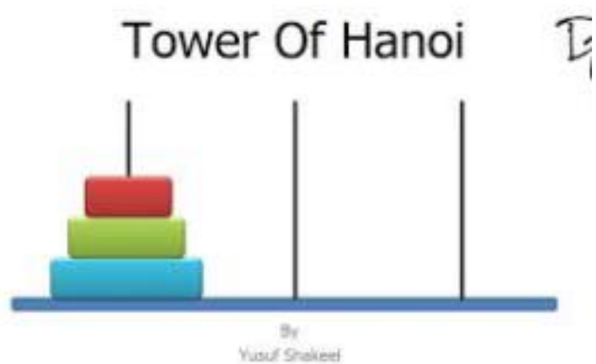
Sequences & Series

Recurrence Relation

A recurrence relation is an equation for a sequence of numbers, where each number (except for the base case) is given in terms of previous numbers in the sequence. These are very useful for determining the complexity of recursive algorithms. Some well know examples of recurrence relations are:

Fibonacci Sequence: $\boldsymbol{F(n) = F(n-1) + F(n-2)\ for\ n > 1, F(0) = 0, F(1) = 1}$

Number of moves to solve the towers of Hanoi: $\boldsymbol{M(n) = 2M(n-1) + 1\ \ for\ n > 1, M(1) = 1}$



Iteration

We will examine two methods of solving these relations, so that we have a closed form solution (not in terms of a previous value). The first method is called **iteration** (like an iterative algorithm). Let's look at a simple case:

$f(n) = f(n - 1) + 10$ for $n > 1$ $\quad f(1) = 5$

For example, $f(4) = f(3) + 10 = f(2) + 10 + 10 = f(1) + 10 + 10 + 10 = 5 + 10 + 10 + 10 = 35$.

In general, $f(n) = 10 + f(n - 1) = 10 + 10 + f(n - 2) = 10 + 10 + \ldots + 10 + f(1)$.
Generalizing, we get: $f(n) = k*10 + f(n - k)$.
This ends when we reach the base case $n = 1$, which occurs when $k = n - 1$. At this point, we have:
$f(n) = (n - 1)*10 + f(1) = 10n - 10 + 5 = 10n - 5$, which is our solution.

We can apply this idea to analysis of the recursive factorial method:

```
// precondition: n >= 0
int factorial(int n) {
    if (n == 0 || n == 1) return 1;
    return n * factorial(n – 1);
}
```

Let's measure the running time in terms of the number of calls to the method, since each call requires O(1) time. The number of calls is:

$f(n) = 1 + f(n - 1)$ for $n > 1$

$f(1) = 1$

Iteration yields: $f(n) = 1 + f(n - 1) = 1 + 1 + f(n - 2) = 1 + 1 + 1 + f(n - 3) = 1 + 1 + 1 + 1 + f(n - 4)$

From here, we can see that the pattern is: $f(n) = k + f(n - k)$

This will end when we reach the stopping case of $n = 1$, so we set $n - k = 1$ and get $k = n - 1$. Substituting this into the pattern we get $f(n) = n - 1 + f(1) = n - 1 + 1 = n$. Thus, we have n recursive calls for $n > 0$ and the overall running time is O(n).

We can do something similar for binary search. We can measure the number of recursive calls as:

$B(n) = 1 + B(n/2)$

$B(1) = 1$

Using iteration gives:

$B(n) = 1 + B(n/2) = 1 + 1 + B(n/4) = 1 + 1 + 1 + B(n/8)$.

This generalizes to:

$B(n) = k + B(n / 2^k)$

The ends when $n / 2^k = 1$ or $k = \log n$. Using $k = \log n$ yields:

$B(n) = (\log n) + B(1) = 1 + \log n$

Now, how about the number of move to solve the **towers of Hanoi** problem:

$M(n) = 2M(n-1) + 1 = 2 * (2M(n-2) + 1) + 1 = 4M(n-2) + 2 + 1 = 4(2M(n-3) + 1) + 2 + 1 = 8M(n-3) + 4 + 2 + 1$

Generalizing, we get:

$M(n) = 2^k * M(n-k) + 2^{k-1} + 2^{k-2} + \ldots + 2 + 1$

This also ends when we reach $k = n - 1$, since $M(1)$ is our base case.

$M(n) = 2^{n-1}M(1) + 2^{n-2} + \ldots + 2^0$

Since $M(1)$ is 1, we get:

$$M(n) = \sum_{i=0}^{n-1} 2^i$$

We will use the following fact in order to simplify this summation.

Fact: $\sum_{i=0}^{x} k^i = 1 + k + k^2 + \ldots + k^x = (k^{x+1} - 1) / (k - 1)$

Substituting $k = 2$ and $x = n - 1$, we get:

$2^{n-1} + 2^{n-2} + \ldots + 2^0 = (2^n - 1) / (2 - 1) = 2^n - 1$

Powers

Powers are the second method of solving recurrence relations. We can write the definition of computing x^n (x to the n^{th} power) in multiple ways. The simplest is:

$$x^0 = 1$$
$$x^n = x * x^{n-1}\ if\ n > 0$$

This leads to the following recursive method:

```
int power2(int x, int n) {
        if (n <= 0) return 1;
        else return x * power2(x, n-1);
}
```

Example 1: Solve the following recurrence relation using iteration:

$$F(n) = F(n-1) + 3, \qquad n > 2$$
$$F(2) = 1$$

Example 2: Solve the following recurrence relation using iteration:

$$a_n = 2a_{n-1}, \qquad n \geq 1$$
$$a_0 = 3$$

Example 1:

$$F(n) = F(n-1) + 3$$

$$F(2) = 1,$$

$$F(3) = F(2) + 3 = 1 + 3 = 4$$

$$F(4) = F(3) + 3 = 4 + 3 = 7$$

$$F(5) = F(4) + 3 = 7 + 3 = 10$$

Generalising

$$F(n) = 3n - 2$$

Example 2:

$$a_n = 2a_{n-1}, \qquad n \geq 1$$

$$a_0 = 3$$

$$a_1 = 2(3) = 6,$$

$$a_2 = 2(6) = 12,$$

$$a_3 = 2(12) = 24$$

Generalising

$$a_n = 3 \cdot 2^n$$

Example 3

Let $n \geq 0$ and find the number s_n of words from the alphabet $\Sigma = \{0, 1\}$ of length n not containing the pattern 11 as a subword.

Solution.
Clearly, $s_0 = 1$(empty word) and $s_1 = 2$. We will find a recurrence relation for $s_n, n \geq 2$. Any word of length n with letters from Σ begins with either 0 or 1. If the word begins with 0, then the remaining $n - 1$ letters can be any sequence of 0's or 1's except that 11 cannot happen. If the word begins with 1 then the next letter must be 0 since 11 can not happen; the remaining $n - 2$ letters can be any sequence of 0's and 1's with the exception that 11 is not allowed. Thus the above two categories form a partition of the set of all words of length n with letters from Σ and that do not contain 11. This implies the recurrence relation

$$s_n = s_{n-1} + s_{n-2}, \quad n \geq 2 \blacksquare$$

Example 4

(i) Find the first 10 terms of the following sequence:

$$U_n = n^2 + 2n + 4$$

<u>Solution:</u>

4, 7, 12, 19, 28, 39, 52, 67, 84, 103.

(ii) Find the first 10 terms of the following sequence:

$$U_n = \frac{3}{n} + 2n^4 + 6$$

<u>Solution:</u>

This sequence is not defined at 0, so start with $n = 1$

11, 39.5, 169, ...

Arithmetic Progressions

An arithmetic progression is a sequence of numbers which has a constant difference between successive terms.

Example :

$$1, 5, 9, 13, 17, \ldots\ldots (\text{common difference} = 4)$$

$$10, 3, -4, -11, \ldots\ldots (\text{common difference} = -7)$$

The general notation for an A.P. is usually

$$a,\ a+d, a+2d, a+3d, \ldots$$

Arithmetic progressions are completely characterised by two numbers :

$a =$ the first term

$d =$ the common difference

In the context of arithmetic progressions, we usually want to calculate one of the following:

(i) The **n**th **term**: is given by

$$a_n = a + (n-1)d$$

(ii) The **sum** of the first n terms is given by:

$$S_n = a_1 + a_2 + ...a_n = \frac{n}{2}[2a + (n-1)d]$$

Note: The sum of the first n positive integers is

$$1+2+3+4+...+n = \frac{n(n+1)}{2}$$

Exercise:

Consider the A.P. 3, 8, 13, ...

Determine the 8th and 19th terms and the sum of the first 100 terms.

