7. Structural Equation Models

Structural equation models (SEM) is a system where causal relationships are modeled between variables.

Variables can be directly observed or latent or a mixture of these.

Path analysis is a special case of SEM.

Notations

The Greek Alphapet			Typical Model Notation	
α	A	alpha	x, y	observed vaiables
β	B	beta	ξ, η	latent variables
γ	Γ	gamma	ζ, δ, ϵ	error variables
δ	Δ	delta		
ϵ	E	epsilon	Λ_x, Λ_y	factor loadings
ζ	Z	zeta	B, Γ	structural parameters
η	H	eta	Φ, Ψ	covariance matrices
θ	Θ	theta	Θ_δ, Θ_ϵ	error covariance
ι	I	iota		matrices
κ	K	kappa	$\hat{\Lambda}$, etc.	estimate of Λ_x, etc.
λ	Λ	lambda		
μ	M	mu	Greek letters	population parameters,
ν	N	nu		latent random variables
ξ	Ξ	xi, ksi	Roman letters	observed random
o	O	omicron		variables
π	Π	pi		
ρ	P	rho		
σ	Σ	sigma		
τ	T	tau		
υ	Υ	upsilon		
ϕ	Φ	phi		
χ	X	chi		
	Ψ	psi		
ω	Ω	omega		

In the most general form with latent variables, the SEM system consists of:

(a) *structural equations* among latent variables: dependent (effect, response, endogenous) variables $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_k)'$ and independent (cause, background, exogenous) variables $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)'$.

$$\boldsymbol{\eta} = B\boldsymbol{\eta} + \Gamma\boldsymbol{\xi} + \boldsymbol{\zeta}, \tag{1}$$

(b) the *measurement models*

$$\mathbf{y} = \Lambda_y \boldsymbol{\eta} + \boldsymbol{\epsilon} \tag{2}$$

$$\mathbf{x} = \Lambda_x \boldsymbol{\xi} + \boldsymbol{\delta}. \tag{3}$$

$\mathbf{x}$ and $\mathbf{y}$ are observed variables, through which $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ (respectively) are measured (or evaluated).

Remark 7.1. Testing of the latent structural equation, i.e., the initial theory, is likely meaningless unless first it is established that the measurement model holds.

Thus, in the latent structural equation models the measurement model must be specified and tested first, after which one can proceed to test structural equation.

Remark 7.2: The covariance matrix of $\boldsymbol{\eta}$ in the measurement model (2) is the same as the covariance matrix of $\boldsymbol{\zeta}$, and is denoted by Ψ. This is due to the fact that if we purely deal with the measurement model, we can set in (1) $B = 0$ and $\Gamma = 0$, implying $\boldsymbol{\eta} = \boldsymbol{\zeta}$ and $\mathbb{C}\text{ov}[\boldsymbol{\eta}] = \mathbb{C}\text{ov}[\boldsymbol{\zeta}] = \Psi$.

Generally, however, with $I - B$ non-singular,

$$\text{(4)} \qquad \mathbb{C}\text{ov}[\boldsymbol{\eta}] = (I - B)^{-1}(\Gamma\Phi\Gamma' + \Psi)(I - B)'^{-1}$$

and

$$\text{(5)} \qquad \mathbb{C}\text{ov}[\boldsymbol{\zeta}] = \Psi.$$

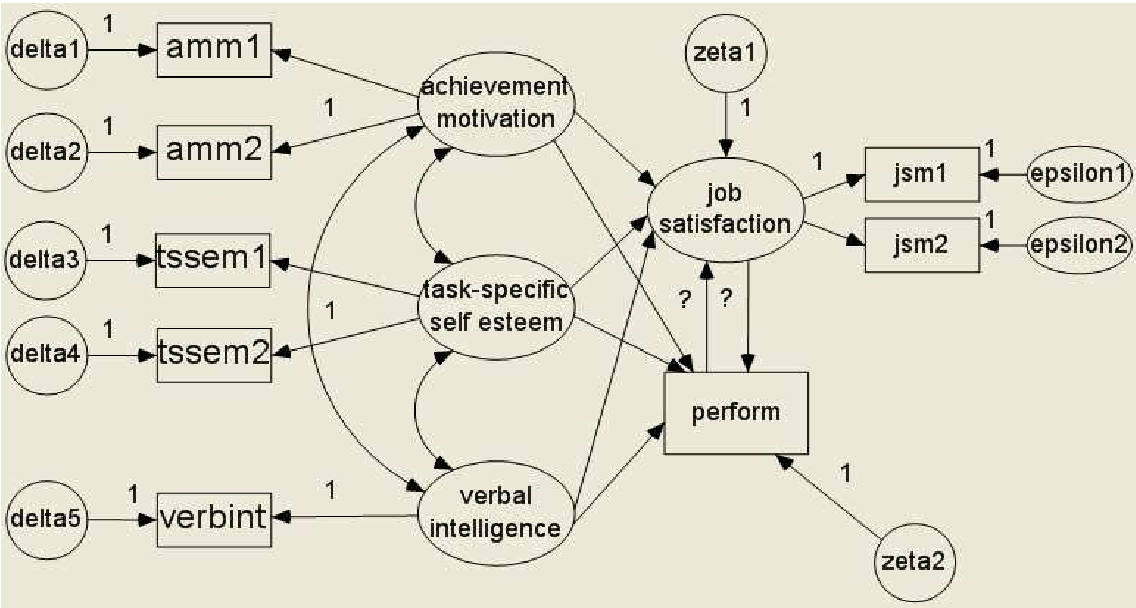
Remark 7.3: (a) Any latent variable that is predicted by other latent (or observed) variable in the structural equation is *latent dependent variable* or *latent endogenous variable*. Thus, at least one arrow must lead to it.

