Set Theory, Functions & Relations

Set Theory

- It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets understand relationships between groups, and to analyze survey data.

Basics

- An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.
- A **set** is a collection of distinct objects, called **elements** of the set
- A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

Example 1

Some examples of sets defined by describing the contents:

1. The set of all even numbers
2. The set of all books written about travel to Chile

Answers

Some examples of sets defined by listing the elements of the set:

1. {1, 3, 9, 12}
2. {red, orange, yellow, green, blue, indigo, purple}

A set simply specifies the contents; order is not important. The set represented by {1, 2, 3} is equivalent to the set {3, 1, 2}.

NOTATION

Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.

- The symbol $\in$ means “is an element of”.
- A set that contains no elements, { }, is called the **empty set** and is notated $\emptyset$.

EXAMPLE 2

Let $A = \{1, 2, 3, 4\}$

- To notate that 2 is element of the set, we’d write $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Prince albums. While Chris's collection is a set, we can also say it is a **subset** of the larger set of all Prince albums.

SUBSET

- A **subset** of a set A is another set that contains only elements from the set A, but may not contain all the elements of A.
- If B is a subset of A, we write $B \subseteq A$.
- A **proper subset** is a subset that is not identical to the original set—it contains fewer elements.
- If B is a proper subset of A, we write $B \subset A$.

Example 3

Consider these three sets:

- A = the set of all even numbers
  $B = \{2, 4, 6\}$
  $C = \{2, 3, 4, 6\}$
- Here $B \subset A$ since every element of B is also an even number, so is an element of A.
- More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$.
- It is also true that $B \subset C$.
- C is not a subset of A, since C contains an element, 3, that is not contained in A

Example 4

Suppose a set contains the films “Finding Dory,” “The Incredibles,” and “Brave.” What is a larger set this might be a subset of?

- There are many possible answers here. One would be the set of films by Pixar Animation Studios.
- This is also a subset of the set of all films ever produced.
- It is also a subset of all animated films.

<u>Exercise:</u>

The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

Union, Intersection, and Complement

- Commonly sets interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.
- The **union** of two sets contains all the elements contained in either set (or both sets). The union is notated $A \cup B$. More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both)
- The **intersection** of two sets contains only the elements that are in both sets. The intersection is notated $A \cap B$. More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$.
- The **complement** of a set A contains everything that is *not* in the set A. The complement is notated A', or A^c, *or sometimes* $\sim A$.

<u>Example 5:</u>

Consider the sets:

A = {red, green, blue}
B = {red, yellow, orange}
C = {red, orange, yellow, green, blue, purple}

Find the following:

1. Find $A \cup B$
2. Find $A \cap B$
3. Find $A^c \cap C$

Answers

1. The union contains all the elements in either set: $A \cup B$ = {red, green, blue, yellow, orange} Notice we only list red once.
2. The intersection contains all the elements in both sets: $A \cap B$ = {red}
3. Here we're looking for all the elements that are *not* in set A and are also in C. $A^c \cap C$ = {orange, yellow, purple}

Exercise

Using the sets from the previous example, find $A \cup C$ and $B^c \cap A$

UNIVERSAL SET

- A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.
- A complement is relative to the universal set, so A^c contains all the elements in the universal set that are not in A.

Example 6:

1. If we were discussing searching for books, the universal set might be all the books in the library.
2. If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
3. If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

Example 7:

Suppose the universal set is U = all whole numbers from 1 to 9. If A = {1, 2, 4}, then A^c = {3, 5, 6, 7, 8, 9}.

As we saw earlier with the expression $A^c \cap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic – to force an order of operations.

Example 8:

Suppose H = {cat, dog, rabbit, mouse}, F = {dog, cow, duck, pig, rabbit}, and W = {duck, rabbit, deer, frog, mouse}

1. Find $(H \cap F) \cup W$
2. Find $H \cap (F \cup W)$
3. Find $(H \cap F)^c \cap W$

Solutions

1. We start with the intersection: $H \cap F$ = {dog, rabbit}. Now we union that result with W: $(H \cap F) \cup W$ = {dog, duck, rabbit, deer, frog, mouse}
2. We start with the union: $F \cup W$ = {dog, cow, rabbit, duck, pig, deer, frog, mouse}. Now we intersect that result with H: $H \cap (F \cup W)$ = {dog, rabbit, mouse}
3. We start with the intersection: $H \cap F$ = {dog, rabbit}. Now we want to find the elements of W that are *not* in $H \cap F$. $(H \cap F)^c \cap W$ = {duck, deer, frog, mouse}

Venn Diagrams

- To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles. These illustrations now called **Venn Diagrams**.

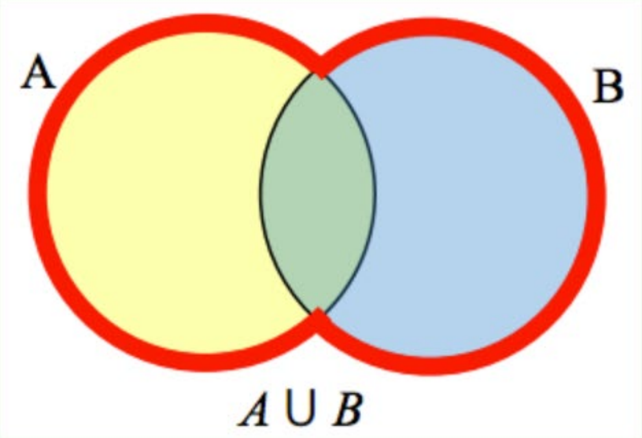
- A Venn diagram represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

- Basic Venn diagrams can illustrate the interaction of two or three sets.

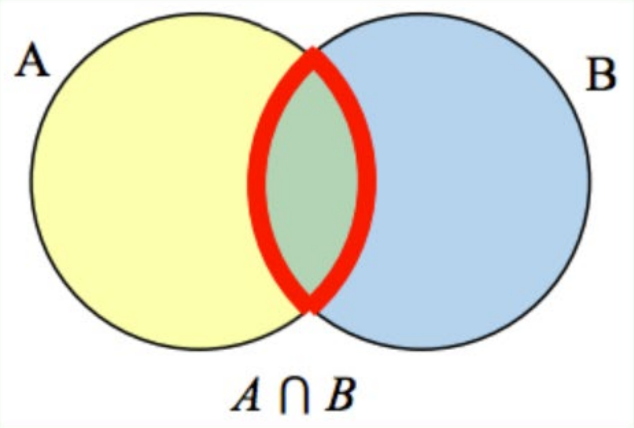
Example 9:

Create Venn diagrams to illustrate $A \cup B$, $A \cap B$, and $A^c \cap B$

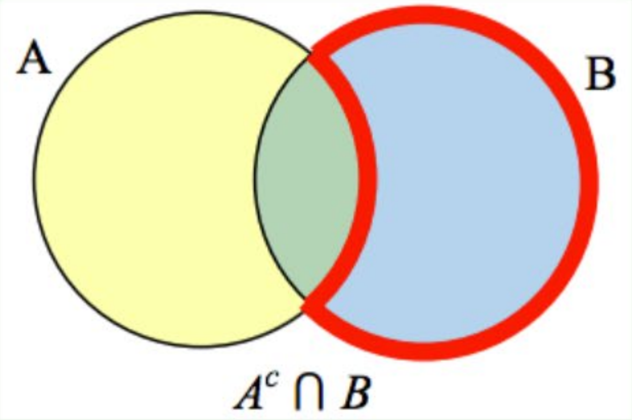
$A \cup B$ contains all elements in *either* set.



$A \cap B$ contains only those elements in both sets—in the overlap of the circles.



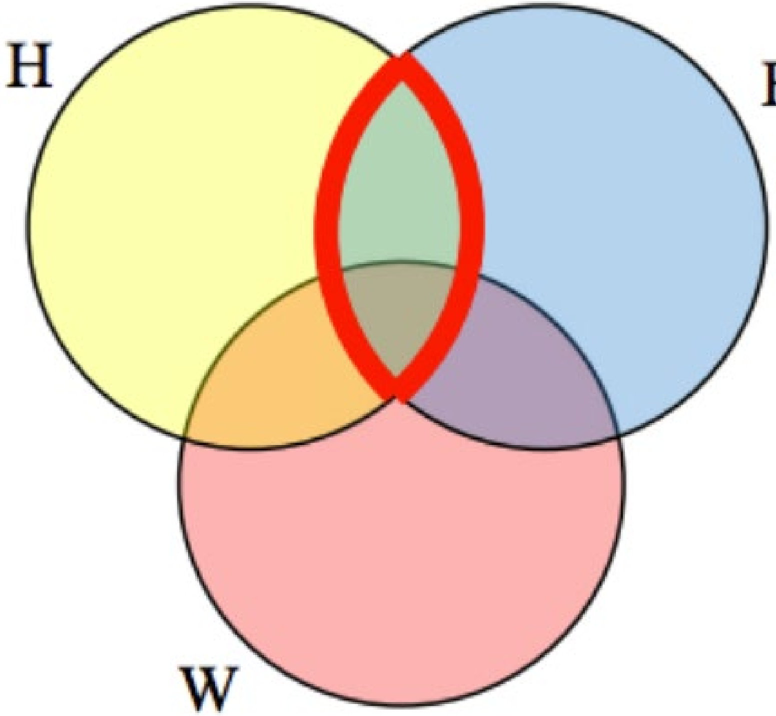
A^c will contain all elements *not* in the set A. $A^c \cap B$ will contain the elements in set B that are not in set A.



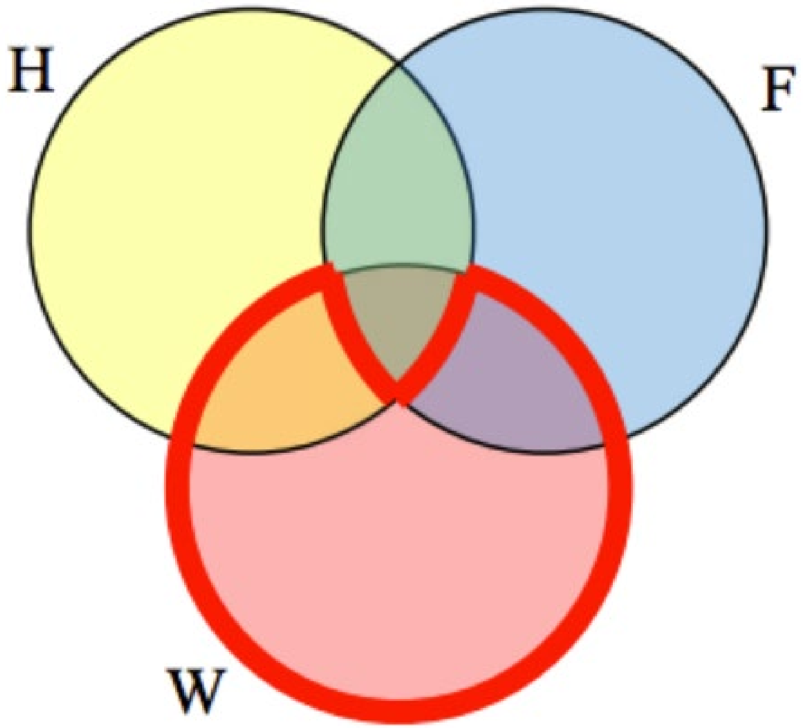
Example 10:

Use a Venn diagram to illustrate $(H \cap F)^c \cap W$

- We'll start by identifying everything in the set $H \cap F$

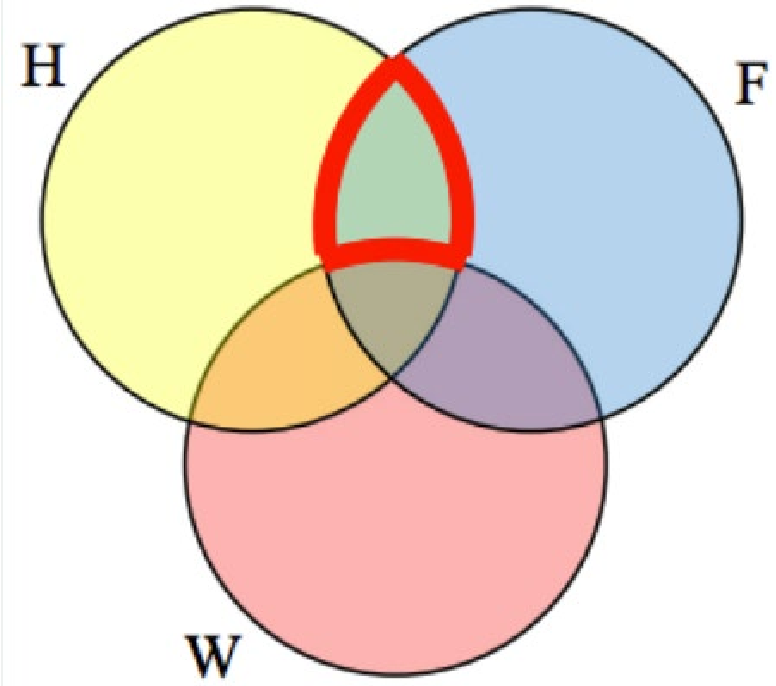


Now, $(H \cap F)^c \cap W$ will contain everything *not* in the set identified above that is also in set W.



