Propositional Logic

- Logic is the study of reasoning. The British mathematician George Boole (1815-1864) is the man who made logic mathematical. Logic can be used in programming and it be applied to the analysis and automation of reasoning about software and hardware. This is why it is considered a part of theoretical computer science. Since reasoning plays an important role in intelligent behaviour, logic is closely related to artificial intelligence.

Truth Tables

The goal of this section is to understand both mathematical conventions and the basics of mathematical reasoning. We will learn how to think and write like a mathematician and we drop some formalisation until we familiarise ourselves with convention. For a given statement P, we will look at the possible truth values of P. Namely, either P is True or P is False.

P
T
F

If we are analyzing two statements P and Q at the same time, we need to consider all possible combinations of truth values.

P	Q
T	T
T	F
F	T
F	F

The first row is the case when both are true. The second row is the case where P is true and Q is false. The third row is the case when P is false and Q is true. The fourth row is the case when both P and Q are false. Using truth tables, we can "define" what certain symbols and words mean in mathematics.

In mathematics, the terms "and", "or", "not" have precise meaning and are often written as symbols instead of words. We will use these symbols to construct more complicated statements from ones that we have.

$\neg$ = "NOT":

$\neg P$ is true when P is false and $\neg P$ is false when P is true.

P	$\neg P$
T	F
F	T

$\wedge$ = "AND":

In order for the statement $P \wedge Q$ to be true, both P and Q must be true. Otherwise, it is false.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Now let's move one step further and write the truth table for $\neg(P \wedge Q)$ which will be true exactly when $(P \wedge Q)$ is false.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

It is crucial to note the placement of the parenthesis and how this is crucial to the meaning of the sentence. Without the parenthesis, we read from left to right. So $\neg P \wedge Q$ is understood to mean $(\neg P) \wedge (Q)$. Notice the difference in the truth tables:

P	Q	$\neg P$	$\neg P \wedge Q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

$\vee$ = "OR":

In mathematics, we use an inclusive "or". This means that for $P \vee Q$ to be true, we require that one of them be true but allow for the possibility that both are true. Therefore, the truth table is as follows:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

$\Rightarrow$ = "IMPLIES"

We will now discuss what it means for a statement to imply another statement. The symbol we use for this is $\Rightarrow$ and $P \Rightarrow Q$ is often read "P implies Q". Logically, this statement is equivalent to $\neg P \vee Q$ and to understand the mathematical implication consider the following: If your mother tells you "If I go to the candy store, I'll buy you a candy", when has she told the truth? Well, there is really only one way that she has lied - namely if she goes to the candy store and doesn't buy you a candy. Notice that if she doesn't go to the candy store, it doesn't really matter what happens later since her promise was conditional on her going to the candy store.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$\iff$ = "IF AND ONLY IF":

We now discuss what it means for two statements to be equivalent. The symbol we use for this is $\iff$ and $P \iff Q$ is often read "P if and only if Q". Logically, this statement is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ and means that the two statements have exactly the same truth values or truth tables. Let us look at the truth table for $P \iff Q$.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \iff Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

TAUTOLOGIES and CONTRADICTIONS:

In mathematics, a *tautology* is a statement that is always true regardless of the "circumstances". Here is a simple example:

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

The next example is very common in mathematics. We consider the implication $P \Rightarrow Q$ and the *contrapositive* $\neg Q \Rightarrow \neg P$. It turns out that these two are equivalent, i.e. the statement $(P \Rightarrow Q) \iff (\neg Q \Rightarrow \neg P)$ is a tautology.

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$	$(P \Rightarrow Q) \iff (\neg Q \Rightarrow \neg P)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

In mathematics, a *contradiction* is the assertion of a statement and its negation, or equivalently, a statement that can never be true. A contradiction is equivalent to the negation of a tautology.

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Exercises

1. Use a truth table to verify that the following are equivalent formulas:
   1. $F \wedge (G \vee H) \sim (F \wedge G) \vee (F \wedge H)$
   2. $F \vee (G \wedge H) \sim (F \vee G) \wedge (F \vee H)$

   <u>De Morgan's Law</u>

   3. $\neg (F \wedge G) \quad \sim \quad \neg F \quad \vee \quad \neg G$
   4. $\neg (F \vee G) \quad \sim \quad \neg F \quad \wedge \quad \neg G$

(1)

$F \wedge (G \vee H) \sim (F \wedge G) \vee (F \wedge H)$

F	G	H	$(G \vee H)$	$F \wedge (G \vee H)$	$(F \wedge G)$	$(F \wedge H)$	$(F \wedge G) \vee (F \wedge H)$
F	F	F	F	F	F	F	F
F	F	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	T	T	T	F	F	F	F
T	F	F	F	F	F	F	F
T	F	T	T	T	F	T	T
T	T	F	T	T	T	F	T
T	T	T	T	T	T	T	T

(2)

$F \vee (G \wedge H) \sim (F \vee G) \wedge (F \vee H)$

F	G	H	$(G \wedge H)$	$F \vee (G \wedge H)$	$(F \vee G)$	$(F \vee H)$	$(F \vee G) \wedge (F \vee H)$
F	F	F	F	F	F	F	F
F	F	T	F	F	F	T	F
F	T	F	F	F	T	F	F
F	T	T	T	T	T	T	T
T	F	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	T	F	F	T	T	T	T
T	T	T	T	T	T	T	T

(3) $\neg(F \land G) \sim \neg F \lor \neg G$

F	G	$(F \land G)$	$\neg(F \land G)$	$\neg F$	$\neg G$	$\neg F \lor \neg G$
F	F	F	T	T	T	T
F	F	F	T	T	T	T
F	T	F	T	T	F	T
F	T	F	T	T	F	T
T	F	F	T	F	T	T
T	F	F	T	F	T	T
T	T	T	F	F	F	F
T	T	T	F	F	F	F

(4) $\neg(F \vee G) \sim \neg F \wedge \neg G$

F	G	$(F \vee G)$	$\neg(F \vee G)$	$\neg F$	$\neg G$	$\neg F \vee \neg G$
F	F	F	T	T	T	T
F	F	F	T	T	T	T
F	T	T	F	T	F	F
F	T	T	F	T	F	F
T	F	T	F	F	T	F
T	F	T	F	F	T	F
T	T	T	F	F	F	F
T	T	T	F	F	F	F

Exercise: Construct truth tables for the following propositions:

$$q \wedge r \wedge \neg p$$

$$p \vee \neg q \vee \neg r$$

$$(q \wedge \neg p) \Rightarrow r$$

$$(q \vee \neg p) \Rightarrow \neg r$$

$$\neg r \Rightarrow (q \wedge \neg p)$$