\sqrt{3}}{2} - \frac{i}{2}$

9) $(-1+\sqrt{3}i)^{1/2}$ $\quad r=2 \quad n=2 \quad \theta = \pi - \pi/3 = \frac{2\pi}{3}$



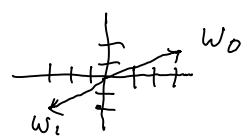
$w_0 = \sqrt{2}\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$

$w_1 = \sqrt{2}\left[\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right] = -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$

12) $(5+12i)^{1/2}$ $\quad n=2 \quad r=13 \quad \theta = \tan^{-1}\left(\frac{12}{5}\right) = 1.176$

$w_0 = \sqrt{13}\left[\cos\frac{1.176}{2} + i\sin\frac{1.176}{2}\right] = 3. + 2i$

$w_1 = \sqrt{13}\left[\cos\frac{7.459}{2} + i\sin\frac{7.459}{2}\right] = -3 - 2i$



15) Verify $(4+3i)^2 = 7+24i$

a) $(4+3i)(4+3i) = 16 + 24i - 9 = 7 + 24i$

b) find two values $(7+24i)^{1/2}$ $\quad n=2 \quad \theta = \tan^{-1}\left(\frac{24}{7}\right) = 1.287$

$r = \sqrt{7^2 + 24^2} = 25$

$w_0 = 5\left[\cos\frac{1.287}{2} + i\sin\frac{1.287}{2}\right] = 4 + 3i$

$w_1 = 5\left[\cos\frac{7.57}{2} + i\sin\frac{7.57}{2}\right] = -4 - 3i$

18) $8i = (2+2i)^2$ use to find solutions to $z^2 - 8z + 16 = 8i$

$z^2 - 8z + 16 = 8i = (2+2i)^2$

$(2+2i)$ $\quad r = \sqrt{8} \quad \theta = \tan^{-1}(1) = \pi/4$

$(z-4)^2 = \sqrt{8}\left[\cos\pi/4 + i\sin\pi/4\right]$

$z+4 = \sqrt[4]{8}\left[\cos\pi/4 + i\sin\pi/4\right]^{1/2}$ $\quad n=2$

$w_0 = \sqrt[4]{8}\left[\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right] = 5.5 + i.643$

$w_1 = \sqrt[4]{8}\left[\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right] = 2.44 - i.643$

21) $K=1$ $\quad w_n = \cos\frac{2\pi K}{n} + i\sin\frac{2\pi K}{n}$

21) $K=1$ $\quad w_n = \cos\frac{2\pi K}{n} + i\sin\frac{2\pi K}{n}$

Explain why the n nth roots of unity can be written

$$1, w_n, w_n^2, w_n^3, \cdots, w_n^{n-1}$$

the set of roots will always start with 1 because when $K=0$ $\sin\frac{2\pi K}{n}$ will be zero and $\cos(0)$ will be 1

24) for the n nth roots of unity given in 21, show that

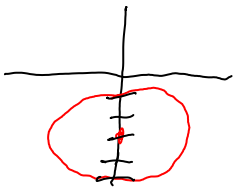
$$1 + w_n + w_n^2 + w_n^3 + \cdots + w_n^{n-1} = 0$$

I am not sure how to do 21 and 24

1.5

Wednesday, June 03, 2009
9:31 AM

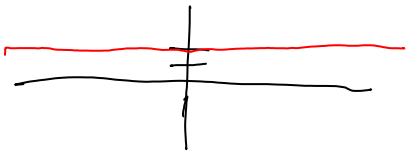
3) $|z+3i| = 2$ radius $= 2$ center $(0,-3)$



6) $\mathrm{Im}(z) = -2$

$-y = -2$

$y = 2$



9) $|\mathrm{Re}(1+iz)| = 3$

$|\mathrm{Re}(1+i(x-yi))| = 3$

$|\mathrm{Re}(1+xi+y)| = 3$

$|1-y| = 3$

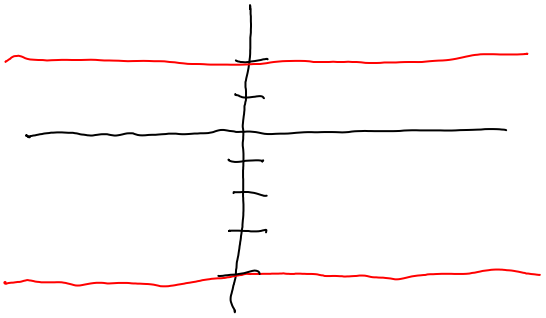
$\sqrt{(1+y)^2} = 3$

$(1+y)^2 = 9$

$1+y = \pm 3$

$y = \pm 3 - 1$

$y = -4$ or $y = 2$

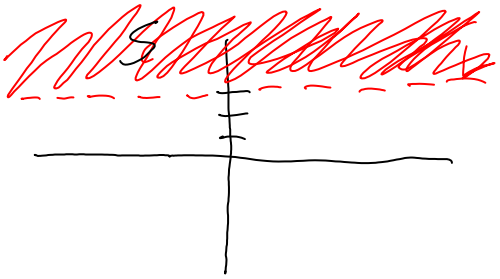


12) $\arg(z) = \pi/4$

15) $\mathrm{Im}(z) > 3$

$y > 3$

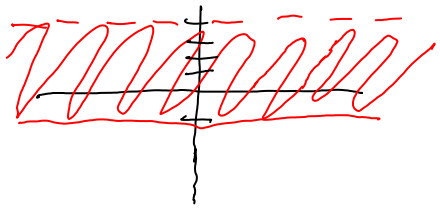
is open
not closed
its a domain
not bounded



not bounded
is connected.

(18) $-1 \le \operatorname{Im}(z) < 4$

$-1 \le y < 4$



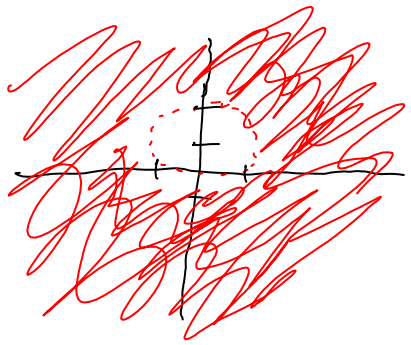
not open
not closed
not a domain
not bounded
is connected.

21) $|z - i| > 1$

$|x + yi - i| > 1$

$|x + i(y-1)| > 1$

$x^2 + (y-1)^2 > 1$



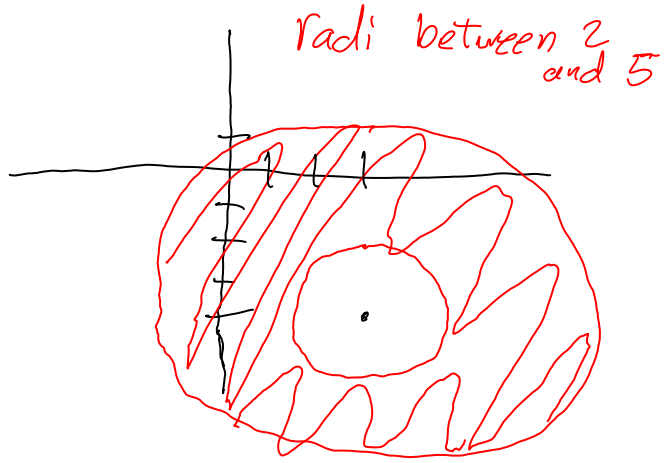
it is opened
not closed
it is domain
not bounded
it is connected

24) $2 \le |z - 3 + 4i| \le 5$

$2 \le |z - (3 - 4i)| \le 5$

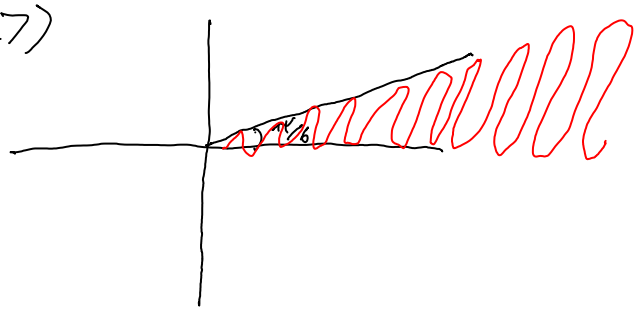
Center $(3, -4)$

radi between 2 and 5



not open
not closed
not a domain
it is bounded
it is connected.

27)



30) $\pi/2 \le \operatorname{Arg}(z) \le -3\pi/2$

2.1

Thursday, June 04, 2009
9:37 AM

Aaron Morrisett
Math 459

3) $f(z) = \ln|z| + i\,\mathrm{Arg}(z)$

a) $\ln 1 + i0 = 0$

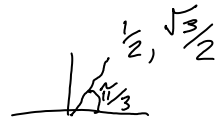
b) $\ln 4 + i\frac{\pi}{2}$

c) $\ln\sqrt{2} + i\pi/4 = \frac{1}{2}\ln 2 + i\pi/4$

6) $f(z) = e^z = e^x\cos y + ie^x\sin y = e^x e^{iy}$

a) $2 - \pi i$

$e^2[\cos -\pi + i\sin -\pi]$



b) $\frac{\pi}{3}i \quad \cos\pi/3 + i\sin\pi/3 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

c) $\log_e 2 - \frac{5\pi}{6}i$

$2\cos(-5\pi/6) + i2\sin(-5\pi/6) = \frac{-2\sqrt{3}}{2} - \frac{1}{2}i$

9) $f(z) = 6z - 5 + 9i$

$= 6x + 6yi - 5 + 9i = 6x - 5 + i(6y + 9)$

$U(x,y) = 6x - 5$

$V(x,y) = 6y + 9$

12) $f(z) = z^2 + \bar{z}^2$

$= x^2 - y^2 + 2xyi + x^2 - y^2 - 2xyi$

$= 2x^2 - 2y^2$

$U(x,y) = 2x^2 - 2y^2$

$V(x,y) = 0$

15) $f(z) = e^{2z+i} = e^i e^{2x+2yi} = e^{2x}e^{i(1+2y)}$

$\theta = \tan^{-1}(\frac{2}{1}) =$

$= e^{2x}[\cos(1+2y) + i\sin(1+2y)]$

$U(x,y) = e^{2x}\cos(2y+1)$

$V(x,y) = e^{2x}\sin(2y+1)$

18) $f(z) = |z| = r$

$U(r,\theta) = r$

$V(r,\theta) = 0$

21) $f(z) = e^z$

$= e^{r[\cos\theta + i\sin\theta]} = e^{r\cos\theta}e^{ir\sin\theta} = e^{r\cos\theta}[\cos(r\sin\theta) + i\sin(r\sin\theta)]$

$U(x,y) = e^{r\cos\theta}\cos(r\sin\theta)$

$v(x,y) = e^{r\cos\theta}\sin(r\sin\theta)$

24) $f(z) = \dfrac{3z+2i}{z^3+4z^2+z} = \dfrac{3z+2i}{z(z\pm\sqrt{3}+2)}$

All z such that $z \neq 0, \pm\sqrt{3}-2$

27) a) $\arg(z)$ defines a angle, not a complex number.

b) $\mathrm{Arg}(z)$ defines a angle, not a complex number.

c) $\cos(\arg(z)) + i\sin(\arg(z))$ is a complex number with a real and Imaginary part.

d) $z^{1/2}$ Yes, the domain and range are are subsets of $\mathbb{C}$.

e) $|z|$ Yes, the domain and range are subsets of $\mathbb{C}$.

f) $\mathrm{Re}(z)$ Yes because $\mathbb{R}$ is a subset of $\mathbb{C}$.

30) $f(z) = \dfrac{1}{z(z-1-i)(z-1+i)}$

2.2

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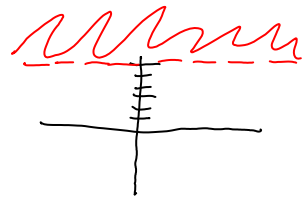
3) for $w = f(z)$

$f(z) = 3z$; S is $\operatorname{Im}(z) > 2$

Becomes $\operatorname{Im}(z) > 6$



6) $f(z) = (1-i)z$; S is $y = 2x+1$

$(1-i)(x+yi)$

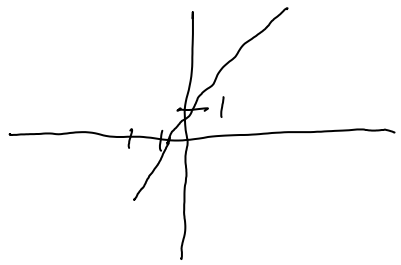
$x+y + i(y-x)$

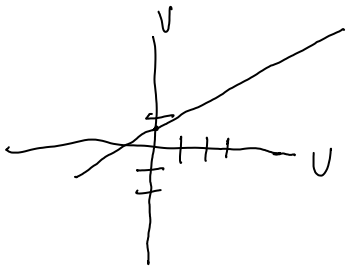
$u = x + (2x+1) = 3x+1$

$v = (2x+1) - x = x+1$

$x = v-1 \Rightarrow u = 3(v-1)+1 = 3v-3+1 = 3v-2$

$v = \frac{1}{3}u + \frac{2}{3}$





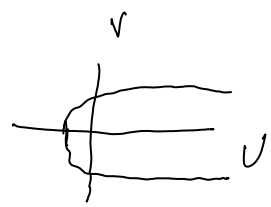
9) $w = z^2 = (x+yi)(x+yi) = x^2 - y^2 + i2xy$

$u = x^2 - y^2$ $\quad v = 2xy$

$u = x^2 - 1$ $\quad v = 2x \Rightarrow x = \frac{v}{2}$

$u = \left(\frac{v}{2}\right)^2 - 1 = \frac{v^2}{4} - 4$

$y = 1$



12) $y = 0$

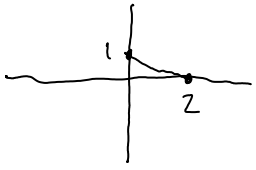
$u = x^2$ $\quad v = 2x(0) \Rightarrow \frac{v}{2(0)} = x \Rightarrow x = \infty$

the ray $\infty < u \le 0$, $v = 0$

15) $z(t) = 2(1-t) + it$, $0 \le t \le 1$; $f(z) = 3z$

a) Plot curve C and describe.

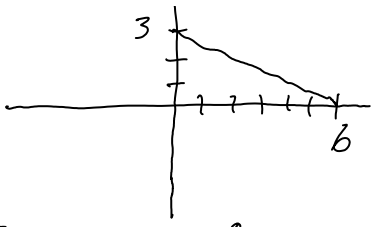
straight line from $(0,1)$ to $(2,0)$



b) $3f(z) = 6(1-t) + i3t$

c)

the y and x intercepts were multiplied by 3.



by 3.

18) $z(t) = i + e^{it}$, $0 \le t \le \pi$ $\quad f(z) = (z-i)^2$

a) using $z(t) = z_0 + re^{it}$ $\quad z_0 =$ center $\quad r =$ radius

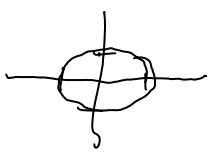
a semi circle with radius 1 and center $(0,1)$

b) $(i + e^{it} - i)^2 = e^{2it}$

$t = \frac{1}{2}s \qquad 0 \le t \le \pi$

$w(s) = e^{is}, \quad 0 \le s \le 2\pi$

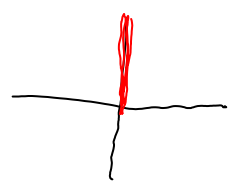
c)



full circle at the origin with a radius of one.

21) Mapping $w = f(z)$

$f(z) = z^3$; C is the positive imaginary Axis.



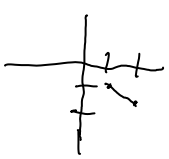
$z(t) = it, \quad 0 \le t \le \infty$

$(it)(it)(it) = (-t^2)\,it = -it^3$

becomes a Line on the negative imaginary Axis.

24) $f(z) = \frac{1}{z}$ $\quad C$ is the Line segment from $1-i$ to $2-2i$

$0 \le t \le 1$



$z = (1-t)z_1 + tz_2$

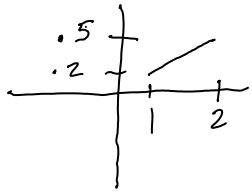
$z = (1-t)(1-i) + t(2-2i)$

$z = (1 - i - t + ti) + 2t - 2ti$

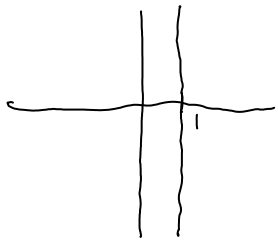
$z = (1+t) + i(-t-1)$

$f(z) = \dfrac{1}{(1+t) + i(-t-1)} \qquad 0 \le t \le 1$

$\dfrac{1}{(1+t)^2 + (-t-1)^2}\left[(1+t) - i(t-1)\right] = \dfrac{.5}{t+1} + \dfrac{.5}{t+1}\,i$



27) $X=1 \qquad w=1/2$



$f(2)=\frac{1}{2} \qquad \frac{1}{1+iy} = \frac{1}{1+y^2}[1-iy]$

a) $U=\frac{1}{1+y^2} \qquad V=\frac{-y}{1+y^2}$

b) $U(1+y^2)=1$

$U+Uy^2=1$

$Uy^2=1-U$

$y^2=\frac{1}{U}-1 \Rightarrow y=\sqrt{\frac{1}{U}-1}$

$V=\frac{-\sqrt{\frac{1}{U}-1}}{1+\frac{1}{U}-1} = \frac{-\sqrt{\frac{1}{U}-1}}{1/U}$

$V^2=U-U^2$

$V^2+U^2-U=0$

$V^2+(U-\frac{1}{2})^2-\frac{1}{4}=0$

$V^2+(U-\frac{1}{2})^2=\frac{1}{4}$

c) semi circle centered

30) $Y=mx+b$

a) Give a Parametrization $z(t)$ for the line.

$z(t)=z_0(1-t)+z_1t \quad , -\infty \le t \le \infty$

$z_0=x_0+iy_0, x_0=0 \qquad z_1=x_1+iy_1 \quad , x_1=1$

$y_0=mx_0+b=b \qquad y_1=mx_1+b=m+b$

$z_0=ib$ and $z_1=1+i(m+b)$

$z(t)=ib(1-t)+(1+i(m+b))$

$=(ib-ibt)+(t+imt+ibt)=t+ib+imt$

$x=t$ and $y=b+mt$

b) image is translated down 3 and right 2

c) image is magnified by a factor of 3.

2.3

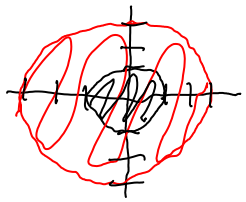
Thursday, June 04, 2009
2:07 PM

3) $f(z) = 3iz$ $\qquad \text{Arg}(3i) = \pi/2 \qquad |3i| = 3$

a) find image of $|z| \le 1$ under the mapping $f(z)$

$|z| \le 3$

b)



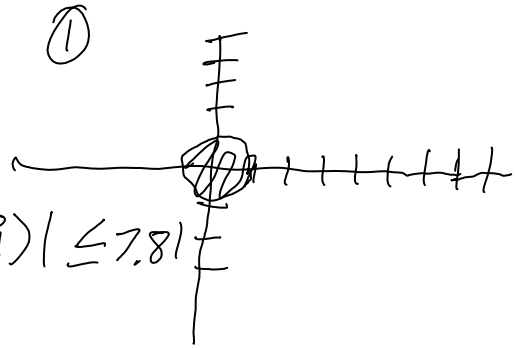
6) $f(z) = (6-5i)z + 1 - 3i$

$\text{Arg}\left(-\frac{5}{6}\right) = -.694$ radians

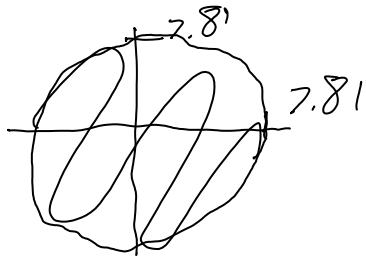
$\sqrt{6^2+5^2} = 7.810$

translations: $1-3i$

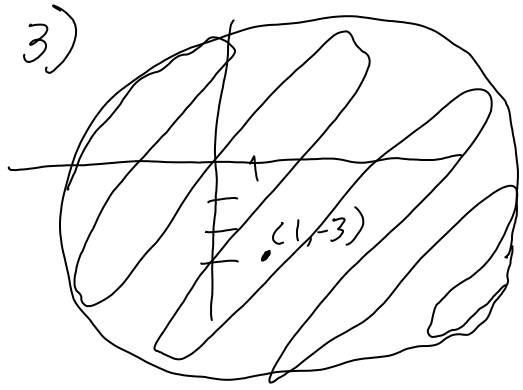
① $|w-(1-3i)| \le 7.81$



②



3)



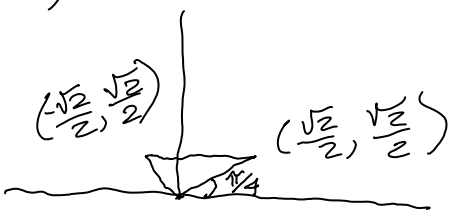
9)

$f(z) = e^{i\frac{\pi}{4}} z$

$= \left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] z$

$= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

9)

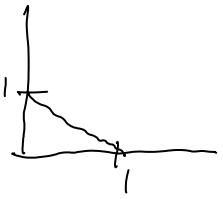


12) $f(z) = (1-i)z - 2$

$\text{Arg}(-1) = -\pi/4$

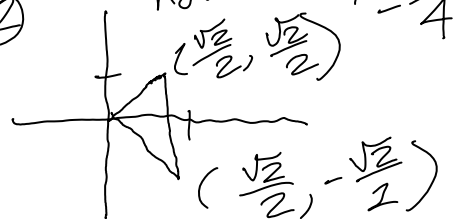
Magnification $\sqrt{2}$

①

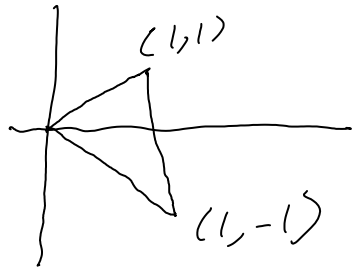


Translation -2

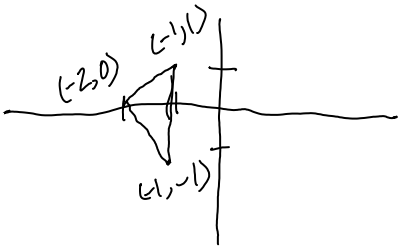
② Rotation by $-\frac{\pi}{4}$



③ Magnification by $\sqrt{2}$



④ translation



15) $f(z) = -\frac{1}{2}z + 1 - \sqrt{3}i$ $\qquad w = f(z)$

(6) says $f(z) = az + b = |a|\left(\frac{a}{|a|}z\right) + b$

$f(z) = T \circ M \circ R(z)$ $\qquad R(z) = e^{\pi i}$

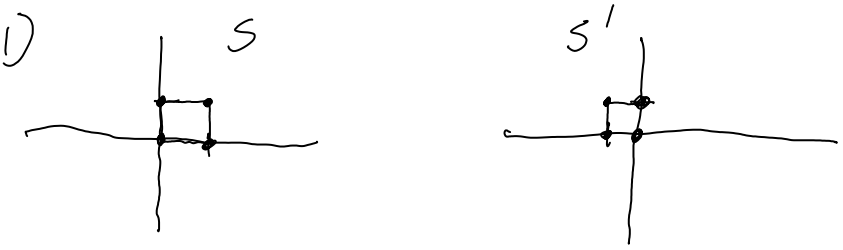
$\theta = \tan^{-1}\left(\frac{0}{-\frac{1}{2}}\right) = \pi$ $\qquad m = \frac{1}{2}z$

Magnification $= \sqrt{(\frac{1}{2})^2} = \frac{1}{2}$ $\qquad T = 1 - \sqrt{3}i$

18) $s = |2-1| = 3$ $\qquad s' = |w + i| = 5$

$f(z) = \frac{5}{3}z - 1 - 9$ $\qquad |z - z_0| = r$

21)



$f_1(z) = iz$ $\qquad f_2(z) = z - 1$

24) $z(t) = z_0 + re^{it}$ $\qquad 0 \le t \le 2\pi$

a) $w(t) = (z_0 + b) + (re^{it} + b) = z_0 + 2b + re^{it}$

the center is moved by $2b$

b) $w(t) = az_0 + are^{it}$ $\qquad |a| = 1$

if $a=-1$ then the center of the circle can be reflected.

c) $az_0 + are^{it}$ $\quad a>0$

The circle can be magnified by a.
the center can be moved by a factor of a.

27) a) $T_1(z) = z + b_1 \quad b_1 \neq 0$

$T_2(z) = z + b_2 \quad b_2 \neq 0$

$T_2(T_1(z)) = z + b_1 + b_2$

$T_1(T_2(z)) = z + b_1 + b_2$

order of composition doesnt matter,

b) $R_1(z) = a_1 z \quad |a_1| = 1$

$R_2(z) = a_2 z \quad |a_2| = 1$

$R_2(R_1(z)) = a_2 a_1 z$

$R_1(R_2(z)) = a_2 a_1 z$

Order of composition doesnt matter.

c) $M_1(z) = a_1 z \quad a_1 > 0$

$M_2(z) = a_2 z \quad a_2 > 0$

$M_1(M_2(z)) = a_1 a_2 z$

$M_2(M_1(z)) = a_1 a_2 z$

Order of the composition does not matter.

2.6
Monday, June 08, 2009
3:43 PM

Aaron Morrisett

3) $\lim_{z\to 1-i} (|z|^2 - iz) = \boxed{3-i}$

$f(z) = |z|^2 - iz = (\sqrt{x^2+y^2})(\sqrt{x^2+y^2}) - ix + y = x^2+y^2-ix+y$

$U_0 = \lim_{(x,y)\to(1,-1)} [x^2+y^2+y] = 1+1+1 = 3$

$V_0 = \lim_{(x,y)\to(1,-1)} [-x] = -1$

6) $\lim_{z\to 0} \frac{z^2+\bar{z}^2}{Re(z)+Im(z)} = 0+0i$

$f(z) = \frac{1}{x+y}[x^2-y^2+2xyi+x^2-2xyi-y^2] = \frac{1}{x+y}[2x^2-2y^2] = \frac{2x^2-2y^2}{x+y}$

$U_0 = \lim_{(x,y)\to(0,0)} \left[\frac{2x^2-2y^2}{x+y}\right] = 0$

9) $\lim_{z\to 2-i} [z^2 - z] = (\lim_{z\to 2-i}(z)) \cdot (\lim_{z\to 2-i}(z)) - \lim_{z\to 2-i}(z)$

$= (2-i)(2-i) - 2-i = 4-2i-1-2-i = \boxed{1-3i}$

12) $\lim_{z\to 1+i} \frac{z^2+1}{z^2-1} = \frac{1+2i}{2i} = \frac{1}{4}[-2i-4] = \boxed{-1-\frac{1}{2}i}$

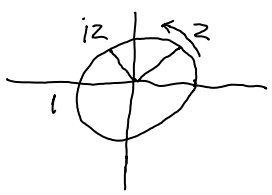
$\lim_{z\to 1+i} (z^2+1) = (1+i)(1+i)+1 = 1+2i-1+1 = 1+2i$

$\lim_{z\to 1+i} (z^2-1) = 2i$

15) $\lim_{z\to z_0} \frac{(az-b)-(az_0+b)}{z-z_0} = \frac{a(z-z_0)-b+b}{z-z_0} = \frac{a(z-z_0)}{z-z_0} = \boxed{a}$

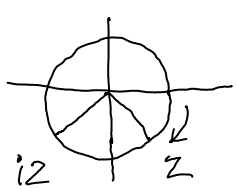
18) $\lim_{z\to i} (|z| + i\,Arg(iz)) = 1 + \lim_{z\to i} i\,Arg(ie^{i\theta})$

a) Value approached on the unit circle as $z\to i$ $\quad |z|=1$ is the unit circle.



$\lim_{z\to i} (|z| + i\,Arg(iz)) = 1 + i\pi$

b) Same question Approached from the second quadrant.



$\lim_{z\to i} (|z| + i\,Arg(iz)) = 1 - i\pi$

c) the Limit from two Paths approach different values so the function is discontinuous at i.

21) $\lim_{z\to\infty} \frac{z^2+iz-2}{(1+2i)z^2} = \frac{1+i/z-\frac{2}{z^2}}{1+2i} = \lim_{z\to\infty} \frac{1}{5}\left[1+i/z-\frac{2}{z^2}\right]\left[1-2i\right] = \frac{1-2i}{5} = \frac{1}{5}-\frac{2}{5}i$

24) $\lim_{z\to -1/2} \frac{(1-i)z+i}{2z+1} = \infty$

27) $f(z) = z^2 - iz + 3 - 2i \qquad z_0 = 2-i$

27) $f(z) = z^2 - iz + 3 - 2i \qquad z_0 = 2 - i$

$\lim_{z \to 2-i} f(z) = (2-i)(2-i) - i(2-i) + 3 - 2i = (4 - 1 - 4i) - 2i - 1 + 3 - 2i$

$= 5 - 8i$

$f(z_0) = 5 - 8i$

$\lim_{z \to z_0} f(z) = f(z_0)$ this is continuous.

30) $f(z) = \dfrac{z - 3i}{z^2 + 2z - 1} \qquad z_0 = 1 + i$

$\lim_{z \to 1+i} f(z) = \dfrac{1 + i - 3i}{(1+i)(1+i) + 2 + 2i - 1} = \dfrac{1 - 2i}{1 + 2i - 1 + 1 + 2i} = \dfrac{1 - 2i}{1 + 4i} = \dfrac{1}{17}(1 - 2i)(1 - 4i)$

$= \frac{1}{17}(1 - 6i - 8) = -\frac{7}{17} - \frac{6}{17}i$

$f(z_0) = -\frac{7}{17} - \frac{6}{17}i$

$\lim_{z \to z_0} f(z) = f(z_0)$ this is continuous.

33) $f(z) = \bar{z} - 3\,\mathrm{Re}(z) + i \qquad z_0 = 3 - 2i$

$\lim_{z \to 3-2i} f(z) = 3 + 2i - 9 + i = -6 + 3i$

$f(z_0) = -6 + 3i$

$\lim_{z \to 3-2i} f(z) = f(z_0)$ this is continuous.