Example 11

Create an expression to represent the outlined part of the Venn diagram shown.



Soln

The elements in the outlined set *are* in sets H and F, but are not in set W. So we could represent this set as $H \cap F \cap W^c$.

CARDINALITY

The number of elements in a set is the cardinality of that set.

The cardinality of the set A is often notated as $|A|$ or n(A)

EXAMPLE 12

- Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$.
- What is the cardinality of B? $A \cup B$, $A \cap B$?

Solns

- The cardinality of B is 4, since there are 4 elements in the set.
- The cardinality of $A \cup B$ is 7, since $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements.
- The cardinality of $A \cap B$ is 3, since $A \cap B = \{2, 4, 6\}$, which contains 3 elements.

Example 14

A survey asks 200 people "What beverage do you drink in the morning", and offers choices:

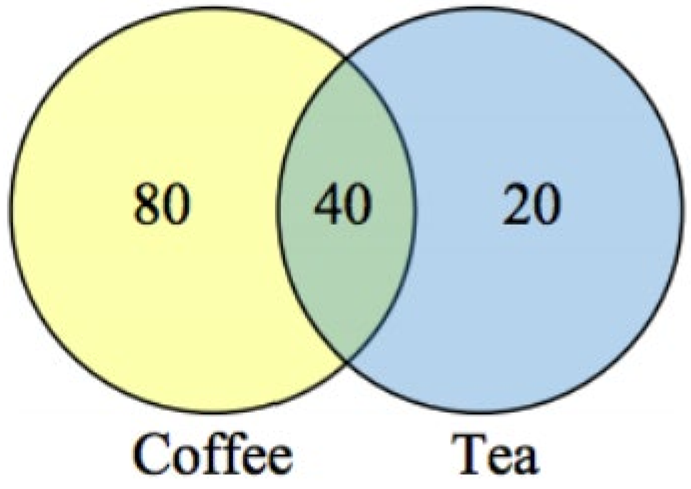
- Tea only
- Coffee only
- Both coffee and tea

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?

Solns

This question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.

- We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.



- 200 – 20 – 80 – 40 = 60 people who drink neither.

Example 15:

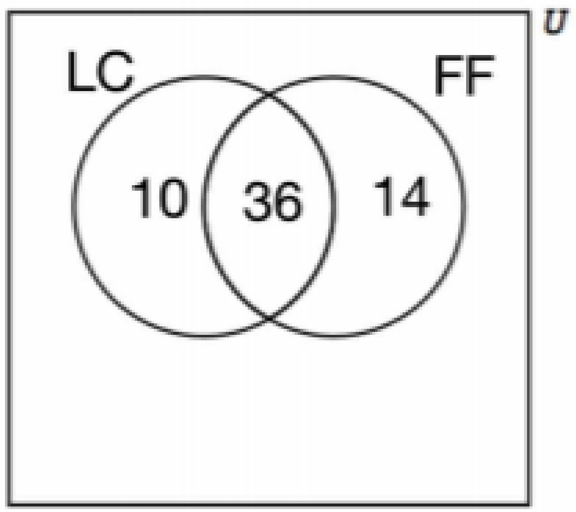
A survey of a group of students revealed that 60 of them liked at least one of the cereals – Cheerios or Wheetos. If 50 of them liked Cheerios and 46 of them liked Wheetos,

i. How many of them liked both cereals?

ii. How many liked Cheerios but did not like Wheetos?

Soln

$50 + 46 - 60 = 36.$



Example 16:

A survey of a sample of 60 computers sold from a popular electronics shop found that 15 had wireless cards, 12 had detachable keyboards and 11 had built-in tablets. 5 had wireless and tablet, 9 had wireless and detachable keyboard, 4 had detachable keyboard and tablet and 3 had all three options.

(i) Draw a Venn diagram to illustrate this problem.

(ii) Determine how many computers had at least one of the options.

(iii) Determine how many computers had *exactly* one of the options.

Example 10

The **symmetric difference** of A and B, denoted by $A\Delta B$, is the set containing those elements in either A or B but not both. Find $A\Delta B$ if $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$.

Solution.
$A\Delta B = \{2, 5\}$ ■

The notation $(a_1, a_2, \cdots, a_n)$ is called an **ordered n-tuples**. We say that two n-tuples $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ are equal if and only if $a_1 = b_1, a_2 = b_2, \cdots, a_n = b_n$.
Given n sets $A_1, A_2, \cdots, A_n$ the **Cartesian product** of these sets is the set

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) : a_1 \in A_1, a_2 \in A_2, \cdots, a_n \in A_n\}$$

Example 11

Let $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$. Find

a. $A \times B \times C$.

b. $(A \times B) \times C$.

Solution.

a.

$$\begin{aligned} A \times B \times C = \; & \{(x,1,a),(x,2,a),(x,3,a),(y,1,a),(y,2,a), \\ & (y,3,a),(x,1,b),(x,2,b),(x,3,b),(y,1,b) \\ & (y,2,b),(y,3,b)\} \end{aligned}$$

b.

$$\begin{aligned} (A \times B) \times C = \; & \{((x,1),a),((x,2),a),((x,3),a),((y,1),a),((y,2),a), \\ & ((y,3),a),((x,1),b),((x,2),b),((x,3),b),((y,1),b) \\ & ((y,2),b),((y,3),b)\} \blacksquare \end{aligned}$$

Let A be a set. The **power set** of A, denoted by $\mathcal{P}(\mathrm{A})$, is the empty set together with all possible subsets of A.

Example 16

Find the power set of $A = \{a, b, c\}$.

