Lab 8

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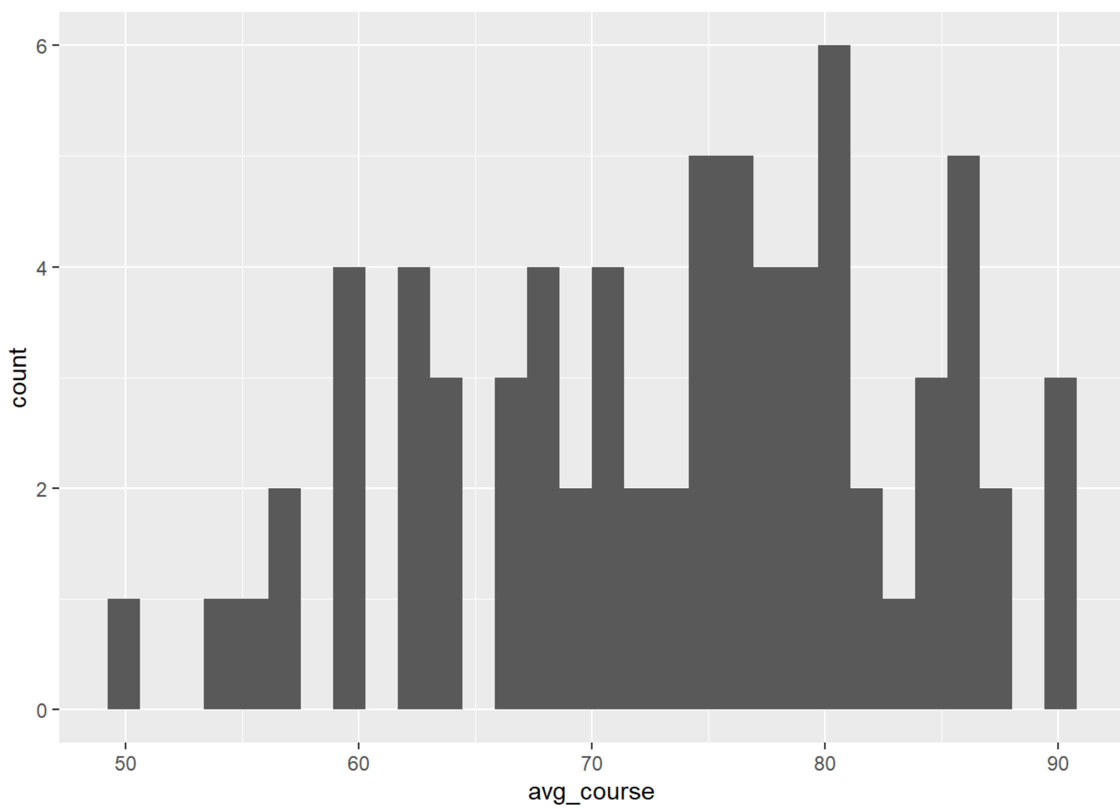
Exercise 1: Simple EDA: for each school, calculate the sample average of the course scores and plot the distribution of these averages in a histogram. What does this plot reveal?

```
data(Gcsemv, package = "mlmRev")

# Make Male the reference category and rename variable
Gcsemv$female <- relevel(Gcsemv$gender, "M")

# Use only total score on coursework paper
GCSE <- subset(x = Gcsemv,
               select = c(school, student, female, course))

# Count unique schools and students
m <- length(unique(GCSE$school))
N <- nrow(GCSE)
```

```
GCSE %>%
  dplyr::group_by(school) %>%
  na.omit()%>%
  dplyr::summarise(avg_course = mean(course)) %>%
  dplyr::ungroup() %>%
  ggplot(aes(x = avg_course))+
  geom_histogram()
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```



This plot reveals that the school average of course falls within the range 50 to 90 with a mean about 75. Distribution of the school average is approximately normal.

```
# Pooled and unpooled models
pooled <- stan_glm(course ~ 1 + female, data = GCSE, refresh = 0)
unpooled <- stan_glm(course ~ -1 + school + female,data=GCSE, refresh = 0)
```

Model1: Variance components model

```
# Model1: Variance components model
mod1 <- stan_lmer(formula = course ~ 1 + (1 | school),
                  data = GCSE,
                  seed = 349,
                  refresh = 0)

prior_summary(object = mod1)
```

```
## Priors for model 'mod1'
## ------
## Intercept (after predictors centered)
##   Specified prior:
##     ~ normal(location = 0, scale = 10)
##   Adjusted prior:
##     ~ normal(location = 0, scale = 163)
##
## Auxiliary (sigma)
##   Specified prior:
##     ~ exponential(rate = 1)
##   Adjusted prior:
##     ~ exponential(rate = 0.061)
##
## Covariance
##  ~ decov(reg. = 1, conc. = 1, shape = 1, scale = 1)
## ------
## See help('prior_summary.stanreg') for more details
```

```
# observed standard deviation of the course variable
sd(GCSE$course, na.rm = T)
```