36) $f(z) = \dfrac{1}{|z| - 1} \qquad z_0 = i$

$f(i)$ is not defined.

39) $f(z) = \begin{cases} \dfrac{z^3 - 1}{z - 1}, & |z| \neq 1 \\ 3, & |z| = 1 \end{cases} \; ; \; z_0 = i$

$\lim_{z \to i} \dfrac{z^3 - 1}{z - 1} = \dfrac{i^3 - 1}{i - 1} = \frac{1}{2}[(-i-1)(-1-i)] = \frac{1}{2}[1 + 2i - 1] = i$

$f(i) = \begin{cases} i, & |z| \neq 1 \\ 3, & |z| = 1 \end{cases}$

$\lim_{z \to i} f(z) \neq f(i)$

42) $f(z) = \bar{z}$ 2.6.3 To find the Largest region f is continuous.

$U = x \qquad \lim_{(x,y) \to (x_0, y_0)} U(x,y) = x_0$

$V = -y \qquad \lim_{(x,y) \to (x_0, y_0)} V(x,y) = -y_0$

So U and V are continuous and $z_0 = x_0 + y_0 i$

therefore $f(z) = \bar{z}$ is continuous on C.

3.1

Tuesday, June 09, 2009
8:08 AM

Aaron Morrisett

3) $f(z) = iz^3 - 7z^2$

$$f'(z) = \lim_{\Delta z \to 0} \frac{i(z+\Delta z)^3 - 7(z+\Delta z)^2 - iz^3 + 7z^2}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{i(\cancel{z^3} + 3z^2\Delta z + 3z\Delta z^2 + \Delta z^3) - 7(\cancel{z^2} + 2z\Delta z + \Delta z^2) + \cancel{iz^3} + \cancel{7z^2}}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{i(3z^2\Delta z + 3z\Delta z^2 + \Delta z^3) - 7(2z\Delta z + \Delta z^2)}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\cancel{\Delta z}(i3z^2 + i3z\Delta z + i\Delta z^3 - 14z - 7\Delta z)}{\cancel{\Delta z}} = i3z^2 - 14z$$

6) $f(z) = -z^{-2} = 2z^{-3}$

$$f(z+\Delta z) = -(z+\Delta z)^{-2} = -\frac{1}{(z+\Delta z)^2} = -\frac{1}{(z^2 + 2z\Delta z + \Delta z^2)}$$

$$f(z+\Delta z) - f(z) = -\frac{1}{(z^2+2z\Delta z+\Delta z^2)} - \left(-\frac{1}{z^2}\right) = -\frac{1}{(z^2+2z\Delta z+\Delta z^2)} + \frac{1}{z^2}$$

$$= \frac{-z^2 + z^2 + 2z\Delta z + \Delta z^2}{z^2(z^2 + 2z\Delta z + \Delta z^2)} = \frac{2z\Delta z + \Delta z^2}{z^4 + 2z^3\Delta z + z^2\Delta z^2}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{2z\Delta z + \Delta z^2}{\Delta z(z^4 + 2z^3\Delta z + z^2\Delta z^2)} = \lim_{\Delta z \to 0} \frac{2z + \Delta z}{z^4 + 2z^3\Delta z + z^2\Delta z^2}$$

$$= \frac{2z}{z^4} = \frac{2}{z^3}$$

9) $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

$f(z) = z^4 - z^2 \Rightarrow f'(z) = 4z^3 - 2z$

$f(z) - f(z_0) = z^4 - z^2 - z_0^4 + z_0^2$

$$\lim_{z \to z_0} \frac{z^4 - z^2 - z_0^4 + z_0^2}{z - z_0}$$

$$f'(g(x)) = f'(g(x))\, g'(z)$$

12) $f(z) = 5(iz)^3 - 10z^2 + 3 - 4i$

$f'(z) = 15(iz)^2 i - 10z^2 = -15iz^2 - 20z$

15) $f(z) = \frac{iz^2 - 2z}{3z + 1 - i}$

$$f'(z) = \frac{(3z+1-i)(2iz-2) - (iz^2-2z)(3)}{(3z+1-i)^2}$$

$$= \frac{6iz^2 - 6z + 2iz - 2 + 2z + 2i - 3iz^2 + 6z}{(3z+1-i)^2}$$

$$= \frac{3iz^2 + 2iz + 2z - 2 + 2i}{(3z+1-i)^2}$$

18) $f(z) = \left(\frac{(4+2i)z}{(2-i)z^2+9i}\right)^3$

$f'(z) = 3\left(\frac{(4+2i)z}{(2-i)z^2+9i}\right)^2 \cdot \left[\frac{((2-i)z^2+9i)(4+2i) - (4+2i)z \cdot 2(2-i)z}{[(2-i)z^2+9i]^2}\right]$

$= 3\left(\frac{(4+2i)z}{(2-i)z^2+9i}\right)^2 \cdot \left[\frac{(8-4i)z^2+36i+(4i+2)z^2-18 \quad -(4z+2iz)(4z-2iz)}{[(2-i)z^2+9i]^2}\right]$

21) $f(z) = \bar{z}$

$f(z+\Delta z) - f(z) = (\cancel{x}+\Delta x) - i(\cancel{y}+\Delta y) - \cancel{x} + \cancel{y}i = \Delta x - \Delta y i$

$\lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$

① Approach on x $\quad \lim_{\Delta z \to 0} \frac{\Delta x}{\Delta x} = 1$

② Approach on y $\quad \lim_{\Delta z \to 0} \frac{-i\Delta y}{i\Delta y} = -1$

f is not differentiable at any point.

24) $\lim_{z \to \sqrt{2}+\sqrt{2}i} \frac{z^4+16}{z^2-2\sqrt{2}z+4} = \frac{4z^3}{2z-2\sqrt{2}}$

$\lim_{z \to \sqrt{2}+\sqrt{2}i} \frac{4(\sqrt{2}+\sqrt{2}i)^3}{2\sqrt{2}+2\sqrt{2}i-2\sqrt{2}} = \frac{4(2^2+3\cdot 2\cdot\sqrt{2}i+3\cdot 2\cdot(-2)+(-i)2^2)}{+2\sqrt{2}i}$

$= 8+8i$

27) $f(z) = \frac{iz^2-2z}{3z+1-i}$ at $z = -\frac{1}{3}+\frac{1}{3}i$ it's not analytic.

30) $f(z) = \left(\frac{(4+2i)z}{(2-i)z^2+9i}\right)^3$ $\quad z^2 = \frac{-9}{(2-i)} = \frac{1}{3}(-18+9i)$

$z = \pm\sqrt{\frac{1}{3}(-18+9i)}$

3.2

Thursday, June 11, 2009
8:05 AM

Aaron Morrisett

3) $f(z) = Re(z) = x$

$$\boxed{\begin{array}{c} 1 \overset{?}{=} 0 \\ U_x = V_y \end{array}}$$

function isnt analytic at any point.

6) $f(z) = \bar{z}^2 = x^2 - y^2 - 2xyi$

$U = x^2 - y^2$
$V = -2xy$

$$\boxed{\begin{array}{c} U_x \overset{?}{=} V_y \\ 2x = -2x \end{array}}$$

function isnt analytic at any point.

9) $f(z) = e^{-x}\cos y - ie^{-x}\sin y$

$U = e^{-x}\cos y \qquad V = -e^{-x}\sin y$

a)
$$\boxed{\begin{array}{c} U_x \overset{?}{=} V_y \\ -\cos y e^{-x} = -\cos y e^{-x} \ \checkmark \\ U_y \overset{?}{=} -V_x \\ -\sin y e^{-x} = -\sin y\, e^{-x} \ \checkmark \end{array}}$$

this function is analytic.

b) $U_x + iV_x = V_y - iU_y$

$f'(z) = -\cos y e^{-x} + i\sin y e^{-x} = -\cos y e^{-x} + i\sin y e^{-x}$

12) $f(z) = 4x^2 + 5x - 4y^2 + 9 + i(8xy + 5y - 1)$

$U = 4x^2 + 5x - 4y^2 + 9 \qquad V = 8xy + 5y - 1$

a)
$$\boxed{\begin{array}{c} U_x \overset{?}{=} V_y \\ 8x + 5 = 8x + 5 \\ U_y = -V_x \\ -8y = -8y \end{array}}$$

function is analytic

b) $f'(z) = U_x + iV_x = 8x + 5 + i8y$

15) $f(z) = \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}$

$U = r^{-1}\cos\theta \qquad V = \frac{-\sin\theta}{r} \qquad V_\theta = \frac{-\cos\theta}{r}$

$\frac{dU}{dr} = -r^{-2}\cos\theta \qquad dV/dr = r^{-2}\sin\theta \qquad U_\theta = -r^{-1}\sin\theta$

a)
$$\boxed{\begin{array}{c} U_r \overset{?}{=} \frac{1}{r}V_\theta \\ -r^{-2}\cos\theta = -r^{-2}\cos\theta \\ V_r \overset{?}{=} -\frac{1}{r}U_\theta \\ r^{-2}\sin\theta = r^{-2}\sin\theta \end{array}}$$

function is analytic

b) $f'(z) = e^{-i\theta}[-r^{-2}\cos\theta + ir^{-2}\sin\theta]$

18) $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$

$U_x = V_y$
$U_y = -V_x$

$U_x = 2x + ay \qquad V_x = 2cx + dy$
$U_y = ax + 2by \qquad V_y = dx + 2y$

$2x + ay = dx + 2y$ and $ax + 2by = -2cx - dy$

$a = 2, \; d = 2$

$a = 2 \quad d = 2$

$2x + 2by = -2cx - 2y$

$b = -1, \; c = -1$

$$\boxed{\begin{aligned} a &= 2 \\ b &= -1 \\ c &= -1 \\ d &= 2 \end{aligned}}$$

21) $f(z) = x^3 + 3xy^2 - x + i(y^3 + 3x^2y - y)$

a)
$$\boxed{\begin{aligned} &U_x \overset{?}{=} V_y \\ &3x^2 + 3y^2 - 1 = 3y^2 + 3x^2 - 1 \\ &U_y \overset{?}{=} -V_x \\ &6xy = -6xy \end{aligned}}$$

✓

✗

Not analytical

b) Coordinate Axis

in $6xy = -6xy$ if either x or y is $= 0$ then $U_y = -V_x$ becomes a true equation, then it is differentiable.

24) $z^{1/2} = \sqrt{r} e^{i\theta/2} \quad -\pi < \theta < \pi$

a) use equation 10 to show it's analytic $-\pi < \theta < \pi$

$f(z) = \sqrt{r}\left[\cos\theta/2 + i\sin\theta/2\right] \qquad U = \sqrt{r}\cos\theta/2 \qquad V = \sqrt{r}\sin\theta/2$

$$\boxed{\begin{aligned} &U_r \overset{?}{=} \tfrac{1}{r} V_\theta \\ &\tfrac{1}{2} r^{-1/2}\cos\theta/2 = \tfrac{1}{r}\sqrt{r}\,\tfrac{1}{2}\cos\theta/2 \\ &\tfrac{1}{2} r^{-\frac{1}{2}}\cos\theta/2 = \tfrac{1}{2} r^{-\frac{1}{2}}\cos\theta/2 \\ &V_r \overset{?}{=} -\tfrac{1}{r} U_\theta \\ &\tfrac{1}{2} r^{-1/2}\sin\theta/2 = +\tfrac{1}{r}\cdot\tfrac{1}{2}\cdot\sqrt{r}\sin\theta/2 = \tfrac{1}{2} r^{-\frac{1}{2}}\sin\theta/2 \end{aligned}}$$

✓

✓

this is analytic in the domain $-\pi < \theta < \pi$

b) $f'(z) = e^{-i\theta}\left[\tfrac{1}{2} r^{-\frac{1}{2}}\cos\theta/2 + i\,\tfrac{1}{2} r^{-\frac{1}{2}}\sin\theta/2\right]$

27) $f(z)$ is analytic. is $g(z) = \overline{f(z)}$ analytic

this can happen, take z = a constant, Lets say $8 + 0i$

$$\boxed{\begin{aligned} &U_x \overset{?}{=} V_y \\ &0 = 0 \\ &U_y \overset{?}{=} -V_x \\ &0 = 0 \end{aligned}}$$

therefor it can be analytic.

4.3
Wednesday, June 17, 2009
9:47 AM

3K<37 Aaron Morrisett

3) $\cos(2-4i) = \cos 2\cosh(-4) - i\sin 2\sinh(-4) = \cos 2\cosh 4 - i\sin 2\sinh 4$

6) $\cot(\pi+2i) = \dfrac{\cancel{\cos\pi}^{-1}\cosh(2) - i\cancel{\sin\pi}^{0}\sinh 2}{\cancel{\sin\pi}^{0}\cosh(2) + i\cancel{\cos\pi}^{-1}\sinh 2} = \dfrac{\cosh 2}{i\sinh 2} = \boxed{-i\coth 2}$

9) $\sin z = i$

$\dfrac{e^{iz}-e^{-iz}}{2i} = i \Rightarrow e^{iz}-e^{-iz} = -2 \Rightarrow e^{iz}-e^{-iz}+2=0$

$e^{2iz}+2e^{iz}-1=0$

$w^2+2w-1=0 \qquad \dfrac{2\pm\sqrt{4-4(-1)}}{2} = 1\pm\sqrt{2}$

$e^{iz} = 1\pm\sqrt{2}$

$iz = \ln(1+\sqrt{2}) = \log_e(1\pm\sqrt{2}) + 2k\pi i$

$z = \dfrac{\log_e(1\pm\sqrt{2}) + 2k\pi i}{i} = \underline{-i\log_e(1+\sqrt{2}) + 2k\pi}$

or

$iz = \ln(1-\sqrt{2}) = \log_e(1-\sqrt{2}) + i(\pi+2\pi k)$

$z = \dfrac{\log_e(1-\sqrt{2}) - i(\pi+2\pi k)}{i} = \underline{-i(\log_e(1-\sqrt{2})) + (\pi+2k\pi)}$

12) $\cos z = i\sin z$

$\dfrac{e^{iz}+e^{-iz}}{2} = i\cdot\dfrac{e^{iz}-e^{-iz}}{2i} = \dfrac{e^{iz}-e^{-iz}}{2}$

$\dfrac{e^{iz}+e^{-iz}-e^{iz}+e^{-iz}}{2} = 0 \Rightarrow \dfrac{e^{-2iz}}{2} = 0 \Rightarrow 2e^{-iz} = 0$

$e^{-iz} = 0 \Rightarrow -iz = \ln(0) =$ undefined. so z is undefined.