Solution.

$$\begin{aligned} \mathcal{P}(\mathrm{A}) &= \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \\ &\quad \{b, c\}, \{a, b, c\}\} \quad \blacksquare \end{aligned}$$

Computer Representation of Sets

- There are many ways to represent sets using a computer. Usually it is less work with ordered sets (set operations are less time consuming).
- We will study one of the methods that allows us to store elements using an arbitrary ordering of the elements of the universal set.

  Let's assume that set *U is a finite and of reasonable size.*

- First, we'll specify an arbitrary ordering of elements of *U, for instance*

$$a_1, a_2, a_3, \ldots. a_n$$

- Then, we'll represent a subset **A** of *U with the bit string of length **n**, where*

  the ith bit in this string is:

  1, if $a_i \in \mathbf{A}$ and

  0, if $a_i \notin \mathbf{A}$.

Example:

Let *U = {0,1,2,3,4,5,6}. Represent sets with bit strings.*

a) A = {2,4,5,6}

b) B is the set of all odd integers, B$\subseteq$*U*

c) C is a subset of *U, containing all integers greater than 4.*

Solution

a) A = {2,4,5,6}

bit string: 0010111

b) B is the set of all odd integers, B$\subseteq$*U*

B = {1,3,5}, therefore the bit string is

010 1010

c) C is a subset of *U, containing all integers greater than 4.*

C = {5,6}, therefore the bit string is 000 0011

Using bit strings to represent sets, it is easy to find complements of sets, unions, intersections, and differences of sets.

Example:

Let $U = \{0,1,2,3,4,5,6,7\}$, $A = \{1,3,5,6,7\}$, $B = \{0,1,2,3,4,5\}$.

Use bit strings to find

$$(a)\ \ \overline{\mathbf{A}} \qquad (b)\ \mathbf{A} \cap \mathbf{B} \qquad (c)\ \mathbf{A} \cup \mathbf{B}$$

Solutions:

(a) The bit string for A: 0101 0111, then

$$\overline{\mathbf{A}} = \overline{01010111} = 1010\ 1000$$

(therefore $\overline{A} = \{0,2,4\}$)

(b) $A \cap B$

Bit string for A : 0101 0111

Bit string for B : 1111 1100

$$
\begin{array}{rl}
\text{Then } A \cap B = & 0101\ 0111 \\
 & \underline{1111\ 1100} \quad \leftarrow \text{AND} \\
 & 0101\ 0100
\end{array}
$$

$(A \cap B = \{1,3,5\})$

$$
\begin{array}{rl}
(c)\ A \cup B = & 0101\ 0111 \\
 & \underline{1111\ 1100} \quad \leftarrow \text{OR} \\
 & 1111\ 1111
\end{array}
$$

$(A \cup B = U)$

Function

Let *X* and *Y* be sets. A **function** *f* from *X* to *Y* is a rule that assigns every element x of *X to a unique y in Y*.
We write ***f*: *X* → *Y*** and ***f*(*x*) = *y***

$$(\forall x \in X\ \exists y \in Y, y = f(x)) \wedge (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \rightarrow x_1 \neq x_2)$$

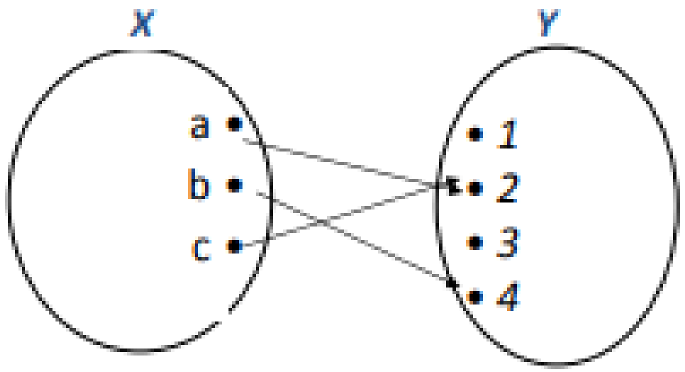
X = domain, Y = codomain
y = image of x under f,
x = preimage of y under f
range = subset of Y with preimages

Example 1

$$(\forall x \in X\ \exists y \in Y, y = f(x)) \wedge (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \rightarrow x_1 \neq x_2)$$

Arrow Diagram of *f*:

Domain X={a,b,c},
Co-domainY={1,2,3,4}
f={(a,2),(b,4),(c,2)},
preimage of 2 is{a,c}
Range={2,4}



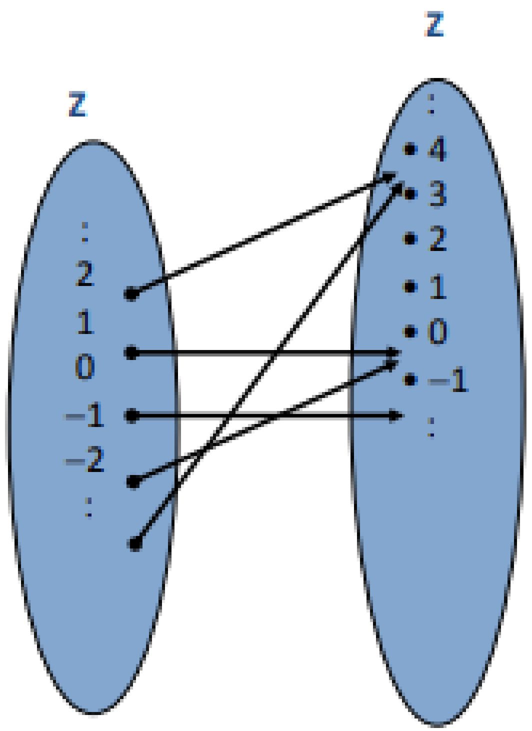
Example 2

Let f be the function from **Z** to **Z** that assigns the square of an integer to this integer.