(b) Any latent variable that has not an arrow leading to it is *latent independent variable* or *latent endogenous variable*

(c) A laten variable can be *mediating* if an arrow leads to it from another latent variable and an arrow from it leads to some other latent variable. For example, η_1 is mediating in $\xi \to \eta_1 \to \eta_2$. Both η_1 and η_2 are endogenous.

Example 7.1: (Example 3.1) In the original paper the basic interest was to investigate the question does job satisfaction creates better performance or is it vice versa.

So model setup is:



Verbal intelligence is measured only by one indicator.

Because it is an explanatory variable, the associated error cannot be estimated.

The possibilities are to consider as if the measurement had a 100% reliability.

This is considered fallible and in this example it assumed that a better choice is to approximate the reliability being 0.85. This implies an error variance of

$$0.15 \times \widehat{\mathbb{V}\mathrm{ar}[x_3]} = 0.15 \times (3.65)^2 = 1.998.$$

Next we test a series of hypotheses:

`job satisfaction` and `performance` are correlated because:

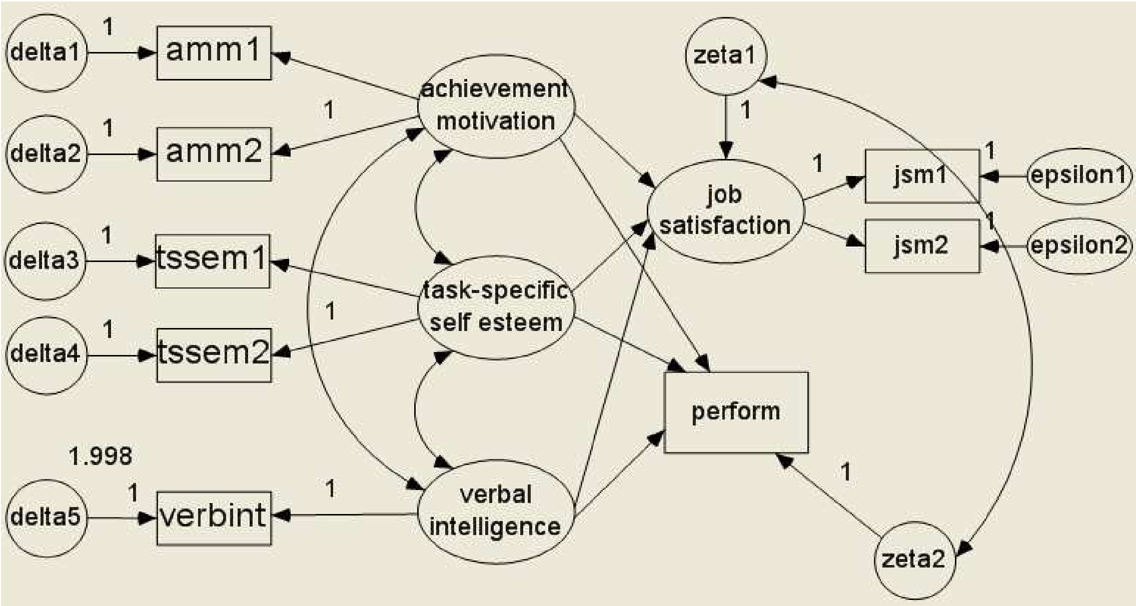
(a) `achievement motivation`, `task-specific self esteem`, and `verbal intelligence` are common causes of them (i.e., the correlation is spurious)

(b1) `job satisfaction` influences `performance`

(b2) `performance` influences `job satisfaction`

(c) `job satisfaction` and `performance` influence each other reciprocally.

In order to investigate hypothesis (a), we estimate model:

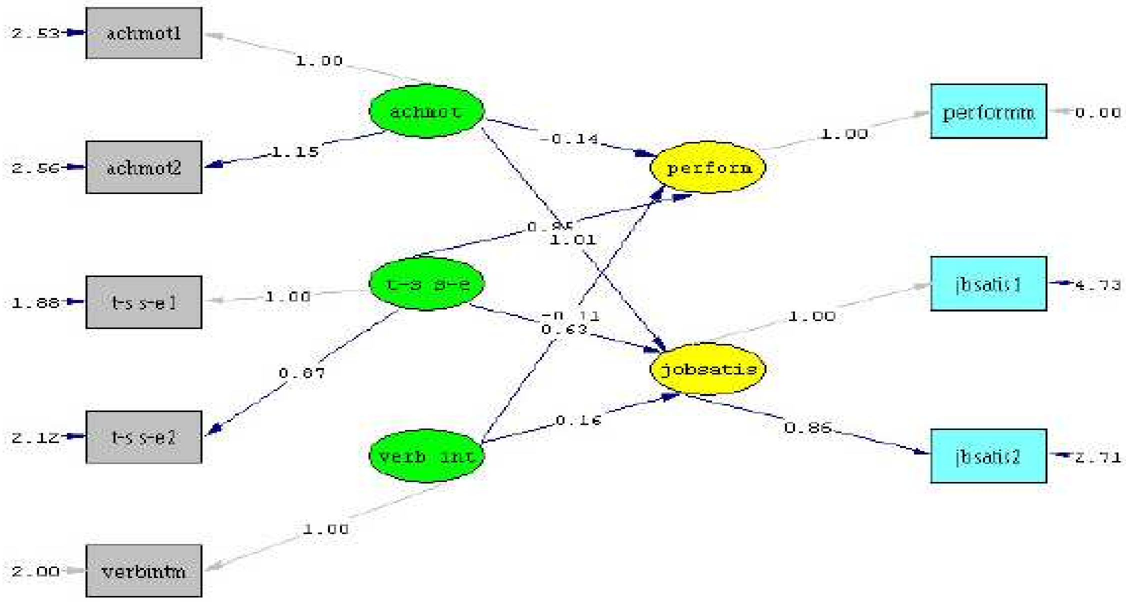


and test the hypothesis

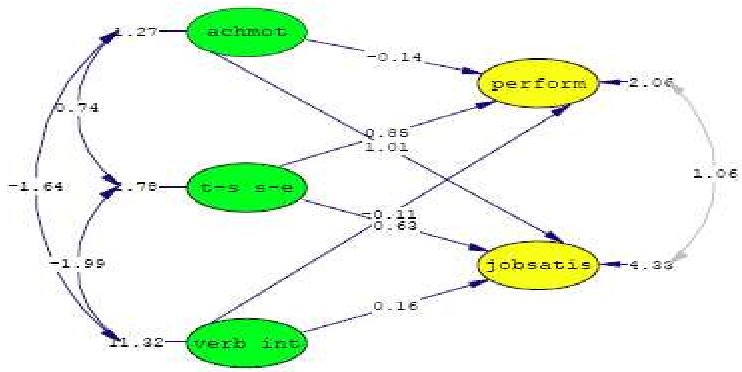
$$H_a : {}_{21} = 0. \tag{6}$$

Remark 7.4: ${}_{21} = \mathbb{C}\text{ov}[\zeta_2, \zeta_1]$ is the covariance between η_1 (`job satisfaction`) and η_2 (`perform`) not explained by the background factors.

Estimation yields (LISREL output)



The structural equation part of which is



$\chi^2 = 10.73$, $df = 12$, $p = 0.552$, $\mathtt{RMSEA} = 0.000$.

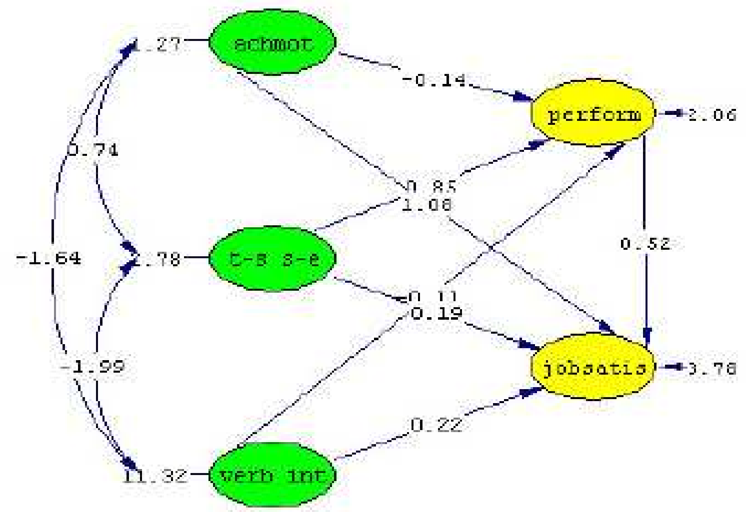
However, $\hat{\ }_{2,1} = 1.06$, $t = 2.23$, and $p = 0.023$, which rejects H_a at 5% level.

Thus the correlation between `jobsatis` (η_1) and `perform` (η_2) cannot be considered spurious.

Next we proceed to test hypothesis (b1) which implies a statistical hypothesis

$$H_{b1} : \beta_{12} = 0. \tag{7}$$

In the previous model, we fix $_{21} = 0$ and estimate free β_{12} (causal link `perform` $(\eta_2) \rightarrow$ `jobsatis` (η_1)).



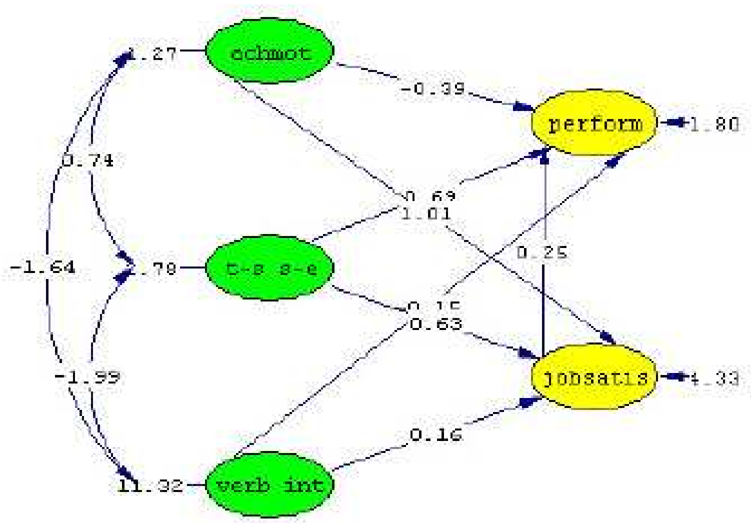
$\chi^2 = 10.73$, $df = 12$, $p = 0.552$, $RMSEA = 0.000$.

$\widehat{\beta}_{12} = 0.52$, $t = 2.47$, $p = 0.014$

Estimating the other way round `job satisfaction` $\rightarrow$ `perform` in order to test

$$H_{b2} : \beta_{21} = 0. \tag{8}$$

yields



$\chi^2 = 10.73$, $df = 12$, $p = 0.552$, $RMSEA = 0.000$.

$\widehat{\beta}_{21} = 0.25$, $t = 2.39$, $p = 0.017$.

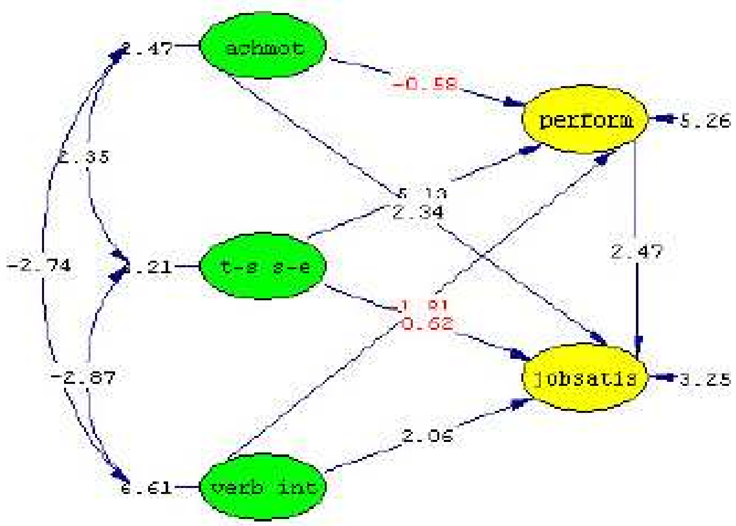
In both cases we the β-estimates are statistically significant, implying that both of the hypotheses (7) and (8) are rejected.

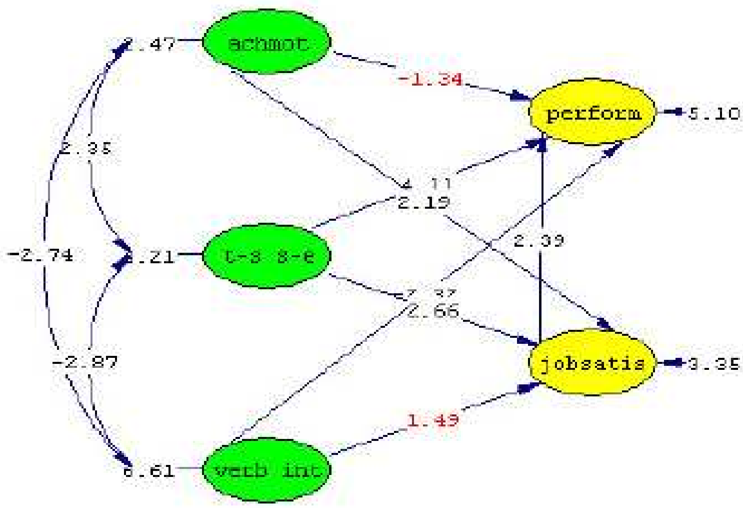
Finally we test hypothesis (c), i.e. whether the relationship is reciprocal.

Unfortunately, adding the link `perform` $\rightarrow$ `jobsatis` makes the model unidentified (the estimated cannot be uniquely solved).

We need to impose additional restrictions (delete some causal links).

Considering the models corresponding to (7) and (8) and checking the t-values

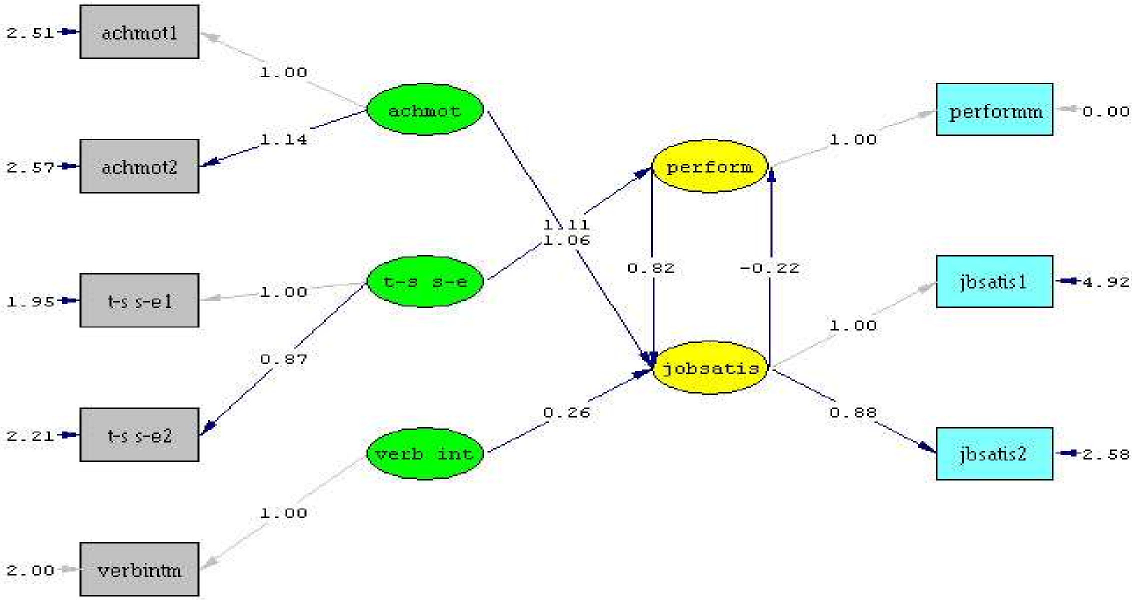




the coefficient estimates indicated by red t-values (**achmot** $\rightarrow$ **perform**, **verb int** $\rightarrow$ **jobsatis**, and **t-s s-e** $\rightarrow$ **jobsatis**) are not statistically significant.

Thus, these links are good candidates to be deleted.

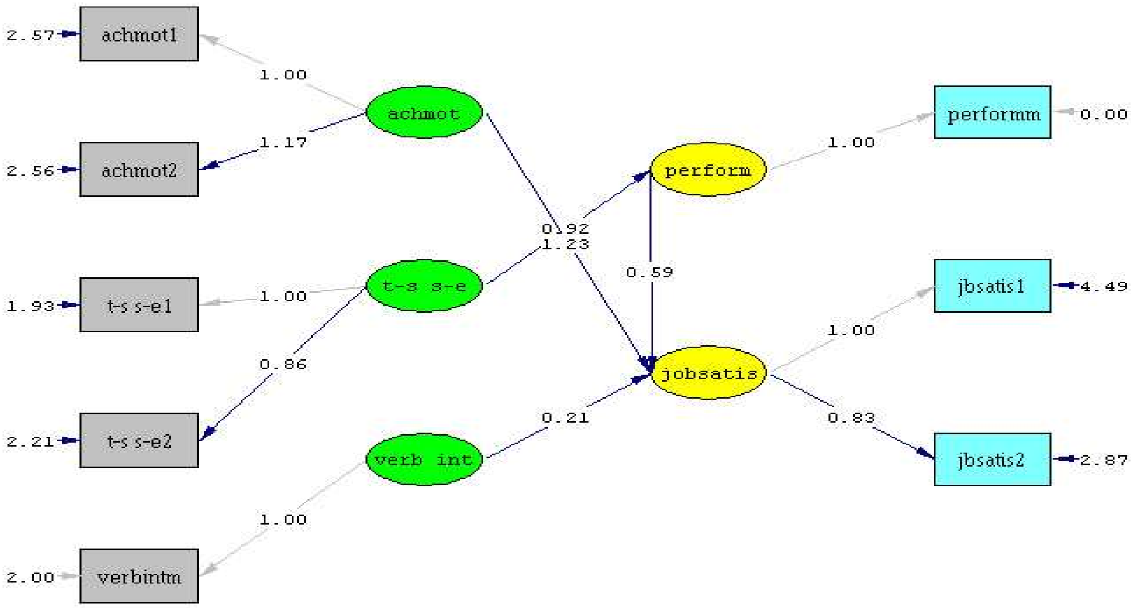
With these restrictions the structural equation part of the estimated model is



$\chi^2 = 13.07$, $df = 14$, $p = 0.521$, $RMSEA = 0.000$.

Now $\widehat{\beta}_{12} = 0.82$ ($\texttt{perform} \rightarrow \texttt{jobsatis}$) is statistically significant with $t = 3.89$ while $\widehat{\beta}_{21} = -0.22$ with $t = -1.36$ is not statistically significant.

Finally we estimate the model by dropping the insignificant causal link `jobsatis` $\rightarrow$ `perform` to obtain



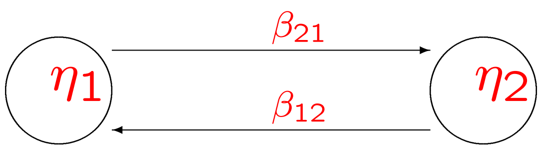
$\chi^2 = 15.31$, $df = 15$, $p = 0.429$, $RMSEA = 0.013$.

Thus, the conclusion on the basis of this empirical analysis is that the causal link between `job satisfaction` and `performance` seems to be indeed one way rather than reciprocal.

Stability of the model

In a non-recursive model with feedback, the total effect is in fact a result of an infinite loop of partial effect.

Consider the following simple case



Keeping other things constant, a one unit change in η_1 causes a change of β_{21}in η_2, which cause a $\beta_{12}\beta_{21}$ change in η_1.