15) $\overline{\cos z} = \cos\bar{z} \qquad \dfrac{e^{iz}+e^{-iz}}{2} = \cos z$

$\overline{\cos z} = \dfrac{e^{-i(x-yi)}+e^{i(x-yi)}}{2}$

$= \dfrac{e^{-ix-y}+e^{ix+y}}{2}$

$=$

$\cos\bar{z} = \dfrac{e^{i(x-yi)}+e^{-i(x-yi)}}{2}$

$= \dfrac{e^{ix+y}+e^{-ix-y}}{2}$

18) $(\cos(ie^z))' = -\sin(ie^z)\cdot ie^z$

21) $\cosh(\pi i) = \dfrac{e^{\pi i}+e^{-\pi i}}{2} \quad \dfrac{-1+(-1)}{2} = -1$

24)

$\tanh(2+3i) = \dfrac{\sin h(2+3i)}{\cosh(2+3i)} = \dfrac{\sin(-3+2i)}{\cos(-3+2i)} = \dfrac{\sin(-3)\cos(2i)+\cos(-3)\sin(2i)}{\cos(-3)\cos(2i)-\sin(-3)\sin(2i)}$

$$= \frac{-\sin 3 \cosh 2 + \cos 3 \sinh 2}{\cos 3 \cosh 2 + \sin 3 \sin 2} = \boxed{1.28}$$

27) $\sinh z = \cosh z$ $\quad \frac{e^{iz} - e^{-iz}}{2} = \frac{e^{iz} + e^{-iz}}{2} \Rightarrow e^{iz} - e^{-iz} = e^{iz} + e^{-iz}$

$-2e^{-iz} = 0 \quad \Rightarrow -iz = \ln(0)$ undefined, therefore there is no solutions.

30) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

$-i\sin(iz_1 + iz_2) = (-i \sin iz_1)\cos iz_2 + \cos iz_1 (-i \sin iz_2)$

$= \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

33) $f(z) = \sin z \sinh z$

$f'(z) = \cos z \sinh z + \sin z \cosh z$

36) $f(z) = \cosh(iz + e^{iz})$

$f'(z) = \sinh(iz + e^{iz}) \cdot (i + ie^{iz})$

5.1

Thursday, June 18, 2009
9:31 AM

Aaron Morrisett
Math 459

$U = 2\pi x \quad dU = 2\pi dx$

3) $\int_{1/2}^{1} \sin 2\pi x \, dx = \frac{1}{2\pi}\int_{1/2}^{1} \sin U \, dU = \frac{1}{2\pi}\left[-\cos U\right]_{1/2}^{1} = \frac{1}{2\pi}\left[-\cos 2\pi x\right]_{1/2}^{1}$

$= \frac{1}{2\pi}\left[-\cos 2\pi(1) + \cos 2\pi(\frac{1}{2})\right] = \frac{1}{2\pi}\left[-1 - 1\right] = \frac{-2}{2\pi} = \frac{-1}{\pi}$

6) $\int_{\ln 2}^{\ln 3} e^{-x} dx = -e^{-x}\Big|_{\ln 2}^{\ln 3} = -\frac{1}{e^{\ln 3}} + \frac{1}{e^{\ln 2}} = -\frac{1}{3} + \frac{1}{2} = -\frac{4}{12} + \frac{6}{12} = -\frac{2}{6} + \frac{3}{6}$

$\boxed{= \frac{1}{6}}$

9) $\int_2^4 \frac{dx}{x^2 - 6x + 5} = \int \frac{dx}{(x-5)(x-1)} = \frac{A}{x-5} + \frac{B}{x-1} = \frac{1/4}{x-5} - \frac{1/4}{x-1}$

$\frac{1}{(1-5)}\Big| = -\frac{1}{4} = B \qquad \frac{1}{(5-1)}\Big| = \frac{1}{4} = A$

$\int_2^4 \frac{1/4}{x-5} dx + \int_2^4 \frac{-1/4}{x-1} dx = \frac{1}{4}\int_2^4 \frac{1}{x-5} dx - \frac{1}{4}\int_2^4 \frac{1}{x-1}$

$\frac{1}{4}\int_2^4 \frac{1}{U} dU - \frac{1}{4}\int_2^4 \frac{1}{U} dU = \frac{1}{4}\ln(x-5)\Big|_2^4 - \frac{1}{4}\ln(x-1)\Big|_2^4$

$= \frac{1}{4}\left[\left[\ln(-1) - \ln(-3)\right] - \left[-\ln 3 + \cancel{\ln 1}^{0}\right]\right] = \frac{-2\ln(3)}{4} = -\frac{1}{2}\ln(3)$

12) $G(x,y) = x^3 + 2xy^2 + 2x$; $x = 2t$, $y = t^2$, $0 \le t \le 1$

a) $\int_C G(x,y)dx = \int_a^b G(x(t), y(t))\, x'(t)\, dt$

$\int_0^1 \left[(2t)^3 + 2(2t)(t^2)^2 + 2(2t)\right] dt = \int_0^1 \left[8t^3 + (4t)(t^2)^2 + 4t\right] dt$

$= \int_0^1 \left[8t^3 + (4t)(t^4) + 4t\right] dt = \int_0^1 \left[8t^3 + 4t^5 + 4t\right] dt$

$= \left[2t^4 + \frac{2}{3}t^6 + 2t^2\right]_0^1 = 2(1)^4 + \frac{2}{3}(1)^6 + 2(1)^2 = 2 + \frac{2}{3} + 2 = 4 + \frac{2}{3} = \frac{12}{3} + \frac{2}{3}$

$= 14/3 \cdot x'(t) = 14/3 \cdot 2 = \boxed{28/3}$

b) $\int_C G(x,y)dy = \int_a^b G(x(t), y(t)) \cdot y'(t)\, dt$

$\int_0^1 (x^3 + 2xy^2 + 2x)dx \quad x = 2t, \; y = t^2, \; 0 \le t \le 1$

$\int_0^1 \left(\left((2t)^3 + 2(2t)(t^2)^2 + 2(2t)\right) \cdot 2t\right) dt = \int_0^1 \left[16t^4 + 8t^6 + 8t^2\right] dt$

$= \left[\frac{16}{5}t^5 + \frac{8}{7}t^7 + \frac{8}{3}t^3\right]_0^1 = \frac{16}{5} + \frac{8}{7} + \frac{8}{3} = 736/105$

c) $\int_a^b G(x(t), y(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

c) $\int_a^b G(x(t),y(t))\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$

$$\int_0^1 \left((8t^3+4t^5+4t)\cdot\sqrt{4+4t^2}\right)dt = \frac{8\cdot(t^2+1)^{7/2}}{7}\Big|_0^1$$

$$= \frac{64\sqrt{2}}{7} - \frac{8}{7}$$

15) evaluate $\int_C (2x+y)dx + xy\,dy$ from $(-1,2)$ to $(2,5)$

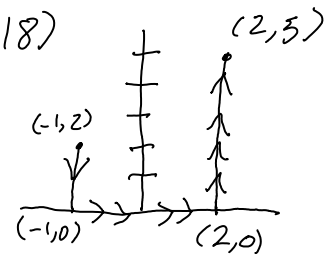
$y = x+3$

$$\int_{-1}^{2}(2x+x+3)dx + \int_{-1}^{2} x(x+3)dx = \left[x^2+\tfrac{1}{2}x^2+3x\right]_{-1}^{2} + \left[\tfrac{1}{3}x^3+\tfrac{3}{2}x^2\right]_{-1}^{2}$$

$$\left[\left[4+2+6\right]-\left[1+\tfrac{1}{2}-3\right]\right] + \left[\left[\tfrac{8}{3}+6\right]-\left[-\tfrac{1}{3}+\tfrac{3}{2}\right]\right]$$

$$\left[12+\tfrac{3}{2}\right]+\left[\tfrac{26}{3}-\tfrac{7}{6}\right]=21$$

18)



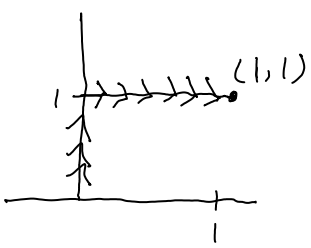
$$\int_2^0 (-1)y\,dy + \int_{-1}^{2} 2x\,dx + \int_0^5 2y\,dy$$

$$= \left[\frac{-y^2}{2}\right]_2^0 + \left[x^2\right]_{-1}^{2} + \left[y^2\right]_0^5$$

$$2 + [4-1] + 25 = 2+3+25 = 30$$

21) C consists of the line segment from $(0,0)$ to $(0,1)$, $(0,1)$, $(1,1)$

$\int_C y\,dx + x\,dy$ from $(0,0)$ to $(1,1)$



$$\int_0^1 0\,dy = 0 + \int_0^1 1\,dx = x\Big|_0^1 = 1$$

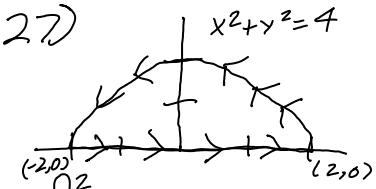
$\boxed{\text{Answer} = 1}$

24) $\int_C -y^2dx + xy\,dy$ C is $x = 2t$, $y = t^3$, $0 \le t \le 2$

$$\int_0^2 -t^6\cdot 2 + (2t)(t^3)\cdot 3t^2 = \int_0^2 \left[-2t^6 + 6t^6\right]dt = \left[-\tfrac{2}{7}t^7 + \tfrac{6}{7}t^7\right]_0^2$$

$$-\tfrac{2}{7}(2)^7 + \tfrac{6}{7}(2)^7 = \tfrac{2}{7}(128) - \tfrac{6}{7}(128) = \frac{512}{7}$$

27)



$$\oint (x^2+y^2)dx - 2xy\,dy$$

$$\oint_C = \int_{C_1} + \int_{C_2}$$

① $\int_{-2}^{2} x^2 dx = \frac{1}{3}x^3\Big|_{-2}^{2} = \frac{16}{3}$

$$\int_{2}^{-2} (4\cos^2 t + 4\sin^2 t)(-2\sin t) - 2(2\cos t)(2\sin t)(2\cos t)$$

$$\int_{2}^{-2} 4(1)(-2\sin t) - 16\sin 2t\cos t$$

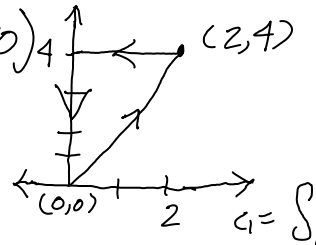
$$\int_{2}^{-2} -8\sin t - 16\sin 2t\cos t$$

② $\int_0^{\pi} -8\sin t = \left[8\cos t\right]_0^{\pi} = -16$

③ $-\int_0^{\pi} 16\sin 2t\cos t = -16\int_0^{\pi}\sin 2t\cos t = -16\left[\frac{-(\cos(t))^3}{3}\right]_0^{\pi} = -\frac{32}{3}$

$$\int_1 + \int_2 + \int_3 = \frac{16}{3} - 16 - \frac{32}{3} = \frac{-64}{3}$$

30)



$\oint_C x^2y^3dx - xy^2dy$ $\qquad y = 2x$

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$C_1 = \int_0^2 x^2(2x)^3dx - x(2x)^2\cdot 2dx = \int_0^2 8x^5dx - 8x^3dx$

$C_1 = \left[\frac{8}{6}x^6 - 2x^4\right]_0^2 = \frac{160}{3}$

$C_2 = \int_2^0 x^2 4^3 dx = \int_2^0 64x^2dx = \left[\frac{64}{3}x^3\right]_2^0 = -\frac{512}{3}$

$C_3 = \int_4^0 -0\cdot y^2dy = 0$

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{160}{3} - \frac{512}{3} + 0 = \frac{-352}{3}$$

33) Verify that the Line integral $\oint_C y^2dx + xy\,dy$

C_1: $x = 2t+1$, $y = 4t+2$, $0 \le t \le 1$
C_2: $x = t^2$, $y = 2t^2$, $1 \le t \le \sqrt{3}$
C_3: $x = \ln t$, $y = 2\ln t$, $e \le t \le e^3$

$\int_{C_1} y^2dx + xy\,dy = \int_0^1 (4t+2)^2 2dx + (2t+1)(4t+2)\cdot 4dt$

$= \int_0^1 (16t^2 + 16t + 4)\cdot 2dt + (8t^2 + 8t + 2)\cdot 4dt$

$= \int_0^1 (32t^2 + 32t + 8)dt + (32t^2 + 32t + 8)dt$

$= \left[\frac{32}{3}t^3 + 16t^2 + 8t + \frac{32}{3}t^3 + 16t^2 + 8t\right]_0^1 = \boxed{\frac{208}{3}}$

$\int_{C_2} y^2dx + xy\,dy = \int_1^{\sqrt{3}} (2t^2)^2\cdot 2t + t^2\cdot(2t^2)\cdot 4t$

$\int_1^{\sqrt{3}} 8t^5 + 8t^5 = \left[\frac{8t^6}{6} + \frac{8t^6}{6}\right]_1^{\sqrt{3}} = \boxed{\frac{208}{3}}$

$$\int_1^{\sqrt{3}} 8t^5 + 8t^5 = \left[\frac{8}{6}t^6 + \frac{8}{6}t^6\right]_1^{\sqrt{3}} = \boxed{\frac{208}{3}}$$

$$\int_{C_3} y^2 dx + xy\,dy = \int_e^{e^3} (2\ln t)^2\left(\frac{1}{t}\right) + (2\ln t)(\ln t)\left(\frac{2}{t}\right)$$

$$= 4\int_e^{e^3} u^2 du + u^2 du = 4\left[\frac{1}{3}u^3 + \frac{1}{3}u^3\right]_e^{e^3} = 4\left[\frac{2}{3}u^3\right]_e^{e^3}$$

$$= 4\left[\frac{2}{3}(\ln t)^3\right]_e^{e^3} = \boxed{\frac{208}{3}}$$

5.2
Friday, June 19, 2009
9:14 AM

3K < 28 Aaron Morrisett

3) $\int_C z^2\,dz$, where C is $z(t) = 3t + 2it \quad -2 \le t \le 2$

$$\int_{-2}^{2}(3t+2it)(3t+2it)(3+2i)\,dt = \int_{-2}^{2}(9t^2+12it^2-4t^2)(3+2i)\,dt$$

$$= \int_{-2}^{2}(27t^2+36it^2-12t^2+18it^2-24t^2-8it^2)\,dt$$

$$= \int_{-2}^{2}(21t^2+46it^2)\,dt = \int_{-2}^{2}9t^2\,dt + i\int_{-2}^{2}46t^2\,dt$$

$$= \left[-3t^3\right]_{-2}^{2} + i\left[\frac{46}{3}t^3\right]_{-2}^{2} = \boxed{-48 + i\,\frac{236}{3}}$$

6) $\int_C |z|^2\,dz$, where C is $x = t^2,\ y = \frac{1}{t},\ 1 \le t \le 2$

$$\int_1^2 \sqrt{(t^2)^2+\left(\tfrac{1}{t}\right)^2}\cdot\sqrt{(t^2)^2+\left(\tfrac{1}{t}\right)^2}\left(2t - i\tfrac{1}{t^2}\right)dt$$

$$\int_1^2\left(t^4+\frac{1}{t^2}\right)\left(2t-i\frac{1}{t^2}\right)dt = \int_1^2\left(2t^5 - \frac{it^4}{t^2} + \frac{2t}{t^2} - i\frac{1}{t^4}\right)dt$$

$$= \int_1^2\left(2t^5+\frac{2}{t}\right)dt + i\int_1^2\left(-\frac{1}{t^4}-t^2\right)dt$$

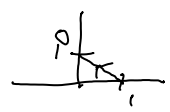
$$= \left[\frac{1}{3}t^6 + 2\ln t\right]_1^2 + i\left[\frac{1}{3t^3} - \frac{1}{3}t^3\right]_1^2 = 2\ln 2 + 21 - i\,\frac{21}{8}$$