Then, f: $\mathbf{Z} \rightarrow \mathbf{Z}$, $f(x) = x^2$

Domain and co-domain of f: **Z**

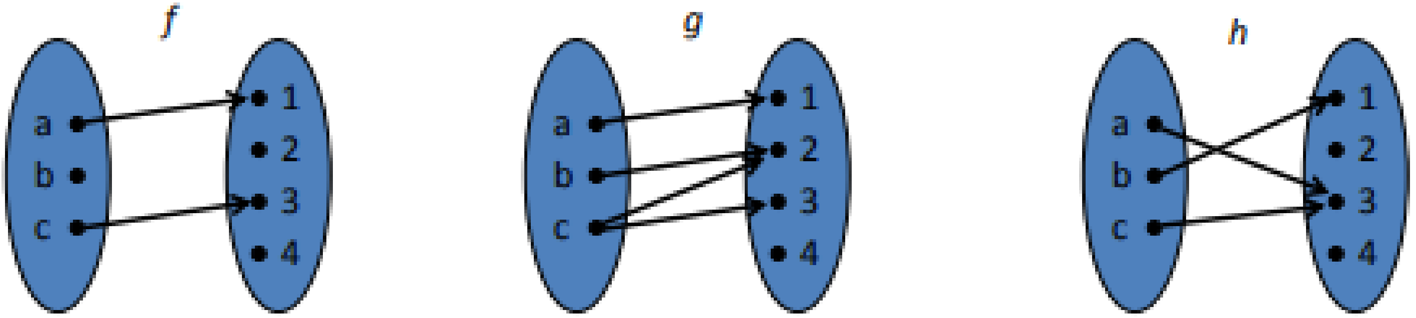
Range(f) = {0, 1, 4, 9, 16, 25, ...}



Functions Vs Non-functions

$$(\forall x \in X\ \exists y \in Y, y = f(x)) \wedge (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \rightarrow x_1 \neq x_2)$$

X = {a,b,c} to Y = {1,2,3,4}



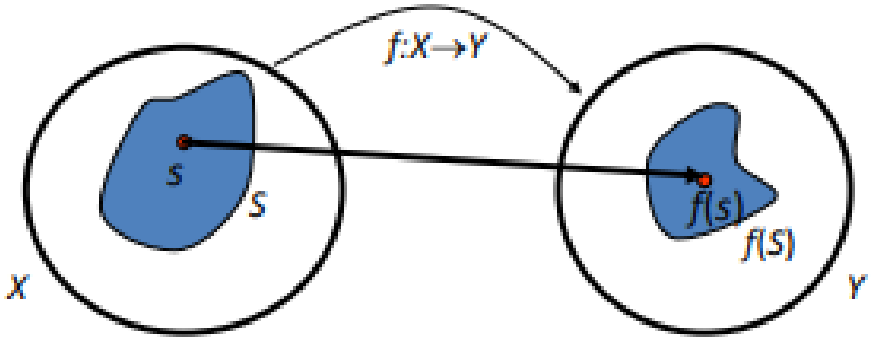
No,
b has no image

No,
c has two images

Yes,
each element of *X* has exactly one image

Image of a Set

Let f be a functions from X to Y and $S \subseteq X$. The **image of S** is the subset of Y that consists of the images of the elements of S: $f(S) = \{f(s) \mid s \in S\}$



One-To-One Function

A function f is **one-to-one** (or **injective**), if and only if $f(x) = f(y)$ implies $x = y$ for all x and y in the domain of f.

In words:

"All elements in the domain of f have different images"

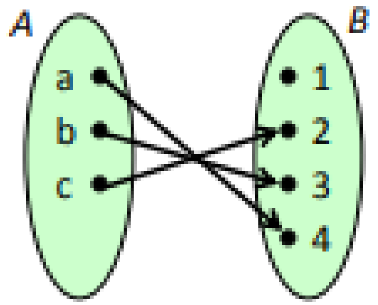
Mathematical Description:

$f:A \rightarrow B$ is **one-to-one** $\Leftrightarrow \forall x_1, x_2 \in A \ (f(x_1)=f(x_2) \rightarrow x_1 = x_2)$

or

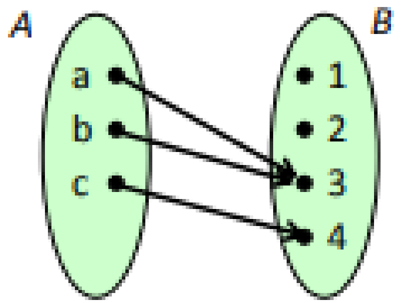
$f:A \rightarrow B$ is **one-to-one** $\Leftrightarrow \forall x_1, x_2 \in A \ (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$

Example: One-to-One (Injective)



one-to-one

(all elements in *A* have a different image)



not one-to-one

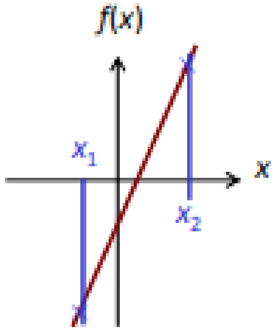
(*a* and *b* have the same image)

Example: One-To-One (Injective)

f: $\mathbf{R} \to \mathbf{R}$, $f(x) = 4x-1$

g: $\mathbf{R} \to \mathbf{R}$, $g(x) = x^2$

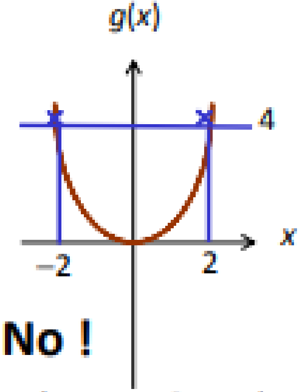
("Does each element in **R** have a different image ?")



Yes !

To show: $\forall x_1, x_2 \in \mathbf{R}\ (f(x_1)=f(x_2) \Rightarrow x_1 = x_2)$

Take some $x_1, x_2 \in \mathrm{R}$ with $f(x_1) = f(x_2)$.
Then $4x_1-1 = 4x_2-1 \Rightarrow 4x_1 = 4x_2 \Rightarrow x_1=x_2$



No !

Take $x_1 = 2$ and $x_2 = -2$.

Then $g(x_1) = 2^2 = 4 = g(x_2)$
and $x_1 \neq x_2$

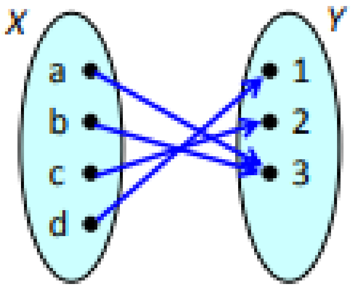
Onto Functions

A function f from X to Y is **onto** (or **surjective**), if and only if for every element $y \in Y$ there is an element $x \in X$ with $f(x)=y$.

