1. Hoff 6.1

Since $\theta \sim gamma(a_\theta, b_\theta)$, $\gamma \sim gamma(a_\gamma, b_\gamma)$

θ and γ are indep

a. $$\begin{aligned}Cov(\theta_A, \theta_B) &= E[\theta_A\theta_B] - E[\theta_A]\cdot E[\theta_B] \\ &= E[\theta^2\gamma] - E[\theta]\cdot E[\theta\gamma] \\ &= E[\theta^2]\cdot E[\gamma] - E[\theta]\cdot E[\theta]\cdot E[\gamma] \\ &= E[\theta^2]\cdot E[\gamma] - (E[\theta])^2\cdot E[\gamma] \\ &= \left(E[\theta^2] - (E[\theta])^2\right)\cdot E[\gamma] \\ &= Var[\theta]\cdot E[\gamma] = \frac{a_\theta}{b_\theta^2}\cdot\frac{a_\gamma}{b_\gamma} \\ &\neq 0\end{aligned}$$

Since $Cov(\theta_A, \theta_B) \neq 0$, θ_A and θ_B are independent.

The prior is justified if we have reason to believe that θ_B is some product of θ_A + random Gamma-distributed noise

b. The sampling model for y_A and y_B are:

$$\begin{cases} P(y_{A1}, \ldots, y_{An}|\theta) \propto \theta^{\sum_{i=1}^{n} y_{Ai}} e^{-n\theta} \\ P(y_{B1}, \ldots, y_{Bn}|\theta,\gamma) \propto (\theta\gamma)^{\sum_{i=1}^{n} y_{Bi}} e^{-n(\theta\gamma)} \end{cases} \qquad \begin{cases} \theta \sim Ga(a_\theta, b_\theta) \\ \gamma \sim Ga(a_\gamma, b_\gamma) \end{cases}$$

Joint distribution:

$$P(y_A, y_B, \theta, \gamma | a_\theta, b_\theta, a_\gamma, b_\gamma) = P(y_A|\theta)\cdot P(y_B|\theta,\gamma)\cdot P(\theta|a_\theta, b_\theta)\cdot P(\gamma|a_\gamma, b_\gamma)$$

Full conditional of θ:

$$\begin{aligned}P(\theta|y_A, y_B, \gamma) &\propto P(y_A|\theta)\cdot P(y_B|\theta,\gamma)\cdot P(\theta|a_\theta, b_\theta) \\ &\propto \theta^{\sum y_{Ai}} e^{-n_A\theta}\cdot(\theta\gamma)^{\sum y_{Bi}}\cdot e^{-n_B(\theta\gamma)}\cdot\theta^{a_\theta-1}\cdot e^{-b_\theta\theta} \\ &\propto \theta^{\sum y_{Ai}+\sum y_{Bi}+a_\theta-1} e^{-(n_A+n_B\gamma+b_\theta)\cdot\theta}\end{aligned}$$

$$\therefore \theta|y_A, y_B, \gamma \sim Gamma\left(\sum_{i=1}^{n_A} y_{Ai} + \sum_{i=1}^{n_B} y_{Bi} + a_\theta,\ n_A + n_B\gamma + b_\theta\right)$$

c. Full conditional of γ:

$$\begin{aligned}P(\gamma|y_A, y_B, \theta) &\propto P(y_B|\theta,\gamma)\cdot P(\gamma|a_\gamma, b_\gamma) \\ &\propto (\theta\gamma)^{\sum y_{Bi}} e^{-n_B(\theta\gamma)}\cdot\gamma^{a_\gamma-1}\cdot e^{-b_\gamma\cdot\gamma} \\ &\propto \gamma^{\sum y_{Bi}}\cdot e^{-n_B(\theta\gamma)}\cdot\gamma^{a_\gamma-1}\cdot e^{-b_\gamma\cdot\gamma} \\ &\propto \gamma^{\sum y_{Bi}+a_\gamma-1} e^{-(n_B\theta+b_\gamma)\gamma}\end{aligned}$$

$$\therefore \gamma|y_A, y_B, \theta \sim Gamma\left(\sum_{i=1}^{n_B} y_{Bi} + a_\gamma,\ n_B\theta + b_\gamma\right)$$

see d) in r portion of homework (page 3).