9) $\int_C (x^2+iy^3)\,dz$ $\quad$ C is the line from $z=1$ to $z=i$

$y = -x+1 \quad 0 \le x \le 1$

$z = x + i(-x+1)$

$dz = 1 - i$



$$\int_1^0 (x^2+i(-x+1)^3)(1-i)\,dx$$

1
1 1
1 2 1
1 3 3 1

$$= \int_1^0 (x^2+i(-x^3+3x^2-3x+1))(1-i)\,dx$$

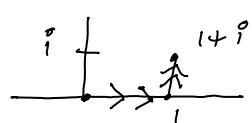
$$= \int_1^0 (x^2+1-x^3+3x^2-3x+1+i(-x^3+2x^2-3x+1))\,dx$$

$$= \int_1^0(-x^3+4x^2-3x+1)\,dx + i\int_1^0(-x^3+2x^2-3x+1)\,dx$$

$$= \left[-\frac{x^4}{4}+\frac{4x^3}{3}-\frac{3x^2}{2}+x\right]_1^0 + i\left[\frac{-x^4}{4}+\frac{2x^3}{3}-\frac{3x^2}{2}+x\right]_1^0$$

$$= -\frac{7}{12} + \frac{1}{12}i$$

12) $\int_C \sin z\,dz$ $\quad$ C is Polygonal Path



$\int_C = \int_{C_1} + \int_{C_2}$

for C_1 $0 \le x \le 1$ $\quad z = x + 0i$

for C_2 $0 \le y \le 1$ $\quad z = 1 + yi$

$$\int_{C_1}\sin z\,dz = \int_0^1 \sin x\,dx = -\cos x\Big|_0^1 = -\cos(1)+1 = \boxed{1-\cos(1)}$$

$$\int_{C_2}\sin z\,dz = \int_0^1 \sin(1+yi)\cdot i\,dy$$

$$\int_{C_2} \sin z\,dz = \int_0^1 \sin(1+yi)\cdot i\,dy$$

$$= i\int_0^1 (\sin(1)\cdot\cosh(y) + i\cos(1)\sinh y)\,dy$$

$$= i\sin(1)\int_0^1 \cosh(y)\,dy - \cos(1)\int_0^1 \sinh y\,dy$$

$$= -\cos(1)[\cosh y]_0^1 + i\sin(1)[\sinh y]_0^1$$

$$= -\cos(1)[\cosh(1) - \cosh(0)] + i\sin(1)[\sinh(1) - \sinh(0)]$$

$$= -\cos(1)[\cosh(1) - 1] + i\sin(1)\cdot\sinh(1) = \boxed{-\cos(1)\cosh(1) + \cos(1) + i\sin(1)\cdot\sinh(1)}$$

$$\int_C = C_1 + C_2 = 1 - \cos(1)\cosh(1) + i\sin(1)\cdot\sinh(1)$$

15) $\oint_C z e^z\,dz$ where

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

C_1 $0 \le x \le 1$ $z = x + 0i$

$$C_1 = \int_0^1 x e^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x\Big|_0^1 = \underline{1}$$

C_2 $0 \le y \le 1$ $z = 1 + yi$ $dz = i\,dy$

$$C_2 = \int_0^1 (1+yi)e^{(1+yi)}\,i\,dy = \int_0^1 (i - y)e^1[\cos y + i\sin y]\,dy$$

$$= \int_0^1 -e\cdot y\cdot\cos y - e\cdot\sin y + i(e\cos y - e\cdot y\cdot\sin y)$$

$$= \underline{-e\cdot\sin(1) + i\,e\cos(1)}$$

C_3 $0 \le x \le 1$ $z = x + i$ $dz = 1$

$$\int z e^z\,dz = ze^z - \int z' e^z\,dz$$

$$\int_1^0 (x+i)e^{(x+i)}\,dx = (x+i)e^{(x+i)} - \int e^{(x+i)}\,dx$$

$$= \left[(x+i)e^{(x+i)} - [\cos(1)\cdot e^x + \sin(1)e^x\,i]\right]_0^1$$

$$= \underline{-(\cos(1) - (e-1)\cdot\sin(1)) - i((e-1)\cdot\cos(1) + \sin(1))}$$

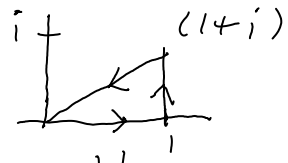
C_4 $0 \le y \le 1$ $z = 0 + yi$ $z' = i\,dy$

$$\int_{C_4} z e^z\,dz = ze^z - \int z' e^z\,dz = yie^{yi}\Big|_1^0 - \int_1^0 i e^{yi}\,i\,dy$$

$$= \underline{\cos(1) + \sin(1) - 1 - i(\cos(1) - \sin(1))}$$

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0$$

18) $\oint_C (2z - 1)\,dz$



C_1 $0 \le x \le 1$ $z = x + i0$

$$C_1 = \int_0^1 (2(x) - 1)\,dx = \int_0^1 (2x - 1)\,dx = x^2 - x\Big|_0^1 = 1 - 1 = \underline{0}$$

$C_2 \quad 0 \le y \le 1 \qquad z = 1 + iy$

$$C_2 = \int_0^1 (2(1+iy)-1)\, i\, dy = i\int_0^1 ((2+2iy)-1)\,dy = i\int_0^1 (1+2iy)\,dy$$

$$= i\left[y + iy^2\right]_0^1 = \underline{i-1}$$

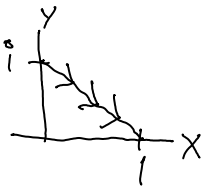
$C_3 \quad y = x \qquad z = x + xi \qquad 0 \le x \le 1$

$$\int_C (2z-1)\,dz = z^2 - z$$

$$(x+xi)(x+xi) - (x+xi)\Big|_1^0 = (x^2 + 2ix^2 - x^2) - (x+xi)\Big|_1^0$$

$$-[(1+2i-1) - (1+i)] = -[-1+i] = \underline{1-i}$$

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = \boxed{0}$$

20) 

$\int_C (z^2 - z + 2)\,dz$ from i to 1

$$\int_C (z^2 - z + 2)\,dz = \frac{z^3}{3} - \frac{z^2}{2} + 2z$$

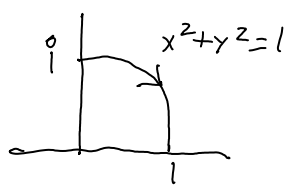
$y = -x + 1 \qquad z = x + i(-x+1) \qquad dz = -dx$

$0 \le x \le 1$

$$\left[\frac{(x+i(-x+1))^3}{3} - \frac{(x+i(-x+1))^2}{2} + 2(x+i(-x+1))\right]_0^1$$

$$\left[\frac{(1+i0)^3}{3} - \frac{(1+i0)^2}{2} + 2(1+i(0))\right] - \left[\frac{(0+i)^3}{3} - \frac{(0+i)^2}{2} + 2(0+i)\right]$$

$$= \frac{11}{6} - \frac{1}{2} - \frac{5}{3}i = \underline{\frac{4}{3} - \frac{5}{3}i}$$

24) 

$x = \cos t \qquad y = \sin t \qquad 0 \le t \le \pi/2$

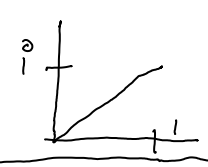
$z = \cos t + i\sin t$

$$\int_C (z^2 - z + 2)\,dz = \frac{z^3}{3} - \frac{z^2}{2} + 2z$$

$$\frac{(\cos t + i\sin t)^3}{3} - \frac{(\cos t + i\sin t)^2}{2} + 2(\cos t + i\sin t)\Big|_{\pi/2}^0$$

$$\left[\frac{1}{3} - \frac{1}{2} + 2\right] - \left[\frac{i^3}{3} - \frac{i^2}{2} + 2i\right] = \frac{4}{3} - \frac{5}{3}i$$

27) $\int_C (z^2+4)\,dz \qquad z = 0$ to $z = 1+i$



$|z| = \sqrt{2} \qquad L = \sqrt{2}$

$M = |z|^2 + 4 = 6 \qquad \boxed{ML = 6\sqrt{2}} \qquad \boxed{\left|\int_C (z^2+4)\,dz\right| \le 6\sqrt{2}}$

5.3
Sunday, June 21, 2009
7:15 PM

Aaron Morrisett

3) $f(z) = \frac{z}{2z+3}$ $\quad c = |z| = 1$ is analytic at all points except $z = -3/2$

$-3/2$ is outside the unit circle, so according to the Cauchy-Goursat theorem: $\oint_c f(z)dz = 0$

6) $f(z) = \frac{e^z}{2z^2+11z+15}$ is analytic everywhere accept:

$z = -3,\ z = -5/2$

these points are outside $|z| = 1$ so $\oint_c f(z)dz = 0$

9) evaluate $\oint_c \frac{1}{z} dz$ where c is

function is analytic excluding the point $z=0$

Using equation (6) $\boxed{\oint_c \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n=1 \\ 0 & n \neq 1 \end{cases}}$

Yields $\oint_c \frac{dz}{z} = \underline{2\pi i}$

12) $\oint_c \left(z + \frac{1}{z^2}\right) dz;\quad |z| = 2$ countor.

$= \oint_c z\,dz + \oint \frac{dz}{z^2}$

$\oint_{c_1} z\,dz$ is analytic everywhere so $\oint_c z\,dz = 0$

$\oint_{c_2} \frac{dz}{z^2} = 0$ by equation 6.

$\oint_c = \oint_{c_1} + \oint_{c_2} = 0$

15) $\oint_c \frac{2z+1}{z^2+z} dz = \oint_c \frac{2z+1}{z\cdot(z+1)} dz = \oint_c \left(\frac{A}{z} + \frac{B}{z+1}\right) dz$

using heavy side cover up Method of Partial fractions.

$A = \frac{1}{1} = 1 \qquad B = \frac{2(-1)+1}{-1} = \frac{-1}{-1} = 1$

thus $\Rightarrow \oint_c \frac{1}{z} dz + \oint_c \frac{1}{z+1} dz$

a) countor is $|z| = \frac{1}{2}$

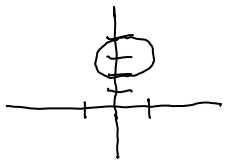
$\oint_{c_1} \frac{dz}{z} = 2\pi i$ by equation 6 and $\oint_{c_2} \frac{dz}{z+1} = 0$ because it's not analytic at $z=-1$, which is outside the countor.

$\oint_c = \oint_{c_1} + \oint_{c_2} = \boxed{2\pi i}$

b) countor $|z| = 2$

$\oint_{c_1} \frac{dz}{z} = 2\pi i \qquad \oint_{c_2} \frac{dz}{z+1} = 2\pi i \qquad \oint_c = \oint_{c_1} + \oint_{c_2} = \boxed{4\pi i}$

c) countor $|z - 3i| = 1$



$\oint_{c_1} \frac{dz}{z} = 0 \qquad \oint_{c_2} \frac{dz}{z+1} = 0$

both are not analytic at points outside the countor. $\oint_C = \oint_{C_1} + \oint_{C_2} = \boxed{0}$

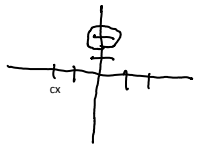
18) $\oint_C \left(\frac{3}{z+2} - \frac{1}{z-2i}\right) dz$

a) countor $|z|=5$

$\oint_{C_1} \frac{3dz}{z+2} = 3\oint_{C_1} \frac{dz}{z+2} = (3)\cdot 2\pi i = 6\pi i$

$-\oint_{C_2} \frac{dz}{z-2i} = -2\pi i \qquad \oint_C = \oint_{C_1} + \oint_{C_2} = \boxed{4\pi i}$

b) $|z-2i| = \frac{1}{2}$



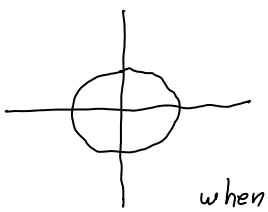
$3\oint_{C_1} \frac{dz}{z+2} = 0 \qquad -\oint_{C_2} \frac{dz}{z-2i} = -2\pi i$

21) $\oint_C \ln(z+10)\,dz$ countor $|z|=2$.

$z=-10$ it is not analytic, which is outside the counter

$\oint_C = \boxed{0}$

24) $\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i, & n=1 \\ 0, & n>1 \end{cases}$



Let the countor be the unit circle and $z_0=0$

$\int_0^{2\pi} \frac{ie^{it}dt}{e^{it}} = \int_0^{2\pi} i\,dt = it\Big|_0^{2\pi} = \boxed{2\pi i}$ when $n=1$

when $n=2$

$\int_0^{2\pi} \frac{ie^{it}dt}{e^{2it}} = \int_0^{2\pi} ie^{-it} = -e^{it}\Big|_0^{2\pi} = \boxed{0}$

thus when $n>1$ $\oint_C = 0$ and when $n=1$ $\oint_C = 2\pi i$

27) a) $f(z) = (5iz^4 - 4z^2 + 2 - 6i)^9$

$\oint_C f(z) = 0$ because it is analytic everywhere, Cauchy-goursat

b) $f(z) = (z^2 - 3iz)e^{5z}$

$\oint_C f(z) = 0$ because it is analytic everywhere, Cauchy-goursat

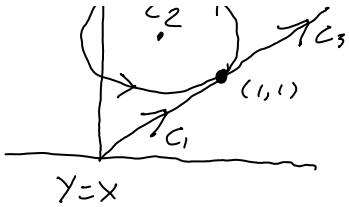
c) $f(z) = \frac{\sin z}{e^{z^2}}$

$\oint_C f(z) = 0$ because it is analytic everywhere, Cauchy-goursat

d) $f(z) = z\cos^2 z$

$\oint_C f(z) = 0$ because it is analytic everywhere, Cauchy-goursat

30) $\int_C e^z dz$ evaluate from $z=0$ to $z=2+2i$



C_1 is the countor $Y=X$ from 0 to 1

$Z = X + Xi \quad dZ = 1+i$

$$\int_0^1 e^u du = e^u \Big|_0^1 = e^{x+xi}\Big|_0^1$$

$$= [e^{(1)}(\cos 1 + i\sin 1)] - [\cos(0) + i\sin(0)]$$

$$C_1 \quad = e^1[\cos(1) + i\sin(1)] - 1$$

C_2: $\oint_{C_2} e^z dz = 0$, does not fall inside the countor.