In words:
"Each element in the co-domain of f has a pre-image"

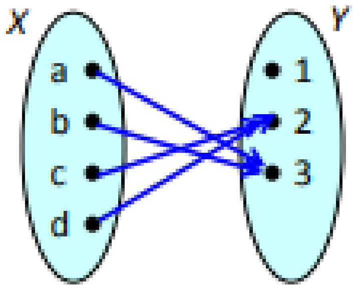
Mathematical Description:
$f:X \rightarrow Y$ is **onto** $\Leftrightarrow \forall y \, \exists x, \; f(x) = y$

Example: Onto (Surjective)



onto

(all elements in Y have a pre-image)



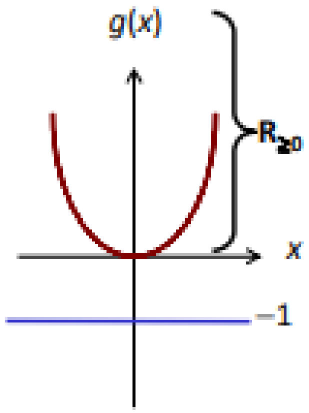
not onto

(1 has no pre-image)

Example: Onto (Surjective)

g:**R**$\rightarrow$**R**, $g(x)=x^2$

("Does each element in **R** have a pre-image ?")



No !

To show: $\exists y \in \mathbf{R}$ such that $\forall x \in \mathbf{R}$ $g(x) \neq y$

Take $y = -1$

Then any $x \in \mathbf{R}$ holds $g(x) = x^2 \neq -1 = y$

But g:**R**$\rightarrow$**R**$_{\geq 0}$, $g(x)=x^2$,(where $\mathbf{R}_{\geq 0}$ denotes the set of non-negative real numbers) is onto !

One-to-one Correspondence

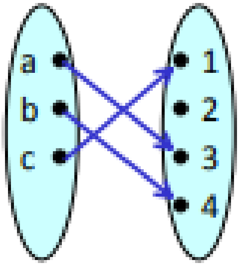
A function f is a **one-to-one correspondence** (or **bijection**), if and only if it is both one-to-one and onto

In words:

"No element in the co-domain of f has two (or more) pre-images" (*one-to-one*) **and**

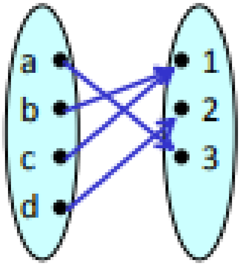
"Each element in the co-domain of f has a pre-image" (*onto*)

Example: Bijection



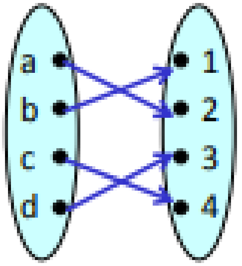
No

(not onto, 2 has no pre-image)



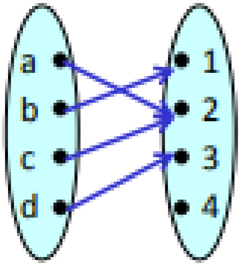
No

(not one-to-one, 1 has two pre-images)



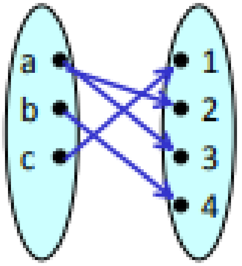
Yes

(each element has exactly one pre-image)



No

(neither one-to-one nor onto)



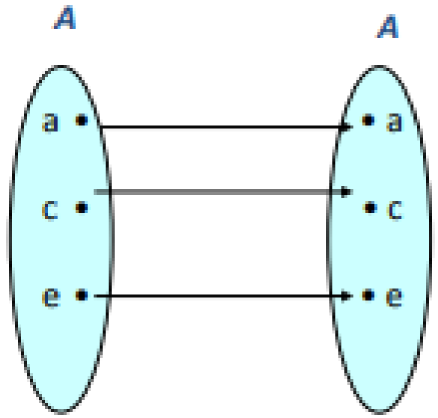
No

(not a function, *a* has two images)

Identity Function

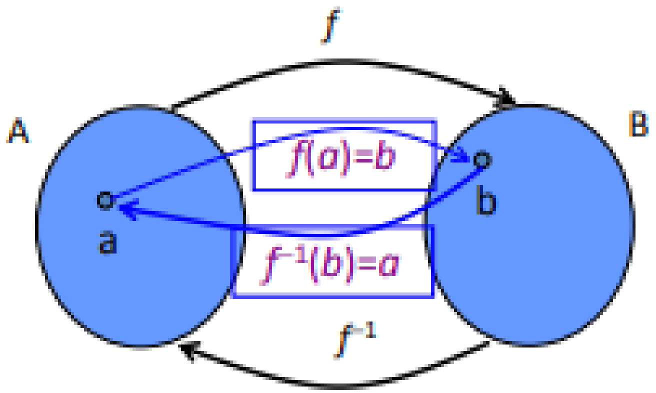
The **identity function** on a set A is defined as:
i_A:$A \rightarrow A$, $i_A(x) = x$.

Example. Any identity function is a bijection.
e.g. for A = {a, c, e}:



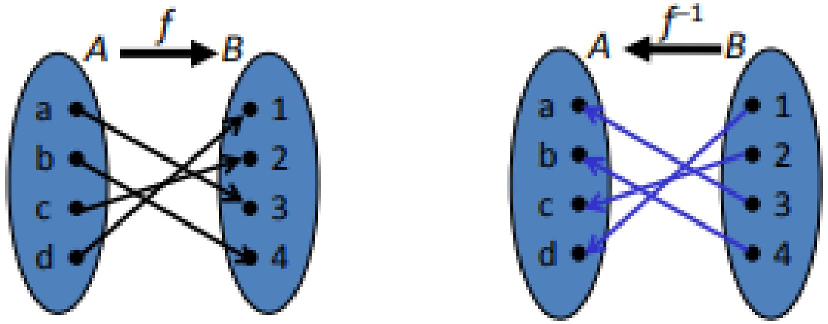
Inverse Function

Let f:$A \rightarrow B$ be a *one-to-one correspondence (bijection).*
Then the inverse function of f, f^{-1}:$B \rightarrow A$, is defined by:
$f^{-1}(b)$ = that unique element $a \in A$ such that $f(a)=b$.
We say that f is invertible.



Example 1

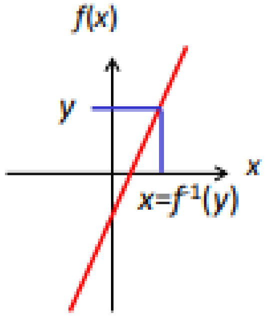
Find the inverse function of the following function:



Let $f: A \rightarrow B$ be a one-to-one correspondence and $f^{-1}: B \rightarrow A$ its inverse. Then $\forall b \in B\ \forall a \in A\ (f^{-1}(b) = a \Leftrightarrow b = f(a))$

Example 2

What is the inverse of f:**R**→**R**, $f(x)=4x-1$?