Exercise:

The 5th term of an A.P. is 18 and the 12th term is 46. Determine the 17th term.

Geometric Progressions

A sequence which has a constant ratio between successive terms is called a **geometric** progression (G.P.). The constant is called the **common ratio**.

Examples :

$$1, 2, 4, 8, ... (\text{ratio} = 2)$$

$$20, 10, 5, \frac{5}{2}, \frac{5}{4}, ... \quad (\text{ratio} = \frac{1}{2})$$

General form :

$$a,\ ar,\ ar^2,\ ar^3, ...$$

where a is the **first term** and r is the **common ratio**.

The **n**$^{\text{th}}$ **term** of a geometric progression:

$$a_n = ar^{n-1}$$

Sum of the first n terms in a geometric progression:

$$S_n = a + ar + ar^2 + .. + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} \quad for\ r > 1$$

$$s_n = \frac{a(1-r^n)}{1-r} \text{ for } r < 1$$

Exercise

Determine the 9th term of the G.P : 2, 4, 8, 16…..and the sum of the first 10 terms.

Exercise

In a G.P. the 5th term is 8 times the 2nd term and the sum of the 6th and 8th terms is 160. Determine the common ratio, the first term and the sum of the 4th to 10th terms inclusive.

Exercise:

Which term of the series 729, 243, 81, ...is $\frac{1}{27}$?

Exercise:

Three numbers are in arithmetic progression. Their sum is 15 and product is 45. Determine the numbers.

NB

Exercise: Find a formula for $A_n = 1+2+3+4+...+n$

Solution: Observe that $k^2-(k-1)^2 = 2k-1$

From this $1^2-0^2 = 2\cdot 1-1$

$$2^2-1^2 = 2\cdot 2-1$$

$$3^2-2^2 = 2\cdot 3-1$$

. .

. .

$$n^2-(n-1)^2 = 2\cdot n-1$$

Adding both columns:

$$n^2-0^2 = 2(1+2+3+...+n)-n$$

Solving for the sum,

$$1+2+3+4+....n = \frac{n^2}{2}+\frac{n}{2} = \frac{n(n+1)}{2}$$

Exercise: Find a formula for $A_n = 1^2 + 2^2 + 3^2 + 4^2 + ... + n^2$

Solution: Observe that $k^3 - (k-1)^3 = 3k^2 - 3k + 1$

From this $1^3 - 0^3 = 3 \cdot 1^2 - 3 \cdot 1 + 1$

$$2^3 - 1^3 = 3 \cdot 2^2 - 3 \cdot 2 + 1$$

$$3^3 - 2^3 = 3 \cdot 3^2 - 3 \cdot 3 + 1$$

. .

. .

$$n^3 - (n-1)^3 = 3 \cdot n^2 - 3 \cdot n + 1$$

Adding both columns:

$$n^3 - 0^3 = 3(1^2 + 2^2 + 3^2 + ... + n^2) - 3(1 + 2 + 3 + ... + n) + n$$

From above we know that,

$$1 + 2 + 3 + 4 + ...n = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$$

So $n^3 - 0^3 = 3(1^2 + 2^2 + 3^2 + ... + n^2) - 3\left(\frac{n(n+1)}{2}\right) + n$

Solving for the sum, $1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Summing a Geometric Series

When we need to sum a Geometric Sequence, there is a handy formula.

To sum:

$$a + ar + ar^2 + ... + ar^{(n-1)}$$

Each term is ar^k, where k starts at 0 and goes up to n-1

Use this formula:

$$\sum_{k=0}^{n-1}(ar^k) = a\left(\frac{1-r^n}{1-r}\right)$$

a is the first term
r is the **"common ratio"** between terms
n is the number of terms

1. Express this series of five terms using sigma notation: 3, 5, 9, 17, 33.

- $\sum_{n=1}^{5} 3 + 2n$
- $\sum_{n=1}^{5} 2^n + 1$
- $\sum_{n=1}^{\infty} 3 + 2n$
- $\sum_{n=1}^{\infty} 2^n + 1$

2. Find the rule for this sequence: 0, 1, 1, 2, 3, 5, 8, 13.

- $a_n = a_{n-1} \cdot a_{n-2}$
- $a_n = n \cdot a_{n-1}$
- $a_n = a_{n-1} + a_{n-2}$
- $a_n = 2a_{n-1} - 1$