```
## [1] 16.32096
```

```
print(mod1, digits = 3)
```

```
## stan_lmer
##  family:       gaussian [identity]
##  formula:      course ~ 1 + (1 | school)
##  observations: 1725
## ------
##             Median MAD_SD
## (Intercept) 73.749  1.137
##
## Auxiliary parameter(s):
##       Median MAD_SD
## sigma 13.823  0.230
##
## Error terms:
##  Groups   Name        Std.Dev.
##  school   (Intercept)  8.917
##  Residual             13.824
## Num. levels: school 73
##
## ------
## * For help interpreting the printed output see ?print.stanreg
## * For info on the priors used see ?prior_summary.stanreg
```

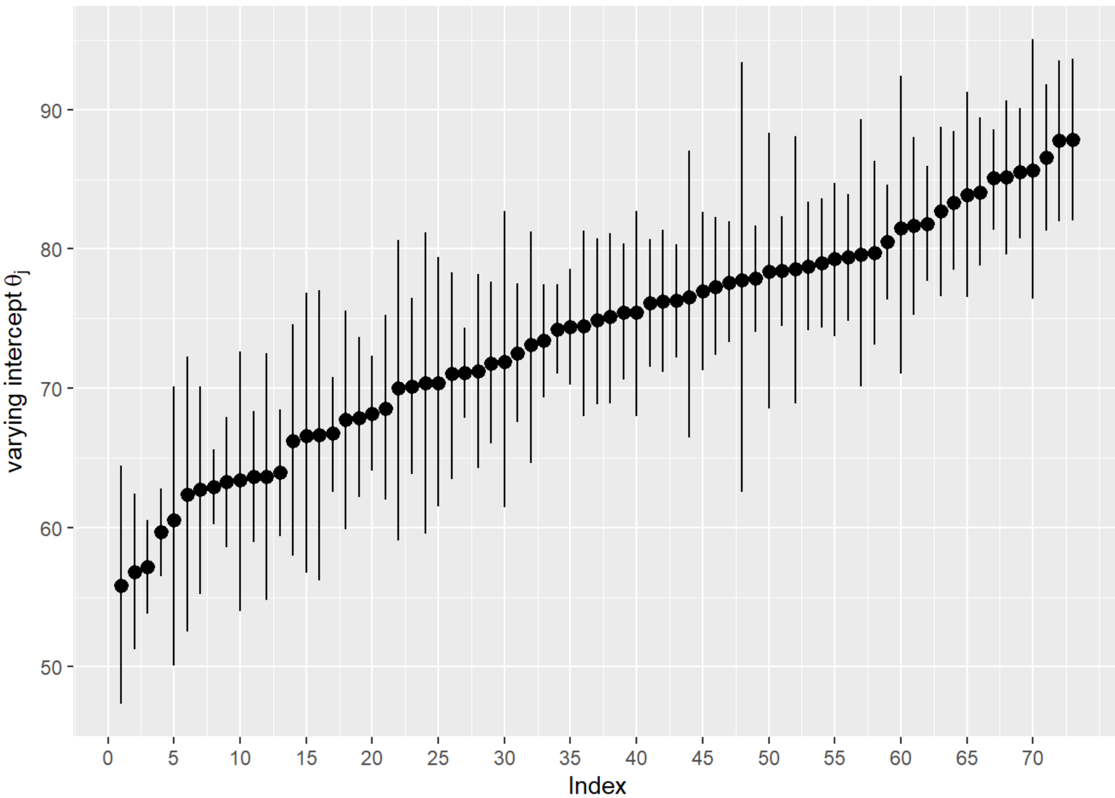
```
summary(mod1,
        pars = c("(Intercept)", "sigma", "Sigma[school:(Intercept),(Intercept)]"),
        probs = c(0.025, 0.975),
        digits = 3)
```

```
##
## Model Info:
##  function:     stan_lmer
##  family:       gaussian [identity]
##  formula:      course ~ 1 + (1 | school)
##  algorithm:    sampling
##  sample:       4000 (posterior sample size)
##  priors:       see help('prior_summary')
##  observations: 1725
##  groups:       school (73)
##
## Estimates:
##                                         mean    sd      2.5%    97.5%
## (Intercept)                            73.761   1.167  71.468  75.985
## sigma                                  13.824   0.236  13.362  14.292
## Sigma[school:(Intercept),(Intercept)]  79.514  15.553  54.263 114.027
##
## MCMC diagnostics
##                                       mcse  Rhat  n_eff
## (Intercept)                           0.043 1.005  745
## sigma                                 0.003 1.000 5793
## Sigma[school:(Intercept),(Intercept)] 0.490 1.004 1006
##
## For each parameter, mcse is Monte Carlo standard error, n_eff is a crude measure of effective sample size, and Rhat is th
e potential scale reduction factor on split chains (at convergence Rhat=1).
```

Exercise 2: Report on the posterior estimates of μ_θ, σ and τ^2.

$E(\mu_\theta|Y) = 73.761$, $E(\sigma|Y) = 13.824$, $E(\tau^2|Y) = 79.514$.

```r
## Posterior draws
mod1_sims <- as.matrix(mod1)
par_names <- colnames(mod1_sims)

# obtain draws for mu_theta
mu_theta_sims <- as.matrix(mod1, pars = "(Intercept)")

# obtain draws for each school's contribution to intercept
theta_sims <- as.matrix(mod1,
                        regex_pars ="b\\[\\(Intercept\\) school\\:")

# to finish: obtain draws for sigma and tau^2
sig_sims <- as.matrix(mod1,
                      pars = "sigma")
tau2_sims <- as.matrix(mod1,
                       pars = "Sigma[school:(Intercept),(Intercept)]")

# 73 school's varying intercept
int_sims <- as.numeric(mu_theta_sims) + theta_sims

# posterior mean
int_mean <- apply(int_sims, MARGIN = 2, FUN = mean)

# credible interval
int_ci <- apply(int_sims, MARGIN = 2, FUN = quantile, probs = c(0.025, 0.975))
int_ci <- data.frame(t(int_ci))

# combine into a single df
int_df <- data.frame(int_mean, int_ci)
names(int_df) <- c("post_mean","Q2.5", "Q97.5")

# sort DF according to posterior mean
int_df <- int_df[order(int_df$post_mean),]

# create variable "index" to represent order
int_df <- int_df %>% mutate(index = row_number())

# plot posterior means of school-varying intercepts, along with 95 CIs
ggplot(data = int_df, aes(x = index, y = post_mean))+
  geom_pointrange(aes(ymin = Q2.5, ymax = Q97.5))+
  scale_x_continuous("Index", breaks = seq(0,m, 5)) +
  scale_y_continuous(expression(paste("varying intercept ", theta[j])))
```

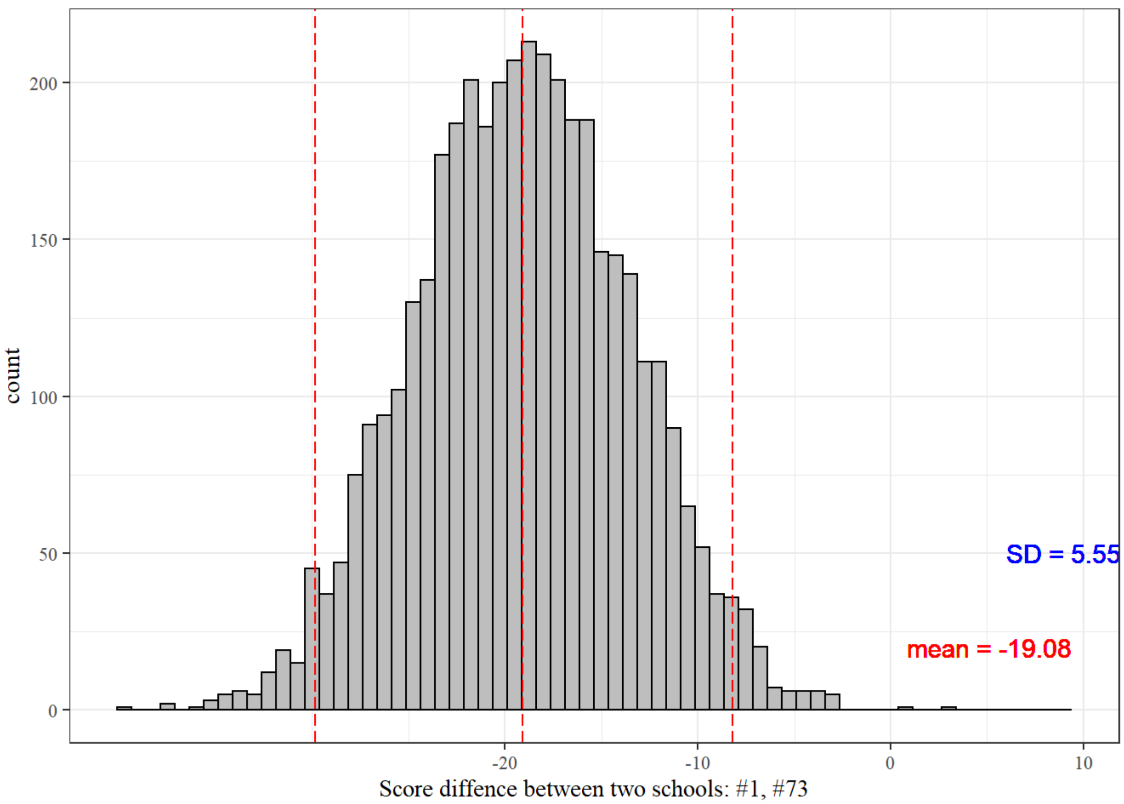


Exercise 3: Choose two schools and report on their difference in average scores with descriptive statistics a histogram and interpretation

```
# the difference between the two school averages (school #1 and #73)
school_diff = int_sims[,1] - int_sims[,73]

# investigate differences of two distributions
mean = mean(school_diff)
sd = sd(school_diff)
quantile = quantile(school_diff, probs = c(0.025, 0.50, 0.975))
names(quantile) = c("Q2.5",'Q50','Q97.5')
diff_df = data.frame(mean, sd, quantile)
round(diff_df,2)
```

```
##         mean   sd quantile
## Q2.5  -19.08 5.55   -29.82
## Q50   -19.08 5.55   -19.07
## Q97.5 -19.08 5.55    -8.21
```

```
# histogram of the differences
ggplot(data = data.frame(school_diff),
       aes(x = school_diff)) +
  geom_histogram(color = "black",
                 fill = "gray",
                 binwidth = 0.75) +
  scale_x_continuous("Score diffence between two schools: #1, #73",
                     breaks = seq(from = -20,
                                  to = 20,
                                  by = 10)) +
  geom_vline(xintercept = c(mean(school_diff),
                            quantile(school_diff,
                                     probs = c(0.025, 0.975))),
             colour = "red",
             linetype = "longdash") +
  geom_text(aes(5.11, 20, label = "mean = -19.08"),
            color = "red",
            size = 4) +
  geom_text(aes(9, 50, label = "SD = 5.55"),
            color = "blue",
            size = 4) +
  theme_bw( base_family = "serif")
```



Interpretation: The expected difference comes to -19.08 with a standard deviation of 5.55 and a wide range of uncertainty. The 95% credible interval is $[-29.82, -8.21]$, so we are 95% certain that the true value of the difference between the two schools lies within the range, given the data.

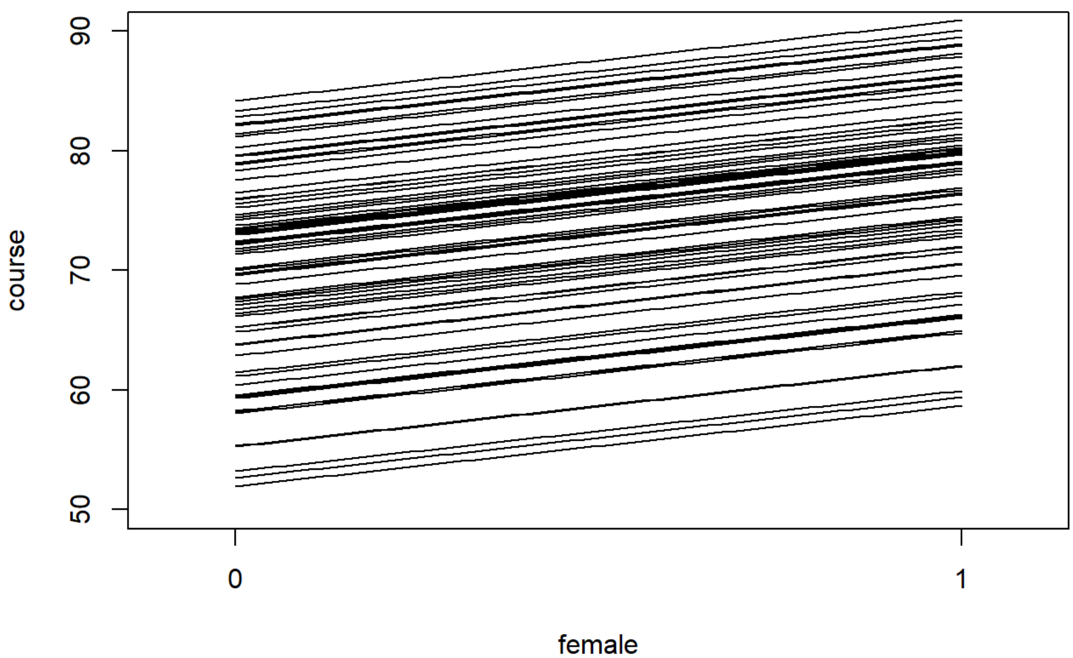
```
# we also can get the proportion of the time that school 20920 has a lower mean than scholl 68255:
prop.table(table(int_sims[,1]<int_sims[,73]))
```

```
## 
##  FALSE   TRUE 
## 0.0005 0.9995
```

This means that the posterior probability that school 20920 is worse than school 68225 is 99.9%.

Model 2: Varying intercept with a single individual-level predictor

```
mod2 <- stan_lmer(formula = course ~ 1 + female + (1 | school),
                  data = GCSE,
                  prior = normal(location = 0,
                                 scale = 100,
                                 autoscale = FALSE),
                  prior_intercept = normal(location = 0,
                                           scale = 100,
                                           autoscale = F),
                  seed = 349,
                  refresh = 0)

# plot varying intercepts
mod2.sims <- as.matrix(mod2)
group_int <- mean(mod2.sims[,1]) #mu_theta
mp <- mean(mod2.sims[,2]) #female F
bp <- apply(mod2.sims[, 3:75], 2, mean)
xvals <- seq(0,1,.01)
plot(x = xvals, y = rep(0, length(xvals)),
     ylim = c(50, 90), xlim = c(-0.1,1.1), xaxt = "n", xlab = "female", ylab = "course")
axis(side = 1, at = c(0,1))
for (bi in bp){
  lines(xvals, (group_int + bi)+xvals*mp)
}
```



Exercise 4: What are the posterior means and credible intervals of μ_θ, β, σ and τ^2?

```
par_names2 = colnames(mod2.sims)
head(par_names2)
```

```
## [1] "(Intercept)"                 "femaleF"
## [3] "b[(Intercept) school:20920]" "b[(Intercept) school:22520]"
## [5] "b[(Intercept) school:22710]" "b[(Intercept) school:22738]"
```

```
tail(par_names2)
```

```
## [1] "b[(Intercept) school:76631]"
## [2] "b[(Intercept) school:77207]"
## [3] "b[(Intercept) school:84707]"
## [4] "b[(Intercept) school:84772]"
## [5] "sigma"
## [6] "Sigma[school:(Intercept),(Intercept)]"
```

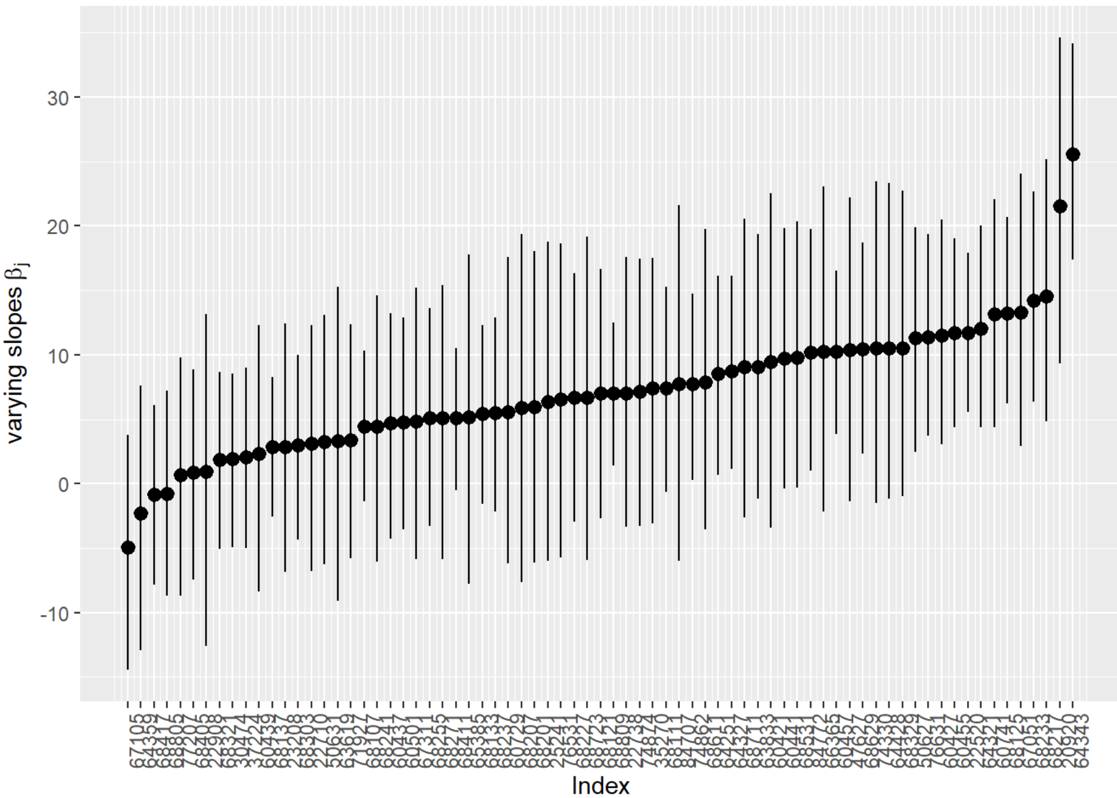
```
summary(mod2,
        pars = c("(Intercept)", "femaleF","sigma", "Sigma[school:(Intercept),(Intercept)]"),
        probs = c(0.025, 0.975),
        digits = 3)
```

```
##
## Model Info:
##  function:     stan_lmer
##  family:       gaussian [identity]
##  formula:      course ~ 1 + female + (1 | school)
##  algorithm:    sampling
##  sample:       4000 (posterior sample size)
##  priors:       see help('prior_summary')
##  observations: 1725
##  groups:       school (73)
##
## Estimates:
##                                          mean    sd      2.5%    97.5%
## (Intercept)                             69.710   1.233  67.271  72.059
## femaleF                                  6.733   0.680   5.393   8.052
## sigma                                   13.424   0.234  12.964  13.890
## Sigma[school:(Intercept),(Intercept)]  81.289  16.558  54.788 121.203
##
## MCMC diagnostics
##                                        mcse  Rhat  n_eff
## (Intercept)                            0.052 1.009  555
## femaleF                                0.010 1.001 4351
## sigma                                  0.004 1.000 3861
## Sigma[school:(Intercept),(Intercept)] 0.715 1.005  536
##
## For each parameter, mcse is Monte Carlo standard error, n_eff is a crude measure of effective sample size, and Rhat is th
e potential scale reduction factor on split chains (at convergence Rhat=1).
```

$E[\mu_\theta|Y] = 69.710$, 95% CI: $[69.271, 72.059]$
$E[\beta|Y] = 6.733$, 95% CI: $[5.393, 8.052]$
$E[\sigma|Y] = 13.424$, 95% CI: $[12.964, 13.890]$
$E[\tau^2|Y] = 81.289$, 95% CI: $[54.788, 121.203]$

Model 3: Allowing for varying slopes across schools

```
mod3 <- stan_lmer(formula = course~ 1+ female + (1 + female | school),
                  data = GCSE,
                  seed = 349,
                  refresh = 0)
mod3_sims <- as.matrix(mod3)

# obtain draws for mu_theta
mu_theta_sims <- as.matrix(mod3, pars = "(Intercept)")

fem_sims <- as.matrix(mod3, pars = "femaleF")
# obtain draws for each school's contribution to intercept
theta_sims <- as.matrix(mod3,
                        regex_pars ="b\\[\\(Intercept\\) school\\:")
beta_sims <- as.matrix(mod3,
                       regex_pars ="b\\[femaleF school\\:")

int_sims <- as.numeric(mu_theta_sims) + theta_sims
slope_sims <- as.numeric(fem_sims) + beta_sims

# posterior mean
slope_mean <- apply(slope_sims, MARGIN = 2, FUN = mean)

# credible interval
slope_ci <- apply(slope_sims, MARGIN = 2, FUN = quantile, probs = c(0.025, 0.975))
slope_ci <- data.frame(t(slope_ci))

# combine into a single df
slope_df <- data.frame(slope_mean, slope_ci, levels(GCSE$school))
names(slope_df) <- c("post_mean","Q2.5", "Q97.5", "school")

# sort DF according to posterior mean
slope_df <- slope_df[order(slope_df$post_mean),]

# create variable "index" to represent order
slope_df <- slope_df %>% mutate(index = row_number())

# plot posterior means of school-varying slopes, along with 95% CIs
ggplot(data = slope_df, aes(x = index, y = post_mean))+
  geom_pointrange(aes(ymin = Q2.5, ymax = Q97.5))+
  scale_x_continuous("Index", breaks = seq(1,m, 1),
                     labels = slope_df$school) +
  scale_y_continuous(expression(paste("varying slopes ", beta[j])))+
  theme(axis.text.x = element_text(angle = 90))
```

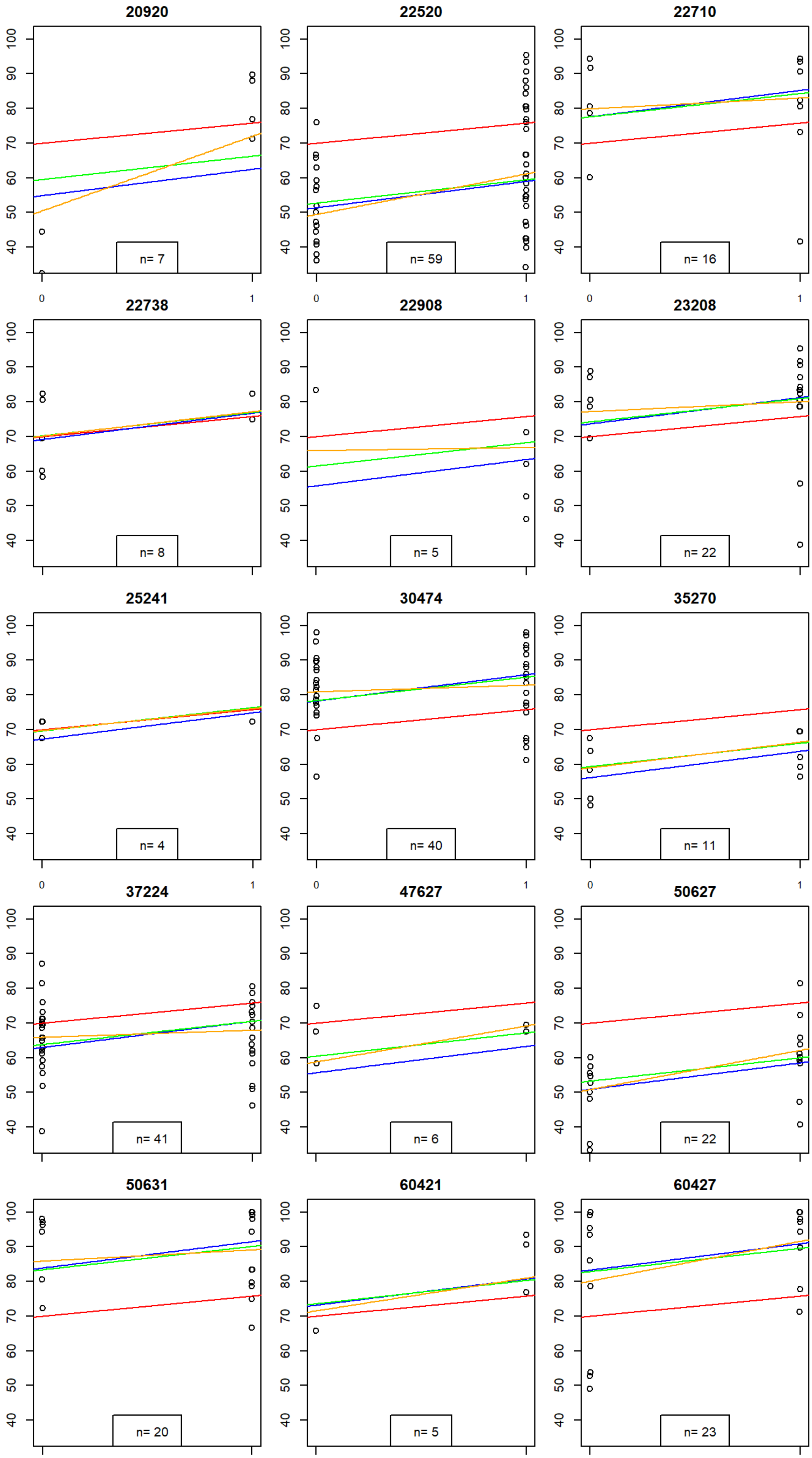


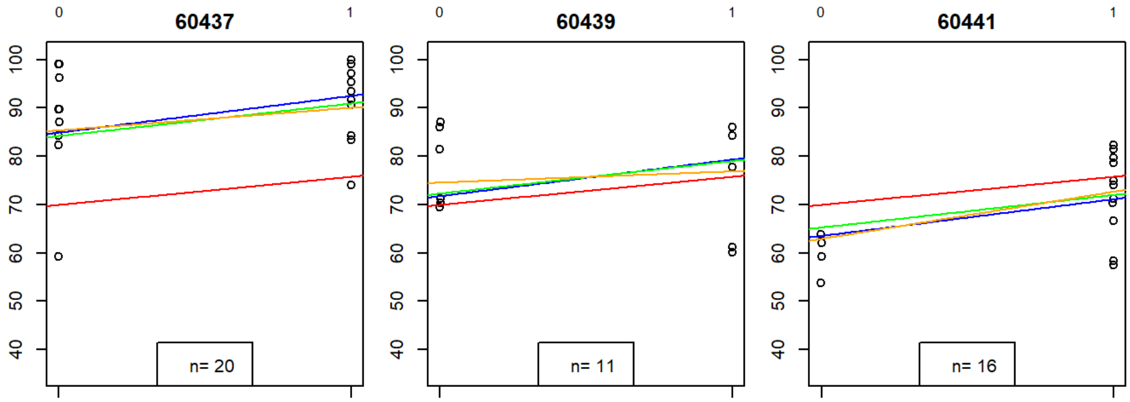
Model Comparison

```
loo1 <- loo(mod1)
loo2 <- loo(mod2)
loo3 <- loo(mod3)
loo_compare(loo1,loo2,loo3)
```

```
##      elpd_diff se_diff
## mod3   0.0       0.0  
## mod2 -29.6       9.9  
## mod1 -78.5      15.1  
```

```r
pooled.sim = as.matrix(pooled)
unpooled.sim = as.matrix(unpooled)
m1.sim = as.matrix(mod1)
m2.sim = as.matrix(mod2)
m3.sim = as.matrix(mod3)
schools = unique(GCSE$school)

alpha2 = mean(m2.sim[,1])
alpha3 = mean(m3.sim[,1])

partial.fem2 = mean(m2.sim[,2])
partial.fem3 = mean(m3.sim[,2])
unpooled.fem = mean(unpooled.sim[,74])

par(mfrow = c(2,3), mar = c(1,2,2,1))
for (i in 1:18){
  temp = GCSE %>% filter(school == schools[i]) %>%
  na.omit()
  y = temp$course
  x = as.numeric(temp$female) - 1
  plot(x + rnorm(length(x))*0.001, y, ylim = c(35,101), xlab = "female", main = schools[i],xaxt = "n", ylab = "course" )
  axis(1,c(0,1), cex.axis = 0.8)

  # no pooling
  b = mean(unpooled.sim[,i])

  # plot lines and data
  xvals = seq(-0.1,1.1,0.01)
  lines(xvals, xvals*mean(pooled.sim[,2]) + mean(pooled.sim[,1]),col = "red") #pooled
  lines(xvals, xvals * unpooled.fem + b, col = "blue") #unpooled
  lines(xvals, xvals * partial.fem2 + (alpha2 + mean(m2.sim[,i+2])), col = "green") # varying int
  lines(xvals, xvals*(partial.fem3 + mean(m3.sim[, 2 + i*2])) + (alpha3 + mean(m3.sim[, 1 + i*2])), col = "orange") # varyin
g int and slope
  legend("bottom", legend = paste("n=", length(y), " "))
}
```


20920
n= 7
22520
n= 59
22710
n= 16
22738
n= 8
22908
n= 5
23208
n= 22
25241
n= 4
30474
n= 40
35270
n= 11
37224
n= 41
47627
n= 6
50627
n= 22
50631
n= 20
60421
n= 5
60427
n= 23



Exercise 5: Compare and contrast the regression lines estimated using these different methods. Also, based on the output of loo_compare, which model would you recommend?

On the graphs, we can see that pooled model performs significantly worse than unpooled and partial-pooling models, which are unpooled, mod2 and mod3. Since pooled ignore differences between groups and sometimes it's underfitting the group data. I would recommmend the models with varying intercept and varying slope (mod3). Based on results in loo_compare, mod3 has the highest elpd_diff. And elpd_diff is the difference in elpd_loo for 2 models (in this case each elpd_loo is compared with the model with highest elpd_loo, which is mod3). elpd_loo measures the bayesian Loo estimate of expected log pointwise predictive density. On the graph, we can see that the orange is neither overfitting nor underfitting.

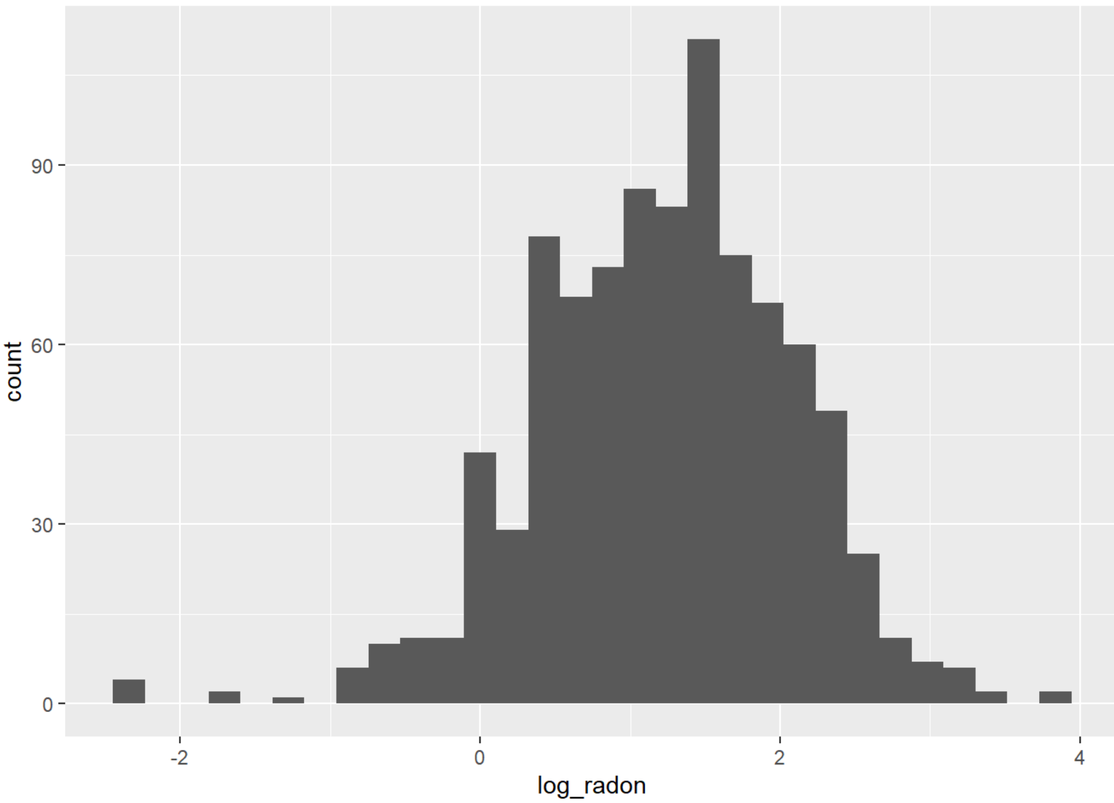
Multilevel modelling exercise

```
radon <- read.csv("radon.txt", header = T,sep="")
radon$county <- as.factor(radon$county)
```

Exercise 6: Do you think a hierachical model is warranted here? Do some EDA! Look for differences across counties.

```
# distribution of log_radon
ggplot(radon, aes(x=log_radon))+ geom_histogram()
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```

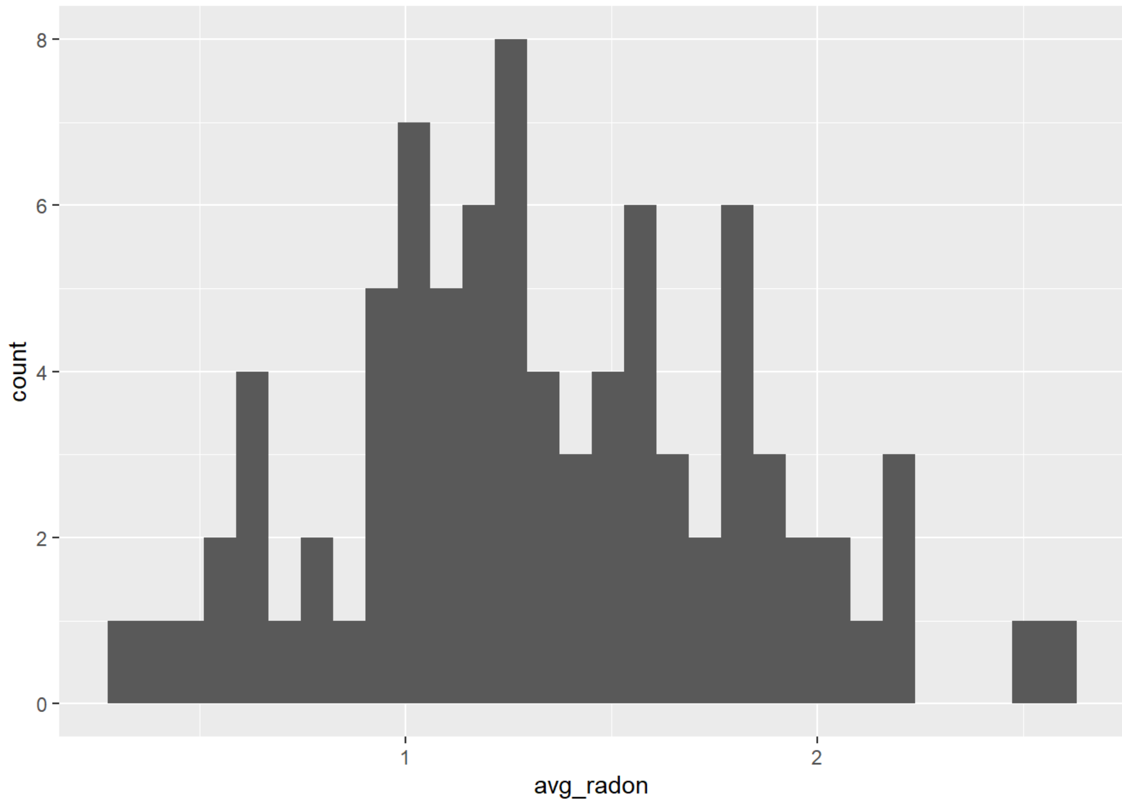



```
# for each county, calculate the sample average of the log_radon and plot the distribution of averages in a histogram
radon %>%
  group_by(county)%>%
  summarise(avg_radon = mean(log_radon)) %>%
  arrange(desc(avg_radon))%>%
  filter(row_number() ==1 | row_number() == n())
```

```
## # A tibble: 2 x 2
##   county        avg_radon
##   <fct>             <dbl>
## 1 LAC QUI PARLE     2.60
## 2 LAKE              0.322
```

```
radon %>%
  dplyr::group_by(county) %>%
  na.omit()%>%
  dplyr::summarise(avg_radon = mean(log_radon)) %>%
  dplyr::ungroup() %>%
  ggplot(aes(x = avg_radon))+
  geom_histogram()
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```



I think hierachical model is warranted here, since distribution of log_radon is normal and distribution of average log_radon by county is also approximately normal.

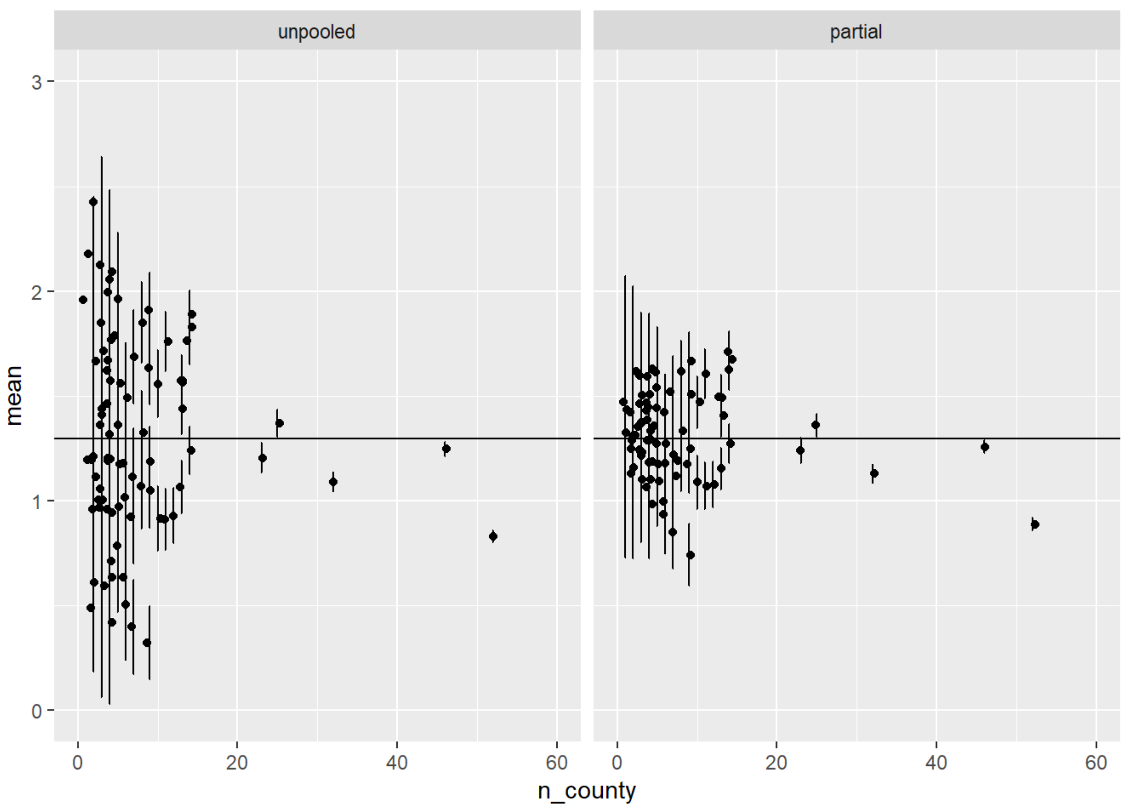
Exercise 7: Begin by creating an unpooled model. Call that model radon.unpooled. Then create a hierachical/partially-pooled model where we model each county’s intercept hierachically, again with no other predictors. Call that model radon.mod1.

```
radon.unpooled =  stan_glm(data=radon,
                           formula = log_radon ~ -1 + county,
                           refresh = 0)
radon.mod1 = stan_glmer(data =radon,
                      formula = log_radon ~ 1 + (1|county),
                      refresh = 0)
```

```r
n_county <- as.numeric(table(radon$county))
create_df <- function(sim,model){
  mean <- apply(sim,2,mean)
  sd <- apply(sim,2,sd)
  df <- cbind(n_county, mean, sd) %>%
    as.data.frame()%>%
    mutate(se = sd/ sqrt(n_county), model = model)
  return(df)
}

unpooled.sim <- as.matrix(radon.unpooled)
unpooled.df <- create_df(unpooled.sim[,1:85], model = "unpooled")


mod1.sim <- as.matrix(radon.mod1)[,1:86]
mod1.sim <- (mod1.sim[,1] + mod1.sim)[,-1]
partial.df <- create_df(mod1.sim, model = "partial")

ggplot(rbind(unpooled.df, partial.df)%>% mutate(model = factor(model, levels = c("unpooled", "partial"))), aes(x= n_county,
 y = mean)) +
  #draws the means
  geom_jitter() +
  #draws the CI error bars
  geom_errorbar(aes(ymin=mean-2*se, ymax= mean+2*se), width=.1)+
  ylim(0,3)+
  xlim(0,60)+
  geom_hline(aes(yintercept= mean(coef(radon.unpooled))))+
  facet_wrap(~model)
```

```
## Warning: Removed 6 rows containing missing values (geom_point).
```

```
## Warning: Removed 12 rows containing missing values (geom_errorbar).
```



Exercise 8:Fit a varying-intercept model, but now add the variable floor as a fixed slope. Call the model radon.mod2. For another model radon.mod3, fit a varying-intercept and varying-slope mode, again with floor as the predictor. Once you have fit these 5 models, report on the differences in model performance. Recall that we have a fourth variable which gives the county-level log uranium measurements. A really powerful aspect of hierarchical/multilevel modeling is the ability to incorporate data at different levels of coarseness. Fit a varying-intercept model, but include both floor and log_uranium in your model as well. So now we have both an individual/house-level covariate, as well as a group/county-level covariate. Group-level predictors help reduce group-level variation, which induces stronger pooling effects. Call this model radon.mod4.

```
radon.mod2 = stan_glmer(data=radon,
                        formula = log_radon ~ 1+ floor + (1|county),
                        refresh = 0)
radon.mod3 = stan_glmer(data = radon,
                        formula = log_radon ~ 1 + floor + (1+floor|county),
                        refresh = 0)
radon.mod4 = stan_glmer(data = radon,
                        formula = log_radon ~  1 + floor + log_uranium +
                          (1 + floor + log_uranium | county),
                        refresh = 0)
```

```
## Warning: There were 1 divergent transitions after warmup. Increasing adapt_delta above 0.95 may help. See
## http://mc-stan.org/misc/warnings.html#divergent-transitions-after-warmup
```

```
## Warning: Examine the pairs() plot to diagnose sampling problems
```

```
loo.unpooled = loo(radon.unpooled)
```

```
## Warning: Found 6 observation(s) with a pareto_k > 0.7. We recommend calling 'loo' again with argument 'k_threshold = 0.7'
in order to calculate the ELPD without the assumption that these observations are negligible. This will refit the model 6 ti
mes to compute the ELPDs for the problematic observations directly.
```

```
loo1 = loo(radon.mod1)
loo2 = loo(radon.mod2)
loo3 = loo(radon.mod3)
```

```
## Warning: Found 1 observation(s) with a pareto_k > 0.7. We recommend calling 'loo' again with argument 'k_threshold = 0.7'
in order to calculate the ELPD without the assumption that these observations are negligible. This will refit the model 1 ti
mes to compute the ELPDs for the problematic observations directly.
```

```
loo4 = loo(radon.mod4)
```

```
## Warning: Found 1 observation(s) with a pareto_k > 0.7. We recommend calling 'loo' again with argument 'k_threshold = 0.7'
in order to calculate the ELPD without the assumption that these observations are negligible. This will refit the model 1 ti
mes to compute the ELPDs for the problematic observations directly.
```

```
compare_models(loo.unpooled, loo1, loo2, loo3, loo4)
```

```
## Warning: 'compare_models' is deprecated.
## Use 'loo_compare' instead.
## See help("Deprecated")
```

```
## Warning: 'loo::compare' is deprecated.
## Use 'loo_compare' instead.
## See help("Deprecated")
```

```
## Model formulas:
##  :  NULL
##  :  NULL
##  :  NULL
##  :  NULL
##  :  NULL               elpd_diff se_diff
## radon.mod4       0.0       0.0
## radon.mod2      -7.3       5.6
## radon.mod3      -8.9       5.3
## radon.mod1     -54.8      12.3
## radon.unpooled -79.8      14.3
```

randon.mod4 has the highest elpd_diff and randon.unpooled has the lowest.