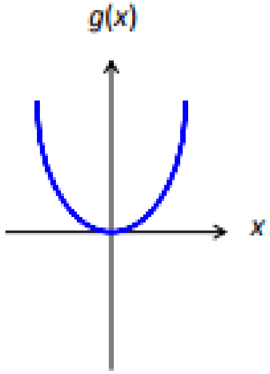


Let $y \in$ **R.** Calculate x with $f(x)=y$:

$y = 4x-1 \Leftrightarrow (y+1)/4 = x$

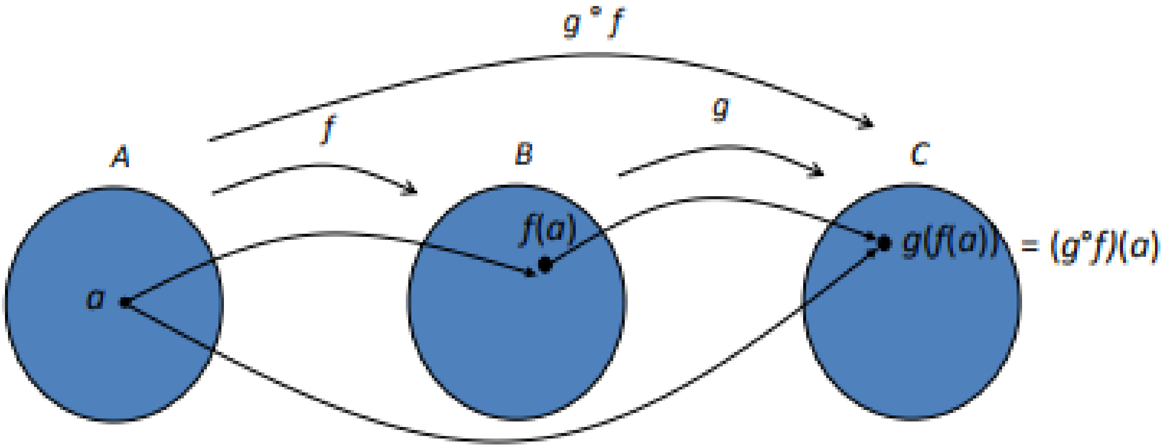
Hence, $f^{-1}(y) = (y+1)/4$

What is the inverse of g:**R**→**R**, $g(x)=x^2$?



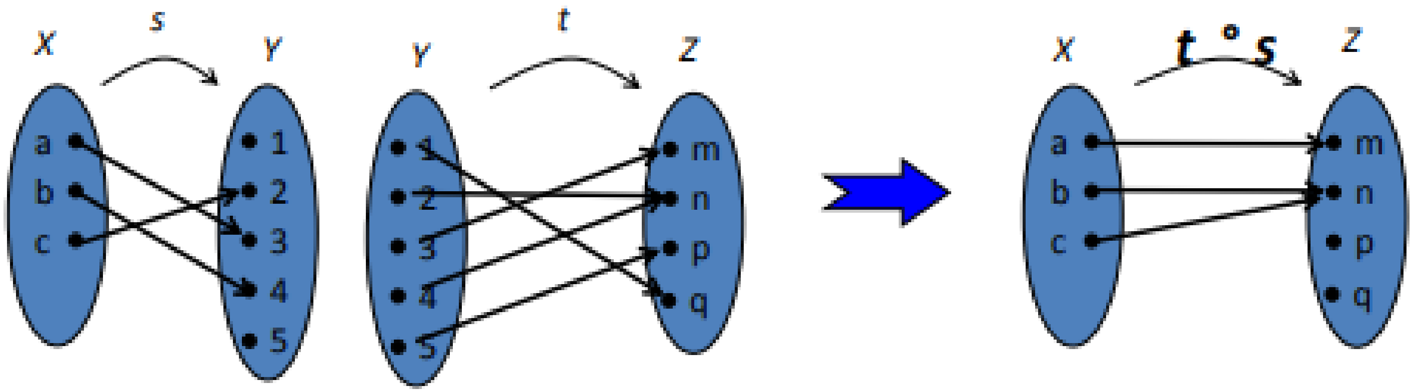
Composition of Functions

Let f:$A \rightarrow B$ and g:$B \rightarrow C$ be functions. The **composition** of the functions f and g, denoted as $g \circ f$, is defined by:
$g \circ f$: $A \rightarrow C$, $(g \circ f)(a) = g(f(a))$



Examples

Example :Given functions $s{:}X \rightarrow Y$ and $t{:}Y \rightarrow Z$. Find $t \circ s$ and $s \circ t$.



Example $f{:}\mathbf{Z}\rightarrow\mathbf{Z}$, $f(n)=2n+3$, $g{:}\mathbf{Z}\rightarrow\mathbf{Z}$, $g(n)=3n+2$. What is $g \circ f$ and $f \circ g$?

$(f \circ g)(n) = f(g(n)) = f(3n + 2) = 2(3n+2) + 3 = 6n + 7$

$(g \circ f)(n) = g(f(n)) = g(2n + 3) = 3(2n+3) + 2 = 6n + 11$

$\boldsymbol{f \circ g \neq g \circ f}$ (no **commutativity** for the composition of functions !)

Example:

Let $f(x) = x^2$ and $g(x) = 2x+1$

$$(i)\ (g \circ f)(x) = g(f(x))$$
$$= g(x^2)$$
$$= 2(x^2)+1 = \boxed{2x^2+1}$$

$$(ii)\ (f \circ g)(x) = f(g(x))$$
$$= f(2x+1)$$
$$= (2x+1)^2 = \boxed{4x^2+4x+1}$$

Relations

- There are various kinds of **relations** between mathematical objects – e.g. +, /, x^y (exponent), =, ≠, ≥, ∧, ~, →, ≡, ∩
- Formal definition of (Binary) Relation:

> **• Definition**
>
> Let A and B be sets. A **relation R from A to B** is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, **x is related to y by R**, written $x\ R\ y$, if, and only if, (x, y) is in R. The set A is called the domain of R and the set B is called its co-domain.

So, **xRy** means **(x,y)** ∈ **R**.