C_3: $Z = X + Xi \quad dZ = 1+i$

$$\int_1^2 e^u du = e^{(x+xi)}\Big|_1^2 = [e^2(\cos(2) + i\sin(2)] - e^1(\cos(1) + i\sin(1))$$

$$\oint_C = \oint_{C_1} + \oint_{C_2} + \oint_{C_3} = e^2[\cos(2) + i\sin(2)] - 1$$

5.4

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3) $\int_C 2z\,dz$ $\quad$ C is $z(t) = 2t^3 + i(t^4 - 4t^3 + 2)$, $-1 \le t \le 1$

at -1 $\quad z(-1) = -2 + 7i$

at 1 $\quad z(1) = 2 - i$

$= (2-i)^2 - (-2+7i)^2 = \boxed{48 + 24i}$

6) $\int_{-2i}^{1} (3z^2 - 4z + 5i)\,dz$ $\quad F(z) = z^3 - 2z^2 + 5iz + C$

$= [1^3 - 2(1)^2 + 5i(1)] - [(-2i)^2 - 2(-2i)^2 + 5i(-2i)] = \boxed{-19 - 3i}$

9) $\int_{-i/2}^{1-i} (2z+1)^2\,dz = \int_{-i/2}^{1-i} (4z^2 + 4z + 1)\,dz$ $\quad F(z) = \frac{4}{3}z^3 + 2z^2 + z + C$

$= \left[\frac{4}{3}(1-i)^3 + 2(1-i)^2 + (1-i)\right] - \left[\frac{4}{3}(-\frac{i}{2})^3 + 2(-\frac{i}{2})^2 + (-\frac{i}{2})\right] = -\frac{7}{6} - \frac{22}{3}i$

12) $\int_{1-i}^{1+2i} z e^{z^2}\,dz$ $\quad U = z^2$ $\quad dU = 2z\,dz$

$\frac{1}{2}\int_{1-i}^{1+2i} e^U\,dU = \frac{1}{2}\left[e^{(1+2i)^2} - e^{(1-i)^2}\right] = \frac{1}{2}\left[e^{(-3+4i)} - e^{-2i}\right]$

$= \frac{e^{-3}}{2}[\cos(4) + i\sin(4)] - \frac{1}{2}[\cos(2) + \sin(2)i]$

$= \frac{1}{2}[e^{-3}\cos(4) - \cos(2)] + i\frac{1}{2}[e^{-3}\cos(4) - \sin(2)]$

15) $\int_{\pi i}^{2\pi i} \cosh z\,dz$ $\quad F(z) = \sinh z + C$

$= \sinh(2\pi i) - \sinh(2\pi i) = 0$

18) $\int_C \frac{dz}{z}$, C is the line segment between $1+i$ and $4+4i$

$F(z) = \ln(z)$

$= \ln(4+4i) - \ln(1+i) = \ln\left(\frac{4+4i}{1+i}\right) = 2\ln(2)$

21) $\int_{\pi}^{i} \overset{U}{e^z} \overset{dV}{\cos z\,dz} = \overset{U}{e^z} \overset{V}{\sin z} - \int \overset{U}{e^z} \overset{dV}{\sin z\,dz}$

$= e^z \sin z - \left[\overset{dU}{e^z} \overset{V}{\cos z} + \int e^z \cos z\,dz\right]$

$2\int e^z \cos z\,dz = e^z \sin z + e^z \cos z$

$\int e^z \cos dz = \frac{e^z \sin z + e^z \cos z}{2}\Big|_{\pi}^{i} = \boxed{11.49 + .966i}$

24) $\int_0^{\pi i} \overset{u}{z^2} \overset{dv}{e^z} dz = z^2 e^z - \int \overset{u}{2z} \overset{dv}{e^z} dz$

$= z^2 e^z - \left[2z e^z - 2\int e^z dz \right]$

$= z^2 e^z - 2z e^z + 2e^z \Big|_0^{\pi i} = \pi^2 - 4 + 2\pi i$

$= \boxed{5.87 + 6.28i}$

27) $f(z) = \sin z^2 \qquad u = z^2 \quad du = 2dz$

$\frac{1}{2}\int \sin u \, du = \boxed{-\frac{1}{2}\cos z^2 + C}$

30) find $\int e^{ax} \sin bx \, dx$

5.5

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3) $\boxed{\text{theorem } 5.5.1 \quad f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz}$

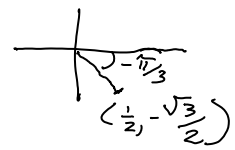
$\boxed{\text{theorem } 5.5.2 \quad f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$

$\oint_C \frac{e^z}{z-\pi i}\,dz \; ; \; |z|=4$

thus $\frac{1}{2\pi i}\oint_{|z|=4} \frac{e^z}{z-\pi i}\,dz = f(\pi i) = e^{\pi i}$

$\oint_C \frac{e^z}{z-\pi i}\,dz = 2\pi i\, e^{\pi i} = \boxed{-2\pi i}$

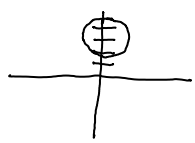
b) $\oint_{|z|=1.1} \frac{\cos z}{3z-\pi}\,dz \qquad z_0 = -\pi/3 \qquad f(z_0) = \cos(-\pi/3)$



$= 2\pi i \cdot \cos(-\pi/3) = 2\pi i \cdot (\frac{1}{2}) = \boxed{\pi i}$

9) $\oint_C \frac{z^2+4}{z^2-5iz-4}\,dz \quad C \Rightarrow |z-3i| = 1.3$

$= \oint_C \frac{z^2+4}{(z-4i)(z-i)}\,dz$



$z=4i$ is the only term that is non-analytic inside the countor.

$= \oint_C \frac{\frac{z^2+4}{z-i}}{z-4i}\,dz = 2\pi i\left(\frac{(4i)^2+4}{4i-i}\right) = \boxed{-8\pi}$

12) $\oint_C \frac{z}{(z+i)^4}\,dz \; ; \; |z| = 2$

Using: $\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$

$= \frac{2\pi i}{3!}(-i) = \frac{2\pi i}{6}(-i) = \boxed{\frac{\pi}{3}}$

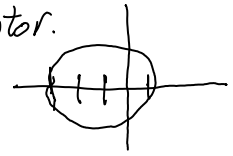
15) $\oint_C \frac{2z+5}{z^2-2z}\,dz = \oint_C \frac{2z+5}{z(z-2)}\,dz$

a) $|z| = \frac{1}{2}$ $\quad z=0$ is only factor not analytic in countor.

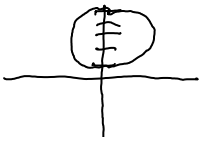
$\oint_C \frac{\frac{2z+5}{z-2}}{z}\,dz = 2\pi i\left[\frac{5}{-2}\right] = \boxed{-5\pi i}$ $\quad z_0 = 0$

b) $|z+1| = 2$ $\quad z=0$ is not analytic inside the countor.

$\oint_C \frac{\frac{2z+5}{z-2}}{z} = \boxed{-5\pi i}$



c) $|z-3| = 2$



$z=2$ is not analytic inside the countor.

$$\oint_C \frac{2z+5}{\frac{z}{z-2}}\,dz = 2\pi i\left[\frac{4+5}{2}\right] = \boxed{9\pi i}$$

d) $|z+2i|=1$

all Points are analytic inside the countor.

$$\oint_C \frac{2z+5}{z^2-2z}\,dz = \boxed{0}$$

18) $\oint_C \frac{1}{z^3(z-4)}\,dz$

a) $|z|=1$ Point $z=0$ is not analytic

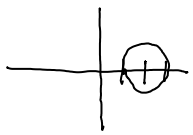
$\oint_C \frac{\frac{1}{z-4}}{z^3}\,dz$ $\quad f(z)=\frac{1}{z-4}$ $\quad f'(z)=\frac{z-4-1}{(z-4)^2}=\frac{z-3}{(z-4)^2}$

$$f''(z)=\frac{(z-4)^2-(z-3)(2(z-4))}{(z-4)^4}=\frac{(z-4)^2-2(z^2-7z+12)}{(z-4)^4}$$

$z_0=0$

$$\oint_C \frac{\frac{1}{z-4}}{z^3}\,dz = \frac{2\pi i}{2}\left[\frac{(-4)^2-2(0^2-7\cdot 0+12)}{(-4)^4}\right]=\boxed{\frac{\pi i}{64}}$$

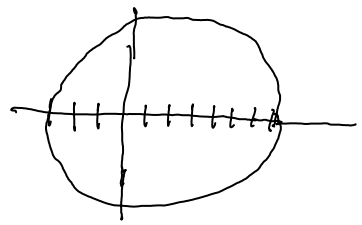
b) $|z-2|=1$



all Points are analytic inside the countor.

$$\oint_C \frac{1}{z^3(z-4)}\,dz = \boxed{0}$$

21) $\oint_C \frac{1}{z^3(z-1)^2}\,dz$ $\quad |z-2|=5$



$z=0$ and $z=1$ are not analytic.

$$\oint_C \frac{\frac{1}{(z-1)^2}}{z^3}\,dz+\oint_C \frac{\frac{1}{z^3}}{(z-1)^2}\,dz$$

$f_1(z)=(z-1)^{-2}$, $f_1'(z)=-2(z-1)^{-3}$, $f_1''(z)=6(z-1)^{-4}$

$f_1''(0)=6(-1)^{-4}=6$

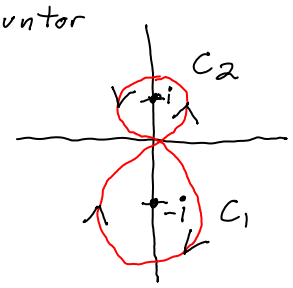
$f_2(z)=z^{-3}$, $f_2'(z)=-3z^{-4}$

$f_2'(z_0)=-3(1)^{-4}=-3$

$$\oint_C \frac{1}{z^3(z-1)^2}\,dz = \frac{2\pi i}{2}(6)+\frac{2\pi i}{1!}(-3) = 6\pi i-6\pi i=\boxed{0}$$

24) $\oint_C \frac{e^{iz}}{(z^2+1)^2}\,dz$

Countor



$$=\oint_{C_1}\frac{e^{iz}}{(z^2+1)^2}\,dz+\oint_{C_2}\frac{e^{iz}}{(z^2+1)^2}\,dz$$

$$=-\oint_{-C_1}\frac{e^{iz}}{(z^2+1)^2}\,dz+\oint_{C_2}\frac{e^{iz}}{(z^2+1)^2}\,dz=-I_1+I_2$$

$I_1=\oint_{-C_1}\frac{e^{iz}}{(z^2+1)^2}\,dz$ $\quad f_1=e^{iz}$, $f_1'=ie^{iz}$, $f_1'(-i)=ie^{1}$

$z_0=-i$ $\quad I_1=\frac{2\pi i}{1}\left[ie^{1}\right]=-2\pi e^{1}$

$I_2=\oint_{C_2}\frac{e^{iz}}{(z^2+1)^2}\,dz$ $\quad f_2=e^{iz}$, $f_2'(i)=ie^{-1}$ $\quad I_2=\frac{2\pi i}{1}\left[ie^{-1}\right]=-2\pi e^{-1}$

$$-I_1 + I_2 = 2\pi e^{1} - 2\pi e^{-1} \simeq 4\sinh(1)\cdot\pi = 14.768$$

6.2

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8:08 AM

Aaron Morrisett

$$\boxed{\text{Taylor} \quad f(z) = \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k}$$

5) $f(z) = e^{-2z} = \sum_{k=0}^{\infty} \frac{(-2z)^k}{k!} = \sum_{n=0}^{\infty} \frac{(-1)^k}{k!}(2z)^k$

there are no isolated singularities, $R = \infty$

10) $f(z) = \sin 3z = \sum_{k=0}^{\infty} (-1)^k \frac{(3z)^{2k+1}}{(2k+1)!}$ $\quad R = \infty$

15) $f(z) = \frac{1}{z}$ $\quad z_0 = 1$

$f(z) = \frac{1}{1+(z-1)}$ $\quad \boxed{\sum_{k=0}^{\infty} (-1)^k (z-1)^k}$ $\quad |z-1| < 1$

$\boxed{\therefore R = 1}$

20) $f(z) = \frac{1+z}{1-z}$ $\quad z_0 = i$ $\quad R = \sqrt{1^2+1^2} = \boxed{\sqrt{2}}$

$$(1+z)\left(\frac{1}{1-z}\right) = (1+(z-i)+i)\left(\frac{1}{1-(z-i)-i}\right) = (1+z)\left(\frac{1}{1-i-(z-i)}\right)$$

$$= \frac{(1+z)}{1-i}\left[\frac{1}{1-\left[\frac{z-i}{1-i}\right]}\right] = \left(\frac{1+z}{1-i}\right)\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n = \boxed{\sum_{n=0}^{\infty}(1+z)\left[\frac{z-i}{1-i}\right]^n}$$

$$f(z) = \frac{i}{(z-i)(z-2i)} = \frac{A}{z-i} + \frac{B}{z-2i} \quad A = -1 \quad B = 1$$

$$= \frac{1}{z-2i} - \frac{1}{z-i} = \frac{1}{-2i}\left[\frac{1}{z/-2i+1}\right] - \frac{1}{-i}\left[\frac{1}{z/-i+1}\right] = \left(\frac{1}{-2i}\right)\cdot\left[\frac{1}{1-z/2i}\right] + \left(\frac{1}{i}\right)\cdot\left[\frac{1}{1-z/i}\right]$$

$$= \boxed{\frac{i}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2i}\right)^n - i\sum_{n=0}^{\infty}\left(\frac{z}{i}\right)^n} \qquad \boxed{R = \sqrt{(0-1)^2} = 1}$$

30) Radius from $z=0$ to $z=-1$ is 1, so $R = 1$

35) $$\frac{d}{dz}\left(\frac{(-1)^k \cdot z^{2k+1}}{(2k+1)!}\right) = \left(\frac{(-1)^k}{(2k+1)!}\right)\frac{d}{dz}\left(z^{2k+1}\right) = \left(\frac{(-1)^k}{(2k+1)!}\right)\left((2k+1)z^{2k}\right)$$

$$= \boxed{\frac{(-1)^k \cdot z^{2k}}{(2k)!}}$$

6.3

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11:42 AM

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3) $f(z) = e^{-1/z^2}, \; 0 < |z|$ $\qquad e^z = \sum_{n=0}^{\infty} \frac{z^k}{k!}$

$$= \sum_{n=0}^{\infty} \frac{(-1/z^2)^k}{k!} = \sum_{n=0}^{\infty} (-1)^k \frac{(\frac{1}{z^2})^k}{k!} = 1 - \frac{1}{1!\cdot z^2} + \frac{1}{2!\cdot z^4} - \frac{1}{3!\cdot z^6} + \cdots$$

6) $z \cos\frac{1}{z}, \quad 0 < |z|$ $\qquad \cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$

$$= z \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{z})^{2k}}{(2k)!} = \boxed{\sum_{k=0}^{\infty} \frac{z(-1)^k (\frac{1}{z})^{2k}}{(2k)!}}$$

9) $0 < |z-3| < 3$

$$f(z) = \frac{1}{z(z-3)} = \frac{1}{(z-3+3)} \frac{1}{z-3} = \frac{1}{z-3} \cdot \frac{1}{3+(z-3)} = \frac{1}{z-3} \cdot \frac{1}{3} \cdot \frac{1}{1+\frac{z-3}{3}}$$

$$f(z) = \frac{1}{3(z-3)} \left[1 - \frac{z-3}{3} + \frac{(z-3)^2}{3^2} - \frac{(z-3)^3}{3^3} + \cdots \right]$$

$$\boxed{= \frac{1}{3(z-3)} - \frac{1}{3^2} + \frac{z-3}{3^3} - \frac{(z-3)^2}{3^4} + \cdots}$$

12) $1 < |z+1| < 4$

$$f(z) = \frac{1}{z(z-3)} = \frac{1}{3(z-3)} - \frac{1}{3z} = f_1(z) + f_2(z)$$

first find a series for $\left|\frac{1}{z+1}\right| < 1$ using $f_2(z)$

$$-\frac{1}{3z} = -\frac{1}{3} \cdot \frac{1}{z+1-1} = \frac{-1}{3(z+1)} \cdot \frac{1}{1-(\frac{1}{z+1})} = \frac{-1}{3(z+1)} \left[1 + \frac{1}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots \right]$$

$$= \boxed{-\frac{1}{3(z+1)} - \frac{1}{3(z+1)^2} - \frac{1}{3(z+1)^3} - \cdots \quad \text{Converges for } 1 < |z+1|}$$