Thus, a one unit change in η_1 causes a change in η_2, which due to the link causes a change back in η_1 of magnitude $\beta_{21}\beta_{12}$.

This again causes an additional change in η_2 of magnitude $\beta_{21}(\beta_{12}\beta_{21})$.

The total effect of unit change in η_1 on η_2 after one loop is

$$\beta_{21} + \beta_{21}^2\beta_{12}. \tag{9}$$

The limit is

$$\sum_{j=1}^{\infty} \beta_{21}^{j}\beta_{12}^{j-1} = \beta_{21} \sum_{k=0}^{\infty} (\beta_{12}\beta_{21})^k. \tag{10}$$

If $|\beta_1\beta_2| < 1$ then (10) is a converging geometric series with end result

$$(11)\quad \beta_{21}\sum_{k=0}^{\infty}(\beta_{12}\beta_{21})^k = \frac{\beta_{21}}{1-\beta_{12}\beta_{21}}.$$

In the same manner, a unit change in η_2 causes in η_1 a total change of

$$(12)\quad \text{total effect } \eta_2 \to \eta_1 = \frac{\beta_{12}}{1-\beta_{12}\beta_{21}}.$$

Again in the same manner, the total effect of η_1 or η_2 on itself can be calculated to be

$$(13)\quad \text{total effect } \eta_i \to \eta_i = \frac{\beta_{12}\beta_{21}}{1-\beta_{12}\beta_{21}},$$

$i = 1, 2$.

If these exist, we say that *the system is stable*.

LISREL produces a stability index (which is the largest eigen value of the matrix of β-coefficients).

The index should fall between -1 and 1 in order the model to be stable.

If the index is (on absolute value) ≥ 1, it implies that the model is wrong or the sample is too small for reliable estimates.

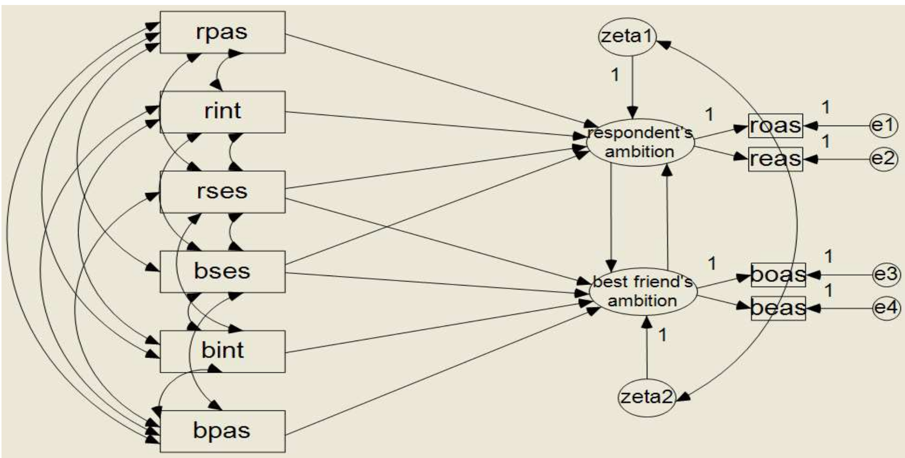
Example 7.2: Peer influences on ambition. Source: Duncan, O.D., A.O. Haller, and A. Portes (1968). Peer influences on aspirateions: A reinterpretation. *American Journal of Sociology*, 74, 119–137.

`rpas`: respondent's parental aspiration
`rint`: respondent's intelligence
`rses`: respondent's socioeconomic status
`bses`: best friend's socioeconomic status
`bint`: best friend's intelligence
`bpas`: best friend's parental aspiration

`roas`: respondent's occupational aspiration
`reas`: respondent's educational aspiration
`boas`: best friend's occupational aspiration
`beas`: best friend's educational aspiration

`rambition`: respondent's ambition (η_1)
`bambition`: best friend's ambition (η_2).

The model:



Correlation matrix:

	rint	rpas	rses	roas	reas	bint	bpas	bses	boas	beas
rint	1.0000									
rpas	0.1839	1.0000								
rses	0.2220	0.0489	1.0000							
roas	0.4105	0.2137	0.3240	1.0000						
reas	0.4043	0.2742	0.4047	0.6247	1.0000					
Best friend										
bint	0.3355	0.0782	0.2302	0.2995	0.2863	1.0000				
bpas	0.1021	0.1147	0.0931	0.0760	0.0702	0.2087	1.0000			
bses	0.1861	0.0186	0.2707	0.2930	0.2407	0.2950	-0.0438	1.0000		
boas	0.2598	0.0839	0.2786	0.4216	0.3275	0.5007	0.1988	0.3607	1.0000	
beas	0.2903	0.1124	0.3054	0.3269	0.3669	0.5191	0.2784	0.4105	0.6404	1.0000

