

z-Test Approximation of the Binomial Test

A binary random variable (e.g., a coin flip), can take one of two values. If we arbitrarily define one of those values as a success (e.g., heads=success), then the following formula will tell us the probability of getting k successes from n observations of the random variable when the probability of a success equals p .

$$P(k | n, p) = \binom{n}{k} p^k (1 - p)^{n-k} \quad (1)$$

The first part of the formula can be read as “ n choose k ”, and is computed as follows:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (2)$$

where ! means factorial.

For example, we can use the formula to compute the probability of getting exactly two heads in three flips of a fair coin.

$n=3$ (i.e., there are three flips)

$k=2$ (i.e., heads comes up twice)

$p=.5$ (i.e., the probability of the coin landing heads up is 50%)

$$P(k = 2 | n = 3, p = .5) = \binom{3}{2} (.5)^2 (1 - .5)^{3-2} = \frac{3!}{2!(1!)} (.5)^2 (.5)^1 = \frac{6}{2(1)} (.5)^3 = 3(.125) = .375$$

However, when n becomes large enough the factorials in Equation 2 become too large to compute (e.g., my computer can't handle $n > 69$). Fortunately, thanks to the central limit theorem, for such large values of n (e.g., $n > 30$) we can accurately approximate the binomial distribution defined by Equation 1 with a normal distribution with the following mean and standard deviation:

$$\mu = np, \sigma = \sqrt{np(1-p)}$$

This enables us to approximate binomial tests for a large number of observations with z -tests. Consider. for example. the following problem:

Nixon thinks that he's a better at the game “rock, paper, scissors” than his friend Kissinger. To find out if this is the case, he challenges Kissinger to 49 bouts of rock, paper, scissors. Nixon wins 31 of these bouts. Can we reject the null hypothesis that the two men are equally good at the game (i.e., $P(\text{Nixon wins}) = P(\text{Nixon loses}) = .5$) at an alpha level of .05?

Because our measurements are binary (Nixon wins or loses), the null hypothesis is binomially distributed with the following parameters: $n=49$, $p=.5$. Because n is large we can approximate the distribution with a normal distribution with a mean of 24.5 and standard deviation of 3.5. We can now conduct a z -test.

Null Hypothesis	$\mu=24.5, \sigma=3.5$
Alternative Hypothesis	$\mu>24.5$
Tail of Test	upper tailed
Type of Test	z -test
Alpha level	$\alpha=.05$
Critical Value(s) of Test Statistic	1.65
Test Statistic of Data	1.86
p -value of Data	$p\approx.0314$
Conclusion	Reject the Null Hypothesis

1.65 is our critical z -score because just less than 5% of the area under the standard normal distribution lies between it and positive infinity (4.95% of it to be more exact). To get out test statistic, we convert the number of Nixon's victories, 31, into a z -score:

$$z = \frac{x - \mu}{\sigma} = \frac{31 - 24.5}{3.5} \approx 1.86$$

Since the z -score of our sample exceeds our critical z -score, we reject the null hypothesis that Nixon and Kissinger are equally good in favor of the alternative hypothesis that Nixon is better. To get a sense of how significant 31 victories are, we compute the p -value of our sample, 3.14%, which is the percentage of the area under the standard normal distribution that lies between 1.86 and positive infinity. Our hypothesis test is thus concluded: