Oddtown problem: For each subset S_i of the town, we can create a 0/1 vector v_i of length n, where the jth entry of v_i is 1 if and only if $j \epsilon S_i$. The oddtown rules can be expressed in terms of inner products of the vectors v_i. The rules state that $v_i v_i$=1(mod2) for all i, and $v_i v_j$=0(mod2) for all $i \neq j$. Then we can see that the vectors v_i are linearly independent. If the vectors v_i were linearly dependent, then there would be a non trivial linear combination $\sum_{i=1}^{m} \lambda_i . v_i$=0. But this would imply that $\sum_{i=1}^{m} \lambda_i \; v_i . v_i$=0, which contradicts the first rule. Therefore, the vectors v_i must be linearly independent. In the end the number of linearly independent vectors in F_2^n cannot exceed n. Therefore, the number of subsets in oddtown cannot exceed n. In other words, this shows that the maximum number of clubs in Oddtown is equal to the number of people in the town which is n.

Eventown problem: Let C = C_1,C_2,...,C_m be a family of subsets of [n]=1,2,...,n such that (i) C_i is even for all i, and (ii) $C_i \cap C_j$ is even for all $i \neq j$. Let V be the vector space of all functions f:[n]→0,1. For each i, define the vector $v_i \epsilon V$ by $v_i(j)$=1 if and only if $j \epsilon C_i$. Then the condition (i) above is equivalent to the condition that $v_i v_i$=0 for all i, where the vectors belong to V. The condition (ii) above is equivalent to the condition that $v_i v_j$=0 for all $i \neq j$. Let W be the subspace of V spanned by the vectors v_1,v_2,...,v_m. Then the conditions (i) and (ii) above imply that W is an even subspace of V. It is well known that the dimension of an even subspace of V is at most $2^{[n/2]}$. Therefore, dimW $\leq 2^{[n/2]}$. Thus the maximum number of clubs in an Eventown with n residents is $2^{[n/2]}$

2)Let A be Hermitian. Then for any vectors x,y ϵ V, we have ⟨x,Ay⟩=⟨x,Ay⟩=⟨Ax,y⟩. Taking y=x, we get ⟨x,Ax⟩=⟨Ax,x⟩. Let ⟨x,Ax⟩=⟨Ax,x⟩ for all vectors xV. Then for any vectors x,yV, we have

⟨x,Ay⟩=⟨x,A(y+x)Ax⟩

=⟨x,Ay+Ax⟩⟨x,Ax⟩

=⟨Ax,y+x⟩⟨Ax,x⟩

=⟨Ax,y⟩.

Thus we can say that the operator A in V is hermitian if these conditions suffice

3)It is clear that in all 3 cases, the matrices consist of a Kronecker product of two individual matrices:

(i) $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \bigotimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bigotimes \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \bigotimes \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$

$Eigenvalues for \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} are (7\pm \sqrt{5}3)/2$ and eigenvalues for $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} are \pm 1$

We are aware that the eigenvalues of the given matrices are just products of the eigenvalues of individual matrices whose Kronecker product produces it. Thus the eigenvalues for (i) which is the same for (ii) are $\pm$ $(7\pm \sqrt{5}3)/2$ and the eigenvalues for (iii) are -2, $(53\pm7\sqrt{57})/2$

4)To show that |x|=⟨x,x⟩ is a valid norm, we need to show that it satisfies the

following properties.
It is non negative and zero only for the zero vector.
It is homogeneous, meaning that |ax|=|a||x| for all scalars a and vectors x.
It satisfies the triangle inequality, meaning that |x+y|≤ |x|+|y| for all vectors x and y.
1: Since ⟨x,x⟩≥0 for all vectors x, it follows that |x|= ⟨x,x⟩≥0. Furthermore, |x|=0 if and only if ⟨x,x⟩=0, which is true if and only if x=0.
2: For any scalar a and vector x, we have |ax|=⟨ax,ax⟩=a$^2 <x,x> = |a| <x,x> = |a||x|$.
$3 : Foranyvectorsxandy, wehave|x+y| = <x+y, x+y> = <x,x> + 2<x,y> + <y,y>$≤⟨x,x⟩+2⟨x,y⟩+⟨y,y⟩=|x|+|y|. The first inequality follows from the Cauchy-Schwarz inequality.
Therefore, |x|=⟨x,x⟩ is a valid norm.
To show that d(x,y)=|x-y| is a valid metric, we need to show that it satisfies the following properties:
It is non negative and zero only for x=y.
It is symmetric, meaning that d(x,y)=d(y,x) for all vectors x and y.
It satisfies the triangle inequality, meaning that d(x,z)≤d(x,y)+d(y,z) for all vectors x, y, and z.
1: Since |x-y|≥0 for all vectors x and y, it follows that d(x,y)=|x-y|≥0. Furthermore, d(x,y)=0 if and only if x=y.
2: For any vectors x and y, we have d(x,y)=|x-y|=|y-x|=d(y,x).
3: For any vectors x, y, and z, we have d(x,z)=|x-z|≤|x-y|+|y-z|=d(x,y)+d(y,z).
Therefore, d(x,y)=|x-y| is a valid metric.

Let V be the set of all continuous real functions on [1,1]. We will show that V is a vector space over R with the operations (f+g)(t)=f(t)+g(t) and (αf)(t)=αf(t).
Let f,g ϵ V. Then f and g are both continuous on [1,1]. Therefore, their sum f+g is also continuous on [1,1]. This is because the sum of two continuous functions is continuous. Therefore, f+g ϵ V.
Let f ϵ V and α ϵ R. Then f is continuous on [1,1]. Therefore, αf is also continuous on [1,1]. This is because the product of a continuous function and a real number is continuous. Therefore, αf ϵ V.
Let f, g ϵ V and α, β ϵ R. Then (α+β)f=(αf)+(βf) and (α β)f=α(βf). This is because the distribution property holds for continuous functions. Thus scalar multiplication distributes over vector addition in V.
Therefore, V is a vector space over R with the operations (f+g)(t)=f(t)+g(t) and (αf)(t)=αf(t)

Let V be the set of all continuous real functions on [1,1] with the inner product ⟨f,g⟩ defined as the given integral. We will show that $<$f$_n, f_m > \rightarrow \epsilon$ as n,m$\rightarrow$ ∞.Let n>m. We integrate this function by integrating separately on the 3 limits.
From this we obtain ⟨f$_n, f_m$>$\rightarrow$ ϵ ≈0 as n$\rightarrow$ ∞ and conclude f$_n$ is Cauchy. Let f(t)=lim$_n$$\rightarrow$ ∞ f$_n(t).However, f_n$ does not converge in V. This is because f$_n$(0)=1 for all n, but f(0) is not defined for any f ϵ V. Thus the inner product

space is not complete.
5)Let T:L(A,B)$\rightarrow$A$\bigotimes$B be defined by U=$\sum_{i,j}$ αji|w$_j$><v_i|$\rightarrow$u= $\sum_{i,j}$ αij|v$_i$>$\bigotimes$|w$_j$|.$WeshouldthatTisbijec$
αij|v$_i$>$\bigotimes$|w$_j$|ϵA$\bigotimes$B. Then we can define the linear transformation U:A$\rightarrow$B by
U(x)=$\sum_{i,j}$ αij$\langle$v$_i$|x>|w_j>$.WecanseethatT(U) = u.Implyingsurjectivity.NowtoshowinjectivityLetU,V\epsilon$
such that T(U)=T(V). Then we have $\sum_{i,j}$ αij|v$_i$>$\bigotimes$|w$_j$| =$\sum_{i,j}$ βij|v$_i$>$\bigotimes$|w$_j$|$.Therefore,$$\alpha$ij
= βij for all i,j. This means that U=V. Which proves T is injective and hence
bijective.
Now, let {<U|V>$_{HS}$:=$\langle$TU|TV$\rangle$=$\langle$u|v$\rangle$ on L(A,B). We will show that {<U|V>$_{HS}$
reduces to tr(UV).
Let U=$\sum_{i,j}$ αji|w$_j$><v_i|$andV$ =$\sum_{i,j}$ ji|w$_j$><v_i|$.Thenwehave${$\langle$U|V$\rangle_{HS}$=$\langle$TU|TV$\rangle$
=$\langle\sum_{i,j}$ αji|w$_j$><v_i|$\bigotimes$|w$_j$><v_i|$\sum_{i,j}$ βji|w$_j$><v_i|> = <$\sum_{i,j}$ αji|w$_j$><v_i|$\sum_{i,j}$
βji|w$_j$><v_i|> =$\sum_{i,j}$ αji βji $\langle$|w$_j$||w_j|><|v_i||v_i|> =$\sum_{i,j}$ αji βji =tr(U*V). Thus
$\langle$U|V$\rangle$HS reduces to tr(UV).