Second find a series for $|z+1| < 4$ using $f_1(z)$

$$\frac{1}{3(z-3)} = \frac{1}{3} \cdot \frac{1}{z+1-4} = \frac{-1}{3} \cdot \frac{1}{4-(z+1)} = \frac{-1}{12} \cdot \frac{1}{1-(\frac{z+1}{4})}$$

$$= -\frac{1}{12} \left[1 + \frac{z+1}{4} + \frac{(z+1)^2}{4^2} + \frac{(z+1)^3}{4^3} + \cdots \right]$$

$$= \boxed{-\frac{1}{12} - \frac{z+1}{48} - \frac{(z+1)^2}{12 \cdot 4^2} - \frac{(z+1)^3}{12 \cdot 4^3} - \cdots \quad \text{Converges for } |z+1| < 4}$$

Combining the results for $f_1(z)$ and $f_2(z)$ Yields:

$$\boxed{\cdots - \frac{1}{3(z+1)^3} - \frac{1}{3(z+1)^2} - \frac{1}{3(z+1)} - \frac{1}{12} - \frac{z+1}{48} - \frac{(z+1)^2}{12 \cdot 4^2} - \frac{(z+1)^3}{12 \cdot 4^3} - \cdots}$$

15) $f(z) = \frac{1}{(z-1)(z-2)} \qquad 0 < |z-1| < 1$

$$= \frac{1}{(z-1)} \cdot \frac{1}{z-1-1} = \frac{1}{z-1} \cdot \frac{-1}{1-(z-1)}$$

$$= \frac{-1}{z-1} \left[1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots \right]$$

$$= \boxed{\frac{-1}{z-1} - 1 - (z-1) - (z-1)^2 - \cdots}$$

18) $f(z) = \frac{z}{(z+1)(z-2)}$ $\quad |z+1| > 3$ or $\left|\frac{1}{z+1}\right| < 3$ or $\left|\frac{3}{z+1}\right| < 1$

$= \frac{z}{z+1} \cdot \frac{1}{(z+1)-3} = \frac{z}{(z+1)^2} \cdot \frac{1}{1-\left(\frac{3}{z+1}\right)}$

$\Rightarrow \frac{z}{(z+1)^2}\left[1 + \frac{3}{z+1} + \frac{3^2}{(z+1)^2} + \frac{3^3}{(z+1)^3} + \cdots\right]$

$$\boxed{\frac{z}{(z+1)^2} + \frac{3z}{(z+1)^3} + \frac{3^2 z}{(z+1)^4} + \frac{3^3 z}{(z+1)^5} + \cdots}$$

21) $f(z) = \frac{1}{z(1-z)^2}$ $\quad 0 < |z| < 1$

Maclaurin series of $\frac{1}{(1-z)^2} = 0 + 1 + 2z + 3z^2$

$\therefore \frac{1}{z}\left[0 + 1 + 2z + 3z^2 + \cdots\right]$

$\Rightarrow \boxed{\frac{1}{z} + 2 + 3z + 4z^2 + \cdots \quad \text{Converges for } |z| < 1}$

24) $f(z) = \frac{1}{(z-2)(z-1)^3}$ $\quad 0 < |z-1| < 1$

$= \frac{1}{(z-1)-1} \cdot \frac{1}{(z-1)^3} = \frac{-1}{1-(z-1)} \cdot \frac{1}{(z-1)^3}$

$\Rightarrow \frac{-1}{(z-1)^3}\left[1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots\right]$

$= \boxed{\frac{-1}{(z-1)^3} - \frac{1}{(z-1)^2} - \frac{1}{(z-1)} - 1 - (z-1) - (z-1)^2 - \cdots}$

27) $f(z) = \frac{z^2 - 2z + 2}{z-2}$ $\quad 1 < |z-1|$ or $\left|\frac{1}{z-1}\right| < 1$

$\frac{z(z-2)+2}{z-2} = z + \frac{2}{z-2} = z + \frac{2}{z-1-1} = z + \frac{2}{z-1} \cdot \frac{1}{1-\left(\frac{1}{z-1}\right)}$

$= 1 + (z-1) + \frac{2}{z-1} \cdot \frac{1}{1-\left(\frac{1}{z-1}\right)}$

$\Rightarrow 1 + (z-1) + \frac{2}{z-1}\left[1 + \frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \cdots\right]$

$= 1 + (z-1) + \frac{2}{z-1} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)^3} + \cdots$

$= \boxed{\cdots + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{z-1} + 1 + (z-1)}$ Converges for $|z-1| > 1$

6.4

Monday, June 29, 2009
9:39 AM

Aaron Morrisett

3) $f(z) = \frac{\sin 4z - 4z}{z^2} = \frac{-4z + 4z - \frac{(4z)^3}{3!} + \frac{(4z)^5}{5!}}{z^2}$

$f(z) = \boxed{\frac{-4^3 z}{3!} + \frac{4^5 z^3}{5!} - \frac{4^7 z^5}{7!} + \cdots}$ has a Removable singularity

$f(0) = 0$

6) $f(z) = z^4 - 16$ $\quad z = \pm 2$, $z = \pm 2i$ order 4

9) $f(z) = e^{2z} - e^z = e^z(e^z - 1)$

$e^z = 0$ or $e^z = 1$

$z = \ln(0) =$ undefined or $z = \ln(1) = \cancel{\log_e(1)}^{0} + 2k\pi i$

$\boxed{2k\pi i}$

12) $f(z) = z - \sin z$; $z = 0$

$z - z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$

$-\frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \Rightarrow z^3\left[\frac{-1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots\right]$

$\boxed{\text{order is } 3}$

15) $f(z) = \frac{3z - 1}{z^2 + 2z + 5}$

$\frac{-2 \pm \sqrt{2^2 - 4(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \boxed{-1 \pm 2i}$ are zeros

18) $f(z) = \frac{z - 1}{(z+1)(z^3+1)}$ $\quad z = -1$ is a pole of order 1 simple pole

$z^3 = -1 \Rightarrow z = (-1)^{1/3}$ $\quad \theta = \pi$ $\quad r = 1$

$K_0 = 1\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$K_1 = 1\left[\cos\pi + i\sin\pi\right] = -1$

$K_2 = 1\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right] = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

of order 3

21) $1 - \cosh z$ $\quad 1 - \frac{e^z + e^{-z}}{}$

$$\frac{\quad}{z^4} = \frac{\quad 2}{z^4}$$

$$\frac{1-\sum_{K=0}^{\infty}\frac{z^K}{K!}+\sum_{K=0}^{\infty}\frac{(-z)^K}{K!}}{z^4} = \frac{1-\frac{1}{2}\left[\left[1+z+\frac{z^2}{2!}+\cdots\right]+\left[1-z+\frac{z^2}{2!}-\cdots\right]\right]}{z^4}$$

$$= \frac{1-\frac{1}{2}\left[2+2\frac{z^2}{2!}+2\frac{z^4}{4!}+\cdots\right]}{z^4} = \frac{-\frac{z^2}{2!}-\frac{z^4}{4!}-\frac{z^6}{6!}-\cdots}{z^4}$$

$$= \boxed{-\frac{1}{z^2 2!}-\frac{1}{4!}-\frac{z^2}{6!}-\cdots}$$ Pole of order 2 at $z=0$

24) $f(z) = \frac{e^z-1}{z^2} = \frac{\sum_{K=0}^{\infty}\frac{z^K}{K!}-1}{z^2} = \frac{\left[1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right]-1}{z^2}$

$$= \boxed{\frac{1}{z}+\frac{1}{2!}+\frac{z}{3!}+\cdots \quad \text{Simple Pole at } 0.}$$

27) $f(z) = z^3\sin\left(\frac{1}{z}\right);\ z=0$ $\qquad \sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-$

$$z^3\left[\frac{1}{z}-\frac{1}{z^3 3!}+\frac{1}{z^5 5!}-\cdots\right] = \boxed{z^2-\frac{1}{3!}+\frac{1}{z^2 5!}-\frac{1}{z^4 7!}+\cdots}$$

the order of the poles continues increasing, is a essential singularity.

30) $z=0$ $\quad f(z) = \tan\left(\frac{1}{z}\right) = \frac{\sin\frac{1}{z}}{\cos\frac{1}{z}}$ $\quad$ $\boxed{\text{has Poles at } \frac{2}{\pi}+\pi K}$

6.5

Monday, June 29, 2009
11:55 AM

$3K < 35$ Aaron Morrisett

3) $f(z) = \frac{4z-6}{z(2-z)} \simeq \frac{-6}{2} = \boxed{-3}$ $z_0 = 0$ $Res(f(z); z_0)$

6) $f(z) = \frac{e^{-z}}{(z-2)^2} = e^{-z} \cdot (z-2)^{-2}$ $Res(f(z), 2)$

$\frac{e^{-z}}{(z-2)(z-2)}$ $\lim_{z\to 2} \frac{d}{dz}\left[(z-2)^2 \cdot \frac{e^{-z}}{(z-2)^2}\right] = \lim_{z\to 2} -e^{-z}$

$= -e^{-2} = \boxed{\frac{-1}{e^2} = Res(f(z), 2)}$

9) $f(z) = \frac{1}{z^4 + z^3 - 2z^2} = \frac{1}{z^2(z^2+z-2)}$

$= \frac{1}{z^2(z-1)(z+2)}$

$Res(f(z); 1) = \frac{1}{4z^3+3z^2-4z}\Big|_1 = \boxed{\frac{1}{3}}$

$Res(f(z); -2) = \frac{1}{4z^3+3z^2-4z}\Big|_{-2} = \boxed{-\frac{1}{12}}$

$Res(f(z); 0)$:

$\lim_{z\to 0} \frac{d}{dz}\left[z^2(z^2(z-1)(z+2))^{-1}\right] = \lim_{z\to 0} \frac{d}{dz}\left[(z^2+z-2)^{-1}\right]$

$= \lim_{z\to 0}\left[-1(z^2+z-2)^{-2}(2z+1)\right] = \boxed{-\frac{1}{4}}$

12) $f(z) = \frac{2z-1}{(z-1)^4(z+3)}$

$Res(f(z), -3) = \frac{2(-3)-1}{(z-1)^4 + 4(z-1)^3(z+3)} = \boxed{\frac{-7}{256}}$

$Res(f(z), 1)$:

$\frac{1}{3!} \lim_{z\to 1} \frac{d^3}{d^3 z} \frac{2z-1}{z+3} = \frac{1}{3!} \lim_{z\to 1} \frac{42}{(z+3)^4} = \boxed{\frac{7}{256}}$

15) $f(z) = \sec z = \frac{1}{\cos z}$ $z = \frac{\pi}{2} + K\pi$

$Res(f(z), \frac{\pi}{2}+K\pi) = \frac{1}{-\sin z} = \frac{-1}{\sin(\pi/2 + K\pi)} = \boxed{-1, 1}$

18) $\oint_C \frac{z+1}{z^2(z-2i)}\,dz$

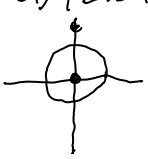
Residues: $Res(f(z), 2i) = \frac{z+1}{2z}\Big|_{2i} = \frac{2i+1}{4i} = -4i(2i+1) = 8-4i$

$Res(f(z), 0) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{z+1}{z-2i}\right) = \lim_{z\to 0} \frac{(z-2i)-(z+1)}{(z-2i)^2} = \frac{-1-2i}{-4}$

$= \frac{1}{4} + \frac{1}{2}i$

$\oint_C f(z)\,dz = 2\pi i \sum_{K=1}^{n} Res(f(z), z_K)$ // cauchy integral formula

a) $|z|=1$



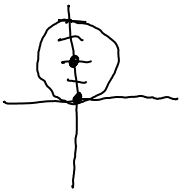
$\oint_{|z|=1} f(z)\,dz = 2\pi i\left[\frac{1}{4} + \frac{1}{2}i\right] = \boxed{-\pi + \frac{\pi}{2}i}$

$$\oint_{|z|=1} f(z)\,dz = 2\pi i\left[\tfrac{1}{4}+\tfrac{1}{2}i\right] = \boxed{-\pi + \tfrac{\pi}{2}i}$$

b) $|z-2i|=1$

$$\oint_{|z-2i|=1} f(z)\,dz = 2\pi i[8-4i] = \boxed{8\pi + 16\pi i}$$

c) $|z-2i|=4$



$$\oint_{|z-2i|=4} f(z)\,dz = 2\pi i\left[8-4i+\tfrac{1}{4}+\tfrac{1}{2}i\right] = 16\pi i + 8\pi + \tfrac{\pi}{2}i - \pi$$

$$= \boxed{7\pi + 16.5\pi i}$$

21) $\displaystyle\oint_{|z-3i|=3} \frac{1}{z^2+4z+13}\,dz$

$$\frac{-4\pm\sqrt{4^2-4(13)}}{2} = -2\pm 3i$$

$$= \oint_C \frac{1}{(z+2+3i)(z+2-3i)}\,dz$$

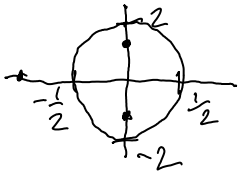
$$\mathrm{Res}(f(z);-2+3i) = \left.\frac{1}{2z+4}\right|_{-2+3i} = \frac{1}{-4+6i+4} = \boxed{-\tfrac{1}{6}i}$$

$$\mathrm{Res}(f(z);-2-3i) = \boxed{\tfrac{1}{6}i}$$

$$\oint_C f(z)\,dz = 2\pi i\left[-\tfrac{1}{6}i\right] = \boxed{\pi/3}$$

24) $\displaystyle\oint_C \frac{z}{(z+1)(z^2+1)}\,dz$ $\quad C: 16x^2+y^2=4$

$z = \pm\sqrt{-1} = \pm i$



$$\mathrm{Res}(f(z);\,i) = \lim_{z\to i}\frac{z}{(z+1)(z+i)} = \frac{i}{(i+1)2i} = \frac{i}{-2+2i}\left[\frac{-2-2i}{-2-2i}\right]$$

$$= \frac{2-2i}{8} = \frac{1}{4}-\frac{1}{4}i$$

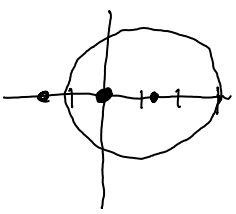
$$\mathrm{Res}(f(z);-i) = \lim_{z\to -i}\frac{z}{(z+1)(z-i)} = \frac{-i}{(-i+1)-2i} = \frac{-i}{-2-2i}\left[\frac{-2+2i}{-2+2i}\right]$$

$$= \frac{2+2i}{8} = \frac{1}{4}+\frac{1}{4}i$$

$$2\pi i\left[\tfrac{1}{4}-\tfrac{1}{4}i+\tfrac{1}{4}+\tfrac{1}{4}i\right] = 2\pi i\left(\tfrac{1}{2}\right) = \boxed{\pi i}$$

27) $\displaystyle\oint_{|z-1|=2} \frac{\tan z}{z}\,dz = \oint \frac{\sin z}{z\cos z}\,dz$