Estimation results:

```
Notes for Group/Model
Stability index for the following variables is .043
bamb
ramb

Chi-square = 26.697
Degrees of freedom = 15
Probability level = .031

Regression Weights:
               Estimate S.E.  C.R.   P
ramb <--- rpas     .161 .039 4.156  ***
ramb <--- rint     .250 .044 5.676  ***
ramb <--- rses     .218 .044 4.942  ***
bamb <--- rses     .058 .048 1.196 .232
ramb <--- bses     .072 .050 1.445 .148
bamb <--- bses     .213 .042 5.104  ***
bamb <--- bint     .325 .044 7.456  ***
bamb <--- bpas     .148 .036 4.070  ***
roas <--- ramb    1.000
reas <--- ramb    1.063 .090 11.789 ***
boas <--- bamb    1.000
beas <--- bamb    1.076 .081 13.229 ***
ramb <--- bamb     .198 .102 1.937 .053
bamb <--- ramb     .219 .111 1.968 .049
```

Covariances:

	Estimate	S.E.	C.R.	P
rpas <--> rint	.183	.056	3.276	.001
rpas <--> rses	.049	.055	.885	.376
rpas <--> bses	.019	.055	.337	.736
rpas <--> bint	.078	.055	1.412	.158
rpas <--> bpas	.114	.055	2.064	.039
rint <--> rses	.221	.056	3.925	***
rint <--> bses	.186	.056	3.314	***
rint <--> bint	.334	.058	5.761	***
rint <--> bpas	.102	.055	1.840	.066
rses <--> bses	.270	.057	4.732	***
rses <--> bint	.230	.056	4.063	***
rses <--> bpas	.093	.055	1.679	.093
bses <--> bint	.294	.057	5.124	***
bses <--> bpas	-.044	.055	-.792	.428
bint <--> bpas	.208	.056	3.700	***
zeta2 <--> zeta1	-.021	.047	-.442	.659

The model does not fit well with p-value 0.034.

The model can be improved, not by releasing additional parameters, but imposing additional restrictions.

Testing $\mathbb{C}\mathrm{ov}[\mathtt{zeta}_1, \mathtt{zeta}_2] = \ {}_{12} = 0$, gives

$$\chi^2 = 26.893 - 26.697 = 0.186$$

with one degree of freedom, and a test of $\beta_{12} = \beta_{21}$, given $\ {}_{12} = 0$, yields

$$\chi^2 = 26.899 - 26.893 = 0.006,$$

again with one degree of freedom. Obviously these hypothese cannot be rejected.

The overall goodness-of-fit, given $\ {}_{21} = 0$ and $\beta_{12} = \beta_{21} = 0$, is $\chi^2 = 26.89$ with 17 degrees of freedom and p-value 0.060 (borderline accepted).

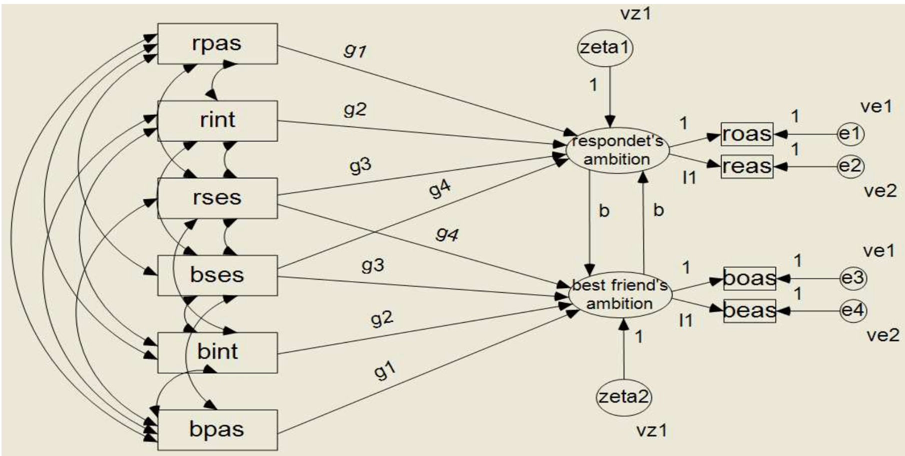
To proceed the analysis, from the path coefficient estimates it is seen that the corresponding estimates of the respondent and the best friend are close to each other.

We next test whether the model is completely symmetric between the respondent and the best friend.

This is done by equating the estimates of the corresponding bath coefficients of the respondent and the best friend.

Furthermore, the corresponding variances are equated.