Example:

Define a relation L from R (real numbers) to R as follows:

For all real numbers x and y, $x \, L \, y \Leftrightarrow x < y$.

a. Is $57 \, L \, 53$?
b. Is $(-17) \, L \, (-14)$?
c. Is $143 \, L \, 143$?
d. Is $(-35) \, L \, 1$?

- **N-ary Relations** – A relation defined on several sets.

> **• Definition**
>
> Given sets $A_1, A_2, \ldots, A_n$, an ***n*-ary relation** R on $A_1 \times A_2 \times \cdots \times A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The special cases of 2-ary, 3-ary, and 4-ary relations are called **binary, ternary,** and **quaternary relations,** respectively.

<u>Example</u>: A simple database

Define a quaternary relation R on A1 x A2 x A3 x A4 as follows:

(a1, a2, a3, a4) ∈ R ⇔ a patient with patient ID number a1, named a2, was admitted on date a3, with primary diagnosis a4.

Example instances/tuples:

(011985, John Schmidt, 020710, asthma)
(574329, Tak Kurosawa, 0114910, pneumonia)
(466581, Mary Lazars, 0103910, appendicitis)

Reflexivity, Symmetry & Transitivity

(Important properties of general relations)

• Definition

Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for all $x \in A$, $x\ R\ x$.
2. R is **symmetric** if, and only if, for all $x, y \in A$, *if* $x\ R\ y$ then $y\ R\ x$.
3. R is **transitive** if, and only if, for all $x, y, z \in A$, *if* $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$.

- Informal definitions:
  - ✓ **Reflexive**: Each element is related to itself.
  - ✓ **Symmetric**: If any one element is related to any other element, then the second element is related to the first.
  - ✓ **Transitive**: If any one element is related to a second and that second element is related to a third, then the first element is related to the third.

[Definitions for Non-relation]

1. R is not reflexive ⇔ there is an element x in A such that x R x [that is, such that (x, x) ∉ R].
2. R is not symmetric ⇔ there are elements x and y in A such that x R y but y R x [that is, such that (x, y) ∈ R but (y, x) ∉ R].
3. R is not transitive ⇔ there are elements x, y and z in A such that x R y and y R z but x R z [that is, such that (x, y) ∈ R and (y, z) ∈ R but (x, z) ∉ R].

Example 1:

Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T on A as follows:

$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\},$$
$$S = \{(0, 0), (0, 2), (0, 3), (2, 3)\},$$
$$T = \{(0, 1), (2, 3)\}.$$

a. Is R reflexive? symmetric? transitive?
b. Is S reflexive? symmetric? transitive?
c. Is T reflexive? symmetric? transitive?

Example 2:

The following binary relation R on the given set S is defined:

$$S = \{1,2,3,4\}, \qquad R = \{(x, y): x \geq y\}$$

(i) Write down the elements of R.

(ii) Test R for reflexivity, symmetry and transitivity.

(iii) Is R an equivalence relation?

Example 3:

The following binary relation R on the given set S is defined:

$$S = \{20,40,60,80,100\}, R = \{(a,b): a\ /\ b\}$$

Note: $a\ /\ b$ means that a divides into b evenly.

(i) Write down the elements of R.

(ii) Test R for reflexivity, symmetry and transitivity.

Equivalence Relations

- A relation on a set that satisfies the three properties of reflexivity, symmetry, and transitivity is called an **equivalence relation**.

> • **Definition**
>
> Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

- Example:
  - ✓ Consider the relation R on a set {1,2,3,4,5}.

    R = {(1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5)}

    is an equivalence relation because:

    - R is reflexive because (1,1), (2,2), (3,3), (4,4), (5,5) are in R.
    - R is symmetric because whenever (x,y) is in R, (y,x) is in R as well.
    - R is transitive because whenever (x,y) and (y,z) are in R, (x,z) is in R as well.

  - ✓ Consider the relation R on a set {1,2,3,4}.

    R = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}

    is NOT an equivalence relation because R is not symmetric.

Application to databases

Databases have been a staple of business computing from the very beginning of the digital era. In fact, the process of relational databases was born in 1970 (by EM Codd who worked in IBM). Since then, relational databases have grown in popularity to become the standard.

Originally, databases were **flat**. This means that the information was stored in one long text file, called a **tab delimited file**. Each entry in the tab delimited file is separated by a special character, such as a vertical bar (**|**). Each entry contains multiple pieces of information (**fields**) about a particular object or person grouped together as a **record**. The text file makes it difficult to search for specific information or to create reports that include only certain fields from each record. Here's an example of the file created by a flat database:

```
Lname, FName, Age, Salary|Smith, John, 35, $280|Doe, Jane,
28, $325|Brown, Scott, 41, $265|Howard, Shemp, 48,
$359|Taylor, Tom, 22, $250
```

You can see that you have to search sequentially through the entire file to gather related information, such as age or salary. A relational database allows you to easily find specific information. It also allows you to sort based on any field and generate reports that contain only certain fields from each record. Relational databases use **tables** to store information. The standard fields and records are represented as columns (fields) and rows (records) in a table. Look at this example:

LName	FName	City	Age	Salary
Smith	John	3	35	$280
Doe	Jane	1	28	$325
Brown	Scott	3	41	$265
Howard	Shemp	4	48	$359
Taylor	Tom	2	22	$250

In the relational database example, you can quickly compare salaries and ages because of the arrangement of data in columns. The relational database model takes advantage of this uniformity to build completely new tables out of required information from existing tables. In other words, it uses the relationship of similar data to increase the speed and versatility of the database.

The "relational" part of the name comes into play because of mathmatical relations. A typical relational database has anywhere from 10 to more than 1,000 tables. Each table contains a column or columns that other tables can key on to gather information from that table. Look at the table below that matches the number in the City column of the above table with the name of a city.

City #	City Name
1	Boston
2	London
3	New York
4	Los Angeles

By storing this information in another table, the database can create a single small table with the locations that can then be used for a variety of purposes by other tables in the database. A typical large database, like the one a big Web site, such as Amazon would have, will contain hundreds or thousands of tables like this all used together to quickly find the exact information needed at any given time.

- Relational databases are created using a special computer language, **structured query language (SQL)**, that is the standard for database interoperability.
- Relational databases conceptually organise data into sets of two-dimensional tables.
- There are many relational DB software packages
  - Open Source: MySQL, Postgres
  - Proprietary: Oracle, Microsoft, IBM, Sybase and others