Points are $0, \pi/2$



$$\mathrm{Res}(f(z);0) = \frac{\sin 0}{\cos 0} = 0$$

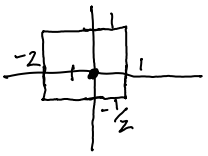
$$\mathrm{Res}(f(z);\pi/2) = \frac{1}{\cos\frac{\pi}{2}-\frac{\pi}{2}\sin\frac{\pi}{2}} = \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi}$$

$$2\pi i\left[0-\tfrac{2}{\pi}\right] = -4i$$

30) $\displaystyle\oint \frac{2z-1}{z^2(z^3+1)}\,dz$ $\quad z=0$ and $z=(-1)^{1/3}$

30) $\oint_C \frac{2z-1}{z^2(z^3+1)}\,dz$ $\qquad z=0$ and $z=(-1)^{1/3}$

$\theta=\pi$



$w_0=\cos\frac{\pi}{3}+i\sin\pi/3=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$w_1=\cos\pi+i\sin\pi=-1$

$w_2=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}=\frac{1}{2}-\frac{\sqrt{3}}{2}i$

$$\oint_C \frac{2z-1}{z^2(z-(\frac{1}{2}+\frac{\sqrt{3}}{2}i))(z+1)(z-(\frac{1}{2}-\frac{\sqrt{3}}{2}i))}\,dz$$

inside the countor is $z=0,\ z=-1,\ z=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$\text{Res}(f(z);0) \Rightarrow \lim_{z\to 0}\frac{d}{dz}\frac{2z-1}{z^3+1}=\lim_{z\to 0}\frac{(z^3+1)(2)-(2z-1)3z^2}{(z^3+1)^2}=\underline{2}$

$\text{Res}(f(z);-1) \Rightarrow \lim_{z\to -1}\frac{2z-1}{z^2(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i)(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i)}=\frac{-3}{3}=\underline{-1}$

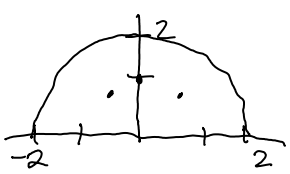
$\text{Res}(f(z);\frac{1}{2}+\frac{\sqrt{3}}{2}i) \Rightarrow \lim_{z\to\frac{1}{2}+\frac{\sqrt{3}}{2}i}\frac{2z-1}{z^2(z+1)(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i)}$

$=\frac{\sqrt{3}i}{-\frac{3}{2}-\frac{3\sqrt{3}}{2}i}\left[\frac{-\frac{3}{2}+\frac{3\sqrt{3}}{2}i}{-\frac{3}{2}+\frac{3\sqrt{3}}{2}i}\right]=-\frac{1}{2}-\frac{\sqrt{3}}{6}i$

$2\pi i\left[1-\frac{1}{2}-\frac{\sqrt{3}}{6}i\right]=2\pi i\left[-\frac{1}{2}-\frac{\sqrt{3}}{6}i\right]=\boxed{\frac{\pi\sqrt{3}}{3}-\pi i}$

33) $\oint_C \frac{1}{z^6+1}\,dz$ $\qquad C: y=\sqrt{4-x^2} \Rightarrow y^2+x^2=4$

$(-1)^{1/6}$ $\qquad \theta=\pi$



$w_0=\cos\pi/6+i\sin\pi/6=\sqrt{3}/2+\frac{1}{2}i$ ✓
$w_1=\cos\pi/2+i\sin\pi/2=0+i$ ✓
$w_2=\cos 5\pi/6+i\sin 5\pi/6=-\sqrt{3}/2+\frac{1}{2}i$ ✓
are in the countor

$w_3=\cos 7\pi/6+i\sin 7\pi/6=-\sqrt{3}/2-\frac{1}{2}i$ ✗
$w_4=\cos 9\pi/6+i\sin 9\pi/6=-i$ ✗
$w_5=\cos 11\pi/3+i\sin 11\pi/3=\sqrt{3}/2-\frac{1}{2}i$ ✗

$\text{Res}(f(z);\frac{\sqrt{3}}{2}+\frac{1}{2}i)=\lim_{z\to\frac{\sqrt{3}}{2}+\frac{1}{2}i}\frac{1}{(z-i)(z+\frac{\sqrt{3}}{2}-\frac{1}{2}i)(z+\frac{\sqrt{3}}{2}+\frac{1}{2}i)(z+i)(z-\frac{\sqrt{3}}{2}+\frac{1}{2}i)}$

$=\frac{1}{-3\sqrt{3}+3i}=\frac{-\sqrt{3}}{12}-\frac{1}{12}i$

$\text{Res}(f(z);i)=\lim_{z\to i}\frac{1}{(z-\sqrt{3}/2-\frac{1}{2}i)(z+\sqrt{3}/2-\frac{1}{2}i)(z+\sqrt{3}/2+\frac{1}{2}i)(z+i)(z-\sqrt{3}/2+\frac{1}{2}i)}$

$=\frac{1}{6i}=-\frac{1}{6}i$

$\text{Res}(f(z);-\sqrt{3}/2+\frac{1}{2}i)=\lim_{z\to-\sqrt{3}/2+\frac{1}{2}i}\frac{1}{(z-\sqrt{3}/2-\frac{1}{2}i)(z-i)(z+\sqrt{3}/2+\frac{1}{2}i)(z+i)(z-\sqrt{3}/2+\frac{1}{2}i)}$

$=\frac{1}{3\sqrt{3}+3i}=\frac{\sqrt{3}}{12}-\frac{1}{12}i$

$2\pi i\left[\cancel{\frac{-\sqrt{3}}{12}}-\frac{1}{12}i-\frac{1}{6}i+\cancel{\frac{\sqrt{3}}{12}}-\frac{1}{12}i\right]=2\pi i\left[-\frac{1}{3}i\right]=\boxed{\frac{2\pi}{3}}$

6.6
Tuesday, June 30, 2009
9:20 AM

$d\theta = \frac{dz}{iz}$ $\quad 3K \le 24$ $\quad$ Aaron Morrisett

3) $\int_0^{2\pi} \frac{\cos\theta}{3+\sin\theta}\, d\theta = \int_C \frac{\frac{1}{2}(z+z^{-1})}{3+\frac{1}{2i}(z-z^{-1})} \frac{dz}{iz} = \int_C \frac{-i\frac{1}{2}(z+z^{-1})}{3z+\frac{z^2}{2i}-\frac{1}{2i}}\, dz$

$= \int_C \frac{-i\frac{1}{2}(z^2+1)}{z(\frac{z^2}{2i}+3z-\frac{1}{2i})}\, dz = \int_C \frac{(z^2+1)}{2z(z^2+6iz-1)}\, dz$

$z_0 = 0$
$z_1 = -(2\sqrt{2}+3)i$
$z_2 = (2\sqrt{2}-3)i$

$(z^2+6iz-1) = (z-(2\sqrt{2}-3)i)(z+(2\sqrt{2}+3)i)$

$= \int_C \frac{(z^2+1)}{2z(z+(2\sqrt{2}+3)i)(z-(2\sqrt{2}-3)i)}\, dz$

$Res(f(z);0) \Rightarrow$
$\lim_{z\to 0} \frac{z^2+1}{(z+(2\sqrt{2}+3)i)(z-(2\sqrt{2}-3)i)} = -1$

$Res(f(z);z_2) \Rightarrow$
$\lim_{z\to(2\sqrt{2}-3)i} \frac{z^2+1}{z(z+(2\sqrt{2}+3)i)} = 1$

$2\pi i[-1+1] = \boxed{0}$

6) $\int_0^{\pi} \frac{1}{1+\sin^2\theta}\, d\theta = \int_C \frac{1}{1-\frac{(z^2+1)^2}{4z^2}} \frac{dz}{iz} = \int_C \frac{4iz}{z^4-6z^2+1}\, dz$

$\boxed{\sin^2\theta = \frac{-(z^2+1)^2}{4z^2}}$ $= \int_C \frac{4iz}{(z^2-2z-1)(z^2+2z-1)}\, dz$

$\frac{2\pm\sqrt{4-4(-1)}}{2} = \frac{2\pm\sqrt{8}}{2} = 1\pm\sqrt{2}$

$\frac{-2\pm\sqrt{4-4(-1)}}{2} = \frac{-2\pm\sqrt{8}}{2} = -1\pm\sqrt{2}$

$z = -1+\sqrt{2}$ in countor

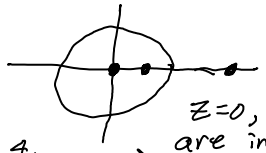
$Res(f(z), -1+\sqrt{2}) = \frac{4iz}{4z^3-12z}\Big|_{z=-1+\sqrt{2}} = \frac{-\sqrt{2}}{4}i$

$2\pi i\left[\frac{-\sqrt{2}}{4i}\right] = \boxed{\frac{\pi\sqrt{2}}{2}}$

9) $\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta}$ $\quad \cos 2\theta = \frac{1}{2}(z^2+z^{-2})$

$\int_C \frac{\frac{1}{2}(z^2+z^{-2})}{5-4(\frac{z+z^{-1}}{2})} \frac{dz}{iz} = \int_C \frac{i(z^4+1)}{2z^2(2z^2-5z+2)}\, dz$

$= \int_C \frac{i(z^4+1)}{2z^2(z-2)(2z-1)}\, dz$



$z=0,\ z=\frac{1}{2}$ are inside the circle.

$Res(f(z),0)$:
$\lim_{z\to 0} \frac{d}{dz}\left(z^2 \cdot \frac{i(z^4+1)}{2z^2(z-2)(2z-1)}\right) = \lim_{z\to 0} \frac{d}{dz}\left(i \cdot \frac{z^4+1}{2(2z^2-5z+2)}\right)$

$= \lim_{z\to 0}\left(i\,\frac{4z^3(2z^2-5z+2)-(4z-5)(z^4+1)}{2(2z^2-5z+2)^2}\right) = \frac{5}{8}i$

$Res(f(z),\frac{1}{2})$:
$\lim_{z\to\frac{1}{2}} \frac{i(z^4+1)}{4z^2(z-2)} = \frac{i((\frac{1}{2})^4+1)}{4(\frac{1}{2})^2(\frac{1}{2}-2)} = -\frac{17}{24}i$

$$\int_0^{2\pi}\frac{\cos 2\theta}{5-4\cos\theta} = 2\pi i\left[\frac{5}{8}i-\frac{17}{24}i\right] = 2\pi i\left[-\frac{i}{12}\right] = \boxed{\frac{\pi}{6}}$$

12) $\displaystyle\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}\,d\theta \qquad \cos 3\theta = \frac{1}{2}(z^3+z^{-3})$

$$\int_C \frac{\frac{1}{2}(z^3-z^{-3})}{5-4\left(\frac{z+z^{-1}}{2}\right)}\frac{dz}{iz} = \int_C \frac{i(z^6+1)}{2z^3(2z^2-5z+2)}dz = \int_C \frac{i(z^6+1)}{2z^3(z-2)(2z-1)}dz$$

$z=0$ and $z=\frac{1}{2}$ are inside the countor.

$\mathrm{Res}(f(z),0)$:

$$\lim_{z\to 0}\frac{1}{2}\frac{d^2}{d^2z}\left(\frac{i(z^6+1)}{2\cdot(z-2)(2z-1)}\right) = \frac{21}{16}i$$

$\mathrm{Res}(f(z),\frac{1}{2})$: $\quad (2z^4-4z^3)(2z-1) = 4z^5-8z^4-2z^4+4z^3 = 4z^5-10z^4+4z^3$

$$\lim_{z\to\frac{1}{2}}\left(\frac{i(z^6+1)}{20z^3-40z^3+12z^2}\right) = -\frac{65}{48}$$

$$\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta = 2\pi i\left[\frac{21}{16}i-\frac{65}{48}i\right] = \boxed{\frac{\pi}{12}}$$

15) $\displaystyle\int_{-\infty}^{\infty}\frac{1}{x^2-2x+2}dx \Rightarrow f(z) = \frac{1}{z^2-2z+2}dz = \frac{1}{(z-1+i)(z-1-i)}dz$

$$\frac{2\pm\sqrt{4-4(2)}}{2} = \frac{2\pm\sqrt{-4}}{2} = 1\pm i$$

$$\mathrm{Res}(f(z),1+i) = \frac{1}{2z-2}\Big|_{z=1+i} = \frac{1}{2+2i-2} = \frac{1}{2i}$$

$$2\pi i\left(\frac{1}{2i}\right) = \boxed{\pi}$$

18) $\displaystyle\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)^2}dx \Rightarrow f(z) = \frac{z^2}{(z^2+1)^2}dz = \frac{z^2}{[(z+i)(z-i)]^2}dz$

$z^2+1=0$

$z^2=-1 \Rightarrow z=i$

$\mathrm{Res}(f(z);i)$

$$\lim_{z\to i}\frac{d}{dz}\left(\frac{z^2}{(z+i)^2}\right) = \frac{2z(z+i)^{\cancel{2}}-z^2(2\cancel{(z+i)})}{(z+i)^{\cancel{4}3}} = \frac{1}{4i}$$

$$2\pi i\left(\frac{1}{4i}\right) = \boxed{\frac{\pi}{2}}$$

21) $\displaystyle\int_{-\infty}^{\infty}\frac{2x^2-1}{x^4+5x^2+4}dx \Rightarrow f(z) = \frac{2z^2-1}{z^4+5z^2+4}dz = \frac{2z^2-1}{(z^2+1)(z^2+4)}dz$

$z^2=1 \Rightarrow z=\pm i \qquad z^2=-4 \Rightarrow z=\pm 2i$

$$\mathrm{Res}(f(z),i) = \lim_{z\to i}\frac{2z^2-1}{(z+i)(z+2i)(z-2i)} = \frac{1}{2}i$$

$$\mathrm{Res}(f(z),2i) = \lim_{z\to 2i}\frac{2z^2-1}{(z+i)(z-i)(z+2i)} = -\frac{3}{4}i$$

$$2\pi i\left(\frac{1}{2}i-\frac{3}{4}i\right) = 2\pi i\left(-\frac{1}{4}i\right) = \boxed{\frac{-\pi}{2}}$$

24) $\displaystyle\int_0^{\infty}\frac{1}{x^6+1}dx \Rightarrow f(z) = \frac{1}{z^6+1}dz$

$z^6=-1 \Rightarrow z=(-1)^{1/6} \qquad \theta=\pi \quad r=1$

Poles in countor $z=\frac{\sqrt{3}}{2}+\frac{1}{2}i$, $z=i$, $z=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$

$$\operatorname{Res}\left(f(z), \frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = \lim_{z \to \frac{\sqrt{3}}{2}+\frac{1}{2}i} \frac{1}{6z^5} = -\frac{\sqrt{3}}{12} - \frac{1}{12}i$$

$$\operatorname{Res}(f(z), i) = \lim_{z \to i} \frac{1}{6z^5} = -\frac{1}{6}i$$

$$\operatorname{Res}\left(f(z), -\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = \lim_{z \to -\frac{\sqrt{3}}{2}+\frac{1}{2}i} \frac{1}{6z^5} = \frac{\sqrt{3}}{12} - \frac{1}{12}i$$

$$2\pi i\left[-\frac{2}{12}i - \frac{1}{6}i\right] = \boxed{\frac{2\pi}{3}}$$