With these restriction the model looks the following



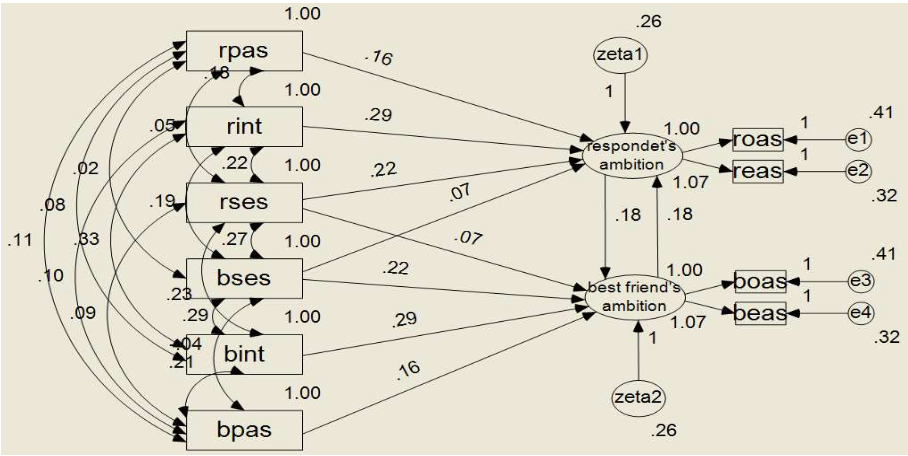
The estimation results are:

```
Chi-square = 30.757
Degrees of freedom = 25
Probability level = .197

Stability index for the following variables is .032
bamb
ramb

Regression Weights:
                Estimate S.E.   C.R.  P    Label
ramb <--- rpas      .158 .026  5.985  ***   g1
ramb <--- rint      .292 .029 10.021  ***   g2
ramb <--- rses      .222 .029  7.600  ***   g3
bamb <--- rses      .073 .030  2.462 .014   g4
ramb <--- bses      .073 .030  2.462 .014   g4
bamb <--- bses      .222 .029  7.600  ***   g3
bamb <--- bint      .292 .029 10.021  ***   g2
bamb <--- bpas      .158 .026  5.985  ***   g1
roas <--- ramb     1.000
reas <--- ramb     1.067 .060 17.677  ***   l1
boas <--- bamb     1.000
beas <--- bamb     1.067 .060 17.677  ***   l1
ramb <--- bamb      .179 .039  4.584  ***   b
bamb <--- ramb      .179 .039  4.584  ***   b
```

The overall chi-square for this model is 30.76 with 25 degrees of freedom and a p-value of 0.197.

Thus, this model is more parsimonious and has a better fit than the other models.

The model diagram with (non-standardized) estimates



Direct, Indirect, and Total Effect

Considering the general SEM system given in (1)–(3), the direct, indirect, and total effect are as follows

	$\boldsymbol{\xi} \rightarrow \boldsymbol{\eta}$	$\boldsymbol{\eta} \rightarrow \boldsymbol{\eta}$
Direct	Γ	B
Indirect	$(I - B)^{-1}\Gamma - \Gamma$	$(I - B)^{-1} - I - B$
Total	$(I - B)^{-1}\Gamma$	$(I - B)^{-1} - I$
	$\boldsymbol{\xi} \rightarrow \mathbf{y}$	$\boldsymbol{\eta} \rightarrow \mathbf{y}$
Direct	$\mathbf{0}$	Λ_y
Indirect	$\Lambda_y(I - B)^{-1}\Gamma$	$\Lambda_y(I - B)^{-1} - \Lambda_y$
Total	$\Lambda_y(I - B)^{-1}\Gamma$	$\Lambda_y(I - B)^{-1}$

Example 7.3: In the previous example:

Total Effects

	bpas	bint	bses	rses	rint	rpas	bamb	ramb
bamb	.163	.302	.243	.117	.054	.029	.033	.185
ramb	.029	.054	.117	.243	.302	.163	.185	.033
beas	.174	.322	.259	.124	.058	.031	1.102	.198
boas	.163	.302	.243	.117	.054	.029	1.033	.185
reas	.031	.058	.124	.259	.322	.174	.198	1.102
roas	.029	.054	.117	.243	.302	.163	.185	1.033

Direct Effects

	bpas	bint	bses	rses	rint	rpas	bamb	ramb
bamb	.158	.292	.222	.073	.000	.000	.000	.179
ramb	.000	.000	.073	.222	.292	.158	.179	.000
beas	.000	.000	.000	.000	.000	.000	1.067	.000
boas	.000	.000	.000	.000	.000	.000	1.000	.000
reas	.000	.000	.000	.000	.000	.000	.000	1.067
roas	.000	.000	.000	.000	.000	.000	.000	1.000

Indirect Effects

	bpas	bint	bses	rses	rint	rpas	bamb	ramb
bamb	.005	.010	.021	.044	.054	.029	.033	.006
ramb	.029	.054	.044	.021	.010	.005	.006	.033
beas	.174	.322	.259	.124	.058	.031	.035	.198
boas	.163	.302	.243	.117	.054	.029	.033	.185
reas	.031	.058	.124	.259	.322	.174	.198	.035
roas	.029	.054	.117	.243	.302	.163	.185	.033

Modification Indices

Modification index is a measure is a measure of predicted change in χ^2 if a single fixed parameter restriction is relaxed and the model is re-estimated.

Thus, if a fitted model is not satisfactory, modification indices can be used to evaluate which relaxed restrictions would most improve the model.

Example 7.4: In the previous example, the largest modification indices are the following:

```
Covariances:
            M.I.   Par Change
e1 <--> e4  7.664      -.080
e1 <--> e3 12.331       .105

Regression Weights:
                M.I. Par Change
boas <--- roas 5.660        .097
roas <--- boas 6.206        .101
```

Largest gain in χ^2 would be obtained if the covariance between the error terms ϵ_1 and ϵ_2 were relaxed.

Re-estimating the model with this modification produces a $\chi^2 = 17.1$ with 24 degrees of freedom and p-value of 0.84.

Thus the true drop in χ^2 is 13.7, which is a little larger than the predicted.

However, because the original model indicates a satisfactory fit there is no need to adopt the change.

Remark 7.5: Relaxing parameters should not be driven purely by data. Additional free parameters should have sound substance based interpretation.

A data driven model improvement leads easily to a capitalization of change, not true dependencies.